

LOCAL-GLOBAL PRINCIPLES FOR CONSTANT REDUCTIVE GROUPS OVER SEMI-GLOBAL FIELDS

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ABSTRACT. We study local-global principles for torsors under reductive linear algebraic groups over semi-global fields; i.e., over one variable function fields over complete discretely valued fields. We provide conditions on the group and the semiglobal field under which the local-global principle holds, and we compute the obstruction to the local-global principle in certain classes of examples. Using our description of the obstruction, we give the first example of a semisimple simply connected group over a semi-global field where the local-global principle fails. Our methods include patching and R-equivalence.

INTRODUCTION

In this paper, we study local-global principles for the triviality of torsors under connected linear algebraic groups over semi-global fields; i.e., over one-variable function fields F over a complete discretely valued field K . This parallels the classical study over global fields, and in particular number fields. As in the case of number fields, there is a local-global principle that holds over semi-global fields for torsors under a connected linear algebraic group that is rational as an F -variety ([HHK09], [HHK15]), but this is not the case for torsors under general nonrational connected linear algebraic groups (e.g. nonrational tori, [CPS16]). Nevertheless, it was shown in [CHHKPS20, Theorem 7.3] that local-global principles do hold for torsors over a semi-global field F if the group is a torus over the valuation ring R of K , and the closed fiber of a regular projective model \mathcal{X} of F over R satisfies a certain property related to trees. The results in that work relied on two key properties of tori: that they are commutative and that they have flasque resolutions.

In the present paper, we use a quite different strategy to treat local-global principles for torsors over F under more general reductive groups G over R , where commutativity might not hold and where flasque tori do not control the group in the same way. The obstruction to such a local-global principle is a Tate-Shafarevich set $\text{III}(F, G)$; viz., the set of isomorphism classes of G -torsors over F that become trivial over each completion F_v of F at a divisorial discrete valuation v . The following is a special case of our Theorem 4.2:

Theorem 0.1. *Let K be a complete discretely valued field, R its ring of integers, and F a semi-global field over K . Let \mathcal{X} be a regular projective model of F over R . Assume that the residue field k of R is of characteristic zero; that the closed fiber \mathcal{X}_k of \mathcal{X} is reduced and has normal crossings; and that the reduction graph associated to \mathcal{X}_k is a tree and remains*

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so under all finite extensions k'/k . Then for any reductive linear algebraic group G over R we have $\text{III}(F, G) = 1$.

The central technical tool in proving this theorem is a factorization result (Proposition 3.4). It implies that under these hypotheses, any torsor over F that is locally trivial with respect to discrete valuations of F can be lifted to a torsor over a model \mathcal{X} of F that is trivial along the reduced closed fiber of \mathcal{X} (Theorem 3.5). We then invoke recent work of Gille, Parimala and Suresh [GPS21] to conclude that the given torsor over F is in fact trivial, thereby obtaining Theorem 4.2. In addition we show that under appropriate hypotheses, we may even drop the assumptions that the group is reductive and connected.

In certain situations where the reduction graph does not satisfy the tree hypothesis of the above theorem, we compute an explicit quotient of $\text{III}(F, G)$ in Proposition 5.2. This quotient can be viewed as a “lower bound” for $\text{III}(F, G)$, and sometimes it turns out to give $\text{III}(F, G)$ exactly. In particular, we have the following result, which computes $\text{III}(F, G)$ in terms of the group $G(k)/R$ of R -equivalence classes of k -points of G under appropriate hypotheses (see Theorem 6.5 for the precise statement):

Theorem 0.2. *In the above situation (but allowing arbitrary characteristic), if the closed fiber of \mathcal{X} is reduced and consists of copies of \mathbb{P}_k^1 meeting at k -points and forming m cycles, and if $\text{char}(k)$ is not one of the bad primes for the reductive group G over R , then $\text{III}(F, G)$ is in bijection with the set of uniform conjugacy classes of $(G(k)/R)^m$.*

Using this, we answer an open question concerning local-global principles for torsors under semisimple simply connected linear algebraic groups. Such principles hold in the case of global fields (by work of Eichler, Kneser, Harder, and Chernousov), and it seemed reasonable to expect that they would hold in the semi-global case. This was conjectured for semi-global fields over p -adic fields in [CPS12]; and in that situation the conjecture has been proven in many cases (see [Pre13], [Hu14], [PPS18], [PS20]). Here we provide the first counterexamples to such a local-global principle over a more general semi-global field (Examples 7.6 and 7.7), thus proving the following:

Theorem 0.3. *For G a semisimple simply connected group over a field k , and F a semi-global field over $k((t))$, the local-global principle for G -torsors over F does not always hold. In particular, there are counterexamples in which k has the form $E(x, y)$ or $E((x))((y))$, where E is either a number field or a field of the form $\kappa_0(u, v)$ or $\kappa_0((u))((v))$ for some field κ_0 .*

In our examples, $\text{cd}(k) \geq 4$, and $\text{cd}(F) \geq 6$. For an arbitrary semisimple simply connected group G over a semi-global field F , it is unknown whether there exist counterexamples of smaller cohomological dimension.

Our examples are obtained by using our descriptions of $\text{III}(F, G)$ given in Proposition 5.2 and Corollary 6.5 as well as by relying on classical results of Platonov and of Voskresenskiĭ. Building on that, we even obtain examples where $\text{III}(F, G)$ is infinite. We also show that if one considers just the discrete valuations on F that are trivial on K , then the corresponding local-global principle can fail even when K is p -adic (Example 7.10). For a further discussion of our examples and the general context, see Section 7.

Structure of the manuscript: Following a discussion of background and preliminary results in Section 1, we prove local factorization results in Section 2 for reductive groups in

the patching setup. Section 3 builds on those results to prove global factorization statements, from which we obtain Theorem 3.5 mentioned above. In Section 4, we use that lifting result to obtain Theorem 4.2, which asserts a local-global principle under the hypothesis that the reduction graph is a “monotonic tree” (a notion introduced in [CHHKPS20]). We also obtain extensions of that theorem to groups that need not be reductive or connected. Afterwards, to treat cases where the monotonic tree hypothesis does not hold, we find an explicit quotient of $\text{III}(F, G)$ in Section 5 in cases where the components of the closed fiber are projective lines; and we show in Section 6 that this is an exact computation of $\text{III}(F, G)$ when those lines and their intersection points are defined over k . These computations, which are given in terms of R -equivalence classes in $G(k)$, rely on the double coset presentation of $\text{III}(F, G)$ arising from patching. The descriptions of $\text{III}(F, G)$ given in Sections 5 and 6 are then used in Section 7 to obtain our examples of semisimple simply connected groups G for which $\text{III}(F, G) \neq 1$. We conclude with an Appendix that establishes a “specialization map” $G(K)/R \rightarrow G(k)/R$ on R -equivalence classes that extends work of Gille (see [Gil04]) and is used especially in Sections 5 and 6.

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1. PATCHING AND LOCAL-GLOBAL PRINCIPLES

In this section, we begin by recalling the patching setup for semi-global fields. Afterwards, we discuss local-global principles over such fields and their models, and we recall the relationships among obstruction sets to such principles.

1.1. The patching framework. Let K be a complete discretely valued field, with valuation ring R , uniformizing parameter $t \in R$, and residue field k . A *semi-global field over K* (or *over R*) is a one-variable function field F over K ; i.e., a finitely generated extension of K of transcendence degree one in which K is algebraically closed. A *normal model* of F is an integral R -scheme \mathcal{X} with function field F that is flat and projective over R of relative dimension one, and that is normal as a scheme. Its closed fiber is denoted by \mathcal{X}_k . If \mathcal{X} is a normal model that is regular as a scheme, we say that \mathcal{X} is a *regular model*. Such a regular model exists by the main theorem in [Lip78] (see also [Sta20, Theorem 0BGP]). Moreover, by [Lip75, page 193], there exists a regular model \mathcal{X} for which the reduced closed fiber $\mathcal{X}_k^{\text{red}}$ is a union of regular curves, with normal crossings. We refer to such a regular model as a (strict) *normal crossings model of F* .

Let \mathcal{X} be a normal model of a semi-global field F . Let U be an affine open subset of $\mathcal{X}_k^{\text{red}}$, and assume that U is irreducible (or equivalently, is contained in an irreducible component of $\mathcal{X}_k^{\text{red}}$). We consider the ring $R_U \subset F$ consisting of the rational functions on \mathcal{X} that are regular at all points of U . The t -adic completion \widehat{R}_U of R_U is an I -adically complete domain, where I is the radical of the ideal generated by t in \widehat{R}_U . The quotient \widehat{R}_U/I equals $k[U]$, the ring of regular functions on the integral affine curve U . We write F_U for the field of fractions of \widehat{R}_U . Also, for a (not necessarily closed) point P of $\mathcal{X}_k^{\text{red}}$, we let F_P denote the field of fractions of the complete local ring $\widehat{R}_P := \widehat{\mathcal{O}}_{\mathcal{X}, P}$ of \mathcal{X} at P . Note that if $P \in U$ then F_P contains F_U .

Finally, for a closed point $P \in \mathcal{X}_k^{\text{red}}$, we consider the height one primes \wp of the complete local ring \widehat{R}_P that contain the uniformizing parameter $t \in R$. For each such \wp , we let

R_\wp be the localization of \widehat{R}_P at \wp , and we let \widehat{R}_\wp be its t -adic (or equivalently, its \wp -adic) completion; this is a complete discrete valuation ring. We write F_\wp for the fraction field of \widehat{R}_\wp , and k_\wp for the residue field. We call such a \wp a *branch at P on $\mathcal{X}_k^{\text{red}}$* (or *on U* , if $P \in \overline{U}$ for $U \subset \mathcal{X}_k^{\text{red}}$ as above if \wp contains the ideal of \widehat{R}_P defining \overline{U}).

We will often choose sets \mathcal{P} and \mathcal{U} as follows:

We let \mathcal{P} be a finite nonempty set of closed points of $\mathcal{X}_k^{\text{red}}$ that contains all the points of $\mathcal{X}_k^{\text{red}}$ at which $\mathcal{X}_k^{\text{red}}$ is not unibranched. (If \mathcal{X} is a normal crossings model, these are the points at which $\mathcal{X}_k^{\text{red}}$ is singular; i.e., not regular.) We let \mathcal{U} be the set of irreducible components of the affine curve $\mathcal{X}_k^{\text{red}} \setminus \mathcal{P}$.

In this situation, the fields of the form F_P, F_U for $P \in \mathcal{P}, U \in \mathcal{U}$ are called *patches* for F and the rings $\widehat{R}_P, \widehat{R}_U$ are called *patches* on \mathcal{X} . Let \mathcal{B} denote the set of all branches at points $P \in \mathcal{P}$ (each of which lies on some $U \in \mathcal{U}$). The fields F_\wp (resp., rings \widehat{R}_\wp) are referred to as the *overlaps* of the corresponding patches F_P, F_U (resp., $\widehat{R}_P, \widehat{R}_U$). For a branch \wp at P on U , there is an inclusion $F_P \subset F_\wp$ induced by the inclusion $\widehat{R}_P \subset \widehat{R}_\wp$, and also an inclusion $F_U \subset F_\wp$ that is induced by the inclusion $\widehat{R}_U \hookrightarrow \widehat{R}_\wp$. (See [HHK11], beginning of Section 4.) We also have an associated *reduction graph* Γ , a bipartite graph whose vertices correspond to the elements of $\mathcal{P} \cup \mathcal{U}$, and whose edges correspond to the elements of \mathcal{B} ; here an edge $\wp \in \mathcal{B}$ connects vertices $P \in \mathcal{P}$ and $U \in \mathcal{U}$ if \wp is a branch at P on U . (See [HHK15, Section 6] for more details.)

Given sets $\mathcal{P}, \mathcal{U}, \mathcal{B}$ as above, we may consider a *refinement* $\mathcal{P}', \mathcal{U}', \mathcal{B}'$, by choosing a finite set of closed points \mathcal{P}' of $\mathcal{X}_k^{\text{red}}$ that contains \mathcal{P} . Thus the new points of \mathcal{P}' (i.e., those that are not in \mathcal{P}) are all regular points of $\mathcal{X}_k^{\text{red}}$. The set \mathcal{B}' consists of \mathcal{B} together with one additional element \wp' for each new element P' of \mathcal{P}' (viz., the unique branch at the regular point P' of $\mathcal{X}_k^{\text{red}}$). The set \mathcal{U}' is in bijection with the set \mathcal{U} ; viz., for each $U \in \mathcal{U}$, the corresponding element of \mathcal{U}' is obtained by deleting from U the finitely many new points of \mathcal{P}' that lie on U .

1.2. Local-global principles over semi-global fields. In this manuscript we will consider smooth affine group schemes G over a ring A , or over a scheme Z ; we will require these to be of finite type. In the case that the ring A is a field F , we will speak of a *linear algebraic group* G ; and again we will require G to be smooth in this manuscript.

If G is a linear algebraic group over a field F , then the isomorphism classes of G -torsors over F are in natural bijection with the pointed set $H^1(F, G)$ in Galois cohomology. Given a family $\{F_\omega\}_{\omega \in \Omega}$ of overfields of F , we can consider the local-global principle for G -torsors over F with respect to these overfields; this asserts the triviality of a G -torsor over F provided that it becomes trivial over each F_ω . As in the classical case of local-global principles over number fields, there is then the associated obstruction set

$$\text{III}_\Omega(F, G) := \ker(H^1(F, G) \rightarrow \prod_{\omega \in \Omega} H^1(F_\omega, G))$$

to the validity of this local-global principle. In particular, if Ω is a set of discrete valuations on F , then for the overfields we take the completions F_v of F with respect to v , for $v \in \Omega$.

In the case that F is a semi-global field, we may let Ω be the set of divisorial discrete valuations on F ; i.e., the discrete valuations associated to prime divisors on regular models

of F . In this situation we simply write

$$\mathbb{I}\mathbb{I}(F, G) := \ker(H^1(F, G) \rightarrow \prod_{v \in \Omega} H^1(F_v, G))$$

for $\mathbb{I}\mathbb{I}_\Omega(F, G)$. If instead we choose a normal crossings model \mathcal{X} of F over the valuation ring R of the underlying complete discretely valued field K , and if we consider the set of overfields F_P , where P ranges over all the points of the reduced closed fiber $X = \mathcal{X}_k^{\text{red}}$ of \mathcal{X} , then we obtain the obstruction

$$\mathbb{I}\mathbb{I}_X(F, G) := \ker(H^1(F, G) \rightarrow \prod_{P \in X} H^1(F_P, G)).$$

Finally, with \mathcal{P} and \mathcal{U} as in Section 1.1, we may consider the obstruction

$$\mathbb{I}\mathbb{I}_{\mathcal{P}}(F, G) := \ker(H^1(F, G) \rightarrow \prod_{\zeta \in \mathcal{P} \cup \mathcal{U}} H^1(F_\zeta, G))$$

to the local-global principle with respect to the set of overfields F_P and F_U , ranging over $P \in \mathcal{P}$ and $U \in \mathcal{U}$. (Since the set \mathcal{P} determines \mathcal{U} , the notation mentions just \mathcal{P} .)

Recall that a connected linear algebraic group G over an algebraically closed field k is *reductive* if its unipotent radical (the maximal smooth connected unipotent normal subgroup) is trivial. More generally, by a *reductive group* over a ring A or a scheme Z we will mean a connected smooth affine group scheme G over A (resp., Z) such that G is reductive in the previous sense over every geometric point.

For convenience, we recall the following:

Proposition 1.1. *Let F be a semi-global field, let \mathcal{X} be a normal model of F , and let $X = \mathcal{X}_k^{\text{red}}$ be its reduced closed fiber. Let G be a linear algebraic group over F .*

- (a) *If $\mathcal{P} \subset X$ is as above then $\mathbb{I}\mathbb{I}_{\mathcal{P}}(F, G) \subseteq \mathbb{I}\mathbb{I}_X(F, G)$.*
- (b) *If $\mathcal{P} \subset X$ is as above and $\mathcal{P}' \subset X$ is a finite set of closed points that contains \mathcal{P} , then $\mathbb{I}\mathbb{I}_{\mathcal{P}}(F, G) \subseteq \mathbb{I}\mathbb{I}_{\mathcal{P}'}(F, G)$.*
- (c) *$\bigcup_{\mathcal{P}} \mathbb{I}\mathbb{I}_{\mathcal{P}}(F, G) = \mathbb{I}\mathbb{I}_X(F, G)$, where the union is taken over all subsets \mathcal{P} as above.*
- (d) *If \mathcal{X} is a regular model of F then $\mathbb{I}\mathbb{I}_X(F, G) \subseteq \mathbb{I}\mathbb{I}(F, G)$.*
- (e) *In part (d), the containment is an equality if G is reductive over the model \mathcal{X} , or if G is a finite linear algebraic group over F .*

Proof. Parts (a) and (b) are shown at the beginning of [HHK15, Section 5]; part (c) is shown in [HHK15, Corollary 5.9]; and part (d) is shown in [HHK15, Proposition 8.2]. The first assertion in (e) is shown in [HHK15, Theorem 8.10(ii)]. Concerning the second assertion in (e), the proof of [HHK15, Lemma 8.6] shows that $\mathbb{I}\mathbb{I}(F_P, G)$ is trivial for every closed point P on \mathcal{X} . (The statement of that lemma assumed that G is a finite constant group, but that additional hypothesis was not used in the proof.) The asserted equality then follows from [HHK15, Proposition 8.4]. \square

See also Theorem 3.2 below, where a stronger conclusion is shown under additional hypotheses.

It will be convenient to consider all patches of a given type and all overlaps together. For this purpose, we define F -algebras:

$$(1) \quad F_{\mathcal{U}} = \prod_{U \in \mathcal{U}} F_U, \quad F_{\mathcal{P}} = \prod_{P \in \mathcal{P}} F_P, \quad F_{\mathcal{B}} = \prod_{\varphi \in \mathcal{B}} F_{\varphi},$$

and rings

$$(2) \quad \widehat{R}_{\mathcal{U}} = \prod_{U \in \mathcal{U}} \widehat{R}_U, \quad \widehat{R}_{\mathcal{P}} = \prod_{P \in \mathcal{P}} \widehat{R}_P, \quad \widehat{R}_{\mathcal{B}} = \prod_{\wp \in \mathcal{B}} \widehat{R}_{\wp}.$$

We note that we have diagonal inclusions $F \subset F_{\mathcal{U}}, F_{\mathcal{P}} \subset F_{\mathcal{B}}$ and $\widehat{R}_{\mathcal{U}}, \widehat{R}_{\mathcal{P}} \subset \widehat{R}_{\mathcal{B}}$, and we can write

$$(3) \quad \text{III}_{\mathcal{P}}(F, G) = \ker(H^1(F, G) \rightarrow H^1(F_{\mathcal{U}}, G) \times H^1(F_{\mathcal{P}}, G)).$$

By Corollary 3.6 of [HHK15], this has a double coset description:

$$(4) \quad \text{III}_{\mathcal{P}}(F, G) \simeq G(F_{\mathcal{U}}) \backslash G(F_{\mathcal{B}}) / G(F_{\mathcal{P}}).$$

Namely, this isomorphism of pointed sets sends each G -torsor $\xi \in \text{III}_{\mathcal{P}}(F, G)$ to the double coset corresponding to the induced patching problem (see the proof of [HHK15, Theorem 2.4]). To see this, choose trivializations $\xi_{F_U} \simeq G_{F_U}$ and $\xi_{F_P} \simeq G_{F_P}$ for each $U \in \mathcal{U}$ and $P \in \mathcal{P}$, where ξ_{F_U}, ξ_{F_P} are the base changes of ξ to the fields F_U, F_P . Then for each branch $\wp \in \mathcal{B}$ on $U \in \mathcal{U}$ at $P \in \mathcal{P}$, we obtain an induced isomorphism $G_{F_U} \otimes_{F_U} F_{\wp} \rightarrow G_{F_P} \otimes_{F_P} F_{\wp}$ of trivial torsors, given by left multiplication by some element $g_{\wp} \in G(F_{\wp})$. The double coset associated to ξ is then the one represented by $(g_{\wp})_{\wp \in \mathcal{B}}$.

We can reinterpret Proposition 1.1 in these terms. If $\mathcal{P}', \mathcal{U}', \mathcal{B}'$ is a refinement of $\mathcal{P}, \mathcal{U}, \mathcal{B}$, then the containment $\text{III}_{\mathcal{P}}(F, G) \subseteq \text{III}_{\mathcal{P}'}(F, G)$ corresponds to the inclusion of the corresponding double coset spaces. This inclusion takes the class of $(g_{\wp})_{\wp \in \mathcal{B}} \in G(F_{\mathcal{B}})$ to the class of $(g'_{\wp})_{\wp \in \mathcal{B}'} \in G(F_{\mathcal{B}'})$, where $g'_{\wp} = g_{\wp}$ if $\wp \in \mathcal{B}$ and where $g'_{\wp} = 1$ otherwise, by the above explicit description of the identification in (4). As \mathcal{P} varies, these double coset spaces form a direct system. If \mathcal{X} is a regular model, then the direct limit is identified with $\text{III}_{\mathcal{X}}(F, G)$, and also with $\text{III}(F, G)$ if in addition G is reductive over \mathcal{X} . See also the comment after Theorem 3.2 below.

In the sequel, it will be useful to consider the spectra of the fields and rings above. The inclusions $F \subset F_{\mathcal{U}}, F_{\mathcal{P}} \subset F_{\mathcal{B}}$ and $\widehat{R}_{\mathcal{U}}, \widehat{R}_{\mathcal{P}} \subset \widehat{R}_{\mathcal{B}}$ yield commutative diagrams

$$(5) \quad \text{Spec } F_{\bullet} = \boxed{\begin{array}{ccccc} & & \text{Spec } F_{\mathcal{U}} & \rightarrow & \text{Spec } F \\ \text{Spec } F_{\mathcal{B}} & \rightarrow & & & \\ & & \text{Spec } F_{\mathcal{P}} & \rightarrow & \end{array}}, \quad \mathcal{X}_{\bullet} = \boxed{\begin{array}{ccccc} & & \text{Spec } \widehat{R}_{\mathcal{U}} & \rightarrow & \mathcal{X} \\ \text{Spec } \widehat{R}_{\mathcal{B}} & \rightarrow & & & \\ & & \text{Spec } \widehat{R}_{\mathcal{P}} & \rightarrow & \end{array}}.$$

Passing to the reduced closed fiber, we also obtain the commutative diagram

$$(6) \quad (\mathcal{X}_k^{\text{red}})_{\bullet} = \boxed{\begin{array}{ccccc} & & \text{Spec } \widehat{R}_{\mathcal{U}} / \sqrt{(t)} & \rightarrow & \mathcal{X}_k^{\text{red}} \\ \text{Spec } \widehat{R}_{\mathcal{B}} / \sqrt{(t)} & \rightarrow & & & \\ & & \text{Spec } \widehat{R}_{\mathcal{P}} / \sqrt{(t)} & \rightarrow & \end{array}},$$

where we write $\sqrt{(t)}$ for the radical of the ideal (t) in the respective ring; and there are morphisms of diagrams:

$$(7) \quad \begin{array}{ccc} & \mathcal{X}_{\bullet} & \\ & \nearrow & \nwarrow \\ \text{Spec } F_{\bullet} & & (\mathcal{X}_k^{\text{red}})_{\bullet} \end{array}$$

We will use this viewpoint in the next subsection.

1.3. Local-global principles over models. We would like to extend definition (3) and equation (4) from the situation of groups over a semi-global field F to groups over a model \mathcal{X} of F . The latter will be done in Corollaries 1.4 and 1.5. In order to extend (3), we will first consider a more abstract framework for local-global principles.

For a scheme Y , a group scheme G over Y , and a Y -scheme Z , let $\mathcal{T}_G(Z)$ be the category of (right) G -torsors over Z . Note that for $Z \rightarrow Z'$ a morphism of Y -schemes, there is a functor $\mathcal{T}_G(Z') \rightarrow \mathcal{T}_G(Z)$ defined by base change (well defined up to natural isomorphism). We will abuse notation in the case that $Z = \text{Spec } A$, and write $\mathcal{T}_G(A)$ for $\mathcal{T}_G(Z) = \mathcal{T}_G(\text{Spec } A)$. Note that $\mathcal{T}_G(\coprod Z_i)$ is naturally equivalent to the product of categories $\prod \mathcal{T}_G(Z_i)$, and we consider this as an identification.

