Index and exponent: Lichtenbaum's theorems for curves over p-adic fields.

The following text is a variation on Lichtenbaum's paper *Duality Theorems for curves over p-adic fields.*

Let k be perfect field, $\overline{k}$ an algebraic closure, $G = Gal(\overline{k}/k)$ and X/k smooth, projective, geometrically integral curve.

One has the exact sequence

$$0 \to \mathrm{Pic}(X) \to \mathrm{Pic}(\overline{X})^G \to \mathrm{Br}(k) \to \mathrm{Br}(X).$$

Let $B(X/k)$: Ker[Br(k) $\to$ Br(X)].

Let ∂ denote the map $\mathrm{Pic}(\overline{X})^G \to \mathrm{Br}(k)$ which factorizes through the surjection $\mathrm{Pic}(\overline{X})^G \to B(X/k)$.

If there is a rational point, then $\partial_k = 0$.

By definition, the index $I = I(X/k)$ is the gcd of the degrees of closed points.

One has $I.B(X) = 0 \subset \mathrm{Br}(k)$.

The degree map induces a Galois equivariant map $\mathrm{Pic}(\overline{X}) \to \mathbf{Z}$. One defines the period $P = P(X/k)$ as the positive generator of the image of $\mathrm{Pic}(\overline{X})^G \to \mathbf{Z}$.

Lemma 1. *Let k be a perfect field, let $\pi : X \to Y$ be a finite morphism of degree d of smooth projective geometrically connected curves.*

(i) The period P_X divides dP_Y.

(ii) If Y is a conic, then P_X divides d.

Proof. The pull-back map $\pi^* : \mathrm{Pic}(\overline{C}) \to \mathrm{Pic}(\overline{X})$ is g-equivariant and induces multiplication by d on $\mathbf{Z}$ after taking degree maps. This gives (i). As for (ii), for a smooth conic Y one has $\mathrm{Pic}(\overline{Y}) = \mathbf{Z}$ with trivial action, hence $P_Y = 1$. QED

Let $J_X = \mathrm{Pic}^0_{X/k}$. One has the exact sequence

$$0 \to J_X \to \mathrm{Pic}(\overline{X}) \to \mathbf{Z} \to 0.$$

This induces an exact sequence

$$\mathrm{Pic}(\overline{X})^G \to \mathbf{Z} \to H^1(k, J_X).$$

The image of $1 \in \mathbf{Z}$ in $H^1(k, J_X)$ is the class of $\mathrm{Pic}^1_{X/k}$ and $P(X)$ is the order of the image of 1 in $H^1(k, J_X)$.

Since the sequence

$$0 \to J_X \to \mathrm{Pic}(\overline{X}) \to \mathbf{Z} \to 0$$

is G-split if $I(X) = 1$, one sees that in general P divides I.

Let $L \in \mathrm{Pic}(\overline{X})^G$. By [BGG, Thm. 7.1.15, p. 190], one has $\chi(L).\partial(L) = 0 \in B(X) \subset \mathrm{Br}(k)$. Let $d = deg(L)$. The Riemann-Roch theorem on the curve X gives $\chi(L) = d + 1 - g$. We thus have

$$(d + 1 - g).\partial(L) = 0 \in B(X) \subset \mathrm{Br}(k).$$

Let K be the canonical bundle. Note that $2g-2 = deg(K)$ is a multiple of the index I and I annihilates $\partial(L) \in B(X)$. Thus $(d + 1 - g, 2g - 2)$ annihilates $\partial(L)$.

The image of $\mathrm{Pic}(\overline{X})^G \to \mathbf{Z}$ is $\mathbf{Z}.P$. Since one may find an element of $\mathrm{Pic}(\overline{X})^G$ of degree P, any element of $\mathrm{Pic}(\overline{X})^G$ may be written as

$$L = \sum_i L_i - \sum L_j$$

with each L_i and L_j of degree P.

This implies that

$$\partial(L) = \sum_i \partial(L_i) - \sum \partial(L_j) \in B(X)$$

is annihilated by $(P + 1 - g, 2g - 2)$. Recall that $B(X)$ is annihilated by I, which is a multiple of P. To summarize:

Theorem (Lichtenbaum). *Let X be a smooth projective geometrically connected curve over a field k. Let g be its genus, I its index, and P its period. The period P divides the index I. Let $I = rP$. The group*

$$B(X) = \mathrm{Ker}[\mathrm{Br}(k) \to \mathrm{Br}(X)]$$

is annihilated by $(P, g-1)$ if r is odd, and it is annihilated by $(2P, P+1-g)$ if r is even.

