Zero-cycles on del Pezzo surfaces
(Variations upon a theme by Daniel Coray)

Iskovskikh seminar
Moscow
June 11th, 2020
(online)

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A preprint is available on arXiv (May 14th).

An improved version is on my webpage.

T. A. Springer (1952) proved :

Let $q(x_0, \ldots, x_n)$, be a quadratic form over a field k. If it has a nontrivial zero over a field extension K/k of odd degree, then it has a nontrivial zero over k.

Proof.

May assume K/k simple, $K = k[t]/P(t)$. Then may write

$$q(R_0(t), \ldots, R_n(t)) = P(t)Q(t)$$

with $deg(R_i) < deg(P)$, not all zero. Then $deg(Q) < deg(P)$, then pick up an irreducible factor of Q of odd degree and reduce modulo this factor.

Exercise in the same spirit.

Let $n \geq 1$ et $Q(x_0, \ldots, x_n)$ be a nondegenerate quadratic form over a field k. Let $P(t) \in k[t]$ be a polynomial of degree $2r$.
Assume

$$P(t) - Q(x_0, \ldots, x_n) = 0 \qquad (1)$$

has a solution in an odd degree extension K/k.
Then
– Either the leading coefficients of P is represented by Q over k (which essentially gives a rational point above $t = \infty$)
– Or equation (1) has a solution in a field extension K/k of odd degree $d \leq r$.

Theorem. *If the intersection of two projective quadrics defined by*

$$\Phi(x_0, \ldots, x_n) = 0 = \Psi(x_0, \ldots, x_n)$$

has a rational point over an odd degree extension of k, then it has a rational point over k.
Special case : del Pezzo surfaces of degree 4 (Coray 1977).

The theorem is an immediate consequence of Springer's result and the following theorem (Amer 1976, Brumer 1978, Leep)
Theorem. *Let $\Phi(x_0, \ldots, x_n)$ et $\Psi(x_0, \ldots, x_n)$ be two quadratic forms with coefficients in the field k. They have a common nontrivial zero over k if and only if the quadratic form $\Phi + t\Psi$ has a nontrivial zero over the rational function field $k(t)$.*

One would like to produce classes of algebraic varieties X with the property :
(*) If X has a zero-cycle of degree one, i.e. if the gcd of degrees of the finite field extensions K/k with $X(K) \neq \emptyset$ is one, then $X(k) \neq \emptyset$.

For instance, (*) holds for arbitrary plane cubics in $\mathbb{P}^2_k$ over any field k. It holds for principal homogeneous spaces of an abelian variety A/k, indeed we have norm maps on $H^1_{gal}(k, A)$.

(*) has been proved for principal homogeneous spaces of connected reductive linear algebraic groups in many cases, but the general case is open.
There are counterexamples for projective homogeneous spaces of such groups (Parimala 2005).

(*) is an open question for cubic hypersurfaces (question raised by Cassels and Swinnerton-Dyer).

In this case, the question reads : if a cubic hypersurface in $\mathbb{P}^n_k$ has a rational point in a finite field extension K/k of degree prime to 3, does it have a rational point in k ?

The question was studied by Daniel Coray (1947-2015) in his thesis (Cambridge, UK 1974).

ACTA ARITHMETICA
XXX (1976)

Algebraic points on cubic hypersurfaces

by

D. F. Coray* (Cambridge, Mass.)

Introduction. The following conjecture was apparently first formulated by Cassels and Swinnerton-Dyer (in the case $n = 3$), and it is closely connected with some of the problems mentioned by B. Segre in [15], p. 2:

Conjecture (CS). *Let $f(x_0, \ldots, x_n)$ be a cubic form with coefficients in a field k. Suppose f has a non-trivial solution in an algebraic extension K/k, of degree d prime to 3. Then f also has a non-trivial solution in the ground field k.*

Of course the crucial condition in this statement is that d should be prime to 3. The case $n \leqslant 2$ was already known to Henri Poincaré [11]; his proof will be given in §2, since the geometrical ideas it involves are fundamental in the study of the case $n = 3$ and will be used throughout this paper. We begin with a few rather dry lemmas on the rationality of cycles on an algebraic variety (§1), which are necessary if we want to proceed on firm ground when using algebraic geometry over an arbitrary field. The use of these lemmas is exemplified in §2, which therefore gives not only Poincaré's proof, but also a few other applications of the same type of argument. In §3 we discuss some of the first attempts made at proving the conjecture when $n = 3$, including a very interesting descent argument due to Cassels (unpublished). This result implies in particular that (CS) holds when $n = 3$ and k is a local field. But the argument fails when the characteristic of the residue class field is equal to 2.

