Strong approximation for the total space of certain quadric fibrations

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Abstract

Soit $q(x, y, z)$ une forme quadratique sur un corps de nombres k, isotrope en une place v, et soit $P(t)$ un polynôme non nul à coefficients dans k. Si $P(t)$ est séparable, on établit l'approximation forte en dehors de la place v pour les solutions de $q(x, y, z) = P(t)$. Pour $P(t)$ quelconque, on montre que sur le lieu lisse de la variété définie par $q(x, y, z) = P(t)$ l'obstruction de Brauer-Manin entière est la seule obstruction à l'approximation forte hors de v.

Let $q(x, y, z)$ be a quadratic form over a number field k, isotropic at a place v, and let $P(t)$ be a nonzero polynomial with coefficients in k. If $P(t)$ is separable, we show that strong approximation away from v holds for the solutions of $q(x, y, z) = P(t)$. For $P(t)$ arbitrary, we show that the integral Brauer-Manin obstruction is the only obstruction to strong approximation away from v for the smooth locus of the variety given by $q(x, y, z) = P(t)$.

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1 Introduction

Let X be a variety over a number field F. For simplicity, let us assume in this introduction that the set $X(F)$ of rational points is not empty. Let S be a finite set of places of F. One says that *strong approximation holds for X off S* if the diagonal image of the set $X(F)$ of rational points is dense in the space of S-adèles $X(\mathbb{A}_F^S)$ (these are the adèles where the places in S have been omitted) equipped with the adelic topology. If this property holds for X, it in particular implies a local-global principle for the existence of integral points on integral models of X over the ring of S-integers of F.

For X projective, $X(\mathbb{A}_F^S) = \prod_{v \notin S} X(F_v)$, and the adelic topology coincides with the product topology. A projective variety satisfies strong approximation off S if and only if weak approximation for the rational points holds off S.

For open varieties, strong approximation has been mainly studied for linear algebraic groups and their homogeneous spaces. A classical case is m-dimensional affine space $\mathbf{A}_F^m$ off any nonempty set S, a special case being the Chinese Remainder Theorem. For a semisimple, almost simple, simply connected linear algebraic group G such that $\prod_{v \in S} G(F_v)$ is not compact, strong approximation off S was established by Eichler, Kneser, Shimura, Platonov, Prasad.

Strong approximation does not hold for groups which are not simply connected, but one may define a Brauer-Manin set. In our paper [CTX], we started the investigation of the Brauer-Manin obstruction to strong approximation for homogeneous spaces of linear algebraic groups. For such varieties, this was quickly followed by works of Harari [H], Demarche [D], Borovoi and Demarche [BD] and Wei and Xu [WX].

Few strong approximation results are known for open varieties which are not homogeneous spaces. Computations of the Brauer-Manin obstruction for some such varieties have been recently performed (Kresch and Tschinkel [KT], Colliot-Thélène et Wittenberg [CTW]).

Just as for problems of weak approximation, it is natural to ask whether strong approximation holds for the total space of a family $f : X \to Y$ when it is known for the basis Y, for many fibres of f, and some algebraico-geometric assumption is made on the map f.

In the present paper, we investigate strong approximation for varieties X/F defined by an equation

$$q(x_1, \ldots, x_n) = p(t),$$

where q is a quadratic form of rank n in $n \geq 3$ variables and $p(t)$ is a nonzero polynomial.

In [Wat], Watson investigated integral points on affine varieties which are the total space of families of quadrics over affine space $\mathbf{A}^m_F$. When restricted to equations as above, in particular $m = 1$, and with coefficients in the ring $\mathbb{Z}$ of integers, under a noncompacity assumption, his Theorems 1 and 2 establish the local-global principle for integral points when $n \geq 4$ ([Wat, Thm. 1, Thm. 2]). Under some additional condition, he also establishes a local-global principle when $n = 3$ ([Wat, Thm. 3], see Remark 6.6 in the present paper).

The paper is organized as follows.

In §2 we recall some definitions related to strong approximation and the Brauer-Manin obstruction.

In §3, we give a simple general method for proving strong approximation for the total space of a fibration. We apply it to varieties defined by an equation $q(x_1, \ldots, x_n) = p(t)$, for $n \geq 4$.

In §4 we detail results of [CTX] on the arithmetic of affine quadrics $q(x, y, z) = a$.

In the purely algebraic §5, we compute the Brauer group of the smooth locus, and of a suitable desingularisation, of a variety defined by an equation $q(x, y, z) = p(t)$.

The most significant results are given in §6. The results of §4 and §5 are combined to study the strong approximation property off S for certain smooth models of varieties defined by an equation $q(x, y, z) = p(t)$, under the assumption that the form q is isotropic at some place in S. For these smooth models, when there is no Brauer-Manin obstruction, we establish strong approximation off S. We give the precise conditions under which strong approximation fails.

In §7 we give two numerical counterexamples to the local-global principles for existence of integral points: this represents a drastic failure of strong approximation in the cases where this is allowed by the results of §6.

Concrete varieties often are singular. In that case the appropriate properties are "central strong approximation" and its Brauer-Manin variant. This is shortly discussed in §8.

2 Basic definitions and properties

Let F be a number field, $\mathfrak{o}_F$ be the ring of integers of F and Ω_F be the set of all primes in F. For each $v \in \Omega_F$, let F_v be the completion of F at v. Let ∞_F be the set of archimedean primes in F and write $v < \infty_F$ for $v \in \Omega_F \setminus \infty_F$. For each $v < \infty_F$, let $\mathfrak{o}_v$ be the completion of $\mathfrak{o}_F$ at v and let π_v be a uniformizer of $\mathfrak{o}_v$. Write $\mathfrak{o}_v = F_v$ for $v \in \infty_F$.

For any finite subset S of Ω_F, let $F_S = \prod_{v \in S} F_v$. For any finite subset S of Ω_F containing ∞_F, the S-integers are defined to be elements in F which are integral outside S. The ring of S-integers is denoted by $\mathfrak{o}_S$. Let $\mathbb{A}_F \subset \prod_{v \in \Omega_F} F_v$ be the adelic group of F with its usual topology. For any finite subset S of Ω_F, one defines $\mathbb{A}_F^S \subset (\prod_{v \notin S} F_v)$ equipped with the analogous adelic topology. The natural projection which omits the S-coordinates defines a homomorphism of rings $\mathbb{A}_F \to \mathbb{A}_F^S$. For any variety X over F this induces a map

$$pr^S : X(\mathbb{A}_F) \to X(\mathbb{A}_F^S)$$

which is surjective if $\prod_{v \in S} X(F_v) \neq \emptyset$.

Definition 2.1. Let X be a geometrically integral F-variety. One says that strong approximation holds for X off S if the image of the diagonal map

$$X(F) \to X(\mathbb{A}_F^S)$$

is dense in $pr^S(X(\mathbb{A}_F)) \subset X(\mathbb{A}_F^S)$.

The statement may be rephrased as:

Given any nonempty open set $W \subset X(\mathbb{A}_F^S)$, if $X(\mathbb{A}_F) \neq \emptyset$, then the diagonal image of $X(F)$ in $X(\mathbb{A}_F)$ meets $W \times \prod_{v \in S} X(F_v)$.

If X satisfies strong approximation off S, and $X(\mathbb{A}_F^S) \neq \emptyset$, then we have $X(F) \neq \emptyset$ and, for any finite set T of places of F away from S, the diagonal image of $X(F)$ is dense in $\prod_{v \in T} X(F_v)$. In other words, X satisfies the Hasse principle, and X satisfies weak approximation off S.

Proposition 2.2. *Assume $X(\mathbb{A}_F) \neq \emptyset$. If X satisfies strong approximation off a finite set S of places, then it satisfies strong approximation off any finite set S' with $S \subset S'$.* □

Proposition 2.3. *Let $U \subset X$ be a dense open set of a smooth geometrically integral F-variety X. If strong approximation off S holds for U, then strong approximation off S holds for X.*

Proof. This follows from the following statement: for X/F as in the proposition, the image of $U(\mathbb{A}_F)$ in $X(\mathbb{A}_F)$ is dense. That statement itself follows from two facts. Firstly, for a given place v, $U(F_v)$ is dense in $X(F_v)$ (smoothness of X). Secondly, U admits a model $\mathbf{U}$ over a suitable $\mathfrak{o}_T$ such that $\mathbf{U}(\mathfrak{o}_v) \neq \emptyset$ for all $v \notin T$ (because U/F is geometrically integral). □

As explained in [CTX], one can refine definition 2.1 by using the Brauer–Manin set. Let X be an F-variety. Let $\mathrm{Br}(X) = H^2_{ét}(X, \mathbb{G}_m)$ and define

$$X(\mathbb{A}_F)^{\mathrm{Br}(X)} = \{\{x_v\}_{v\in\Omega_F} \in X(\mathbb{A}_F) : \ \forall \xi \in \mathrm{Br}(X), \sum_{v\in\Omega_F} \mathrm{inv}_v(\xi(x_v)) = 0\}.$$

This is a closed subset of $X(\mathbb{A}_F)$. Class field theory implies

$$X(F) \subset X(\mathbb{A}_F)^{\mathrm{Br}(X)} \subset X(\mathbb{A}_F).$$

Let

$$X(\mathbb{A}_F^S)^{\mathrm{Br}(X)} := pr^S(X(\mathbb{A}_F)^{\mathrm{Br}(X)}) \subset X(\mathbb{A}_F^S).$$

Definition 2.4. Let X be a geometrically integral variety over the number field F. If the diagonal image of $X(F)$ in $(X(\mathbb{A}_F^S))^{\mathrm{Br}(X)} \subset X(\mathbb{A}_F^S)$ is dense, we say that strong approximation with Brauer-Manin obstruction holds for X off S.

As above, the statement may be rephrased as :

Given any open set $W \subset X(\mathbb{A}_F^S)$, if $[W \times \prod_{v\in S} X(F_v)]^{\mathrm{Br}(X)} \neq \emptyset$, then there is a point of the diagonal image of $X(F)$ in $W \times \prod_{v\in S} X(F_v) \subset X(\mathbb{A}_F)$.

Proposition 2.5. *Assume $X(\mathbb{A}_F) \neq \emptyset$. If strong approximation with Brauer-Manin obstruction holds for X off a finite set S of places, then it holds off any finite set S' with $S \subset S'$.* □

Proposition 2.6. *Let F be a number field. Let $U \subset X$ be a dense open set of a smooth geometrically integral F-variety X. Assume:*

(i) $X(\mathbb{A}_F) \neq \emptyset$;

(ii) the quotient $\mathrm{Br}(U)/\mathrm{Br}(F)$ is finite.

Let S be a finite set of places of F. If strong approximation with Brauer-Manin obstruction off S holds for U, then it holds for X.

Proof. There exists a finite subgroup $B \subset \mathrm{Br}(U)$ such that B generates $\mathrm{Br}(U)/\mathrm{Br}(F)$ and $B \cap \mathrm{Br}(X)$ generates $\mathrm{Br}(X)/\mathrm{Br}(F)$. There exists a finite set T of places of k containing S and all the archimedean places, and smooth $\mathfrak{o}_T$-schemes $\mathbf{U} \subset \mathbf{X}$ with geometrically integral fibres over the points of $\mathrm{Spec}(\mathfrak{o}_T)$ such that

(a) The restriction $\mathbf{U} \subset \mathbf{X}$ over $\mathrm{Spec}(F) \subset \mathrm{Spec}(\mathfrak{o}_T)$ is $U \subset X$.

(b) $B \subset \mathrm{Br}(\mathbf{U})$.

(c) $B \cap \mathrm{Br}(X) \subset \mathrm{Br}(\mathbf{X})$.

(d) For each $v \notin T$, $\mathbf{U}(\mathfrak{o}_v) \neq \emptyset$ (this uses the fact that $U \to \mathrm{Spec}(\mathfrak{o}_T)$ is smooth with geometrically integral fibres, the Weil estimates and the fact that we took T big enough).

To prove the proposition, it is enough to show:

Given any finite set T as above and given, for each place $v \in T \setminus S$, an open set $W_v \subset X(F_v)$ such that the set

$$[\prod_{v\in S} X(F_v) \times \prod_{v\in T\setminus S} W_v \times \prod_{v\notin T} \mathbf{X}(\mathfrak{o}_v)]^{\mathrm{Br}(X)}$$

is not empty, then this set contains a point of the diagonal image of $X(F)$ in $X(\mathbb{A}_F)$.

Each $\alpha \in B \cap \mathrm{Br}(X)$ vanishes when evaluated on $\mathbf{X}(\mathfrak{o}_v)$. For any element $\alpha \in \mathrm{Br}(X)$ and any place v, the map $X(F_v) \to \mathrm{Br}(F_v) \subset \mathbb{Q}/\mathbb{Z}$ given by evaluation of α is locally constant. Since X is smooth, for each place v, the set $U(F_v)$ is dense in $X(F_v)$ for the local topology. In particular, for $v \notin T$, the set $\mathbf{X}(\mathfrak{o}_v) \cap U(F_v)$ is not empty. There thus exists a point $\{M_v\} \in X(\mathbb{A}_F)$ which lies in the above set such that $M_v \in U(F_v)$ for $v \in T$ and $M_v \in \mathbf{X}(\mathfrak{o}_v) \cap U(F_v)$ for $v \notin T$.

We now use Harari's formal lemma in the version given in [CT]. According to the proof of [CT, Théorème 1.4], there exist a finite set T_1 of places of k, $T_1 \cap T = \emptyset$, and for $v \in T_1$ points $N_v \in \mathbf{X}(\mathfrak{o}_v) \cap U(F_v)$, such that

$$\sum_{v\in T} \beta(M_v) + \sum_{v\in T_1} \beta(N_v) = 0$$

for each $\beta \in B$.

