Rational points and zero-cycles on fibred varieties :

Schinzel's hypothesis and Salberger's device

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Introduction

Let $\mathbf{P}^1_k$ be the projective line over a number field k. In a series of earlier papers, rational points and zero-cycles on the total space of certain fibrations $f : X \to \mathbf{P}^1_k$ have been studied.

A first series builds upon Schinzel's hypothesis (H). This starts with [CT/S82] and goes on with [SwD94], [Se94], [CT/SwD94], [CT/Sk/SwD97b]. Taking hypothesis (H) for granted, one obtains sufficient conditions for the existence of rational points on X, as well as density results for these points (weak approximation).

A second line of investigation starts with Salberger's article [Sal88], whose work was continued in [CT/SwD94]. Here one obtains unconditional results on the existence of zero-cycles of degree one, as well as quantitative information on the Chow group of zero-cycles on X.

These techniques have originally been used in the context of pencils of conic bundles ([CT/S82], [Sal88]), pencils of Severi-Brauer varieties (Serre [Se94]) and generalizations thereof ([SwD94], [CT/SwD94]), and they have recently been applied to pencils of curves of genus one ([SwD95], [CT/Sk/SwD97b]).

In the present paper, we spell out general theorems which lie at the heart of these various papers. The proof of these theorems does not require much more effort than has already been made in [CT/SwD94], but it is interesting to see how far one can get. The main results are Theorem 1.1 (rational points, assuming (H)), Theorems 2.2.1 and 2.2.2 (rational points), Theorem 4.1 (zero-cycles of degree one) and Theorem 4.8 (Chow groups of zero-cycles). A key technical assumption has to do with the reducible fibres : for each closed point $M \in \mathbf{P}^1_k$, we want the fibre $X_M = f^{-1}(M)$ to contain at least one irreducible component Y_M of multiplicity one such that the algebraic closure of the residue field $k(M)$ in the function field of Y_M is abelian over $k(M)$.

In Section 3, we give a self-contained version of a device originally due to Salberger [Sal88]. Our presentation makes it clear that this device is a perfect analogue of Schinzel's hypothesis [Sch/Sie58], leading to actual proofs of existence of zero-cycles of degree one ([Sal88]), whereas Schinzel's hypothesis leads to conditional proofs of the existence of rational points ([CT/S82], [CT/SwD94]). The parallel between the two methods was blatant in [CT/SwD94], but it is only in the present paper that the quintessence of Salberger's device is revealed.

Since much of this text is an elaboration of the previous paper [CT/SwD94], we refer to that paper, and in particular to its introduction, for motivation, historical background and some standard arguments. We also refer to that paper for the statement of Schinzel's hypothesis (H) (over the rationals) and its extension (H_1) to number fields (pointed out by Serre, see [CT/SwD94], Section 4).

Notation and preliminaries

Let k be a field and $\overline{k}$ an algebraic closure of k. We let $\mathbf{A}^1_k = \mathrm{Spec}(k[t])$, resp. $\mathbf{P}^1_k$, denote the affine line, resp. the projective line over k. Given X a k-variety, i.e. a separated k-scheme of finite type, we let $\overline{X} = X \times_k \overline{k}$. Given a field extension K/k, we let $X_K = X \times_k K$. If X is integral, we let $k(X)$ denote its function field.

Following [Sk96], we shall say that a k-variety Y is split if it contains a non-empty smooth open set U which is geometrically integral over k (i.e. U is integral and k is algebraically closed in $k(U)$). The k-variety Y is geometrically split if $\overline{Y}/\overline{k}$ is split : such is the case if and only if Y contains a non-empty smooth open set.

Let X/k be a smooth, proper, geometrically integral k-variety, and let $f : X \to \mathbf{P}^1_k$ be a flat k-morphism. Its generic fibre X_η is regular; if char$(k) = 0$, it is therefore smooth over the function field $k(t)$ of $\mathbf{P}^1_k$. Given M a closed point of $\mathbf{P}^1_k$, with residue field $k(M)$, we let $X_M = f^{-1}(M)/k(M)$ denote the fibre of f at M. The $k(M)$-variety X_M is split if and only if if there exists at least one irreducible component Y_M of X_M which is generically reduced (i.e. of multiplicity one in the divisor $X_M \subset X$) and such that $k(M)$ is algebraically closed in the function field of Y_M.

For basics on the Brauer group, the reader is referred to [Gr68] and to Section 1 of [CT/SwD94]. The vertical subgroup $\mathrm{Br}_{\mathrm{vert}}(X)$ of the Brauer group $\mathrm{Br}(X)$ of X with respect to f is defined by the equality :

$$\mathrm{Br}_{\mathrm{vert}}(X) = \mathrm{Br}(X) \cap f^*(\mathrm{Br}(k(\mathbf{P}^1))) \subset \mathrm{Br}(X) \subset \mathrm{Br}(k(X)).$$

Assume char(k)=0. If all fibres of f are geometrically split, i.e. if each fibre of $\overline{f} : \overline{X} \to \mathbf{P}^1_{\overline{k}}$ contains a component of multiplicity one, then the quotient of $\mathrm{Br}_{\mathrm{vert}}(X)$ by the image of $\mathrm{Br}(k)$ is *finite* (cf. [Sk96], Cor. 4.5). It is actually enough to assume that all geometric fibres but one possess a component of multiplicity one.

Given a number field k, we let $\Omega = \Omega_k$ denote the set of places of k and we let k_v be the completion of k at the place v. Given a finite place v of k, we let O_v be the ring of integers of k_v and $\mathbf{F}_v$ be the (finite) residue field. For a proper k-variety Y, the topological space of adèles $Y(\mathbb{A}_k)$ of Y coincides with the product $\prod_{v\in\Omega} Y(k_v)$ equipped with the product topology.

For any element $A \in \mathrm{Br}(X)$, the map which sends the adèle $\{p_v\}$ to $\sum_{v\in\Omega} \mathrm{inv}_v(A(p_v)) \in \mathbf{Q}/\mathbf{Z}$ is a continuous function $\theta_A : X(\mathbb{A}_k) \to \mathbf{Q}/\mathbf{Z}$ with finite image. Given $B \subset \mathrm{Br}(X)$, we let $X(\mathbb{A}_k)^B \subset X(\mathbb{A}_k)$ denote the closed subset which is the intersection of the kernels $\theta_A^{-1}(0)$ for $A \in B$. By global reciprocity, we have $X(k) \subset X(\mathbb{A}_k)^B$. The (Brauer-Manin) B-obstruction to the existence of a rational point is the condition $X(\mathbb{A}_k)^B = \emptyset$. For $f : X \to \mathbf{P}^1_k$ as above, the $\mathrm{Br}_{\mathrm{vert}}(X)$-obstruction $X(\mathbb{A}_k)^{\mathrm{Br}_{\mathrm{vert}}} = \emptyset$ is simply called the vertical obstruction. If the image of B in the quotient $\mathrm{Br}(X)/\mathrm{Br}(k)$ is finite, then $X(\mathbb{A}_k)^B$ is open in $X(\mathbb{A}_k)$. For a more detailed discussion of the Brauer-Manin obstruction, see [CT/SwD94], Section 3, and [CT97].

§1. Local-global properties for rational points, conditional on Schinzel's hypothesis

The following theorem extends Theorem 4.2 of [CT/SwD94], to which we shall refer for some standard arguments.

Theorem 1.1 *Let X be a smooth, projective, geometrically integral variety over a number field k, equipped with a flat k-morphism $f : X \to \mathbf{P}^1_k$ with (smooth) geometrically integral generic fibre. Assume :*

(i) For each closed point $M \in \mathbf{P}^1_k$, there exists a multiplicity one irreducible component $Y_M \subset X_M$ such that the algebraic closure of $k(M)$ in the function field of Y_M is an abelian extension of $k(M)$.

(ii) Schinzel's hypothesis (H) holds.

Let $\mathcal{R} \subset \mathbf{P}^1(k)$ be the set of k-points m with smooth fibre X_m such that $X_m(\mathbb{A}_k) \neq \emptyset$. Then :

(a) The closure of $\mathcal{R}$ in $\mathbf{P}^1(\mathbb{A}_k)$ coincides with $f(X(\mathbb{A}_k)^{\mathrm{Br}_{\mathrm{vert}}})$.

(b) Assume $X(\mathbb{A}_k)^{\mathrm{Br}_{\mathrm{vert}}} \neq \emptyset$. Then for any finite set $S \subset \Omega$, the closure of $\mathcal{R}$ in $\prod_{v\in S} \mathbf{P}^1(k_v)$ contains a non-empty open set. In particular $\mathcal{R}$ is Zariski-dense in $\mathbf{P}^1_k$.

(c) Assume $X(\mathbb{A}_k)^{\mathrm{Br}_{\mathrm{vert}}} \neq \emptyset$. Then there exists a finite set $S_0 \subset \Omega$ such that for any finite set $S \subset \Omega$ with $S \cap S_0 = \emptyset$, the closure of $\mathcal{R}$ under the diagonal embedding $\mathbf{P}^1(k) \to \prod_{v \in S} \mathbf{P}^1(k_v)$ coincides with $\prod_{v\in S} f(X(k_v))$.

(d) Assume that the Hasse principle holds for smooth fibres of f. Then $f(X(k))$ is dense in $f(X(\mathbb{A}_k)^{\mathrm{Br}_{\mathrm{vert}}})$: the vertical Brauer-Manin obstruction to the existence of a rational point on X is the only obstruction. Assume moreover $X(\mathbb{A}_k)^{\mathrm{Br}_{\mathrm{vert}}} \neq \emptyset$. Then for any finite set $S \subset \Omega$, the closure of $f(X(k))$ in $\prod_{v\in S} \mathbf{P}^1(k_v)$ contains a non-empty open set. Moreover there exists a finite set $S_0 \subset \Omega$ such that for any finite set $S \subset \Omega$ with $S \cap S_0 = \emptyset$, the closure of $f(X(k))$ in $\prod_{v\in S} \mathbf{P}^1(k_v)$ coincides with $\prod_{v\in S} f(X(k_v))$. In particular, $f(X(k))$ is Zariski-dense in $\mathbf{P}^1_k$.

(e) Assume that the Hasse principle and weak approximation hold for smooth fibres of f. Then the vertical Brauer-Manin obstruction to weak approximation on X is the only obstruction : The closure of $X(k)$ in $X(\mathbb{A}_k)$ coincides with the open and closed set $X(\mathbb{A}_k)^{\mathrm{Br}_{\mathrm{vert}}}$.

Proof. Let $U \subset \mathbf{P}^1_k$ be the complement of the set of points M whose fibre X_M is singular. Thus the restriction $f_U : X_U \to U$ is smooth. Given $A \in \mathrm{Br}_{\mathrm{vert}}(X)$, by definition one may find $\beta_A \in \mathrm{Br}(k(\mathbf{P}^1))$ such that $f^*(\beta_A) = A \in \mathrm{Br}(k(X))$. An easy computation with residues ([CT/SwD94], Prop. 1.1.1) shows that β_A lies in $\mathrm{Br}(U)$. Let $m \in U(k)$, assume $X_m(\mathbb{A}_k) \neq \emptyset$, and let $\{p_v\} \in X_m(\mathbb{A}_k)$ be an arbitrary adèle. We then have

$$\sum_{v\in\Omega} \mathrm{inv}_v(A(p_v)) = \sum_{v\in\Omega} \mathrm{inv}_v(\beta(m))$$

and this last sum is zero by the reciprocity law of class field theory. We thus have $\mathcal{R} \subset f(X(\mathbb{A}_k)^{\mathrm{Br}_{\mathrm{vert}}})$. This last set, which is the image under f of the compact set $X(\mathbb{A}_k)^{\mathrm{Br}_{\mathrm{vert}}}$, is closed in $\mathbf{P}^1(\mathbb{A}_k)$. Thus the closure of $\mathcal{R}$ in $\mathbf{P}^1(\mathbb{A}_k)$ lies in $f(X(\mathbb{A}_k)^{\mathrm{Br}_{\mathrm{vert}}})$. The main result in the above theorem is (a), which claims that under hypotheses (i) and (ii), this inclusion is an equality.