At this point, the exposition breaks into two parts: in §§4 and 5, we use a different method to prove the conjecture in full generality over any local field (i.e. for all n and without any restriction on the characteristic). This is done by purely arithmetic means, and the reader who is more interested in the geometrical aspect of the problem may proceed

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* This paper forms the substance of a dissertation presented to the University of Cambridge [3]. I wish to express my gratitude to the Research Committee of the University of Geneva and to the Société Académique (Turettini Fund) for financial support.

There are two main theorems in this thesis.

Theorem (Coray). If k is the field of fractions of a complete DVR with residue field κ, if Property (*) holds for cubic hypersurfaces over κ in any dimension, then (*) holds for cubic hypersurfaces over k in any dimension.

This is proved by a delicate study of possible bad reduction of cubic hypersurfaces, extending earlier work of Demjanov (1950), Lewis (1952), Springer (1955), who had proved that over a p-adic field k, for any $n \geq 9$, any cubic hypersurface in $\mathbb{P}^n_k$ has a rational point.

Corollary (Coray). Property (*) holds for arbitrary cubic hypersurfaces over a p-adic field.
Indeed, (*) for cubic hypersurfaces is easy over a finite field.

This talk is concerned with the second main theorem in Coray's thesis.

Theorem A

Let $X \subset \mathbb{P}^3_k$ be a smooth cubic surface. If it has a rational point in a field extension of degree prime to 3, then it has a rational point in an extension of degree 1, or of degree 4, or of degree 10.

("or "not exclusive)

I shall describe the main points of Coray's proof. It uses curves of low genus lying on the surface. One does not know whether the curves are smooth or even irreducible. One must then envision possible degeneracy cases.

I shall then explain a general method to dodge this part of the argument, and, with the added flexibility, produce new results without much pain.

Zero-cycle on a k-variety X : finite linear combination with integral coefficients of closed points $\sum_P n_P P$, $n \in \mathbb{Z}$
Effective cycle : all $n_P \geq 0$

Degree of the zero-cycle (over k) : $\sum_P n_P [k(P) : k] \in \mathbb{Z}$

Rational equivalence on the group $Z_0(X)$ of zero-cycles :
for any proper morphism $p : C \to X$ from a normal integral k-curve and any rational function $f \in k(C)^*$, mod out by $p_*(div_C(f))$.
If X/k is proper, then induced degree map

$$CH_0(X) = Z_0(X)/\mathrm{rat} \to \mathbb{Z}$$

from the Chow group of degree zero-cycles to $\mathbb{Z}$.
The image is $\mathbb{Z}.I(X) \subset \mathbb{Z}$, where $I(X)$ is the gcd of degrees of closed points.
The kernel $A_0(X)$ is the reduced Chow group of zero-cycles.

Curves

Riemann-Roch on a smooth, projective, geometrically connected curve Γ/k of genus g implies :

• Any zero-cycle of degree at least equal to g on Γ is rationally equivalent to an effective cycle.

• For $g > 1$, if there exists a zero-cycle of degree prime to $2g-2$, then there exist effective zero-cycles of degree g and of degree $g+1$.

Curves on surfaces

Let X be a smooth, projective, geom. connected k-surface. Let K be the class of the canonical divisor. Let $D \subset X$ be an effective divisor. Let

$$p_a(D) := (D.(D+K))/2 + 1.$$

If $D \subset X$ is a smooth, projective curve of genus g, then $p_a(D) = g$.

