Arithmetic upon intersection of two quadrics

Hannover, June 14th, 2023
Göttingen–Hannover Seminar

Jean-Louis Colliot-Thélène
CNRS et Université Paris-Saclay
z. Z. Leibniz Universität Hannover

References

JLCT, J-J. Sansuc, P. Swinnerton-Dyer, Crelle, 1987

JLCT, Retour sur l'arithmétique des intersections de deux quadriques, avec un appendice par A. Kuznetsov, https ://arxiv.org/abs/2208.04121

A. Molyakov, Le principe de Hasse pour les intersections de deux quadriques dans $\mathbb{P}^7$, arXiv 2305.0031

Let k be a number field. Let k_v run through the completions of k. Let $X \subset \mathbb{P}^n_k$, be a smooth complete intersection of two quadrics :

$$f(x_0, \cdots, x_n) = g(x_0, \cdots, x_n) = 0.$$

A well known conjecture asserts :
For $n \geq 5$, for any such X, the Hasse principle holds, namely

$$\prod_v X(k_v) \neq \emptyset \Longrightarrow X(k) \neq \emptyset.$$

When $X(k) \neq \emptyset$, and $n \geq 5$, one knows that $X(k) \subset \prod_v X(k_v)$ is dense.

For $n = 3$, the Hasse principle need not hold. One then has a curve of genus one, the obstruction to the Hasse principle is related to the Tate-Shafarevich group of the jacobian of the curve.

For $n = 4$, the Hasse principle need not hold (first explicit example : Birch and Swinnerton-Dyer 1975). Conjecturally, the defect is controlled by the Brauer-Manin obstruction.

Results were obtained for $n \geq 12$ by Mordell (1959) and for $n = 10$ by Swinnerton-Dyer (1964).

Assume k is totally imaginary, and $n = 12$. Assume $f(x_0, \ldots, x_{12})$ is non-degenerate. Here is Mordell's argument. The quadratic form f may be written as the direct sum of a totally hyperbolic quadratic form in 10 variables and a quadratic form in 3 variables. On a linear space of codimension $5 + 3 = 8$, that is a $\mathbb{P}^4_k$, the form f identically vanishes. The restriction of g to this $\mathbb{P}^4_k$ is given by a quadratic form in 5 variables, it has a nontrivial zero over k.

Formally real fields are handled by an elegant trick over the reals : consider the behaviour of the signature of the quadratic form $af + bg$ as (a, b) varies over $a^2 + b^2 = 1$. One proves the existence of quadratic forms in the pencil over $\mathbb{R}$ with 6 hyperbolics.

The Hasse principle for X *smooth* complete intersection of two quadrics in $\mathbb{P}^n_k$ is known to hold :

For $n \geq 8$ (CT–Sansuc–Swinnerton-Dyer 1987) [Note : for $n \geq 8$, $X(k_v) \neq \emptyset$ for v nonarchimedean].

For $n \geq 4$ if X contains two lines globally defined over k (the case $n = 4$ was known before 1970).

For $n \geq 5$ if X contains a conic (Salberger 1993).

For $n = 7$ (Heath-Brown 2018).

Taking two difficult conjectures (finiteness of $Ш$ of elliptic curves and Schinzel's hypothesis) for granted, Wittenberg (2007) gave a proof of the Hasse principle for any smooth X for $n \geq 5$.

A number of the above results hold for smooth projective models of possibly singular projective models of intersections of two quadrics.

In this talk, I shall discuss the path to the following theorem of A. Molyakov (2023), which completes and encompasses results of Heath-Brown (2018) and myself (2022).

Theorem. *Let k be a number field and $X \subset \mathbb{P}^7_k$ be a nonconical, geom. integral complete intersection of two quadrics. For any smooth projective model Y of X, the Hasse principle holds.*

One useful tool is the theorem : *Over any field, if an intersection of two quadrics $X \subset \mathbb{P}^n_k$ has a rational point over an odd degree extension of k then it has a rational point.*

This is an immediate consequence of Springer's theorem (same statement for one quadric, over any field) and the theorem of Amer and of Brumer :

Let $k(t)$ be the rational function field in one variable. A sytem of two quadratic forms $f = g = 0$ over a field k has a nontrivial zero if and only if the quadratic form $f + tg$ over the field $k(t)$ has a nontrivial zero.

