The Brauer group and beyond : a survey

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Summer School – Quadratic Forms in Chile 2018
First part : Quadratic forms over function fields
Universitad de Santiago de Chile
Instituto de Matemática y Física
2-5 January 2018

The Brauer group of varieties may detect nonrationality of varieties. It may also prevent local-global principles for rational points of projective varieties and for integral points of affine varieties. There are “higher” analogues of the Brauer group.

Let X be an algebraic variety, or more generally a scheme.
On X, we have the Zariski topology. We also have the étale topology.
The functor which to a sheaf of abelian groups F for one of these topologies associates the group of global sections $H^0(X, F)$ is only left exact for short exact sequences of sheaves.
This gives rise to higher cohomology groups $H^i_{Zar}(X, F)$ and $H^i_{\text{ét}}(X, F)$.

On any scheme X we have the sheaf $\mathbb{G}_m$ of invertible elements which to an open set U (Zariski or étale) associates the group $H^0(U, \mathbb{G}_m)$ of invertible functions on U.
Grothendieck's Hilbert's theorem 90 asserts
$H^1_{Zar}(X, \mathbb{G}_m \xrightarrow{\simeq} H^1_{\text{ét}}(X, \mathbb{G}_m)$.
This group, which is also the group of isomorphism classes of invertible line bundles on X, is the Picard group of X, it is denoted $\mathrm{Pic}(X)$ and has been the object of much study.
For regular integral varieties over a field, it is also the quotient of the group of Weil divisors (free abelian group on irreducible codimension 1 closed subvarieties) by the subgroup of principal divisors, i.e. divisors of nonzero rational functions on X.

For X a regular scheme, one has $H^2_{Zar}(X, \mathbb{G}_m) = 0$.

Definition (Grothendieck).
Let X be a scheme. The Brauer group $H^2_{\text{ét}}(X, \mathbb{G}_m)$ is called the Brauer group of X and is denoted $\mathrm{Br}(X)$.

If $X = \mathrm{Spec}(k)$ for k a field, then $\mathrm{Br}(X)$ is the classical Brauer group of the field k, built from classes of central simple algebras over k.

If X is a smooth connected variety over a field, with function field $k(X)$, then there is a natural embedding

$$\mathrm{Br}(X) \subset \mathrm{Br}(k(X))$$

and $\mathrm{Br}(X)$ is a torsion group.

Two reasons for being interested in the Brauer group of schemes

• Variations on the Lüroth problem : which complex varieties are rational ?

• The Brauer-Manin obstruction : trying to decide if a variety defined over a number field has a rational point (cf. Hilbert's 10th problem over a number field)

Variations on the Lüroth problem over $\mathbb{C}$

A smooth, projective, connected, complex algebraic variety X is called rational if if it birational to projective space, i.e. if it field of rational functions $\mathbb{C}(X)$ is purely transcendental over $\mathbb{C}$.

It is called unirational if there is a dominant rational map from some projective space $\mathbb{P}^n$ to X, i.e. if there is an inclusion $\mathbb{C}(X) \subset \mathbb{C}(t_1, \ldots, t_n)$.

A smooth, projective, connected, complex algebraic variety X is called **rationally connected** if any two general points of $X(\mathbb{C})$ may be connected by a chain of curves of genus zero (actually, for given two points, one curve is then enough). This important notion came up in the classification of higher dimensional varieties in the late 80s.

Unirational clearly implies rationally connected. The converse is a big open question.

Connected linear algebraic groups are rational.
Homogeneous spaces of connected linear algebraic groups are unirational.
Smooth cubic surfaces are rational.
Smooth cubic hypersurfaces of dimension at least 2 are unirational.
Rationality for odd dimensional cubic hypersurfaces is a totally open problem.
Smooth hypersurfaces $X \subset \mathbb{P}^n$ of degree d are known to be unirational if n is much bigger than the degree d. It is an open question whether $n \geq d$ implies unirationality.

In general, proving rationality or unirationality of a variety can only be done by ad hoc methods.

Rational connectedness, on the other hand, has been established for vast classes of naturally defined varieties (Campana, Kollár-Miyaoka-Mori).