Recall that if we are given categories C_0, C_1, C_2 , and functors $\iota_1 : C_1 \rightarrow C_0$, and $\iota_2 : C_2 \rightarrow C_0$, then we define the 2-fiber product of categories $C_1 \times_{C_0} C_2$, as follows. The objects of $C_1 \times_{C_0} C_2$ are triples (x_1, x_2, ϕ) , where x_i is an object in C_i , and $\phi : \iota_1 x_1 \rightarrow \iota_2 x_2$ is an isomorphism. Morphisms $(x_1, x_2, \phi) \rightarrow (x'_1, x'_2, \phi')$ are defined to be pairs of morphisms (f_1, f_2) with $f_i : x_i \rightarrow x'_i$ such that we have a commutative diagram:

$$\begin{array}{ccc} \iota_1 x_1 & \xrightarrow{\iota_1 f_1} & \iota_1 x'_1 \\ \phi \downarrow & & \downarrow \phi' \\ \iota_2 x_2 & \xrightarrow{\iota_2 f_2} & \iota_2 x'_2 \end{array}$$

If we have a commutative diagram of schemes

$$\begin{array}{ccccc} & & Y_1 & & \\ & \searrow & \rightarrow & \searrow & \\ Y_0 & & & & Y \\ & \searrow & Y_2 & \rightarrow & \\ & & & & \end{array}$$

and a group scheme G over Y , base change induces a natural functor

$$\begin{aligned} \mathcal{T}_G(Y) &\rightarrow \mathcal{T}_G(Y_1) \times_{\mathcal{T}_G(Y_0)} \mathcal{T}_G(Y_2) \\ P &\mapsto (P_{Y_1}, P_{Y_2}, 1), \end{aligned}$$

where 1 stands for the natural identification $(P_{Y_1})_{Y_0} = (P_{Y_2})_{Y_0}$, coming from commutativity of the diagram of schemes. We will refer to this functor as the one *induced by base change*.

Definition 1.2. Given a commutative diagram of schemes

$$Y_\bullet = \boxed{\begin{array}{ccccc} & & Y_1 & & \\ & \searrow & \rightarrow & \searrow & \\ Y_0 & & & & Y \\ & \searrow & Y_2 & \rightarrow & \\ & & & & \end{array}}$$

and a smooth affine group scheme G over Y , we let

$$\text{III}(Y_\bullet, G) = \ker(H^1(Y, G) \rightarrow H^1(Y_1, G) \times H^1(Y_2, G)).$$

Here $H^1(Z, G)$ denotes the first étale cohomology set of G over a scheme Z ; this classifies isomorphism classes of G -torsors over Z .

As a special case of the definition, if we take $Y_\bullet = \text{Spec } F_\bullet$, where F and $\text{Spec } F_\bullet$ are as in Equation (5), then

$$\text{III}(\text{Spec } F_\bullet, G) = \text{III}_\mathcal{D}(F, G) = G(F_\mathcal{U}) \backslash G(F_\mathcal{D}) / G(F_\mathcal{D}).$$

Proof. By [HKL, Corollary 3.1.5], base change induces an equivalence of categories

$$\mathcal{T}_G(\mathcal{X}) \rightarrow \mathcal{T}_G(\widehat{R}_{\mathcal{U}}) \times_{\mathcal{T}_G(\widehat{R}_{\mathcal{B}})} \mathcal{T}_G(\widehat{R}_{\mathcal{P}}).$$

The assertion then follows from Proposition 1.3. \square

Corollary 1.5. *Let \mathcal{X} be a normal model of a semi-global field F , and let $(\mathcal{X}_k^{\text{red}})_{\bullet}$ be as in diagram (6). Then for any smooth affine group scheme G over \mathcal{X} ,*

$$\text{III}((\mathcal{X}_k^{\text{red}})_{\bullet}, G) = G(\widehat{R}_{\mathcal{U}}/\sqrt{(t)}) \backslash G(\widehat{R}_{\mathcal{B}}/\sqrt{(t)}) / G(\widehat{R}_{\mathcal{P}}/\sqrt{(t)}).$$

Proof. By [HKL, Corollary 3.1.2], base change induces an equivalence of categories

$$\mathcal{T}_G(\mathcal{X}_k^{\text{red}}) \rightarrow \mathcal{T}_G(\widehat{R}_{\mathcal{U}}/\sqrt{(t)}) \times_{\mathcal{T}_G(\widehat{R}_{\mathcal{B}}/\sqrt{(t)})} \mathcal{T}_G(\widehat{R}_{\mathcal{P}}/\sqrt{(t)}).$$

The assertion then follows from Proposition 1.3. \square

Note that the identifications in Equation (4), Corollary (1.4), and Corollary (1.5) are compatible, because each is given by base change. Thus we have the following:

Corollary 1.6. *In the notation of Corollaries 1.4 and 1.5, let G be a smooth affine group scheme over \mathcal{X} . Then we have the following commutative diagram of sets*

$$\begin{array}{ccccc} H^1(F, G) & \supseteq & \text{III}_{\mathcal{P}}(F, G) & \rightarrow & G(F_{\mathcal{U}}) \backslash G(F_{\mathcal{B}}) / G(F_{\mathcal{P}}) \\ \uparrow & & \uparrow & & \uparrow \\ H^1(\mathcal{X}, G) & \supseteq & \text{III}(\mathcal{X}_{\bullet}, G) & \rightarrow & G(\widehat{R}_{\mathcal{U}}) \backslash G(\widehat{R}_{\mathcal{B}}) / G(\widehat{R}_{\mathcal{P}}) \\ \downarrow & & \downarrow & & \downarrow \\ H^1(\mathcal{X}_k^{\text{red}}, G) & \supseteq & \text{III}((\mathcal{X}_k^{\text{red}})_{\bullet}, G) & \rightarrow & G(\widehat{R}_{\mathcal{U}}/\sqrt{(t)}) \backslash G(\widehat{R}_{\mathcal{B}}/\sqrt{(t)}) / G(\widehat{R}_{\mathcal{P}}/\sqrt{(t)}) \end{array}$$

where the upper and lower right vertical maps are respectively induced by the inclusion and reduction maps.

Corollary 1.7. *Let \mathcal{X} be a normal model of a semi-global field F , let $\mathcal{P}, \mathcal{U}, \mathcal{B}$ be as in Section 1.1, and let G be a smooth affine group scheme over \mathcal{X} .*

- (a) *If $g, g' \in G(\widehat{R}_{\mathcal{B}})$ are congruent modulo $\sqrt{(t)}$, then the corresponding G -torsors over \mathcal{X} (as in Corollary 1.4) restrict to isomorphic G -torsors over $\mathcal{X}_k^{\text{red}}$.*
- (b) *In particular, if $g \in G(\widehat{R}_{\mathcal{B}})$ is trivial modulo $\sqrt{(t)}$, then the corresponding G -torsor over \mathcal{X} restricts to the trivial G -torsor over $\mathcal{X}_k^{\text{red}}$.*

Proof. This is immediate from the compatible identifications in Corollaries 1.4-1.6. \square

If L is any field, then an L -torus is a linear algebraic group T over L such that $T \times_L L^{\text{sep}}$ is isomorphic to a product of copies of $\mathbb{G}_{m, L^{\text{sep}}}$, where L^{sep} is a separable algebraic closure of L . More generally, if Z is a scheme, then by a Z -torus (or *torus over Z*) we mean a smooth affine group scheme T over Z for which there exists a finite étale morphism $Y \rightarrow Z$ such that $T \times_Z Y$ is isomorphic to a product of copies of $\mathbb{G}_{m, Y}$. A torus is *split* if it is isomorphic to a product of copies of \mathbb{G}_m over the given base.

Proposition 1.8. *Let R be a complete discrete valuation ring with fraction field K and residue field k . Let \mathcal{X} be a normal model of a semi-global field F over K , and let $\mathcal{X}_k^{\text{red}}$ be its reduced closed fiber. Let T be a torus over \mathcal{X} . Then the image of $\ker(H^1(\mathcal{X}, T) \rightarrow H^1(\mathcal{X}_k^{\text{red}}, T))$ in $H^1(F, T)$ is trivial.*

Proof. Let $\zeta \in \ker(H^1(\mathcal{X}, T) \rightarrow H^1(\mathcal{X}_k^{\text{red}}, T))$. The image of ζ in $H^1(\mathcal{X}_k^{\text{red}}, T)$ is trivial, hence so is the image of ζ in $H^1(\widehat{R}_{\mathcal{U}}/\sqrt{(t)}, T)$. Since $\widehat{R}_{\mathcal{U}}$ is (t) -adically complete and since G is smooth, it follows that the natural map $H^1(\widehat{R}_{\mathcal{U}}, T) \rightarrow H^1(\widehat{R}_{\mathcal{U}}/\sqrt{(t)}, T)$ is injective (see [Str83], Theorem 1). Hence the image of ζ in $H^1(\widehat{R}_{\mathcal{U}}, T)$ is trivial. Similarly the image of ζ in $H^1(\widehat{R}_{\mathcal{P}}, T)$ is trivial. Thus $\zeta \in \text{III}(\mathcal{X}_{\bullet}, T) \subseteq H^1(\mathcal{X}, T)$; moreover, its image in $H^1(F, T)$ lies in $\text{III}_{\mathcal{P}}(F, T)$, by the top left part of the diagram in Corollary 1.6.

Under the isomorphism $\text{III}(\mathcal{X}_{\bullet}, T) \simeq T(\widehat{R}_{\mathcal{U}}) \backslash T(\widehat{R}_{\mathcal{B}}) / T(\widehat{R}_{\mathcal{P}})$ in Corollary 1.4, ζ corresponds to the double coset of an element $(g_{\varphi}) \in T(\widehat{R}_{\mathcal{B}})$. Since $\zeta \in \ker(H^1(\mathcal{X}, T) \rightarrow H^1(\mathcal{X}_k^{\text{red}}, T))$, it follows from Corollary 1.6 that the image of (g_{φ}) in $T(\widehat{R}_{\mathcal{B}}/\sqrt{(t)})$ is in $T(\widehat{R}_{\mathcal{U}}/\sqrt{(t)})T(\widehat{R}_{\mathcal{P}}/\sqrt{(t)})$. By formal smoothness, the maps $T(\widehat{R}_{\mathcal{U}}) \rightarrow T(\widehat{R}_{\mathcal{U}}/\sqrt{(t)})$ and $T(\widehat{R}_{\mathcal{P}}) \rightarrow T(\widehat{R}_{\mathcal{P}}/\sqrt{(t)})$ are surjective; so after adjusting the choice of double coset representative of ζ we may assume that $(g_{\varphi}) \in T(\widehat{R}_{\mathcal{B}})$ has trivial image in $T(\widehat{R}_{\mathcal{B}}/\sqrt{(t)})$. By Proposition 2.2 of [Gil04], it follows from the description of $\text{III}_{\mathcal{P}}(F, T)$ in [CHHKPS20, Theorem 3.1(b)] that the image of ζ in $\text{III}_{\mathcal{P}}(F, T) \subseteq H^1(F, T)$ is the trivial class. (In the notation of [CHHKPS20], $F_{U,P}$ denotes F_{φ} , where φ is the branch on U at P . For more about the notion of R-equivalence used there, see the discussion following the proof of Lemma 2.3 below.) \square

2. LOCAL FACTORIZATION

In this section, we prove local factorization results for reductive groups, in the context of semi-global fields. Our main results here are Proposition 2.4 and Corollary 2.5, which will permit us afterwards to reduce the study of general reductive groups to those that are anisotropic, in order to obtain results about local-global principles.

We let \mathcal{X} be a regular model of a semi-global field F over a complete discretely valued field K having valuation ring R and residue field k . We choose a non-empty finite set \mathcal{P} of closed points satisfying the conditions given in Section 1.1, and as in that earlier discussion we obtain associated sets \mathcal{U} and \mathcal{B} . In addition to the notation from Section 1.1, we now introduce some other terminology and notation that will be used throughout the remainder of the manuscript.

For a closed point $P \in \mathcal{X}_k^{\text{red}}$, we write $\kappa(P)$ for the residue field of P . If U is a non-empty affine open subset of an irreducible component of $\mathcal{X}_k^{\text{red}}$, we write $\kappa(U)$ for the ring of constants on U ; i.e., $\kappa(U) = \mathcal{O}(\bar{U})$, the ring of functions on the closure of U in $\mathcal{X}_k^{\text{red}}$. The field $k' = \kappa(U)$ is a finite extension of k ; and we write $k'[U]$ and $k'(U)$ for the ring of regular functions and the field of rational functions on U .

If $\varphi \in \mathcal{B}$ is a branch at $P \in \mathcal{P}$, then φ is a prime ideal of \widehat{R}_P , and we may consider the residue ring \widehat{R}_P/φ . This discrete valuation ring is the complete local ring at P of the irreducible component of $\mathcal{X}_k^{\text{red}}$ on which φ lies. The fraction field of \widehat{R}_P/φ is the residue field $k_{\varphi} = \widehat{R}_{\varphi}/\varphi$ of \widehat{R}_{φ} ; and the residue field of \widehat{R}_P/φ is $\kappa(P)$.

For a smooth affine group scheme G over \widehat{R}_{φ} , we write $G_{\varphi}(\widehat{R}_{\varphi})$ for the kernel of $G(\widehat{R}_{\varphi}) \rightarrow G(k_{\varphi})$; i.e., for the set of elements that become trivial on the closed fiber.

Our first lemmas deal with reductive groups that are anisotropic; i.e., that contain no copy of \mathbb{G}_m .

Lemma 2.1. *Let \mathcal{X} be a normal crossings model of a semi-global field and let P be a singular point of the reduced closed fiber. Let G be a reductive group over \widehat{R}_P which is anisotropic over $\ell := \kappa(P)$. Let \wp_1, \wp_2 be the two branches at P , and consider the compositions*

$$s_i : G(F_{\wp_i}) = G(\widehat{R}_{\wp_i}) \rightarrow G(k_{\wp_i}) = G(\widehat{R}_P/\wp_i) \rightarrow G(\ell)$$

for $i = 1, 2$. Suppose we are given $g_i \in G(F_{\wp_i})$ such that $s_1(g_1) = s_2(g_2)$. Then there exists $g' \in G(\widehat{R}_P)$ such that $g'g_i \in G_{\wp_i}(\widehat{R}_{\wp_i})$ for $i = 1, 2$.

Proof. Since G is anisotropic over ℓ , it is anisotropic over k_{\wp_i} by Proposition A.2 in the Appendix. So by Proposition A.3, we have $G(k_{\wp_i}) = G(\widehat{R}_P/\wp_i)$ and $G(F_{\wp_i}) = G(\widehat{R}_{\wp_i})$. Thus $g_i \in G(\widehat{R}_{\wp_i})$, and the compositions are defined. We may write $\wp_i = t_i \widehat{R}_P$, for some $t_1, t_2 \in \widehat{R}_P$. Here t_1, t_2 form a regular system of parameters for \widehat{R}_P , since \mathcal{X} is a normal crossings model. Thus $(t_1 t_2) = (t_1) \cap (t_2)$, and there is a short exact sequence

$$0 \rightarrow \widehat{R}_P/\wp_1\wp_2 \rightarrow \widehat{R}_P/\wp_1 \times \widehat{R}_P/\wp_2 \xrightarrow{\bar{\cdot}} \widehat{R}_P/(\wp_1 + \wp_2) \rightarrow 0.$$

That is, we have a pullback square of rings

$$(8) \quad \begin{array}{ccc} \widehat{R}_P/\wp_1\wp_2 & \longrightarrow & \widehat{R}_P/\wp_1 \\ \downarrow & & \downarrow \\ \widehat{R}_P/\wp_2 & \longrightarrow & \widehat{R}_P/(\wp_1 + \wp_2), \end{array}$$

where the lower right ring is $\kappa(P) = \ell$. By hypothesis, the images $\bar{g}_i \in G(\widehat{R}_P/\wp_i)$ of g_i agree in $G(\ell)$, for $i = 1, 2$. Since g_1, g_2 have the same image in $\widehat{R}_P/(\wp_1 + \wp_2)$, they are induced by a common element $\bar{g}'' \in G(\widehat{R}_P/\wp_1\wp_2)$. But G is smooth and \widehat{R}_P is $\wp_1\wp_2$ -adically complete, so by formal smoothness we may lift \bar{g}'' to an element $g'' \in G(\widehat{R}_P)$. Let $g' := (g'')^{-1}$. Then $g'g_i \in G(\widehat{R}_{\wp_i})$, and in fact the product is in $G_{\wp_i}(\widehat{R}_{\wp_i})$ because g'' and g_i have the same image in $G(\widehat{R}_P/\wp_i)$. \square

For the next lemma and some later results, we need to define rings associated with somewhat more general (not necessarily open) subsets of the reduced closed fiber. Following [HHK15a, Section 3.1], let W be a non-empty affine open subset of an irreducible component of the reduced closed fiber $\mathcal{X}_k^{\text{red}}$ of \mathcal{X} . Thus W is connected. Note that W is not necessarily open in $\mathcal{X}_k^{\text{red}}$; viz., it will not be open if W contains a singular point of $\mathcal{X}_k^{\text{red}}$ (i.e., one that lies on two distinct components of $\mathcal{X}_k^{\text{red}}$). Let R_W be the subring of F consisting of those elements that are regular at every point of W , and let J_W be the Jacobson radical of R_W . Thus $\text{Spec}(R_W/J_W)$ is the reduced closed fiber W of $\text{Spec}(R_W)$ (see [HHK15a, Lemma 3.3] and the paragraph before that); hence $R_W/J_W = \mathcal{O}(W)$, the affine coordinate ring of W .

Let \widehat{R}_W be the J_W -adic completion of R_W . The extension of J_W to \widehat{R}_W (which we again call J_W) is the Jacobson radical of \widehat{R}_W . Since W is connected, the ring \widehat{R}_W is a domain (by [HHK15a, Proposition 3.4]). Also, if W' is a non-empty open subset of W , then R_W is contained in $R_{W'}$, and J_W^n is the restriction of $J_{W'}^n$ to R_W for every $n \geq 1$; so \widehat{R}_W is contained in $\widehat{R}_{W'}$.

Note that in the special case that W is a non-empty affine open subset U of the regular locus of $\mathcal{X}_k^{\text{red}}$, the definition above of \widehat{R}_W agrees with the prior definition of \widehat{R}_U , since J is the radical of the ideal (t) in this case.

Lemma 2.2. *Let \mathcal{X} be a regular model of a semi-global field over a complete discrete valuation ring with residue field k . Choose a non-empty finite set \mathcal{P} as in Section 1.1, with associated sets \mathcal{U}, \mathcal{B} . Let W be a non-empty affine open subset of an irreducible component of the reduced closed fiber $\mathcal{X}_k^{\text{red}}$. Let $P \in \mathcal{P} \cap W$ be a closed point such that $U := W \setminus \{P\}$ is in \mathcal{U} , and suppose that $\kappa(U)$ is equal to $\kappa(P) =: \ell$. Consider a reductive group G over \widehat{R}_W that is anisotropic over ℓ , and take $h \in G(\ell)$. Let \wp be a branch at P along U , and let $g \in G(F_\wp)$. Then there exists $g' \in G(\widehat{R}_U) \subset G(\widehat{R}_\wp)$ such that gg' maps to h under the composition*

$$s : G(F_\wp) = G(\widehat{R}_\wp) \rightarrow G(k_\wp) = G(\widehat{R}_P/\wp) \rightarrow G(\ell).$$

Proof. Since $\kappa(U) = \ell$, we have an inclusion $\ell \hookrightarrow \mathcal{O}(U) = \ell[U]$, which induces an inclusion $G(\ell) \hookrightarrow G(\ell[U])$. Let $g'_0 \in G(\ell[U])$ be the image of $s(g)^{-1}h \in G(\ell)$ under $G(\ell) \hookrightarrow G(\ell[U])$. Since \widehat{R}_U is complete and G is smooth, there is an element $g' \in G(\widehat{R}_U)$ whose image under the map $G(\widehat{R}_U) \rightarrow G(\ell[U])$ is g'_0 , by formal smoothness. Viewing $G(\widehat{R}_U)$ as a subgroup of $G(\widehat{R}_\wp)$, we have that $s(g') = s(g)^{-1}h$ by the commutativity of the following diagram:

$$\begin{array}{ccc} G(\widehat{R}_U) & \hookrightarrow & G(\widehat{R}_\wp) \\ \downarrow & & \downarrow \\ G(\ell[U]) & \hookrightarrow & G(k_\wp) = G(\widehat{R}_P/\wp) \\ \uparrow & & \downarrow \\ G(\ell) & \xrightarrow{\text{id}} & G(\ell) \end{array}$$

Hence $s(gg') = h$ as asserted. □

Consider a smooth affine group scheme G over \widehat{R}_P , and let \wp be a branch on $U \in \mathcal{U}$ at $P \in \mathcal{P}$. Then $G(\widehat{R}_P)$ is a subgroup of $G(\widehat{R}_\wp)$, and $G_\wp(\widehat{R}_\wp)$ is a normal subgroup of $G(\widehat{R}_\wp)$. So we may take the product $G_s(\widehat{R}_\wp) := G_\wp(\widehat{R}_\wp)G(\widehat{R}_P) = G(\widehat{R}_P)G_\wp(\widehat{R}_\wp)$. This is a subgroup of $G(\widehat{R}_\wp)$, and it is the inverse image of $G(\widehat{R}_P/\wp) = G(\widehat{\mathcal{O}}_{\bar{U},P})$ under the reduction map $G(\widehat{R}_\wp) \rightarrow G(k_\wp) = G(k(U)_P)$; here $k(U)$ is the function field of U , and $k(U)_P$ is its completion with respect to the discrete valuation defined by P . Note that $G_s(\widehat{R}_\wp)$ and $G(\widehat{R}_P)$ have the same image in $G(k_\wp)$. Let \mathfrak{m}_P be the maximal ideal of \widehat{R}_P . Composing the reduction maps $G_s(\widehat{R}_\wp) \rightarrow G(\widehat{R}_P/\wp)$ and $G(\widehat{R}_P/\wp) \rightarrow G(\widehat{R}_P/\mathfrak{m}_P) = G(\kappa(P))$ yields a *specialization* map $\Theta_\wp : G_s(\widehat{R}_\wp) \rightarrow G(\kappa(P))$; and we refer to the elements of $G_s(\widehat{R}_\wp)$ as *specializable*. (This is related to the specialization maps that appear in Sections 5 and 6 and in the Appendix.)

Lemma 2.3. *Let \mathcal{X} be a normal crossings model of a semi-global field over a complete discrete valuation ring R , and consider be a finite set \mathcal{P} of points on the reduced closed fiber as in Section 1.1. Let G be a reductive group over R , and let $P \in \mathcal{P}$.*

- (a) If P is a regular point of the reduced closed fiber with \wp its unique branch, and if g_\wp, g'_\wp are elements of $G_s(\widehat{R}_\wp)$, then there exists $\beta_P \in G(\widehat{R}_P)$ such that the elements $g_\wp\beta_P$ and g'_\wp of $G_s(\widehat{R}_\wp)$ have equal images in $G(k_\wp)$.
- (b) If P is a singular point of the reduced closed fiber with branches \wp_1, \wp_2 , and if $g_{\wp_i}, g'_{\wp_i} \in G_s(\widehat{R}_{\wp_i})$ are elements satisfying $\Theta_{\wp_i}(g_{\wp_i}) = \Theta_{\wp_i}(g'_{\wp_i}) \in G(\kappa(P))$ for $i = 1, 2$, then there exists $\beta_P \in G(\widehat{R}_P)$ such that the elements $g_{\wp_i}\beta_P$ and g'_{\wp_i} of $G_s(\widehat{R}_{\wp_i})$ have equal images in $G(\widehat{R}_{\wp_i}/\wp_i)$ for $i = 1, 2$.

