The integral Hodge conjecture for the total space of one-parameter families of cubic threefolds

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$X/\mathbb{C}$ projective and smooth.

Let $H^{2i}_{Hodge}(X(\mathbb{C}), \mathbb{Z}(i)) \subset H^{2i}_{Betti}(X(\mathbb{C}), \mathbb{Z}(i))$ be the inverse image of Hodge classes in $H^{2i}_{Betti}(X(\mathbb{C}), \mathbb{Q}(i))$.

The cycle map $cl_i : CH^i(X) \to H^{2i}_{Betti}(X(\mathbb{C}), \mathbb{Z}(i))$ lands in $H^{2i}_{Hodge}(X(\mathbb{C}), \mathbb{Z}(i))$.

The (rational) Hodge conjecture asserts that the group

$$Z^{2i}(X) := H^{2i}_{Hodge}(X(\mathbb{C}), \mathbb{Z}(i))/\mathrm{Im}(CH^i(X))$$

is a finite group, equal to the torsion subgroup of $H^{2i}_{Betti}(X(\mathbb{C}), \mathbb{Z}(i))/\mathrm{Im}(CH^i(X))$.

This is so for $i = 1$ (Lefschetz).
If $Z^{2i}(X) = 0$, one says that the integral Hodge conjecture holds for cycles of codimension i on X, or in (cohomological) degree $2i$.
Atiyah and Hirzebuch gave examples with $Z^{2i}(X) \neq 0$.

In this talk we shall concentrate on $Z^4(X)$.
In that degree, $Z^4(X)$ is a birational invariant in X (Voisin).
Kollár gave examples of hypersurfaces $X \subset \mathbb{P}^4$ of suitable (high) degree for which $Z^4(X) \neq 0$.
For X a threefold, Voisin (2006) proved $Z^4(X) = 0$ if X is uniruled or X is Calabi-Yau.

For rationally connected varieties, the rational Hodge conjecture for cycles of codimension 2 holds (Bloch-Srinivas). Voisin (2007) asked whether $Z^4(X) = 0$ holds for any such variety.

For X of any dimension at least 6, a negative answer was given by CT-Voisin 2012, using work of CT-Ojanguren 1988.

The question whether $Z^4(X) = 0$ holds for rationally connected varieties of dimension 4 and 5 is open.

In dimension 4, the following special results are known :

Theorem *The integral Hodge conjecture for cycles of codimension 2 holds for smooth projective fourfolds X with a dominant morphism $X \to Y$ to a smooth projective Y if :*
(i) dim$(Y) = 2$, and the generic fibre is a del Pezzo surface of degree d at least 5 (CT-Voisin 2012 when fibre is a quadric surface ; CAO Yang 2017).
(ii) dim$(Y) = 1$, and the generic fibre is a smooth complete intersection of two quadrics in $\mathbb{P}^5$ (Voisin 2010 when bad fibres have ord. quad. singularities ; CT 2012).

Theorem (Voisin)

The integral Hodge conjecture for cycles of codimension 2 holds for :

(a) (2007) smooth cubic hypersurfaces in $\mathbb{P}^5$ *;*

(b) (2010, J. Alg. Geom. 2013) varieties X *with a fibration* $X \to \Gamma$ *to a curve* Γ*, whose generic fibre is a smooth cubic in* $\mathbb{P}^4$ *and whose special fibres have ordinary quadratic singularities (implies a)).*

In this talk, I will describe the proof of :

Theorem (CT-Pirutka, 2017)
The integral Hodge conjecture for cycles of codimension 2 holds for fourfolds X with a fibration $X \to \Gamma$ to a curve Γ, fibration whose generic fibre is a smooth cubic in $\mathbb{P}^4$.

The improvement over Voisin's result is that we impose no restriction on the singular fibres. Our proof combines geometric ideas in her paper with tools from algebraic K-theory (motivic homology, unramified cohomology).