Assumption (i) implies that all fibres of f are geometrically split, hence, as recalled in the introduction, the quotient $\mathrm{Br}_{\mathrm{vert}}(X)/\mathrm{Br}(k)$ is finite. Let $A_l \in \mathrm{Br}(X)$, $l = 1, \ldots, n$ be a finite set of representatives for the elements of $\mathrm{Br}_{\mathrm{vert}}(X)/\mathrm{Br}(k)$. By a good reduction argument, there exists a finite set S_0 of places of k, which we may assume to contain all archimedean places, such that for any $v \notin S_0$, any point $p_v \in X(k_v)$, and any $l \in \{1, \ldots, n\}$, we have $A_l(p_v) = 0 \in \mathrm{Br}(k_v)$.

If $X(\mathbb{A}_k)^{\mathrm{Br}_{\mathrm{vert}}} = \emptyset$, there is nothing to prove. Hence we shall assume $X(\mathbb{A}_k)^{\mathrm{Br}_{\mathrm{vert}}} \neq \emptyset$. Let $\{p_v\} \in X(\mathbb{A}_k)^{\mathrm{Br}_{\mathrm{vert}}}$, i.e. assume that for all $A \in \mathrm{Br}_{\mathrm{vert}}(X)$ we have

$$\sum_{v\in\Omega} \mathrm{inv}_v(A(p_v)) = 0. \tag{1.1}$$

This is equivalent to assuming

$$\forall l \in \{1, \ldots, n\}, \quad \sum_{v\in\Omega} \mathrm{inv}_v(A_l(p_v)) = 0, \tag{1.2}$$

which in turn is equivalent to

$$\forall l \in \{1, \ldots, n\}, \quad \sum_{v\in S_0} \mathrm{inv}_v(A_l(p_v)) = 0. \tag{1.3}$$

Note that in (1.1), (1.2) and (1.3) we are free to replace p_v for $v \notin S_0$ by an arbitrary point in $X(k_v)$. Set $\gamma_v(l) = A_l(p_v) \in \mathrm{Br}(k_v)$.

Let $m_v = f(p_v) \in \mathbf{P}^1(k_v)$ for $v \in \Omega$, and let $S \subset \Omega$ be a finite set of places containing S_0. For each $v \in S$, let $N_v \subset \mathbf{P}^1(k_v)$ be a neighbourhood of m_v. We now use the continuity of the evaluation maps $X(k_v) \to \mathbf{Q}/\mathbf{Z}$ given by $p \mapsto \mathrm{inv}_v(A(p))$ and the implicit function theorem for the map $X_U(k_v) \to U(k_v)$ induced by the smooth map $f_U : X_U \to U$. For each place $v \in \Omega$, this enables us to find a non-empty open subset $W_v \subset N_v \cap U(k_v)$ (which need not contain m_v) such that the following properties hold : the map $X_U(k_v) \to U(k_v)$ admits an analytic section $\sigma_v : W_v \to X_U(k_v)$ over W_v, hence in particular $f^{-1}(W_v) \neq \emptyset$; on W_v, each β_{A_l} ($l \in \{1, \dots, n\}$) takes the constant value $\gamma_v(l)$, hence each A_l takes the constant value $\gamma_v(l)$ on $f^{-1}(W_v) \subset X(k_v)$.

We shall show that there exists $m \in U(k)$ satisfying $X_m(\mathbb{A}_k) \neq \emptyset$ and such that m lies in W_v for $v \in S$. This will prove (a).

At this point, in (1.1), (1.2), (1.3), we may replace p_v by an arbitrary point in $f^{-1}(W_v)$ for $v \in S$, and by an arbitrary point in $X(k_v)$ for $v \notin S_0$. The proof below will therefore give statements (b) and (c) at the same stroke. Statement (d) is an obvious reformulation of the previous statements in the case where the Hasse principle holds for the fibres of f. As for (e), let $\{p_v\} \in X(\mathbb{A}_k)^{\mathrm{Br}_{\mathrm{vert}}}$, and let S be a finite set of places containing S_0. For each place $v \in S$, let us choose W_v and the analytic section σ_v above such that $\sigma_v(W_v)$ contains a new point p_v very close to the original point p_v. The proof of (a) will enable us to find a point $m \in U(k)$ such that $X_m(\mathbb{A}_k) \neq \emptyset$ and such that for each $v \in S$ the point $\sigma_v(m)$ and the new point p_v are very close on $X(k_v)$. If the Hasse principle and weak approximation hold for X_m, one may then find a k-point on X_m which is very close to each $\sigma_v(m) \in X_m(k_v)$ for each $v \in S$, hence is very close to the original point $p_v \in X(k_v)$ for each $v \in S$.

Let us now proceed with the proof of (a).

For each real place v, we may assume that W_v is a connected open interval of $k_v \simeq \mathbf{R}$. For each such real place v, let us choose three distinct points $a_v, b_v, c_v \in W_v \setminus m_v$ such that m_v lies between a_v and c_v, and such that b_v lies between a_v and m_v. For each complex place v, let us choose $c_v \in W_v \setminus m_v$. Using the weak approximation theorem on $\mathbf{P}^1_k$, we may find four distinct k-points a, b, m, c in $\mathbf{P}^1(k)$ lying in each W_v for v non-archimedean and close enough to each a_v, b_v, m_v, c_v for v real that the same inclusions hold between the images of a, b, m, c in k_v as for a_v, b_v, m_v, c_v.

We now choose c as the point at infinity on $\mathbf{P}^1_k$, and normalize the coordinate t on $\mathrm{Spec}(k[t]) = \mathbf{A}^1_k = \mathbf{P}^1_k \setminus \infty$ so that $a = 0$ and $b = 1$. Since the places $v \notin S_0$ do not play a rôle in (1.1), (1.2), (1.3), while keeping $f(p_v)$ in W_v for each place $v \in \Omega$ we may and shall assume $m_v = f(p_v) \neq \infty$ for all $v \in \Omega$. For later use, let us replace U by $U \setminus \infty$.

Let $P(t)$ be the monic separable polynomial which describes all the closed points $M \in \mathbf{A}^1_k$ such that the fibre X_M is not split (note that the fibre X_∞ is smooth, hence split). This polynomial may be written as a product of irreducible monic polynomials $P(t) = \prod_{i=1}^n P_i(t)$. Each of them corresponds to a closed point M_i of the affine line. Use assumption (i) : for each such M_i, let $k_i = k(M_i)$ and let Z_i/k_i be a multiplicity one irreducible component of X_{M_i} such that the algebraic closure K_i of k_i in the function field of Z_i is an abelian extension of k_i. Write K_i/k_i as a composite of cyclic extensions $K_{i,j}/k_i$. Let $e_i \in k_i$ denote the class of t under the identification $k_i = k[t]/P_i(t)$. Let $k(t)$ be the function field of $\mathbf{A}^1_k = \mathrm{Spec}(k[t])$. Define $A_{i,j} \in \mathrm{Br}(k(t))$ by

$$A_{i,j} = \mathrm{Cores}_{k_i/k}(K_{i,j}/k_i, t - e_i).$$

We are making here a slight abuse of notation : in order to define an element in the Brauer group, one needs to fix a generator of the Galois group of the cyclic extension $K_{i,j}/k_i$. Here and later on we shall always assume that this has been done.

Let us now increase the finite set S of places so that it also includes finite primes where some P_i is not integral, finite primes where all P_i are integral but the reduction of the product $P(t) = \prod_i P_i(t)$ is not separable, finite places above a prime p less than or equal to the degree of the polynomial $N_{k/\mathbf{Q}}(\prod_i P_i(t))$ (this will be required when applying (H_1), see [CT/SwD94], beginning of Section 4) and primes ramified in one of the extensions $K_{i,j}/k$.

Let O denote the ring of integers of k with primes in S inverted, and let O_i denote the integral closure of O in K_i. By enlarging S we may also assume that the fibration $f: X \to \mathbf{P}^1_k$ extends to a projective, flat fibration $g: \mathcal{X} \to \mathbf{P}^1_O$. Let $T \subset \mathbf{P}^1_O$ be the closed subset defined by $P(t) = 0$ in $\mathbf{A}^1_O$, and let T_i be the analogous closed subset defined by $P_i(t) = 0$. By enlarging S, we may assume that the points of $\mathbf{P}^1_O$ with non-split fibre are all contained in T (note that the set of points of $\mathbf{P}^1_O$ whose fibre is geometrically split is the projection of the smooth locus of the morphism g).

Lemma 1.2 *Let k be a number field and let $\mathrm{Spec}(O)$ be an open set of the ring of integers of k. Let $g: \mathcal{X} \to \mathbf{P}^1_O$ be a flat, projective morphism with $\mathcal{X}$ regular and smooth over O. Let $f: X \to \mathbf{P}^1_k$ be the restriction of g over $\mathrm{Spec}(k) \subset \mathrm{Spec}(O)$. Let $T \subset \mathbf{P}^1_O$ be a closed subset, finite and étale over O, such that fibres of g above points not in T are split. Let $T = \cup_{i \in I} T_i$ be the decomposition of T into irreducible closed subsets, and let k_i be the field of fractions of T_i.*

After inverting finitely many primes in O, the following holds.

(a) Given any closed point $u \in \mathbf{P}^1_O$, if the fibre $\mathcal{X}_u$ over the finite field $\kappa(u)$ is split, then it contains a smooth $\kappa(u)$-point.

(b) Given any rational point $m \in \mathbf{P}^1(k)$, with closure $\mathrm{Spec}(O) \simeq \tilde{m} \subset \mathbf{P}^1_O$, if $u \in \tilde{m} \subset \mathbf{P}^1_O$ is a closed point such that $\mathcal{X}_u/\kappa(u)$ is split, then X_m contains a smooth k_v-point.

(c) Let u belong to one of the T_i's, thus defining a place v_i of k_i. Assume that there exists a component Z of the fibre of f at $T_i \times_O k = \mathrm{Spec}(k_i)$ which has multiplicity one. Let K_i denote the algebraic closure of k_i in the function field of Z. If the place v_i splits completely in the ring of integers $O_i \subset K_i$, then $\mathcal{X}_u/\kappa(u)$ is split.

(d) Assume that for each i there exists at least one component of $f^{-1}(\mathrm{Spec}(k_i))$ which has multiplicity one. Then given any finite field extension L/k, there exist infinitely many places v of k which are totally decomposed in L and are such that for any finite extension K_w of k_v the induced map $f: X(K_w) \to \mathbf{P}^1(K_w)$ is surjective.

The proof is postponed. In the case under consideration here, we have $T_i = \mathrm{Spec}(O[t]/P_i(t))$. We shall henceforth assume that S is chosen big enough so that the statements in Lemma 1.2 hold for $O = O_S$ the ring of integers of k where finite primes of S have been inverted.

By assumption, for all $A \in \mathrm{Br}_{\mathrm{vert}}(X)$, we have

$$\sum_{v \in \Omega} \mathrm{inv}_v(A(p_v)) = 0.$$

In particular, for the adèle $\{p_v\} \in X(\mathbb{A}_k)$, there is no Brauer-Manin obstruction to weak approximation attached to the intersection of the Brauer group of X with the subgroup of $\mathrm{Br}(k(X))$ spanned by the images of the $A_{i,j}$'s under the natural map $\mathrm{Br}(k(t)) \to \mathrm{Br}(k(X))$. A key result in Harari's thesis ([Ha94], Cor. 2.6.1, reproduced as [CT/SwD94], Thm. 3.2.1), then ensures that there exist a *finite* set of places S_1 of k containing S, and points $p'_v \in X_U(k_v)$ for $v \in S_1$, with $p'_v = p_v$ for $v \in S$, such that for all $\{i,j\}$,

$$\sum_{v \in S_1} \mathrm{inv}_v(A_{i,j}(m'_v)) = 0 \tag{1.4}$$

where as above $m'_v = f(p'_v)$.

At this point let us modify the open sets W_v for $v \in S_1 \setminus S$. For such v, we choose open neighbourhoods $W_v \subset U(k_v)$ of m'_v such that the map $X_U(k_v) \to U(k_v)$ admits an analytic section over W_v.

