The index of algebraic varieties

J.-L. Colliot-Thélène
CNRS, Université Paris-Sud

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Let k be a field and X an algebraic variety over k, that is, a system of polynomial equations with coefficients in k

$$f_1(x_1 \cdots, x_n) = \cdots = f_r(x_1, \ldots, x_n) = 0.$$

We write $X(k)$ for the set of solutions with coordinates in k. The sytem may have no solution with coordinates in k, but unless it defines the empty variety, there are solutions in finite extension fields K/k.

Definition *The index $I(X/k)$ of a k-variety X is the gcd of the degrees $[K : k]$ of finite extensions K/k such that $X(K) \neq \emptyset$.*

Clearly,

$$X(k) \neq \emptyset \Longrightarrow I(X/k) = 1.$$

Two problems.

Problem I. Describe classes of varieties X which over any field k have the property : if $I(X/k) = 1$, then $X(k) \neq \emptyset$.

Problem II. Given a field of number theoretic interest (finite field, the real field, a p-adic field, a global field (number field, or function field in one variable over a finite field), and a given class $\mathcal{C}$ of varieties over that field, give necessary and sufficient conditions for an X in $\mathcal{C}$ to satisfy $I(X/k) = 1$.

If $I(X/k) = 1$, do we have $X(k) \neq \emptyset$?

Springer's theorem (1952). *If a quadratic form over a field k has a nontrivial zero over an odd degree extension K of k, then it has a solution in k.*

Proof : Induction on the degree $[K:k]$. One may assume $K = k[t]/P(t)$ where P is a monic polynomial of odd degree d. If there is a solution in K then there is an equation

$$q(a_1(t), \cdots, a_n(t)) = p(t)r(t) \in k[t]$$

where the $a_i \in k[t]$ are coprime as a whole and have degree $< d$, hence the LHS has degree $\leq 2d-2$, hence $r(t)$ has degree $\leq d-2$. If q has no zero over k, then the LHS is of even degree, hence $r(t)$ has an irreducible factor of odd degree $\leq d-2$. Reduce modulo that factor.

Already for curves of genus 2 over a (small) finite field, $I(X/k) = 1$ does not imply $X(k) = \emptyset$. The question is whether there are interesting classes of varieties for which the implication holds.

Starting from quadrics, one may try to go into two different directions :

– Homogeneous spaces of connected linear algebraic groups

– Hypersurfaces of degree d in projective space $\mathbf{P}^n$, with $n \geq d$, or possibly n bigger than some function of d.

Homogeneous spaces of connected linear algebraic groups

$$I(X/k) = 1 \Longrightarrow X(k) \neq \emptyset \ ?$$

For homogeneous spaces of *commutative* algebraic groups, one can use a norm formalism to show that the answer is affirmative.

For X/k a *principal homogeneous spaces* of connected linear algebraic groups, Serre asked whether $I(X/k) = 1$ implies $X(k) \neq \emptyset$. This has been established in many cases, but the question is still open.

Over p-adic fields and number fields, a positive answer was given by Sansuc (1981).

Over an arbitrary field, this is known for the orthogonal group (variant of Springer's result), for the norm 1 group of an algebra with involution (Bayer-Fluckiger and Lenstra, 1989). A more general result is due to Gille and Semenov (2010).

For arbitrary homogeneous spaces of connected linear algebraic groups, there are examples with $I(X/k) = 1$ but $X(k) = \emptyset$.

There are examples over $k = \mathbf{Q}_p$ (Florence, 2004). Affine homogenous spaces with finite, noncommutative geometric isotropy groups.

Projective homogeneous spaces over $k = \mathbf{Q}_p((t))$ (Parimala, 2005).

Hypersurfaces, complete intersections

$$I(X/k) = 1 \Longrightarrow X(k) \neq \emptyset \ ?$$

The answer is positive for an arbitrary intersection of two quadrics.

$$q_1(x_0, \cdots, x_n) = q_2(x_0, \cdots, x_n) = 0.$$

This is an immediate consequence of Springer's theorem and the elegant result (Amer 1976, Brumer 1978) :
The above system has a solution in k if and only if the quadratic form $q_1 + tq_2$ has a zero over $k(t)$.

These results do not extend to the intersection of three quadrics.