For $v \in T$, let $N_v = M_v$. For $v \notin T \cup T_1$, let $N_v \in \mathbf{U}(\mathfrak{o}_v)$ be an arbitrary point. The adèle $\{N_v\}$ of X belongs to

$$[\prod_{v\in S} X(F_v) \times \prod_{v\in T\setminus S} W_v \times \prod_{v\notin T} \mathbf{X}(\mathfrak{o}_v)]^{\mathrm{Br}(X)}.$$

It is the image of an adèle of U which lies in

$$[\prod_{v\in S} U(F_v) \times \prod_{v\in T\setminus S} W_v \cap U(F_v) \times \prod_{v\in T_1} U(F_v) \cap \mathbf{X}(\mathfrak{o}_v) \times \prod_{v\notin T\cup T_1} \mathbf{U}(\mathfrak{o}_v)]^{\mathrm{Br}(U)}.$$

Using the finiteness of B and the continuity of the evaluation map $U(F_v) \to \mathrm{Br}(F_v)$ attached to each element of B, we find that there exist open sets $W'_v \subset U(F_v)$ for $v \in T \cup T_1$, with $W'_v \subset W_v$ for $v \in T \setminus S$, such that the subset

$$[\prod_{v\in T\cup T_1} W'_v \times \prod_{v\notin T\cup T_1} \mathbf{U}(\mathfrak{o}_v)]^{\mathrm{Br}(U)}$$

of the adèles of U is nonempty. Since strong approximation with Brauer-Manin obstruction off S holds for U, hence off $T \cup T_1$ since $S \subset T$, there exists a point in the diagonal image of $U(F)$ in $U(\mathbb{A}_F)$ which lies in this set.

Since this set maps into

$$[\prod_{v\in S} X(F_v) \times \prod_{v\in T\setminus S} W_v \times \prod_{v\notin T} \mathbf{X}(\mathfrak{o}_v)]^{\mathrm{Br}(X)}$$

via the inclusion $U \subset X$, this concludes the proof. □

Lemma 2.7. *Let F be a number field. Let $U \subset X$ be a dense open set of a smooth geometrically integral F-variety X. Assume $X(\mathbb{A}_F) \neq \emptyset$. Let $\alpha_1, \ldots, \alpha_n \in \mathrm{Br}(X)$. Let S be a finite set of places of F. The image of the evaluation map $U(\mathbb{A}_F^S) \to (\mathbb{Q}/\mathbb{Z})^n$ defined by the sum of the invariants of each α_i on the $U(F_v)$ for $v \notin S$ coincides with the image of the analogous evaluation map $X(\mathbb{A}_F^S) \to (\mathbb{Q}/\mathbb{Z})^n$.*

Proof. There is a natural map $U(\mathbb{A}_F^S) \to X(\mathbb{A}_F^S)$ which is compatible with evaluation of elements of $\mathrm{Br}(X)$, hence one direction is clear. Let $\{M_v\} \in X(\mathbb{A}_F^S)$. There exist a finite set T of places containing S and regular integral models $\mathbf{U} \subset \mathbf{X}$ of $U \subset X$ over $\mathfrak{o}_T$ such that $\alpha_i \in \mathrm{Br}(\mathbf{X}) \subset \mathrm{Br}(\mathbf{U})$ for each $i = 1, \ldots, n$, such that $M_v \in \mathbf{X}(\mathfrak{o}_v)$ for each $v \notin T$, and such that moreover $\mathbf{U}(\mathfrak{o}_v) \neq \emptyset$ for $v \notin T$. For $v \in T \setminus S$, let $N_v \in U(F_v), v \in T \setminus S$ be close enough to $M_v \in X(F_v)$ that $\alpha_i(N_v) = \alpha_i(M_v)$ for each $i = 1, \ldots, n$ (such points exist since X is smooth). For $v \notin T$, let N_v be an arbitrary point of $\mathbf{U}(\mathfrak{o}_v)$.

Then

$$\sum_{v\notin S} \alpha_i(M_v) = \sum_{v\in T, v\notin S} \alpha_i(M_v) = \sum_{v\in T, v\notin S} \alpha_i(N_v) = \sum_{v\notin S} \alpha_i(N_v).$$

□

Proposition 2.8. *Let F be a number field. Let $U \subset X$ be a dense open set of a smooth geometrically integral F-variety X. Assume $X(\mathbb{A}_F) \neq \emptyset$.*

(i) Assume $\mathrm{Br}(X)/\mathrm{Br}(F)$ finite. If $pr_S(X(\mathbb{A}_F)^{\mathrm{Br}(X)})$ is strictly smaller than $X(\mathbb{A}_F^S)$, then $pr_S(U(\mathbb{A}_F)^{\mathrm{Br}(U)})$ is strictly smaller than $U(\mathbb{A}_F^S)$.

(ii) If $\mathrm{Br}(X) \to \mathrm{Br}(U)$ is an isomorphism, if $pr_S(U(\mathbb{A}_F)^{\mathrm{Br}(U)})$ is strictly smaller than $U(\mathbb{A}_F^S)$, then $pr_S(X(\mathbb{A}_F)^{\mathrm{Br}(X)})$ is strictly smaller than $X(\mathbb{A}_F^S)$.

Proof. (i) Let $\alpha_i \in \mathrm{Br}(X)$, $i = 1, \ldots, n$, generate $\mathrm{Br}(X)/\mathrm{Br}(F)$.

If $pr_S(X(\mathbb{A}_F)^{\mathrm{Br}(X)})$ is strictly smaller than $X(\mathbb{A}_F^S)$, then there exists an adèle $\{M_v\} \in X(\mathbb{A}_F^S)$ such that for each $\{N_v\} \in \prod_{v\in S} X(F_v)$ there exists α_i such that

$$\sum_{v\notin S} \alpha_i(M_v) + \sum_{v\in S} \alpha_i(N_v) \neq 0 \in \mathbb{Q}/\mathbb{Z}.$$

In other words, the image of the map $\prod_{v\in S} X(F_v) \to (\mathbb{Q}/\mathbb{Z})^n$ given by $\{N_v\} \mapsto \sum_{v\in S} \alpha_i(N_v)$ does not contain $\{-\sum_{v\notin S} \alpha_i(M_v)\} \in (\mathbb{Q}/\mathbb{Z})^n$. By Lemma 2.7, there exists an adèle $\{M'_v\} \in U(\mathbb{A}_F^S)$ such that:

$$\{-\sum_{v\notin S} \alpha_i(M'_v)\} = \{-\sum_{v\notin S} \alpha_i(M_v)\} \in (\mathbb{Q}/\mathbb{Z})^n.$$

Thus for each $\{N'_v\} \in \prod_{v\in S} U(F_v)$ there exists some i such that

$$\sum_{v\notin S} \alpha_i(M'_v) + \sum_{v\in S} \alpha_i(N'_v) \neq 0 \in \mathbb{Q}/\mathbb{Z}.$$

Hence $\{M'_v\} \in U(\mathbb{A}_F^S)$ does not belong to $pr_S(U(\mathbb{A}_F)^{\mathrm{Br}\, U})$.

(ii) Let $\{M_v\} \in U(\mathbb{A}_F^S)$ be an adèle such that for each $\{N_v\} \in \prod_{v\in S} U(F_v)$ there exists $\alpha \in \mathrm{Br}(U)$ such that

$$\sum_{v\notin S} \alpha(M_v) + \sum_{v\in S} \alpha(N_v) \neq 0 \in \mathbb{Q}/\mathbb{Z}.$$

The adèle $\{M_v\} \in U(\mathbb{A}_F^S)$ defines an adèle $\{M_v\} \in X(\mathbb{A}_F^S)$. By hypothesis $\mathrm{Br}(X) = \mathrm{Br}(U)$. For each $\alpha \in \mathrm{Br}(X) = \mathrm{Br}(U)$, the image of the evaluation map of $\alpha \in \mathrm{Br}(X)$ on $U(F_v)$ coincides with the image of the evaluation map on $X(F_v)$. We conclude that for each $\{N_v\} \in \prod_{v\in S} X(F_v)$ there exists an element $\alpha \in \mathrm{Br}(X)$ such that

$$\sum_{v\notin S} \alpha(M_v) + \sum_{v\in S} \alpha(N_v) \neq 0 \in \mathbb{Q}/\mathbb{Z}.$$

□

3 The easy fibration method

Proposition 3.1. *Let F be a number field and $f : X \to Y$ be a morphism of smooth quasi-projective geometrically integral varieties over F. Assume that all geometric fibres of f are nonempty and integral. Let $W \subset Y$ be a nonempty open set such that $f_W : f^{-1}(W) \to W$ is smooth.*

Let S be a finite set of places of F. Assume *(i) Y satisfies strong approximation off S.*

(ii) The fibres of f above F-points of W satisfy strong approximation off S.

(iii) For each $v \in S$ the map $f^{-1}(W)(F_v) \to W(F_v)$ is onto.

Then X satisfies strong approximation off S.

Proof. There exist a finite set T of places containing all archimedean places and a morphism of smooth quasiprojective $\mathfrak{o}_T$-schemes $\phi : \mathcal{X} \to \mathcal{Y}$ which restricts to $f : X \to Y$ over F, and such that:

(a) All geometric fibres of ϕ are geometrically integral.

(b) For any closed point m of $\mathcal{Y}$, the fibre at m, which is a variety over the finite field $\kappa(m)$, contains a smooth $\kappa(m)$-point.

(c) For any $v \notin T$, the induced map $\mathcal{X}(\mathfrak{o}_v) \to \mathcal{Y}(\mathfrak{o}_v)$ is onto.

The proof of this statement combines standard results from EGA IV 9 and the Lang-Weil estimates for the number of points of integral varieties over a finite field. Many variants have already appeared in the literature.

To prove the proposition, it is enough to show:

Given any finite set T as above, with $S \subset T$, and given, for each place $v \in T \setminus S$, an open set $U_v \subset X(F_v)$ such that the open set

$$\prod_{v\in S} X(F_v) \times \prod_{v\in T\setminus S} U_v \times \prod_{v\notin T} \mathbf{X}(\mathfrak{o}_v)$$

of $X(\mathbb{A}_F)$ is not empty, then this set contains a point of the diagonal image of $X(F)$ in $X(\mathbb{A}_F)$.

The Zariski open set $f^{-1}(W) \subset X$ is not empty. For each $v \in T \setminus S$, we may thus replace U_v by the nonempty open set $U_v \cap f^{-1}(W)(F_v)$. Since f is smooth on $f^{-1}(W)$, $f(U_v) \subset Y(F_v)$ is an open set. By hypothesis (i), there exists a point $N \in Y(F)$ whose diagonal image lies in the open set

$$\prod_{v\in S} Y(F_v) \times \prod_{v\in T\setminus S} f(U_v) \times \prod_{v\notin T} \mathbf{Y}(\mathfrak{o}_v)$$

of $Y(\mathbb{A}_F)$. Let $Z = X_N = f^{-1}(N)$. The point N comes from a point $\mathbf{N}$ in $\mathcal{Y}(\mathfrak{o}_T)$. The $\mathfrak{o}_T$-scheme $\mathcal{Z} := \phi^{-1}(\mathbf{N})$ is thus a model of Z. For $v \notin T$, statement (c) implies $\mathcal{Z}(\mathfrak{o}_v) \neq \emptyset$. By assumption (iii), we have $Z(F_v) \neq \emptyset$ for each $v \in S$. For $v \in T \setminus S$, the intersection $U_v \cap Z(F_v)$ by construction is a nonempty open set of $Z(F_v)$. Assumption (ii) now guarantees that the product

$$\prod_{v\in S} Z(F_v) \times \prod_{v\in T\setminus S} U_v \cap Z(F_v) \times \prod_{v\notin T} \mathcal{Z}(\mathfrak{o}_v)$$

contains the diagonal image of a point of $Z(F)$. This defines a point in $X(F)$ which lies in the given open set of $X(\mathbb{A}_F)$. $\square$

Let us recall a well known fact.

Proposition 3.2. *Let F be a number field. Let $q(x_1, \ldots, x_n)$ be a nondegenerate quadratic form over F and let $c \in F^{\times}$. Assume $n \geq 4$. Let X be the smooth affine quadric defined by $q(x_1, \ldots, x_n) = c$. Suppose $X(F_v) \neq \emptyset$ for each real completion F_v. Then $X(F) \neq \emptyset$. Let v_0 be a place of F such that the quadratic form q is isotropic at v_0. Then X satisfies strong appproximation off any finite set $S \subset \Omega_F$ containing v_0.*

Proof. This goes back to Eichler and Kneser. See [CTX] Thm. 3.7 (b) and Thm. 6.1. $\square$

Lemma 3.3. *Let $q(x_1, \ldots, x_n)$ ($n \geq 1$) be a nondegenerate quadratic form over a field k of characteristic different from 2. Let $p(t) \in k[t]$ be a nonzero polynomial. Let X be the affine k-scheme defined by $q(x_1, \ldots, x_n) = p(t)$. The singular points of X are the points defined by $x_i = 0$ (all i) and $t = \theta$ with θ a multiple root of $p(t)$. In particular, if $p(t)$ is a separable polynomial, then X is smooth over k.* $\square$

Proposition 3.4. *Let F be a number field and X be an F-variety defined by an equation*

$$q(x_1, \ldots, x_n) = p(t)$$

where $q(x_1, \ldots, x_n)$ is a nondegenerate quadratic form with $n \geq 4$ over F and $p(t) \neq 0$ is a polynomial in $F[t]$. Let $\tilde{X}$ be any smooth geometrically integral variety which contains the smooth locus X_{smooth} as a dense open set. Assume $X_{smooth}(F_v) \neq \emptyset$ for each real place v of F.

(1) $\tilde{X}(F)$ is Zariski-dense in $\tilde{X}$.

(2) $\tilde{X}$ satisfies weak approximation.

Let v_0 be a place of F such that q is isotropic over F_{v_0}.

(3) $\tilde{X}$ satisfies strong approximation off any finite set S of places which contains v_0.