For each $v \in S_1$, let us denote by $\lambda_v \in k_v$ the t-coordinate of m'_v. We can now copy most of the proof of Theorem 4.2 from [CT/SwD94]. Applying Schinzel's hypothesis (H), or rather its consequence (H_1) ([CT/SwD94], Prop. 4.1), we produce a parameter $\lambda \in k$ lying in each

W_v for $v \in S_1$ (for v a complex place, this holds as soon as the absolute value $|\lambda|_v$ is big enough), satisfying $\lambda_v > 0$ for v real, integral away from S_1, such that each $P_i(\lambda)$ is a unit in the completion k_v for each finite place $v \notin S_1$ except perhaps in one finite place v_i where $P_i(\lambda)$ is a uniformizing parameter. Let $m \in U(k)$ be the point with t-coordinate λ.

The fibre X_m is smooth. For $v \in S_1$, it contains the k_v-point $\sigma_v(m)$.

If $v \notin S_1$ is a finite place distinct from the place v_i, then $P_i(\lambda)$ is a unit at v. For any place $v \notin S_1 \cup v_1 \cup \ldots \cup v_n$, Lemma 1.2 (b) then shows that $X_m(k_v)$ is not empty.

By good reduction one gets $\mathrm{inv}_v(A_{i,j}(\lambda)) = 0$ for each finite place $v \notin (S_1 \cup v_i)$. By the choice we made of the three points $0, 1, \infty$, for each real place v, none of the $A_{i,j}(t)$ has a pole on the open connected interval $0 < t < \infty$ of $k_v = \mathbf{R}$. In particular for each such v, each $\mathrm{inv}_v(A_{i,j}(t))$ is constant on $]0, \infty[\subset k_v$, and for λ as above, we have $\mathrm{inv}_v(A_{i,j}(\lambda)) = \mathrm{inv}_v(A_{i,j}(m'_v))$. (The present choice of $0, 1, \infty$ and the argument just given should have been made in [CT/SwD94] to ensure that (iii) p. 76 op. cit. holds at the real places.)

Using (1.4) above, the remarks just made and the global reciprocity law of class field theory as in [CT/SwD94, p. 76], one then gets $\mathrm{inv}_{v_i}(A_{i,j}(\lambda)) = 0$, i.e. :

$$\mathrm{inv}_{v_i}(\mathrm{Cores}_{k_i/k}(K_{i,j}/k_i, \lambda - e_i)) = 0,$$

which is equivalent to :

$$\sum_{w \in \Omega_{k_i}, w | v_i} \mathrm{inv}_w(K_{i,j}/k_i, \lambda - e_i) = 0.$$

Now $(\lambda - e_i) \in k_i$ is integral at all places w above v_i, and the norm of that element from k_i to k has v_i-adic valuation one in k. This implies that in the decomposition of $k_i \otimes_k k_{v_i}$ into a product of local fields $k_{i,w}$, $\lambda - e_i$ goes to a unit into all local fields $k_{i,w}$ but one, call it k_{i,w_i}, which is of degree one over k_{v_i} and in which $\lambda - e_i$ becomes a uniformizing parameter. In the above sum, all terms but one then vanish. Thus the remaining one $\mathrm{inv}_{w_i}(K_{i,j}/k_i, \lambda - e_i)$ also vanishes. Since $\lambda - e_i$ is a uniformizing parameter of k_{i,w_i} and since $K_{i,j}/k_i$ is unramified at w_i, this implies that w_i is totally split in the Galois extension $K_{i,j}/k_i$. Applying this for each j, we conclude that w_i is totally split in the composite extension K_i/k_i. By (b) and (c) of Lemma 1.2 we thus have $X_m(k_{v_i}) \neq \emptyset$.

Thus X_m has points in all completions of k, which completes the proof since m lies in W_v for each $v \in S$. □

Proof of Lemma 1.2 Let $T_1 \subset \mathbf{P}^1_O$ be a proper closed subset, containing T and all the points with singular fibre, equipped with its reduced subscheme structure. By shrinking $\mathrm{Spec}(O)$, we may assume that T_1 is finite and étale over O, and by Lemma 1.3 below we may assume that (a) holds for all closed points in T_1. The restriction h of the fibration $g : \mathcal{X} \to \mathbf{P}^1_O$ over the complement of T_1 is projective and smooth, hence flat. The Hilbert polynomial is constant in projective flat families with connected parameter space. Hence all the fibres of h are geometrically integral, projective and smooth of fixed dimension d and degree n in a projective space of a fixed dimension r. The fibre over a closed point $u \notin T_1$ is a projective, smooth, geometrically integral variety over the finite field $\kappa(u)$. By the Lang-Weil estimates [L/W54], if the order of $\kappa(u)$ is bigger than a constant depending only on d, n, r, such a fibre has a (smooth) $\kappa(u)$-point. We now invert the finitely many primes in O whose residue field is smaller than this constant. This proves (a). Statement (b) then follows by an application of Hensel's lemma. Statement (c) is a purely algebraic statement which follows from the definition of split fibres.

With $\mathrm{Spec}(O)$ as above, let us prove (d). Tchebotarev's density theorem implies that there exist infinitely many primes v in $\mathrm{Spec}(O)$ which are totally split in the compositum over k of L and the fields K_i. Given any closed point $u \in \mathbf{P}^1_O$ above such a place v, the fibre $\mathcal{X}_u/\kappa(u)$ is split by (c), hence contains a smooth $\kappa(u)$-point by (a). Let K_w/k_v be a finite extension,

O_w the ring of integers of K_w and $\mathbf{F}_w/\mathbf{F}_v$ the extension of residue fields. Let m be any point in $\mathbf{P}^1(K_w) = \mathbf{P}^1(O_w)$. Let $u \in \mathbf{P}^1_O$ be the closed point above v which is the image of the specialization of m at w under the map $\mathbf{P}^1_{O_w} \to \mathbf{P}^1_O$. The K_w-variety X_m has a proper model over the ring of integers O_w whose special fibre $\mathcal{X}_u \times_{\mathbf{F}_v} \mathbf{F}_w$ has a smooth $\mathbf{F}_w$-point. By Hensel's lemma, X_m contains a (smooth) K_w-point : the map $X(K_w) \to \mathbf{P}^1(K_w)$ is surjective. □

It remains to prove :

Lemma 1.3 *Let* Spec(O) *be a non-empty open set of the spectrum of the ring of integers of a number field* k*, and let* $\mathcal{X} \to$ Spec(O) *be a flat, quasiprojective* O*-scheme. Let* X/k *be the generic fibre of* $\mathcal{X} \to$ Spec(O)*. Then there exists a finite set* S *of points of* Spec(O) *such that the following holds. If* v *is a closed point of* Spec(O) *not in* S*, and if the fibre* $\mathcal{X}_{\kappa(v)}/\kappa(v)$ *is split, then* $\mathcal{X}_{\kappa(v)}$ *contains a smooth* $\kappa(v)$*-point and* X *contains a smooth* k_v*-point.*

Proof Let $\mathcal{U} \subset \mathcal{X}$ be the biggest open set of $\mathcal{X}$ on which the induced map $\mathcal{U} \to$ Spec(O) is smooth. Then $\mathcal{U}$ is regular. Let us decompose it into a finite union of disjoint integral open sets : $\mathcal{U} = \bigcup_{i \in I} \mathcal{U}_i$. For each $i \in I$, let $\mathcal{U}_i \to$ Spec$(O_i) \to$ Spec(O) be the Stein factorization of the map $\mathcal{U}_i \to$ Spec(O). Since $\mathcal{U}_i$ is regular, each ring O_i is a Dedekind domain and is quasifinite over O. By definition, the quasiprojective map $\mathcal{U}_i \to$ Spec(O_i) has smooth, geometrically integral fibres. Because of the Lang-Weil estimates [L/W54], there exists a finite set S of places of k such that for any closed point w of Spec(O_i) not above S, the fibre of $\mathcal{U}_i \to$ Spec(O_i) has a smooth $\kappa(w)$-point.

Let now $v \notin S$ be a point of Spec(O) whose fibre is split. This implies that the fibre of $\mathcal{U} \to$ Spec(O) above v is not empty, and that there exists at least one $i \in I$ and $v_i \in$ Spec(O_i) above v of degree one. It is then clear that the fibre $\mathcal{X}_v/\kappa(v)$ contains a smooth $\kappa(v)$-point. By Hensel's lemma, such a point can be lifted to a smooth k_v-point of X. □

Remark 1.4 : The geometric conditions on reducible fibres

Hypothesis (i) of Theorem 1.1 implies in particular that each fibre contains a component of multiplicity one. It seems quite unlikely that one can get results without an assumption of that kind (for example, in the case when there are at least 5 double fibres, then the k-rational points cannot be Zariski-dense, see [CT/Sk/SwD97a]). However, in many interesting cases, for example if the generic fibre is $k(t)$-birational to a homogeneous space of a connected linear algebraic group, or if it is a rational surface, it automatically possesses a $\overline{k}(t)$-point, hence the multiplicity one condition is satisfied.

The abelianness condition in hypothesis (i) is a much more serious problem. Can one dispense with it ? The problem is to find a substitute for the algebras $A_{i,j} = \mathrm{Cores}_{k_i/k}(K_{i,j}/k_i, t - e_i)$ in the above proof. If we could get rid of the abelianness condition, then, under (H), we would have a proof that the (vertical) Brauer-Manin obstruction is the only obstruction for $X/\mathbf{P}^1_k$ when the generic fibre is a projective homogenous space under a connected linear algebraic group G over $k(\mathbf{P}^1_k)$. Indeed, the Hasse principle is known to hold for such varieties over a number field (Harder). This would generalize Theorem 4.2 of [CT/SwD94] (see Remark 4.2.1 of [CT/SwD94]). We would have similar results for pencils of Del Pezzo surfaces of degree 6.

Remark 1.5 : The arithmetic hypotheses on smooth fibres

To prove (d) and (e), it is enough to assume that the Hasse principle (resp. weak approximation) hold for fibres of f over rational points in a non-empty Zariski open set $U \subset \mathbf{P}^1_k$.

Let U be a non-empty Zariski open set of $\mathbf{P}^1_k$ Assume hypotheses (i) and (ii) of Theorem 1.1. Suppose that for each $m \in U(k)$ with smooth fibre X_m, the closure of $X_m(k)$ in $X_m(\mathbb{A}_k)$ coincides with $X_m(\mathbb{A}_k)^{\mathrm{Br}(X_m)}$. Does it follow that the closure of $X(k)$ in $X(\mathbb{A}_k)$ coincides with $X(\mathbb{A}_k)^{\mathrm{Br}(X)}$? If there is no Brauer-Manin obstruction to the Hasse principle (resp. weak approximation) for X at all (i.e. for the whole group Br(X)), and if one already knows that the

Brauer-Manin obstruction to the Hasse principle (resp. weak approximation) is the only one for the smooth fibres, under hypotheses (i) and (ii) of Theorem 1.1, can one conclude that $X(k) \neq \emptyset$ (resp. X satisfies weak approximation) ?

Under suitable additional conditions on the geometry of the generic fibre, results of this kind have been obtained by Harari [Ha94] in the special case where *all* fibres over $\mathbf{A}^1_k$ are geometrically integral.

To conclude this remark, and to give a measure of our ignorance, let us point out that even under Schinzel's hypothesis (H), we do not know whether the Brauer-Manin obstruction to the Hasse principle is the only obstruction for (smooth projective models of) varieties given by an affine equation

$$N_{K/k}(x_1\omega_1 + \ldots + \omega_n x_n) = P(t)$$

with K/k a biquadratic extension, i.e. a Galois extension with $\mathrm{Gal}(K/k) = (\mathbf{Z}/2)^2$, and $P(t) \in k[t]$ a polynomial of degree at least two.

Examples 1.6

In order to apply Theorem 1.1 to a pencil $X/\mathbf{P}^1_k$, one needs to check the abelianness condition (i). The assumption and the conclusion of the theorem do not change if f is replaced by a fibration f' which is birationally equivalent to f over $\mathbf{P}^1_k$.

If the generic fibre of f is a Severi-Brauer variety, there exist explicit models $X/\mathbf{P}^1_k$ (Artin models, see Frossard [Fr96b]) : for these models, assumption (i) of Theorem 1.1 is satisfied. From Schinzel's hypothesis we thus get statements (a) to (e) in the theorem for Artin models, hence also for all other models. This special case of Theorem 1.1 was established in Section 4 of [CT/SwD94].