Surfaces

Let X be a smooth, projective, geom. connected k-surface. Let L be (the class of) an invertible sheaf in $\mathrm{Pic}(X)$.
$\chi(X, L) := h^0(X, L) - h^1(X, L) + h^2(X, L)$
Riemann-Roch :

$$\chi(X, L) = (L.(L - K))/2 + \chi(X, O_X).$$

$h^i(X, L) = h^{2-i}(X, K - L)$ (Serre)
If $\mathrm{char}(k) = 0$ and L is ample then $h^1(X, -L) = 0$ (Kodaira) (not really needed in the arguments to follow).
If X is geometrically birational to $\mathbb{P}^2$, then $\chi(X, O_X) = 1$.

From now on, to be on the safe side, I assume $\mathrm{char}(k) = 0$.

To follow fashion, I suppose I should write $\mathbb{K}$.

Coray's method for a smooth cubic surface $X \subset \mathbb{P}^3_k$.
We assume that X has a closed point of degree d prime to 3. Let d be the least such integer.
If $d = 1$, there is nothing to do. If $d = 2$, then taking the line through a quadratic point and its conjugate we get a rational point, thus in fact $d = 1$.
Let us thus assume d prime to 3 and $d \geq 4$. Let $P \in X$ be a closed point of degree d.
On the surface X we find a closed point Q of degree 3 by intersecting with a line $\mathbb{P}^1_k \subset \mathbb{P}^3_k$.

Let $n \geq 1$ be the smallest integer such that there exists a surface $\Sigma \subset \mathbb{P}^3_k$ of degree n cutting out a curve $\Gamma \subset X$ which contains both P and Q.
On the surface X we easily compute

$$h^0(X, O_X(n)) = 3n(n+1)/2 + 1.$$

Assume that the surface Σ of degree n cuts out a curve $\Gamma = D \subset X$ which is geometrically irreducible and smooth. On this curve there is a zero-cycle of degree 1. One computes the genus

$$g = p_a(D) = 3n(n-1)/2 + 1.$$

If on the one hand $3n(n+1)/2 - 3 \geq d$ then

$$3n(n+1)/2 + 1 \geq d + 3 + 1$$

and one may find a surface of degree n cutting out a curve Γ (assumed to be smooth) passing through the closed points P (of degree d) and Q (of degree 3).
If on the other hand $d \geq 3n(n-1)/2 + 4$, then

$$d - 3 \geq 3n(n-1)/2 + 1 = g(\Gamma),$$

thus on the smooth curve Γ, the zero-cycle $P - Q$ is rationally equivalent to an effective zero-cycle of degree $d - 3 < d$. Thus there exists a closed point of degree prime to 3 and smaller than d, contradiction.

This argument works for any integer d prime to 3 which lies in an interval

$$3n(n+1)/2 - 3 \geq d \geq 3n(n-1)/2 + 4.$$

For other values of d, a complementary argument is needed.
In particular, for integers of the shape $d = 3n(n-1)/2 + 1$, one uses a curve Γ which is the normalisation of a curve $\Gamma_0 \subset X$ cut out by a surface of degree n passing through P and having a double point at the point Q of degree 3. The genus of the curve drops down by 3, and the dimension of the linear system of interest drops down by 9.

For $d = 3n(n-1)/2 + 1$ with $n \geq 4$, there is enough room. But there is not enough room in the case $n = 2, d = 4$ and in the case $n = 3, d = 10$.

CONCLUSION (up to good position argument)

On a smooth cubic surface X/k with a closed point of degree d prime to 3, the least such d lies in $\{1, 4, 10\}$.

45 years old question : Can one eliminate 10, 4, both ?

[A challenge : for any odd integer d, can one produce a field k and a smooth quartic surface in $\mathbb{P}^3_k$ which has a closed point in a field extension of k of odd degree $\geq d$ but no point in a field extension of k of odd degree $< d$?]

The above argument for cubic surfaces assumes that the curves Γ found in the linear system are geometrically irreducible and smooth. In his paper, Coray then discusses the possible singular and even reducible curves which may turn up, and manages to go down to 1, 4 or 10 also in these cases.