Proof. Statements (1) and (2), which are easy, are special cases of Prop. 3.9, p. 66 of [CTSaSD]. Let us prove (3) for $\tilde{X} = X_{smooth}$, the smooth locus of X. Let $f : X_{smooth} \to \mathbf{A}^1_F$ be given by the coordinate t. By Lemma 2.2, it suffices to prove the theorem for $S = \{v_0\}$. Let W be the complement of $p(t) = 0$ in $\mathbf{A}^1_F$. Given Prop. 3.2, Lemma 3.3, statement (3) for $\tilde{X} = X_{smooth}$ is an immediate consequence of Proposition 3.1 applied to the map f. Statement (3) for an arbitrary $\tilde{X}$ is then an immediate application of Proposition 2.3. $\square$

4 The equation $q(x, y, z) = a$

Let $q(x, y, z)$ be a nondegenerate quadratic form over a field k of characteristic zero and let $a \in k^*$. Let Y/k be the affine quadric defined by the equation

$$q(x, y, z) = a.$$

This is an open set in the smooth projective quadric defined by the homogeneous equation

$$q(x, y, z) - au^2 = 0.$$

Let $d = -a. \det(q) \in k^\times$.

Proposition 4.1. *[CTX, §5.6, §5.8] Assume $Y(k) \neq \emptyset$. If d is a square, then $\mathrm{Br}(Y)/\mathrm{Br}(k) = 0$. If d is not a square, then $\mathrm{Br}(Y)/\mathrm{Br}(k) = \mathbb{Z}/2$. For any field extension K/k, the natural map $\mathrm{Br}(Y)/\mathrm{Br}(k) \to \mathrm{Br}(Y_K)/\mathrm{Br}(K)$ is surjective.*

(i) If $\alpha x + \beta y + \gamma z + \delta = 0$ is an affine equation for the tangent plane of Y at a k-point of the projective quadric $q(x, y, z) - au^2 = 0$. then the quaternion algebra $(\alpha x + \beta y + \gamma z + \delta, d) \in \mathrm{Br}(k(Y))$ belongs to $\mathrm{Br}(Y)$ and it generates $\mathrm{Br}(Y)/\mathrm{Br}(k)$.

(ii) Assume $q(x, y, z) = xy - \det(q)z^2$. Then the quaternion algebra $(x, d) \in \mathrm{Br}(k(Y))$ belongs to $\mathrm{Br}(Y)$ and it generates $\mathrm{Br}(Y)/\mathrm{Br}(k)$. □

Lemma 4.2. *Let F be a number field. Let $q(x_1, \ldots, x_n)$ be a nondegenerate quadratic form over F. Let v be a nondyadic valuation of F. Assume $n \geq 3$. If the coefficients of $q(x_1, \ldots, x_n)$ are in $\mathfrak{o}_v$ and the determinant of $q(x_1, \ldots, x_n)$ is a unit in $\mathfrak{o}_v$, then for any $d \in \mathfrak{o}_v$ the equation $q(x_1, \ldots, x_n) = d$ admits a solution $(\alpha_1, \ldots, \alpha_n)$ in $\mathfrak{o}_v$ such that one of $\alpha_1, \ldots, \alpha_n$ is a unit in $\mathfrak{o}_v^\times$.*

Proof. This follows from Hensel's lemma. □

Lemma 4.3. *Let v be a nondyadic valuation of a number field F. Let $q(x, y, z)$ be a quadratic form defined over $\mathfrak{o}_v$ with $v(\det(q)) = 0$. Let $a \in \mathfrak{o}_v$, $a \neq 0$. Let $\mathbf{Y}$ be the $\mathfrak{o}_v$-scheme defined by the equation*

$$q(x, y, z) = a.$$

Let Y be the generic fibre of $\mathbf{Y}$ over F_v. Assume $-a. \det(q) \notin F_v^{\times 2}$. Let

$$\mathbf{Y}^*(\mathfrak{o}_v) = \{(x_v, y_v, z_v) \in \mathbf{Y}(\mathfrak{o}_v) : \text{ one of } x_v, y_v, z_v \in \mathfrak{o}_v^\times\}.$$

An element which represents the nontrivial element of $\mathrm{Br}(Y)/\mathrm{Br}(F_v)$ takes two values over $\mathbf{Y}^*(\mathfrak{o}_v)$ if and only if $v(a)$ is odd.

Proof. After an invertible $\mathfrak{o}_v$-linear change of coordinates, one may write

$$q(x,y,z) = xy - \det(q)z^2$$

over $\mathfrak{o}_v$. In the new coordinates, the set $\mathbf{Y}^*(\mathfrak{o}_v)$ is still defined by the same conditions on the coordinates. By Proposition 4.1, one has

$$\mathrm{Br}(Y)/\mathrm{Br}(F_v) \simeq \mathbb{Z}/2 \quad \text{ for } \quad -a.\det(q) \notin F_v^{\times^2}$$

and the generator is given by the class of the quaternion algebra

$$(x, -a.\det(q)) \in \mathrm{Br}(F_v(Y)).$$

If $v(a) = v(-a.\det(q))$ is odd, one can choose $(x_v, y_v, 0) \in \mathbf{Y}^*(\mathfrak{o}_v)$ where x_v is a square, resp. a nonsquare unit in $\mathfrak{o}_v^\times$. On these points, $(x, -a.\det(q))$ takes the value 0, resp. the value 1/2.

If $v(a) = v(-a.\det(q))$ is even, we claim that for any $(x_v, y_z, z_v) \in \mathbf{Y}^*(\mathfrak{o}_v)$, $v(x_v)$ is even. Indeed, suppose there exists $(x_v, y_v, z_v) \in \mathbf{Y}^*(\mathfrak{o}_v)$ such that $v(x_v)$ is odd. Then y_v or z_v is in $\mathfrak{o}_v^\times$. If we have $z_v \in \mathfrak{o}_v^\times$, then by Hensel's lemma $-a.\det(q) \in F_v^{\times^2}$, which is excluded. We thus have $z_v \notin \mathfrak{o}_v^\times$ and $y_v \in \mathfrak{o}_v^\times$. This implies $v(x_v y_v)$ is odd. Therefore

$$v(-\det(q).z_v^2) = v(a) < v(x_v y_v)$$

and $-a.\det(q) \in F_v^{\times^2}$ by Hensel's lemma. A contradiction is derived and the claim follows. By the claim, the algebra $(x, -a.\det(q))$ vanishes on $\mathbf{Y}^*(\mathfrak{o}_v)$. □

Lemma 4.4. *Let $k = F_v$ be a completion of the number field F. Let $q(x,y,z)$ be a nondegenerate quadratic form over k and let $a \in k^\times$. Let Y be the affine k-scheme defined by the equation*

$$q(x,y,z) = a.$$

Assume $-a.\det(q) \notin k^{\times 2}$. Assume Y has a k-point. One has $\mathrm{Br}(Y)/\mathrm{Br}(k) \simeq \mathbb{Z}/2$. Let ξ be an element of $\mathrm{Br}(Y)$ with nonzero image in $\mathrm{Br}(Y)/\mathrm{Br}(k)$. Then ξ takes a single value over $Y(k)$ if and only if v is a real place and q is anisotropic over F_v.

Proof. By Proposition 4.1, one has

$$\mathrm{Br}(Y)/\mathrm{Br}(k) \simeq \mathbb{Z}/2.$$

Let V be the quadratic space defined by $q(x,y,z)$ over k. Fix a k-point $m \in Y(k)$. To prove the lemma, we may take $\xi \in \mathrm{Br}(Y)$ to be the nonzero element, of order 2, which vanishes at m. Associated to the k-point m we have the map $SO(V) \to Y$ sending g to $g.m$. By a theorem of Witt, this map induces a surjective map $SO(V)(k) \to Y(k)$. By [CTX, p. 331], the composite map

$$SO(V)(k) \to Y(k) \to \mathrm{Br}(k),$$

where the map $Y(k) \to \mathrm{Br}(k)$ is defined by evaluation of ξ, coincides with the composite map

$$SO(V)(k) \to k^\times/k^{\times 2} \to k^\times/N_{K/k}(K^\times) \hookrightarrow \mathrm{Br}(k),$$

where $K = k(\sqrt{-a.\det(q)})$, the map $k^\times/N_{K/k}(K^\times) \hookrightarrow \mathrm{Br}(k)$ sends $c \in k^\times$ to the class of the quaternion algebra $(c, -a.\det(q))$, the map $\theta : SO(V)(k) \to k^\times/k^{\times 2}$ is the spinor map, and $k^\times/k^{\times 2} \to k^\times/N_{K/k}(K^\times)$ is the natural projection. This latter map is onto, and it is by assumption an isomorphism if $k = \mathbb{R}$. For k a nonarchimedean local field, the spinor map is surjective [OM, 91: 6]. For $k = \mathbb{R}$ the reals, the spinor map has trivial image in $\mathbb{R}^\times/\mathbb{R}^{\times 2} \simeq \pm 1$ if and only if the quadratic form q is anisotropic. $\square$

The following proposition does not appear formally in §6 of [CTX], where attention is restricted to schemes over the whole ring of integers. It follows however easily from Thm. 3.7 and §5.6 and §5.8 of [CTX].

Proposition 4.5. *Let F be a number field. Let Y/F be a smooth affine quadric defined by an equation*

$$q(x, y, z) = a.$$

Assume $Y(F) \neq \emptyset$. Let S be a finite set of places of F. Assume there exists $v_0 \in S$ such that q is isotropic over F_{v_0}. Then strong approximation with Brauer-Manin obstruction off S holds for Y. Namely, the closure of the image of $Y(F)$ under the diagonal map $Y(F) \to Y(\mathbb{A}_F^S)$ coincides with the image of $Y(\mathbb{A}_F)^{\mathrm{Br}(Y)} \subset Y(\mathbb{A}_F)$ under the projection map $Y(\mathbb{A}_F) \to Y(\mathbb{A}_F^S)$. $\square$

5 Computation of Brauer groups for the equation $q(x, y, z) = p(t)$

Let k be a field of characteristic zero, $q(x, y, z)$ a nondegenerate quadratic form in three variables over k and $p(t) \in k[t]$ a nonzero polynomial.

Let X be the affine variety defined by the equation

$$q(x,y,z) = p(t). \tag{5.1}$$

The singular points of $X_{\overline{k}}$ are the points $(0,0,0,t)$ with t a multiple root of p (Lemma 3.3). Let $U \subset X_{smooth}$ be the the complement of the closed set of X defined by $x = y = z = 0$.

Let $\pi : \tilde{X} \to X$ a desingularization of X, i.e. $\tilde{X}$ is smooth and integral, the k-morphism π is proper and birational. We moreover assume that the map $\pi : \pi^{-1}(X_{smooth}) \to X_{smooth}$ is an isomorphism. In particular $\pi : \pi^{-1}(U) \to U$ is an isomorphism.

Write $p(t) = c.p_1(t)^{e_1} \dots p_s(t)^{e_s}$, where c is in $k^\times$ and the $p_i(t)$, $1 \leq i \leq s$, are distinct monic irreducible polynomials over k. Let $k_i = k[t]/(p_i(t))$ for $1 \leq i \leq s$.

Let $K = \overline{k}(t)$ where $\overline{k}$ is an algebraic closure of k. The polynomial $p(t)$ is a square in K if and only if all the e_i are even.

In this section we compute the Brauer groups of U and the Brauer group of the desingularization $\tilde{X}$ of X. By purity for the Brauer group [G, Thm. (6.1)], we have $\mathrm{Br}(X_{smooth}) \stackrel{\simeq}{\to} \mathrm{Br}(U)$, and the group $\mathrm{Br}(\tilde{X})$ does not depend on the choice of the resolution of singularities $\tilde{X} \to X$ (see [G, Cor. (7.3) and Thm. (7.4)].)

The following lemma is well known (see [CTSk, Thm. 2.5]).

Lemma 5.1. *Let F be a field,* $\mathrm{char}(F) \neq 2$. *Let $\overline{F}$ be a separable closure of F, and let $g = \mathrm{Gal}(\overline{F}/F)$. Let $f(x,y,z,t)$ be a nondegenerate quadratic form over F. Let $d \in F^\times$ be its discriminant. Let $X \subset \mathbf{P}^3_F$ be the smooth quadric defined by $f = 0$.*

(a) There is an isomorphism of g-lattices $\mathrm{Pic}(\overline{X}) \simeq \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$*, with the following Galois action.*

(b) If $d \in F^{\times 2}$, the action of g on $\mathrm{Pic}(\overline{X})$ *is trivial.*

(c) If $d \notin F^{\times 2}$, the action of g factors through $\mathrm{Gal}(F(\sqrt{d})/F)$*, the non-trivial element of the latter group acting by permutation of e_1 and e_2.*

(d) The class $e_1 + e_2$ belongs to $\mathrm{Pic}(X) \subset \mathrm{Pic}(\overline{X})$*, it is the class of a hyperplane section of the quadric $X \subset \mathbf{P}^3_F$.*

(e) There is a natural exact sequence

$$0 \to \mathrm{Pic}(X) \to \mathrm{Pic}(\overline{X})^g \to \mathrm{Br}(F) \to \mathrm{Br}(X) \to 0.$$

□

Proposition 5.2. *Let $p(t) = c.p_1(t)^{e_1} \dots p_s(t)^{e_s}$, $q(x, y, z)$ and U be as in the beginning of this section. If $p(t)$ is not a square in $K = \overline{k}(t)$, i.e. if not all e_i are even, the natural map $\mathrm{Br}(k) \to \mathrm{Br}(U)$ is an isomorphism.*

Proof. Let Z be the closed subscheme of $\mathbf{P}^3_k \times \mathbf{A}^1_k$ defined by the equation

$$q(x, y, z) = p(t)u^2$$

where (x, y, z, u) are homogeneous coordinates for $\mathbf{P}^3_k$. Then X can be regarded as an open set in Z with $u \neq 0$. The complement of X in Z is given by $u = 0$ and isomorphic to $D = C \times_k \mathbf{A}^1_k$ where C is the projective conic in $\mathbf{P}^2_k$ defined by $q(x, y, z) = 0$. Let $f : \mathbf{P}^3_k \times \mathbf{A}^1_k \to \mathbf{A}^1_k$ be the projection onto $\mathbf{A}^1_k$. We shall abuse notation and also denote by f the restriction of f to Zariski open sets of X.