A case where condition (i) is immediate is that of a suitable model $X/\mathbf{P}^1_k$ of a variety given by an affine equation

$$N_{K/k}(x_1\omega_1 + \ldots + \omega_n x_n) = P(t)$$

with K/k an abelian extension, $\omega_1, \ldots, \omega_n$ a basis of the extension K/k, and $P(t)$ a separable polynomial of degree at least two. We let $f : X \to \mathbf{P}^1_k$ be a model extending the projection to the t coordinate. Schinzel's hypothesis gives statement (a) to (c) in the theorem. If moreover K/k is cyclic, then the Hasse principle and weak approximation hold for the smooth fibres (over the k-points of a non-empty Zariski open set) hence Schinzel's hypothesis also gives (d) and (e). In this case the generic fibre is actually birational to a Severi-Brauer variety, and we recover the case handled in [CT/SwD94].

One may check the abelianness condition in the case of a pencil of 2-dimensional quadrics ([Sk90b]). In this case, Schinzel's hypothesis gives statements (a) to (e) (see [CT/SwD94], Theorem 4.2).

In the paper [SwD95], a special one-parameter family of curves of genus one $X/\mathbf{P}^1_k$ is studied. It has exactly three singular fibres X_M, each of them integral and of multiplicity one; for each such closed point M, the algebraic closure of $k(M)$ in the function field of X_M is a quadratic extension of $k(M)$. Statements (a), (b) and (c) in Theorem 1.1 therefore apply. In the case considered in [SwD95], one has $\mathrm{Br}_{\mathrm{vert}}(X)/\mathrm{Br}(k) = 0$. Thus in this case $X(\mathbb{A}_k) = X(\mathbb{A}_k)^{\mathrm{Br}_{\mathrm{vert}}}$, and Schinzel's hypothesis (H) implies that $\mathcal{R}$ is dense in $p(X(\mathbb{A}_k))$. In particular, if $X(\mathbb{A}_k) \neq \emptyset$, then from (H) one deduces the existence of $m \in \mathbf{P}^1(k)$ with smooth fibre X_m having points in all completions of k.

§2. The case of few non-split fibres : unconditional results

2.1 The case where at most one fibre is non-split

Let $f : X \to \mathbf{P}^1_k$ be as in Theorem 1.1. If all fibres of f except possibly one at a k-rational point are split, Theorem 1.1 yields an unconditional result, since the special case of Schinzel's hypothesis then used is Dirichlet's theorem on primes in an arithmetic progression (the degree of $P(t)$ is one). We have here $\mathrm{Br}_{\mathrm{vert}}(X)/\mathrm{Br}(k) = 0$, and the theorem reads : the set of points $m \in \mathbf{P}^1(k)$ whose fibre X_m is smooth over k and contains points in all completions of k is dense in the product $\prod_{v \in \Omega} f(X(k_v)) \subset \mathbf{P}^1(\mathbb{A}_k)$.

A more general result can however be given a simpler proof, which does not use Dirichlet's theorem. Namely, under the assumption that the fibres X_M for M closed point of $\mathbf{A}^1_k \subset \mathbf{P}^1_k$ are split and that the fibre X_∞ possesses at least one component of multiplicity one, the set of points $m \in \mathbf{P}^1(k)$ whose fibre X_m is smooth and contains points in all completions of k is dense in the product $\prod_{v \in \Omega} f(X(k_v)) \subset \mathbf{P}^1(\mathbb{A}_k)$. The proof uses Lemma 1.2 and weak approximation (when all fibres are split) or strong approximation (when one fibre is non-split). This formalization of the technique used first in [CT/S/SwD87] (in the case where all fibres over $\mathbf{A}^1_k$ are geometrically integral) has been known for some time (see [Sk90], [CT92], [Ha94], [Sk96]).These various papers actually make the additional assumption that f admits a section over an algebraic closure $\overline{k}$ of k, but this assumption may be replaced by the weaker hypothesis that the only possibly non-split fibre is geometrically split (i.e. possesses a component of multiplicity one). Indeed, the assumption that all fibres of $f : X \to \mathbf{P}^1_k$ possess one component of multiplicity one is enough to ensure the existence of infinitely many places v of k such that the induced map $X(k_v) \to \mathbf{P}^1(k_v)$ is surjective (Lemma 1.2 (d)).

2.2 The case where exactly two k-fibres are non-split

In this case, we have the *unconditional* result :

Theorem 2.2.1 *Let k be a number field and X a smooth, projective, geometrically integral variety over k, equipped with a flat k-morphism $f : X \to \mathbf{P}^1_k$ with geometrically integral generic fibre. Assume :*

(i) All fibres of f are split, except for the fibres at two distinct k-points A and B, and the fibres at these two points are geometrically split, i.e. contain at least one irreducible component of multiplicity one.

(ii) There exists a multiplicity one irreducible component $Z \subset X_A$ such that the algebraic closure k_Z of $k = k(A)$ in the function field of Z is an abelian extension of k.

Let $\mathcal{R} \subset \mathbf{P}^1(k)$ be the set of k-points m with smooth fibre X_m such that $X_m(\mathbb{A}_k) \neq \emptyset$. Then :

(a) Given $\{p_v\} \in X(\mathbb{A}_k)^{\mathrm{Br}_{\mathrm{vert}}}$ and a finite set S of places of k, there exists $m \in \mathbf{P}^1(k)$ such that the fibre X_m/k is smooth and has points in all completions of k, and the point m can be chosen as close as one wishes to each $f(p_v) \in \mathbf{P}^1(k_v)$ for $v \in S$.

(b) Assume $X(\mathbb{A}_k)^{\mathrm{Br}_{\mathrm{vert}}} \neq \emptyset$. Then for any finite set S of places of k, the closure of $\mathcal{R}$ in $\prod_{v \in S} \mathbf{P}^1(k_v)$ contains a non-empty open set. In particular $\mathcal{R}$ is Zariski-dense in $\mathbf{P}^1_k$.

(c) Assume $X(\mathbb{A}_k)^{\mathrm{Br}_{\mathrm{vert}}} \neq \emptyset$. Then there exists a finite set $S_0 \subset \Omega$ such that for any finite set $S \subset \Omega$ with $S \cap S_0 = \emptyset$, the closure of $\mathcal{R}$ under the diagonal embedding $\mathbf{P}^1(k) \to \prod_{v \in S} \mathbf{P}^1(k_v)$ coincides with $\prod_{v \in S} f(X(k_v))$.

(d) Assume that the Hasse principle holds for smooth fibres of f. Then the vertical Brauer-Manin obstruction to the existence of a rational point on X is the only obstruction. More precisely, if $X(\mathbb{A}_k)^{\mathrm{Br}_{\mathrm{vert}}} \neq \emptyset$, then for any finite set S of places of k, the closure of $f(X(k))$ in $\prod_{v \in S} \mathbf{P}^1(k_v)$ contains a non-empty open set. In particular $f(X(k))$ is Zariski-dense in $\mathbf{P}^1_k$.

(e) Assume that the Hasse principle and weak approximation hold for smooth fibres of f. Then the Brauer-Manin obstruction to weak approximation on X is the only obstruction : The closure of $X(k)$ in $X(\mathbb{A}_k)$ coincides with the open and closed set $X(\mathbb{A}_k)^{\mathrm{Br}_{\mathrm{vert}}}$.

Proof Let us prove (a). Changing coordinates on $\mathbf{P}^1_k$ and then $\mathbf{A}^1_k$, we may assume that A is given by $t = 0$ and B by $t = \infty$. Let $U \subset \mathbf{A}^1_k$ be the complement of the points with singular fibre. Let $\{p_v\} \in X(\mathbb{A}_k)^{\mathrm{Br}_{\mathrm{vert}}}$, and let $m_v = f(p_v)$. Since all geometric fibres are split, the quotient $\mathrm{Br}_{\mathrm{vert}}(X)/\mathrm{Br}(k)$ is finite. One thus produces finitely many elements $A_l \in \mathrm{Br}_{\mathrm{vert}}(X)$

whose images span the previous quotient, and an associated finite set S_0 of places as in Theorem 1.1. We may assume that S_0 contains all the archimedean places and all the places of k ramified in k_Z, and that the set S in (a) contains S_0. Using continuity of the Brauer pairing, one may assume that $m_v \neq 0$ and $m_v \neq \infty$, and that each p_v lies in X_U. For each $v \in \Omega$ one then produces open sets $W_v \subset U(k_v)$ and sections $\sigma_v : W_v \to X_U(k_v)$ of $X_U(k_v) \to U(k_v)$ as in Theorem 1.1.

Let $Y \subset X_\infty$ be an irreducible component of multiplicity one. One enlarges S so that Lemma 1.2 holds. We here have $T = T_0 \cup T_\infty$, and Lemma 1.2 (c) applies to closed points in T_0, the field K_0 of Lemma 1.2 being k_Z, and to closed points in T_∞, the field K_∞ of Lemma 1.2 being the algebraic closure k_Y of k in the function field $k(Y)$.

Let us write k_Z as a composite of cyclic extensions K_j/k, and define

$$A_j(t) = (K_j/k, t) \in \mathrm{Br}(k(t)).$$

Using Cor. 2.6.1 of [Ha94], one finds a finite set of places S_1 containing S, and points $p'_v \in X_U(k_v)$ for $v \in S_1$, with $p'_v = p_v$ for $v \in S$, such that for each j

$$\sum_{v \in S_1} \mathrm{inv}_v(A_j(m'_v)) = 0,$$

where $m'_v = f(p'_v)$, hence $m_v = m'_v$ for $v \in S_1$. Let $\lambda_v \in k_v^*$ be the t-coordinate of m'_v. Let V be an infinite set of finite places v of k lying over primes of $\mathbf{Q}$ which are totally split in k_Z and in k_Y (such a set exists by Tchebotarov's theorem). We assume $V \cap S_1 = \emptyset$. We now apply a modified form ([San82], Cor. 4.4) of Dirichlet's theorem on primes in an arithmetic progression in the number field case; the modified form, which builds upon a theorem of Waldschmidt, is required if one wants to approximate at the archimedean places.

This enables us to find $\lambda \in k$ as close as we wish to each λ_v for $v \in S_1$ (in particular $\lambda_v \in W_v$), and such that λ is a unit at any place v of k away from $S_1 \cup V$, except at one place w, where it is a uniformizing parameter.

For each j, we thus have :

$$\sum_{v \in S_1} \mathrm{inv}_v((K_j/k, \lambda)) = 0.$$

By the global reciprocity law and the properties of λ, this implies $\mathrm{inv}_w((K_j/k, \lambda)) = 0$. Since λ is a uniformizing parameter at w, this implies that w splits in K_j/k. This holds for each j, hence w splits in the composite k_Z of the fields K_j.

Let $m \in U(k)$ be the k-point with coordinate λ. The fibre X_m is smooth. It has points in k_v for $v \in S_1$ (use the sections σ_v). It has points in k_v for $v \notin S_1 \cup V \cup \{w\}$ by Lemma 1.2, (a) and (b). It has points in k_v for $v \in V$ by Lemma 1.2 ,(b) and (c), since places in V are split in k_Z and in k_Y, and both Z and Y have multiplicity one. Finally, X_m has points in k_w also by Lemma 1.2 (b) and (c), since Z has multiplicity one, w is split in k_Z, and $\mathrm{val}_w(\lambda) = 1 > 0$.

This completes the proof of statement (a). As explained at the beginning of the proof of Theorem 1.1, the other statements follow easily. □

When k is totally imaginary, we have the following variant :

Theorem 2.2.2 *Let k be a totally imaginary number field and X a smooth, projective, geometrically integral variety over k, equipped with a flat k-morphism $f : X \to \mathbf{P}^1_k$ with geometrically integral generic fibre. Assume :*

(i) All fibres of f are split, except for the fibres at two distinct k-points A and B.

(ii) There exists a multiplicity one irreducible component $Z \subset X_A$ such that the algebraic closure k_Z of $k = k(A)$ in the function field of Z is an abelian extension of k.