Proof. For part (a), since $G_s(\widehat{R}_\wp)$ and $G(\widehat{R}_P)$ have the same image in $G(k_\wp)$, there exists a lift $\beta_P \in G(\widehat{R}_P) \subseteq G_s(\widehat{R}_\wp)$ of the image of $g_\wp^{-1}g'_\wp \in G_s(\widehat{R}_\wp)$ in $G(k_\wp)$. Then β_P has the desired property.

For part (b), we will proceed similarly to the proof of Lemma 2.1 (but using the assumption here that the given elements are specializable, to replace the anisotropy hypothesis of Lemma 2.1). Namely, the maximal ideal of \widehat{R}_P is generated by two elements t_1, t_2 , where \widehat{R}_{\wp_i} is the completion of the localization of \widehat{R}_P at (t_i) , and $\wp_i = t_i\widehat{R}_P$. For $i = 1, 2$ let $\bar{g}_{\wp_i}, \bar{g}'_{\wp_i} \in G(\widehat{R}_P/\wp_i)$ be the images of g_{\wp_i}, g'_{\wp_i} and write $\bar{\beta}_{\wp_i} = \bar{g}_{\wp_i}^{-1}\bar{g}'_{\wp_i}$. Since $\Theta_{\wp_i}(g_{\wp_i}) = \Theta_{\wp_i}(g'_{\wp_i}) \in G(\kappa(P))$, the image of $\bar{\beta}_{\wp_i}$ in $G(\kappa(P))$ is equal to 1, for $i = 1, 2$.

We have the pullback diagram of rings (8) as in the proof of Lemma 2.1. Hence there is a well-defined element $\tilde{\beta}_P \in G(\widehat{R}_P/\wp_1\wp_2)$ that maps to $\bar{\beta}_{\wp_i}$ in $G(\widehat{R}_P/\wp_i)$ for $i = 1, 2$. Since G is a smooth R -scheme and the ring \widehat{R}_P is complete with respect to $\wp_1\wp_2$, by formal smoothness we may choose a lift $\beta_P \in G(\widehat{R}_P)$ of $\tilde{\beta}_P$. Thus the elements $g_{\wp_i}\beta_P$ and g'_{\wp_i} of $G_s(\widehat{R}_{\wp_i})$ have equal images in $G(\widehat{R}_{\wp_i}/\wp_i)$ for $i = 1, 2$, as required. \square

Recall that for any field E , two E -points of an E -variety V are *directly R-equivalent* if they are in the image of E -points of \mathbb{A}_E^1 under some rational map $f : \mathbb{A}_E^1 \dashrightarrow V$. The transitive closure of this relation is *R-equivalence*. If V is a linear algebraic group G over a field E , then any two R-equivalent points of $G(E)$ are directly R-equivalent (see [Gil97, II.1.1(b)]). The points of $G(E)$ that are R-equivalent to the identity are called *R-trivial*, and these form a normal subgroup $R = RG(E)$ of $G(E)$. The quotient group $G(E)/R$ is the group of *R-equivalence classes* of $G(E)$.

The following result will be the key ingredient in reducing to the anisotropic case in the proof of Proposition 3.4; and that proposition in turn will lead to the proofs of our local-global results in the later sections, via Theorems 3.5 and 4.2.

Proposition 2.4. *Let \mathcal{X} be a regular model of a semi-global field over a complete discretely valued field having residue field k . Let P be a closed point of $\mathcal{X}_k^{\text{red}}$, and suppose G is a reductive group over \widehat{R}_P . Let $S \subseteq G$ be a maximal split torus in G and let $H = C_G(S)$, the centralizer of S in G . Let $\wp_1, \wp_2, \dots, \wp_r$ be a collection of distinct branches at P . Suppose we are given $g_i \in G(F_{\wp_i})$ for every i . Then there is an element $g \in G(F_P)$ that is R-trivial in each $G(F_{\wp_i})$, together with elements $h_i^* \in H(\widehat{R}_P)$ for all i , such that $h_i^*g_i g \in G_{\wp_i}(\widehat{R}_{\wp_i})$. That is, each $g_i g \in H(\widehat{R}_P)G_{\wp_i}(\widehat{R}_{\wp_i}) \subseteq G(\widehat{R}_P)G_{\wp_i}(\widehat{R}_{\wp_i}) = G_s(\widehat{R}_{\wp_i})$.*

Proof. As in the statement of Proposition A.8 of the Appendix, the group H is reductive, and there are parabolic subgroups of G with unipotent radicals $\mathcal{U}, \mathcal{U}'$ such that the morphism

$H \times \mathcal{U} \times \mathcal{U}' \rightarrow G$ is an open immersion. (See the proof of Proposition A.8 for more details.) Let \mathcal{C} be the image of this open immersion.

Since G is reductive, it follows that G_{F_P} is unirational (see [Bor91, Chap. V, Theorem 18.2]). Choose a dominant morphism $\phi : V \rightarrow G_{F_P}$ where $V \subseteq \mathbb{A}_{F_P}^N$ is a Zariski affine open set. We may assume that $\phi(V)$ contains the identity element of G . For $i = 1, \dots, r$, let $V_i = \phi^{-1}(g_i^{-1}\mathcal{C})$; this is an open subscheme of $V_{F_{\varphi_i}}$. Since these are all nonempty open in affine space, we may choose points $v_i \in V_i(F_{\varphi_i})$. Since V_i is Zariski open, the set $V_i(F_{\varphi_i})$ is open in the φ_i -adic topology on $V(F_{\varphi_i})$. By weak approximation, we may find $v \in \mathbb{A}^N(F_P)$ that is sufficiently close φ_i -adically to v_i for each i , so as to make $v \in V_i(F_{\varphi_i})$ for all i . Since $V \subseteq \mathbb{A}_{F_P}^N$ is an open subset and $\phi(V)$ contains the identity of G , the element $\tilde{g} := \phi(v)$ is R-trivial in $G(F_P)$, and hence in each $G(F_{\varphi_i})$. Moreover $g_i\tilde{g} \in \mathcal{C}(F_{\varphi_i})$ for each i .

Hence we may write $g_i\tilde{g} = h_i u_i u'_i$, where $h_i \in H(F_{\varphi_i})$, $u_i \in \mathcal{U}(F_{\varphi_i})$, and $u'_i \in \mathcal{U}'(F_{\varphi_i})$. By Proposition A.4(b) in the Appendix, we may write $h_i = h'_i s_i$ for some $h'_i \in H(\widehat{R}_{\varphi_i})$ and $s_i \in S(F_{\varphi_i})$. Let \bar{h}'_i be the reduction of h'_i modulo φ_i ; thus $\bar{h}'_i \in H(k_{\varphi_i})$, where k_{φ_i} is the residue field of the complete discrete valuation ring \widehat{R}_{φ_i} . Since k_{φ_i} is also the fraction field of the complete discrete valuation ring \widehat{R}_P/φ_i , it again follows from Proposition A.4(b) that $\bar{h}'_i = \bar{h}''_i \bar{s}'_i$ for some $\bar{h}''_i \in H(\widehat{R}_P/\varphi_i)$ and $\bar{s}'_i \in S(k_{\varphi_i})$. By the smoothness of H and S , we may lift these elements to some $h''_i \in H(\widehat{R}_P)$ and $s'_i \in S(\widehat{R}_{\varphi_i})$. Thus h'_i and $h''_i s'_i$ are elements of $H(\widehat{R}_{\varphi_i})$ that have the same image modulo φ_i . Hence $h'_i = h''_i s'_i h'''_i$ for some $h'''_i \in H_{\varphi_i}(\widehat{R}_{\varphi_i})$ and thus $g_i\tilde{g} = h_i u_i u'_i = h'_i s_i u_i u'_i = h''_i s'_i h'''_i s_i u_i u'_i = h''_i h'''_i s'_i s_i u_i u'_i$.

Consider the map $\psi : S \times \mathcal{U} \times \mathcal{U}' \rightarrow G$ defined by $\psi(s, u, u') = suu'$. Since S is a split torus, S_{F_P} is a rational variety. Moreover, \mathcal{U}_{F_P} and \mathcal{U}'_{F_P} are also rational varieties, by [SGA70c, Exp. XXVI, §2, Corollaire 2.5] (where the notation W in that corollary is defined in [SGA70a, I, Définition 4.6.1]). By weak approximation and continuity we may find $s \in S(F_P)$, $u \in \mathcal{U}(F_P)$, and $u' \in \mathcal{U}'(F_P)$ such that $s'_i s_i u_i u'_i (suu')^{-1} = \psi(s'_i s_i, u_i, u'_i) \psi(s, u, u')^{-1} \in G_{\varphi_i}(\widehat{R}_{\varphi_i})$ for all i . We can rewrite this element as $h'''_i{}^{-1} h''_i{}^{-1} g_i \tilde{g} (suu')^{-1}$, by the above expression for $g_i \tilde{g}$. But $h'''_i \in H_{\varphi_i}(\widehat{R}_{\varphi_i}) \subseteq G_{\varphi_i}(\widehat{R}_{\varphi_i})$. So $h''_i{}^{-1} g_i \tilde{g} (suu')^{-1} \in G_{\varphi_i}(\widehat{R}_{\varphi_i})$, and we can take $g = \tilde{g} (suu')^{-1} \in G(F_P)$ and $h_i^* = h''_i{}^{-1} \in H(\widehat{R}_P)$. We observed that the element \tilde{g} is R-trivial in each $G(F_{\varphi_i})$; and so are s, u, u' , by the rationality of the groups $S_{F_P}, \mathcal{U}_{F_P}, \mathcal{U}'_{F_P}$. Hence g is also R-trivial in each $G(F_{\varphi_i})$, as asserted. \square

Corollary 2.5. *Let \mathcal{X} , P , and G be as in Proposition 2.4, let φ be a branch of the reduced closed fiber of \mathcal{X} at P , and let $g_{\varphi} \in G(F_{\varphi})$. Then there exists $g_P \in G(F_P)$ such that $g_{\varphi} g_P \in G_{\varphi}(\widehat{R}_{\varphi})$.*

Proof. By Proposition 2.4, there is an element $g \in G(F_P)$ such that $g_{\varphi} g \in G_s(\widehat{R}_{\varphi})$. That is, $g_{\varphi} g = g'_{\varphi} g_0$ with $g'_{\varphi} \in G_{\varphi}(\widehat{R}_{\varphi})$ and $g_0 \in G(\widehat{R}_P)$. We may thus set $g_P = g g_0^{-1}$. \square

3. LOCALLY TRIVIAL TORSORS

In this section, we will prove a result on lifting torsors over a semi-global field F to torsors over a model of F that are trivial over the reduced closed fiber; see Theorem 3.5. This theorem will be key to proving the local-global results of Section 4. Throughout this section, we restrict attention to normal crossings models \mathcal{X} of a semi-global field F over a complete discrete valuation ring R . As before, we let K and k be the fraction field and residue field of R , respectively.

We begin this section by proving a local-global result for torsors on a patch (Proposition 3.1). We then use that to prove the equivalence of various notions of a G -torsor being locally trivial. Namely, in Theorem 3.2 we show that for G a reductive group over R , and F a semi-global field over R , the local-global obstruction sets $\text{III}_{\mathcal{P}}(F, G)$ are independent of the choice of \mathcal{P} and even of the choice of normal crossings model \mathcal{X} ; viz., they are all equal to $\text{III}(F, G)$. Hence by Proposition 1.1, if X is the reduced closed fiber of a normal crossings model \mathcal{X} , the sets $\text{III}_{\mathcal{P}}(F, G)$ are also all equal to $\text{III}_X(F, G)$. This assertion is similar to that of [CHHKPS20, Thm. 4.4], which concerned the case of a torus over \mathcal{X} , though the proof there was different (it relied on flasque resolutions of tori).

Afterwards, we turn our attention to a special class of normal crossings models that was defined in [CHHKPS20]; viz., those models whose reduction graph is a monotonic tree (see below). For such a model \mathcal{X} , we show in Theorem 3.5 that any G -torsor over F whose class lies in $\text{III}(F, G)$ can be lifted to a torsor over \mathcal{X} that is trivial along the reduced closed fiber $\mathcal{X}_k^{\text{red}}$. Theorem 3.5 is proven using Proposition 3.4, a global factorization result that relies on the local factorization results of Section 2.

Proposition 3.1. *Let \mathcal{X} be a normal crossings model of a semi-global field F over a discrete valuation ring R , take an irreducible component Z of its reduced closed fiber $\mathcal{X}_k^{\text{red}}$, and let V be a nonempty affine open subset of Z that does not intersect any other irreducible component of $\mathcal{X}_k^{\text{red}}$. Let $P \in V$ be a closed point and let $W = V \setminus \{P\}$. Let $\xi \in H^1(F_V, G)$, where G is a reductive group over R . If ξ is trivial over F_W and F_P , then ξ is trivial over F_V .*

Proof. Let φ be the unique branch at P on V (or equivalently, on W). In the situation of Definition 1.2, let Y_{\bullet} be the commutative diagram of schemes associated to $Y_0 = \text{Spec}(F_{\varphi})$, $Y_1 = \text{Spec}(F_W)$, $Y_2 = \text{Spec}(F_P)$, and $Y = \text{Spec}(F_V)$. Since ξ is trivial over F_W and F_P , it lies in $\text{III}_P(F_V, G) := \text{III}(Y_{\bullet}, G)$.

By [HHK15, Theorem 2.4] and [HHK15a, Proposition 3.9], the natural map

$$\text{III}_P(F_V, G) \rightarrow G(F_W) \backslash G(F_{\varphi}) / G(F_P)$$

is a bijection. Let $g_0 \in G(Y_0) = G(F_{\varphi})$ be a representative of the double coset corresponding to ξ . By Corollary 2.5, there exists $g_2 \in G(Y_2) = G(F_P)$ such that $g_{\varphi} := g_0 g_2 \in G_{\varphi}(\widehat{R}_{\varphi}) \subset G(F_{\varphi})$. The elements g_0, g_{φ} lie in the same double coset, and so g_{φ} also represents ξ .

Let $\mathcal{P} \subset \mathcal{X}_k^{\text{red}}$ be the finite set of closed points consisting of the points in $\overline{W} \setminus W$ together with the points at which irreducible components of $\mathcal{X}_k^{\text{red}}$ intersect. Let \mathcal{U} be the set of connected components of $\mathcal{X}_k^{\text{red}} \setminus \mathcal{P}$, and let \mathcal{B} be the set of branches at the points of \mathcal{P} . By [HHK15, Corollary 3.6], the natural map

$$\text{III}_{\mathcal{P}}(F, G) \rightarrow \prod_{U' \in \mathcal{U}} G(F_{U'}) \backslash \prod_{\varphi' \in \mathcal{B}} G(F_{\varphi'}) / \prod_{P' \in \mathcal{P}} G(F_{P'})$$

is a bijection. For $\varphi' \in \mathcal{B}$ with $\varphi' \neq \varphi$, let $g_{\varphi'} = 1$, and let $\zeta \in \text{III}_{\mathcal{P}}(F, G)$ be the G -torsor over F corresponding to the class of $(g_{\varphi'})_{\varphi' \in \mathcal{B}} \in \prod_{\varphi' \in \mathcal{B}} G(F_{\varphi'})$ in this double coset space. Thus ζ induces ξ under the natural map $H^1(F, G) \rightarrow H^1(F_V, G)$.

Consider the subset $\text{III}_{\mathcal{P}}(\mathcal{X}, G) \subset H^1(\mathcal{X}, G)$ consisting of the G -torsors over \mathcal{X} that are trivial over each $\widehat{R}_{P'}$ and each $\widehat{R}_{U'}$, for $P' \in \mathcal{P}$ and $U' \in \mathcal{U}$. The natural map

$$\text{III}_{\mathcal{P}}(\mathcal{X}, G) \rightarrow \prod_{U' \in \mathcal{U}} G(\widehat{R}_{U'}) \backslash \prod_{\varphi' \in \mathcal{B}} G(\widehat{R}_{\varphi'}) / \prod_{P' \in \mathcal{P}} G(\widehat{R}_{P'})$$

is bijective, by Corollary 1.4. Let $\tilde{\zeta}$ be the G -torsor over \mathcal{X} corresponding to the class of $(g_{\wp'})_{\wp' \in \mathcal{B}} \in \prod_{\wp' \in \mathcal{B}} G(\widehat{R}_{\wp'})$ in this double coset space. Thus $\tilde{\zeta}$ induces ζ under the natural map $H^1(\mathcal{X}, G) \rightarrow H^1(F, G)$, and hence also induces $\xi \in H^1(F_W, G)$.

Consider the restriction of $\tilde{\zeta}$ to a G -torsor on $\mathcal{X}_k^{\text{red}}$. Since each $g_{\wp'}$ is congruent to 1 modulo $\wp' = \sqrt{t\widehat{R}_{\wp'}}$, this restriction is trivial by Corollary 1.5. But the natural map $H^1(\widehat{R}_V, G) \rightarrow H^1(k[V], G)$ is injective, by smoothness of G and completeness of R_V (see [Str83], Theorem 1). Thus the generic fiber ζ of $\tilde{\zeta}$ is a trivial G -torsor over F . Hence its image $\xi \in H^1(F_V, G)$ is the trivial G -torsor over F_V . \square

The following assertion generalizes [CHHKPS20, Theorem 4.4] from the case of tori to the case of reductive groups. In the torus case we had permitted the group to be defined over \mathcal{X} , whereas in the next theorem we require it to be defined over R .

Theorem 3.2. *Let \mathcal{X} be a normal crossings model of a semi-global field F over a complete discrete valuation ring R having residue field k , and let $X = \mathcal{X}_k^{\text{red}}$. As in Section (1.1), let \mathcal{P} be a non-empty finite set of closed points of $\mathcal{X}_k^{\text{red}}$ containing all the singular points. Let G be a reductive group over R . Then $\text{III}_{\mathcal{P}}(F, G) = \text{III}_X(F, G) = \text{III}(F, G)$.*

Proof. Since G is a reductive group over R , it is in particular reductive over \mathcal{X} . So by parts (c) and (e) of Proposition 1.1,

$$\text{III}(F, G) = \text{III}_X(F, G) = \cup_{\mathcal{P}'} \text{III}_{\mathcal{P}'}(F, G),$$

where \mathcal{P}' runs over all finite sets of closed points of X containing \mathcal{P} . Thus it suffices to show that the containment $\text{III}_{\mathcal{P}}(F, G) \subseteq \text{III}_{\mathcal{P}'}(F, G)$ is an equality, for all finite subsets \mathcal{P}' of closed points of \mathcal{X} containing \mathcal{P} .

So let \mathcal{P}' be such a finite subset, and let $\xi \in \text{III}_{\mathcal{P}'}(F, G)$. For each $P \in \mathcal{P}$, we have $P \in \mathcal{P}'$ and hence ξ is trivial over F_P .

To prove $\text{III}_{\mathcal{P}}(F, G) = \text{III}_{\mathcal{P}'}(F, G)$, it remains to show that ξ is trivial over F_U , for each irreducible component U of $X \setminus \mathcal{P}$. Since $\mathcal{P} \subseteq \mathcal{P}'$, there exists a unique irreducible component V of $X \setminus \mathcal{P}'$ such that $V \subseteq U$. Thus ξ is trivial over F_V , since $\xi \in \text{III}_{\mathcal{P}'}(F, G)$.

Suppose $V \neq U$, and let $P \in U \setminus V$. Thus $P \in \mathcal{P}' \setminus \mathcal{P}$. In particular, P is a regular point of X . Also, ξ is trivial over F_P , since $\xi \in \text{III}_{\mathcal{P}'}(F, G)$. Hence, by Proposition 3.1, ξ is trivial over $F_{V \cup \{P\}}$. Since $U \setminus V$ is finite, it follows by induction that ξ is trivial over F_U . Thus $\xi \in \text{III}_{\mathcal{P}}(F, G)$. \square

Thus each of the double coset spaces $\text{III}_{\mathcal{P}}(F, G) \simeq G(F_{\mathcal{U}}) \backslash G(F_{\mathcal{B}}) / G(F_{\mathcal{P}})$ in (4) of Section 1.2, and not just their direct limit, becomes identified with $\text{III}(F, G)$ under the hypotheses of Theorem 3.2.

We now turn to preparation for proving Theorem 3.5. Let F be a semi-global field with normal crossings model \mathcal{X} . Choose sets $\mathcal{P}, \mathcal{U}, \mathcal{B}$ as in Section 1.1, and let Γ be the corresponding reduction graph as defined there. To each vertex $v \in \mathcal{P} \cup \mathcal{U}$ of Γ we associate the field $\kappa(v)$ as before (viz., $\kappa(P)$ or $\kappa(U)$; see the beginning of Section 2). Following [CHHKPS20], Section 7, we say that the reduction graph Γ is a *monotonic tree* if it is a tree and there is some vertex v_0 (the *root*) with the following property: If v, w are vertices of Γ and v is the parent of w (i.e., v, w are adjacent, and v lies on the unique (non-repeating) path connecting v_0 to w) then $\kappa(v) \subseteq \kappa(w)$.

If $P \in \mathcal{P}$ lies on the closure of $U \in \mathcal{U}$, then $\kappa(P)$ necessarily contains $\kappa(U) = \mathcal{O}(\bar{U})$; hence if P is the parent of U on a monotonic tree then necessarily $\kappa(P) = \kappa(U)$. For that reason, the root of a monotonic tree can always be chosen so as to correspond to some $U_0 \in \mathcal{U}$; and we will always do so from now on. For every vertex v of Γ , we may consider its *distance* from v_0 ; viz., the number of edges in the unique path from v_0 to v . Since each edge connects an element of \mathcal{U} to an element of \mathcal{P} , the vertices corresponding to elements of \mathcal{U} have even distance, while those corresponding to elements of \mathcal{P} have odd distance.

As shown in [CHHKPS20, Proposition 7.6], if Γ is a monotonic tree, then for every finite extension k'/k of the residue field k , the graph associated to the base change from k to k' of the reduced closed fiber is also a tree. Moreover, the converse holds if k is perfect.

Remark 3.3. Enlarging the set \mathcal{P} preserves the monotonic tree property; so since any two choices of \mathcal{P} have a common refinement, the property is independent of the choice of \mathcal{P} on a given normal crossings model. Moreover, the monotonic tree property is preserved under blow-up, and as a consequence it is independent of the choice of normal crossings model of F (see [CHHKPS20, Remark 7.1]). Thus it is meaningful to say that the reduction graph of a semi-global field F is a monotonic tree, without referring to \mathcal{X} or \mathcal{P} .

As at the end of Section 1.2, given a refinement $\mathcal{P}', \mathcal{U}', \mathcal{B}'$ of $\mathcal{P}, \mathcal{U}, \mathcal{B}$, to each element of $G(F_{\mathcal{B}}) = \prod_{\wp \in \mathcal{B}} G(F_{\wp})$ there is an associated element of $G(F_{\mathcal{B}'})$, obtained by inserting the identity of G at each entry indexed by an element of $\mathcal{B}' \setminus \mathcal{B}$.

Proposition 3.4. *Let \mathcal{X} be a normal crossings model of a semi-global field F over a complete discrete valuation ring R . Let $\mathcal{P}, \mathcal{U}, \mathcal{B}$ as in Section 1.1, and assume that the associated reduction graph is a monotonic tree. Let G be a reductive group over R , and let $(g_{\wp})_{\wp \in \mathcal{B}} \in G(F_{\mathcal{B}})$. Then possibly after refining $\mathcal{P}, \mathcal{U}, \mathcal{B}$ (and replacing $(g_{\wp})_{\wp}$ by the associated element of $G(F_{\mathcal{B}'})$), we may find $(g_U)_{U \in \mathcal{U}} \in G(F_{\mathcal{U}})$ and $(g_P)_{P \in \mathcal{P}} \in G(F_{\mathcal{P}})$ such that $g_U g_{\wp} g_P \in G_{\wp}(\hat{R}_{\wp})$ whenever \wp is a branch at P along U .*

Proof. As above, we assume that the root of the reduction graph Γ is a vertex v_0 corresponding to some $U_0 \in \mathcal{U}$. We need to define elements g_v for all vertices v , and we will do this by induction on the distance from v_0 . Write $\mathcal{P}^{\text{sing}}$ for the subset of \mathcal{P} consisting of singular points of $\mathcal{X}_k^{\text{red}}$ and write $\mathcal{P}^{\text{reg}} \subseteq \mathcal{P}$ for the subset of regular points. Then the vertices corresponding to elements of \mathcal{P}^{reg} are leaves of the tree Γ , connected to the rest of the tree by just one edge; while for those in $\mathcal{P}^{\text{sing}}$, there are two edges at the vertex, of which exactly one is on the path connecting the vertex to v_0 .