The next case after quadrics is that of *cubic hypersurfaces* X given by

$$C(x_0, \cdots, x_n) = 0$$

in projective space $\mathbf{P}^n$. The problem is open. Some cases are known.

For $n = 2$, the answer is affirmative. If $X \subset \mathbf{P}^2$ is a nonsingular cubic curve, this follows from the Riemann-Roch theorem. One may also use the fact that such a curve is a principal homogeneous space of its jacobian, which is a commutative algebraic group.

For $n \geq 3$, Coray (1974) gave an affirmative answer over p-adic fields. In the case $n = 3$, over an arbitrary field, he proved the answer is affirmative if the surface X is singular. When X is nonsingular, Coray showed that $I(X/k) = 1$ implies the existence of a point in a field extension of k of degree either 1, 4 or 10.

Over $k = \mathbf{Q}(t)$, Kollár (2002) has shown that there exist nonsingular *quartic hypersurfaces* in projective space *of arbitrary dimension* with $X(k) = \emptyset$ and $I(X/k) = 1$.

This completely dashed a naive hope that for "rationally connected" hypersurfaces $I(X/k) = 1 \Longrightarrow X(k) \neq \emptyset$.

Deciding if $I(X/k) = 1$

Finite fields

Theorem (Chevalley, Warning, 1935). *Over a finite field, any form of degree d in $n > d$ variables has a nontrivial zero.*

Theorem (Esnault 2003) *Over a finite field, any nice "rationally connected" variety has a rational point.*

Theorem (Lang-Weil 1954) *For any absolutely irreducible variety X over a finite field $\mathbf{F}$, there exists an integer $n_0(X)$ such that over any field extension of $\mathbf{F}$ of degree at least $n_0(X)$, the variety X has a point. In particular $I(X/\mathbf{F}) = 1$.*

p-adic fields

Over a p-adic field, there was a conjecture of Artin that any form of degree d in $n > d^2$ variables has a nontrivial zero. This was disproved by Terjanian and others. In the Terjanian examples, for the associated hypersurfaces X, one may check $I(X/k) = 1$. Kato and Kuzumaki (1986) made the :

Conjecture *For any hypersurface $X \subset \mathbf{P}^n$ of degree d over a p-adic field, with $n \geq d^2$, one has $I(X/k) = 1$.*

They proved this for d prime.
For a complete intersection of quadrics, the natural generalisation of this conjecture was proved by Heath-Brown in 2009.

In the rest of the talk, I shall use the Brauer group $\mathrm{Br}(X)$ of an algebraic variety X.
Natural extensions of finite dimensional vector spaces over a field are finite rank vector bundles over a variety.
Over a field, we have the notion of central simple algebras, which are twisted forms of algebras of matrices (e.g. the Hamilton quaternions over the reals). Stable isomorphy classes of such algebras over a field k build up an abelian group $\mathrm{Br}(k)$ (Brauer, 1935).
Over a variety (and more generally a scheme) X, the natural generalisations of central simple algebras are called Azumaya algebras. Their stable isomorphy classes build up an abelian group $\mathrm{Br}(X)$ (Auslander-Goldman, Grothendieck 1968).
This group is functorial contravariant in X.

Over a p-adic field k, a formula for the index of any smooth projective (geometrically connected) *curve* X was given by Lichtenbaum (1969) :

$$\mathbf{Z}/I(X/k) = Ker[\mathrm{Br}(k) \to \mathrm{Br}(X).]$$

The Brauer group of a p-adic field k is isomorphic to $\mathbf{Q}/\mathbf{Z}$.

Over the reals, the analogous result is due to Witt (1934) : If a real curve has no real point, then -1 is the sum of two squares in the function field of X.

For X smooth, projective, of arbitrary dimension, there is an analogous result (CT/S. Saito 1996; K. Sato/S. Saito 2009 for the p-part). In Lichtenbaum's formula, one must replace $\mathrm{Br}(X)$ by the quotient of $\mathrm{Br}(X)$ by the image of the Brauer group $\mathrm{Br}(\mathcal{X})$ of a regular proper model $\mathcal{X}$ of X over the ring of integers of k ($\mathrm{Br}(\mathcal{X})$ vanishes for X a curve).

Over the reals, the analogue in higher dimension of Witt's result is due to Pfister (1967) : if $X/\mathbf{R}$, of dimension d, has no real point, then -1 is a sum of 2^d squares in the function field of X.