Let $\mathcal{R} \subset \mathbf{P}^1(k)$ be the set of k-points m with smooth fibre X_m such that $X_m(\mathbb{A}_k) \neq \emptyset$. Then :

(a) Given $\{p_v\} \in X(\mathbb{A}_k)^{\mathrm{Br}_{\mathrm{vert}}}$ and a finite set S of finite places of k, there exists $m \in \mathbf{P}^1(k)$ such that the fibre X_m/k is smooth and has points in all completions of k, and the point m can be chosen as close as one wishes to each $f(p_v) \in \mathbf{P}^1(k_v)$ for $v \in S$.

(b) Assume $X(\mathbb{A}_k)^{\mathrm{Br}_{\mathrm{vert}}} \neq \emptyset$. Then for any finite set S of finite places of k, the closure of $\mathcal{R}$ in $\prod_{v\in S} \mathbf{P}^1(k_v)$ contains a non-empty open set. In particular $\mathcal{R}$ is Zariski-dense in $\mathbf{P}^1_k$.

(c) Assume $X(\mathbb{A}_k)^{\mathrm{Br}_{\mathrm{vert}}} \neq \emptyset$. Then there exists a finite set $S_0 \subset \Omega$ such that for any finite set $S \subset \Omega$ with $S \cap S_0 = \emptyset$, the closure of $\mathcal{R}$ under the diagonal embedding $\mathbf{P}^1(k) \to \prod_{v\in S} \mathbf{P}^1(k_v)$ coincides with $\prod_{v\in S} f(X(k_v))$.

(d) Assume that the Hasse principle holds for smooth fibres of f. Then the vertical Brauer-Manin obstruction to the existence of a rational point on X is the only obstruction. More precisely, if $X(\mathbb{A}_k)^{\mathrm{Br}_{\mathrm{vert}}} \neq \emptyset$, then for any finite set S of finite places of k, the closure of $f(X(k))$ in $\prod_{v\in S} \mathbf{P}^1(k_v)$ contains a non-empty open set. In particular $f(X(k))$ is Zariski-dense in $\mathbf{P}^1_k$.

(e) Assume that the Hasse principle and weak approximation hold for smooth fibres of f. Given $\{p_v\} \in X(\mathbb{A}_k)^{\mathrm{Br}_{\mathrm{vert}}}$, and S a finite set of finite places of k, there exists a point $p \in X(k)$ as close as one wishes to each p_v for $v \in S$.

Proof We only give the points where the proof of (a) differs from the previous ones, and we leave the details of the other statements to the reader. We may assume that A is given by $t = 0$ and B by $t = \infty$. Let $U \subset \mathbf{A}^1_k$ be the complement of the points with singular fibre. Let $\{p_v\} \in X(\mathbb{A}_k)^{\mathrm{Br}_{\mathrm{vert}}}$, and let $m_v = f(p_v)$. We define a set S_0 as in the previous theorem, containing all the places of k ramified in k_Z. We may assume that S contains the finite places in S_0. One may assume that $m_v \neq 0$ and $m_v \neq \infty$, and that each p_v lies in X_U. For each $v \in \Omega$ one then produces open sets $W_v \subset U(k_v)$ and sections $\sigma_v : W_v \to X_U(k_v)$ of $X_U(k_v) \to U(k_v)$. One now enlarges S so that Lemma 1.2 (a) to (c) holds. We here have $T = T_0 \cup T_\infty$, and Lemma 1.2 (c) applies to closed points in T_0, the field K_0 of Lemma 1.2 being k_Z.

Let us write k_Z as a composite of cyclic extensions K_j/k, and define

$$A_j(t) = (K_j/k, t) \in \mathrm{Br}(k(t)).$$

Using Cor. 2.6.1 of [Ha94], one finds a finite set of places S_1 containing S, and points $p'_v \in X_U(k_v)$ for $v \in S_1$, with $p'_v = p_v$ for $v \in S$, such that for each j

$$\sum_{v\in S_1} \mathrm{inv}_v(A_j(m'_v)) = 0,$$

where $m'_v = f(p'_v)$, hence $m_v = m'_v$ for $v \in S_1$. Let $\lambda_v \in k_v^*$ be the t-coordinate of m'_v. We then apply Dirichlet's theorem on primes in an arithmetic progression in the number field case, as in [CT/S82, p. 39]. This produces $\lambda \in k$, λ as close as we wish to λ_v for v finite in S_1 and such that λ is a unit away from S_1, except at one place $w \notin S_1$, where λ is a uniformizing parameter. For each j, we thus have

$$\sum_{v\in S_1} \mathrm{inv}_v(A_j(\lambda)) = 0$$

and by the law of reciprocity this implies $\mathrm{inv}_w(A_j(\lambda)) = 0$, hence w splits in K_j/k for each j. Thus w splits in the composite k_Z.

Let $m \in \mathbf{P}^1(k)$ be the point with affine coordinate λ. The smooth k-variety X_m has points in the completions k_v for finite places $v \in S_1$ by the implicit function theorem, it has k_v-points for finite places $v \notin S_1$ by Lemma 1.2 (b), it has k_v-points for v archimedean, hence complex, for trivial reasons. Finally it has k_w-points because Z is a multiplicity one component of the fibre at $t = 0$, the place w splits in k_Z and $\mathrm{val}_w(\lambda) = 1 > 0$: we may thus apply Lemma 1.2, (b) and (c). □

Remarks 2.2.3

(a) In contrast with Theorem 1.1, the results here are unconditional : indeed, Dirichlet's theorem on primes in an arithmetic progression is the special case of hypothesis (H) which is known.

(b) Theorem 2.2.1 and 2.2.2 should be compared with [Sk96], which deals with applications of the descent method to one-parameter families of varieties, in the case where the sum of the degrees of the closed points with non-split fibres is at most *two*, and the Hasse principle and weak approximation hold for the fibres. In that case, an unconditional proof of statements (d) and (e) of Theorem 1.1 is given in [Sk96] (no mention of Schinzel's hypothesis), under the further assumptions that *all* components of *all* fibres are of multiplicity one, and that there is a section of the fibration f over an algebraic closure $\overline{k}$ of k (this last assumption can presumably be dispensed with). Thus [Sk96] does not require the abelianness assumption which we make in 2.2.1 and 2.2.2. But the theorems above do not require that *all* components of the fibres X_A and X_B be of multiplicity one, and when k is totally imaginary, at the expense of losing control of approximation at the archimedean (complex) places, we impose no multiplicity one condition at all on X_B.

(c) Let us come back to the case considered in Section 2.1, namely the case where just the fibre X_∞ is non-split. In Section 2.1, we assumed X_∞ to be geometrically split, in other words we assumed that X_∞ contains a component of multiplicity one. Let us drop this assumption, but restrict attention to the case where k is totally imaginary. Using Dirichlet's theorem as in Theorem 2.2.2 (rather than strong approximation as in 2.1), we conclude that for any finite set S of finite places of k, the image of $f(X(k))$ in the product $\prod_{v\in S} f(X(k_v))$ is dense (in particular the Hasse principle holds for X). If we also assume weak approximation for the smooth fibres, then for any finite S of finite places of k, the diagonal map $X(k) \to \prod_{v\in S} X(k_v)$ has dense image : weak approximation holds at the finite places.

§3. Salberger's device

Salberger's device [Sal88] can be described in the following free-standing fashion, which clearly displays the fact that it is a substitute for Schinzel's hypothesis, or rather for its variant (H_1).

Theorem 3.1 *Let k be an algebraic number field. Let $P_i(t)$, $i = 1,\dots,n$ be distinct irreducible monic polynomials in $k[t]$. Let S be a finite set of places of k which contains all the archimedean places, all the finite places v of k for which some polynomial $P_i(t)$ has some coefficient not integral at v, and all the finite places v such that all $P_i(t)$ have v-integral coefficients, but such that the product $R(t) = \prod_i P_i(t)$ does not remain separable when reduced modulo v.*

Let L be a finite extension of k in which all the polynomials P_i split completely, and which is Galois over $\mathbf{Q}$. Let V be an infinite set of finite primes of k lying over primes in $\mathbf{Q}$ which are totally split in L (the existence of such a set is guaranteed by Tchebotarov's density theorem). Suppose that we are given an integer

$$N \geq \sum_i \deg(P_i)$$

and for each $v \in S$ a non-empty open set U_v of separable monic polynomials in $k_v[t]$ of degree N. Then we can find an irreducible monic polynomial $G(t) \in k[t]$ of degree N such that if $k(\theta) = k[t]/G(t)$,

(i) the class θ of t is an integer in $k(\theta) = k[t]/G(t)$, except perhaps at some of the primes in $k(\theta)$ above those in $S \cup V$;

(ii) $G(t)$ is in U_v for each $v \in S$;

(iii) for each i there is a finite prime w_i in $k(\theta)$, of absolute degree one, such that $P_i(\theta)$ is a uniformizing parameter for w_i and a unit except at w_i and possibly at primes above those in $S \cup V$.

Proof Let $R_i(t) = \prod_{j \neq i} P_j(t)$. Any polynomial $G(t) \in k[t]$ may be written in a unique way as

$$G(t) = R(t)Q(t) + \sum_i R_i(t)\psi_i(t)$$

with

$$\deg(\psi_i) < \deg(P_i).$$

If $G(t)$ is monic and of degree at least equal to that of $R(t)$, then $Q(t)$ is non-zero and monic. This corresponds to the isomorphism

$$k[t]/R(t) \simeq \prod_i k[t]/P_i(t).$$

We have a similar isomorphism if we go over to each k_v. For each $v \in S$, let $G_v(t)$ be a polynomial of degree N in U_v. We may thus write

$$G_v(t) = R(t)Q_v(t) + \sum_i R_i(t)\psi_{i,v}(t) \in k_v[t]$$

with each $Q_v(t)$ monic of degree $N - \deg(R)$ and $\deg(\psi_{i,v}) < \deg(P_i)$.

We build a polynomial $G(t)$ close to each $G_v(t)$ for $v \in S$ in the following manner.

We choose a prime $v_0 \in V$ away from S. We also fix a monic irreducible polynomial $G_{v_0} \in k_{v_0}[t]$, prime to $R(t) = \prod_i P_i(t)$, and of degree N (irreducible polynomials of all degrees exist over a p-adic field).

We choose a second prime $v_0' \in V$ away from $S \cup v_0$. We may find a monic polynomial $Q(t)$ as close as needed to each $Q_v(t)$ for $v \in S \cup \{v_0\}$, in such a way that all coefficients of Q have positive valuation at all places $v \notin S \cup v_0 \cup v_0'$: this is achieved by applying strong approximation in k to the coefficient of each power of t.

We then choose each $\psi_i(t)$ as follows. Recall $k_i = k[t]/P_i(t)$. Consider the diagonal map

$$k_i^* \longrightarrow \prod_{v \in S \cup v_0} (k_i \otimes_k k_v)^*.$$

For each place $v \in S \cup v_0$ and each i let $\gamma_{i,v}$ be the class of $\psi_{i,v}(t)$ in $k_v[t]/P_i(t) = k_i \otimes_k k_v$. According to Cor. 4.4 of [San82], we may find a $\gamma_i \in k_i^*$ such that its image under this diagonal map is very close to each element $\gamma_{i,v}$ for $v \in S \cup v_0$ and such that the decomposition of γ_i into prime ideals in the ring of integers of k_i reads

$$(\gamma_i) = \mathfrak{p}_i . \prod_{w \in (V \cup S)_{k_i}} \mathfrak{p}_w^{n_w},$$

i.e. consists of primes in the set $(V \cup S)_{k_i}$ of places of k_i lying above S or above primes in V (the integers n_w lie in $\mathbf{Z}$), and of one prime $\mathfrak{p}_i$ of absolute degree 1, away from S_{k_i}. Let v_i be the place of k lying below the place $\mathfrak{p}_i$ of k_i. (If we were only to approximate at the finite places, Dirichlet's theorem on primes in arithmetic progression would be enough. But if we want to approximate at the archimedean places too, then we need Cor. 4.4 of [San82].)