Some important theorems

• (Graber, Harris and Starr) If $f : X \to Y$ is a dominant morphism of irreducible complex varieties, if Y is rationally connected, and if the general fibre of f is rationally connected, then X is rationally connected.

Easy case, modulo Tsen : if $X \to \mathbb{P}^n$ is a family whose general fibre is a smooth quadric of dimension at least one, then X is rationally connected.

• (Kollár–Miyaoka–Mori, Campana) If $X \subset \mathbb{P}^n$, $n \geq 2$, is a smooth Fano variety ($-K_X$ ample), then X is rationally connected.

Thus if $X \subset \mathbb{P}^n$, $n \geq 2$, is a smooth hypersurface of degree $d \leq n$, then X is rationally connected.
Other examples are given by cyclic covers of $\mathbb{P}^n$ whose ramification locus has low enough degree.

The Lüroth problem asked whether unirational (complex) varieties are rational.

The generalized Lüroth problem asks whether specific classes of rationally connected (complex) varieties are rational.

Rather than asking for rationality, one may ask for stable rationality.
An irreducible complex variety X is called stably rational if there exists positive integers n, m such that $X \times \mathbb{P}^n$ is birational to $\mathbb{P}^m$.
It is known since 1985 that stable rationality does not imply rationality.

The answer to the Lüroth problem is negative, as was shown around 1972 by
• Clemens–Griffiths (cubic threefolds, none rational, all unirational)
• Iskovskikh–Manin (quartic threefolds, none rational, some of them unirational, all rationally connected)
• Artin–Mumford (special conic bundles over $\mathbb{P}^2$, also double covers of $\mathbb{P}^3$ ramified along a singular quartic surface, some of them unirational, all rationally connected)

The question then became : **For specific classes of rationally connected varieties, can we disprove (stable) rationality ?**
In many cases, one has to be satisfied with the weaker answer that "very general elements" in the class are not (stably) rational.

How the Brauer group (and higher variants) comes in.

The Brauer group is a stable birational invariant :
If X and Y are *smooth*, connected projective complex varieties and $X \times \mathbb{P}^n$ is birational to $Y \times \mathbb{P}^m$ then $\mathrm{Br}(X) \simeq \mathrm{Br}(Y)$. In particular, if X is stably rational, then $\mathrm{Br}(X) = 0$.
The Artin-Mumford example may be interpreted from this point of view : for any smooth, projective model X of the threefold they consider, one has $\mathrm{Br}(X) \neq 0$. More below.

At first sight, the Brauer group is useless for the other two 1972 counterexamples, since we have :

Theorem. *For any smooth hypersurface $X \subset \mathbb{P}^n$, if $n \geq 4$, then* $\mathrm{Br}(X) = 0$.

This is a consequence of a general theorem.

Theorem. *For X a smooth, projective, complex variety X, the Brauer group of X is an exension of the finite group* $H^3_{Betti}(X(\mathbb{C}), \mathbb{Z})_{tors}$ *by the divisible group* $(\mathbb{Q}/\mathbb{Z})^{b_2-\rho}$, *where b_2 is the second Betti number and ρ is the rank of the Néron-Severi group of X.*

But it is difficult to produce explicit smooth projective models of :

- homogeneous spaces of connected linear algebraic groups
- singular hypersurfaces in projective space
- singular quadric bundles over projective space
- singular cyclic coverings of projective space.

In each of these cases, the "Brauer group of a smooth projective model" is a natural group attached to the variety, if it is nonzero, it tells us that the variety is not rational.

Can one compute this group without actually producing a smooth projective model ?

A further reason why one is interested in such question is that in the last four years, a whole industry has developed which aims at proving nonrationality of some classes of (very general) smooth projective varieties by letting them degenerate to singular projective varieties which admit a (nice enough) desingularization with e.g. nontrivial Brauer group, whereas the general variety in the family has trivial Brauer group.

This specific specialization technique was started by C. Voisin, who applied it to double covers of $\mathbb{P}^3$ ramified along a smooth quartic surface. A. Pirutka and I developed it shortly thereafter and applied it to smooth quartic hypersurfaces in $\mathbb{P}^4$ – the Iskovskikh–Manin example – and proved nonstable rationality for very general such quartic hypersurfaces. In both cases, the computation of the Brauer group of a smooth model of a singular variety is crucial.