Unramified cohomology of degree 3 and integral Hodge conjecture

F a field, $\mathrm{char}(F) = 0$

X/F variety , $n \in \mathbb{N}$, $n > 0$, $i \in \mathbb{N}$, $j \in \mathbb{Z}$. Let $\mathcal{H}^i(\mu_n^{\otimes j})$ be the Zariski sheaf associated to the presheaf $U \mapsto H^i_{et}(U, \mu_n^{\otimes j})$.
Definition of unramified cohomology of X

$$H^i_{nr}(X, \mu_n^{\otimes j}) := H^0_{Zar}(X, \mathcal{H}^i(\mu_n^{\otimes j}))$$

Let $\mathbb{Q}/\mathbb{Z}(j) = colim_n \mu_n^{\otimes j}$.
Assume X/F smooth. $H^1_{nr}(X, \mathbb{Q}/\mathbb{Z}) = H^1_{et}(X, \mathbb{Q}/\mathbb{Z})$.
$H^2_{nr}(X, \mathbb{Q}/\mathbb{Z}(1)) = \mathrm{Br}(X)$.
We shall be concerned with $H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2))$.

Theorem (Bloch-Ogus 1974, Gersten's conjecture for étale cohomogy) *For X/F smooth connected, with field of functions $F(X)$, there is an exact sequence*

$$0 \to H^i_{nr}(X, \mu_n^{\otimes j}) \to H^i(F(X), \mu_n^{\otimes j}) \to \oplus_{x \in X^{(1)}} H^{i-1}(F(x), \mu_n^{\otimes j-1}).$$

Here, for any point x of codimension 1 of X, the map

$$H^i(F(X), \mu_n^{\otimes j}) \to H^{i-1}(F(x), \mu_n^{\otimes j-1})$$

is the residue map associated to the DVR $O_{X,x}$ (with residue field $\kappa(x)$).

Corollary *For X/F connected, smooth and proper, the group $H^i_{nr}(X, \mu_n^{\otimes j})$ is an F-birational invariant, hence sometimes denoted $H^i_{nr}(F(X)/F, \mu_n^{\otimes j})$. If X is stably F-rational, then $H^i(F, \mu_n^{\otimes j}) \overset{\simeq}{\to} H^i_{nr}(F(X)/F, \mu_n^{\otimes j})$.*

For X/F smooth and proper, $H^i_{nr}(X, \mu_n^{\otimes j}) \subset H^i(F(X), \mu_n^{\otimes j})$ may be computed by using all DVR's R of $F(X)$ with $F \subset R$.

For algebraically closed fields $k \subset K$, and X/k smooth and projective, for torsion coefficients, one has the "rigidity theorem" : $H^i_{nr}(X, \bullet) = H^i_{nr}(X_K, \bullet)$ (CT, Jannsen ; Suslin's method)

Using results of Merkurjev–Suslin, Rost, Voevodsky, one proves :

Theorem (CT-Voisin 2012) *Let $X/\mathbb{C}$ be a smooth projective variety. The quotient of $H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2))$ by its maximal divisible subgroup is finite and it is equal to the torsion subgroup of $H^4_{Betti}(X(\mathbb{C}), \mathbb{Z}(2))/Im[CH^2(X)]$, hence also to the torsion subgroup of $Z^4(X)$.*

[Compare : the quotient of $H^2_{nr}(X, \mathbb{Q}/\mathbb{Z}(1))$ by its maximal divisible subgroup is finite and is equal to the torsion subgroup of $H^3_{Betti}(X, \mathbb{Z})$.]

Corollary *If X is a rationally connected variety, or a uniruled threefold, there is an isomorphism of finite groups*

$$Z^4(X) \simeq H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2)).$$

Starting from theorems obtained via geometry and Hodge theory, one gets results on $H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2))$ such as its vanishing for uniruled threefolds.
Starting from results in algebraic K-theory (e.g. computation of unramified cohomology of quadrics over arbitrary ground fields, Kahn, Rost, Sujatha), one gets results on the integral Hodge conjecture.
This accounts for some of the results mentioned above.

Galois descent on Chow groups : some long exact sequences

Let F be a field, $\mathrm{char.}(F) = 0$. Let $\overline{F}$ be an algebraic closure of F and $\mathfrak{g} = \mathrm{Gal}(\overline{F}/F)$. For a Galois module M, let $H^i(\mathfrak{g}, M)$ be its Galois cohomology groups.
Let X be a smooth, geometrically integral F-variety. We write $\overline{X} = X \times_F \overline{F}$.
Recall that Quillen has associated groups $K_i(A)$ ($i \geq 0$) to any commutative ring A. On a given scheme X, sheafification of the Zariski presheaf $U \mapsto K_i(\Gamma(U, \mathcal{O}_U))$ defines the Zariski sheaf $\mathcal{K}_i$ on X. For X smooth over a field, a celebrated formula, conjectured by Bloch and proved by Quillen, identifies $H^i_{Zar}(X, \mathcal{K}_i)$ with the Chow group $CH^i(X)$ of codimension i cycles modulo rational equivalence.