The element $\gamma_i \in k_i^*$ is represented in a unique way as the class of a polynomial $\psi_i(t) \in k[t]$ of degree strictly smaller than $P_i(t)$. Let

$$G(t) = R(t)Q(t) + \sum_i R_i(t)\psi_i(t) \in k[t].$$

Because the polynomial $G(t)$ may be made arbitrarily close to $G_{v_0}(t) \in k_{v_0}[t]$, and because this last polynomial was chosen irreducible, we may ensure that $G(t)$ is irreducible in $k_{v_0}[t]$ (Krasner's lemma), hence also in $k[t]$. We now let K be the field $k[t]/G(t)$. This is a field extension of k of degree N.

Note that the coefficients of the monic polynomial $G(t)$ are integral at finite places of k which do not belong to $S \cup V$. In the field $K = k(\theta) = k[t]/G(t)$, the element θ which is the class of t is therefore integral at finite places of K which do not lie over those of $S \cup V$, as claimed in (i). Because of the approximation conditions we put on the places in S, we can certainly ensure condition (ii). It only remains to check (iii). If the trace on k of a place w of K does not belong to $S \cup V$, then clearly $P_i(\theta) \in K$ is integral at w. Let us compute the norm $N_{K/k}(P_i(\theta))$. The formula for the resultant of two polynomials shows that

$$N_{K/k}(P_i(\theta)) = \pm N_{k_i/k}(G(\theta_i)),$$

where $k_i = k[t]/P_i(t) = k(\theta_i)$ (here θ_i denotes the class of t in $k_i = k[t]/P_i(t)$). A glance at the equation defining $G(t)$ shows :

$$G(\theta_i) = R_i(\theta_i)\gamma_i \in k_i^*.$$

Thus

$$N_{k_i/k}(G(\theta_i)) = N_{k_i/k}(R_i(\theta_i)).N_{k_i/k}(\gamma_i) \in k^*.$$

Given the enlargement of S we made at the outset, the element $N_{k_i/k}(R_i(\theta_i)) \in k^*$ is a unit at any place $v \notin S$. As for $N_{k_i/k}(\gamma_i)$, it is a unit at all places of k outside $S \cup V$, except at the place v_i (which has absolute degree one over $\mathbf{Q}$), where it has valuation exactly one (recall that $\mathfrak{p}_i$ is a place of absolute degree one of k_i). We thus conclude that $N_{K/k}(P_i(\theta)) \in k^*$ is a unit away from $S \cup V \cup \{v_i\}$, and that $v_i(N_{K/k}(P_i(\theta))) = 1$. Since $P_i(\theta) \in K$ is integral away from the places above $S \cup V$, we conclude that $P_i(\theta)$ is a unit at places of K not above $S \cup V$, except for just one place w_i of absolute degree one (lying over v_i), where it is a uniformizing parameter. □

§4. Local-global properties for zero-cycles

Let X be a variety over a field k. We refer to [CT/SwD94], Section 3.1, for the definition and properties of the pairing between the Brauer group of X and zero-cycles on X, sending a class $A \in \mathrm{Br}(X)$ and a zero-cycle z on X to $< A, z > \in \mathrm{Br}(k)$. We also refer to [CT/SwD94], Section 3.1, for the definition of the Brauer-Manin obstruction in this context.

Proceeding as in Section 5 of [CT/SwD94], we shall now replace Schinzel's hypothesis by Salberger's device and prove the (unconditional) generalization of Theorem 1.1 (d) for zero-cycles of degree one instead of rational points. We then go on and prove the generalization of one of the local-global exact sequences for zero-dimensional Chow groups in Section 6 of [CT/SwD94].

Theorem 4.1 *Let X be a smooth, projective, geometrically integral variety over a number field k, equipped with a flat morphism $f : X \to \mathbf{P}^1_k$ with geometrically integral generic fibre. Assume :*

(i) If M is a closed point of $\mathbf{P}^1_k$ and X_M denotes the fibre over M, then there exists a multiplicity one irreducible component $Z \subset X_M$ such that the algebraic closure of $k(M)$ in the function field of Z is an abelian extension of $k(M)$.

(ii) There is no vertical Brauer-Manin obstruction to the existence of a zero-cycle of degree one on X.

(iii) The Hasse principle for the existence of zero-cycles of degree one holds for all smooth fibres of f (over all closed points with smooth fibre).

Then there exists a zero-cycle of degree one on X.

Proof. We start just as in the proof of Theorem 1.1. We repeat some of the preliminaries. We may assume that the fibre of f at infinity is smooth. Let $V \subset \mathbf{A}^1_k$ be the complement of the set of closed points of $\mathbf{A}^1_k$ with singular fibre and let $U = f^{-1}(V) \subset X$. Its complement $F \subset X$ is the union of the fibre at infinity of f and of the singular fibres of f.

Let $P(t)$ be the monic separable polynomial which describes all the closed points M such that the fibre X_M is not split. This polynomial may be written as a product of irreducible monic polynomials $P(t) = \prod_{i=1}^n P_i(t)$. Each of them corresponds to a closed point M_i of the affine line. Use assumption (i) : for each such M_i, let $k_i = k(M_i)$ and let Z_i/k_i be a multiplicity one irreducible component of X_{M_i} such that the algebraic closure K_i of k_i in the function field of Z_i is an abelian extension of k_i. Write K_i/k_i as a composite of cyclic extensions $K_{i,j}/k_i$. Let $e_i \in k_i$ denote the class of t under the identification $k_i = k[t]/P_i(t)$. Let $k(t)$ be the function field of $\mathbf{A}^1_k = \mathrm{Spec}(k[t])$. Define $A_{i,j} \in \mathrm{Br}(k(t))$ by

$$A_{i,j} = \mathrm{Cores}_{k_i/k}(K_{i,j}/k_i, t - e_i).$$

Let s be the least common multiple of the orders of the $A_{i,j} \in \mathrm{Br}(k(t))$.

Let S_0 be a finite set of places of k containing all the archimedean places and all the bad finite primes in sight : finite primes where one P_i is not integral, finite primes where all P_i are integral but the reduction of the product $P(t) = \prod_i P_i(t)$ is not separable, primes ramified in one of the extensions $K_{i,j}/k$. We want S_0 to be big enough for the fibration $f : X \to \mathbf{P}^1_k$ to extend to a (projective, flat) fibration $g : \mathcal{X} \to \mathbf{P}^1_O$, where O denotes the ring of integers of k with primes in S_0 inverted, and $\mathcal{X}$ is regular. We also enlarge the finite set S_0 so that Lemma 1.2 applies.

Having repeated these preliminaries, we now proceed as follows. Let N_0 be a closed point of $U \subset X$ and let $d = [k(N_0) : k]$ be its degree over k.

Since we assume that there is no vertical Brauer-Manin obstruction to the existence of a zero-cycle of degree one, an obvious variant of Harari's result ([CT/SwD94], Thm. 3.2.2) implies that we may find a finite set S_1 of places of k containing S_0 and for each $v \in S_1$ a zero-cycle z_v of degree one with support in $U \times_k k_v \subset X \times_k k_v$ such that

$$\forall \, \{i,j\}, \quad \sum_{v \in S_1} \mathrm{inv}_v(< A_{i,j}, z_v >) = 0. \tag{4.1}$$

Let us write the zero-cycle z_v as $z_v^+ - z_v^-$, with z_v^+ and z_v^- effective cycles. Let z_v^1 be the effective cycle $z_v^+ + (ds-1)z_v^-$. We have $z_v = z_v^1 - dsz_v^-$, hence $< A_{i,j}, z_v > = < A_{i,j}, z_v^1 >$ since each $A_{i,j}$ is killed by s. We thus have

$$\forall \, \{i,j\}, \quad \sum_{v \in S_1} \mathrm{inv}_v(< A_{i,j}, z_v^1 >) = 0. \tag{4.2}$$

Similarly $< A_{i,j}, sN_0 > = 0$. The degree of z_v^1 is congruent to 1 modulo ds. The cycle sN_0 has degree ds. Adding suitable multiples of sN_0 to each z_v^1 for v in the finite set S_1, we then find effective cycles z_v^2, all of the same degree $1 + Dsd$ for some $D > 0$, and such that

$$\forall \, \{i,j\}, \quad \sum_{v \in S_1} \mathrm{inv}_v(< A_{i,j}, z_v^2 >) = 0. \tag{4.3}$$

We claim that in (4.3), for each $v \in S_1$, each effective cycle z_v^2 may be assumed to be a sum of distinct closed points (i.e. there are no multiplicities) whose images under $f_{k_v} : U_{k_v} \to \mathbf{P}^1_{k_v}$ are also distinct. Indeed, if P is a closed point of U_{k_v}, with residue field $F = k_v(P)$, and A is a class in $\mathrm{Br}(U)$, the map $U(F) \to \mathrm{Br}(F) \subset \mathbf{Q}/\mathbf{Z}$ given by evaluation of A is continuous. Since X is smooth, the point P defines a non-singular F-point of $U \times_k F$. The statement now follows from the implicit function theorem.

We claim that while keeping (4.3) we can moreover assume that, for each z_v^2 and each closed point P in the support of z_v^2, the field extension map $k_v(f(P)) \subset k_v(P)$ is an isomorphism. Once more, it is enough to replace P by a suitable, close enough, $k_v(P)$-rational point on $U_{k_v(P)}$: the argument for this appears on the bottom of p. 89 of [CT/SwD94].

Each of the zero-cycles $f(z_v^2)$ is now given by a separable monic polynomial $G_v(t) \in k_v[t]$ of degree $1 + Dds$, prime to $P(t)$ and with the property that the (smooth) fibres of f above the roots of G_v have points rational over their field of definition.

According to Krasner's lemma, any monic polynomial $H(t)$ close enough to $G_v(t)$ for the v-adic topology on the coefficients will be separable, with roots 'close' to those of G_v. Thus the fibres above the roots of the new polynomial are still smooth and still possess points rational over their field of definition.

Let L be the Galois closure, over $\mathbf{Q}$, of the composite of the field extensions $K_i/\mathbf{Q}$. By Lemma 1.2 (d) (here we are using hypothesis (i) in Theorem 4.1, at least to the extent that each fibre contains a component of multiplicity one), there exists an infinite set S_2 of places v of k, away from S_1, lying over primes totally decomposed in L, such that for any $v \in S_2$ and any finite field extension K_w/k_v the map $f_{K_w} : X(K_w) \to \mathbf{P}^1(K_w)$ is surjective.

We now choose the irreducible polynomial $G(t)$ as given by Salberger's device (with V of Section 2 being the present S_2, and S of Section 2 being the present S_1). Thus $G(t)$ is very close to each $G_v(t)$ for $v \in S_1$. The irreducible polynomial G defines a closed point M of degree $1 + Dds$ on $\mathbf{A}^1_k$. Let $K = k(M) = k[t]/G(t)$. For each polynomial $P_i(t)$ there is an associated finite place w_i of K.

We claim that the fibre X_M/K has points in all completions of K. Talking about the reduction of this variety makes sense at finite primes of K which do not lie above primes in $S_1 \cup S_2$.

At primes w of K above a prime v of S_2, the existence of a K_w-point on X_M is clear, since f_{K_w} is surjective on K_w-points. At primes w above primes in S_1, the existence of K_w-points on X_M follows from the fact that G is very close to each G_v for $v \in S_1$. At finite primes w of K which do not lie above $S_1 \cup S_2$ and are distinct from $w_1, \ldots, w_n$, the reduction of X_M over the finite field $\kappa(M)_w$ is split, hence possesses a smooth rational point, hence X_M has a K_w-point (Lemma 1.2 (b) is stated for rational points of $\mathbf{P}^1_k$ but also holds for closed points).

Let us now consider a prime w_i (compare [CT/SwD94], proof of Theorem 5.1, p. 83 to 85). We let $E_i = k_i \otimes_k K$ and $F_{i,j} = K_{i,j} \otimes_k K$. These Artinian algebras need not be fields, but this does not matter. Using equation (4.3), the continuity of the local invariant of class field theory, its behaviour under corestriction and global reciprocity, we end up with

$$\mathrm{inv}_{w_i} \mathrm{Cores}_{E_i/K}(F_{i,j}/E_i, \theta - e_i) = 0 \qquad (4.4)$$

while $N_{E_i/K}(\theta - e_i) = P_i(\theta)$ has w_i-valuation

$$w_i(N_{E_i/K}(\theta - e_i)) = 1. \qquad (4.5)$$

From equality (4.5) follows that in the decomposition of $E_i \otimes_K K_{w_i}$ as a product of field extensions of K_{w_i}, there is just one of these field extensions, call it $E_{i,w}$, in which $(\theta - e_i) \in E_i$ is not sent to a unit, and in that particular extension $(\theta - e_i)$ becomes a uniformizing parameter. Moreover, $E_{i,w}$ is a trivial extension of K_{w_i}.