Let $U_{\overline{k}} = U \times_k \overline{k}$. Let $U_K = U \times_{\mathbf{A}^1_k} \mathrm{Spec}(K)$ and $Z_K = Z \times_{\mathbf{A}^1_k} \mathrm{Spec}(K)$. Any invertible function on $U_K \subset Z_K$ has its divisor supported in $u = 0$, which is an irreducible curve over K. Hence such a function is a constant in $K^\times$. Since the fibres of $f : U \to \mathbf{A}^1_k$ are nonempty, any invertible function on $U_{\overline{k}}$ is the inverse image of a function in $K[U]^\times = K^\times$ which is invertible on $\mathbf{A}^1_{\overline{k}}$, hence is in $\overline{k}^\times$. Thus

$$\overline{k}[U]^\times = \overline{k}^\times.$$

Let $V = Z_{smooth}$ and $V_{\overline{k}} = V \times_k \overline{k}$. Since $p(t)$ is not a square in K, the K-variety

$$V_K = V \times_{\mathbf{A}^1_k} \mathrm{Spec}(K) \subset \mathbf{P}^3_K$$

is a smooth projective quadric defined by a quadratic form whose discriminant is not a square. By Lemma 5.1 (c) (e) together with $\mathrm{Br}(K) = 0$ (Tsen's theorem), this implies that the abelian group $\mathrm{Pic}(V_K)$ is free of rank one and is spanned by the class of a hyperplane section of V_K. Since $U_K \subset V_K$ is the complement of the hyperplane section $u = 0$, this implies $\mathrm{Pic}(U_K) = 0$. Since U is smooth, $\mathrm{Pic}(\mathbf{A}^1_{\overline{k}}) = 0$ and all the fibres of $f : U \to \mathbf{A}^1_k$ are geometrically integral, the restriction map $\mathrm{Pic}(U_{\overline{k}}) \to \mathrm{Pic}(U_K)$ is an isomorphism. Thus

$$\mathrm{Pic}(U_{\overline{k}}) = 0.$$

Lemma 5.1 (e) and $\mathrm{Br}(K) = 0$ then yields $\mathrm{Br}(V_K) = 0$. Moreover, since $V_{\overline{k}}$ is regular, the natural map $\mathrm{Br}(V_{\overline{k}}) \to \mathrm{Br}(V_K)$ is injective. Therefore $\mathrm{Br}(V_{\overline{k}}) = 0$.

Let

$$C_{\overline{k}} = C \times_k \overline{k} \quad \text{and} \quad D_{\overline{k}} = D \times_k \overline{k}.$$

Since $D = C \times_k \mathbf{A}^1_k$ and $C_{\overline{k}} \simeq \mathbf{P}^1_{\overline{k}}$, we have $H^1_{ét}(D_{\overline{k}}, \mathbb{Q}/\mathbb{Z}) = 0$. Since $D_{\overline{k}}$ is a smooth divisor in the smooth variety $V_{\overline{k}}$, we have the exact localization sequence

$$0 \to \mathrm{Br}(V_{\overline{k}}) \to \mathrm{Br}(U_{\overline{k}}) \to H^1_{ét}(D_{\overline{k}}, \mathbb{Q}/\mathbb{Z}).$$

One concludes

$$\mathrm{Br}(U_{\overline{k}}) = 0.$$

The Hochschild-Serre spectral sequence for étale cohomology of the sheaf $\mathbf{G}_m$ and the projection morphism $U \to \mathrm{Spec}(k)$ yields a long exact sequence

$$\mathrm{Pic}(U_{\overline{k}})^g \to H^2(g, \overline{k}[U]^\times) \to \ker[\mathrm{Br}(U) \to \mathrm{Br}(U_{\overline{k}})] \to H^1(g, \mathrm{Pic}(U_{\overline{k}}))$$

where $g = \mathrm{Gal}(\overline{k}/k)$. Combining it with the displayed isomorphisms, we get

$$\mathrm{Br}(k) \simeq \mathrm{Br}(U).$$

□

Let us now consider the case where $p(t)$ is a square in $K = \overline{k}(t)$.

Proposition 5.3. *Let $p(t) = c.p_1(t)^{e_1} \dots p_s(t)^{e_s}$, $q(x, y, z)$ and U be as above. Assume all e_i are even, i.e. $p(t) = c.r(t)^2$ with $c \in k^\times$ and $r(t) \in k[t]$ nonzero. Let $d = -c. \det(q)$.*

The following conditions are equivalent:

(i) d is not a square in k and the natural map $H^3_{ét}(k, \mathbf{G}_m) \to H^3_{ét}(U, \mathbf{G}_m)$ is injective;

(ii) $\mathrm{Br}(U)/\mathrm{Br}(k) = \mathbb{Z}/2$.

If they are not satisfied then $\mathrm{Br}(U)/\mathrm{Br}(k) = 0$.

Proof. We keep the same notation as that in the proof of Proposition 5.2, in particular $g = \mathrm{Gal}(\overline{k}/k)$. Let $M = k(\sqrt{d})$. If $d \notin k^{\times 2}$, let $\widetilde{\mathbb{Z}}_d$ be the rank one g-lattice defined by the $\mathrm{Gal}(M/k)$-lattice such that $\sigma.x = -x$ for σ the nontrivial element in $\mathrm{Gal}(M/k)$. If $d \in k^{\times 2}$, let $\widetilde{\mathbb{Z}}_d = \mathbb{Z}$ with trivial g-action.

Since $p(t)$ is a square in $K = \overline{k}(t)$, one has $\mathrm{Pic}(V_K) \cong \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ (cf. Lemma 5.1). The Galois group $g = \mathrm{Gal}(\overline{k}/k)$ acts on $\mathrm{Pic}(V_K)$ trivially if $d \in k^{\times 2}$. If $d \notin k^{\times 2}$, then $\mathrm{Gal}(\overline{k}/k)$ acts on $\mathrm{Pic}(V_K)$ through $\mathrm{Gal}(M/k)$ with permutation action on the two generators e_1 and e_2. We thus have an isomorphism of g-modules $\mathrm{Pic}(U_K) \cong \widetilde{\mathbb{Z}}_d$.

By the same argument as those in the proof of Proposition 5.2, one has

$$\overline{k}^\times = \overline{k}[U]^\times, \quad \mathrm{Pic}(U_{\overline{k}}) \simeq \mathrm{Pic}(U_K) \simeq \widetilde{\mathbb{Z}}_d \quad \text{and} \quad \mathrm{Br}(U_{\overline{k}}) = 0.$$

Using $\operatorname{Br}(U_{\overline{k}}) = 0$, we deduce from the Hochschild-Serre spectral sequence a long exact sequence

$$\operatorname{Br}(k) \to \operatorname{Br}(U) \to H^1(g, \operatorname{Pic}(U_{\overline{k}})) \to H^3_{\acute{e}t}(k, \mathbf{G}_m) \to H^3_{\acute{e}t}(U, \mathbf{G}_m).$$

If $d \in k^{\times 2}$, one has

$$H^1(g, \operatorname{Pic}(U_{\overline{k}})) = \operatorname{Hom}_{cont}(g, \mathbb{Z}) = 0$$

and the long exact sequence yields $\operatorname{Br}(U)/\operatorname{Br}(k) = 0$.

Assume $d \notin k^{\times 2}$. From

$$H^1(g, \operatorname{Pic}(U_{\overline{k}})) = H^1(g, \widetilde{\mathbb{Z}}_d) = \mathbb{Z}/2$$

one gets an inclusion $\operatorname{Br}(U)/\operatorname{Br}(k) \subset \mathbb{Z}/2$, which is an equality if and only if $H^3_{\acute{e}t}(k, \mathbf{G}_m) \to H^3_{\acute{e}t}(U, \mathbf{G}_m)$ is injective. □

Remark 5.4. The natural map $H^3_{\acute{e}t}(k, \mathbf{G}_m) \to H^3_{\acute{e}t}(U, \mathbf{G}_m)$ is injective under each of the following hypotheses:

(i) the open set U has a point over a finite, odd degree extension of k;

(ii) the field k is a number field (in which case $H^3_{\acute{e}t}(k, \mathbf{G}_m) = 0$).

Proposition 5.5. *Keep notation as in Proposition 5.3. Assume that we have* $\operatorname{Br}(U)/\operatorname{Br}(k) = \mathbb{Z}/2$. *Then:*

(a) For any field extension L/k, the map $\operatorname{Br}(U)/\operatorname{Br}(k) \to \operatorname{Br}(U_L)/\operatorname{Br}(L)$ *is onto.*

(b) For any field extension L/k and any $\alpha \in \mathbf{A}^1(L)$ such that $p(\alpha) \neq 0$, the evaluation map $\operatorname{Br}(U)/\operatorname{Br}(k) \to \operatorname{Br}(U_\alpha)/\operatorname{Br}(L)$ *on the fibre $q(x, y, z) = p(\alpha)$ is onto.*

Proof. The long exact sequence

$$\operatorname{Br}(k) \to \operatorname{Br}(U) \to H^1(g_k, \operatorname{Pic}(U_{\overline{k}})) \to H^3_{\acute{e}t}(k, \mathbf{G}_m) \to H^3_{\acute{e}t}(U, \mathbf{G}_m).$$

is functorial in the base field k. The assumption $\operatorname{Br}(U)/\operatorname{Br}(k) = \mathbb{Z}/2$ and the possible Galois actions of the Galois group on $\operatorname{Pic}(U_{\overline{k}})$ (as discussed in the proof of the previous proposition) imply that the map $\operatorname{Br}(U)/\operatorname{Br}(k) \to H^1(g_k, \operatorname{Pic}(U_{\overline{k}}))$ is an isomorphism.

(a) Let $\overline{L}$ be an algebraic closure of L extending $k \subset \overline{k}$. If we have $\operatorname{Br}(U_L)/\operatorname{Br}(L) = 0$, the assertion is obvious. If $\operatorname{Br}(U_L)/\operatorname{Br}(L) \neq 0$, then $d \notin L^{\times 2}$ and $\operatorname{Br}(U_L)/\operatorname{Br}(L) = H^1(g_L, \operatorname{Pic}(U_{\overline{L}})) = \mathbb{Z}/2$. The natural map $\operatorname{Pic}(U_{\overline{k}}) \to \operatorname{Pic}(U_{\overline{L}})$ is an isomorphism of free rank one abelian groups which moreover is Galois-equivariant. Under the hypothesis $d \notin L^{\times 2}$, it is an isomorphism of $\mathrm{Gal}(M/k)$-modules. Thus the natural map $H^1(g_k, \mathrm{Pic}(U_{\overline{k}})) \to H^1(g_L, \mathrm{Pic}(U_{\overline{L}}))$ is an isomorphism. This implies that the map

$$\mathrm{Br}(U)/\mathrm{Br}(k) \to \mathrm{Br}(U_L)/\mathrm{Br}(L)$$

is an isomorphism, as claimed in (a).

(b) If d is a square in L, then $\mathrm{Br}(U_\alpha)/\mathrm{Br}(L) = 0$. Assume $d \notin L^{\times 2}$. Let $\overline{L}$ be an algebraic closure of L extending $k \subset \overline{k}$. By the functoriality of the Hochschild-Serre spectral sequence for the morphism $U_\alpha \to U$, we have a commutative diagram of exact sequences

$$(5.2)\qquad \begin{array}{ccccccccc} \mathrm{Br}(k) & \to & \mathrm{Br}(U) & \to & H^1(g_k, \mathrm{Pic}(U_{\overline{k}})) & \to & H^3_{\acute{e}t}(k, \mathbf{G}_m) & \to & H^3_{\acute{e}t}(U, \mathbf{G}_m) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathrm{Br}(L) & \to & \mathrm{Br}(U_\alpha) & \to & H^1(g_L, \mathrm{Pic}(U_{\alpha,\overline{L}})) & \to & H^3_{\acute{e}t}(L, \mathbf{G}_m) & \to & H^3_{\acute{e}t}(U_\alpha, \mathbf{G}_m) \end{array}$$

One readily verifies that the evaluation map $\mathrm{Pic}(U_{\overline{k}}) \to \mathrm{Pic}(U_{\alpha,\overline{L}})$ is an isomorphism of Galois modules (split by a quadratic extension), hence the map $H^1(g_k, \mathrm{Pic}(U_{\overline{k}})) \to H^1(g_L, \mathrm{Pic}(U_{\alpha,\overline{L}}))$ is an isomorphism $\mathbb{Z}/2 = \mathbb{Z}/2$. From the diagram we conclude that $\mathrm{Br}(U) \to \mathrm{Br}(U_\alpha)/\mathrm{Br}(L)$ is onto. $\square$

Proposition 5.6. *Let $p(t) = c.\prod_{i\in I} p_i(t)^{e_i}$, $q(x, y, z)$, X, U and the map $\pi : \tilde{X} \to X$ be as above. Assume $H^3_{\acute{e}t}(k, \mathbf{G}_m) \to H^3_{\acute{e}t}(U, \mathbf{G}_m)$ is injective. Let $d = -c.\det(q)$.*

Consider the following conditions:

(i) All e_i are even, i.e. $p(t) = c.r(t)^2$ for $c \in F^\times$ and some $r(t) \in k[t]$.

(ii) $d \notin k^{\times 2}$.

(iii) For each $i \in I$, $d \in k_i^{\times 2}$.

We have:

(a) If (i) or (ii) or (iii) is not fulfilled, then $\mathrm{Br}(\tilde{X})/\mathrm{Br}(k) = 0$.

(b) Assume $U(k) \neq \emptyset$. If (iii) is fulfilled, then $\mathrm{Br}(\tilde{X}) \xrightarrow{\simeq} \mathrm{Br}(U)$.

(c) If (i), (ii) and (iii) are fulfilled, then

$$\mathrm{Br}(\tilde{X})/\mathrm{Br}(k) \xrightarrow{\simeq} \mathrm{Br}(U)/\mathrm{Br}(k) = \mathbb{Z}/2.$$

In that case, for any field extension L/k and any $\alpha \in L$ such that $p(\alpha) \neq 0$, the evaluation map $\mathrm{Br}(\tilde{X})/\mathrm{Br}(k) \to \mathrm{Br}(X_\alpha)/\mathrm{Br}(L)$ is surjective.

Proof. One has $\mathrm{Br}(\tilde{X}) \subset \mathrm{Br}(U)$.