Global fields

Given a variety X over a number field k, a necessary condition for $X(k) \neq \emptyset$ is that for each completion k_v of k, p-adic or real, $X(k_v) \neq \emptyset$ (this is the modern form of checking congruences). We let Ω denote the set of "places" v of k.

Analogously, a necessary condition for $I(X/k) = 1$ is $I(X_v/k_v) = 1$ for all v, where X_v denotes the variety X considered over k_v.

Let us restrict attention to “nice” varieties (smooth, projective, absolutely irreducible).
The implication

$$X(k_v) \neq \emptyset \ \text{ for all } \ v \in \Omega \Longrightarrow X(k) \neq \emptyset$$

holds for some families of algebraic varieties (example : quadrics) but it does not hold in general (example : curves of genus one).

The implication

$$I(X_v/k_v) = 1 \ \text{ for all } \ v \in \Omega \Longrightarrow I(X/k) = 1$$

does not hold either for all classes of varieties (example : curves of genus one).

In 1970, Manin showed how the notion of Brauer group of schemes together with class field theory gave a unified explanation for most counterexamples to the Hasse principle for rational points then available in the literature.
This was later named "the Brauer–Manin obstruction."
The necessary condition for $X(k) \neq \emptyset$ coined by Manin reads :

There exists a family of local points $\{M_v\} \in \prod_{v \in \Omega} X(k_v)$ *such that for any element* $\alpha \in \mathrm{Br}(X)$, *the sum*

$$\sum_{v \in \Omega} \alpha(M_v) = 0 \in \mathbf{Q}/\mathbf{Z}.$$

There were good reasons not to believe this should be the whole story. Nevertheless, for nearly thirty years after Manin's paper, this necessary condition accounted for all the known counterexamples to the Hasse principle.

In 1999, Skorobogatov produced a counterexample to the Hasse principle which could not be accounted for in this manner. Harari and Skorobogatov (2002) then developed a general framework ("descent by torsors under arbitrary linear groups") which accounts for Skorobogatov's counterexample, and, as it later turned out, also accounts for any Brauer–Manin obstruction.

In 2009, Poonen produced new counterexamples to the Hasse principle. These are not accounted for by the Harari-Skorobogatov paradigm.

There are still classes of varieties for which one wonders whether Manin's necessary condition for $X(k) \neq \emptyset$ is sufficient :

– rational surfaces (very likely)

– curves of genus one (very likely)

– curves of arbitrary genus (?)

– Nonsingular hypersurfaces of degree d in $\mathbf{P}^n$ with $n \geq 4$ and $d \leq n$.

– $K3$-surfaces, e.g. nonsingular quartic surfaces in $\mathbf{P}^3$ (??)

For arbitrary (nice) varieties, finding a cohomological substitute for the Hasse principle seems a very hard problem.

What about the analogous problem for the weaker statement $I(X/k) = 1$?

It is easy to use the Brauer group $\mathrm{Br}(X)$ to define a Brauer-Manin obstruction for the implication

$$I(X_v/k_v) = 1 \text{ for all } v \in \Omega \Longrightarrow I(X/k) = 1.$$

The necessary condition for $I(X/k) = 1$ here reads :

There exists a family $\{z_v\}_{v\in\Omega}$ of "zero-cycles of degree one", z_v on X_v, such that for any element $\alpha \in \mathrm{Br}(X)$, the sum

$$\sum_{v\in\Omega} \alpha(z_v) = 0 \in \mathbf{Q}/\mathbf{Z}.$$

We here hope that this is also a sufficient conditiion :

Conjecture *For any smooth, projective, connected variety over k, if there exists a family $\{z_v\}_{v \in \Omega}$ of "zero-cycles of degree one", where z_v is on X_v, such that for any element $\alpha \in \mathrm{Br}(X)$, the sum*

$$\sum_{v \in \Omega} \alpha(z_v) = 0 \in \mathbf{Q}/\mathbf{Z}.$$

then $I(X/k) = 1$.