It is clear that such cases may occur : consider the simpler question of finding a smooth plane conic through a closed point of degree 3 in $\mathbb{P}^2_k$. If the closed point happens to lie on a $\mathbb{P}^1_k \subset \mathbb{P}^2_k$, this is not possible.

I now explain how to avoid such a discussion of degenerate cases.

Ideas :

• When available, use results of the type : if there is a k-rational point on a k-variety X of the type under study, then the k-rational points are Zariski dense.

• use the Bertini theorems (not very original !)

• replace k by the "large" field $F = k((t))$, so that there are many F-points on whichever smooth variety appears in the process (the original variety, or some parameter space) as soon as there is at least one F-point.

• For the problems under consideration here, to prove a result for a k-variety X, it is enough to prove if for the $k((t))$-variety $X \times_k k((t))$.

Theorem (a variation on the Bertini theorems, as found in Jouanolou's book)

Let X be a smooth, projective, geom. connected k-variety. Let $f : X \to \mathbb{P}^n_k$ be a k-morphism. Assume its image has dimension at least 2 and generates $\mathbb{P}^n_k$.
Let $r \leq n$ be an integer. There exists a nonempty open set $U \subset X^r$ such that, for any field L containing k and any L-point $(P_1, \ldots, P_r) \in U(L)$, there exists a hyperplane $h \subset \mathbb{P}^n_L$ whose inverse image $f^{-1}(h) \subset X_L$ is a smooth, geometrically integral L-variety which contains the points $\{P_1, \ldots, P_r\}$.

Here we just say : " If there is a point in $U(L)$, then ...". But for a given L, $U(L)$ could be empty.

Let X be a smooth k-variety and $m > 0$ be an integer. Consider the open set W of X^m consisting of m-tuples $(x_1, \ldots, x_m)$ with $x_i \neq x_j$ for $i \neq j$.

The symmetric group $\mathfrak{S}_m$ acts on W, the quotient is a smooth k-variety $Sym^m_{sep}X$. It parametrizes effective zero-cycles of degre m which are "separable".

Theorem (zero-cycles version of previous theorem)

Let X be a smooth, projective, geom. connected k-variety. Let $f : X \to \mathbb{P}^n_k$ be a k-morphism. Assume its image has dimension at least 2 and generates $\mathbb{P}^n_k$. Let $s_1, \ldots, s_t$ be natural integers such that $\sum_i s_i \leq n$. There exists a nonempty open set U of the product $\mathrm{Sym}^{s_1}_{sep} X \times \cdots \times \mathrm{Sym}^{s_t}_{sep} X$ such that, for any field L containing k and any L-point of U, corresponding to a family $\{z_i\}$ of separable effective zero-cycles of respective degrees s_i, there exists a hyperplane $h \subset \mathbb{P}^n_L$ whose inverse image $X_h = f^{-1}(h) \subset X_L$ is a smooth, geometrically integral L-variety which contains the points of the supports of the cycle $\sum_i z_i$.

Same comment as before on $U(L)$ being possibly empty.
Note : Let $s = s_1 + \cdots + s_t$. For the proofs of Theorems A,B,C, we use $\mathrm{Sym}^{s_1}_{sep} X \times \cdots \times \mathrm{Sym}^{s_t}_{sep} X$ and not only $\mathrm{Sym}^{s}_{sep} X$.

Let k be a field, $\mathrm{char}(k) = 0$. Let X be a smooth, projective, geom. connected k-variety.

In this talk, we say that X has the *density property* if it satisfies : for any finite field extension L/k with $X(L) \neq \emptyset$, the set $X(L)$ is Zariski dense in X_L.

R-equivalence on $X(k)$ is the equivalence relation generated by the elementary relation : $A, B \in X(k)$ both lie in the image of $\mathbb{P}^1(k)$ under a k-morphism $\mathbb{P}^1_k \to X$.

In this talk, we say that X has the *R-density property* if it satisfies : for any finite field extension L/k and $P \in X(L)$, the set of points of $X(L)$ which are R-equivalent to P on X_L is Zariski dense on X_L.