Theorem A . *Let* X/F *be as above. Assume that* $H^1(X, O_X) = 0$ *and* $\mathrm{Pic}\,(\overline{X}) = NS(\overline{X})$ *has no torsion. Assume* $X(F) \neq \emptyset$*. Then there is an exact sequence*

$$0 \to \mathrm{Ker}[CH^2(X) \to CH^2(\overline{X})^{\mathfrak{g}}] \to H^1(\mathfrak{g}, H^1(\overline{X}, \mathcal{K}_2)) \to$$

$$N(X) \to \mathrm{Coker}[CH^2(X) \to CH^2(\overline{X})^{\mathfrak{g}}] \to H^2(\mathfrak{g}, H^1(\overline{X}, \mathcal{K}_2)),$$

where

$$N(X) = \mathrm{Ker}[H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2))/H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \to H^3_{nr}(\overline{X}, \mathbb{Q}/\mathbb{Z}(2))].$$

Special case of a general sequence without assumption on the Picard group. Parts of this sequence : Bloch, 1970's ; CT-Sansuc and CT-Raskind 1980's. Whole sequence : **B. Kahn 1996** (via Lichtenbaum's $\mathbb{Z}(2)$'s) CT-Kahn 2013 and CT 2013.

The proof relies on theorems of Merkurjev and Suslin and on the Gersten conjecture (Quillen for K-theory, Bloch-Ogus for étale cohomology). Most of the sequence may be established in a pedestrian manner, combining 1993 results of B. Kahn with the Galois cohomology of the complex

$$K_2(\overline{F}(X)) \to \oplus_{x \in \overline{X}^{(1)}} \overline{F}(x)^\times \to \oplus_{x \in \overline{X}^{(2)}} \mathbb{Z},$$

the homology groups of which are precisely the groups $H^i(\overline{X}, \mathcal{K}_2)$.

There is a natural map

$$\mathrm{Pic}\,(\overline{X}) \otimes_{\mathbb{Z}} \overline{F}^{\times} \to H^1(\overline{X}, \mathcal{K}_2).$$

A special case of CT-Raskind results (1985) – building on works of Bloch, Merkurjev, Suslin – gives that this map is an isomorphism if $\overline{X}$ is rationally connected and $\mathrm{Br}\,(\overline{X}) = 0$. We thus get :

Theorem B. *Let X/F be a smooth, projective, geometrically connected variety. Assume $X(F) \neq \emptyset$ and :*
(a) $\overline{X}$ *rationally connected, hence* $\mathrm{Pic}\,(\overline{X})$ *is a lattice ;*
(b) $\mathrm{Br}(\overline{X}) = 0$ *;*
(c) $H^3_{nr}(\overline{X}, \mathbb{Q}/\mathbb{Z}(2)) = 0$.
Then there is an exact sequence

$$0 \to \mathrm{Ker}[CH^2(X) \to CH^2(\overline{X})^{\mathfrak{g}}] \xrightarrow{\alpha} H^1(\mathfrak{g}, \mathrm{Pic}\,(\overline{X}) \otimes \overline{F}^{\times}) \to$$
$$\to H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2))/H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \to$$
$$\to \mathrm{Coker}[CH^2(X) \to CH^2(\overline{X})^{\mathfrak{g}}] \xrightarrow{\beta} H^2(\mathfrak{g}, \mathrm{Pic}\,(\overline{X}) \otimes \overline{F}^{\times}).$$

Corollary. *Let* $K = \mathbb{C}(\Gamma)$ *be the function field of a curve. Let* X/K *be smooth, projective, geometrically rationally connected variety of dimension at most 3. Assume* $\mathrm{Br}(\overline{X}) = 0$*. Then*

$$H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2)) \stackrel{\simeq}{\to} \mathrm{Coker}[CH^2(X) \to CH^2(\overline{X})^{\mathfrak{g}}].$$

Corollary. *Let* $K = \mathbb{C}(\Gamma)$ *be the function field of a curve, and let* Y/K *be a smooth, projective, geometrically rational surface, for example a smooth cubic surface in* $\mathbb{P}^3_K$*. Then*

$$H^3_{nr}(Y, \mathbb{Q}/\mathbb{Z}(2)) = 0.$$

An aside on some birational invariants

Extending the ground field $\mathbb{C}$ *of* X *to an arbitrary overfield* F

Theorem B and rigidity of unramified cohomology imply :