Let us repeat.

Can one compute the Brauer group of a smooth projective model of a variety without actually producing a smooth projective model ?

Given a discrete valuation ring A with field of fractions K and residue field κ of char. zero, there is a *residue map* $\partial_A : \mathrm{Br}(K) \to H^1(\kappa, \mathbb{Q}/\mathbb{Z})$ with values in the first Galois cohomology group of κ with coefficients in $\mathbb{Q}/\mathbb{Z}$: this associates to an element of $\mathrm{Br}(K)$ a finite cyclic extension of κ and a choice of generator of the Galois group. This invariant has a long history, in particular in class field theory. It measures how an element $\alpha \in \mathrm{Br}(K)$ is "ramified" with respect to the valuation. It fits into an exact sequence

$$0 \to \mathrm{Br}(A) \to \mathrm{Br}(K) \to H^1(\kappa, \mathbb{Q}/\mathbb{Z}) \to 0.$$

Unramified cohomology

Given a smooth projective irreducible variety X over $\mathbb{C}$, the following subgroups of $\mathrm{Br}(\mathbb{C}(X))$ coincide :

- $\mathrm{Br}(X)$
- $\mathrm{Br}_{nr}(\mathbb{C}(X)/X) := \mathrm{Ker}[\mathrm{Br}(\mathbb{C}(X)) \to \oplus_{x \in X^{(1)}} H^1(\kappa(x), \mathbb{Q}/\mathbb{Z})]$, where the maps are given by the residue maps ∂_x associated to the valuation rings defined by the local rings $O_{X,x}$ at points x of codimension 1 of X, whose residue fields are denoted $\kappa(x)$.
- $\mathrm{Br}_{nr}(\mathbb{C}(X)) := \mathrm{Ker}[\mathrm{Br}(\mathbb{C}(X) \to \prod_{A \subset \mathbb{C}(X))} H^1(\kappa_A, \mathbb{Q}/\mathbb{Z})]$, where A runs through all discrete, rank one, valuation rings of the function field $\mathbb{C}(X)$, and ∂_A is the residue map.

Note that
$\mathrm{Br}_{nr}(\mathbb{C}(X)) := \mathrm{Ker}[\mathrm{Br}(\mathbb{C}(X)) \to \prod_{A \subset \mathbb{C}(X))} H^1(\kappa_A, \mathbb{Q}/\mathbb{Z})]$
is defined purely in terms of the function field of X, hence is obviously a birational invariant. By defining variants over an arbitrary field, and studing the specific case of the projective line over an arbitrary field, one checks that his group vanishes if $\mathbb{C}(X)$ is purely transcendental over $\mathbb{C}$.

The unramified Brauer group was put to gut use by D. Saltman and by F. Bogomolov in the 80s.

Saltman found the first examples of a linear action of a finite group G on a finite dimensionsal complex vector space V with field of invariants $\mathbb{C}(V)^G$ not purely transcendental. Bogomolov gave a close-cut formula for $\mathrm{Br}_{nr}(\mathbb{C}(V)^G)$.
Bogomolov proved that $\mathrm{Br}_{nr}(\mathbb{C}(X)) = 0$ for a homogeneous space X of a complex, connected linear algebraic group with connected isotropy groups. The question of rationality of any such X is still an open question.

There are higher degree variants of the unramified Brauer group (joint work with Ojanguren, 1988, at the time of my previous visit to Santiago).
For K a field of char. zero, a positive integer $n > 0$ and μ_n the group of n-th roots of unity, Kummer theory identifies the n-torsion subgroup of $\mathrm{Br}(K)$ with the second Galois cohomology group of K with values in μ_n :

$$H^2(K, \mu_n) = \mathrm{Br}(K)[n].$$

If K is the field of fractions of a dvr A with residue field κ of char. zero, and $i > 0$ is an integer there is a residue map

$$\partial_A : H^{i+1}(K, \mu_n^{\otimes i}) \to H^i(\kappa, \mu_n^{\otimes i-1})$$

which generalizes the residue map for the Brauer group.