When discussing a complete intersection of two quadrics $X \subset \mathbb{P}^n_k$ over a field k (char. not 2) given by a system $f = g = 0$, one is quickly led to consider the pencil of quadrics $\lambda f + \mu g = 0$ containing X.

Ignoring subtle points with the singular forms in the pencil, there is a close relation between the following statements, where we assume $r \geq 1$:

- There exists a form $\lambda f + \mu g$ in the pencil which splits off $r+1$ hyperbolic planes.
- There exists a quadric in the pencil which contains a linear space $\mathbb{P}^r_k \subset \mathbb{P}^n_k$.
- The variety X contains an $(r-1)$-dimensional quadric $Y \subset \mathbb{P}^r_k \subset \mathbb{P}^n_k$.

Theorem (CT 2022) *Let k be a p-adic field. Let $X \subset \mathbb{P}^3_k$ be an intersection of two quadrics given by a system*

$$f(x_0, x_1, x_2, x_3) = 0, \ g(x_1, x_2, x_3) = 0.$$

Then there exists a quadratic extension K/k with $X(K) \neq \emptyset$.

Proof. When X is not a smooth complete intersection, this is proven by a case-by-case discussion. Assume X is a smooth complete intersection. Then X is a genus one curve.

Let $\overline{k}$ be an algebraic closure of k, and $G := \mathrm{Gal}(\overline{k}/k)$. The period of a curve X is defined as the positive generator of the image of the degree map $\mathrm{Pic}(X \times_k \overline{k})^G \to \mathbb{Z}$.

The assumption that $g(x_1, x_2, x_3)$ involves only three variables implies that the "period" of the curve X divides 2. This one sees by using the fact any conic has period 1 and that the curve X is a double cover of the conic $g(x_1, x_2, x_3) = 0$.

For a curve of genus one, it is a theorem of Lichtenbaum (1969) that the period coincides with the index. Thus the index divides 2. By Riemann-Roch, this implies that there exists a field K/k of degree at most 2 with $X(K) \neq \emptyset$.

Theorem (Creutz–Viray 2021) *Let k be a p-adic field. Let $X \subset \mathbb{P}^n_k$, $n \geq 4$ be an intersection of two quadrics. There exists a field K/k of degree at most 2 with $X(K) \neq \emptyset$.*

(Alternate) proof. It is enough to handle the case $n = 4$. Singular cases are handled by a case by case analysis. Assume X is a smooth complete intersection. It is then given by a system

$$h(x_0, x_1, x_2) + x_3 x_4 = 0 = g(x_0, \cdots, x_4).$$

The section by $x_4 = 0$ is an intersection of two quadrics in $\mathbb{P}^3_k$ as in the previous theorem. QED

Theorem (Creutz–Viray 2021). *Let k be a number field and $X \subset \mathbb{P}^n_k$ be a smooth complete intersection of two quadrics. For $n \geq 4$, the index $I(X)$ divides 2.*

The proof is very elaborate.

Theorem (CT 2022) *Let k be a number field and $X \subset \mathbb{P}^n_k$ be a smooth complete intersection of two quadrics. For $n \geq 5$ there exists a quadratic extension K/k with $X(K) \neq \emptyset$.*

The question whether this holds for $n = 4$ remains open. Partial results are given by Creutz–Viray.

Proof. By Bertini it is enough to prove the case $n = 5$. In this case the variety $F_1(X)$ of lines on X is geometrically integral – it is actually a principal homogeneous space under an abelian variety. Hence there exists a finite set S of places of k such that $F_1(X)(k_v) \neq \emptyset$ for $v \notin S$. Thus for almost all v, any $\lambda f + \mu g$ splits off 2 hyperbolics over k_v.