Theorem C. *Let $X/\mathbb{C}$ be a connected smooth projective variety. Assume*
(a) X is rationally connected, hence $\mathrm{Pic}\,(X)$ is a lattice.
(b) $\mathrm{Br}(X) = 0$.
(c) $H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2)) = 0$.
Then for any field $F/\mathbb{C}$, with algebraic closure $\overline{F}$, letting $\mathfrak{g} = \mathrm{Gal}(\overline{F}/F)$, we have an exact sequence

$$0 \to H^3_{nr}(X_F, \mathbb{Q}/\mathbb{Z}(2))/H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \to$$
$$\to \mathrm{Coker}[CH^2(X_F) \to CH^2(X_{\overline{F}})^{\mathfrak{g}}] \xrightarrow{\beta} H^2(\mathfrak{g}, \mathrm{Pic}\,(X) \otimes \overline{F}^{\times}).$$

One way to look at this theorem is as an ordered search to establish the nonrationality of X.

Indeed, if any of the hypotheses (a), (b) or (c) is not fulfilled, then X is not rational.

For any field $F/\mathbb{C}$, the group $\mathrm{Coker}[CH^2(X_F) \to CH^2(X_{\overline{F}})^{\mathfrak{g}}]$ is a birational invariant of $X/\mathbb{C}$.

For any field $F/\mathbb{C}$, the first two groups in the sequence are birational invariants of $X/\mathbb{C}$, and vanish if X is rational.

Theorem D. *Let $X \subset \mathbb{P}^5_{\mathbb{C}}$ be a smooth Fano hypersurface. Assume that the integral Hodge conjecture holds for codimension 2 cycles on X. Then for any field $F/\mathbb{C}$, one has*

$$H^3(F, \mathbb{Q}/\mathbb{Z}(2)) = H^3_{nr}(F(X)/F, \mathbb{Q}/\mathbb{Z}(2)).$$

Proof. The hypothesis implies $H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2)) = 0$, hence (rigidity) $H^3_{nr}(X \times_{\mathbb{C}} \overline{F}, \mathbb{Q}/\mathbb{Z}(2)) = 0$. Theorem C then gives an embedding

$$H^3_{nr}(F(X)/F, \mathbb{Q}/\mathbb{Z}(2))/H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \hookrightarrow Coker[CH^2(X_F) \to CH^2(X_{\overline{F}})^{\mathfrak{g}}]$$

For codimension 2 cycles over X as above over an algebraically closed field, rational equivalence, algebraic equivalence and homological equivalence coincide. Thus $CH^2(X) \stackrel{\simeq}{\to} CH^2(X_{\overline{F}})$. Hence $CH^2(X_F) \to CH^2(X_{\overline{F}})^{\mathfrak{g}}$ is onto. Hence $H^3(F, \mathbb{Q}/\mathbb{Z}(2)) = H^3_{nr}(F(X)/F, \mathbb{Q}/\mathbb{Z}(2))$. QED

For $X \subset \mathbb{P}^5_{\mathbb{C}}$ a cubic hypersurface, the integral Hodge conjecture in degre 4 holds (Voisin 2007) and the conclusion of the above theorem is due to her (2015).

What about *smooth cubic hypersurfaces* $X \subset \mathbb{P}^4_{\mathbb{C}}$?

None is rational (Clemens-Griffiths) but could some, all, be stably rational ? Could none be ?

For any such X and any field $F/\mathbb{C}$, using Theorem C one may show :

$$H^3_{nr}(F(X)/F, \mathbb{Q}/\mathbb{Z}(2))/H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \stackrel{\simeq}{\to} Coker[CH^2(X_F) \to CH^2(X_{\overline{F}})^{\mathfrak{g}}]$$

Could this group be non zero for some suitable F (hence X not be stably rational) ?

The relevant field F to look at is the field of rational functions of the intermediate jacobian $J^3(X)$. This is an abelian variety which parametrizes cycles of codimension 2 on X which are homologically equivalent to zero. There is an obvious class in $CH^2(X_{\overline{F}})^{\mathfrak{g}}$ and one wonders whether it comes from $CH^2(X_F)$, thus defining on $X \times J^3(X)$ what C. Voisin (2014) calls a "universal codimension 2 cycle". She shows that the existence of a universal codimension 2 cycle on X is actually equivalent to $H^3(F, \mathbb{Q}/\mathbb{Z}(2)) = H^3_{nr}(F(X)/F, \mathbb{Q}/\mathbb{Z}(2))$ for any field $F/\mathbb{C}$.