If $\mathbb{C}(X)$ is the field of rational functions of an irreducible complex variety X, for any $i > 0$ and any integer $n > 1$, one is thus led to define

$$H^i_{nr}(\mathbb{C}(X), \mathbb{Z}/n) := \mathrm{Ker}[H^i(\mathbb{C}(X), \mu_n^{\otimes i-1}) \to \prod_{A \subset \mathbb{C}(X)} H^{i-1}(\kappa_A, \mu_n^{\otimes i-2})]$$

This is obviously birational invariant. As above, there is a more general definition for varieties over a field, and by computations for the projective line, one shows that $H^i_{nr}(\mathbb{C}(X), \mathbb{Z}/n) = 0$ is $\mathbb{C}(X)$ is purely transcendental.

Ojanguren and I (1989) reinterpreted the Artin-Mumford example from the point of view of the unramified Brauer group, produced more examples, then produced new examples of unirational, nonrational varieties by using H^3_{nr}. Our examples are fibrations into quadrics over $\mathbb{P}^3_{\mathbb{C}}$, with general fibre defined by a 3-Pfister neighbour.

For the Artin-Mumford example, the argument runs as follows. One has a fibration $X \to Y = \mathbb{P}^2_{\mathbb{C}}$ of complex varieties whose generic fibre is a conic Z over the function field $\mathbb{C}(Y)$. Assume this conic does not have a rational point. Let $\beta \neq 0$ be the class of the associated quaternion algebra in $\mathrm{Br}(\mathbb{C}(Y))[2]$.

• The kernel of the map $\mathrm{Br}(\mathbb{C}(Y) \to \mathrm{Br}(\mathbb{C}(X))$ is just $\mathbb{Z}/2.\beta$ (Witt).

• One produces an element $\alpha \in \mathrm{Br}(\mathbb{C}(Y))[2]$ whose degeneracy on Y is "smaller" than the degeracy of α, while not being empty.

Then the image of α in $\mathrm{Br}(\mathbb{C}(X))$ becomes unramified, and is nonzero.

Hence $\mathrm{Br}_{nr}(\mathbb{C}(X)) \neq 0$, and X is not stably rational.

One may try to produce similar examples using unramified Galois cohomology H^i with coefficients $\mathbb{Z}/n$ for fibrations $X \to Y$ when one can control the kernel of the restriction map

$$H^i(\mathbb{C}(Y), \mathbb{Z}/n) \to H^i(\mathbb{C}(X), \mathbb{Z}/n).$$

Such is the case for
• $i = 2$ and the generic fibre is a Severi-Brauer variety (Châtelet, Amitsur)
• $i = 3$, $n = 2$ and the generic fibre is a quadric defined by a neighbour of a 3-Pfister form (Arason) (used by CT-Ojanguren)
• $i > 1$ arbitrary, $n = 2$ and the generic fibre is a quadric defined by a neighbour of an i-Pfister form (Orlov-Vishik-Voevodsky 2007).

The idea in CT-Ojanguren was later used by Asok, and most recently by S. Schreieder. This author produced new types of quadric bundles over projective space $\mathbb{P}^n_{\mathbb{C}}$ with general fibre a quadric defined by a (very special) n-Pfister neighbour. He managed to combine this with the specialisation method to show that "very general" quadric bundles of many different "types" over $\mathbb{P}^n_{\mathbb{C}}$ with fibre a quadric defined by a quadratic form in at most 2^n variables and with total space a smooth variety flat over $\mathbb{P}^n_{\mathbb{C}}$ are not (stably) rational.
Note that 2^n is the maximum not excluded by Tsen's theorem.

The theorem below computes the exact value of $\mathrm{Br}_{nr}(\mathbb{C}(X))$.
When the ramification locus $\bigcup_i C_i$ is smooth, the result has been known for a long time (Iskovskikh).
To produce nonrational conic bundles X over $\mathbb{P}^2$, as in the Artin-Mumford case, the full strength of the theorem is not needed, since all one needs if $\mathrm{Br}_{nr}(\mathbb{C}(X)) \neq 0$.