Proof. One has $I = rP$ for some integer r. The group $B(X)$ is annihilated by $(rP, P+1-g, 2g-2)$. Now

$$(rP, P+1-g, 2g-2) = (rP, P+1-g, 2P, 2g-2).$$

If r is odd, this is equal to $(P, 1-g)$. If r is even, this is equal to $(2P, P+1-g)$. QED

One thus gets:

If $g = 0$, then X is a smooth conic. In this case $P = 1$ and I is 1 or 2.
If $g = 1$, then in all cases one finds $P.B(X) = 0$.
For arbitrary g, one has $2P.B(X) = 0$.
Suppose X is a hyperelliptic curve of genus g. Then $I(X)$ is 1 or 2. Then $2.B(X) = 0$ and $P(X)$ is 1 or 2. If $I(X) = 1$ then $B(X) = 0$. Suppose $I(X) = 2$. If $P(X) = 1$ and g is odd, then $B(X) = 0$. If $P(X) = 2$ and g is even, then $B(X) = 0$.

For more results for k arbitrary, see Theorem 8 of Lichtenbaum's paper.

For the rest of the note, let k be a p-adic field. Then we have the isomorphism $\mathrm{Br}(k) \simeq \mathbf{Q}/\mathbf{Z}$.

Theorem (Roquette, Lichtenbaum) *For X a smooth, projective, geometrically connected curve over a p-adic field, the group $B(X) \subset \mathrm{Br}(k) \simeq \mathbf{Q}/\mathbf{Z}$ is isomorphic to $\mathbf{Z}/I \subset \mathbf{Q}/\mathbf{Z}$.*

Proof. This is Lichtenbaum's Theorem 3. This theorem may be obtained as a combination of an index computation for varieties of arbitrary dimension over a p-adic field ([BGG], Thm. 10.5.8) and a specific theorem for curves over the field of fractions of a complete discrete valuation ring ([BGG], Thm. 10.3.1).

One now concludes that I divides $(P+1-g, 2g-2)$ hence I divides $2P$. Thus P divides I which divides $2P$.
If $g = 0$ then $P = 1$ and I is 1 or 2.
If $g = 1$ then I divides P so $I = P$.
Let g be arbitrary. Then $I = P$ or $I = 2P$. Recall that I divides $2g-2$. If $I = 2P$ then P divides $g-1$. If $I = P$ then $B(X)$ is annihilated by $(P, P+1-g)$ hence by $g-1$, and then the Roquette-Lichtenbaum theorem gives that I divides $g-1$, hence P also divides $g-1$.
All these results are contained in Lichtenbaum's Theorem 7. He also mentions the implication: If $I = 2P$, then $(g-1)/P$ is odd.

Here are curious corollaries of the property $I = P$ for curves of genus one over a p-adic field.

Proposition 1. *Let k be a p-adic field. Let $X \subset \mathbf{P}^3_k$ be a smooth complete intersection of two quadrics given by the vanishing of of two quadratic forms.*

$$f(x_1, x_2, x_3, x_4) = 0 = g(x_0, x_1, x_2, x_3) = 0.$$

If there exists a rational point on the curve C of genus one given by the equation $y^2 = det(\lambda f + \mu g)$, then there exists a rational point or a closed point of degree 2 on X.

Proof. By Theorem 2.25 of Wang [W], over any field, the assumption on the curve C implies that the period of X is 1 or 2. Since $I = P$ for curve of genus one over a p-adic field, the result follows. QED

Here is a special case, for which we offer an alternate proof.

Proposition 2. *Let k be a p-adic field. Let $X \subset \mathbf{P}^3_k$ be a smooth complete intersection of two quadrics given by the vanishing of of two quadratic forms, one of which is of rang 3:*

$$f(x_1, x_2, x_3) = 0 = g(x_0, x_1, x_2, x_3) = 0.$$

Then there exists a rational point or a closed point of degree 2 on X.

Proof. Sending (x_0, x_1, x_2, x_3) to (x_1, x_2, x_3) makes X into a double cover $\pi : X \to C$ of the conic C with equation $f(x_1, x_2, x_3) = 0$. Lemma 1 gives that $P = P(X)$ divides 2. Using Lichtenbaum's theorem over the p-adics, one concludes that the index of X is 1 or 2, which for the genus one curve X ensures the existence of a rational point or a closed point of degree 2. QED

Challenge: *Give a down-to-earth proof of the second Proposition, purely in terms of quadratic forms.*

References

[BGG] JLCT et A. N. Skorobogatov, The Brauer–Grothendieck group, Springer Ergebnisse, 2021.

[L] S. Lichtenbaum, Duality Theorems for Curves over p-adic Fields, Invent. math. **7** (1969) 120–136.

[W] Xiaoheng Wang, Maximal linear spaces contained in the base loci of pencils of quadrics. Alg. Geom. **5** (3) (2018) 359–397.

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