Various versions of the conjecture :
CT/Sansuc 1981 (rational surfaces, with evidence from various directions), Kato/Saito 1986 (arbitrary varieties, because the world should be beautiful), Saito 1989, CT 1999

Why should we hope for such a statement when the analogous result for rational points fails ?
Because the non-Brauer-Manin proofs of nonexistence of a rational point often involve noncommutative finite Galois covers of the varieties (as seen in the Harari-Skorobogatov argument; this also comes up in Grothendieck's section conjecture for curves).
But torsors under noncommutative groups do not behave well with respect to 0-cycles, there are no norm maps (= transfer maps), two rational points may be rationally equivalent and the evaluations of the torsors on these two points may be different.
So we do not fear too much contradiction from this side !

What do we know ?

For curves of arbitrary genus, the conjecture follows from a famous conjecture, the finiteness of Tate-Shafarevich groups.

Rational surfaces break up into two big classes: conic bundles and del Pezzo surfaces.

For conic bundles, the conjecture was proved by Salberger (1988). Various extensions of his result have been obtained.

The case of Del Pezzo surfaces of low degree, for instance cubic surfaces, is open.

In the Poonen examples, $I(X/k) = 1$ (CT, 2009).

The original Skorobogatov example is given by the system of (affine) equations

$$(x^2 + 1)y^2 = (x^2 + 2)z^2 = 3(t^4 - 54t^2 - 117t - 243) \neq 0$$

If the above conjecture on the index is true, this system, which has no solution in $\mathbf{Q}$, should have a solution in an extension of $\mathbf{Q}$ of odd degree. This is an open question.

One may consider the analogue of the above conjecture over the other class of global fields : function fields of curves over a finite field.

A variety X over such a field $\mathbf{F}(C)$ may be viewed as the "generic fibre" of a fibration $f : W \to C$ of varieties over the finite field F.

Let us assume that W and the curve C are nice varieties over $\mathbf{F}$.

Curves on W induce zero-cycles on the generic fibre $X/\mathbf{F}(C)$.

S. Saito (1989) showed:

A integral version of the Tate conjecture for curves on the $\mathbf{F}$*-variety* W *implies the conjecture for* $I(X/\mathbf{F}(C))$*.*

The integral Tate conjecture is an l-adic analogue of the integral Hodge conjecture for algebraic cycles. The usual conjectures are with cohomology with rational coefficients. One knows that both integral versions are wrong in general.
However, for algebraic cycles *of dimension 1*, the integral Tate conjecture is open, and modulo the standard Tate conjecture for divisors on surfaces, a weak version of it for varieties of arbitrary dimension was proven by Schoen (1998).

For W a threefold, work in motivic cohomology (Merkurjev-Suslin, Lichtenbaum, Kahn) enables one to relate *the integral Tate conjecture for cycles of dimension 1 on* W to *the vanishing of the third unramified cohomology group of* W.

The third unramified cohomology group (with coefficients in $\mathbf{Q}/\mathbf{Z}(2)$) is the natural analogue in cohomological degree 3 of the Brauer group, which is the second unramified cohomology group (with coefficients in $\mathbf{Q}/\mathbf{Z}(1)$).

For W fibered into conics over a surface over a finite field, Parimala and Suresh (2010) have proved the vanishing of the third unramified cohomology group. This leads to function fields versions of the Salberger theorem mentioned above, and to the next question :

For $W \to C$ a family of cubic surfaces over a curve C over a finite field $\mathbf{F}$, is the third unramified cohomology group of W zero ?

The analogous statement for families of cubic surfaces over a complex curve is known, it is in fact a special case of a theorem of C. Voisin on the integral Hodge conjecture for rationally connected threefolds.

Theorem (CT, Swinnerton-Dyer 2010). *Let* $\mathbf{F}$ *be a finite field,* d *an integer prime to the characteristic,* $k = \mathbf{F}(t)$*, and* $X \subset \mathbf{P}^3$ *a nice surface of degree* d *over* k *defined by an equation*

$$f + tg = 0$$

where f *and* g *are forms of degree* d *over* $\mathbf{F}$*.*
(a) The Brauer-Manin condition for $I(X/k) = 1$ *is a sufficient condition.*
(b) If $d = 3$*, the Brauer-Manin condition for* $X(k) \neq \emptyset$ *is a sufficient condition.*

With $W/\mathbf{P}^1$ a nice model of $X/\mathbf{F}(t)$, the key point in the proof of (a) is that $W/\mathbf{F}$ is birational to $\mathbf{P}^3$ over $\mathbf{F}$, hence the unramified cohomology of W is trivial.