Proof of (a)

By Proposition 5.2, resp. Proposition 5.3, if (i), resp. (ii), is not fulfilled, then $\mathrm{Br}(U)/\mathrm{Br}(k) = 0$. Assume (i) and (ii) are fulfilled. Proposition 5.3 then gives $\mathrm{Br}(U)/\mathrm{Br}(k) \simeq \mathbb{Z}/2$.

Let F be the function field of the smooth projective conic C defined by $q(x, y, z) = 0$. Assume (iii) does not hold. Let $i \in I$ such that $d \notin k_i^{\times 2}$. Let F_i be the composite field $F.k_i$. Since k is algebraically closed in F, so is k_i in F_i. Thus d is not a square in F_i.

By the same argument as in Proposition 5.5, the map

$$\mathbb{Z}/2 = \mathrm{Br}(U)/\mathrm{Br}(k) \to \mathrm{Br}(U_{F_i})/\mathrm{Br}(F_i)$$

is an isomorphism. Over the field F_i, one may rewrite the equation of X_{F_i} as

$$xy - \det(q)z^2 = c.r(t)^2$$

and assume that $t = 0$ is a root of $r(t)$. After restriction to the generic fibre of $U_{F_i} \to \mathrm{Spec}(F_i[t])$, the quaternion algebra $(x, d) \in \mathrm{Br}(F_i(X))$ defines a generator modulo $\mathrm{Br}(F_i(t))$. This follows from Proposition 4.1. Now the algebra $(x, d) = (y.\det(q), d)$ is unramified on the complement of the closed set $\{x = y = 0\}$ on U_{F_i}, of codimension 2 in U_{F_i}, thus (x, d) belongs to $\mathrm{Br}(U_{F_i})$. It thus generates $\mathrm{Br}(U_{F_i})/\mathrm{Br}(F_i)$.

Define $h(T) \in k[T]$ by $Th(T) = cr(T)^2$. Consider the morphism

$$\sigma : \mathrm{Spec}(F_i[[T]]) \to X$$

defined by

$$(x, y, z, t) = (T, h(T), 0, T).$$

The induced morphism $\mathrm{Spec}(F_i((T))) \to X$ has its image in U. Since $\pi : \tilde{X} \to X$ is proper, we conclude that the morphism σ lifts to a morphism $\tilde{\sigma} : \mathrm{Spec}(F_i[[T]]) \to \tilde{X}$. Suppose $(x, d) \in \mathrm{Br}(U_{F_i})$ is in the image of $\mathrm{Br}(\tilde{X}_{F_i}) \to \mathrm{Br}(U_{F_i})$. Then $\tilde{\sigma}^*((x, d)) = (T, d)$ belongs to $\mathrm{Br}(F_i[[T]])$. But the residue of $(T, d) \in \mathrm{Br}(F_i(T))$ at $T = 0$ is $d \neq 1 \in F_i^{\times}/F_i^{\times 2}$. This is a contradiction. Taking into account Proposition 5.5, we conclude that the embedding $\mathrm{Br}(\tilde{X})/\mathrm{Br}(k) \hookrightarrow \mathrm{Br}(U)/\mathrm{Br}(k) = \mathbb{Z}/2$ is not onto, hence $\mathrm{Br}(\tilde{X})/\mathrm{Br}(k) = 0$.

Proof of (b)

Let $E = k(\sqrt{d})$. By Proposition 5.3, we have $\mathrm{Br}(U_E)/\mathrm{Br}(E) = 0$. Using the hypothesis $U(k) \neq \emptyset$, we see that any element of $\mathrm{Br}(U) \subset \mathrm{Br}(k(U))$ may be represented as the sum of an element of $\mathrm{Br}(k)$ and the class of a quaternion algebra (g, d) for some $g \in k(U)^{\times}$.

Assume (iii) is fulfilled. Let x be a point of codimension 1 of $\tilde{X}$ which does not belong to $p^{-1}(U)$. Let v be the associated discrete rank one valuation on the function field of X. We then have $v(p_i(t)) > 0$ for some $i \in I$. We thus have $k \subset k_i \subset \kappa_v$, where $\kappa_v = \kappa(x)$ is the residue field of v. If assumption (iii) is fulfilled we conclude that d is a square in κ_v.

But then the residue of (g, d) at x, which is a power of d in $\kappa_v^\times/\kappa_v^{\times 2}$, is trivial. By purity for the Brauer group, we conclude $\mathrm{Br}(\tilde{X})/\mathrm{Br}(k) = \mathrm{Br}(U)/\mathrm{Br}(k)$. This proves (b).

Proof of (c)

This follows from Proposition 5.3 and Proposition 5.5. □

Let Q be the smooth affine quadric over k defined by $q(x, y, z) = c$. For simplicity, let us assume $Q(k) \neq \emptyset$. In the situation of Proposition 5.3, with $d = -c.\det(q) \notin (k^\times)^2$, one may give an explicit generator in $\mathrm{Br}(U)$ for $\mathrm{Br}(U)/\mathrm{Br}(k) = \mathbb{Z}/2$.

The assumption $Q(k) \neq \emptyset$ implies $U(k) \neq \emptyset$. By Prop. 4.1, we have $\mathrm{Br}(Q)/\mathrm{Br}(k) = \mathbb{Z}/2$. Let $\alpha x + \beta y + \gamma z + \delta = 0$ define the tangent plane of Q at some k-point. Not all α, β, γ are zero. As recalled in Proposition 4.1,

$$A = (\alpha x + \beta y + \gamma z + \delta, d) \in \mathrm{Br}(k(Q))$$

belongs to $\mathrm{Br}(Q)$ and generates $\mathrm{Br}(Q)/\mathrm{Br}(k)$.

Given a nonzero $r(t) \in k[t]$, let $W = Q \times_k (\mathbf{A}^1_k \setminus \{r(t) = 0\})$. Consider the birational k-morphism

$$f : Q \times_k \mathbf{A}^1_k \to X \subset \mathbf{A}^4_k; \quad (x, y, z, t) \mapsto (r(t)x, r(t)y, r(t)z, t).$$

This map induces an isomorphism between $W = Q \times_k \{\mathbf{A}^1 \setminus \{r(t) = 0\}$ and the open set V of $U = X_{smooth}$ defined by $r(t) \neq 0$. Let A_V be the image of A inside $\mathrm{Br}(V)$ under the composition map

$$\mathrm{Br}(Q) \to \mathrm{Br}(W) \cong \mathrm{Br}(V).$$

Proposition 5.7. *Let $p(t) = c.r(t)^2$ with $c \in k^\times$ and $r(t) \in k[t]$ nonzero. Assume*

$$d = -c.\det(q) \notin k^{\times^2}.$$

Assume $Q(k) \neq \emptyset$. With notation as above, the element

$$B = A_V + (r(t), d) = (\alpha x + \beta y + \gamma z + \delta r(t), d) \in \mathrm{Br}(V)$$

can be extended to $\mathrm{Br}(U)$ and it generates the group $\mathrm{Br}(U)/\mathrm{Br}(k) \simeq \mathbb{Z}/2$.

Proof. On $V \subset U = X_{smooth}$, we have

$$A_V = (\alpha x/r(t)+\beta y/r(t)+\gamma z/r(t)+\delta, d) = (\alpha x+\beta y+\gamma z+\delta r(t), d)-(r(t), d).$$

Thus

$$B = A_V + (r(t), d) = (\alpha x + \beta y + \gamma z + \delta r(t), d) \in \mathrm{Br}(k(V))$$

is unramified on V. To check that it is unramified on U, it is enough to compute the residue at the generic point of each component of $r(t) = 0$ on U. These are defined by a system $p_i(t) = 0, q(x, y, z) = 0$. But at such a point, $\alpha x+\beta y+\gamma z+\delta r(t)$ is a unit since it induces the class of $\alpha x+\beta y+\gamma z$ on the residue field, and this is not zero since $\alpha x + \beta y + \gamma z$ is not divisible by $q(x, y, z)$. Since d is clearly a unit, we conclude that B is not ramified at such points. The natural map $\mathrm{Br}(Q)/\mathrm{Br}(k) \to \mathrm{Br}(Q_{k(t)})/\mathrm{Br}(k(t))$ is the identity on $\mathbb{Z}/2$. It sends the nontrivial class A to the class of B. The image of B in $\mathrm{Br}(U)/\mathrm{Br}(k) = \mathbb{Z}/2$ is thus nontrivial. □

One may use this proposition to give a more concrete description of specialization of the Brauer group, as discussed in Propositions 5.5 and 5.6.

6 Arithmetic of the equation $q(x, y, z) = p(t)$

Let F be a number field, $q(x, y, z)$ a nondegenerate quadratic form in three variables over F and $p(t) \in F[t]$ a nonzero polynomial. Let X be the affine variety over F defined by the equation

$$q(x, y, z) = p(t). \tag{6.1}$$

The singular points of $X_{\overline{F}}$ are the points $(0, 0, 0, t)$ with t a multiple root of p (Lemma 3.3). Let $U \subset X_{smooth}$ be the complement of the closed set of X defined by $x = y = z = 0$.

Let $\pi : \tilde{X} \to X$ a desingularization of X, i.e. $\tilde{X}$ is smooth and integral, the map π is proper and birational. We assume that $\pi : \pi^{-1}(X_{smooth}) \to X_{smooth}$ is an isomorphism. Thus $\pi : \pi^{-1}(U) \to U$, is an isomorphism. This allows us to view U as an open set of $\tilde{X}$.

Write $p(t) = c.p_1(t)^{e_1} \dots p_s(t)^{e_s}$, with c is in $F^\times$ and the $p_i(t)$, $1 \leq i \leq s$ distinct monic irreducible polynomials over F. Let $F_i = F[t]/(p_i(t))$ for $1 \leq i \leq s$.

Under some local isotropy condition for q, we investigate strong approximation for the F-variety $\tilde{X}$.

This variety is equipped with an obvious fibration $\tilde{X} \to \mathbf{A}^1_F = \mathrm{Spec}(F[t])$.

We begin with two lemmas.

Lemma 6.1. *If $r(t)$ is an irreducible polynomial over a number field F, then there are infinitely many valuations v of F for which there exist infinitely many $t_v \in \mathfrak{o}_v$ with $v(r(t_v)) = 1$.*

Proof. By Chebotarev's theorem, there are infinitely many valuations v of F which are totally split in the field $F[t]/(r(t))$. Let d denote the degree of $r(t)$. For almost all such v, we may write

$$r(t) = c\prod_{i=1}^{d}(t-\xi_i) \in F_v[t]$$

with all ξ_i in $\mathfrak{o}_v$ and c and all $\xi_i - \xi_j$ $(i \neq j)$ units in $\mathfrak{o}_v$. Since there are infinitely many elements of $\mathfrak{o}_v$ with v-valuation 1, there exist infinitely many $t_v \in \mathfrak{o}_v$ such that $v(t_v - \xi_1) = 1$. Then $v(r(t_v)) = 1$. □

Lemma 6.2. *Let F be a number field, and $q(x, y, z)$ and $p(t)$ be as above. If not all e_i are even, then there exist infinitely many valuations w of F for which there exists $t_w \in \mathfrak{o}_w$ with $w(p(t_w))$ odd and $-p(t_w).\det(q) \notin F_w^{\times 2}$.*

Proof. Assume e_{i_0} is odd for some $i_0 \in \{1, \cdots, s\}$. If $s = 1$, the result immediately follows from Lemma 6.1. Assume $s > 1$.

For any $j \neq i_0$, there are polynomials $a_j(t)$ and $b_j(t)$ over F such that

$$a_j(t)p_j(t) + b_j(t)p_{i_0}(t) = 1 \tag{6.2}$$

holds.

Let S be a finite set of primes such that each of the following conditions hold:

(i) the coefficients of q are integral away from S;

(ii) $w(c) = w(\det(q)) = 0$ for all $w \notin S$;

(iii) the coefficients of $a_j(t), b_j(t)$ for $j \neq i_0$ and of $p_i(t)$ for $1 \leq i \leq s$ are in $\mathfrak{o}_w$ for all $w \notin S$.

By applying Lemma 6.1 to $p_{i_0}(t)$, we see that there exist infinitely many primes $w \notin S$ and $t_w \in \mathfrak{o}_w$ such that $w(p_{i_0}(t_w)) = 1$. By equation (6.2), one has $w(p_j(t_v)) = 0$ for any $j \neq i_0$. This implies $w(p(t_w)) = e_{i_0}$ is odd. Therefore $-p(t_w) \cdot \det(q) \notin F_w^{\times 2}$. □

Proposition 6.3. *Let F be a number field and X be an F-variety defined by an equation*

$$q(x,y,z) = p(t)$$

where $q(x,y,z)$ is a nondegenerate quadratic form over F and $p(t)$ is a nonzero polynomial in $F[t]$. Assume $X_{smooth}(F_v) \neq \emptyset$ for each place v of F. Then

(1) $X_{smooth}(F)$ is Zariski-dense in X.

(2) X_{smooth} satisfies weak approximation.

Proof. This is a special case of Thm. 3.10, p. 66 of [CTSaSD]. □

Theorem 6.4. *Let F be a number field. Let $U \subset \tilde{X}$ be as above. Assume $U(\mathbb{A}_F) \neq \emptyset$. Let S be a finite subset of Ω_F which contains a place v_0 such that the quadratic form $q(x,y,z)$ is isotropic over F_{v_0}. Then strong approximation off S with Brauer-Manin condition holds for any open set V with $U \subset V \subset \tilde{X}$, in particular for X_{smooth}.*

Since $\tilde{X}$ is smooth and geometrically integral, the hypotheses $U(\mathbb{A}_F) \neq \emptyset$, $X_{smooth}(\mathbb{A}_F) \neq \emptyset$ and $\tilde{X}(\mathbb{A}_F) \neq \emptyset$ are all equivalent.

Taking into account the isomorphism $\mathrm{Br}(X_{smooth}) \xrightarrow{\simeq} \mathrm{Br}(U)$, the finiteness of $\mathrm{Br}(U)/\mathrm{Br}(F)$ (§5) and Proposition 2.6, this theorem is an immediate consequence of the following more precise statement.