For any place v, Theorem 2 gives a point of X in an extension of k_v of degree 2, hence there exists a $\lambda_v f + \mu_v g$ in the pencil over k_v which splits off two hyperbolics.

Using weak approximation, we find $(\lambda, \mu) \in \mathbb{P}^1(k)$ such that $\lambda f + \mu g$ splits off 2 hyperbolics over each k_v. By a result of Hasse (1924) it splits off 2 hyperbolics over k. Thus X contains a point over a quadratic extension of k.

Theorem (Salberger 1993 + ε) *Let k be a number field and $X \subset \mathbb{P}^n_k$, $n \geq 4$, be a geometrically integral, nonconical, complete intersection of two quadrics, and let Y/k be a smooth projective model of X.* **Assume that X contains a conic $C \subset \mathbb{P}^2_k \subset \mathbb{P}^n_k$.** *Then*
(a) The set $Y(k)$ is dense in the Brauer-Manin set $Y(\mathbb{A}_k)^{\mathrm{Br}(Y)} \subset Y(\mathbb{A}_k)$.
(b) For $n \geq 6$, the Hasse principle and weak approximation hold for Y.
(c) For $n = 5$ and X smooth, the Hasse principle and weak approximation hold for X.

The proof of the theorem relies in part on various works (CTSaSD 87, Coray-Tsfasman 88). Salberger's proof of the case $n = 4$ builds upon his very original work on zero-cycles.

Theorem 7 (Heath-Brown 2018) *Let k be a local field. Let $X \subset \mathbb{P}^7_k$ be a smooth complete intersection of two quadrics given by $f = g = 0$. If $X(k) \neq \emptyset$, then there exists a nondegenerate form $\lambda f + \mu g$ in the pencil which splits off three hyperbolics.*

Proof (CT 2022) Let $P \in X(k)$. The intersection C of X with the tangent $\mathbb{P}^5_k$ at P is a cone with vertex P over an intersection of two quadrics $Y \subset \mathbb{P}^4_k$. By Theorem 2 (Creutz–Viray) there exists a point on Y in a quadratic extension K/k. This defines a line over K on C passing through the vertex P of the cone. One thus gets a pair of lines in $C \subset X$ passing through P and globally defined over k. Fix a k-point Q in the plane $\mathbb{P}^2_k$ defined by these two lines, outside of the two lines. The form $\lambda f + \mu g$ vanishing at Q vanishes on the plane $\mathbb{P}^2_k$ spanned by the two lines. If nondegenerate, this form splits off 3 hyperbolics. There is a simple way to handle the case where the form is of rank 7.

Theorem 8 (Heath-Brown, 2018) *Let k be a number field. Let $X \subset \mathbb{P}^7_k$ be a smooth complete intersection of two quadrics given by $f = g = 0$. The Hasse principle holds for X.*

Hasse principle for smooth $X \subset \mathbb{P}^7_k$

Proof (CT 2022, some ingredients from HB's proof). The variety $F_2(X)$ of planes $\mathbb{P}^2_k \subset X \subset \mathbb{P}^7_k$ is a geometrically integral variety – it is actually a principal homogeneous spaces under an abelian variety. Hence there exists a finite set S of places of k such that $F_2(X)(k_v) \neq \emptyset$ for $v \notin S$. Thus each $v \notin S$, any nondegenerate $\lambda f + \mu g$ splits off 3 hyperbolics over k_v. By Theorem 7, for each $v \in S$ the assumption $X(k_v) \neq \emptyset$ implies that there exists a point $(\lambda_v, \mu_v) \in \mathbb{P}^1(k_v)$ such that $\lambda_v f + \mu_v g$ is nondegenerate and contains 3 hyperbolics. By weak approximation on $\mathbb{P}^1_k$, there exists $(\lambda, \mu) \in \mathbb{P}^1(k)$ such that $\lambda f + \mu g$ is nondegenerate and contains 3 hyperbolics over each k_v. By Hasse 1924 it contains 3 hyperbolics over k. Thus X contains a conic. Theorem 5 (Salberger) and the hypothesis $\prod_v X(k_v) \neq \emptyset$ then give $X(k) \neq \emptyset$.