Smooth cubic hypersurfaces in $\mathbb{P}^n_k$, $n \geq 3$, satisfy both properties.

Theorem (Bertini for varieties with density properties)

Let k be a field, $\mathrm{char}(k) = 0$. Let X be a smooth, projective, geom. connected k-variety. Let $f : X \to \mathbb{P}^n_k$ be a k-morphism. Assume its image has dimension at least 2 and generates $\mathbb{P}^n_k$. Let $P_1, \dots, P_t$ be closed points of X of respective degrees $s_1, \dots, s_t$ such that $\sum_i s_i \leq n$.

(a) If X satisfies the density property, then there exists a hyperplane $h \subset \mathbb{P}^n_k$ defined over k such that $X_h = f^{-1}(h) \subset X$ is smooth, geom. integral and contains effective zero-cycles $z_1, \dots, z_t$ of respective degrees $s_1, \dots, s_t$.

(b) If X is satisfies the R-density property, then one may moreover ensure that, for each i, the zero-cycle z_i is rationally equivalent to the zero-cycle P_i.

Definition (F. Pop)
A field F is said to be a *large field* (in French, *corps fertile*, in Russian плодотворное поле) if, for any smooth connected variety X over F, if $X(F) \neq \emptyset$ then the set $X(F)$ of F-rational points is Zariski dense in X.

If a field F is large, then any finite field extension of F is large.

Thus any smooth geom. connected variety over a large field satisfies the density property.

The formal power series field $F = k((t))$ over any field k is a large field.

Theorem (Bertini over a large field)

Let F be a large field, $\mathrm{char}(F) = 0$. Let X be a smooth, projective, geom. connected F-variety. Let $f : X \to \mathbb{P}^n_F$ be an F-morphism. Assume its image has dimension at least 2 and generates $\mathbb{P}^n_F$. Let $P_1, \dots, P_t$ be closed points of X of respective degrees $s_1, \dots, s_t$ such that $\sum_i s_i \leq n$.

(a) There exists a hyperplane $h \subset \mathbb{P}^n_F$ defined over F such that $X_h = f^{-1}(h) \subset X$ is smooth, geom. integral and contains effective zero-cycles $z_1, \dots, z_t$ of respective degrees $s_1, \dots, s_t$.

(b) If X is geometrically rationally connected, then one may moreover ensure that, for each i, the zero-cycle z_i is rationally equivalent to the zero-cycle P_i.

For the proof of (a) :
The family $P_1, \ldots, P_t$ defines an F-point of the smooth, connected k-variety $\mathrm{Sym}^{s_1}_{sep}X \times \cdots \times \mathrm{Sym}^{s_t}_{sep}X$. Since F is large, any nonempty Zariski open set of that k-variety contains an F-point.

For the proof of (b), one moreover uses a result due to Kollár (1999) (deformation method) : for any F-point P on a smooth, projective geometrically (separably) rationally connected variety X over a large field F, the set of F-points which are R-equivalent to P, hence in particular are rationally equivalent to P, is Zariski dense in X.

(Easy) Proposition

Let k be a field and $F = k((t))$. Let X be a proper k-variety.

(a) The gcd of degrees of closed points coincides for X/k and X_F/F.

(b) For any integer $r \geq 1$, the smallest degree of a closed point of degree prime to r, which is also the smallest degree of an effective zero-cycle of degree prime to r, coincides for X/k and X_F/F.

(c) Let I be a set of natural integers. If the Chow group of zero-cycles on X_F may be generated by the classes of effective cycles of degree $d \in I$, then the same holds for X.

(d) Let $d \geq 0$ be an integer. If every zero-cycle on X_F of degree at least d is rationally equivalent to an effective cycle, then the same holds for X.

One may then run Coray's proof using only smooth projective curves in the linear systems of interest. There are two ways to do this.

One may use the density property of smooth cubic surfaces and apply Bertini's theorem (a) for varieties with this property.

Or one may reduce to the case of large fields F via replacing k by $k((t))$, use Bertini theorem (a) for large fields, and then use the fact that the statement of the theorem for $X_{k((t))}$ over $k((t))$ implies it for X over k.