Theorem 6.5. *Let F be a number field. Let $p(t) = c.p_1(t)^{e_1} \dots p_s(t)^{e_s}$, $q(x,y,z)$, X, U and $\tilde{X}$ be as above. Let $d = -c.\det(q)$. Let S be a finite subset of Ω_F which contains a place v_0 such that the quadratic form $q(x,y,z)$ is isotropic over F_{v_0}. Assume $U(\mathbb{A}_F) \neq \emptyset$.*

Then $U(F) \neq \emptyset$ is Zariski dense in U.

(i) If at least one e_i is odd, then

$$\mathrm{Br}(\tilde{X})/\mathrm{Br}(F) = \mathrm{Br}(U)/\mathrm{Br}(F) = 0,$$

and strong approximation off S holds for U and for $\tilde{X}$.

(ii) If all e_i are even and $d \in F^{\times 2}$, then

$$\mathrm{Br}(\tilde{X})/\mathrm{Br}(F) = \mathrm{Br}(U)/\mathrm{Br}(F) = 0,$$

and strong approximation off S holds for U and for $\tilde{X}$.

(iii) If all e_i are even and there exists i such that $d \notin F_i^{\times 2}$, then

$$\mathrm{Br}(\tilde{X})/\mathrm{Br}(F) = 0, \ \ \mathrm{Br}(U)/\mathrm{Br}(F) = \mathbb{Z}/2,$$

strong approximation off S with Brauer-Manin condition holds for U and for any open set V with $U \subset V \subset \tilde{X}$. Strong approximation holds for $\tilde{X}$ and for any open set V with $U \subset V \subset \tilde{X}$ which satisfies $\mathrm{Br}(\tilde{X}) \stackrel{\simeq}{\to} \mathrm{Br}(V)$.

(iv) If all e_i are even, $d \notin F^{\times 2}$, and for all i, $d \in F_i^{\times 2}$, then

$$\mathrm{Br}(\tilde{X})/\,\mathrm{Br}(F) = \mathrm{Br}(U)/\,\mathrm{Br}(F) = \mathbb{Z}/2,$$

and strong approximation off S with Brauer-Manin condition holds for U and for $\tilde{X}$.

(v) Strong approximation off S fails for U, resp. for $\tilde{X}$, if and only if the following two conditions simultaneously hold:

(a) $\mathrm{Br}(U)/\,\mathrm{Br}(F) = \mathbb{Z}/2$, *resp.* $\mathrm{Br}(\tilde{X})/\,\mathrm{Br}(F) = \mathbb{Z}/2$;

(b) d is a square in F_v for each finite place $v \in S$ and also for each real place $v \in S$ such that either $q(x,y,z)$ is isotropic over F_v or $r(t)$ has a root over F_v.

Proof. By Proposition 6.3, $U(F) \neq \emptyset$ and $U(F)$ is Zariski dense in U. The various values of $\mathrm{Br}(U)$ and $\mathrm{Br}(X)$ have been computed in §5. By Proposition 2.3 and Proposition 2.6, to prove (i) to (iv), it is enough to prove the strong approximation statements (with Brauer-Manin obstruction) for U.

We fix a finite set T of places, which contains S, the infinite primes, the dyadic primes and all the finite places v where $q(x,y,z)$ has bad reduction. We also assume that $p(t)$ has coefficients in $\mathfrak{o}_T$ and that its leading coefficient c is invertible in $\mathfrak{o}_T$. We denote by $\mathbf{X}$ the $\mathfrak{o}_T$-scheme given by

$$q(x,y,z) = p(t).$$

We let $\mathbf{U} \subset \mathbf{X}$ be the complement of the closed set defined by the ideal (x,y,z). We may extend T so that there is a smooth integral $\mathfrak{o}_T$-scheme $\tilde{\mathbf{X}}$ equipped with a proper birational $\mathfrak{o}_T$-morphism $\tilde{\mathbf{X}} \to \mathbf{X}$ extending the map $\pi : \tilde{X} \to X$.

For any $v \notin T$, $\mathbf{U}(\mathfrak{o}_v)$ is the set of points (x_v, y_v, z_v, t_v) with all coordinates in $\mathfrak{o}_v$, $q(x_v,y_v,z_v) = p(t_v)$ and one of (x_v,y_v,z_v) a unit. By Lemma 4.2, given any $t_v \in \mathfrak{o}_v$, this set is not empty.

To prove the statements (i) to (iv), after possibly increasing T, we have to prove that for any such finite set T containing S, a nonempty open set of $U(\mathbb{A}_F)$ of the shape

$$W_U = [\prod_{v \in S} U(F_v) \times \prod_{v \in T \setminus S} U_v \times \prod_{v \notin T} \mathbf{U}(\mathfrak{o}_v)]^{\mathrm{Br}(U)}$$

with U_v open in $U(F_v)$, contains a point in $U(F)$.

Given $t_0 \in \mathfrak{o}_T = \mathbf{A}^1(\mathfrak{o}_T)$ with $p(t_0) \neq 0$, we let $\mathbf{U}_{t_0}/\mathrm{Spec}(\mathfrak{o}_T)$ be the fibre of $\mathbf{U}/\mathbf{A}^1_{\mathfrak{o}_T}$ above t_0. This is the $\mathfrak{o}_T$-scheme defined by $q(x,y,z) = p(t_0)$. We let $U_{t_0} = \mathbf{U}_{t_0} \times_{\mathfrak{o}_T} F$.

It is enough to show that in each of the cases under consideration:

There exists $t_0 \in \mathfrak{o}_T$ such that the set

$$[\prod_{v\in S} U_{t_0}(F_v) \times \prod_{v \in T\setminus S} U_v \cap U_{t_0}(F_v) \times \prod_{v \notin T} \mathbf{U}_{t_0}(\mathfrak{o}_v)]^{\mathrm{Br}(U_{t_0})}$$

is nonempty.

Indeed, Proposition 4.5 implies that such a nonempty set contains an F-rational point.

We have $\mathrm{Br}(U)/\mathrm{Br}(F) \subset \mathbb{Z}/2$. If $\mathrm{Br}(U)/\mathrm{Br}(F)$ is nonzero, we may represent the group by an element ξ of order 2 in $\mathrm{Br}(U)$. To prove the result, we may extend T. After doing so, we may assume that ξ vanishes identically on each $\mathbf{U}(\mathfrak{o}_v)$ for $v \notin T$.

We start with a point $\{M_v\} = \{(x_v, y_v, z_z, t_v)\}_{v\in\Omega_F}$ in W_U such that $p(t_v) \neq 0$ for each $v \in \Omega_F$.

We have

$$\sum_v \xi(M_v) = 0 \in \mathbb{Z}/2.$$

In case (i), we choose a $w \notin T$ and a $t'_w \in \mathfrak{o}_w$ with $w(p(t'_w))$ odd and $-p(t'_w).\det(q) \notin F_w^{\times 2}$. The existence of such w, t'_w is guaranteed by Lemma 6.2.

Using the strong approximation theorem, we find a $t_0 \in \mathfrak{o}_T$ which is very close to each t_v for $v \in T \setminus \{v_0\}$ and is also very close to t'_w in case (i).

By Lemma 4.2, as recalled above, for each $v \notin S$, the projection map $\mathbf{U}(\mathfrak{o}_v) \to \mathbf{A}^1(\mathfrak{o}_v)$ is onto. By assumption, q is isotropic at $v_0 \in S$, hence $U(F_{v_0}) \to \mathbf{A}^1(F_{v_0})$ is onto.

Combining this with the implicit function theorem, we find an adèle $\{P_v\} \in U_{t_0}(\mathbb{A}_F) = \tilde{X}_{t_0}(\mathbb{A}_F)$ with the following properties:

- For $v \in T \setminus \{v_0\}$, P_v is very close to M_v in $U(F_v)$, hence belongs to $U_v \cap \mathbf{U}_{t_0}(F_v)$ for $v \in T \setminus S$. Moreover $\xi(M_v) = \xi(P_v)$.
- For $v \notin T$, $P_v \in \mathbf{U}_{t_0}(\mathfrak{o}_v)$, hence $\xi(P_v) = 0 = \xi(M_v)$.

By the Hasse principle, there exists an F-point on the affine F-quadric $U_{t_0} = \tilde{X}_{t_0}$.

Consider case (i). By the definition of w, $w(p(t_0))$ is odd, $-p(t_0).\det(q) \notin F_w^{\times 2}$, hence $-p(t_0).\det(q) \notin F^{\times 2}$, thus

$$\mathbb{Z}/2 = \mathrm{Br}(U_{t_0})/\mathrm{Br}(F) \simeq \mathrm{Br}(U_{t_0,F_w})/\mathrm{Br}(F_w)$$

by Proposition 4.1. Let $\rho \in \mathrm{Br}(U_{t_0})$ be an element of order 2 which generates these groups.

If $\sum_v \rho(P_v) = 0$, the adèle $\{P_v\} \in U_{t_0}(\mathbb{A}_F)$ belongs to the Brauer-Manin set of U_{t_0}.

Suppose $\sum_v \rho(P_v) = 1/2$. By Lemma 4.3 , ρ takes two distinct values on $\mathbf{U}_{t_0}(\mathfrak{o}_w)$. We may thus choose a new point $P_w \in \mathbf{U}_{t_0}(\mathfrak{o}_w)$ such that now $\sum_v \rho(P_v) = 0$, that is the new adèle $\{P_v\} \in U_{t_0}(\mathbb{A}_F)$ belongs to the Brauer-Manin set of U_{t_0}, which completes the proof in this case.

Consider case (ii). In this case $-\det(q).p(t_0) \in F^{\times 2}$, hence we have $\mathrm{Br}(U_{t_0})/\mathrm{Br}(F) = 0$ by Proposition 4.1. Thus the adèle $\{P_v\} \in U_{t_0}(\mathbb{A}_F)$ is trivially in the Brauer-Manin set of U_{t_0}, which completes the proof in this case.

Let us consider (iii) and (iv). In these cases, $-c.\det(q) \notin F^{\times 2}$, hence $-\det(q).p(t) \notin F(t)^{\times 2}$ and $-\det(q).p(t_0) \notin F^{\times 2}$ for any $t_0 \in F$. We have $\mathrm{Br}(U)/\mathrm{Br}(F) = \mathbb{Z}/2$ and $\mathrm{Br}(U_{t_0})/\mathrm{Br}(F) = \mathbb{Z}/2$ for any t_0 with $p(t_0) \neq 0$. The element $\xi \in \mathrm{Br}(U)$ has now exact order 2. It generates $\mathrm{Br}(U)/\mathrm{Br}(F)$. The restriction of this element to $\mathrm{Br}(U_{t_0})/\mathrm{Br}(F) = \mathbb{Z}/2$ is the generator of that group (Propositions 5.3 and 5.5).

By hypothesis, $\sum_v \xi(M_v) = 0$. We then have

$$\sum_v \xi(P_v) = \xi(P_{v_0}) + \sum_{v \in T \setminus \{v_0\}} \xi(P_v) = \xi(P_{v_0}) + \sum_{v \in T \setminus \{v_0\}} \xi(M_v) = \xi(P_{v_0}) - \xi(M_{v_0}).$$

If $d \in F_{v_0}^{\times 2}$, then $\mathrm{Br}(U_{F_{v_0}})/\mathrm{Br}(F_{v_0}) = 0$ (Prop. 5.3), from which we deduce $\xi(P_{v_0}) - \xi(M_{v_0}) = 0$. We thus get $\sum_v \xi(P_v) = 0$. The adèle $\{P_v\}$ is in the Brauer-Manin set of U_{t_0}.

Assume $d \notin F_{v_0}^{\times 2}$. Then $\mathrm{Br}(U_{F_{v_0}})/\mathrm{Br}(F_{v_0}) \xrightarrow{\simeq} \mathrm{Br}(U_{t_0,F_{v_0}})/\mathrm{Br}(F_{v_0}) = \mathbb{Z}/2$ (Propositions 5.3 and 5.5). The image of ξ in $\mathrm{Br}(U_{t_0,F_{v_0}})/\mathrm{Br}(F_{v_0})$ generates this group. By Lemma 4.4, the class ξ takes two distinct values on $U_{t_0}(F_{v_0})$. This holds whether v_0 is real or not, because by assumption q is isotropic at the place v_0. We may then change $P_{v_0} \in U_{t_0}(F_{v_0})$ in order to ensure that $\xi(P_{v_0}) - \xi(M_{v_0}) = 0$, which yields $\sum_v \xi(P_v) = 0$. The adèle $\{P_v\}$ is in the Brauer-Manin set of U_{t_0}.

This proves (iii) and (iv) for U.

It remains to establish (v).

Assume (a) and (b). Under (a), all e_i are even and $d \notin F^{\times 2}$. We let ξ be an element of exact order 2 in $\mathrm{Br}(U)$, resp. $\mathrm{Br}(\tilde{X})$, which generates $\mathrm{Br}(U)/\mathrm{Br}(F)$, resp. $\mathrm{Br}(\tilde{X})/\mathrm{Br}(F)$. Under (b), at each finite place $v \in S$, by

Proposition 5.3 we have $\xi_{F_v} \in \mathrm{Br}(F_v)$, hence ξ is constant on $U(F_v)$, resp. $\tilde{X}(F_v)$. The same holds at a real place v such that $d \in F_v^{\times 2}$. At a real place $v \in S$ such that $d \notin F_v^{\times 2}$, the form $q(x,y,z)$ is anisotropic over F_v and $r(t)$ has no real root. At such v, the equation after suitable transformation reads $x^2+y^2+z^2 = (r(t))^2$ and $U(F_v) = U(\mathbb{R})$ is connected. Then ξ is constant on $U(\mathbb{R})$.