The unique vertex of distance zero is v_0 , corresponding to U_0 , and we set $g_{v_0} = g_{U_0}$ equal to $1 \in G(F_{U_0})$. For the inductive step, let $n \geq 0$ and assume that g_v has been defined for every vertex v of distance at most $2n$. We will define g_v for all vertices v of distance $2n+1$ and $2n+2$ (which are respectively in \mathcal{P} and in \mathcal{U}). In the process, we may refine $\mathcal{P}, \mathcal{U}, \mathcal{B}$ by adding finitely many regular points of $\mathcal{X}_k^{\text{red}}$ to \mathcal{P} . These points correspond to new terminal vertices of distance $2n+3$ from the root, and they are handled at the next step of the induction. Note that the inductive process will still terminate, since refining $\mathcal{P}, \mathcal{U}, \mathcal{B}$ multiple times can increase the maximum distance of vertices from the root by at most one in total.

Let $P \in \mathcal{P}$ correspond to a vertex of distance $2n+1$ from v_0 . We first consider the case that $P \in \mathcal{P}^{\text{reg}}$. Thus P is unbranched on the reduced closed fiber, and is a leaf of the tree Γ . The unique component $V \in \mathcal{U}$ on which P lies is the parent of P in Γ , having

distance $2n$ from v_0 (so that g_V is already defined). Let \wp be the unique branch at P , and set $g'_\wp = g_V g_\wp \in G(F_\wp)$. By Corollary 2.5, there exists $g_P \in G(F_P)$ such that $g'_\wp g_P \in G_\wp(\widehat{R}_\wp)$. So then $g_V g_\wp g_P = g'_\wp g_P \in G_\wp(\widehat{R}_\wp)$, as desired.

We next consider the case that $P \in \mathcal{P}^{\text{sing}}$. Then P is the parent of a unique $U \in \mathcal{U}$, since P is not a leaf and since \mathcal{X} is a normal crossings model. Here U has distance $2n + 2$ from v_0 ; and every vertex of distance $2n + 2$ arises in this manner for a unique $P \in \mathcal{P}^{\text{sing}}$ (since each vertex other than v_0 in the tree Γ has a unique parent). Let $V \in \mathcal{U}$ be the unique parent of P ; thus V has distance $2n$, and g_V is already defined by the inductive hypothesis. Let \wp_1 be the branch at P lying on V , and let \wp_2 be the branch at P lying on U .

Let $W = U \cup \{P\}$. If needed, we delete a regular point from $U \subset W$, so that we may assume that W is affine. In this case we add to \mathcal{P} the point deleted from W , thereby refining $\mathcal{P}, \mathcal{U}, \mathcal{B}$; and we set $g_\wp = 1$ for \wp the (unique) branch at that deleted point. Note that this new point added to \mathcal{P} has distance $2n + 3$ from v_0 in the enlarged graph. Since W is a non-empty affine open subset of an irreducible component of $\mathcal{X}_k^{\text{red}}$, there is an associated ring \widehat{R}_W , as in the discussion before Lemma 2.2.

Set $g_1 := g_V g_{\wp_1} \in G(F_{\wp_1})$ and $g_2 := g_{\wp_2} \in G(F_{\wp_2})$, where the branches \wp_1 and \wp_2 at P are as above. We wish to define $g_P \in G(F_P)$ and $g_U \in G(F_U)$ such that $g_V g_{\wp_1} g_P \in G_{\wp_1}(\widehat{R}_{\wp_1})$ and $g_U g_{\wp_2} g_P \in G_{\wp_2}(\widehat{R}_{\wp_2})$.

Let \bar{S} be maximal among the split tori in $G_{\kappa(P)} = G \times_R \kappa(P)$. Since P is the parent of U in a monotonic tree, $\kappa(P) = \kappa(U) = \kappa(W)$. Here $\kappa(W)$ is contained in the coordinate ring $k[W]$ of W . So \bar{S} defines a torus \bar{S}_W in $G_W := G \times_R k[W]$ via base change, and \bar{S}_W remains split. By Proposition A.1(a) in the Appendix, we may therefore lift \bar{S}_W to a split torus S over \widehat{R}_W . Here S is also split over \widehat{R}_P since $\widehat{R}_W \subset \widehat{R}_P$; and $S_{\kappa(P)} = \bar{S}$. Consider the group $H = C_G(S)$ over \widehat{R}_W , and write $\bar{H} = H_{\kappa(P)}$. Then $(H/S)_{\kappa(P)} = \bar{H}/\bar{S}$ is anisotropic over $\kappa(P)$ by applying Proposition A.4(a) over $\kappa(P)$, since \bar{S} is a maximal split torus in $G_{\kappa(P)}$ and since $\bar{H} = C_{G_{\kappa(P)}}(\bar{S})$. By Proposition A.2, H/S is therefore anisotropic over F_P and \widehat{R}_P , and thus over \widehat{R}_W .

By Proposition 2.4, there exist $g \in G(F_P)$ and $h_i \in H(F_{\wp_i})$ for $i = 1, 2$ such that $h_i g_i g \in G_{\wp_i}(\widehat{R}_{\wp_i})$ for $i = 1, 2$. Let \bar{h}_i denote the image of h_i in $(H/S)(F_{\wp_i})$. By Lemma 2.2 applied to the anisotropic reductive group H/S , there exists $\bar{h}_U \in (H/S)(\widehat{R}_U)$ so that \bar{h}_1 and $\bar{h}_2 \bar{h}_U$ map to the same element in $(H/S)(\kappa(P))$. By Lemma 2.1, there is an element $\bar{h}_P \in (H/S)(\widehat{R}_P)$ so that $\bar{h}_{\pi_1} := \bar{h}_P \bar{h}_1 \in (H/S)_{\wp_1}(\widehat{R}_{\wp_1})$ and $\bar{h}_{\pi_2} := \bar{h}_P \bar{h}_2 \bar{h}_U \in (H/S)_{\wp_2}(\widehat{R}_{\wp_2})$. Since \widehat{R}_P is local and S is a split torus, $H^1(\widehat{R}_P, S) = 0$ by Hilbert 90. So the cohomology exact sequence associated to $1 \rightarrow S \rightarrow H \rightarrow H/S \rightarrow 1$ asserts that $H(\widehat{R}_P)$ surjects onto $(H/S)(\widehat{R}_P)$, and we may lift \bar{h}_P to an element $h_P \in H(\widehat{R}_P)$. For analogous reasons, we can lift \bar{h}_{π_i} to some $h_{\pi_i} \in H_{\wp_i}(\widehat{R}_{\wp_i})$.

We claim that after refining $\mathcal{P}, \mathcal{U}, \mathcal{B}$ by deleting some finite collection of points from U (and hence also from W), the element \bar{h}_U may be lifted to some $h_U \in H(\widehat{R}_U)$. To see this, consider the image ξ of \bar{h}_U under the coboundary map $(H/S)(\widehat{R}_U) \rightarrow H^1(\widehat{R}_U, S)$. Let R_η be the local ring of \mathcal{X} at the generic point η of U . Then ξ has trivial image in $H^1(R_\eta, S)$, since R_η is local and S is a split torus; so there is a Zariski affine open neighborhood $\mathcal{N} = \text{Spec}(E)$ of η in \mathcal{X} such that ξ maps to the trivial element in $H^1(E, S)$. Now $U' := \mathcal{N} \cap U$ is a dense

open subset of U , and $E \subset R_{U'} \subset \widehat{R}_{U'}$. Thus \bar{h}_U , or equivalently its image $\bar{h}_{U'} \in (H/S)(\widehat{R}_{U'})$, maps to the trivial element in $H^1(\widehat{R}_{U'}, S)$; hence $\bar{h}_{U'}$ can be lifted to some $h_{U'} \in H(\widehat{R}_{U'})$, as claimed. We now proceed, replacing U by U' (and W by $U' \cup \{P\}$). We add the deleted points to \mathcal{P} , and set $g_\wp = 1$ for \wp the corresponding branches. (As before, these new points of \mathcal{P} have distance $2n + 3$ from v_0 , since U' has distance $2n + 2$.)

Since the two elements $h_1, h_P^{-1}h_{\pi_1} \in H(\widehat{R}_{\wp_1})$ have the same image in $(H/S)(\widehat{R}_{\wp_1})$, there exists $s_1 \in S(\widehat{R}_{\wp_1})$ such that $h_1 = s_1h_P^{-1}h_{\pi_1}$. Similarly, there exists $s_2 \in S(\widehat{R}_{\wp_2})$ such that $h_2 = s_2h_P^{-1}h_{\pi_2}h_U^{-1}$. Hence $s_1h_P^{-1}h_{\pi_1}g_1g = h_1g_1g \in G_{\wp_1}(\widehat{R}_{\wp_1})$ and $s_2h_P^{-1}h_{\pi_2}h_U^{-1}g_2g = h_2g_2g \in G_{\wp_2}(\widehat{R}_{\wp_2})$. Since the split torus S is rational, weak approximation yields elements $s_P \in S(\widehat{R}_P)$ and $s_{\pi_i} \in S_{\wp_i}(\widehat{R}_{\wp_i})$ for $i = 1, 2$ such that $s_i = s_{\pi_i}s_P$ for $i = 1, 2$. Thus $G_{\wp_1}(\widehat{R}_{\wp_1})$ contains $s_{\pi_1}s_Ph_P^{-1}h_{\pi_1}g_1g$, hence also $s_Ph_P^{-1}h_{\pi_1}g_1g$. Since $G(\widehat{R}_P)$ normalizes $G_{\wp_1}(\widehat{R}_{\wp_1})$, the group $G_{\wp_1}(\widehat{R}_{\wp_1})$ also contains $h_{\pi_1}g_1gs_Ph_P^{-1}$ and thus $g_1gs_Ph_P^{-1}$. Let $g_P = gs_Ph_P^{-1} \in G(F_P)$. Thus $g_Vg_{\wp_1}g_P = g_1g_P \in G_{\wp_1}(\widehat{R}_{\wp_1})$, as desired. Similarly, setting $g_U = h_U^{-1}$, we find that $g_Ug_{\wp_2}g_P = h_U^{-1}g_2gs_Ph_P^{-1} \in G_{\wp_2}(\widehat{R}_{\wp_2})$, also as desired. This completes the inductive step for points of \mathcal{P} of distance $2n + 1$ and elements of \mathcal{U} of distance $2n + 2$, and thus completes the proof. \square

Combined with the results in Section 1.3, and in the notation of Section 1.2, the above proposition gives the following.

Theorem 3.5. *Let \mathcal{X} be a normal crossings model of a semi-global field F over a discrete valuation ring R , and assume that the reduction graph is a monotonic tree. Let G be a reductive group over R , and let Z be a G -torsor over F whose class is in $\text{III}(F, G)$. Then Z extends to a torsor over \mathcal{X} which is trivial when restricted to the reduced closed fiber $\mathcal{X}_k^{\text{red}}$.*

Proof. Consider an element ξ in $\text{III}(F, G)$. Consider a non-empty collection of closed points \mathcal{P} on a normal crossings model \mathcal{X} of F , with \mathcal{P} containing all the singular points of the reduced closed fiber $\mathcal{X}_k^{\text{red}}$. By Theorem 3.2, ξ lies in $\text{III}_{\mathcal{P}}(F, G)$. By the double coset description of $\text{III}_{\mathcal{P}}(F, G)$ given in Corollary 3.6 of [HHK15], the cocycle ξ is represented by a collection of elements $g_\wp \in G(F_\wp)$, where \wp runs over the branches at elements of \mathcal{P} . By Proposition 3.4, this collection gives rise to a collection of elements $g'_\wp \in G_\wp(\widehat{R}_\wp)$ representing the same double coset class, and hence the same torsor over F , if \mathcal{P} is sufficiently large (i.e., possibly after refining the initial choice of \mathcal{P}). By Corollary 1.7(b), this new collection defines a torsor over \mathcal{X} whose reduction to the closed fiber is trivial. \square

Remark 3.6. (a) It was already shown in [CPS12, Theorem 4.2] that every G -torsor over F arises from a G -torsor over \mathcal{X} , even without assuming that the class of the given torsor lies in $\text{III}_{\mathcal{P}}(F, G)$ or assuming that the reduction graph is a monotonic tree. But the key new point of Theorem 3.5 is that under the hypotheses stated there, the torsor over \mathcal{X} may be chosen so as to be trivial along the reduced closed fiber. This will be essential in the next section.

(b) In the above proof, Proposition 1.1(c) could have been cited instead of Theorem 3.2, since the set \mathcal{P} is allowed to grow in the proof. But the full strength of Theorem 3.2 will be needed in the proof of Proposition 5.2, and hence in the results in Sections 6 and 7 that build on that.

4. LOCAL-GLOBAL PRINCIPLES FOR MONOTONIC TREES

In this section, building on Theorem 3.5, we prove local-global principles for torsors in the context of monotonic trees. First, in Theorem 4.2, we obtain such a result for reductive groups over a complete discrete valuation ring R , extending [CHHKPS20, Theorem 7.3], which had considered the case of tori. Afterwards we prove related results for groups that are not necessarily connected or reductive, especially in the case where R is of equal characteristic zero.

The proof of Theorem 4.2 below relies on [GPS21], and to apply that we need additional mild assumptions. Given a reductive group G over a scheme S , there is a canonical semisimple normal subgroup G^{ss} , called the derived subgroup of G , such that the quotient G/G^{ss} is a torus (see [SGA70c, Exp. XXII, Théorème 6.2.1, Remarque 6.2.2]). The semisimple group G^{ss} has a simply connected cover $G^{\text{sc}} \rightarrow G^{\text{ss}}$; this is the universal central isogeny to G^{ss} . (See [Har67, beginning of Section 1.2].) The kernel of this isogeny is a finite commutative S -group scheme $\mu(G^{\text{ss}})$ of multiplicative type, called the *algebraic fundamental group* of G^{ss} (e.g., see [Gon13, Remark 2.3]); this agrees with the étale fundamental group (as in [SGA71]) over a field of characteristic zero. More generally, write

$$\mu(G) := \ker(G^{\text{sc}} \times_S \text{rad}(G) \rightarrow G),$$

where $\text{rad}(G)$ is the radical of the reductive group G (this is trivial if G is semisimple). The group $\mu(G)$ is an extension of $\ker(G^{\text{ss}} \times_S \text{rad}(G) \rightarrow G)$ by $\ker(G^{\text{sc}} \rightarrow G^{\text{ss}})$; and so $\mu(G)$, like the latter two groups, is a finite S -group scheme of multiplicative type (see [SGA70c, Exp. XXII, 6.2.3]).

Now let R be a complete discrete valuation ring having residue field k , let $S = \text{Spec}(R)$, and let G be a reductive group over R (or equivalently, over S). The group $\mu(G)$ is étale provided $\mu(G_k) = \mu(G)_k$ is; and that condition is automatic if $\text{char}(k) = 0$ since then every k -group scheme is smooth. More generally, since $\mu(G)$ is a finite central subgroup of G^{sc} , it is étale if $\text{char}(k)$ does not divide the order of the center of G^{sc} . As the list on [PR94, page 332] shows, this includes the cases where G^{ss} is of type B_n , C_n , or D_n , with $\text{char}(k) \neq 2$; or of type 1A_n or 2A_n if $\text{char}(k)$ does not divide $n + 1$.

The following proposition is a consequence of the main theorem (Theorem 1.1) of the paper [GPS21] by Gille, Parimala and Suresh.

Proposition 4.1. *Let F be a semi-global field over a complete discrete valuation ring R with residue field k , let \mathcal{X} be a regular model of F over R , and let G be a reductive group over \mathcal{X} such that $\mu(G)$ is étale. If an element $\zeta \in H^1(\mathcal{X}, G)$ has trivial image in $H^1(\mathcal{X}_k, G)$, then its image in $H^1(F, G)$ is also trivial. More generally, if two elements $\zeta, \zeta' \in H^1(\mathcal{X}, G)$ have equal images in $H^1(\mathcal{X}_k, G)$, then their images in $H^1(F, G)$ are also equal.*

Proof. Let $T = \text{rad}(G)$ and write $\mu = \mu(G)$. As in the proof of [GPS21, Theorem 7.1], there is a commutative diagram

$$\begin{array}{ccccccc} H^1(\mathcal{X}, \mu) & \longrightarrow & H^1(\mathcal{X}, G^{\text{sc}}) \times H^1(\mathcal{X}, T) & \longrightarrow & H^1(\mathcal{X}, G) & \longrightarrow & H^2(\mathcal{X}, \mu) \\ \downarrow & & \downarrow \alpha & & \downarrow \beta & & \downarrow \\ H^1(\mathcal{X}_k, \mu) & \longrightarrow & H^1(\mathcal{X}_k, G^{\text{sc}}) \times H^1(\mathcal{X}_k, T) & \longrightarrow & H^1(\mathcal{X}_k, G) & \longrightarrow & H^2(\mathcal{X}_k, \mu) \end{array}$$

with exact rows. Here the left and right vertical arrows are bijective by proper base change (see [SGA73, Exp. XII, Corollaire 5.5]). A diagram chase then implies that the natural map $\ker(\alpha) \rightarrow \ker(\beta)$ is surjective. Here $\ker(\alpha) = \ker(\alpha_1) \times \ker(\alpha_2)$, where $\alpha_1 : H^1(\mathcal{X}, G^{\text{sc}}) \rightarrow H^1(\mathcal{X}_k, G^{\text{sc}})$ and $\alpha_2 : H^1(\mathcal{X}, T) \rightarrow H^1(\mathcal{X}_k, T)$ are the natural maps. Every element of $\ker(\alpha_1)$ has trivial image in $H^1(F, G^{\text{sc}})$ by [GPS21, Theorem 1.1]. Moreover, every element of $\ker(\alpha_2)$ is contained in the kernel of $H^1(\mathcal{X}, T) \rightarrow H^1(\mathcal{X}_k^{\text{red}}, T)$ and thus has trivial image in $H^1(F, T)$ by Proposition 1.8. So by the surjectivity of $\ker(\alpha) \rightarrow \ker(\beta)$, it follows that every element in $\ker(\beta)$ has trivial image in $H^1(F, G)$. This proves the first assertion.

The second assertion then follows by twisting G by an element of $Z^1(\mathcal{X}, G)$ (see [Gir71, Sections 2.3, 2.6], generalizing the field case in [Ser00, Section I.5.3]), since each twist of G also satisfies the hypotheses above. \square

The next result is an analog of Theorem 7.3 of [CHHKPS20], where the group was required to be a torus. Recall the definition of a “monotonic tree” from just after the proof of Theorem 3.2, and also recall that whether this property holds for the reduction graph of a normal crossings model of a semi-global field F is independent of the choice of the model \mathcal{X} and finite set \mathcal{P} (Remark 3.3).

Theorem 4.2. *Let \mathcal{X} be a normal crossings model of a semi-global field F over a complete discrete valuation ring R . Assume that the closed fiber of \mathcal{X} is reduced and that the associated reduction graph Γ is a monotonic tree. If G is any reductive group over R such that $\mu(G)$ is étale, then $\text{III}(F, G) = 1$.*

Proof. Since Γ is a monotonic tree, Theorem 3.5 applies. Thus each element ξ in $\text{III}(F, G)$ corresponds to a torsor over F that extends to \mathcal{X} and is trivial over the reduced closed fiber; and this is the same as the closed fiber since the closed fiber is reduced. Since $\mu(G)$ is étale, we can apply Proposition 4.1, and conclude that ξ is isomorphic to the trivial G -torsor over F . That is, an arbitrary element $\xi \in \text{III}(F, G)$ is trivial. \square

In the special case of an equal characteristic zero ring R , i.e., $R = k[[t]]$ for some parameter t , and G a group over k , the reductivity hypothesis can be dropped (and $\mu(G)$ is automatically étale):

Corollary 4.3. *Let G be a connected linear algebraic group over a field k of characteristic zero, and let F be a semi-global field over $R := k[[t]]$. Suppose that F has a normal crossings model over $R := k[[t]]$ whose closed fiber is reduced and whose reduction graph is a monotonic tree. Then $\text{III}(F, G)$ is trivial.*

Proof. Let $\mathcal{R}_u(G)$ be the unipotent radical of G . Thus the group $G/\mathcal{R}_u(G)$ is reductive over k . The exactness of $1 \rightarrow \mathcal{R}_u(G) \rightarrow G \rightarrow G/\mathcal{R}_u(G) \rightarrow 1$ yields that $H^1(F, \mathcal{R}_u(G)) \rightarrow H^1(F, G) \rightarrow H^1(F, G/\mathcal{R}_u(G))$ is exact. But $H^1(F, \mathcal{R}_u(G))$ is trivial, because $\mathcal{R}_u(G)$ is unipotent and F is perfect (see [Ser00, III §2.1, Prop. 6]). So the map $H^1(F, G) \rightarrow H^1(F, G/\mathcal{R}_u(G))$ has trivial kernel. But this map sends $\text{III}(F, G)$ to $\text{III}(F, G/\mathcal{R}_u(G))$, and the latter is trivial by Theorem 4.2, using that $\mu(G)$ is étale in residue characteristic zero. It thus follows that $\text{III}(F, G)$ is also trivial. \square

So far, we have been focusing on connected groups, but new issues arise for local-global principles for groups that are disconnected. For example, in [HHK15], it was shown that if a (smooth) linear algebraic group G over a semi-global field F is connected and rational

as an F -variety then $\text{III}_X(F, G)$ is trivial, where X is the reduced closed fiber; whereas if G is disconnected and each connected component of the F -variety G is rational, then the vanishing of $\text{III}_X(F, G)$ depends on the reduction graph. Below, in Theorems 4.9 and 4.11, we prove analogs of Corollary 4.3 in the disconnected case.

In preparation, we consider the finite case, in Lemma 4.4 and Proposition 4.5. Namely, let G be a finite étale group scheme over a semi-global field F over a complete discretely valued field K with valuation ring R . Thus $G_{F^{\text{sep}}}$ is a finite constant group over the separable closure F^{sep} of F , equipped with an action of $\text{Gal}(F^{\text{sep}}/F)$. There is then a finite Galois extension L/F that splits G ; i.e., such that G_L is a constant finite group over L . (In fact, the minimal such finite extension is the fixed field $L_0 := (F^{\text{sep}})^N$ under the kernel N of the Galois action on $G_{F^{\text{sep}}}$.)

Pick such a field L and let $\Delta = \text{Gal}(L/F)$. Take a normal crossings model \mathcal{X} for F over R , and let \mathcal{Y} be the normalization of \mathcal{X} in L . Here \mathcal{Y} need not be regular; but $\pi : \mathcal{Y} \rightarrow \mathcal{X}$ is a Δ -Galois branched cover of normal R -curves, with reduced closed fibers $\mathcal{Y}_k^{\text{red}}$ and $\mathcal{X}_k^{\text{red}}$. If a closed point $Q \in \mathcal{Y}_k^{\text{red}}$ lies over $P \in \mathcal{X}_k^{\text{red}}$, then the set of branches of $\mathcal{Y}_k^{\text{red}}$ at Q surjects onto the set of branches of $\mathcal{X}_k^{\text{red}}$ at P , by the going-down theorem applied to the extension $\widehat{R}_P \subseteq \widehat{R}_Q$. Let \mathcal{P} be a non-empty finite set of closed points on $\mathcal{X}_k^{\text{red}}$, and assume that

$$\mathcal{P}' := \pi^{-1}(\mathcal{P}) \quad \text{contains the set of points where } \mathcal{Y}_k^{\text{red}} \text{ is not regular.} \quad (*)$$

Thus \mathcal{P}' is a non-empty Δ -stable finite set of closed points of $\mathcal{Y}_k^{\text{red}}$. Note that \mathcal{P} contains all the non-unibranched points of $\mathcal{X}_k^{\text{red}}$, by the above surjectivity on branches, since \mathcal{P}' contains all the non-unibranched points of $\mathcal{Y}_k^{\text{red}}$. Consider the set \mathcal{U} of connected components of the complement of \mathcal{P} in $\mathcal{X}_k^{\text{red}}$, and similarly the set \mathcal{U}' with respect to \mathcal{P}' and $\mathcal{Y}_k^{\text{red}}$.