In any case, an important point has been to be able to move the effective zero-cycles through which one wants curves of a given linear system to pass and simultaneously be smooth.

The gained flexibility enables one to prove the next two theorems by Coray's method without too much effort.

"Bertini over a large field" (a) is enough to prove :

Theorem B

Let X be a del Pezzo surface of degree 2, i.e. a double cover of $\mathbb{P}^2_k$ ramified along a smooth quartic. If there exists a closed point of odd degree on X, then there exists a closed point of degre 1, or 3, or 7.

In the proof, just as for cubic surfaces, in certain cases, one needs to blow up points on X. To apply the Bertini types of results, one needs to know that certain invertible sheaves are very ample. Here one may use Reider's criteria (1988).

For del Pezzo surfaces of degree 2 with a k-rational point not in a very special situation, k-unirationality is known. But the trick with large fields enables us to handle our problem without using k-unirationality. .

Remark (Kollár-Mella 2017). There exist examples of del Pezzo surfaces of degree 2 with a closed point of degree 3 but no rational point.

Suppose k is a field with a quadratic field extension $k(\sqrt{a})/k$, a cubic field extension and a quintic field extension.
Let $C \subset \mathbb{P}^2_k$ a conic with a smooth k-point.
Let $Q \subset \mathbb{P}^2_k$ be a smooth quartic with $Q \cap C = \{A, B\}$, with A closed point of degree 3 and B closed point of degree 5.
Let $F = k(t)$ be the smooth del Pezzo surface of degree 2 defined by the equation

$$z^2 - aC(u, v, w)^2 + tQ(u, v, w) = 0.$$

It has obvious points of degree 3 and 5.
However congruences modulo powers of t show it has no F-point.

Using either "Bertini over a large field" (b) or "Bertini for varieties with the R-density property" (b), one proves :

Theorem C

Let $X \subset \mathbb{P}^3_k$ be a smooth cubic surface and $Q \in X(k)$.

(a) Every effective zero-cycle of degree at least 4 on X is rationally equivalent to an effective zero-cycle $z_1 + rQ$ with $r \geq 0$ and z_1 effective of degree ≤ 3.

(b) Every zero-cycle of degree zero is rationally equivalent to the difference of two effective cycles of degree 3.

(c) The Chow group of zero-cycles on X is generated by the classes of rational points and of closed points of degree 3.

(d) Every zero-cycle on X of degree ≥ 10 is rationally equivalent to an effective zero-cycle.

Let k be a p-adic field.
For cubic surfaces over a p-adic field, using k-unirationality and symmetries one may prove that the set $X(k)/R$ of R-equivalence classes is finite.
Since there are only finitely many field extensions of a given degree for a p-adic field, for X a cubic surface as above, together with Theorem C (c), this implies that the reduced Chow group $A_0(X)$ is a finite group.

The finiteness result for $X(k)/R$ holds for any smooth, projective geometrically rationally connected variety X/k (Kollár 1999).
Hence the interest in trying to prove analogues of Theorem C for other classes of rationally connected varieties.

However, quite a few results are already known.
For any smooth, projective geometrically rationally connected variety X/k over a field k, the group $A_0(X)$ is killed by some positive integer.

Via algebraic K-theory, results on the finiteness of $A_0(X)$ for X smooth, projective geometrically rationally connected variety over a p-adic field k are known :

• If X/k is a surface, the group $A_0(X)$ is finite (CT, 1983, proof based on Merkurjev-Suslin theorem in algebraic K-theory).

• In arbitrary dimension the prime-to-p torsion is finite (special case of theorem of Saito–Sato 2010 for arbitrary smooth projective varieties over a p-adic field).

• In the good reduction case, with sep. rat. connected specialization, $A_0(X) = 0$ (Kollár 2004).

Analogues of part of Theorem C for the other del Pezzo surfaces ?

(For the next two slides I still have to double-check the details of the computation.)