From (4.4) now follows

$$(F_{i,j}/E_i, \theta - e_i) \otimes_{E_i} E_{i,w} = 0.$$

Since $(\theta - e_i)$ is a uniformizing parameter in $E_{i,w}$, this implies that the cyclic extension $F_{i,j}/E_i$ becomes trivial when tensored with $E_{i,w}$. Since this holds for all j, we conclude that the reduction of X_M at w_i is split. Hence X_M/K possesses a K_{w_i}-point (use the closed point variant of Lemma 1.2, (b) and (c)).

This completes the proof that X_M has points in all completions of $K = k(M)$. Since we had postulated the Hasse principle for zero-cycles of degre one on smooth fibres (hypothesis (iii)), we conclude that X_M/K contains a zero-cycle of degree one. The degree of K over k is $1+Dds$, hence X/k contains a zero-cycle of degree $1+Dds$. Since it also contains the zero-cycle N_0 which is of degree d, we conclude that there exists a zero-cycle of degree one on X. □

Remarks 4.2

(a) If we drop assumption (iii) in the theorem, i.e. if we do not assume the Hasse principle in the fibres, but keep assumptions (i) and (ii), the above proof shows that, given any positive integer d, there exists a closed point M of $\mathbf{P}^1_k$ of degree prime to d such that the fibre $X_M/k(M)$ is smooth and has points in all completions of $k(M)$. In other words, the greatest common divisor of the degrees $[k(M) : k]$ of such points is equal to one.

(b) The (simple) techique used at the beginning of the proof to reduce the problem to effective zero-cycles of a fixed degree represents an improvement upon the technique used in [CT/SwD94], p. 79 (which in turn improved upon Salberger's original argument for conic bundles, see Remark 5.1.1 of [CT/SwD94]).

(c) An obvious but seemingly difficult question is whether one can get rid of the abelianness condition in the theorem above.

(d) The result raises the question what families of varieties over k have the property that if a variety V in the family contains a zero-cycle of degree one defined over k then it possesses a k-point. This is known to hold, for example, for any intersection of two quadrics in projective space, but it is in general false for pencils of conics. For cubic surfaces the question is open.

We now address the generalization of Section 6 in [CY/SwD94]. We shall need some lemmas.

Lemma 4.3 *Let X be a geometrically integral variety over a number field k, let $A \in \mathrm{Br}(X)$. Then for almost all places v of k there exists $P_v \in X(k_v)$ such that $A(P_v) = 0 \in \mathrm{Br}(k_v)$.*

Proof We may assume that X/k is affine. There exists a non-empty open set $\mathrm{Spec}(O)$ of the ring of integers of k and an affine integral scheme $\mathcal{X}$ over $\mathrm{Spec}(O)$ such that the projection $\mathcal{X} \to \mathrm{Spec}(O)$ is flat with non-empty smooth geometrically integral fibres, and with generic fibre X/k, and such that moreover there exists $\mathcal{A} \in \mathrm{Br}(\mathcal{X})$ which restricts to A over X. By Lemma 1.3 for all closed points $v \in \mathrm{Spec}(O)$ away from a finite set S the corresponding closed fibre of $\mathcal{U}/\mathrm{Spec}(O)$ has a rational point which lifts to a k_v-point on U. By Lemma 1.3 for all closed points $v \in \mathrm{Spec}(O)$ away from a finite set S the corresponding closed fibre of $\mathcal{X}/\mathrm{Spec}(O)$ has a rational point, and such a point lifts to an O_v-point M_v on $\mathcal{X}$, giving rise to a k_v-point $P_v \in X(k_v)$. Now $A(P_v) \in \mathrm{Br}(k_v)$ is the restriction of $\mathcal{A}(M_v) \in \mathrm{Br}(O_v) = 0$. □

Lemma 4.4 *Let X be a smooth geometrically integral variety over a number field k, let $U \subset X$ be a non-empty open set, and let $A \in \mathrm{Br}(U)$. Assume $A \notin \mathrm{Br}(X)$. Then :*

(a) There exist infinitely many places v of k such that the map $U(k_v) \to \mathrm{Br}(k_v)$ given by evaluating A at k_v-points of U takes at least two different values, one of them zero.

(b) Given any integer $r \geq 1$, for infinitely many places v, the map from effective zero-cycles of degree r on $U \times_k k_v$ to $\mathrm{Br}(k_v)$ given by evaluation of A contains both zero and a non-zero value in its image.

Proof Thm. 2.1.1 of [Ha94] asserts that the map $U(k_v) \to \mathrm{Br}(k_v)$ takes a non-zero value for infinitely many places v. Statement (a) then follows from the previous lemma. As for (b), it follows from (a) by considering effective zero-cycles defined by a suitable sum of r points of $U(k_v)$. □

This lemma allows us to get rid of the restriction on the degree in the 0-cycle variant of Harari's theorem ([CT/SwD94], Thm. 3.2.2).

Lemma 4.5 *Let X be a smooth, projective, geometrically integral variety over a number field k, let $U \subset X$ be a non-empty open set of X and $\{A_1, \dots, A_n\} \in \mathrm{Br}(U) \subset \mathrm{Br}(k(X))$. Let $r \geq 1$ be a positive integer. Assume that X contains 0-cycles (resp. effective 0-cycles) u_v of degree r over each completion k_v of k. Let B be the intersection of $\mathrm{Br}(X)$ with the subgroup generated by the A_i's, $i = 1, \dots, n$, in $\mathrm{Br}(k(X))$. Assume that the subgroup $B \subset \mathrm{Br}(X)$ creates no obstruction to the Hasse principle for the existence of 0-cycles (resp. effective 0-cycles) of degree r on X. Then for any finite set of places S one can find a bigger finite set of places S_1, $S \subset S_1$, and 0-cycles (resp. effective 0-cycles) z_v of degree r with support in U_{k_v}, for $v \in S_1$, such that for each $i = 1, \dots, n$ one has*

$$\sum_{v \in S_1} \mathrm{inv}_v(< A_i, z_v >) = 0 \in \mathbf{Q}/\mathbf{Z}.$$

If B creates no obstruction to weak approximation for the family of 0-cycles (resp. effective 0-cycles) $\{u_v\}_{v \in \Omega}$ of degree r, then one can moreover choose $z_v = u_v$ for $v \in S$.

Proof This follows from Lemma 4.4 by Harari's formal argument (Cor. 2.6.1 of [Ha94]). □

Given X/k a projective variety, one denotes by $\mathrm{CH}_0(X)$ the Chow group of zero-cycles of X, and by $A_0(X) \subset \mathrm{CH}_0(X)$ the subgroup of classes of zero-cycles of degree zero. For any field extension K/k, the Brauer pairing defines a homomorphism $\mathrm{CH}_0(X_K) \to \mathrm{Hom}(\mathrm{Br}(X_K), \mathrm{Br}(K))$.

Lemma 4.6 *Let k be a number field. Let X be a smooth, projective, geometrically integral variety over a number field k, equipped with a flat morphism $f : X \to \mathbf{P}^1_k$ with geometrically integral generic fibre. Assume that all geometric fibres of f are split. For $v \in \Omega_k$, let $X_v = X \times_k k_v$. Then for almost all places v of k, the pairing $A_0(X_v) \times \mathrm{Br}_{\mathrm{vert}}(X_v) \to \mathrm{Br}(k_v)$ is trivial.*

Proof Let $\mathfrak{g} = \mathrm{Gal}(\overline{k}/k)$. Let $\hat{T}$ be the $\mathfrak{g}$-submodule of $\mathrm{Pic}(\overline{X})$ which is spanned by the classes of vertical divisors on $\overline{X}$, i.e. divisors which are components of the fibres of $\overline{f}$. This is a torsion-free abelian group of finite type (cf. [Sk96], Prop. 3.2.3). Let $\lambda : \hat{T} \subset \mathrm{Pic}(\overline{X})$ be the obvious inclusion. Let T be the k-torus whose character group is $\hat{T}$. We may consider the subset of $H^1(X,T) = H^1_{\text{ét}}(X,T)$ consisting of isomorphism classes of torsors over X under the k-torus T which have image λ under the natural map $H^1(X,T) \to \mathrm{Hom}_{\mathfrak{g}}(\hat{T}, \mathrm{Pic}(\overline{X}))$. By [CT/S87] (Prop. 2.2.8 (v) and Thm. 2.3.1) and [Sk96] (Lemma 3.2.1), this subset is not empty. Let $\mathcal{T}/X$ be a torsor whose class lies in this subset. There exists an open set $\mathrm{Spec}(O)$ of the ring of integers of k over which T extends to an O-torus $\tilde{T}$, the k-variety X to a (smooth) projective O-scheme $\tilde{X}$ and $\mathcal{T}$ to a $\tilde{T}$-torsor over $\tilde{X}$. Since H^1 of a ring O_v with values in an O_v-torus is trivial (Hensel's lemma together with Lang's theorem on principal homogeneous spaces of connected algebraic groups over a finite field), we conclude that for $v \in \mathrm{Spec}(O)$, the map $\mathrm{CH}_0(X_v) \to H^1(k_v, T)$ defined by $\mathcal{T}$ is zero.

Let K be any overfield of k. We may define the composite map

$$\eta_1 : A_0(X_K) \to H^1(K,T) \to (H^1(K, \hat{T}))^{\sim},$$

where for any abelian group C, we let $C^{\sim} = \mathrm{Hom}(C, \mathrm{Br}(K))$. The first map is defined by $\mathcal{T}$. The second map comes from the cup-product $H^1(K,T) \times H^1(K,\hat{T}) \to \mathrm{Br}(K)$.

Let $\mathrm{Br}_1(X_K) = \mathrm{Ker}[\mathrm{Br}(X_K) \to \mathrm{Br}(X_K \times_K \overline{K})]$. If K is a number field or a local field there is a well-known natural isomorphism

$$\mathrm{Br}_1(X_K)/\mathrm{Br}(K) \stackrel{\simeq}{\to} H^1(K, \mathrm{Pic}(X_K \times_K \overline{K}))$$

(this uses the vanishing of $H^3(K, \overline{K}^*)$ provided by class field theory). Let us consider the inverse isomorphism $H^1(K, \mathrm{Pic}(X_K \times_K \overline{K})) \stackrel{\simeq}{\to} \mathrm{Br}_1(X_K)/\mathrm{Br}(K)$. Under this isomorphism, the image of $H^1(K,\hat{T})$ under λ maps onto the group $\mathrm{Br}_{\mathrm{vert}}(X_K)/\mathrm{Br}(K)$ ([Sk96], Cor. 4.5). We thus get an injective map $\rho : (\mathrm{Br}_{\mathrm{vert}}(X_K)/\mathrm{Br}(K))^{\sim} \hookrightarrow (H^1(K,\hat{T}))^{\sim}$. Pairing of $A_0(X_K)$ with the Brauer group induces a map $A_0(X_K) \to (\mathrm{Br}_{\mathrm{vert}}(X_K)/\mathrm{Br}(K))^{\sim}$ which composed with the map ρ defines $\eta_2 : A_0(X_K) \to (H^1(K,\hat{T}))^{\sim}$. Proposition 2.7.10 in [CT/S87] (stated for rational points, but the proof immediately extends to zero-cycles) shows that η_1 and η_2 coincide (up to a sign). Since ρ is injective, we conclude that the kernel of η_1 (defined by the torsor $\mathcal{T}$) is contained in the kernel of the map $A_0(X_K) \to (\mathrm{Br}_{\mathrm{vert}}(X_K)/\mathrm{Br}(K))^{\sim}$. We now apply this to $K = k_v$ for each $v \in \mathrm{Spec}(O)$. □

As already mentioned in the proof of Theorem 1.1, a consideration of residues ([CT/SwD94], Prop. 1.1.1; [Sk96], Lemma 4.4) gives the easy :

Lemma 4.7 *Let X be a smooth, projective, geometrically integral variety over a field k, equipped with a flat morphism $f : X \to \mathbf{P}^1_k$ with geometrically integral generic fibre. Let $W \subset \mathbf{P}^1_k$ be the Zariski open set which is the complement of points with singular fibre. If the image of a class $A \in \mathrm{Br}(k(t))$ in the Brauer group of $k(X)$ lies in $\mathrm{Br}(X)$, then A belongs to $\mathrm{Br}(W)$.* □

The following theorem generalizes Theorem 6.2 (i) of [CT/SwD94].