What about singular complete intersections of two quadrics ?

Let k be a number field and $X \subset \mathbb{P}^n_k$ a possibly singular complete intersection of two quadrics. Assume it is geometrically integral and not a cone. One is interested in the Hasse principle for a smooth projective model Y of X.

In CT-Sansuc-Swinnerton-Dyer 1987, we proved the Hasse principle for Y under the assumption $n \geq 8$. We proposed :

Conjecture. *For $n = 6$ and $n = 7$, the Hasse principle holds for Y.*

For such n, one has $\mathrm{Br}(Y)/\mathrm{Br}(k) = 0$ so there is no Brauer-Manin obstruction. Under various additional hypotheses on Y, the conjecture is proved in CT–S–SD 1987. As we saw, Salberger 1993 proves it when X contains a conic.

A. Molyakov recently proved the above conjecture for $n = 7$.

Theorem (Molyakov 2023) *Let k be a number field. Let $X \subset \mathbb{P}^7_k$ be a nondegenerate geom. integral complete intersection of two quadrics. Then the Hasse principle holds for X_{smooth}.*

I sketch the main steps of his proof.

A local result

Theorem *Let k be a local field. Let $X \subset \mathbb{P}^7_k$ be a nondegenerate geom. integral complete intersection of two quadrics given by $f = g = 0$. If $X_{smooth}(k) \neq \emptyset$, and there is no form of rank ≤ 5 in the geometric pencil $\lambda f + \mu g$ then there exists a nondegenerate form $\lambda f + \mu g$ in the pencil which splits off three hyperbolics.*

The proof is similar to the proof in the smooth case. Namely, one finds a smooth k-point $P \in X(k)$ such that the intersection of the tangent space T_P at X in the point P is a cone over a reasonable intersection of two quadrics $Y \subset \mathbb{P}^4$. Then there exists a quadratic point on Y over the p-adic field, which leads to a (degenerate conic) lying in $T_P \cap X$. A quadric in the pencil containing a conic is defined by a quadratic form which splits off three hyperbolics.

Global result, the regular case

Theorem *Let k be a number field. Let $X \subset \mathbb{P}^7_k$ be a nondegenerate geom. integral complete intersection of two quadrics given by $f = g = 0$. Assume there is no form of rank ≤ 6 in the geometric pencil $\lambda f + \mu g$. Then the Hasse principle holds for X_{smooth}.*

Proof. Under the geometric hypothesis one knows that the variety parametrizing the planes $\mathbb{P}^2 \subset X$ is a generalized jacobian (X. Wang) and in particular is **geometrically integral**.
Via Lang-Weil and Hensel this shows there is a finite set S of places such that for $v \notin S$, there exists a $\mathbb{P}^2_{k_v} \subset X_{k_v}$. Thus any form $\lambda f + \mu g$ contains 3 hyperbolics over k_v for $v \notin S$.
The previous theorem and weak approximation then produce a $\lambda f + \mu g$ over k with 3 hyperbolics over each k_v hence over k by Hasse, hence we have a conic lying on X and may conclude by Salberger's theorem.

Global result, the irregular case

We now allow the existence a form of rank ≤ 6 in the geometric pencil. In this case the variety parametrizing the $\mathbb{P}^2 \subset X \subset \mathbb{P}^7$ need not be geometrically connected.

There is an interesting case by case discussion. A number of the cases were handled in [CT/Sa/SD].

But two cases require a new, specific argument.
– The geometric pencil contains two conjugate forms of rank 6.
– The geometric pencil contains 4 forms of rank 6.
One uses the **fibration method for zero-cycles** (Harpaz-Wittenberg), which is more flexible than the fibration method for rational points. In the second case, one ends up with a fibration over $\mathbb{P}^1$ whose generic fibre is a principal homogeneous space under a torus. And one concludes by an application of the Amer-Brumer theorem which gives that existence of a rational point on an intersection of two quadrics follows from the existence of a point in an extension of odd degree.