Let Γ_F and Γ_L be the reduction graphs associated to these models and sets, as in Section 1.1 (where there is no regularity requirement). Up to homotopy equivalence, these are determined by F and L ; i.e., they do not depend on the choice of \mathcal{P} and \mathcal{P}' . (See [HHK15, Remark 6.1(b)] and the sentence just before Corollary 6.5 in that paper. Note that these also do not require \mathcal{Y} to be a regular model.) Observe that Δ acts on Γ_L , and $\Gamma_F = \Gamma_L/\Delta$. This is because the action of Δ on \mathcal{Y} over \mathcal{X} induces actions of Δ on \mathcal{P}' and on \mathcal{U}' , with the elements of \mathcal{P} and \mathcal{U} being the orbits of this action (because $\mathcal{X}_k^{\text{red}} = \mathcal{Y}_k^{\text{red}}/\Delta$).

Let $\mathcal{V}(\Gamma)$ denote the set of vertices of a graph Γ . For $\xi \in \mathcal{V}(\Gamma_F)$, let $\mathcal{V}_\xi(\Gamma_L) \subseteq \mathcal{V}(\Gamma_L)$ be the set of vertices that map to ξ . The tensor product $F_\xi \otimes_F L$ is a Δ -Galois étale F_ξ -algebra consisting of a direct product of the fields L_ζ (ranging over $\zeta \in \mathcal{V}_\xi(\Gamma_L)$). So if $\zeta \in \mathcal{V}_\xi(\Gamma_L)$, then L_ζ is a finite Galois extension of F_ξ , and its Galois group Δ_ζ is a subgroup of Δ . More precisely, $\Delta_\zeta \subseteq \Delta$ is the stabilizer of the factor L_ζ in the direct product (or equivalently, the stabilizer of ζ) under the action of Δ .

Lemma 4.4. *As above, let G be a finite étale group scheme over a semi-global field F and let L be a finite Galois extension of F that splits G . Let \mathcal{X} be a normal crossings model for F , let \mathcal{Y} be the normalization of \mathcal{X} in L , and let Γ_F, Γ_L be the reduction graphs for \mathcal{X}, \mathcal{Y} associated to finite subsets $\mathcal{P}, \mathcal{P}'$ of their reduced closed fibers such that \mathcal{P}' is the inverse image of \mathcal{P} . If Γ_L is a tree, then*

$$\text{III}_{\mathcal{P}}(F, G) = \ker(H^1(L/F, G(L))) \rightarrow \prod_{\substack{\xi \in \mathcal{V}(\Gamma_F) \\ \zeta \in \mathcal{V}_\xi(\Gamma_L)}} H^1(L_\zeta/F_\xi, G(L_\zeta)).$$

Proof. Consider the commutative diagram

$$\begin{array}{ccc} \mathbb{III}_{\mathcal{P}}(F, G) & \longrightarrow & H^1(F, G) \\ \downarrow & & \downarrow \\ \mathbb{III}_{\mathcal{P}'}(L, G_L) & \longrightarrow & H^1(L, G_L), \end{array}$$

where the horizontal arrows are inclusions. Since Γ_L is a tree and since G_L is a constant finite group, it follows from [HHK15, Corollary 6.5] that $\mathbb{III}_{\mathcal{P}'}(L, G_L)$ is trivial. Thus $\mathbb{III}_{\mathcal{P}}(F, G)$ is contained in $H^1(L/F, G) = \ker(H^1(F, G) \rightarrow H^1(L, G_L))$. Hence

$$\begin{aligned} \mathbb{III}_{\mathcal{P}}(F, G) &= \ker(H^1(F, G(F^{\text{sep}})) \rightarrow \prod_{\xi \in \mathcal{V}(\Gamma_F)} H^1(F_\xi, G(F_\xi^{\text{sep}}))) \\ &= \ker(H^1(L/F, G(L)) \rightarrow \prod_{\xi \in \mathcal{V}(\Gamma_F)} H^1(F_\xi, G(F_\xi^{\text{sep}}))) \\ &= \ker(H^1(L/F, G(L)) \rightarrow \prod_{\substack{\xi \in \mathcal{V}(\Gamma_F) \\ \zeta \in \mathcal{V}_\xi(\Gamma_L)}} H^1(L_\zeta/F_\xi, G(L_\zeta))), \end{aligned}$$

as asserted. \square

We then obtain the following sufficient criterion for the vanishing of $\mathbb{III}(F, G)$ if G is a finite étale group scheme over a semi-global field F :

Proposition 4.5. *Let F be a semi-global field over a complete discretely valued field with valuation ring R , and let \mathcal{X} be a normal crossings model of F over R . Let G be a finite étale group scheme over F , and let L be a finite Galois extension of F such that G_L is a constant finite group. Let \mathcal{Y} be the normalization of \mathcal{X} in L , and consider its associated reduction graph Γ_L . If Γ_L is a tree, then $\mathbb{III}(F, G)$ is trivial.*

Proof. Let $X = \mathcal{X}_k^{\text{red}}$. By Proposition 1.1, $\mathbb{III}(F, G) = \mathbb{III}_X(F, G) = \bigcup \mathbb{III}_{\mathcal{P}}(F, G)$, where the right hand side is an increasing union over the non-empty finite sets \mathcal{P} of closed points of X that contain all the singular points of X . So it suffices to show that $\mathbb{III}_{\mathcal{P}}(F, G)$ is trivial for all such sets \mathcal{P} that are sufficiently large; in particular, we may now restrict attention to sets \mathcal{P} that satisfy the condition (*) in the discussion before Lemma 4.4.

As above, let $\Delta = \text{Gal}(L/F)$ and $\Delta_\zeta = \text{Gal}(L_\zeta/F_\xi)$ for $\zeta \in \mathcal{V}(\Gamma_L)$ lying over $\xi \in \mathcal{V}(\Gamma_F)$. For \mathcal{P} as above, Lemma 4.4 applies since Γ_L is a tree, and yields

$$\mathbb{III}_{\mathcal{P}}(F, G) = \ker(H^1(\Delta, G(L)) \rightarrow \prod_{\substack{\xi \in \mathcal{V}(\Gamma_F) \\ \zeta \in \mathcal{V}_\xi(\Gamma_L)}} H^1(\Delta_\zeta, G(L_\zeta))),$$

where each map $H^1(\Delta, G(L)) \rightarrow H^1(\Delta_\zeta, G(L_\zeta))$ is induced by restriction.

As noted before Lemma 4.4, the action of Δ on \mathcal{Y} stabilizes \mathcal{P}' and thus also the associated set \mathcal{U}' . Hence it acts on Γ_L without inversion, in the sense of [Ser03, Section I.3.1]; i.e., no element of Δ can interchange two adjacent vertices of the bipartite tree Γ_L . But by [Ser03, Theorem I.6.1.15], any action of a finite group on a tree without inversion has a global fixed point; i.e., there is a vertex that is fixed by the entire group. So there exists $\zeta \in \mathcal{V}(\Gamma_L)$ such that $\Delta_\zeta = \Delta$. But G_L is a constant finite group, and so the inclusion $G(L) \rightarrow G(L_\zeta)$ is an equality; hence $H^1(\Delta, G(L)) = H^1(\Delta_\zeta, G(L_\zeta))$. Thus $\mathbb{III}_{\mathcal{P}}(F, G)$ is trivial, as required. \square

Corollary 4.6. *Let R, F, \mathcal{X} be as in Proposition 4.5, and let G be a finite étale group scheme over R . Suppose that the reduction graph associated to \mathcal{X} is a monotonic tree. Then $\text{III}(F, G)$ is trivial.*

Proof. By smoothness, there is an étale algebra S/R that splits G . Let ℓ be the residue field of S , and let $L = F \otimes_R S$. Thus G_ℓ and G_L are finite constant groups. Also, $\mathcal{X}_S := \mathcal{X} \times_R S$ is finite étale over \mathcal{X} ; it is the normalization of \mathcal{X} in L ; and it is a normal crossings model of its function field L over S . The reduced closed fiber $(\mathcal{X}_S)_\ell^{\text{red}}$ of \mathcal{X}_S is the base change $X_\ell := X \times_k \ell$ of the reduced closed fiber X of \mathcal{X} (with k the residue field of R); and $X_\ell \rightarrow X$ is étale since S/R is. The reduction graph of \mathcal{X}_S , which is the graph associated to X_ℓ , is a tree because the reduction graph of \mathcal{X} is a monotonic tree (see [CHHKPS20, Proposition 7.6(a)]). By Proposition 4.5, it follows that $\text{III}(F, G)$ is trivial (where we view G as a finite étale group scheme over F here). \square

Remark 4.7. The conclusion of Corollary 4.6 can fail if the reduction graph is not a monotonic tree. For example, suppose the closed fiber of \mathcal{X} consists of two copies of \mathbb{P}_k^1 that meet transversally at a single point P having a bigger residue field k' . (Such a model is given in Example 7.7.) Suppose that G is a finite étale group scheme over R such that $G(k)$ is trivial but $G(k')$ is non-trivial. Take $\mathcal{P} = \{P\}$, so that we then have $\mathcal{U} = \{U_1, U_2\}$ and $\mathcal{B} = \{\wp_1, \wp_2\}$, where U_1, U_2 are affine open subsets of projective k -lines, and \wp_1, \wp_2 are the branches at P . Since G is a finite étale group scheme, $G(A) = G(L)$ for an R -algebra A that is a domain with fraction field L ; and $G(A) = G(A/I)$ if A is I -adically complete. In particular, $G(F_P) = G(\widehat{R}_P) = G(\kappa(P)) = G(k')$, and $G(F_{U_i}) = G(\widehat{R}_{U_i}) = G(k[U_i]) = G(k(x)) = G(k)$. Similarly, $G(F_{\wp_i}) = G(\widehat{R}_{\wp_i}) = G(\widehat{R}_{\wp_i}/\wp_i) = G(\widehat{R}_P/\wp_i) = G(\kappa(P)) = G(k')$. By the double coset description of $\text{III}_{\mathcal{P}}(F, G) \subseteq \text{III}(F, G)$ (Equation (4) of Section 1.2), we have that $\text{III}(F, G) \supseteq \text{III}_{\mathcal{P}}(F, G) = (G(k') \times G(k'))/G(k') \neq 1$.

By combining Corollary 4.3 with Proposition 4.5, we will obtain Theorem 4.9 below, concerning local-global principles for groups that need not be connected. First we prove a lemma that is analogous to [HHK15, Corollary 2.6], though that result applied to $\text{III}_{\mathcal{P}}(F, G)$ rather than to the situation of discrete valuations.

Lemma 4.8. *Let F be a field equipped with a set Ω of discrete valuations. Let G be a linear algebraic group over F , with identity component G^0 , and write $\bar{G} = G/G^0$. Suppose that the associated obstruction sets $\text{III}_{\Omega}(F, G^0)$ and $\text{III}_{\Omega}(F, \bar{G})$ are both trivial. If $G(F_v) \rightarrow \bar{G}(F_v)$ is surjective for all discrete valuations $v \in \Omega$, then $\text{III}_{\Omega}(F, G)$ is trivial.*

Proof. We have the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccc}
& & 1 & & 1 & & 1 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & \text{III}_{\Omega}(F, G^0) & & \text{III}_{\Omega}(F, G) & & \text{III}_{\Omega}(F, \bar{G}) & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & H^1(F, G^0) & \longrightarrow & H^1(F, G) & \longrightarrow & H^1(F, \bar{G}) & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
\prod_v G(F_v) & \longrightarrow & \prod_v \bar{G}(F_v) & \longrightarrow & \prod_v H^1(F_v, G^0) & \longrightarrow & \prod_v H^1(F_v, G) & \longrightarrow & \prod_v H^1(F_v, \bar{G})
\end{array}$$

The assertion now follows by a diagram chase. Namely, if $\xi \in \text{III}_\Omega(F, G) \subseteq H^1(F, G)$, then the commutativity of the right hand square implies that the image of ξ in $H^1(F, \bar{G})$ lies in $\text{III}_\Omega(F, \bar{G})$ and hence is trivial. Thus ξ is the image of some $\xi^0 \in H^1(F, G^0)$. Since $\xi \in \text{III}_\Omega(F, G)$, the image $(\xi_v^0)_v$ of ξ^0 in $\prod_v H^1(F_v, G^0)$ is in the kernel of the map to $\prod_v H^1(F_v, G)$; hence it is the image of an element of $\prod_v \bar{G}(F_v)$. By the surjectivity hypothesis, it follows that $(\xi_v^0)_v$ is trivial. Hence $\xi^0 \in \text{III}_\Omega(F, G^0)$, which is trivial. Thus ξ is trivial. \square

Theorem 4.9. *Let F be a semi-global field over a complete discrete valuation ring R , with a normal crossings model \mathcal{X} . Assume that the closed fiber of \mathcal{X} is reduced and the reduction graph is a monotonic tree. Suppose that either*

- (i) G is a smooth affine group scheme over R such that G^0 is reductive and $\mu(G^0)$ is étale; or
- (ii) $R = k[[t]]$ for some field k of characteristic zero, and G is a linear algebraic group over k .

Let $\bar{G} = G/G^0$. We then have the following conclusions:

- (a) Both $\text{III}(F, G^0)$ and $\text{III}(F, \bar{G})$ are trivial.
- (b) If $G(F_v) \rightarrow \bar{G}(F_v)$ is surjective for all divisorial discrete valuations v , then $\text{III}(F, G)$ is trivial.

Proof. By Corollary 4.6, $\text{III}(F, \bar{G})$ is trivial. But $\text{III}(F, G^0)$ is also trivial, by Theorem 4.2 in case (i) and by Corollary 4.3 in case (ii). So part (a) holds; and part (b) then follows by applying Lemma 4.8 with Ω equal to the set of divisorial discrete valuations on F . \square

Note that the surjectivity hypothesis of Theorem 4.9(b) is satisfied in particular if k is algebraically closed, or if the morphism $G \rightarrow G/G^0$ has a section. Even without that surjectivity hypothesis, we can still obtain a local-global principle, though for $\text{III}_\varphi(F, G)$, or equivalently $\text{III}_X(F, G)$, where X is the reduced closed fiber of a normal crossing model (see Proposition 1.1(c)). We first prove the next lemma.

Lemma 4.10. *Let \mathcal{X} be a normal crossings model of a semi-global field F over a complete discrete valuation ring R . Let $\mathcal{P}, \mathcal{U}, \mathcal{B}$ be as in Section 1.1, with associated reduction graph Γ . Suppose that either*

- (i) G is a smooth affine group scheme over R such that G^0 is reductive; or
- (ii) $R = k[[t]]$ for some field k of characteristic zero, and G is a linear algebraic group over k .

Let $\varphi \in \mathcal{B}$ be a branch of $\mathcal{X}_k^{\text{red}}$ at $P \in \mathcal{P}$, lying on $U \in \mathcal{U}$. Let G_1 be a connected component of G , and suppose that $G_1(F_\varphi)$ is non-empty. Then $G_1(F_P)$ is also non-empty; and if $\kappa(U) = \kappa(P)$ then $G_1(F_U)$ is non-empty.

Proof. We first consider case (i). Since G^0 is reductive over R , it is also reductive over the discrete valuation ring \widehat{R}_φ , whose fraction field is F_φ . It follows that the kernel of $H^1(\widehat{R}_\varphi, G^0) \rightarrow H^1(F_\varphi, G^0)$ is trivial (see [Nis84] and [Gil94, Théorème I.1.2.2]). Hence the G^0 -torsor G_1 is trivial over \widehat{R}_φ ; i.e., it has an \widehat{R}_φ -point. Reducing that point modulo the maximal ideal φ of \widehat{R}_φ , we obtain an E -point of G_1 , where E is the fraction field of the complete local ring A of \bar{U} at P . Since A is a discrete valuation ring, applying the above cited result a second time shows that G_1 has an A -point. Reducing that point modulo the

maximal ideal of A yields a $\kappa(P)$ -point of G_1 . By formal smoothness, the latter lifts to an \widehat{R}_P -point of G_1 . Thus there is also an F_P -point of G_1 , as asserted.

If in addition $\kappa(U) = \kappa(P)$, the above $\kappa(P)$ -point of G_1 is thus also a $\kappa(U)$ -point of G_1 , and that induces a $k[U]$ -point of G_1 . By formal smoothness, this lifts to an \widehat{R}_U -point of G_1 , and so there is an F_U -point on G_1 .

We next consider case (ii). Let $\mathcal{R}_u(G)$ be the unipotent radical of G (or equivalently, of G^0), and let $\tilde{G} = G/\mathcal{R}_u(G)$. Since G_1 has an F_φ -point g_φ , the image \tilde{G}_1 of G_1 in \tilde{G} also contains an F_φ -point, viz. the image \tilde{g}_φ of g_φ . But the identity component \tilde{G}^0 of \tilde{G} is $G^0/\mathcal{R}_u(G)$, which is reductive. So by case (i) of the lemma, \tilde{G}_1 contains an F_P -point \tilde{g}_P , and also an F_U -point \tilde{g}_U if $\kappa(U) = \kappa(P)$. Since $H^1(F_\xi, \mathcal{R}_u(G))$ is trivial for $\xi = P, U$ (using $\text{char}(k) = 0$, as in the proof of Corollary 4.3), it follows from the cohomology exact sequence that the map $G(F_\xi) \rightarrow \tilde{G}(F_\xi)$ is surjective. Pick a point $g_P \in G(F_P)$ that lies over \tilde{g}_P ; and also pick $g_U \in G(F_U)$ that lies over \tilde{g}_U if $\kappa(U) = \kappa(P)$. Since \tilde{g}_P (resp., \tilde{g}_U) lies on the connected component \tilde{G}_1 of $\tilde{G} = G/\mathcal{R}_u(G)$, and since the group $\mathcal{R}_u(G)$ is connected, it follows that g_P (resp., g_U) lies on G_1 , as asserted. \square

We now obtain the following local-global result for groups that need not be connected, but which are given over a field of characteristic zero.

Theorem 4.11. *Let G be a linear algebraic group over a field k of characteristic zero, let F be a semi-global field over $K := k((t))$, and let \mathcal{X} be a normal crossings model of F over $R := k[[t]]$. Assume that the closed fiber X of \mathcal{X} is reduced, and that the associated reduction graph is a monotonic tree. Then $\text{III}_X(F, G)$ is trivial.*

Proof. It suffices by Proposition 1.1(c) to show that if $\mathcal{P} \subset X$ is a finite set as in Section 1.1 then $\text{III}_{\mathcal{P}}(F, G)$ is trivial. Given \mathcal{P} , we have associated sets \mathcal{U}, \mathcal{B} , along with a reduction graph Γ , which is a monotonic tree. As explained before Remark 3.3, we choose the root of this monotonic tree to be an element $U_0 \in \mathcal{U}$. For each element of $\mathcal{P} \cup \mathcal{U}$, we may consider its distance $n \geq 0$ from U_0 in the associated reduction graph Γ ; here $n = 0$ for U_0 itself.

As in Section 1.2, write $F_{\mathcal{U}} = \prod_{U \in \mathcal{U}} F_U$, $F_{\mathcal{P}} = \prod_{P \in \mathcal{P}} F_P$, and $F_{\mathcal{B}} = \prod_{\varphi \in \mathcal{B}} F_\varphi$, and identify $\text{III}_{\mathcal{P}}(F, G)$ with the double coset space $G(F_{\mathcal{U}}) \backslash G(F_{\mathcal{B}}) / G(F_{\mathcal{P}})$, by [HHK15, Corollary 3.6]. It suffices to show that $G(F_{\mathcal{U}}) \backslash G(F_{\mathcal{B}}) / G(F_{\mathcal{P}})$ consists just of the trivial double coset. Since $\text{III}_{\mathcal{P}}(F, G^0) = G^0(F_{\mathcal{U}}) \backslash G^0(F_{\mathcal{B}}) / G^0(F_{\mathcal{P}})$ is trivial by Corollary 4.3, it suffices to show that every element of $G(F_{\mathcal{U}}) \backslash G(F_{\mathcal{B}}) / G(F_{\mathcal{P}})$ contains a representative $(g_\varphi^0)_{\varphi \in \mathcal{B}}$ such that each $g_\varphi^0 \in G^0(F_\varphi)$. Equivalently, we will show that for every $(g_\varphi)_{\varphi \in \mathcal{B}} \in G(F_{\mathcal{B}})$, there exist elements $g_P \in G(F_P)$ and $g_U \in G(F_U)$ for all $P \in \mathcal{P}$ and $U \in \mathcal{U}$ such that $g_U^{-1} g_\varphi g_P \in G^0(F_\varphi)$ for each branch $\varphi \in \mathcal{B}$ at $P \in \mathcal{P}$ lying on $U \in \mathcal{U}$. We will construct these elements g_ξ , for all $\xi \in \mathcal{U} \cup \mathcal{P}$, by induction on the distance in Γ from the root U_0 to ξ .

If the distance is zero, then $\xi = U_0$, and we set $g_\xi = g_{U_0} = 1 \in G^0(F_{U_0})$. Now take $i \geq 1$ and assume that g_ξ has been defined for all $\xi \in \mathcal{U} \cup \mathcal{P}$ of distance less than i from the root. Let ξ be a vertex of Γ that has distance i from the root U_0 .

If i is odd, then ξ is an element $P \in \mathcal{P}$. In this case the vertex of Γ that is adjacent to P and that lies between P and the root is an element $U \in \mathcal{U}$ of distance $i - 1 \geq 0$ from the root. By the inductive hypothesis, $g_U \in G(F_U)$ has been defined. Let g_P be an element of $G(F_P)$ that lies on the same connected component of G as $g_U^{-1} g_U \in G(F_U)$, where $\varphi \in \mathcal{B}$ is the branch at P on U ; such an element exists by Lemma 4.10(ii). Thus $g_U^{-1} g_\varphi g_P \in G^0(F_\varphi)$.

On the other hand, if i is even, then ξ is an element $U \in \mathcal{U}$, and the adjacent vertex between ξ and the root is some $P \in \mathcal{P}$ of distance $i-1$ from U_0 . The element $g_P \in G(F_P)$ has then been inductively defined, and again by Lemma 4.10(ii) there is an element $g_U \in G(F_U)$ that lies on the same connected component of G as $g_\varphi g_P \in G(F_\varphi)$, where $\varphi \in \mathcal{B}$ is the branch at P on U . (Here $\kappa(U) \subseteq \kappa(P)$ since $P \in \bar{U}$, and the reverse containment holds because Γ is monotonic. So Lemma 4.10 indeed applies in this case.) Again, we have $g_U^{-1} g_\varphi g_P \in G^0(F_\varphi)$, as asserted. \square

Remark 4.12. Concerning the relationship between Theorem 4.9(b) and Theorem 4.11, it is not known in general whether the containment $\text{III}_X(F, G) \rightarrow \text{III}(F, G)$ is an equality. Theorem 8.10(ii) of [HHK15] provides a set of conditions under which equality holds, for a linear algebraic group G over F : that G^0 is a reductive group over the model \mathcal{X} ; that $\bar{G} := G/G^0$ is a constant finite group scheme; and that moreover the homomorphism $G((F_P)_v) \rightarrow \bar{G}((F_P)_v)$ is surjective for every point $P \in X$ and every discrete valuation v on F_P .