Theorem 4.8 *Let X be a smooth, projective, geometrically integral variety over a number field k, equipped with a flat morphism $f : X \to \mathbf{P}^1_k$ with geometrically integral generic fibre. Assume :*

(i) If M is a closed point of $\mathbf{P}^1_k$ and X_M denotes the fibre over M, then there exists a multiplicity one irreducible component $Z \subset X_M$ such that the algebraic closure of $k(M)$ in the function field of Z is an abelian extension of $k(M)$.

(ii) The Hasse principle for the existence of zero-cycles of degree one holds for all smooth fibres of f (over all closed points with smooth fibre).

Then there is a natural exact sequence

$$A_0(X) \to \bigoplus_{v \in \Omega} A_0(X_v)/\mathrm{Br}_{\mathrm{vert}} \to \mathrm{Hom}(\mathrm{Br}_{\mathrm{vert}}(X)/\mathrm{Br}(k), \mathbf{Q}/\mathbf{Z})$$

where the second and last groups are finite.

Here the left hand side map is induced by the diagonal map, $A_0(X_v)/\mathrm{Br}_{\mathrm{vert}}$ denotes the image of $A_0(X_v)$ under the natural map $A_0(X_v) \to \mathrm{Hom}(\mathrm{Br}_{\mathrm{vert}}(X_v)/\mathrm{Br}(k_v), \mathbf{Q}/\mathbf{Z})$, and the maps $A_0(X_v)/\mathrm{Br}_{\mathrm{vert}} \to \mathrm{Hom}(\mathrm{Br}_{\mathrm{vert}}(X)/\mathrm{Br}(k), \mathbf{Q}/\mathbf{Z})$ are induced by the Brauer pairing $A_0(X_v) \times \mathrm{Br}_{\mathrm{vert}}(X) \to \mathrm{Br}(k_v) \hookrightarrow \mathbf{Q}/\mathbf{Z}$.

Proof Since all the geometric fibres of f are split, for any field L containing k, the group $\mathrm{Br}_{\mathrm{vert}}(X_L)/\mathrm{Br}(L)$ is finite (cf. [Sk96], Cor. 4.5). The finiteness statement in the theorem now follows from Lemma 4.6. That the composite map is zero is an immediate consequence of the global reciprocity law.

We may assume that the fibre of f over the point ∞ is smooth. Let $F \subset \mathbf{P}^1_k$ be the union of ∞ and the closed points with singular fibre. Let $W \subset \mathbf{P}^1_k$ be the Zariski open set complement of F. Let $U = f^{-1}(W) \subset X$. Let N_0 be a closed point in U and let d be its degree.

By Lemma 4.7 we have

$$\mathrm{Br}_{\mathrm{vert}}(X_L) = \mathrm{Br}(X_L) \cap f^*\mathrm{Br}(W_L) \subset \mathrm{Br}(U_L).$$

For each place $v \in \Omega$, let us fix a finite set $\mathcal{B}_v$ of elements of $\mathrm{Br}(W_v)$ whose images under f^* generate $\mathrm{Br}_{\mathrm{vert}}(X_v)$ modulo algebras coming from $\mathrm{Br}(k_v)$ under the natural map $\mathrm{Br}(k_v) \to \mathrm{Br}(X_v)$.

Keep notation as in Theorem 4.1, but assume that the finite set S also contains all places with $A_0(X_v)/\mathrm{Br}_{\mathrm{vert}} \neq 0$ (see Lemma 4.6).

Let $\{c_v\}$ be a family of zero-cycles of degree zero, and assume that for all $A \in \mathrm{Br}_{\mathrm{vert}}(X)$, one has

$$\sum_{v \in \Omega} \mathrm{inv}_v(A(c_v)) = 0.$$

Moving c_v in its class for rational equivalence, which does not affect the above equality, we may arrange that the support of each c_v lies in U.

Let z_v be the zero-cycle of degree d defined by $z_v = c_v + N_0$. By the global reciprocity law and functoriality of corestrictions, for all $A \in \mathrm{Br}_{\mathrm{vert}}(X)$, we have

$$\sum_{v \in \Omega} \mathrm{inv}_v(A(z_v)) = 0.$$

By the zero-cycle version of Harari's result (Lemma 4.5 above), we may find a finite set S_1 of places of k containing S and zero-cycles z'_v of degree d with support in U, with $z'_v = z_v$ for $v \in S$, such that

$$\forall \, \{i,j\}, \quad \sum_{v \in S_1} \mathrm{inv}_v(< A_{i,j}, z'_v >) = 0.$$

Let s be an integer which kills all the $A_{i,j}$, and also all the elements of the finite set $\mathcal{B}_v$ for $v \in S$. Proceeding as in the proof of Theorem 4.1, for each $v \in S_1$, we find an effective zero-cycle z^2_v, of degree $d + Dds$ (same positive integer D for all $v \in S_1$), with support in U, such that

$$\forall \, \{i,j\}, \quad \sum_{v \in S_1} \mathrm{inv}_v(< A_{i,j}, z^2_v >) = 0$$

and such that the class $z'_v - z^2_v$ is divisible by s in $\mathrm{CH}_0(X_v)$.

Using the implicit function theorem as in the proof of Theorem 4.1, one then replaces z^2_v by an effective zero-cycle z^3_v of the same degree, with multiplicities all equal to one (or zero), close enough to z^2_v, and such that f induces an isomorphism between the subscheme $z^3_v \hookrightarrow X_v$ and $f(z^3_v) \hookrightarrow \mathbf{P}^1_{k_v}$. For z^2_v close enough to z^3_v , we have

$$\forall \, \{i,j\}, \quad \sum_{v \in S_1} \mathrm{inv}_v(< A_{i,j}, z^3_v >) = 0.$$

By Salberger's device, one then produces a closed point M of degree $d+Dds$ on $\mathbf{P}^1_k$ defining a cycle very close to each $f(z^3_v)$ for $v \in S_1$, and such that the fibre X_M has points in all completions of $K = k(M)$. Since we assume that the Hasse principle for zero-cycles of degree one holds on the smooth fibres of f, we ultimately find a zero-cycle ρ of degree 1 on X_M/K, hence of degree $d + Dds$ on X/k, such that $f_*(\rho) = M$.

Let us consider the zero-cycle of degree 0 defined by $c = \rho - (1 + Ds)N_0$, and let v be a place in S. For any $\alpha = f^*(\beta)$ with $\beta \in \mathcal{B}_v \subset \mathrm{Br}(W_v)$ we have the following equalities in $\mathrm{Br}(k_v)$. Firstly,

$$< \alpha, c > = < \alpha, \rho > - < \alpha, N_0 >$$

since s kills all the elements of $\mathcal{B}_v$. Then

$$< \alpha, \rho > = < f^*(\beta), \rho > = < \beta, f_*(\rho) > = < \beta, M > = < \beta, f_*(z_v^3) >,$$

the second equality by functoriality of the Brauer pairing, the last equality because M could be chosen close enough to $f(z_v^3)$ (the elements β belong to the finite set $\mathcal{B}_v$). Then

$$< \beta, f_*(z_v^3) > = < f^*(\beta), z_v^3 > = < \alpha, z_v^3 > = < \alpha, z_v^2 >$$

the last equality because z_v^3 could be chosen close enough to z_v^2 (there are only finitely many α's). Then

$$< \alpha, z_v^2 > = < \alpha, z'_v >$$

since $z_v^2 - z'_v$ is divisible by s in $\mathrm{CH}_0(X_v)$ and $\alpha \in f^*(\mathcal{B}_v)$ is killed by s. Finally

$$< \alpha, z'_v > = < \alpha, z_v >$$

since $z_v = z'_v$ for v in S and

$$< \alpha, z_v > = < \alpha, c_v + N_0 > = < \alpha, c_v > + < \alpha, N_0 > .$$

All in all, we find

$$< \alpha, c > = < \alpha, c_v >$$

for all $\alpha \in f^*(\mathcal{B}_v)$, hence also for all $\alpha \in \mathrm{Br}_{\mathrm{vert}}(X_v)$, since c and c_v have the same degree (zero). In other words, the class of c coincides with the class of c_v in $A_0(X_v)/\mathrm{Br}_{\mathrm{vert}}$. □

Remark 4.9 Let $f : X \to \mathbf{P}^1_k$ be as in Theorem 4.8. Let T be the k-torus described in Lemma 4.6. As in that lemma, there is a natural map

$$\varphi : A_0(X) \to H^1(k, T)$$

induced by a torsor over X of type λ. Similarly, we have a local map

$$\varphi_v : A_0(X_v) \to H^1(k_v, T)$$

for each place $v \in \Omega$. Global class field theory gives rise to an exact sequence (cf. [CT/S87] (3.0.7) or [Sal88], (7.4)) :

$$0 \to Ш^1(k, T) \to H^1(k, T) \to \oplus_{v \in \Omega} H^1(k_v, T) \to \mathrm{Hom}(H^1(k, \hat{T}), \mathbf{Q}/\mathbf{Z}),$$

where the right hand side map is induced by the local pairings

$$H^1(k_v, T) \times H^1(k_v, \hat{T}) \to \mathbf{Q}/\mathbf{Z}$$

Using the same arguments as in Lemma 4.6, together with the fact that each of these local pairings induces a perfect duality of finite abelian groups (local class field theory), one may reformulate Theorem 4.8 as follows :

The above exact sequence induces an exact sequence

$$\mathrm{Im}(\varphi) \to \oplus_{v \in \Omega} \mathrm{Im}(\varphi_v) \to \mathrm{Hom}(H^1(k, \hat{T}), \mathbf{Q}/\mathbf{Z}).$$

In the case of conic bundles, the exactness of this sequence was proved by Salberger [Sal88]. There is an obvious embedding

$$\mathrm{Ker}[\mathrm{Im}(\varphi) \to \oplus_{v\in\Omega}\mathrm{Im}(\varphi_v)] \hookrightarrow Ш^1(k,T).$$

In the case of conic bundles, Salberger [Sal88] also proved that this embedding is an isomorphism, i.e. there is in this case an exact sequence :

$$0 \to Ш^1(k,T) \to \mathrm{Im}(\varphi) \to \oplus_{v\in\Omega}\mathrm{Im}(\varphi_v) \to \mathrm{Hom}(H^1(k,\hat{T}),\mathbf{Q}/\mathbf{Z}).$$

Going back to the ideas in [Sal88], and combining them with the general formalism described in [CT/Sk93] and [Fr96a], one may obtain the same results, provided one makes two additional assumptions on $X/\mathbf{P}^1_k$. For each closed point $M \in \mathbf{P}^1_k$, let k_M be the residue field at M, and let $N_M \subset k_M^*$ be the subgroup $\prod_{Z\subset X_M} N_{k_Z/k_M}(k_Z^*)^{n_Z}$, where Z runs through the irreducible components of X_M, k_Z denotes the algebraic closure of k_M in the function field $k(Z)$ of Z and n_Z denotes the multiplicity of Z in X_M. The two additional assumptions are :

(i) ([Fr96a]) For each closed point $M \in \mathbf{P}^1_k$ whose fibre is not integral, the composite map $A_0(X_M) \to A_0(X) \xrightarrow{\varphi} H^1(k,S)$ is zero (e.g. $A_0(X_M) = 0$).

(ii) For each closed point $M \in \mathbf{P}^1_k$ whose fibre is non-split, there exists a cyclic extension K_M/k_M of the residue field at M such that $N_M \subset k_M^*$ coincides with the norm group $N_{K_M/k_M}(K_M^*)$.

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