C. Voisin also shows that there exist cubic hypersurfaces $X \subset \mathbb{P}^4_{\mathbb{C}}$ for which $H^3(F, \mathbb{Q}/\mathbb{Z}(2)) = H^3_{nr}(F(X)/F, \mathbb{Q}/\mathbb{Z}(2))$ for any field $F/\mathbb{C}$, and indeed for which the *stronger property*

$$CH_0(X_F) \simeq \mathbb{Z}$$

holds for every $F/\mathbb{C}$ (CH_0-triviality, integral decomposition of the diagonal).
I have given a simple proof that this stronger property holds for any cubic hypersurface $X \subset \mathbb{P}^4_{\mathbb{C}}$ given by a partially diagonal equation

$$f(x_0, x_1, x_2) + g(x_3, x_4) = 0.$$

More generally, one has the following result, which applies for instance to the Fermat hypersurface $\sum_{i=0}^{n} X_i^3 = 0$.

Theorem. *Let $X \subset \mathbb{P}^n_{\mathbb{C}}$, $n \geq 4$, be a nonsingular cubic hypersurface given by a form*

$$F(X_0, \ldots, X_n) = \sum_{i=1}^{r} G_i(X_0, \ldots, X_n)$$

where the G_i's are forms in at most three explicit variables and two different G_i's have no explicit variable in common. Then X is CH_0-trivial : for any field extension $F/\mathbb{C}$ the degree map $deg_F : CH_0(X_F) \to \mathbb{Z}$ is an isomorphism.

The proof uses :

Theorem. *Let k be a field and X a smooth, projective, connected variety with a k-point. Assume $H^1(X, O_X) = 0$. If there exists a smooth, projective, connected curve Γ with a k-point and a k-morphism $\Gamma \to X$ such that over any overfield F of k the induced map $CH_0(\Gamma_F) \to CH_0(X_F)$ is onto, then over any overfield F of k, the degree map $deg_F : CH_0(X_F) \to \mathbb{Z}$ is an isomorphism : the k-variety X is universally CH_0-trivial.*

End of the (long) aside on some birational invariants.

The title theorem and its proof

Theorem (CT-Pirutka, 2017) *Let* Γ *be a smooth curve and* $K = \mathbb{C}(\Gamma)$ *its function field. The integral Hodge conjecture for cycles of codimension 2 holds for fourfolds* $Y/\mathbb{C}$ *with a fibration* $Y \to \Gamma$ *whose generic fibre* X/K *is a smooth cubic threefold in* $\mathbb{P}^4_K$*.*

Such a variety Y is dominated by the product of $\mathbb{P}^3_{\mathbb{C}}$ and a curve, hence the rational Hodge conjecture holds in degree 4 (Bloch-Srinivas). Using the CT-Voisin result

$$H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2))/maxdiv \simeq [H^4_{Betti}(X(\mathbb{C}), \mathbb{Z}(2))/Im(CH^2(X))]_{tors},$$

it is thus enough to prove $H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2)) = 0$.

Now $H^3_{nr}(Y, \mathbb{Q}/\mathbb{Z}(2)) \subset H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2))$. The result thus follows from (i) in the following :

Theorem (CT-Pirutka 2017). *Let $K = \mathbb{C}(\Gamma)$ be the function field of a complex curve Γ. Let $X \subset \mathbb{P}^4_K$ be a smooth cubic threefold over K. Then*
(i) One has $H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2)) = 0$.
(ii) The map $CH^2(X) \to CH^2(\overline{X})^{\mathfrak{g}}$ is onto.

Over the cohomological dimension 1 field $K = \mathbb{C}(\Gamma)$, Theorem B gives

$$H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2)) \stackrel{\simeq}{\to} Coker[CH^2(X) \to CH^2(\overline{X})^{\mathfrak{g}}].$$

We shall prove that $CH^2(X) \to CH^2(\overline{X})^{\mathfrak{g}}$ is onto.