Theorem. *Let X be a smooth threefold equipped with a dominant morphism $\pi : X \to \mathbb{P}^2_{\mathbb{C}}$ whose generic fibre is a smooth conic. Let $\alpha \in \mathrm{Br}(\mathbb{C}(S))[2]$ be the associated quaternion algebra class. Assume that $\alpha \neq 0$. Let $C_1, \ldots, C_n$ be all the integral curves in S such that the residue of α at the generic point of C_i is non-zero:*

$$0 \neq \partial_{C_i}(\alpha) \in H^1(\mathbb{C}(C_i), \mathbb{Z}/2) = \mathbb{C}(C_i)^*/\mathbb{C}(C_i)^{*2}.$$

Assume that each C_i is smooth and that the ramification locus $C = \cup_{i=1}^n C_i$ of α is a curve with at most ordinary quadratic singularities. Consider the subgroup $H \subset (\mathbb{Z}/2)^n$ consisting of the elements $(r_1, \ldots, r_n)$ with the property that for $i \neq j$ we have $r_i = r_j$ when there is a point $p \in C_i \cap C_j$, necessarily smooth on C_i and on C_j, with the property that $\partial_p(\partial_{C_i}(\alpha)) = \partial_p(\partial_{C_j}(\alpha)) \in \mathbb{Z}/2$ and the common value is non-zero. Then $\mathrm{Br}(X)$ is the quotient of H by the diagonal element $(1, \ldots, 1)$ which is the image of α.

Starting from this result, one may show the existence of threefolds X with a conic bundle structure over $\mathbb{P}^2_{\mathbb{C}}$ with ramification locus the (singular !) union of 6 lines in general position in $\mathbb{P}^2$, and such that $\mathrm{Br}_{nr}(\mathbb{C}(X)) \neq 0$, hence X is not stably rational.
One breaks the 6 lines into two triples, then on each line L in a triple one considers the class in $\mathbb{C}(L)^*/\mathbb{C}(L)^{*2} = H^1(\mathbb{C}(L), \mathbb{Z}/2)$ of a function with divisor $A_L - B_L$, where A_L and B_L are the points of intersection of L with the other two lines in the triple.
General theory (Bloch-Ogus) shows that the associated family of residues is the total residue of an element α of order 2 in $\mathrm{Br}(\mathbb{C}(\mathbb{P}^2))$. Then Merkurjev's theorem on K_2 of a field mod. 2 combined with the fact that over $\mathbb{C}(\mathbb{P}^2)$ the tensor product of two quaternion algebras is the class of a quaternion algebra gives that α is the class of a quaternion algebra over $\mathbb{C}(\mathbb{P}^2)$, which produces the desired (birational) conic bundle over $\mathbb{P}^2$.

One may check that 6 is the lowest number of lines for which one may produce such a construction.

Starting from the above theorem, and using the specialisation method of Voisin et al., one may show that the very general smooth threefold with a conic bundle structure over $\mathbb{P}^2_{\mathbb{C}}$ and with degeneracy locus a smooth curve of degree at least 6 is not stably rational.
This is a special case of a result of Hassett and Kresch.

For fields of invariants $\mathbb{C}(V)^G$ with a linear action of a finite group G on a finite dimensional vector space, E. Peyre produced a closed cut formula for the odd part of the finite group $H^3_{nr}(\mathbb{C}(V)^G), \mathbb{Z}/n)$.

Merkurjev has studied $H^3_{nr}(\mathbb{C}(X), \mathbb{Z}/n)$ for homogeous spaces of the shape SL_m/G with G a complex, connected linear algebraic group. He proved its vanishing in many cases.

The Brauer group, rational points and integral points

Let k be a number field. For each place v of k, let k_v denote the completion. Class field theory produces embeddings

$$\mathrm{Br}(k_v) \hookrightarrow \mathbb{Q}/\mathbb{Z}$$

(isomorphisms for v non archimedean) and an exact sequence

$$0 \to \mathrm{Br}(k) \to \oplus_v \mathrm{Br}(k_v) \to \mathbb{Q}/\mathbb{Z} \to 0.$$

This sequence contains among others :
- Gauss's reciprocity laws and other reciprocity laws
- Hasse's principle for norms of cyclic extensions of k