5. A LOWER BOUND ON THE TATE-SHAFAREVICH SET

In this section, we give a combinatorial description of an explicit quotient set of the Tate-Shafarevich set associated to a reductive group G over the ground ring R of our semi-global field F . This “lower bound” on $\text{III}(F, G)$, which is given in Proposition 5.2(b), is in the context of a model \mathcal{X} such that each irreducible component of the reduced closed fiber is isomorphic to a projective line (over some extension of the residue field k). In that situation, it will afterwards be used to obtain counterexamples to a local-global principle in Section 7.

Our result on $\text{III}(F, G)$ below relies on the notion of R-equivalence (see the discussion just before Proposition 2.4). By Theorems A.10 and A.14 of the Appendix, if A is a regular local ring of dimension at most two, with fraction field L and residue field ℓ , and for any reductive group G over A , there is an associated homomorphism on R-equivalence classes $\text{sp}_A : G(L)/R \rightarrow G(\ell)/R$, known as the *specialization map*, which is compatible with the natural reduction map $G(A) \rightarrow G(\ell)$. Here the compatibility condition is that the compositions $G(A) \rightarrow G(L) \rightarrow G(L)/R \xrightarrow{\text{sp}_A} G(\ell)/R$ and $G(A) \rightarrow G(\ell) \rightarrow G(\ell)/R$ agree, where $G(A) \rightarrow G(\ell)$ is the natural pullback map. If A is complete, the specialization map is surjective since $G(A) \rightarrow G(\ell)$ is.

We preserve the notation from Section 1.1, with F a semi-global field over a complete discrete valuation ring R having fraction field K , and with \mathcal{X} a normal crossings model of F together with sets $\mathcal{P}, \mathcal{U}, \mathcal{B}$. As in Section 2, we write $\kappa(P)$ for the residue field at a point $P \in \mathcal{P}$ and $\kappa(U)$ for the constant field of $U \in \mathcal{U}$. We write $k(U)$ for the function field of $U \in \mathcal{U}$ and $k[U]$ for the coordinate ring of U ; these each contain $\kappa(U)$. If φ is a branch on $U \in \mathcal{U}$ at $P \in \mathcal{P}$, we define $\kappa(\varphi)$ as $\kappa(\varphi) := \kappa(P)$. This is not to be confused with the residue field k_φ of the discrete valuation ring $\widehat{R}_\varphi \subset F_\varphi$.

Lemma 5.1. *With notation as above, let G be a reductive group over R .*

(a) *The specialization maps on regular local rings induce specialization maps*

$$\theta_\varphi : G(F_\varphi) \rightarrow G(\kappa(\varphi))/R, \quad \theta_P : G(F_P) \rightarrow G(\kappa(P))/R, \quad \theta_U : G(F_U) \rightarrow G(k(U))/R,$$

for each $\varphi \in \mathcal{B}$, $P \in \mathcal{P}$, $U \in \mathcal{U}$, which are homomorphisms that factor through $G(F_\varphi)/R$, $G(F_P)/R$, and $G(F_U)/R$, respectively.

- (b) If $\wp \in \mathcal{B}$ is a branch at $P \in \mathcal{P}$, then the maps θ_P and θ_\wp are surjective, and θ_P is the restriction of θ_\wp to $G(F_P)$.
- (c) The restriction of θ_\wp to $G(F_U)$ factors through θ_U , if $\wp \in \mathcal{B}$ is a branch on $U \in \mathcal{U}$.
- (d) If $U \in \mathcal{U}$ is isomorphic to an open subset of a projective line over a finite field extension of k , then the natural map $G(\kappa(U))/\mathbb{R} \rightarrow G(k(U))/\mathbb{R}$ is an isomorphism, and θ_U is surjective.

Proof. The map θ_P is defined to be the composition of $G(F_P) \rightarrow G(F_P)/\mathbb{R}$ with the specialization map of Theorem A.14 with respect to the local ring \widehat{R}_P ; and this is surjective since \widehat{R}_P is complete. If $\wp \in \mathcal{B}$ is a branch at $P \in \mathcal{P}$, then the map θ_\wp is defined to be the composition

$$G(F_\wp) \rightarrow G(F_\wp)/\mathbb{R} \rightarrow G(k(U)_P)/\mathbb{R} \rightarrow G(\kappa(P))/\mathbb{R} = G(\kappa(\wp))/\mathbb{R},$$

where the second and third maps in this composition are specializations, and $k(U)_P$ is the completion of $k(U)$ at P (which is the same as the residue field k_\wp of F_\wp). The first map is trivially surjective, and the second and third maps are surjective because they are specializations with respect to complete discrete valuation rings. By Theorem A.14(a) together with the definitions of θ_P and θ_\wp , the map θ_P coincides with the composition $G(F_P) \hookrightarrow G(F_\wp) \xrightarrow{\theta_\wp} G(\kappa(\wp))/\mathbb{R} = G(\kappa(P))/\mathbb{R}$. Note that this composition factors through $G(F_P)/\mathbb{R}$ and $G(F_\wp)/\mathbb{R}$ by the properties of the specialization map in Theorem A.14.

For $U \in \mathcal{U}$, let η be the generic point of the curve U , so that the localization $\widehat{R}_{U,\eta}$ of \widehat{R}_U at η is a discrete valuation ring with fraction field F_U , residue field $k(U)$, and completion \widehat{R}_η . We define the map θ_U as the composition of the natural map $G(F_U) \rightarrow G(F_U)/\mathbb{R}$ with the specialization map $G(F_U)/\mathbb{R} \rightarrow G(k(U))/\mathbb{R}$ with respect to the discrete valuation ring $\widehat{R}_{U,\eta}$. By Remark A.11(b), θ_U is the same as the composition $G(F_U) \rightarrow G(F_\eta) \xrightarrow{\text{sr}} G(k(U))/\mathbb{R}$, where F_η is the fraction field of \widehat{R}_η and the second map is specialization with respect to \widehat{R}_η . Note that if $\wp \in \mathcal{B}$ is a branch on U , then the restriction of θ_\wp to $G(F_U)$ is the composition of θ_U with the map $G(k(U))/\mathbb{R} \rightarrow G(k(U)_P)/\mathbb{R} \rightarrow G(\kappa(P))/\mathbb{R}$, where the latter map is given by specialization as above. This completes the proof of parts (a)-(c).

For part (d), suppose that U is isomorphic to an open subset of a projective line $\mathbb{P}_{k'}^1$ over some finite extension k' of k . Consider the natural map $\alpha : G(\kappa(U))/\mathbb{R} \rightarrow G(k(U))/\mathbb{R}$ that is induced by the inclusion of $k' = \kappa(U)$ into $k(U) \simeq k'(x)$, where x is a coordinate function on the affine line. Then $\beta\alpha$ is the identity on $G(k(U))/\mathbb{R}$, where $\beta : G(k(U))/\mathbb{R} \rightarrow G(\kappa(U))/\mathbb{R}$ is given by specialization at a k' -point of \bar{U} . Hence α is injective. For surjectivity, we want that every element of $G(k'(x))$ is \mathbb{R} -equivalent to an element in the image of $G(k')$. An element of $G(k'(x))$ is given by a map $g : S \rightarrow G$, for S an open subset of the x -line over k' that we may assume contains $x = 0$ (after making a change of coordinates on the line). View $S \times S$, with coordinates s, x , as an open subset of a family of x -lines parametrized by the s -line; and consider the rational map $f : S \times S \dashrightarrow G$ given by $f(s, x) = g(sx)$. The restriction of f to $s = 0$ is the constant morphism $S \rightarrow G$ with value $g(0) \in G(k') \subset G(k'(x))$, and the restriction of f to $s = 1$ is $g : S \rightarrow G$. So f defines a rational map $\mathbb{A}_{k'(x)}^1 \dashrightarrow G_{k'(x)}$ that is defined at $s = 0, 1$ and connects g to $g(0)$, proving that α is surjective and hence an isomorphism.

Still under the hypothesis that U is isomorphic to an open subset of a projective line $\mathbb{P}_{k'}^1$, we then want to show that $\theta_U : G(F_U) \rightarrow G(k(U))/\mathbb{R} \simeq G(k')/\mathbb{R}$ is surjective, where

$k' = \kappa(U)$. Every class in $G(k(U))/R$ is represented by an element of $G(k')$ and hence by an element of $G(k[U])$, where we view $G(k') \subseteq G(k[U]) \subseteq G(k(U))$. Also, the reduction map $G(\widehat{R}_U) \rightarrow G(k[U])$ is surjective by formal smoothness. But the composition $G(\widehat{R}_U) \rightarrow G(k[U]) \rightarrow G(k(U)) \rightarrow G(k(U))/R$ is the restriction of θ_U to $G(\widehat{R}_U)$, since the restriction of $G(F_\eta) \xrightarrow{\text{SR}} G(k(U))/R$ to the subgroup $G(\widehat{R}_\eta)$ is induced by the reduction map. Hence θ_U is surjective under the additional hypothesis on U . \square

By Theorem 3.2 above, together with the double coset formula [HHK15, Corollary 3.6], we have bijections

$$\mathbb{I}\mathbb{I}(F, G) \simeq \mathbb{I}\mathbb{I}_{\mathcal{P}}(F, G) \simeq \prod_{U \in \mathcal{U}} G(F_U) \setminus \prod_{\wp \in \mathcal{B}} G(F_\wp) / \prod_{P \in \mathcal{P}} G(F_P).$$

The first part of the next result shows that this double coset description is compatible with the specialization map, while the second part gives our explicit quotient of $\mathbb{I}\mathbb{I}(F, G)$.

Proposition 5.2. *With notation as above, let G be a reductive group over R .*

(a) *The maps θ_\wp together define a surjection*

$$\theta_0 : \mathbb{I}\mathbb{I}(F, G) \rightarrow \prod_{U \in \mathcal{U}} (G(k(U))/R) \setminus \prod_{\wp \in \mathcal{B}} (G(\kappa(\wp))/R) / \prod_{P \in \mathcal{P}} (G(\kappa(P))/R)$$

via the above identification

$$\mathbb{I}\mathbb{I}(F, G) \simeq \prod_{U \in \mathcal{U}} G(F_U) \setminus \prod_{\wp \in \mathcal{B}} G(F_\wp) / \prod_{P \in \mathcal{P}} G(F_P).$$

(b) *The natural surjection*

$$\begin{array}{c} \prod_{U \in \mathcal{U}} (G(\kappa(U))/R) \setminus \prod_{\wp \in \mathcal{B}} (G(\kappa(\wp))/R) / \prod_{P \in \mathcal{P}} (G(\kappa(P))/R) \\ \downarrow \pi \\ \prod_{U \in \mathcal{U}} (G(k(U))/R) \setminus \prod_{\wp \in \mathcal{B}} (G(\kappa(\wp))/R) / \prod_{P \in \mathcal{P}} (G(\kappa(P))/R), \end{array}$$

is an isomorphism if each $U \in \mathcal{U}$ is an open subset of a projective line over a finite field extension of k . Hence in that case θ_0 becomes identified with the surjection

$$\theta = \pi^{-1} \circ \theta_0 : \mathbb{I}\mathbb{I}(F, G) \rightarrow \prod_{U \in \mathcal{U}} (G(\kappa(U))/R) \setminus \prod_{\wp \in \mathcal{B}} (G(\kappa(\wp))/R) / \prod_{P \in \mathcal{P}} (G(\kappa(P))/R).$$

Proof. As \wp ranges over \mathcal{B} , the specialization maps θ_\wp in Lemma 5.1(a) together define a homomorphism $\prod_{\wp} G(F_\wp) \rightarrow \prod_{\wp} G(\kappa(\wp))/R$. Similarly, there are homomorphisms $\prod_U G(F_U) \rightarrow \prod_U G(k(U))/R$ and $\prod_P G(F_P) \rightarrow \prod_P G(\kappa(P))/R$. By parts (b) and (c) of Lemma 5.1, these maps are compatible. So together they induce a homomorphism θ_0 on the double coset space $\prod_{U \in \mathcal{U}} G(F_U) \setminus \prod_{\wp \in \mathcal{B}} G(F_\wp) / \prod_{P \in \mathcal{P}} G(F_P)$, which, as above, we identify with $\mathbb{I}\mathbb{I}(F, G)$, via Theorem 3.2 and [HHK15, Corollary 3.6]. The map θ_0 is surjective by the surjectivity of θ_\wp in Lemma 5.1(b). This proves part (a). (Note here that in writing these quotients, we do not assume that the maps from $\prod_U G(k(U))/R$ and $\prod_P G(\kappa(P))/R$ to $\prod_{\wp} G(\kappa(\wp))/R$ are injective; just that we are quotienting by their images in $\prod_{\wp} G(\kappa(\wp))/R$.)

For part (b), the vertical map is induced by the identity map on $\prod_{\varphi \in \mathcal{B}} (G(\kappa(\varphi))/R)$ via the inclusion $\kappa(U) \hookrightarrow k(U)$, and is therefore surjective. If each $U \in \mathcal{U}$ is an open subset of a projective line, then the stated properties follow from Lemma 5.1(d). \square

Remark 5.3. (a) The above maps θ_0 and θ on

$$\text{III}(F, G) \simeq \prod_{U \in \mathcal{U}} G(F_U) \setminus \prod_{\varphi \in \mathcal{B}} G(F_\varphi) / \prod_{P \in \mathcal{P}} G(F_P)$$

factor through $\prod_U (G(F_U)/R) \setminus \prod_\varphi (G(F_\varphi)/R) / \prod_P (G(F_P)/R)$, since each of the maps θ_φ , θ_P , θ_U factors through the corresponding group of R-equivalence classes by Lemma 5.1(a).

(b) In the case that G is a torus, the maps $G(F_P)/R \rightarrow G(\kappa(P))/R$ and $G(F_\varphi)/R \rightarrow G(\kappa(\varphi))/R$ are isomorphisms, and so is the surjection θ_0 in Proposition 5.2(a). This follows from [CHHKPS20, Proposition 4.6(a,b) and Lemma 6.2] (which are given in terms of flasque resolutions), together with [CTS77, Théorème 2, p.199] (which interprets $T(k)/R$ in terms of flasque resolutions if T is a torus; see also [CTS87, Theorem 3.1]). As a result, if G is a torus, then the vertical map in Proposition 5.2(b) defines an epimorphism

$$\prod_{U \in \mathcal{U}} (G(\kappa(U))/R) \setminus \prod_{\varphi \in \mathcal{B}} (G(\kappa(\varphi))/R) / \prod_{P \in \mathcal{P}} (G(\kappa(P))/R) \rightarrow \text{III}(F, G),$$

which is an isomorphism (inverse to θ) if each $U \in \mathcal{U}$ is an open subset of a projective line over a finite field extension of k . See [CHHKPS20, Proposition 6.3(a,c)].

In the situation of Proposition 5.2(b), if $g \in \prod_{\varphi \in \mathcal{B}} G(F_\varphi)$, we will write $\theta(g)$ for $\theta([g])$, where $[g]$ is the class of g in $\prod_U G(F_U) \setminus \prod_\varphi G(F_\varphi) / \prod_P G(F_P) \simeq \text{III}(F, G)$. This is the same as the class of $(\theta_\varphi(g))_\varphi$ in $\prod_U (G(\kappa(U))/R) \setminus \prod_\varphi (G(\kappa(\varphi))/R) / \prod_P (G(\kappa(P))/R)$.

Recall (from Section 2) that if φ is a branch on U at P , there is the subgroup $G_s(\widehat{R}_\varphi) = G(\widehat{R}_P)G_\varphi(\widehat{R}_\varphi) \subseteq G(\widehat{R}_\varphi)$ of specializable elements, together with the specialization map $\Theta_\varphi : G_s(\widehat{R}_\varphi) \rightarrow G(\kappa(\varphi))$ obtained by composing the reduction maps $G_s(\widehat{R}_\varphi) \rightarrow G(\widehat{R}_P/\varphi) \rightarrow G(\kappa(P)) = G(\kappa(\varphi))$. By Theorem A.14(c), the map $\Theta_\varphi : G_s(\widehat{R}_\varphi) \rightarrow G(\kappa(\varphi))$ lifts the restriction of $\theta_\varphi : G(F_\varphi) \rightarrow G(\kappa(\varphi))/R$ to $G_s(\widehat{R}_\varphi)$. Note also that by Proposition 2.4, for any $(\tilde{g}_\varphi) \in \prod_{\varphi \in \mathcal{B}} G(F_\varphi)$ there is a representative (g_φ) of the same double coset in $\text{III}(F, G)$ such that each g_φ is R-equivalent to \tilde{g}_φ and lies in $G_s(\widehat{R}_\varphi)$.

The next result will be useful in proving Theorem 6.3.

Corollary 5.4. *Let G be a reductive group over R , and suppose that every irreducible component of the reduced closed fiber of the model \mathcal{X} is isomorphic to a projective line over a finite field extension of k . Let $g, g' \in \prod_{\varphi \in \mathcal{B}} G(F_\varphi)$, with $g = (g_\varphi)_\varphi$, $g' = (g'_\varphi)_\varphi$. Suppose that $\theta(g) = \theta(g')$, with θ as in Proposition 5.2(b). Then there is a family $(\alpha_\xi)_{\xi \in \mathcal{U} \cup \mathcal{P}}$ with $\alpha_\xi \in G(F_\xi)$ such that*

$$\alpha_U g'_\varphi \alpha_P \in G_s(\widehat{R}_\varphi) \text{ and } \theta_\varphi(\alpha_U g'_\varphi \alpha_P) = \theta_\varphi(g_\varphi) \in G(\kappa(\varphi))/R$$

for each branch $\varphi \in \mathcal{B}$ at $P \in \mathcal{P}$ lying on $U \in \mathcal{U}$.

Proof. By hypothesis, there exist $g_U \in G(\kappa(U))/R$ for all $U \in \mathcal{U}$, and $g_P \in G(\kappa(P))/R$ for all $P \in \mathcal{P}$, such that $\theta_\varphi(g_\varphi) = g_U \theta_\varphi(g'_\varphi) g_P \in G(\kappa_\varphi)/R$ for every branch $\varphi \in \mathcal{B}$ at $P \in \mathcal{P}$ lying

on $U \in \mathcal{U}$. Since the morphisms θ_P and θ_U are surjective by Lemma 5.1(b),(d), there exist $\alpha_U \in G(F_U)$ and $\alpha'_P \in G(F_P)$ such that $\theta_U(\alpha_U) = g_U$ and $\theta_P(\alpha'_P) = g_P$ (where we identify $G(\kappa(U))/\mathbb{R}$ with $G(k(U))/\mathbb{R}$ via Lemma 5.1(d)). Thus $\theta_\varphi(\alpha_U g'_\varphi \alpha'_P) = \theta_\varphi(g_\varphi) \in G(\kappa(\varphi))/\mathbb{R}$ for each branch $\varphi \in \mathcal{B}$. By Proposition 2.4, there is an element $\tilde{g} = (\tilde{g}_P)_P \in \prod_{P \in \mathcal{P}} G(F_P)$ such that $(\alpha_U g'_\varphi \alpha'_P) \tilde{g}_P \in G_s(\widehat{R}_\varphi)$ and \tilde{g}_P is \mathbb{R} -trivial in $G(F_P)$, for every branch $\varphi \in \mathcal{B}$ at $P \in \mathcal{P}$ on $U \in \mathcal{U}$. Setting $\alpha_P = \alpha'_P \tilde{g}_P$ completes the proof, since $\theta_\varphi(\alpha_U g'_\varphi \alpha_P) = \theta_\varphi(\alpha_U g'_\varphi \alpha'_P) = \theta_\varphi(g_\varphi) \in G(\kappa(\varphi))/\mathbb{R}$ by the \mathbb{R} -triviality of \tilde{g}_P . \square

We can view the above corollary as saying that the adjustment of elements appearing in Proposition 2.4 can be done compatibly with θ .

6. AN EXACT COMPUTATION OF THE TATE-SHAFAREVICH SET

In this section, we build on the results of Section 5 to obtain an explicit computation of $\text{III}(F, G)$ under a somewhat stronger hypothesis; see Theorem 6.5. As before, we let G be a reductive group over a complete discrete valuation ring R ; we let \mathcal{X} be a normal crossings model of a semi-global field F over R ; and we take $\mathcal{P}, \mathcal{U}, \mathcal{B}$ as in Section 1.1. The additional condition that we will need below is the following:

Hypothesis 6.1. Assume that each irreducible component of the reduced closed fiber of \mathcal{X} is a projective line over the residue field k of R , and that each intersection point of components is defined over k . Thus $\kappa(P) = \kappa(U) = k$ for all $P \in \mathcal{P}$ and all $U \in \mathcal{U}$.

Using Theorem 6.5, we will obtain counterexamples to the local-global principle for semisimple simply connected groups in Section 7, beyond those that will arise via Proposition 5.2(b). These will include an example in which $\text{III}(F, G)$ is infinite.

6.1. A double coset description of the Tate-Shafarevich set. Under Hypothesis 6.1, we show in Theorem 6.3 that the surjective specialization map

$$\begin{aligned} \theta : \text{III}(F, G) &\rightarrow \prod_{U \in \mathcal{U}} (G(\kappa(U))/\mathbb{R}) \backslash \prod_{\varphi \in \mathcal{B}} (G(\kappa(\varphi))/\mathbb{R}) / \prod_{P \in \mathcal{P}} (G(\kappa(P))/\mathbb{R}) \\ &= \prod_{U \in \mathcal{U}} (G(k)/\mathbb{R}) \backslash \prod_{\varphi \in \mathcal{B}} (G(k)/\mathbb{R}) / \prod_{P \in \mathcal{P}} (G(k)/\mathbb{R}) \end{aligned}$$

defined in Proposition 5.2(b) is bijective, thereby providing a computable double coset description of $\text{III}(F, G)$. Afterwards, in Corollary 6.5, we make this more explicit, describing $\text{III}(F, G)$ just in terms of the number of loops in the reduction graph.

We first prove a lemma about specialization. Here we preserve the notation from Section 5.

Lemma 6.2. *Under Hypothesis 6.1, let G be a reductive group over R and let $g, g' \in \prod_{\varphi \in \mathcal{B}} G_s(\widehat{R}_\varphi)$, with $g = (g_\varphi)_\varphi, g' = (g'_\varphi)_\varphi$. Suppose that $\theta_\varphi(g_\varphi) = \theta_\varphi(g'_\varphi) \in G(\kappa(\varphi))/\mathbb{R}$ for all branches $\varphi \in \mathcal{B}$. Then there exists a finite set of closed points \mathcal{P}^* containing \mathcal{P} such that the set \mathcal{U}^* of connected components of the complement of \mathcal{P}^* in the reduced closed fiber has the following property: there exists a family $(\alpha_V)_{V \in \mathcal{U}^*}$ of elements $\alpha_V \in G(\widehat{R}_V)$ for $V \in \mathcal{U}^*$ such that if $P \in \mathcal{P}$ is in the closure of $V \in \mathcal{U}^*$ and φ is a branch on V at P , then $\alpha_V g'_\varphi \in G_s(\widehat{R}_\varphi)$ and $\Theta_\varphi(\alpha_V g'_\varphi) = \Theta_\varphi(g_\varphi) \in G(\kappa(\varphi))$.*

Note that in the above lemma, each element of \mathcal{U}^* is a dense open subset of an element of \mathcal{U} , and so the sets \mathcal{U} and \mathcal{U}^* are in natural bijection. Note also that the elements g_φ, g'_φ are defined only for $\varphi \in \mathcal{B}$; i.e., for φ a branch at a point $P \in \mathcal{P}$. For this reason, the conclusion of the above lemma considers just points $P \in \mathcal{P}$.

Proof. Let $U \in \mathcal{U}$ and let $\bar{U} \cap \mathcal{P} = \{P_1, \dots, P_n\}$, with $n \geq 1$. For each $i = 1, \dots, n$, let f_i be a rational function on \bar{U} that has a pole at P_i and is regular elsewhere, with zeros at each P_j for $j \neq i$. Let φ_i be the branch at P_i that lies on U .