Let M be a point of $U(F)$, resp. $\tilde{X}(F)$, with $p(t(M)) \neq 0$. Since we have $d \notin F^{\times 2}$, there are infinitely many finite places $w \notin S$ such that $d \notin F_w^{\times 2}$. At such a place w, ξ takes two distinct values on $U_{t(M)}(F_w) = \tilde{X}_{t(M)}(F_w)$ (use Proposition 5.5 and Lemma 4.4). Pick $P_w \in U_{t(M)}(F_w)$ such that $\xi(P_w) \neq \xi(M)_{F_w} \in \mathbb{Z}/2$. If we let $\{P_v\}$ be the adèle of U, resp. $\tilde{X}$ with $P_v = M$ for $v \neq w$ and P_w as just chosen, then $\sum_v \xi(P_v) \neq 0$, and this adèle lies in an open set of the shape $\prod_{v\in S} U(F_v) \times \prod_{v \in T\setminus S} U_v \times \prod_{v \notin T} \mathbf{U}(\mathfrak{o}_v)$, resp. $\prod_{v\in S} \tilde{X}(F_v) \times \prod_{v\in T\setminus S} U_v \times \prod_{v\notin T} \tilde{\mathbf{X}}(\mathfrak{o}_v)$, which contains no diagonal image of $U(F)$, resp. $\tilde{X}(F)$. Strong approximation off S therefore fails for U, resp. $\tilde{X}$.

Suppose either (a) or (b) fails. Let us prove that strong approximation holds off S. If (a) fails, then $\mathrm{Br}(U)/\mathrm{Br}(F) = 0$, resp. $\mathrm{Br}(\tilde{X})/\mathrm{Br}(F) = 0$, and we have proved that strong approximation holds off S. We may thus assume $\mathrm{Br}(U)/\mathrm{Br}(F) = \mathbb{Z}/2$, resp. $\mathrm{Br}(\tilde{X})/\mathrm{Br}(F) = \mathbb{Z}/2$, hence all e_j are even and $d \notin F^{\times 2}$, and that (b) fails. Then either

(i) there exists a finite place $v \in S$ with $d \notin F_v^{\times 2}$

or

(ii) there exists a real place $v \in S$ with $d \notin F_v^{\times 2}$, i.e. $d < 0$, such that q is isotropic over F_v or $r(t)$ has a root in F_v.

We let ξ be an element of exact order 2 in $\mathrm{Br}(U)$, resp. $\mathrm{Br}(\tilde{X})$ which generates $\mathrm{Br}(U)/\mathrm{Br}(F)$, resp. $\mathrm{Br}(\tilde{X})/\mathrm{Br}(F)$. For any $t_v \in \mathbf{A}^1(F_v)$ with $p(t_v) \neq 0$, ξ generates $\mathrm{Br}(U_{t_v})/\mathrm{Br}(F_v)$, resp. $\mathrm{Br}(\tilde{X}_{t_v})/\mathrm{Br}(F_v)$ (Proposition 5.5). If v is a finite place of S with $d \notin F_v^{\times 2}$ then, by Lemma 4.4, above any point of $t_v \in \mathbf{A}^1(F_v)$ with $p(t_v) \neq 0$, ξ takes two distinct values on $U_{t_v}(F_v) = \tilde{X}_{t_v}(F_v)$. It thus takes two distinct values on $U(F_v)$, resp. $\tilde{X}(F_v)$. The same argument applies if $v \in S$ is a real place with $d \notin F_v^{\times 2}$ and q is isotropic at v. If v is a real place with $d \notin F_v^{\times 2}$ and q is anisotropic at v, then one may write the equation of X over $F_v = \mathbb{R}$ as

$$x^2 + y^2 + z^2 = r(t)^2.$$

The real quadric Q defined by $x^2+y^2+z^2 = 1$ contains the point $(1,0,0)$. Applying the recipe in Proposition 5.7, one finds that the class of the quaternion algebra $(x - r(t), -1)$ in $\mathrm{Br}(F(U))$ lies in $\mathrm{Br}(U)$ and generates

$\mathrm{Br}(U \times_F \mathbb{R})/\mathrm{Br}(\mathbb{R})$. By assumption, $r(t)$ has a real root. One easily checks that $(x - r(t))$ takes opposite signs on $U(\mathbb{R})$ when one crosses such a real root of $r(t)$. Thus $\xi_{\mathbb{R}} = (x - r(t), -1)$ takes two distinct values on $U(F_v)$.

Let now $\{P_v\}$ be an adèle of U, resp. $\tilde{X}$. If $\sum_v \xi(P_v) = 1/2$, then we change P_v at a place $v \in S$ so that the new $\sum_v \xi(P_v) = 0$. We then know that that we can approximate this family off S by a point in $U(F)$, resp. a point in $X(F)$. $\square$

Remark 6.6. Over the ring of usual integers, a special case of Watson's Theorem 3 in [Wat] reads as follows.

Assume the ternary quadratic form $q(x, y, z)$ with integral coefficients is of rank 3 over $\mathbb{Q}$ and isotropic over $\mathbb{R}$. Let $p(t) \in \mathbb{Z}[t]$ be a nonconstant polynomial. Assume

(W) For each big enough prime l, the equation $p(t) = 0$ has a solution in the local field $\mathbb{Q}_l$.

If the equation $q(x, y, z) = p(t)$ has solutions in $\mathbb{Z}_l$ for each prime l, then it has a solution in $\mathbb{Z}$.

Let $k = \mathbb{Q}$ and X/k and $\tilde{X}/k$ be as above. This result is a consequence of Theorem 6.5. Indeed, if $\mathrm{Br}(\tilde{X})/\mathrm{Br}(k) = 0$, strong approximation holds for $\tilde{X}$, hence in particular the local-global principle holds for integral points of $\tilde{X}$. By Proposition 5.6, $\mathrm{Br}(\tilde{X})/\mathrm{Br}(k) \neq 0$ occurs only if all e_i are even, $d \notin k^{\times 2}$ and $d \in k_i^{\times 2}$ for all i. That is to say, for each i, the quadratic field extension $k(\sqrt{d})$ of k lies in k_i. There are infinitely many primes v of k which are inert in $k(\sqrt{d})$. For such primes v, none of the equations $p_i(t) = 0$ admits a solution in k_v. Condition (W) excludes this possibility.

7 Two examples

In this section we give two examples which exhibit a drastic failure of strong approximation: there are integral points everywhere locally but there is no global integral point.

The first example develops [Xu, (6.1), (6.4)].

Proposition 7.1. *Let $\mathbf{X} \subset \mathbb{A}^4_{\mathbb{Z}}$ be the scheme over $\mathbb{Z}$ defined by*

$$-9x^2 + 2xy + 7y^2 + 2z^2 = (2t^2 - 1)^2.$$

Let $\mathbf{U}$ over $\mathbb{Z}$ be the complement of $x = y = z = 0$ in $\mathbf{X}$. Let $X = \mathbf{X} \times_{\mathbb{Z}} \mathbb{Q}$ and $U = \mathbf{U} \times_{\mathbb{Z}} \mathbb{Q}$. Let $\tilde{X} \to X$ be a desingularization of X inducing an *isomorphism over U. Let $\tilde{\mathbf{X}} \to \mathbf{X}$, with $\mathbf{U} \subset \tilde{\mathbf{X}}$, be a proper morphism extending $\tilde{X} \to X$.*

Strong approximation off ∞ fails for U and for $\tilde{X}$. More precisely:

(i)

$$\prod_{p \leq \infty} \mathbf{X}(\mathbb{Z}_p) \neq \emptyset \quad \textit{and} \quad \mathbf{X}(\mathbb{Z}) = \emptyset.$$

(ii)

$$\prod_{p \leq \infty} \mathbf{U}(\mathbb{Z}_p) \neq \emptyset \quad \textit{and} \quad \mathbf{U}(\mathbb{Z}) = \emptyset.$$

(iii)

$$\prod_{p \leq \infty} \tilde{\mathbf{X}}(\mathbb{Z}_p) \neq \emptyset \quad \textit{and} \quad \tilde{\mathbf{X}}(\mathbb{Z}) = \emptyset.$$

Proof. With notation as in Theorem 6.5, we have $F = \mathbb{Q}$, $v_0 = \infty$, $S = \{v_0\}$. One has $\det(q) = -2^7$ and $d = -c.\det(q) = 2^9$. We are in case (iv) of Theorem 6.5. Over $\mathbb{R}$, $q(x, y, z)$ is isotropic. By Theorem 6.5 (iv) we have

$$\mathrm{Br}(\tilde{X})/\mathrm{Br}(F) = \mathrm{Br}(U)/\mathrm{Br}(F) = \mathbb{Z}/2$$

and by Theorem 6.5 (v) we know that strong approximation off S fails for U and $\tilde{X}$.

The equation may be written as

$$(x - y)(9x + 7y) = 2z^2 - (2t^2 - 1)^2. \tag{7.1}$$

Let $Y/\mathbb{Q}$ be the smooth open set defined by

$$(x - y)(9x + 7y) = 2z^2 - (2t^2 - 1)^2 \neq 0. \tag{7.2}$$

Thus $Y \subset U \subset X$. We have $Y(\mathbb{Q}) = U(\mathbb{Q}) = X(\mathbb{Q})$ since 2 is not a square in $\mathbb{Q}$. We also have $Y(\mathbb{Q}_p) = U(\mathbb{Q}_p) = X(\mathbb{Q}_p)$ for any prime p such that 2 is not a square in $\mathbb{Q}_p$.

On the 3-dimensional smooth variety U, the algebra

$$B = (y - x, 2) = (-2(9x + 7y), 2) = (9x + 7, 2) \tag{7.3}$$

is unramified off the codimension 2 curve $x = y = 0$, hence by purity it is unramified on U. One could show by purely algebraic means that it generates $\mathrm{Br}(U)/\mathrm{Br}(F) = \mathbb{Z}/2$ but this will follow from the arithmetic computation below.

Note that $U(\mathbb{Q}) = X(\mathbb{Q})$, since the singular points of X are not defined over $\mathbb{Q}$.

For $p \neq 2$, there is a point of $\mathbf{U}(\mathbb{Z}_p)$ with $t = 1$. For $p \neq 3$, we have the point $(0, 1/3, 1/3, 1)$ in $\mathbf{U}(\mathbb{Z}_p)$. Thus $\prod_{p \leq \infty} \mathbf{U}(\mathbb{Z}_p) \neq \emptyset$.

For $p \neq 2$, and 2 not a square in $\mathbb{Q}_p$, for any solution of (7.2) in $\mathbb{Z}_p$, $y - x$ and $9x + 7y$ are p-adic units. For any $p \neq 2$, equality (7.3) thus implies $B(M_p) = 0$ for any point in $\mathbf{X}(\mathbb{Z}_p) \cap Y(\mathbb{Q}_p)$. Since U is smooth, $Y(\mathbb{Q}_p)$ is dense in $U(\mathbb{Q}_p)$. Since $\mathbf{X}(\mathbb{Z}_p)$ is open in $X(\mathbb{Q}_p)$, this implies that $\mathbf{X}(\mathbb{Z}_p) \cap Y(\mathbb{Q}_p)$ is dense in $\mathbf{X}(\mathbb{Z}_p) \cap U(\mathbb{Q}_p)$, and then that $B(M_p) = 0$ for any point in $\mathbf{X}^*(\mathbb{Z}_p) := \mathbf{X}(\mathbb{Z}_p) \cap U(\mathbb{Q}_p)$. This last set contains $\mathbf{U}(\mathbb{Z}_p)$.

The algebra B trivially vanishes on $\mathbf{X}^*(\mathbb{R}) := U(\mathbb{R})$.

Let us consider a point $M_2 \in \mathbf{X}(\mathbb{Z}_2) \subset Y(\mathbb{Q}_2)$. From (7.2), for such a point with coordinates (x, y, z, t), we have

$$(x - y)(9x + 7y) = \pm 1 \mod. 8.$$

Thus the 2-adic valuation of $y - x$ and of $9x + 7y$ is zero. If B vanishes on M_2 then $y - x = 1 \mod. 4$ and $9x + 7y = 1 \mod. 4$. But then $16x = 2 \mod. 4$, which is absurd. Thus $B(M_2)$ is not zero, that is $B(M_2) = 1/2 \in \mathbb{Q}/\mathbb{Z}$.

We conclude that for any point $\{M_p\} \in \prod_p \mathbf{X}^*(\mathbb{Z}_p) \times X^*(\mathbb{R})$,

$$\sum_p B(M_p) = B(M_2) = 1/2.$$

This implies $\mathbf{X}(\mathbb{Z}) = \mathbf{X}(\mathbb{Z}) \cap U(\mathbb{Q}) = \emptyset$, hence $\mathbf{U}(\mathbb{Z}) = \emptyset$ and $\tilde{\mathbf{X}}(\mathbb{Z}) = \emptyset$, since both sets map to $\mathbf{X}(\mathbb{Z})$.

Since $\tilde{\mathbf{X}} \to \mathbf{X}$ is proper, the map $\tilde{\mathbf{X}}(\mathbb{Z}_p) \to \mathbf{X}(\mathbb{Z}_p)$ contains $\mathbf{X}^*(\mathbb{Z}_p)$ in its image. We thus have $\tilde{\mathbf{X}}(\mathbb{Z}_p) \neq \emptyset$.

One actually has

$$[\prod_{p \leq \infty} \tilde{\mathbf{X}}(\mathbb{Z}_p)]^{Br(\tilde{X})} = \emptyset.$$

Indeed, the algebra $B = (y - x, 2)$ on U extends to an unramified class on $\tilde{X}$. To see this, one only has to consider the points of codimension 1 on $\tilde{X}$ above the closed point $2t^2 - 1 = 0$ of $\mathbb{A}^1_{\mathbb{Q}}$. For the corresponding valuation v on the field $F(\tilde{X})$, one have $v(2t^2 - 1) > 0$, thus 2 is a square in the residue field of v, hence the residue of $(y - x, 2)$ at v is trivial. □

The next example is inspired by an example of Cassels (cf. [CTX, 8.1.1]).