Let X be an algebraic variety over k. We have inclusions

$$X(k) \subset X(\mathbb{A}_k) \subset \prod_v X(k_v)$$

of the set $X(k)$ of rational points into the set $X(\mathbb{A}_k)$ of adèles of X. For X projective, $X(\mathbb{A}_k) = \prod_v X(k_v)$.
There is a natural pairing

$$X(\mathbb{A}_k) \times \mathrm{Br}(X) \to \mathbb{Q}/\mathbb{Z}$$

$$(\{P_v\}, \alpha) \mapsto \sum_v \alpha(P_v) \in \mathbb{Q}/\mathbb{Z}.$$

We let $X(\mathbb{A}_k)^{\mathrm{Br}} \subset X(\mathbb{A}_k)$ denote the left kernel of this pairing. By the above exact sequence, the diagonal image of $X(k)$ in $X(\mathbb{A}_k)$ lies in $X(\mathbb{A}_k)^{\mathrm{Br}}$ (Manin 1970).

We have $X(\mathbb{A}_k) = \emptyset$ if and only if $\prod_v X(k_v) = \emptyset$, if and only if, for some place v, $X(k_v) = \emptyset$. For v non archimedean, the latter condition means that by using suitable v-congruences we may decide that there is no solution.

It may happen that $X(\mathbb{A}_k) \neq \emptyset$ (no congruence impossibility, no real impossibility) and nevertheless $X(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$. This then implies $X(k) = \emptyset$. In that case, we say that there is a Brauer–Manin obstruction to the local-global principle.

There is a natural topology on $X(\mathbb{A}_k)$. The set $X(\mathbb{A}_k)^{\mathrm{Br}}$ is closed in $X(k)$. The closure $X(k)^{cl}$ of $X(k)$ in $X(\mathbb{A}_k)$ thus lies in $X(\mathbb{A}_k)^{\mathrm{Br}}$.

In 1970, one might have asked the general question : do we have equality

$$X(k)^{cl} = X(\mathbb{A}_k)^{\mathrm{Br}} \ ?$$

Is this at least true for smooth, projective varieties X ?

Skorobogatov answered this question in the negative in 1999 (elliptic surface). Since then, simpler negative examples were given (Poonen; Harpaz–Skorobogatov; CT–Pál–Skorobogatov; A. Smeets).

Here is one of the simplest examples (CT–Pál–Skorobogatov).

Let $f : X \to \mathbb{P}^1_{\mathbb{Q}}$ be quadric bundle of relative dimension at least 3, with smooth total space.
For instance it may be given by an equation

$$\sum_{i=1}^{4} X_i^2 + t(t-2)X_0^2 = 0$$

in $\mathbb{P}^4 \times \mathbb{A}^1$. All fibres are smooth except those at $t = 0$ and $t = 2$. For each prime p we have $f(X(\mathbb{Q}_p)) = \mathbb{P}^1(\mathbb{Q}_p)$ because quadratic forms in at least 5 variables over a p-adic field have a nontrivial zero over that field. The image of $X(\mathbb{R})$ is the interval $0 \leq t \leq 2$.

Let $C/\mathbb{Q}$ be a smooth projective curve with only one rational point $A \in C(\mathbb{Q})$. (One knows how to produce elliptic curves with this property). Let $q : C \to \mathbb{P}^1_{\mathbb{Q}}$ be étale at A. There is an open interval $I \subset C(\mathbb{R})$ around A which is sent isomorphically to an open interval $J \subset \mathbb{A}^1(\mathbb{R})$ around $B = q(A)$. One may assume that B is given by $t = -1$ in $\mathbb{A}^1$ and that $t = 1$ is in J. Let $D \in I$ be the inverse image of $t = 1$. One then consider the fibre product $Y := X \times_{\mathbb{P}^1} C$. We have the projection map $g : Y \to C$. Provided q was chosen general enough, Y is smooth. There exists $P_\infty \in Y(\mathbb{R})$ with image $D \in I$. The (smooth) fibre Y_B of g above the point B has $\mathbb{Q}_p$-points P_p for all finite primes p. We have $Y_B(\mathbb{Q}) = \emptyset$ since there is no $\mathbb{R}$-point of X above $t = -1$.

Claim : the adèle $\{P_p, P_\infty\} \in Y(\mathbb{A}_\mathbb{Q})$ is orthogonal to $\mathrm{Br}(Y)$ for the Brauer-Manin pairing.