Let k be a field of char. zero, $\overline{k}$ an algebraic closure, $\mathfrak{g} = \mathrm{Gal}(\overline{k}/k)$. Let $X \subset \mathbb{P}^4_k$ be a smooth cubic hypersurface, let $\overline{X} = X \times_k \overline{k}$. Algebraic and numerical equivalence coincide on $CH^2(\overline{X})$. There is a natural exact sequence of Galois modules

$$0 \to CH^2(\overline{X})_{alg} \to CH^2(\overline{X}) \to \mathbb{Z} \to 0,$$

where the map $CH^2(\overline{X}) \to \mathbb{Z}$ is given by the intersection with a hyperplane.
The group $CH^2(\overline{X})$ is then the disjoint union of the sets $CH^2(\overline{X})_d$ consisting of classes of degree $d \in \mathbb{Z}$.

The group $CH^2(\overline{X})_{alg} = CH^2(\overline{X})_0$ is "the group of $\overline{k}$-points of the intermediate jacobian of $\overline{X}$". It can be given an algebraic structure via the Prym construction (Mumford, Murre, Beauville 1977) attached to a conic bundle structure on a blow-up of $\overline{X}$ along a (general) line. This depends on the choice of a line on $\overline{X}$, and such a line need not exist over k.

For cubic threefolds, another presentation of $CH^2(\overline{X})_{alg}$ goes back to Murre and can be followed over k. Let S/k be the Fano variety of lines on X. It is a smooth projective surface.
Let $V \subset S \times_k X$ be the incidence variety consisting of pairs (L, x) with $x \in X$ lying in the line L. One then has the Galois equivariant map

$$p_{S,*} \circ p_X^* : CH^2(\overline{X}) \to CH^1(\overline{S}) = \mathrm{Pic}_{S/k}(\overline{k}).$$

Let $J := \mathrm{Pic}^0_{S/k}$.
Murre : This map induces an isomorphism of Galois modules

$$CH^2(\overline{X})_{alg} \stackrel{\simeq}{\to} J(\overline{k}).$$

For each d, there exists a principal homogeneous space J_d of J and an isomorphism of Galois sets $CH^2(\overline{X})_d \xrightarrow{\simeq} J_d(\overline{k})$.
One wants to prove that if $k = \mathbb{C}(\Gamma)$ is the function field of a complex curve, then each map $CH^2(X)_d \to CH^2(\overline{X})_d^{\mathfrak{g}}$ is onto.
From this point on, the proof uses many geometric ingredients from the paper of C. Voisin (JAG 2013).
The trace h of a plane on X gives a class in $CH^2(X)_3$. A simple argument (consider $\pm c + nh$) then shows that it is enough to prove the surjectivity statement for $d = 5$ and $d = 6$.

Some moduli spaces of curves of genus 1 lying on a cubic hypersurface $X \subset \mathbb{P}^4_k$

For $d \geq 1$ let $M_{d,1}(X)$ be the union of (reduced, closed) components of the Hilbert scheme of X whose general point parametrizes smooth, connected curves $C \subset X \subset \mathbb{P}^4$ of genus 1, of degree d, and which span $\mathbb{P}^4$. The $\overline{k}$-points of these spaces correspond to (possibly) reductible projective curves of degree d over $\overline{k}$. $M_{5,1}(X)$ is geometrically irreducible (Markushevich, Tikhomirov ; Harris, Roth, Starr). For X general, $M_{6,1}(X)$ is geometrically irreducible (Voisin).

One fixes desingularisations $\tilde{M}_{5,1}(X) \to M_{5,1}(X)$ and $\tilde{M}_{6,1}(X) \to M_{6,1}(X)$. Using the universal family of curves over $M_{d,1}(X)$, for $d = 5, 6$, one gets k-morphisms $\tilde{M}_{d,1}(X) \to J_d$. These will be called Abel-Jacobi morphisms.

Theorem (Iliev, Markushevich, Tikhomirov for $d = 5$; Voisin for $d = 6$) *Let $k = \mathbb{C}$ and $X \subset \mathbb{P}^4_{\mathbb{C}}$ be a very general cubic threefold. Then for $d = 5$ and $d = 6$ the Abel-Jacobi morphisms $\tilde{M}_{d,1}(X) \to J_d$ are surjective and their generic fibre is geometrically rationally connected.*

All the previous constructions were done over an arbitrary field k of characteristic zero.

One may globalize them over the parameter space $P \subset \mathbb{P}^{34}_{\mathbb{C}}$ of cubic threefolds and the universal family $\mathcal{X} \to P$: at the generic point of P we get the above construction over the function field $k = \mathbb{C}(P)$.

