

QUADRIC SURFACES BIRATIONAL TO EACH OTHER

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ABSTRACT. If two quadric surfaces are birational to each other and have no rational point, then they are isomorphic to each other.

1. INTRODUCTION

This note claims no originality. It was written in 2021. At the request of Ivan Cheltsov, I make it more widely available.

For the following well known facts about quadratic forms in 4 variables, I refer the reader to [2], [3, Thm. 2.5], [1, Thm. 3.1] and to the literature quoted there.

Let k be a field of characteristic different from 2. Let q be a nondegenerate quadratic form q in 4 variables. Let $Q \subset \mathbb{P}_k^3$ be the smooth quadric defined by the equation $q = 0$. Let $\delta \in k^\times/k^{\times 2}$ be the discriminant of q . Let K/k be the trivial or quadratic extension of k defined by the square root of δ . There is a canonical quaternion class $\alpha \in \text{Br}(K)$ associated to Q . It generates the kernel of $\text{Br}(K) \rightarrow \text{Br}(K(Q))$. Let C/K be the associated conic. If $\delta = 1$ hence $K = k$, then $Q \simeq C \times_k C$. If $\delta \neq 1 \in k^\times/k^{\times 2}$, then $Q = R_{K/k}(C)$ (Weil restriction of scalars). The quadric surface Q has a rational point over k if and only if it has a rational point over K . It is then k -rational, i.e. k -birational to projective space \mathbb{P}_k^2 .

Let $F := k(Q)$ denote the function field of Q . The kernel of the restriction map $\text{Br}(k) \rightarrow \text{Br}(F)$ is zero, unless $Q(k) = \emptyset$, and $\delta = 1$. In that case, the kernel is $\mathbb{Z}/2$, spanned by the class $\alpha \in \text{Br}(k)$.

2. THE RESULT

Proposition 2.1. *Let k be a field, $\text{char.}(k) \neq 2$. Two smooth quadric surfaces Q_1 and Q_2 over k are k -birational to each other if and only if*

either

(i) Q_1 and Q_2 both have a rational point, in which case they are birational to \mathbb{P}_k^2 ,

or

(ii) none of the smooth quadrics has a rational point, $\text{disc}(Q_1) = \text{disc}(Q_2) = d \in k^\times/k^{\times 2}$, and in $\text{Br}(k(\sqrt{d}))$ the associated Brauer classes α_1 and α_2 coincide. In that case the k -varieties Q_1 and Q_2 are k -isomorphic.

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Proof. Let Q_1 and Q_2 be two smooth quadric surfaces which are k -birational to each other.

If one of the smooth quadrics has a k -rational point, then so has the other one (Nishimura-Lang lemma), each of them is k -birational to \mathbb{P}_k^2 , hence they are k -birational to each other.

Suppose now that the sets of rational points $Q_1(k)$ and $Q_2(k)$ are empty.

Let $F_i = k(Q_i)$, $i = 1, 2$ denote the function fields. Since Q_1 and Q_2 are k -birational to each other,

$$\text{Ker}[\text{Br}(k) \rightarrow \text{Br}(F_1)] = \text{Ker}[\text{Br}(k) \rightarrow \text{Br}(F_2)].$$

Let us call this group B .

If $B \neq 0$, then $\delta_1 = 1$ and $\delta_2 = 1$ and the group B is spanned by the class α attached to a quaternion algebra, defining a conic C over k , then $Q_1 \simeq C \times_k C \simeq Q_2$.

If $B = 0$, then $\delta_1 \neq 1$ and $\delta_2 \neq 1$.

Suppose $\delta_1 \neq \delta_2 \in k^\times/k^{\times 2}$. Let us go over to the field K_1 . We have $Q_1(K_1) = \emptyset$, hence $Q_2(K_1) = \emptyset$. The kernel of $\text{Br}(K_1) \rightarrow \text{Br}(K_1(Q_1))$ is $\mathbb{Z}/2$. The kernel of $\text{Br}(K_1) \rightarrow \text{Br}(K_1(Q_2))$ is 0 since $\delta_2 K_1 \neq 1$. Since Q_1 and Q_2 are birational over k hence over K_1 , this is not possible.

Suppose $B = 0$ and $\delta_1 = \delta_2 \in k^\times/k^{\times 2}$ and this element is not 1. Let K/k be the corresponding quadratic extension. Since Q_1 is k -birational to Q_2 , this also holds over K , and we have $\alpha_1 = \alpha_2 \in \text{Br}(K)$, thus $C_1 \simeq C_2$ over K , thus $Q_1 \simeq R_{K/k}(C_1) \simeq R_{K/k}(C_2) \simeq Q_2$. \square

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