Proposition 7.2. *Let $\mathbf{X} \subset \mathbb{A}^4_{\mathbb{Z}}$ be the scheme over $\mathbb{Z}$ defined by*

$$x^2 - 2y^2 + 64z^2 = (2t^2 + 3)^2.$$

Let $\mathbf{U}$ *over* $\mathbb{Z}$ *be the complement of* $x = y = z = 0$ *in* $\mathbf{X}$*. Let* $X = \mathbf{X} \times_{\mathbb{Z}} \mathbb{Q}$ *and* $U = \mathbf{U} \times_{\mathbb{Z}} \mathbb{Q}$*. Let* $\tilde{X} \to X$ *be a desingularization of* X*. Let* $\tilde{\mathbf{X}} \to \mathbf{X}$ *be a proper morphism extending* $\tilde{X} \to X$*.*

Strong approximation off ∞ *holds for* $\tilde{X}$ *and fails for* U*. More precisely:*

(i) $\tilde{\mathbf{X}}(\mathbb{Z})$ *is dense in* $\prod_{p<\infty} \tilde{\mathbf{X}}(\mathbb{Z}_p)$*.*

(ii) There are solutions (x, y, z, t) *in* $\mathbb{Z}$ *with* $p(t) \neq 0$*, thus we have* $\mathbf{X}(\mathbb{Z}) \cap U(\mathbb{Q}) \neq \emptyset$*.*

(iii) We have $\prod_{p\leq\infty} \mathbf{U}(\mathbb{Z}_p) \neq \emptyset$ *and* $[\prod_{p\leq\infty} \mathbf{U}(\mathbb{Z}_p)]^{Br(U)} = \emptyset$*, hence* $\mathbf{U}(\mathbb{Z}) = \emptyset$ *: there are no solutions* (x, y, z, t) *in* $\mathbb{Z}$ *with* (x, y, z) *primitive.*

Proof. With notation as in Theorem 6.5, we have $F = \mathbb{Q}$, $v_0 = \infty$, $S = \{v_0\}$. We have $d = 2^9$. Over $\mathbb{R}$, $q(x, y, z)$ is isotropic. We are in case (iii) of Theorem 6.5. We have $\mathrm{Br}(\tilde{X})/\mathrm{Br}(F) = 0$ and $\mathrm{Br}(U)/\mathrm{Br}(F) = \mathbb{Z}/2$.

According to Theorem 6.5 (iii), strong approximation off ∞ holds for $\tilde{X}$.

Theorem 6.5 (v) then says that strong approximation off S fails for U. That is, $U(\mathbb{Q})$ is not dense in $U(\mathbb{A}_{\mathbb{Q}}^{\infty})$.

The point $(x, y, z, t) = (3, 0, 0, 0) \in U(\mathbb{Q}) \cap \mathbf{X}(\mathbb{Z})$ provides a point in $\mathbf{U}(\mathbb{Z}_p)$ for each prime $p \neq 3$ and for $p = \infty$. In general, for p odd, we have $\mathbf{U}(\mathbb{Z}_p) \neq \emptyset$ by Lemma 4.2.

Let us prove statement (iii).

Since $1-8z = 0$ is the tangent plane on affine quadric $x^2 - 2y^2 + 64z^2 = 1$ over $\mathbb{Q}$ at the point $(0, 0, \frac{1}{8})$, Proposition 5.7 shows that $B = (2t^2+3-8z, 2)$ is the generator of $\mathrm{Br}(U)/\mathrm{Br}(F)$. We have

$$(2t^2 + 3 - 8z)(2t^2 + 3 + 8z) = x^2 - 2y^2 \tag{7.4}$$

thus

$$B = (2t^2 + 3 - 8z, 2) = (2t^2 + 3 + 8z, 2). \tag{7.5}$$

Let p be an odd prime such that 2 is not a square modulo p. For a point $(x, y, z) \in \mathbf{U}(\mathbb{Z}_p)$, if p divides both $2t^2+3-8z$ and $2t^2+3+8z$, then on the one hand p divides z and on the other hand, by equation (7.4), it divides $x^2 - 2y^2$, which then implies that p divides x and y. Thus p divides x, y, z, which is impossible for a point in $\mathbf{U}(\mathbb{Z}_p)$. We conclude from (7.5) that for any odd prime p, B vanishes on $\mathbf{U}(\mathbb{Z}_p)$.

For $p = 2$, for any t and z in $\mathbb{Z}_2$, we have $2t^2 + 3 - 8z = \pm 3$ modulo 8, hence

$$(2t^2 + 3 - 8z, 2) = (\pm 3, 2) = 1/2 \in \mathrm{Br}(\mathbb{Q}_2).$$

Thus

$$[\prod_{p\leq\infty} \mathbf{U}(\mathbb{Z}_p)]^{\mathrm{Br}(U)} = \emptyset,$$

which implies $\mathbf{U}(\mathbb{Z}) = \emptyset$.

□

8 Approximation for singular varieties

The following lemma is well known.

Lemma 8.1. *Let k be a local field of characteristic zero. Let X be a geometrically integral variety over k. Let $f : \tilde{X} \to X$ be a resolution of singularities for X, i.e. $\tilde{X}$ is a smooth, geometrically integral k-variety and f is a proper birational k-morphism. The following closed subsets of $X(k)$ coincide:*

(a) The closure of $X_{smooth}(k)$ in $X(k)$ for the topology of k.

(b) The set $f(\tilde{X}(k)) \subset X(k)$.

In particular, this set, called the set of central points of X, does not depend on the resolution $f : \tilde{X} \to X$. It will be denoted $X(k)_{\mathrm{cent}}$.

Proof. One uses the fact that for a nonempty open set U of $\tilde{X}$, $U(k)$ is dense in $\tilde{X}(k)$ for the local topology, and that the inverse image of a compact subset of $X(k)$ under f is a compact set in $\tilde{X}(k)$. □

Definition 8.2. Let F be a number field. Let X be a geometrically integral variety over F. Assume $X_{smooth}(F) \neq \emptyset$. Let S be a finite set of places of F. One says that X satisfies central weak approximation at S if either of the following conditions is fulfilled:

(a) $X_{smooth}(F)$ is dense in $\prod_{v \in S} X_{smooth}(F_v)$.

(b) $X_{smooth}(F)$ is dense in $\prod_{v \in S} X(F_v)_{\mathrm{cent}}$.

One says that X satisfies weak approximation if this holds for any finite set S of places of F. □

While discussing the possible lack of weak approximation for a given variety X the natural Brauer-Manin obstruction is defined by means of the Brauer group of a *smooth, projective* birational model of X.

Let us now discuss strong approximation.

Lemma 8.3. *Let F be a number field. Let X be a geometrically integral variety over F. Let $f : \tilde{X} \to X$ be a resolution of singularities for X, i.e. $\tilde{X}$ is a smooth, geometrically integral F-variety and f is a proper birational F-morphism. Let S be a finite set of places of F. The following closed subsets of $X(\mathbb{A}_F^S)$ coincide:*

(a) The intersection of $X(\mathbb{A}_F^S)$ with $\prod_{v \notin S} X(F_v)_{\mathrm{cent}}$.

(b) The image of $\tilde{X}(\mathbb{A}_F^S)$ under $f : \tilde{X}(\mathbb{A}_F^S) \to X(\mathbb{A}_F^S)$.

This set does not depend on the resolution $f : \tilde{X} \to X$. We shall call it the set of central S-adèles of X, and we shall denote it $X(\mathbb{A}_F^S)_{\mathrm{cent}}$.

Proof. There exists a finite set T of places of F containing S and a proper $\mathfrak{o}_T$-morphism of $\mathfrak{o}_T$ schemes $\tilde{\mathbf{X}} \to \mathbf{X}$ extending $\tilde{X} \to X$. For $v \notin T$, one checks that

$$\tilde{\mathbf{X}}(\mathfrak{o}_v) = \mathbf{X}(\mathfrak{o}_v) \times_{X(F_v)} \tilde{X}(F_v).$$

$\square$

Proposition 8.4. *Let X be a geometrically integral variety over the number field F. Assume $X_{smooth}(F) \neq \emptyset$. Let $f : \tilde{X} \to X$ be a resolution of singularities for X. Let S be a finite set of places of F. The following conditions are equivalent:*

(a) The diagonal image of $X_{smooth}(F)$ in $X(\mathbb{A}_F^S)_{\mathrm{cent}}$ is dense.

(b) The diagonal image of $\tilde{X}(F)$ in $\tilde{X}(\mathbb{A}_F^S)$ is dense.

Definition 8.5. If these conditions hold, we say that central strong approximation holds for X off S.

If central strong approximation off S holds for X, it holds off any finite set S' containing S.

Definition 8.6. Let X be a geometrically integral variety over the number field F. Assume $X_{smooth}(F) \neq \emptyset$. Let $f : \tilde{X} \to X$ be a resolution of singularities. Let S be a finite set of places of F. If the diagonal image of $\tilde{X}(F)$ in $(\tilde{X}(\mathbb{A}_F^S))^{\mathrm{Br}(\tilde{X})} \subset \tilde{X}(\mathbb{A}_F^S)$ is dense, we say that central strong approximation with Brauer-Manin obstruction off S holds for X. If central strong approximation with Brauer-Manin obstruction off S holds for X, it holds off any finite set S' containing S.

We leave it to the reader to translate the statement in terms of $X(\mathbb{A}_F^S)_{\mathrm{cent}}$. We insist that the relevant group is the group $\mathrm{Br}(\tilde{X})$, which does not depend on the chosen resolution of singularities $\tilde{X} \to X$.

Example 8.7. Let k be a local field of characteristic zero and X be a k-variety defined by an equation

$$q(x_1, \cdots, x_n) = p(t),$$

where q is a nondegenerate quadratic form and $p(t) \in k[t]$ a nonzero polynomial. Then $X(k) \neq X(k)_{\mathrm{cent}}$ if and only if there is a zero α of $p(t)$ over k of even order r and the quadratic form in $n+1$ variables

$$q(x_1, \cdots, x_n) - p_0(\alpha) x_{n+1}^2$$

is anisotropic over k, where $p(t) = (t-\alpha)^r p_0(t)$.

Proof. By Lemma 3.3, a singular point of $X(k)$ is given by $(0, \cdots, 0, \alpha)$, where α is a zero of $p(t)$ of order $r > 1$. Let $p(t) = (t-\alpha)^r p_0(t)$. We may assume

$$q(x_1, \cdots, x_n) = \sum_{i=1}^{n} a_i x_n^2.$$

Let π denote a uniformizer of k if k is p-adic and some nonzero element with $|\pi| < 1$ when k is archimedean.

Suppose r is odd. Let $\alpha_l = \alpha + p_0(\alpha) a_1 \pi^{2l}$, hence $\lim_{l\to\infty} \alpha_l = \alpha$. For $l \gg 0$, one has $p_0(\alpha_l) = p_0(\alpha)\epsilon_l^2$ with $\epsilon_l \in k^\times$ and $\epsilon_l \to 1$ as $l \to \infty$. Then

$$P_l = (\epsilon_l a_1^{\frac{r-1}{2}} p_0(\alpha)^{\frac{r+1}{2}} \pi^{lr}, 0, \cdots, 0, \alpha_l)$$

are smooth points of $X(k)$ for $l \gg 0$ and $P_l \to (0, \cdots, 0, \alpha)$ when $l \to \infty$. Therefore $(0, \cdots, 0, \alpha) \in X(k)_{\text{cent}}$.

Suppose r is even and the quadratic form $q(x_1, \cdots, x_n) - p_0(\alpha) x_{n+1}^2$ is isotropic. There exists

$$(\theta_1, \cdots, \theta_n, \theta_{n+1}) \neq (0, \cdots, 0, 0)$$

in k^{n+1} such that $q(\theta_1, \cdots, \theta_n) = p_0(\alpha)\theta_{n+1}^2$. If $\theta_{n+1} = 0$, then the smooth points of $X(k)$

$$P_n = (\pi^l \theta_1, \cdots, \pi^l \theta_n, \alpha) \to (0, \cdots, 0, \alpha)$$

as $l \to \infty$. Therefore $(0, \cdots, 0, \alpha) \in X(k)_{\text{cent}}$.

If $\theta_{n+1} \neq 0$, one can assume that $\theta_{n+1} = 1$. Let $t_l = \alpha + \pi^{2l}$. Then $p_0(t_l) = p_0(\alpha)\epsilon_l^2$ with $\epsilon_l \in k^\times$ and $\epsilon_l \to 1$ as $l \to \infty$. The smooth points of $X(k)$

$$P_n = (\pi^{rl}\epsilon_l \theta_1, \cdots, \pi^{rl}\epsilon_l \theta_n, t_l) \to (0, \cdots, 0, \alpha)$$

as $l \to \infty$. Therefore $(0, \cdots, 0, \alpha) \in X(k)_{\text{cent}}$.

Suppose r is even and the quadratic form in $n+1$ variables

$$q(x_1, \cdots, x_n) - p_0(\alpha) x_{n+1}^2$$

is anisotropic over k. Suppose the singular point $P_0 = (0, \cdots, 0, \alpha)$ is the limit of a sequence of smooth k-points. There thus exists a sequence of smooth k-points $P_l, l \in \mathbb{N}$, satisfying $P_l \to P_0$ when $l \to \infty$. Let $P_l = (Q_l, \alpha_l)$ where α_l is the t-coordinate of P_l. Then $p_0(\alpha_l) = p_0(\alpha)\epsilon_l^2 \neq 0$ with $\epsilon_l \in k^\times$ for $l \gg 0$. Therefore

$$q(Q_l) - p(\alpha_l) = q(Q_l) - p_0(\alpha)[(\alpha_l - \alpha)^{\frac{r}{2}}\epsilon_l]^2 = 0$$

for $l \gg 0$, which implies that $q(x_1, \cdots, x_n) - p_0(\alpha)x_{n+1}^2$ is isotropic over k. A contradiction is derived, the point P_0 does not lie in $X(k)_{\mathrm{cent}}$. □

We conclude that $X(k) \neq X(k)_{\mathrm{cent}}$ may happen only in the following cases.

1) The field k is $\mathbb{R}$ and $q(x_1, \cdots, x_n)$ is $\pm$-definite over $\mathbb{R}$ and there is a zero α of $p(t)$ over $\mathbb{R}$ of even order r such that $p_0(\alpha)$ has $\mp$ sign.

2) The field k is p-adic field and $n \leq 3$. One can determine if a quadratic space is anisotropic over k by computing determinants and Hasse invariants, as in [OM, 42:9; 58:6; 63:17].

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