One has the family $\mathcal{S} \to P$ of Fano surfaces of lines, global correspondences, components of the relative Hilbert scheme $Hilb_{\mathcal{X}/P}$, irreducible families $\mathcal{M}_{5,1}$ and $\mathcal{M}_{6,1}$ over P of nondegenerate curves of genus 1 and degree $d = 5, 6$, with the property $\mathcal{M}_{d,1,t} \subset M_{d,1}(\mathcal{X}_t)$ for $t \in P(\overline{k})$. One fixes desingularizations $\tilde{\mathcal{M}}_{d,1} \to \mathcal{M}_{d,1}$ for $d = 5, 6$. One then finds globalized Abel-Jacobi morphisms $(d = 5, 6)$ over P :

$$\varphi_d : \tilde{\mathcal{M}}_{d,1} \to \mathcal{J}_d = \mathrm{Pic}^d_{\mathcal{S}/P}$$

which are proper, surjective and whose geometric generic fibre is smooth and rationally connected (by the above theorem).

A theorem of A. Hogadi and C. Xu 2009 then guarantees that over any schematic point x of the scheme $\mathcal{J}_d$, with residue field $\kappa(x)$, the fibre of φ_d contains a $\kappa(x)$-subvariety which is geometrically rationally connected. If x corresponds to a point of dimension 0 or 1 on $\mathcal{J}_d$, then $\kappa(x) = \mathbb{C}$ or $\mathbb{C}(\Gamma)$ (Γ a curve). By the Graber-Harris-Starr theorem, the fibre at x contains a $\kappa(x)$-point. Since $\mathcal{X} \to P$ is the universal family of cubic fourfolds, we conclude : for $K = \mathbb{C}(\Gamma)$ the function field of a curve and any smooth projective cubic threefold $X \subset \mathbb{P}^4_K$, and $d = 5, 6$, the map

$$CH^2_d(X)(K) \to J_d(X)(K) = CH^2_d(\overline{X})^{\mathfrak{g}}$$

is onto. QED.

Over finite fields

Finite fields may sometimes be viewed as “simpler” analogues of function fields in one variable over $\mathbb{C}$. They both have Galois cohomological dimension 1.

Theorem (CT, Sansuc, Soulé ; K. Kato) (1983)
Let X be a smooth projective connected surface over a finite field $\mathbb{F}$. Then $H^3_{nr}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = 0$.

This is one special result in higher class field theory (Bloch, Kato, Saito in the 80s, later Jannsen, Saito ...).

Theorem (CT-Kahn 2013) *Let X be a smooth, projective, geometrically connected variety over a finite field $\mathbb{F}$. Let $G = Gal(\overline{\mathbb{F}}/\mathbb{F})$. Let M be the finite Galois module*

$$M = \oplus_{\ell} H^3(\overline{X}, \mathbb{Z}_{\ell}(2))_{tors}.$$

(H^3 is for H^3_{et} for $\ell \neq \mathrm{char}(\mathbb{F})$).
There is an exact sequence

$$0 \to \mathrm{Ker}[CH^2(X) \to CH^2(\overline{X})] \to H^1(G, M) \to$$

$$\to \ker[H^3_{nr}(X, \mathbb{Q}_{\ell}/\mathbb{Z}_{\ell}(2)) \to H^3_{nr}(\overline{X}, \mathbb{Q}_{\ell}/\mathbb{Z}_{\ell}(2))] \to$$

$$\to \mathrm{Coker}[CH^2(X) \to CH^2(\overline{X})^G] \to 0.$$

The following theorem parallels the result by CT-Voisin over $\mathbb{C}$.

Theorem (Kahn ; CT-Kahn 2013) *Let $X/\mathbb{F}$ be a smooth projective, geometrically connected variety over a finite field $\mathbb{F}$. The quotient of $H^3_{nr}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$ by its maximal divisible subgroup is finite and is equal to the torsion subgroup of*

$$cyc_X : \mathrm{Coker}[CH^2(X) \otimes \mathbb{Z}_\ell \to H^4(X, \mathbb{Z}_\ell(2)].$$

If Tate conjecture holds (coefficients $\mathbb{Q}_\ell$), that cokernel is torsion.