Since $\theta_{\varphi_i}(g_{\varphi_i}) = \theta_{\varphi_i}(g'_{\varphi_i}) \in G(\kappa(\varphi_i))/R = G(k)/R$, the element $\Theta_{\varphi_i}(g_{\varphi_i})\Theta_{\varphi_i}(g'_{\varphi_i})^{-1} \in G(k)$ is R -trivial. So by [Gil97, Lemma II.1.1] there exists some $\gamma_{U,i} \in G(k(\mathbb{P}^1))$ such that $\gamma_{U,i}(0) = 1 \in G(k)$ and $\gamma_{U,i}(\infty) = \Theta_{\varphi_i}(g_{\varphi_i})\Theta_{\varphi_i}(g'_{\varphi_i})^{-1} \in G(k)$ (viewing $\gamma_{U,i}$ as a rational map from \mathbb{P}_k^1 to G). Pulling back $\gamma_{U,i}$ by f_i , we obtain an element $\beta_{U,i} \in G(k(\bar{U}))$ such that $\beta_{U,i}(P_i) = \Theta_{\varphi_i}(g_{\varphi_i})\Theta_{\varphi_i}(g'_{\varphi_i})^{-1} \in G(\kappa(\varphi_i)) = G(k)$ and $\beta_{U,i}(P_j) = 1 \in G(k)$ for $j \neq i$. Since $\beta_{U,i}$ is a rational map from \bar{U} to G that is defined at $\{P_1, \dots, P_n\}$, there is an affine open subset $W_i \subset \bar{U}$ containing $\{P_1, \dots, P_n\}$ such that $\beta_{U,i} \in G(k[W_i])$. Let $\tilde{\alpha}_{W_i} \in G(\widehat{R}_{W_i})$ be a lift of $\beta_{U,i}$. Note that \widehat{R}_{W_i} is contained in $\widehat{R}_{P_1}, \dots, \widehat{R}_{P_n}$, and also in \widehat{R}_{V_i} where $V_i = W_i \cap U$. Thus $\tilde{\alpha}_{W_i}$ is an element of $G(\widehat{R}_{V_i})$ and of $G(\widehat{R}_{P_j}) \subseteq G_s(\widehat{R}_{\varphi_j}) \subseteq G(\widehat{R}_{\varphi_j})$ for $j = 1, \dots, n$. By the definition of Θ_{φ_i} , we have

$$\Theta_{\varphi_i}(\tilde{\alpha}_{W_i}) = \beta_{U,i}(P_i) = \Theta_{\varphi_i}(g_{\varphi_i})\Theta_{\varphi_i}(g'_{\varphi_i})^{-1} \in G(k);$$

and for $j \neq i$ we have

$$\Theta_{\varphi_j}(\tilde{\alpha}_{W_i}) = \beta_{U,i}(P_j) = 1 \in G(k).$$

Let $W_U = W_1 \cap \dots \cap W_n \subseteq \bar{U}$; let $V_U = V_1 \cap \dots \cap V_n = W_U \cap U \subseteq U$; and let $\alpha_{W_U} = \prod_{i=1}^n \tilde{\alpha}_{W_i} \in G(\widehat{R}_{W_U})$. Then for each i , α_{W_U} is an element of $G(\widehat{R}_{V_i})$ and of $G(\widehat{R}_{P_j}) \subseteq G_s(\widehat{R}_{\varphi_j}) \subseteq G(\widehat{R}_{\varphi_j})$ for all j ; and we have

$$\Theta_{\varphi_i}(\alpha_{W_U}) = \Theta_{\varphi_i}(\tilde{\alpha}_{W_i}) = \Theta_{\varphi_i}(g_{\varphi_i})\Theta_{\varphi_i}(g'_{\varphi_i})^{-1} \in G(k).$$

After performing the above construction (separately) for each $U \in \mathcal{U}$, we have that

$$\Theta_\varphi(\alpha_{W_U} g'_\varphi) = \Theta_\varphi(\alpha_{W_U})\Theta_\varphi(g'_\varphi) = \Theta_\varphi(g_\varphi) \in G(\kappa(\varphi)) = G(k)$$

for every branch $\varphi \in \mathcal{B}$ at a point $P \in \mathcal{P}$ lying on an element $U \in \mathcal{U}$.

Finally, let $\mathcal{P}^* = \mathcal{X}_k^{\text{red}} \setminus (\bigcup_{U \in \mathcal{U}_0} V_U)$. Then the set \mathcal{U}^* of irreducible components of $\mathcal{X}_k^{\text{red}} \setminus \mathcal{P}^*$ is just the collection of the sets V_U , for $U \in \mathcal{U}$. For each $V = V_U \in \mathcal{U}^*$, write $\alpha_V = \alpha_{W_U}$. Let $P \in \mathcal{P}$ be in the closure of $V \in \mathcal{U}^*$ and φ is a branch on V at P ; thus $P \in \mathcal{P} \cap \bar{U}$. By the above display, $\Theta_\varphi(\alpha_V g'_\varphi) = \Theta_\varphi(g_\varphi) \in G(\kappa(\varphi))$, as asserted. \square

Recall (from the beginning of Section 4) that given a reductive group G , there is an associated finite group scheme $\mu(G)$.

Theorem 6.3. *Under Hypothesis 6.1, assume that the closed fiber \mathcal{X}_k is reduced. Let G be a reductive group over R such that $\mu(G)$ is étale (which is automatic if $\text{char}(k) = 0$). Then the natural map*

$$\theta : \text{III}(F, G) \rightarrow \prod_{U \in \mathcal{U}} (G(k)/R) \setminus \prod_{\varphi \in \mathcal{B}} (G(k)/R) / \prod_{P \in \mathcal{P}} (G(k)/R)$$

is a bijection.

Proof. By Proposition 5.2(b), θ is surjective. So it suffices to prove injectivity. Given elements ζ, ζ' of $\text{III}(F, G)$ with the same image under θ , we will show that $\zeta = \zeta'$.

We may identify $\text{III}(F, G)$ with $\prod_{U \in \mathcal{U}} G(F_U) \setminus \prod_{\varphi \in \mathcal{B}} G(F_\varphi) / \prod_{P \in \mathcal{P}} G(F_P)$ for a choice of \mathcal{P} and \mathcal{U} . Pick respective representatives $(g_\varphi)_\varphi, (g'_\varphi)_\varphi \in \prod_{\varphi \in \mathcal{B}} G(F_\varphi)$ for our elements $\zeta, \zeta' \in \text{III}(F, G)$. By Proposition 2.4, after adjusting the representative of ζ on the right by an element of $\prod_{P \in \mathcal{P}} G(F_P)$, we may assume that each $g_\varphi \in G_s(\widehat{R}_\varphi)$. Next, by Corollary 5.4, after adjusting the representative of ζ' on the left by an element of $\prod_{U \in \mathcal{U}} G(F_U)$ and on the right by an element of $\prod_{P \in \mathcal{P}} G(F_P)$, we may assume that $g'_\varphi \in G_s(\widehat{R}_\varphi)$ and $\theta_\varphi(g_\varphi) = \theta_\varphi(g'_\varphi) \in G(k)/R$ for all $\varphi \in \mathcal{B}$.

Now invoking Hypothesis 6.1, we may apply Lemma 6.2 to obtain a finite set \mathcal{P}^* containing \mathcal{P} , together with the associated set \mathcal{U}^* in bijection with \mathcal{U} , and elements $\alpha_V \in G(\widehat{R}_V)$ for $V \in \mathcal{U}^*$, satisfying the condition stated there. Let \mathcal{B}^* be the associated set of branches; this consists of the elements of \mathcal{B} together with an additional element at each point of \mathcal{P}^* that is not in \mathcal{P} . (Note that the closed fiber of our model is unbranched at these latter points.) Under the isomorphism

$$\text{III}(F, G) \simeq \prod_{U \in \mathcal{U}^*} G(F_U) \setminus \prod_{\varphi \in \mathcal{B}^*} G(F_\varphi) / \prod_{P \in \mathcal{P}^*} G(F_P),$$

the elements $\zeta, \zeta' \in \text{III}(F, G)$ respectively have representatives $(g_\varphi)_{\varphi \in \mathcal{B}^*}, (g'_\varphi)_{\varphi \in \mathcal{B}^*}$, where g_φ, g'_φ are as before for $\varphi \in \mathcal{B}$, and are equal to 1 if $\varphi \in \mathcal{B}^*$ is not in \mathcal{B} . (See the end of Section 1.2.) By the conclusion of Lemma 6.2, after replacing $\mathcal{P}, \mathcal{U}, \mathcal{B}$ by $\mathcal{P}^*, \mathcal{U}^*, \mathcal{B}^*$, and after adjusting the representative of ζ' on the left by an element of (the new) $\prod_{U \in \mathcal{U}^*} G(F_U)$, we may assume that $\Theta(g_\varphi) = \Theta(g'_\varphi) \in G(\kappa(\varphi)) = G(k)$ for all branches $\varphi \in \mathcal{B}$ at points of \mathcal{P} at which the closed fiber is not unbranched.

By Lemma 2.3, after adjusting the representative of ζ on the right by some element of $\prod_{P \in \mathcal{P}} G(F_P)$, we may assume that $g_\varphi, g'_\varphi \in G_s(\widehat{R}_\varphi) \subseteq G(\widehat{R}_\varphi)$ have equal images in $G(k_\varphi)$. So by Corollary 1.7(a), the tuples $(g_\varphi), (g'_\varphi)$ define G -torsors over \mathcal{X} whose restrictions to \mathcal{X}_k are isomorphic. By the hypotheses on the closed fiber and on $\mu(G)$, Proposition 4.1 applies. Hence the G -torsors over F defined by these tuples are isomorphic; i.e., $\zeta = \zeta'$. \square

6.2. Explicit description of the obstruction set. Below we make the target space of θ in Theorem 6.3 more concrete, in terms of the structure of the reduction graph. The resulting description of $\text{III}(F, G)$, given in Theorem 6.5, provides a necessary and sufficient condition for a local-global principle to hold under the above hypothesis. This theorem will rely on Lemma 6.4 below, which draws on ideas from [CHHKPS20], where we studied $\text{III}(F, G)$ in the case that G is a torus, by means of the cohomology of decorated graphs. Given a (connected) graph Γ and a *coefficient system* A_\bullet consisting of abelian groups A_v and A_e associated to the vertices v and edges e of the graph, we had defined $H^i(\Gamma, A_\bullet)$ for $i = 0, 1$. We then applied that to the case that Γ is the reduction graph associated to a model of a semi-global field, where the coefficient system depended on the torus. In our current more general situation, our algebraic groups need not be commutative, and so we instead need non-abelian coefficients. Since the reduction graph of a model is a bipartite graph, and since the description of $\text{III}(F, G)$ requires just H^1 , we restrict ourselves here to defining H^1 with non-abelian coefficients just in the case of bipartite graphs, where the description is a bit simpler.

So let Γ be a bipartite graph with edge set E and vertex set V , with V the disjoint union of subsets V_1, V_2 , such that each edge has one vertex in V_1 and the other in V_2 . A *coefficient system* G_\bullet will be a system of groups G_e, G_v associated to the edges and vertices of Γ , together with a homomorphism $G_{(v,e)} : G_v \rightarrow G_e$ for every pair (v, e) such that v is a vertex of an edge e . The homomorphisms $G_{(v,e)}$ for $v \in V_1$ together define a homomorphism $G_1 : \prod_{v \in V_1} G_v \rightarrow \prod_{e \in E} G_e$, whose image acts on $\prod_{e \in E} G_e$ on the left. Similarly, the homomorphisms $G_{(v,e)}$ for $v \in V_2$ together define a homomorphism $G_2 : \prod_{v \in V_2} G_v \rightarrow \prod_{e \in E} G_e$, whose image acts on $\prod_{e \in E} G_e$ on the right. With respect to these actions, we may form the double coset space

$$H^1(\Gamma, G_\bullet) := \prod_{v \in V_1} G_v \backslash \prod_{e \in E} G_e / \prod_{v \in V_2} G_v.$$

When the groups in G_\bullet are abelian, this agrees with the definition in [CHHKPS20, Section 5]. (Also, whether or not the groups are abelian, $H^1(\Gamma, G_\bullet)$ is equal to $H^1(\Gamma, \mathcal{G})$, where Γ is regarded as a topological space together with a sheaf of groups \mathcal{G} that is induced by the groups G_e, G_v .)

Viewing Γ as a one-dimensional simplicial complex, we may also consider its cohomology group $H^1(\Gamma, \mathbb{Z})$. This is a free \mathbb{Z} -module of finite rank.

Given a group G and a positive integer m , two elements $(g_1, \dots, g_m), (g'_1, \dots, g'_m) \in G^m$ are *uniformly conjugate* if there exists $h \in G$ such that $g'_j = hg_jh^{-1}$ for all j . This equivalence relation will be denoted by \sim .

Lemma 6.4. *Let G_\bullet be a coefficient system on a bipartite graph Γ , as above. Suppose that the groups G_e and G_v are all equal to a given group G , and the maps $G_{(v,e)}$ are each the identity. Let m be the number of cycles in the graph Γ ; i.e., the rank of $H^1(\Gamma, \mathbb{Z})$. Then $H^1(\Gamma, G_\bullet) \simeq G^m / \sim$ as pointed sets.*

Proof. First consider the case where $m = 0$; i.e., where Γ is a tree. We proceed by induction on the number of edges of Γ . If there is just one edge (and two vertices), then the assertion is trivial. Now assume the assertion holds for trees with n vertices. If Γ is a tree with $n + 1$ vertices, let v_0 be a terminal vertex, and let e_0 be the unique edge having v_0 as a vertex. Let Γ' be the (bipartite) tree obtained by deleting v_0 and e_0 , with edge set $E' = E \setminus \{e_0\}$ and vertex set $V' = V \setminus \{v_0\}$. Let G'_\bullet be the constant coefficient system G on Γ' . The projection map $\prod_{e \in E} G \rightarrow \prod_{e \in E'} G$ induces a bijection on the respective double coset spaces, since each tuple in $\prod_{e \in E} G$ is in same double coset as one whose e_0 entry is trivial. Since $H^1(\Gamma', G'_\bullet)$ is trivial by inductive hypothesis, so is $H^1(\Gamma, G_\bullet)$.

Now consider the case where $m > 0$. There exists a set $E_0 = \{e_1, \dots, e_m\} \subseteq E$ of m edges such that the graph obtained by deleting these edges (and retaining all the vertices) is a maximal subtree Γ' of Γ . For each $e \in E$ let the vertices of e be $v_{e,1}, v_{e,2}$, with $v_{e,i} \in V_i$. For $e = e_j$ write $v_{j,i} = v_{e_j,i}$.

By the previous case, $H^1(\Gamma', G'_\bullet)$ is trivial, where Γ' is given the induced bipartite structure and G'_\bullet is the constant coefficient system on Γ' as above. Thus every element of $H^1(\Gamma, G_\bullet)$ is a double coset represented by an element of $\prod_{e \in E} G$ whose non-trivial entries all have indices in E_0 . Hence the composition $G^m = \prod_{e \in E_0} G \rightarrow \prod_{e \in E} G \rightarrow H^1(\Gamma, G_\bullet)$ is surjective, where the first map assigns trivial entries for edges not in E_0 . To complete the proof, we will show that two elements $(g_1, \dots, g_m), (g'_1, \dots, g'_m) \in G^m$ have the same image in $H^1(\Gamma, G_\bullet)$ if and only if they are uniformly conjugate by some element $h \in G$.

Let $g_1, \dots, g_m, h \in G$, and for $1 \leq i \leq m$ let $g'_i = hg_i h^{-1}$. Then the double cosets in $H^1(\Gamma, G_\bullet)$ associated to (g_1, \dots, g_m) and to (g'_1, \dots, g'_m) (with 1's in the other entries) are the same, by taking the constant tuple (h) on V_1 and taking the constant tuple (h^{-1}) on V_2 . Conversely, suppose that $(g_1, \dots, g_m), (g'_1, \dots, g'_m) \in G^m$ define the same double coset, where again we set $g_e = g'_e = 1$ for all $e \notin E_0$. That is, there are tuples $(h_v)_{v \in V_1}$ and $(h_v)_{v \in V_2}$ of elements of G taking (g_1, \dots, g_m) to (g'_1, \dots, g'_m) , and taking 1 to 1 in the other entries. More precisely, $g'_j = h_{v_{j,1}} g_j h_{v_{j,2}}$ for each $j = 1, \dots, m$, and $1 = h_{v_{e,1}} \cdot 1 \cdot h_{v_{e,2}}$ for each $e \notin E_0$. Thus $h_{v_2} = h_{v_1}^{-1}$ if $v_1 \in V_1$ and $v_2 \in V_2$ are the vertices of some edge $e \notin E_0$; i.e., some edge of Γ' . Since Γ' is a (connected) tree, and since the vertices of Γ and Γ' are the same, it follows that all the elements h_{v_1} (for $v_1 \in V_1$) are equal to a common element $h \in G$, and that all the elements h_{v_2} (for $v_2 \in V_2$) are equal to h^{-1} . Hence $g'_j = hg_j h^{-1}$ for all j . \square

By Theorem 6.3 and Lemma 6.4, we obtain the following explicit description of $\text{III}(F, G)$ under Hypothesis 6.1. As above, \sim denotes uniform conjugacy, as defined just before Lemma 6.4.

Theorem 6.5. *Under Hypothesis 6.1, assume that the closed fiber of the model \mathcal{X} is reduced and let G be a reductive group over R such that $\mu(G)$ is étale. Then $\text{III}(F, G)$ is in bijection with $(G(k)/R)^m / \sim$ as pointed sets, where m is the number of cycles in the reduction graph of the model. In particular, $\text{III}(F, G)$ is trivial if and only if the reduction graph is a tree or $G(k)/R$ is trivial.*

Proof. Under these hypotheses, the fields $\kappa(U), \kappa(P), \kappa(\wp)$ are all equal to k , and so the groups $G(\kappa(U))/R, G(\kappa(P))/R, G(\kappa(\wp))/R$ are all equal to $G(k)/R$. Thus by Theorem 6.3, we have a bijection $\text{III}(F, G) \rightarrow H^1(\Gamma, G_\bullet)$, where Γ is the reduction graph associated to \mathcal{X} and G_\bullet is the constant coefficient system on Γ given by the group $G(k)/R$. The assertion now follows from Lemma 6.4. \square

It is a well-known open problem whether for any (connected) reductive group G over a field k , the group $G(k)/R$ is abelian. When this holds, conjugation is trivial, and $\text{III}(F, G)$ is in bijection with $(G(k)/R)^m$. This generalizes [CHHKPS20, Theorem 6.4(c)], which treated the case where G is a torus T . (That result was phrased in terms of an abelian group $H^1(k, S)$ rather than $T(k)/R$, but these are isomorphic by [CTS77, Théorème 2, p. 199]; see also [CTS87, Theorem 3.1].)

Remark 6.6. In combination with Theorem 3.2, the statement of Theorem 6.5 says that under the above hypotheses, and for any choice of \mathcal{P} , there is a bijection

$$\text{III}(F, G) \simeq \text{III}_{\mathcal{P}}(F, G) \simeq \text{Hom}(\pi_1(\Gamma), G(k)/R) / \sim,$$

where Γ is the reduction graph of the model, and \sim is the equivalence relation induced by the action of conjugation by $G(k)/R$ on maps to that group. This directly parallels Corollary 6.5 of [HHK15], which concerns linear algebraic groups G over a semiglobal field F , where G is not assumed to be connected or reductive, but for which each connected component of the F -variety G is assumed to be rational. Namely, the result there says that $\text{III}_{\mathcal{P}}(F, G) \simeq \text{Hom}(\pi_1(\Gamma), G/G^0) / \sim$, where G^0 is the identity component of G . By the rationality hypothesis on the group, $H(k)/R$ is trivial for every component H of G ; hence G/G^0 is identified with the finite constant group $G(F)/R$ and we obtain $\text{III}_{\mathcal{P}}(F, G) \simeq \text{Hom}(\pi_1(\Gamma), G(F)/R) / \sim$. In the special case where G is induced by a group over R (and its components are rational), this is equivalent to saying that $\text{III}_{\mathcal{P}}(F, G) \simeq \text{Hom}(\pi_1(\Gamma), G(k)/R) / \sim$.

7. SIMPLY CONNECTED GROUPS G OVER A SEMI-GLOBAL FIELD F WITH $\text{III}(F, G) \neq 1$

In this section we give counterexamples to local-global principles for reductive groups G in the context of semi-global fields F , by using the results of the previous two sections. That is, we obtain examples where $\text{III}(F, G)$ is non-trivial, and in some cases we compute this obstruction explicitly.

As context, we recall that if G is any F -rational linear algebraic group, then $\text{III}_{\mathscr{A}}(F, G) = \text{III}_X(F, G) = 1$ by [HHK15, Theorems 4.2 and 5.10]; and hence $\text{III}(F, G) = 1$ if in addition G is a reductive group over \mathscr{X} , by Proposition 1.1(e). On the other hand, in [CHHKPS20, Section 8], we gave counterexamples to the local-global principles where the group is a torus over R . In [CPS12], it was conjectured that the local-global principle always holds for semisimple simply connected groups for a semi-global field F over a p -adic field K (in which case the residue field of K has cohomological dimension 1, and $\text{cd}(F) = 3$). This has since been proven in a number of cases involving classical groups; see [Pre13], [Hu14], [PPS18], [PS20].

Here we show that the local-global principle does *not* always hold for semisimple simply connected groups in the case of a semi-global field over a general complete discretely valued field. Namely, in Examples 7.6 and 7.7, we provide counterexamples to the local-global principle for semisimple simply connected groups G over a field k of cohomological dimension 4 or more, with the local-global principle being considered over a semiglobal field F over $K = k((t))$. Thus F has cohomological dimension at least 6 in these examples where we show that $\text{III}(F, G)$ is non-trivial.

The fields k that we consider in these examples include ones of the forms $\kappa(x, y)$ and $\kappa((x))((y))$, where κ is either a number field or a field of the form $\kappa_0(u, v)$ and $\kappa_0((u))((v))$ for some field κ_0 . Our groups G are of the form $\text{SL}_1(D)$, where D is a quaternion algebra; here $\text{SL}_1(D)$ is the group of elements g whose reduced norm $\text{Nrd}(g)$ (e.g., see [GS17, Section 2.6]) is equal to 1.

Afterwards, in Proposition 7.8, we show that a counterexample of the type we produce cannot exist if $\text{cd}(k) \leq 3$. Also, whereas it is conjectured for a p -adic field K that $\text{III}(F, G) = 1$ for G a semi-simple simply connected group over a semi-global field F over K , we show in Example 7.10 that the local-global principle for such F and G can fail if one instead considers just those discrete valuations on F that are trivial on K , rather than all the divisorial discrete valuations on F .

A major ingredient in constructing examples where $\text{III}(F, G) \neq 1$ will be Proposition 5.2 and its consequences, Theorems 6.3 and 6.5. Proposition 5.2(b) provides a quotient set of $\text{III}(F, G)$ under certain hypotheses on a model \mathscr{X} of F , and Theorems 6.3 and 6.5 provide an exact computation of $\text{III}(F, G)$ assuming an additional condition on \mathscr{X} . In those situations, to obtain counterexamples to local-global principles we will want to find examples of semisimple simply connected groups G over k such that $G(k)/R \neq 1$, or such that the map $G(k)/R \rightarrow G(k')/R$ is not surjective for some extension k' of k .

7.1. Determining $G(k)/R$. In order to compute $\text{III}(F, G)$ by means of Proposition 5.2 and its consequences, we will need to determine $G(k)/R$. In the case that G is of the form $\text{SL}_1(D)$ for some (central) division algebra D over k , this can be done using a result of Voskresenskiĭ. Recall that if D is a division algebra over a field k , and $G = \text{SL}_1(D)$, then $\text{SK}_1(D)$ is defined to be the quotient of $G(k) = D^{\times 1} := \{g \in D \mid \text{Nrd}(g) = 1\}$ by the commutator

subgroup $[D^\times, D^\times]$; this is an abelian group. Voskresenskii's result says that the natural homomorphism $G(k) = D^{\times 1} \rightarrow \mathrm{SK}_1(D)$ induces an isomorphism $G(k)/\mathrm{R} \xrightarrow{\sim} \mathrm{SK}_1(D)$. (See [Vosk98, §18.2], where this is shown using a result of Bass-Platonov; viz., that the natural map $\mathrm{SK}_1(D) \rightarrow \mathrm{SK}_1(D_{k(t)})$ is an isomorphism.)