Let X/k be a del Pezzo surface of degree 2 with a rational point Q. (To be written) *Every effective zero-cycle of degree d at least 7 on X is rationally equivalent to an effective zero-cycle $z_1 + rQ$ with $r \geq 0$ and z_1 effective of degree ≤ 6. Every zero-cycle of degree zero is rationally equivalent to the difference of two effective zero-cycles of degree 6.*

The cases $d = n^2 - n + 1$, resp. $d = n^2 - n$, require the use of curves with one, resp. two double rational points.

Since we do not know the R-density property for del Pezzo surfaces of degree 2, the proof here relies on Bertini over a large field (b), the combination of the reduction trick from k to $k((t))$ and Kollár's result on R-density for geometrically rationally connected varieties (proved using deformation theory).

Let X/k be a del Pezzo surface of degree 1. Let Q be the fixed point of the anticanonical system.
(To be written) *Every effective zero-cycle of degree at least 16 on* X *is rationally equivalent to an effective zero-cycle* $z_1 + rQ$ *with* $r \geq 0$ *and* z_1 *effective of degree* ≤ 15*. Every zero-cycle of degree zero is rationally equivalent to the difference of two effective zero-cycles of degree 15.*

The cases $d = n(n-1)/2 + 1$, resp. $d = n(n-1)/2$, require the use of curves with one, resp. two double rational points.

Since we do not know the density property and even less the R-density property for del Pezzo surfaces of degree 1, the proof here relies on Bertini over a large field (b), the combination of the reduction trick from k to $k((t))$ and Kollár's result on R-density for geometrically rationally connected varieties.

The proof of the following result is independent of previous arguments.

Theorem D

Let $X \subset \mathbb{P}^3_k$ *be a smooth cubic surface with* $X(k) = \emptyset$*. If any closed point of degree 3 on* X *is cut out by a line* $\mathbb{P}^1_k \subset \mathbb{P}^3_k$*, then to a general line* $\mathbb{P}^1_k \subset \mathbb{P}^3_k$ *we may associate a del Pezzo surface* W *of degree 1 over* k *such that* $W(k)$ *is not Zariski dense in* W*.*

The question whether such cubic surfaces exist was recently raised by Qixiao Ma.

Whether rational points are always Zariski dense on a del Pezzo surface of degree 1 is a well known open question.

Idea of the proof

Take a line $L \subset \mathbb{P}^3_k$. By assumption it cuts out a closed point P of degree 3 on X. Blow up that point. This gives a fibration $Y \to \mathbb{P}^1_k$ whose fibres are the sections of X by planes containing L.
If any closed point of degree 3 on X is cut out by a line, then in particular for any $t \in \mathbb{P}^1(k)$ with smooth fibre Y_t, we have $\mathrm{Pic}(Y_t) = \mathbb{Z}P$ hence $\mathrm{Pic}^0(Y_t) = 0$
The regular, relatively minimal model $g : W \to \mathbb{P}^1_k$ associated to the Jacobian of the generic fibre Y_η of $Y \to \mathbb{P}^1_k$ is the announced del Pezzo surface of degree 1. For $t \in \mathbb{P}^1(k)$ with W_t smooth, any k-point on the elliptic curve W_t is a 3-torsion point. The k-points of W lie in the union of the singular fibres of g and the curve which is the closure of the 3-torsion subscheme of W_η.

Conclusion

Theorems A, B, C raise the following problems, essentially solved in dimension 2, but which already look hard for cubic hypersurfaces in $\mathbb{P}^4_k$ and for conic bundles over $\mathbb{P}^2_k$.

Let k be a field, $\mathrm{char}(k) = 0$, let $\overline{k}$ be an algebraic closure.
Let X be a smooth, projective, geom. connected variety over k.
Assume that $\overline{X} = X \times_k \overline{k}$ is a rationally connected variety.

1. Does there exist an integer $n(\overline{X})$ such that, if X has a zero-cycle of degree 1, then X has closed points of coprime degrees $\leq n(\overline{X})$?
For quartics in $\mathbb{P}^N$, $N \geq 4$, there is no such bound independent of N (Kollár, 2004)

2. Does there exist an integer $m(\overline{X})$ such that the Chow group $CH_0(X)$ is generated by closed points of degree at most $m(\overline{X})$?