Threefolds over a finite field

Finite fields $\mathbb{F}$ and function fields $\mathbb{C}(\Gamma)$ in one variable over the complex field have some properties in common. Here is one more.

Theorem. *Let $X \subset \mathbb{P}^4_{\mathbb{F}}$ be a smooth cubic hypersurface over a finite field $\mathbb{F}$ of odd characteristic p. Let ℓ be a prime, $\ell \neq p$.*
(i) The map $CH^2(X)[1/p] \to CH^2(\overline{X})^G[1/p]$ is an isomorphism.
(ii) One has $H^3_{nr}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = 0$.
(iii)The cycle map $CH^2(X) \otimes \mathbb{Z}_\ell \to H^4(X, \mathbb{Z}_\ell(2))$ is onto.

Proof. Let Y denote the Fano surface of lines on X. It is a smooth, geometrically integral surface, hence contains a zero-cycle of degree 1. For our statements, we may thus assume that there exists a line defined over $\mathbb{F}$ on X. Intersection with a hyperplane thus induces a split exact sequence of Galois modules :

$$0 \to \widetilde{CH}^2(\overline{X}) \to CH^2(\overline{X}) \to \mathbb{Z} \to 0,$$

The universal line induces $CH_0(\overline{Y}) \to CH^2(\overline{X})$ and similarly for $\widetilde{CH}^2$ and for the groups over $\mathbb{F}$.

The map $\widetilde{CH}_0(\overline{Y}) \to \widetilde{CH}^2(\overline{X})$ factorizes as a composite of isomorphisms $\widetilde{CH}_0(\overline{Y}) \stackrel{\simeq}{\to} Alb_Y(\overline{\mathbb{F}}) \stackrel{\simeq}{\to} \widetilde{CH}^2(\overline{X})$ (the first one, Roitman ; the second one, identification of the intermediate jacobian). *Higher class field theory* of the surface $Y/\mathbb{F}$ (Kato, Saito, 1980s) gives that the map $\widetilde{CH}_0(Y) \to \widetilde{CH}_0(\overline{Y})^G$ is onto. One then deduces that $\widetilde{CH}^2(X) \to \widetilde{CH}^2(\overline{X})^G$ is onto. Hence $CH^2(X) \to CH^2(\overline{X})^G$ is onto.

From this one deduces $H^3_{nr}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = 0$ and then $cyc_X : CH^2(X) \otimes \mathbb{Z}_\ell \to H^4(X, \mathbb{Z}_\ell(2))$ onto.

One may also prove the above result by reduction to a conic bundle situation.

Theorem (Parimala & Suresh, 2012) *Let* $\mathbb{F}$ *be a finite field* $\mathbb{F}$ *of odd characteristic. Let* $X \to S$ *be a conic bundle, with* X *and* S *smooth, projective varieties over* $\mathbb{F}$*,* $\dim(X) = 3$ *and* $\dim(S) = 2$*. Then for any prime* $\ell \neq \mathrm{char}(\mathbb{F})$*,* $H^3_{nr}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = 0$*.*

The proof of this theorem relies on *higher class field theory of surfaces* and on many delicate arguments, some of them inspired by work of D. Saltman on unramified cohomology.

Theorem (Pirutka) *Let* $\mathbb{F}$ *be a finite field. Let* $X = C \times_F S$ *be the product of a smooth, projective curve* $C/\mathbb{F}$ *and a geometrically rational smooth projective surface* $S/\mathbb{F}$*. Then for any prime* $\ell \neq \mathrm{char}(\mathbb{F})$*,* $H^3_{nr}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = 0$*.*

[This would be obvious if S was birational to $\mathbb{P}^2_{\mathbb{F}}$ over $\mathbb{F}$.]

Questions (already raised in various conferences and papers) :

(1) Is $H^3_{nr}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$ finite for any smooth projective variety X over a finite field ?

(2) Is $H^3_{nr}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = 0$ for any smooth projective 3-fold or 4-fold X over a finite field ? (Not so for 5-folds – A. Pirutka)

(3) Special case : Let $X \subset \mathbb{P}^5_{\mathbb{F}}$ a smooth cubic hypersurface.
Is $H^3_{nr}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = 0$?
Is $cyc_X : CH^2(X) \otimes \mathbb{Z}_\ell \to H^4(X, \mathbb{Z}_\ell(2))$ onto ?
Known for $\overline{X} = X \times_{\mathbb{F}} \overline{\mathbb{F}}$ (Charles-Pirutka).