Proposition : *The pull-back map* $g^* : \mathrm{Br}(C) \to \mathrm{Br}(Y)$ *is an isomorphism.* (This uses the fact that $Y \to C$ is a quadric bundle of relative dimension (at least) 3.)

Let $\alpha \in \mathrm{Br}(Y)$. By the above proposition, $\alpha = g^*(\beta)$ with $\beta \in \mathrm{Br}(C)$. Now

$$\sum_p \alpha(P_p) + \alpha(P_\infty) = \sum_p \beta(g(P_p)) + \beta(g(P_\infty))$$

$$= \sum_p \beta(B)_p + \beta(D)_\infty.$$

But B and D lie in the same (connected) interval inside $C(\mathbb{R})$. By continuity of the Brauer pairing, $\beta(B)_\infty = \beta(D)_\infty$. Thus the above sum equals $\sum_p \beta(B))_p + \beta(B)_\infty = 0$ by the law of reciprocity.

For smooth, absolutely irreducible projective varieties, one may still ask whether

$$X(k)^{cl} = X(\mathbb{A}_k)^{\mathrm{Br}}$$

(possibly ignoring the archimedean places)
for :

- Curves of arbitrary genus
- Geometrically rationally connected varieties
- $K3$-surfaces

In these three cases, people have been as far as conjecturing $X(k)^{cl} = X(\mathbb{A}_k)^{\mathrm{Br}}$, and even proving it for some geometrically rationally connected varieties.
The most ancient results are on rational points of homogeneous spaces of connected linear algebraic groups. The most recent, impressive results are due to Harpaz and Wittenberg. Some of the proofs involve results from analytic number theory.

To test on a given X whether the left kernel $X(\mathbb{A}_k)^{\mathrm{Br}}$ of the pairing

$$X(\mathbb{A}_k) \times \mathrm{Br}(X) \to \mathbb{Q}/\mathbb{Z}$$

is not empty, one must in principle know *explicit elements in* $\mathrm{Br}(X)$ *which generate the quotient* $\mathrm{Br}(X)/\mathrm{Br}(k)$.

This question is basic both from a theoretical point of view and from a practical point of view : even if we abstractly prove that for a certain class of varieties, $X(\mathbb{A}_k)^{\mathrm{Br}} \neq \emptyset$ implies $X(k) \neq \emptyset$, if we want to prove $X(k) \neq \emptyset$ for a given such X, one must be able to decide if $X(\mathbb{A}_k)^{\mathrm{Br}} \neq \emptyset$, and for this one needs to compute a complete list of explicit representants of $\mathrm{Br}(X)/\mathrm{Br}(k)$ in $\mathrm{Br}(X)$.

For a smooth, projective, rationally connected variety over an arbitrary field k of char. zero, the quotient $\mathrm{Br}(X)/\mathrm{Br}(k)$ is finite, for purely algebraic reasons, and it is relatively easy to compute. Finding representants in $\mathrm{Br}(X)$ may be difficult – already for cubic surfaces.

For a (smooth) $K3$ surface over a number field the quotient $\mathrm{Br}(X)/\mathrm{Br}(k)$ is finite (Skorobogatov and Zarhin). Actually computing the group is very delicate (work of Skorobogatov and others on specific Kummer surfaces).

Integral points

For projective varieties X over a number field k, one is interested in rational points. For e.g. affine varieties X over a number field, a more natural question is that of integral points. For instance, one has a polynomial $P(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n]$. The vanishing of this polynomial defines an affine hypersurface X in $\mathbb{A}^n_{\mathbb{Z}}$. The set $X(\mathbb{Z})$ is the set of integral points. It diagonally lies in the left kernel of the pairing

$$\prod_{p \text{ prime or } \infty} X(\mathbb{Z}_p) \times \mathrm{Br}(X_{\mathbb{Q}}) \to \mathbb{Q}/\mathbb{Z},$$

where $X(\mathbb{Z}_p)$ for p finite is the set of solutions of $P(x_1, \ldots, x_n) = 0$ with coordinates in the ring of p-adic integers and $X(\mathbb{Z}_\infty) = X(\mathbb{R})$.

For some classes of polynomials, one may ask whether $[\prod_{p \text{ prime or } \infty} X(\mathbb{Z}_p)]^{\mathrm{Br}} \neq \emptyset$ implies $X(\mathbb{Z}) \neq \emptyset$.

This is a special case of the general question whether

$$X(k)^{cl} = X(\mathbb{A}_k)^{\mathrm{Br}}$$

for X/k a smooth, not necessarily projective, variety.

The problem has been studied for X a homogeneous space of a connected linear algebraic group, and for closely related varieties. Classical results go back to theorems such as Eichler and Kneser's theorem that the local-global principle holds for integral solutions of equations $q(x_1, \ldots, x_n) = a$, with $a \in \mathbb{Z}$, where q is an integral quadratic form nondegenerate over $\mathbb{Q}$ and indefinite over $\mathbb{R}$, in at least 4 variables.
Over the last 10 years, there have been many works proving Brauer-Manin type results for integral points of homogeneous spaces of the above type (CT-Xu, Harari, Borovoi, Demarche, Xu, Yang Cao).

Here is a very special case.
Let $a, b \in \mathbb{Z}$, both nonzero. Consider the scheme X over $\mathbb{Z}$ defined by the Pell type equation

$$x^2 - ay^2 = b$$

Theorem. If $[\prod_{p \text{ prime or } \infty} X(\mathbb{Z}_p)]^{\mathrm{Br}(X_{\mathbb{Q}})} \neq \emptyset$ then $X(\mathbb{Z}) \neq \emptyset$.
This result, also proved by F. Xu and D.Wei, is just a special case of a theorem of Harari on principal homogeneous spaces of algebraic tori over a number field. That theorem builds upon class field theory.
Warning : to decide whether $X(\mathbb{Z}_p)]^{\mathrm{Br}(X_{\mathbb{Q}})} \neq \emptyset$ one would need to compute $\mathrm{Br}(X_{\mathbb{Q}})/\mathrm{Br}(\mathbb{Q})$. But this is an infinite group !
There are ways to reduce this to a finite amount of computations (see the book by Cox on primes of the shape $x^2 + ny^2$, and papers of Fei and Wei).

Integral points : beyond homogeneous spaces

From a geometric point of view, the Pell type equation above is the complement of 2 points in the projective line. The relevant canonical bundle is zero, just like the canonical bundle of a projective elliptic curve.
In higher dimension, one should first inverstigate the analogue of projective surfaces with trivial canonical bundle.
From this point of view, the complement of a hyperplane section in a cubic surface is such an analogue. They are sometimes called "log-K3 surfaces".
Here are two such cases where people have been interested in integral points on such schemes.

A very ancient one is

$$n = x^3 + y^3 + z^3$$

with $n \neq 0$.
This is the famous question whether any integer n which is not congruent to ± 4 modulo is a sum of three cubes of relative integers.
CT-Wittenberg proved $[\prod_{p \text{ prime or } \infty} X(\mathbb{Z}_p)]^{\mathrm{Br}(X_{\mathbb{Q}})} \neq \emptyset$ for any integer n not congruent to ± 4.
The proof involves the nontrivial task of computing the group $\mathrm{Br}(X_{\mathbb{Q}})/\mathrm{Br}(\mathbb{Q})$ which turns out to be finite – but not zero in general. That computation uses the knowledge of the curves of genus one (defined over $\mathbb{Q}$) isogenous to the curve "at infinity"

$$x^3 + y^3 + z^3 = 0.$$

Another interesting case is that of Markoff type equations

$$x^2 + y^2 + z^2 - xyz = n$$

with $n \in \mathbb{Z}$, $n \neq 0$, $n \neq 4$. This has been recently investigated by Ghosh and Sarnak, who think that the Brauer-Manin condition should not be the only obstruction to existence of integral points. Note that projection to the coordinate z makes this surface into a fibration over $\mathbb{A}^1$ with fibres affine conics.
Other integral equations with such a structure are provided by affine Châtelet surfaces $x^2 - ay^2 = P(z)$ with $a \in \mathbb{Z}$ and $P(x) \in \mathbb{Z}[x]$.
Some families of affine conic bundles have been studied by Harpaz.