We begin with a result that helps us compute $\mathrm{SK}_1(D)$ in some cases, by relating it to the group of R-equivalence classes in an appropriate torus T . This builds on work of Platonov, who had shown the second assertion in part (b) below by a different argument (sketched in [Vosk98, Section 18.3]). Before stating the result, we recall some background.

Given a finite Galois field extension E/κ , the *norm one torus* $T := \mathrm{R}_{E/\kappa}^1 \mathbb{G}_m$ is the kernel of the norm map $\mathrm{R}_{E/\kappa} \mathbb{G}_m \rightarrow \mathbb{G}_m$, where $\mathrm{R}_{E/\kappa} \mathbb{G}_m$ is the Weil restriction of \mathbb{G}_m . The group $T(\kappa)/\mathrm{R}$ coincides with the Tate cohomology group $\widehat{H}^{-1}(\Sigma, E^\times)$, where $\Sigma = \mathrm{Gal}(E/\kappa)$. (See [CTS77, Proposition 15].) This cohomology group (and hence $T(\kappa)/\mathrm{R}$) is the quotient of the group ${}^N L^\times$ of elements of L^\times of norm 1 by the subgroup $I_G L^\times$ generated by the elements $\sigma(x)/x$ for $x \in L^\times$ and $\sigma \in \Sigma$. In particular, $T(\kappa)/\mathrm{R}$ is abelian. Also, if Σ is cyclic with generator σ , and if $z \in \kappa^\times$, there is an associated cyclic algebra $(E/\kappa, \sigma, z)$, which is a central simple κ -algebra split by E ; see [GS17, Section 2.5] for details.

Theorem 7.1. *Let κ be a field, let K/κ and L/κ be linearly disjoint cyclic field extensions of κ with compositum $E = KL$, and let $T = \mathrm{R}_{E/\kappa}^1 \mathbb{G}_m$. Let σ and τ be generators of the Galois groups $\mathrm{Gal}(K/\kappa)$ and $\mathrm{Gal}(L/\kappa)$. Write $k = \kappa(x, y)$ and $k' = \kappa((x))((y))$, and let $D = (K(x, y)/k, \sigma, x) \otimes_k (L(x, y)/k, \tau, y)$ and $D' = D_{k'}$. Then:*

- (a) *There are homomorphisms $T(\kappa)/\mathrm{R} \rightarrow \mathrm{SK}_1(D)$ and $\mathrm{SK}_1(D') \rightarrow T(\kappa)/\mathrm{R}$ such that the composition $T(\kappa)/\mathrm{R} \rightarrow \mathrm{SK}_1(D) \rightarrow \mathrm{SK}_1(D') \rightarrow T(\kappa)/\mathrm{R}$ is the identity map.*
- (b) *The maps $\mathrm{SK}_1(D) \rightarrow T(\kappa)/\mathrm{R}$ and $\mathrm{SK}_1(D') \rightarrow T(\kappa)/\mathrm{R}$ are surjective. Moreover if $\mathrm{char}(\kappa)$ does not divide $[K : \kappa][L : \kappa]$ then the latter map is an isomorphism.*

Proof. For part (a), let $n = [K : \kappa]$ and $m = [L : \kappa]$. Since K and L are linearly disjoint over κ , E/κ is a Galois extension of degree nm and $\mathrm{Gal}(E/\kappa)$ is an abelian group generated by σ and τ (where σ and τ are treated as automorphism of E in the obvious way). There exist commuting elements $\pi, \delta \in D^\times$ such that $\pi^n = x$, $\delta^m = y$, and such that the restrictions of the inner automorphisms given by π and δ to E are σ and τ , respectively. We again write σ, τ for those inner automorphisms of D^\times ; these automorphisms each fix both π and δ .

Since $E(x, y) \subset D$ is a maximal subfield, for $\varepsilon \in E(x, y)$ we have $\mathrm{N}_{E(x, y)/k}(\varepsilon) = \mathrm{Nrd}_D(\varepsilon)$ (see [GS17, Proposition 2.6.3(2)]). Since $E \subset E(x, y)$, we have $T(\kappa) = \mathrm{R}_{E/\kappa}^1 \mathbb{G}_m(\kappa) \subseteq D^{\times 1} = \mathrm{SL}_1(D)(k)$. The group $\mathrm{RT}(\kappa)$ is generated by the elements of the form $\alpha^{-1} \sigma^i \tau^j(\alpha)$ with $\alpha \in E^\times$ and $i, j \in \mathbb{Z}$, by [CTS77, Proposition 15] (as recalled above). Since σ and τ are restrictions of inner automorphisms on D (by π, δ above), it follows that these generators are commutators. Thus $\mathrm{RT}(\kappa) \subseteq [D^\times, D^\times]$, and hence we have a homomorphism $T(\kappa)/\mathrm{R} \rightarrow \mathrm{SK}_1(D)$.

We now define a homomorphism $\mathrm{SL}_1(D')(k') \rightarrow T(\kappa)$. Let Γ be the unique maximal $\kappa((x))[[y]]$ -order of D' ([Re03, Theorem 12.8]). Then $\delta \in \Gamma$, and $\Gamma\delta$ is a 2-sided ideal of Γ with $\Gamma/\Gamma\delta$ isomorphic to the division algebra $D_0 := (K((x))/\kappa((x)), \sigma, x) \otimes_{\kappa((x))} L((x))$. Here σ and τ induce automorphisms $\bar{\sigma}$ and $\bar{\tau}$ on $\Gamma/\Gamma\delta$. Let Γ_0 be the unique maximal $L[[x]]$ -order of D_0 , and let $\bar{\pi}$ be the image of π in Γ_0 . Then $\Gamma_0\bar{\pi}$ is a 2-sided ideal of Γ_0 , and $\Gamma_0/\Gamma_0\bar{\pi}$ is isomorphic to E .

Let $\theta \in \mathrm{SL}_1(D')(k')$. Then $\theta \in \Gamma$, and it is a unit ([Re03, p. 139]). Let $\bar{\theta}$ be the image of θ in $\Gamma/\Gamma\delta \simeq D_0$. Since $\mathrm{Nrd}(\theta) = 1$, we have $\mathrm{N}_{L((x))/\kappa((x))}(\mathrm{Nrd}(\bar{\theta})) = 1$, and so $\mathrm{Nrd}(\bar{\theta}) \in L[[x]]$ is a unit. Thus $\bar{\theta}$ lies in Γ_0 and is a unit. Let $\bar{\bar{\theta}}$ be the image of $\bar{\theta}$ in $\Gamma_0/\Gamma_0\bar{\pi} \simeq E$. Since $\mathrm{Nrd}(\theta) = 1$, it follows that $\bar{\bar{\theta}} \in \mathrm{R}_{E/\kappa}^1 \mathbb{G}_m(\kappa) = T(\kappa)$. Hence we have a map $\mathrm{SL}_1(D')(k') \rightarrow T(\kappa)$ which is a homomorphism. It follows from the definition that the composite map $T(\kappa) \rightarrow \mathrm{SL}_1(D)(k) \rightarrow \mathrm{SL}_1(D')(k') \rightarrow T(\kappa)$ is the identity.

Let $f, g \in D'^{\times}$. Then $f = u\delta^s$ and $g = v\delta^t$ for some u, v units in Γ and $s, t \in \mathbb{Z}$. We have $fgf^{-1}g^{-1} = u\delta^s v\delta^t \delta^{-s} u^{-1} \delta^{-t} v^{-1} = u\delta^s v\delta^{-s} \delta^t u^{-1} \delta^{-t} v^{-1} = u\tau^s(v)\tau^t(u^{-1})v^{-1} \in \Gamma$, using that τ is conjugation by δ . Writing \bar{u}, \bar{v} for the images of u, v in Γ_0 , we have $\bar{u} = u_0\bar{\pi}^{s_0}$ and $\bar{v} = v_0\bar{\pi}^{t_0}$ for some units $u_0, v_0 \in \Gamma_0$ and $s_0, t_0 \in \mathbb{Z}$. Then $\overline{fgf^{-1}g^{-1}} = \overline{u\tau^s(v)\tau^t(u^{-1})v^{-1}} = u_0\bar{\pi}^{s_0}\bar{\tau}^s(v_0\bar{\pi}^{t_0})\bar{\tau}^t(\bar{\pi}^{-s_0}u_0^{-1})\bar{\pi}^{-t_0}v_0^{-1} = u_0\bar{\sigma}^{s_0}\bar{\tau}^s(v_0)\bar{\tau}^t\bar{\sigma}^{t_0}(u_0^{-1})v_0^{-1} \in \Gamma_0$, using that $\bar{\sigma}$ is conjugation by $\bar{\pi}$ and that $\bar{\tau}$ fixes $\bar{\pi}$. The images of $\overline{fgf^{-1}g^{-1}}$ and $(\bar{u}_0\bar{\tau}^t\bar{\sigma}^{t_0}(\bar{u}_0^{-1}))(\bar{v}_0^{-1}\bar{\sigma}^{s_0}\bar{\tau}^s(\bar{v}_0))$ under $\Gamma_0 \rightarrow \Gamma_0/\Gamma_0\bar{\pi} \simeq E$ are thus equal, since E is commutative. So the image of $fgf^{-1}g^{-1}$ under the map $\mathrm{SL}_1(D')(k') \rightarrow T(\kappa)$ is in $\mathrm{RT}(\kappa)$, again by [CTS77, Proposition 15]. Hence we obtain a homomorphism $\mathrm{SK}_1(D') \rightarrow T(\kappa)/\mathrm{R}$ such that the composition $T(\kappa)/\mathrm{R} \rightarrow \mathrm{SK}_1(D) \rightarrow \mathrm{SK}_1(D') \rightarrow T(\kappa)/\mathrm{R}$ is the identity map. This completes the proof of part (a).

The surjectivity assertion in part (b) is immediate from part (a). For the isomorphism assertion under the assumption that $\mathrm{char}(\kappa)$ does not divide nm , let $\theta \in \mathrm{SL}_1(D')(k')$ with $\bar{\theta} = 1 \in T(\kappa)/\mathrm{R}$ (where $\bar{\theta}$ is as in the proof of part (a)). Since $\mathrm{RT}(k) \subseteq [D'^{\times}, D'^{\times}] \subseteq \mathrm{SL}_1(D')$, after multiplying θ by an element in $[D'^{\times}, D'^{\times}]$ we may assume that $\bar{\bar{\theta}}$ is equal to $1 \in T(\kappa)$. Since k' is a complete discretely valued field whose residue field is also a complete discretely valued field, and since $\mathrm{char}(\kappa)$ does not divide nm , by Hensel's lemma there exists $\theta_0 \in D'^{\times}$ with $\theta = \theta_0^{nm}$ and $\bar{\bar{\theta}}_0 = 1$. Since $\mathrm{Nrd}(\theta_0)^{nm} = \mathrm{Nrd}(\theta) = 1$ and $\bar{\bar{\theta}}_0 = 1$, it then follows that $\mathrm{Nrd}(\theta_0) = 1$. Since $\mathrm{SK}_1(D')$ is nm -torsion (see [Tits78, Lemme 2.2(i)]), we get $\theta = 1 \in \mathrm{SK}_1(D')$. Thus the homomorphism $\mathrm{SK}_1(D') \rightarrow T(\kappa)/\mathrm{R}$ is an isomorphism. \square

We will use this theorem to obtain examples in which $G(k)/\mathrm{R}$ is non-trivial by showing that $T(\kappa)/\mathrm{R}$ is non-trivial there, where $T = \mathrm{R}_{L/\kappa}^1 \mathbb{G}_m$ with L/κ Galois. To do this, it will again be useful to describe $T(\kappa)/\mathrm{R}$ as the quotient of the group of norm 1 elements of L^{\times} by the subgroup generated by the elements $\sigma(x)/x$ for $x \in L^{\times}$ and $\sigma \in \mathrm{Gal}(L/\kappa)$. We will take groups G of the form $\mathrm{SL}_1(D)$, where D is taken to be a tensor product of two cyclic algebras (as in the above theorem), with a focus on the case of biquaternion algebras.

Example 7.2. (a) Let L/κ be a degree four field extension of the form $L = \kappa(\sqrt{a}, \sqrt{b})$, where -1 is a square in κ . Suppose in addition that the diagonal quadratic form $\langle 1, a, -b \rangle$ is anisotropic over κ ; or equivalently, the quaternion algebra (a, b) over κ is not split. (For example, we could take $\kappa = \mathbb{Q}(i)$, $a = 3$, $b = 5$. Or we could take κ to be a function field of the form $\kappa_0(u, v)$ or $\kappa_0((u))((v))$ for some field κ_0 , and take $a = u$ and $b = v$.) It was shown in [CHHKPS20, Example 8.1] that ${}^N L^{\times}/I_G L^{\times} = \widehat{H}^{-1}(\Sigma, L^{\times}) = T(\kappa)/\mathrm{R}$ is non-trivial, where $T = \mathrm{R}_{L/\kappa}^1(\mathbb{G}_m)$. Hence so is $G(k)/\mathrm{R}$, where $k = \kappa(x, y)$ or $\kappa((x))((y))$, and where $G = \mathrm{SL}_1(D)$, with D the biquaternion k -algebra $(a, x) \otimes (b, y)$; this is because $G(k)/\mathrm{R} = \mathrm{SK}_1(D) \rightarrow T(\kappa)$ is surjective by Theorem 7.1(b).

- (b) Let L/κ be a degree four field extension of number fields of the form $L = \kappa(\sqrt{a}, \sqrt{b})$, where we do not require -1 to be a square, and let $T = R_{L/\kappa}^1(\mathbb{G}_m)$. Let d_κ be the number of places of κ over which there is a unique place of L . Then, as shown in the proof of [CTS77, Corollaire 2], $T(\kappa)/R$ is non-trivial if and only if $d_\kappa > 1$, and in that case it is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{d_\kappa-1}$. For example, if $\kappa = \mathbb{Q}$, $a = -1$, and $b = 2$, then $d_\kappa = 1$ and $T(\kappa)/R$ is trivial; while if $\kappa = \mathbb{Q}(\sqrt{17})$, then the same choice of a, b yields $T(\kappa)/R \simeq \mathbb{Z}/2\mathbb{Z}$. As in part (a), we can then take $k = \kappa(x, y)$ or $\kappa((x))(y)$; D the biquaternion k -algebra $(a, x) \otimes (b, y)$; and $G = \mathrm{SL}_1(D)$. In the case $k = \kappa(x, y)$ we then find that $G(k)/R$ is non-trivial if $\kappa = \mathbb{Q}(\sqrt{17})$; while in the case $k = \kappa((x))(y)$ we find that $G(k)/R$ is trivial for $\kappa = \mathbb{Q}$ and that it is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ if $\kappa = \mathbb{Q}(\sqrt{17})$.
- (c) In the notation of Theorem 7.1, assume that $[K : \kappa] = [L : \kappa] = p$ for some prime number p . Let $G = \mathrm{SL}_1(D')$ over $k' = \kappa((x))(y)$. Then $G(k')/R \simeq \mathrm{SK}_1(D') \simeq (\mathbb{Z}/p\mathbb{Z})^{d_\kappa-1}$, by [Pla76, Theorem 5.13], if $d_\kappa \geq 1$ (where d_κ is defined as in part (b) above). In the case that $p = 2$ and κ is a number field, this agrees with the conclusion in part (b) above.
- (d) For any example of a field κ and a torus T over κ for which $T(\kappa)/R$ is non-trivial, there exists an overfield $\tilde{\kappa}/\kappa$ such that $T(\tilde{\kappa})/R$ is infinite. An explicit example of this was given in [CHHKPS20, Example 8.2], by taking the union of an increasing tower of number fields κ_n such that $d_{\kappa_n} \rightarrow \infty$ and using part (b) above. The general case was shown in [CHHKPS20, Example 8.4], by iterating [CHHKPS20, Proposition 8.3], which asserts that $T(\kappa(T))$ strictly contains $T(\kappa)$. In particular, this applies to the examples given in parts (a) and (b) above where $T(\kappa)$ is non-trivial. (Note that the group $H^1(\kappa, S)$ in [CHHKPS20] is the same as $T(\kappa)/R$, by [CTS77, Théorème 2, p. 199]; see also [CTS87, Theorem 3.1].) With \tilde{k} equal to $\tilde{\kappa}(x, y)$ or $\tilde{\kappa}((x))(y)$, one then has that $G(\tilde{k})/R$ is infinite for $G = \mathrm{SL}_1(D)$ as above, since $G(\tilde{k})/R \xrightarrow{\sim} \mathrm{SK}_1(D_{\tilde{k}}) \rightarrow T(\tilde{\kappa})/R$ is surjective.

Another approach to computing $\mathrm{SK}_1(D)$ is given in the following result, which is based on a theorem of Rost.

Proposition 7.3. *Let k be a field with $\mathrm{char}(k) \neq 2$, and let D be a biquaternion algebra $(a, b) \otimes (c, d)$ over k .*

- (a) *If $\mathrm{cd}_2(k) \leq 3$, then $\mathrm{SK}_1(D) = 0$.*
(b) *If -1 is a square in k and $(a) \cup (b) \cup (c) \cup (d) \neq 0 \in H^4(k, \mathbb{Z}/2)$, then $\mathrm{SK}_1(D) \neq 0$.*

Proof. By a result of Rost (see [Merk95, Theorem 4]), there is an injection $\mathrm{SK}_1(D) \rightarrow H^4(k, \mathbb{Z}/2)$; and if -1 is a square in k then $\sqrt{-1} \in D^{\times 1}$, and its image under the composition $D^{\times 1} \rightarrow \mathrm{SK}_1(D) \rightarrow H^4(k, \mathbb{Z}/2)$ is the cup product $(a) \cup (b) \cup (c) \cup (d)$.

Part (a) is then immediate from the injectivity of $\mathrm{SK}_1(D) \rightarrow H^4(k, \mathbb{Z}/2)$, and part (b) holds by the non-triviality of the above composition. \square

Part (b) of this result can be used to obtain examples where $G(k)/R = \mathrm{SK}_1(D)$ is non-trivial, where $G = \mathrm{SL}_1(D)$. We would like to go further, enlarging k to obtain examples where $G(k)/R$ is infinite, as in Example 7.2(d). That example had relied on [CHHKPS20, Proposition 8.3], which concerned tori. In the context of the above proposition, we can instead use the following analogous result for reductive groups.

Proposition 7.4. *Let G be a connected linear algebraic k -group, with k perfect or G reductive. Assume $G(k)/R$ is non-trivial. Then:*

- (a) *The natural map $G(k)/R \rightarrow G(k(G))/R$ is injective but not surjective.*
- (b) *There is a field \tilde{k} of infinite transcendence degree over k such that $G(\tilde{k})/R$ is infinite.*

Proof. If k is finite, then $G(k)/R = 1$ ([CTS77, Corollaire 6, p. 202]). Thus k is infinite. Also, if two k -points are R -equivalent, they are directly R -equivalent [Gil97, II.1.1 b)]. Since we have k perfect or G reductive, the k -variety G is k -unirational by [Bor91, Chap. V. Theorem 18.2]; and the map $U(k)/R \rightarrow G(k)/R$ is a bijection for any non-empty Zariski open subset $U \subseteq G$, by [CTS77, Proposition 11, p. 196]. Since k is infinite and G is unirational, $G(k)$ is dense in G , and hence $U(k)$ is non-empty.

To prove injectivity in part (a), suppose that $g, g' \in G(k)$ become R -equivalent over $k' := k(G)$. Then they become R -equivalent over some Zariski dense open subset $U \subseteq G$. Specializing to a k -point of U shows that g, g' are R -equivalent over k .

To prove that $G(k)/R \rightarrow G(k')/R$ is not surjective, suppose otherwise for the sake of contradiction. Then the generic point η in $G(k')$, which is given by the natural map $\text{Spec}(k') \rightarrow G$, is directly R -equivalent on $G_{k'}$ to an element $g_0 \in G(k) \subseteq G(k')$. That is, there exists an open set $S \subseteq \mathbb{A}_{k'}^1$ containing two k' -points A and B and a k' -morphism $S \rightarrow G_{k'}$ such that A maps to $g_0 \times_k k' \in G(k')$ and B maps to $\eta \in G(k')$. Applying an automorphism on $\mathbb{A}_{k'}^1$ we may assume that the points A and B come from k -points of \mathbb{A}_k^1 . There then exist a dense open set $U \subseteq G$, an open set $W \subseteq \mathbb{A}_U^1$ containing $A \times U$ and $B \times U$, and a U -morphism $W \rightarrow G \times_k U$, such that the image of $A \times U$ is $g_0 \times U$ and the image of $B \times U$ is the diagonal in $U \times U$.

Now for any k -point $g \in U(k) \subseteq G(k)$, the fiber of $W \rightarrow U$ over g gives a direct R -equivalence between the k -point g and the point $g_0 \in G(k)$. Thus $G(k)/R = U(k)/R = 1$. This is a contradiction, thereby proving part (a).

Part (b) then follows by infinite iteration of the passage from k to $k(G)$. □

Combining this with Proposition 7.3, we obtain the following example.

Example 7.5. (a) Let κ be a field with $\text{char}(\kappa) \neq 2$, such that -1 is a square in κ . Let $a, b \in \kappa^\times$ be elements such that the quaternion algebra (a, b) over κ is not split (e.g., $\kappa = \mathbb{Q}(i)$, $a = 3$, $b = 5$). Then over the rational function field $k = \kappa(x, y)$, the biquaternion algebra $D = (a, b) \otimes (x, y)$ satisfies $\text{SK}_1(D) \neq 0$, by Proposition 7.3(b). Thus $G(k)/R$ is non-trivial, where $G = \text{SL}_1(D)$.

- (b) In the situation of part (a), there is a field extension \tilde{k}/k such that $G(\tilde{k})/R$ is infinite, by applying Proposition 7.4(b).

7.2. Examples with $\text{III}(F, G) \neq 1$. We now use the above discussion to obtain examples of semi-global fields F and groups G of the form $\text{SL}_1(D)$ for which $\text{III}(F, G)$ is non-trivial. These are the first known counterexamples to the local-global principle for semisimple simply connected groups over a semi-global field.

As before, we take a semi-global field F over a complete discrete valuation ring R having residue field k , and a normal crossings model \mathcal{X} . As shown in Theorem 4.2, if the closed fiber of \mathcal{X} is reduced and the associated reduction graph Γ is a monotonic tree, and if G is a reductive group over R such that $\mu(G)$ is étale, then $\text{III}(F, G) = 1$. We therefore consider examples where either Γ is not a tree, or else it is a tree that is not monotonic. We restrict

our attention here to the equal characteristic case, so that $R = k[[t]]$. By taking $G = \mathrm{SL}_1(D)$ for D a biquaternion algebra over k (and hence over R and \mathcal{X}), we have at our disposal Examples 7.2 and 7.5. Also, in this situation, $G(k)/R$ is isomorphic to $\mathrm{SK}_1(D)$, and so it is commutative.

We begin with an example in which the closed fiber of \mathcal{X} consists of three copies of \mathbb{P}_k^1 such that each pair intersects at a single point. Thus the reduction graph is a cycle, rather than a tree; and by a suitable choice of k and D we can make $\mathrm{III}(F, G)$ non-trivial, and even infinite.

Example 7.6. As in Example 8.7 of [CHHKPS20], let k be a field, let $R = k[[t]]$, let $\mathcal{X} = \mathrm{Proj}(R[u, v, w]/(uvw - t(u + v + w)^3))$, and let F be the function field of \mathcal{X} . Then \mathcal{X} is a normal crossings model of F over R whose closed fiber consists of three copies of \mathbb{P}_k^1 meeting pairwise at k -points to form a triangle. Thus $m = 1$ in Theorem 6.5, and for any reductive group G over R we have a bijection from $\mathrm{III}(F, G)$ to the set of conjugacy classes $(G(k)/R)/\sim$ in the group $G(k)/R$. Take $G = \mathrm{SL}_1(D)$ for D a biquaternion algebra over k . Since $G(k)/R$ is commutative, we have a bijection from $\mathrm{III}(F, G)$ to $G(k)/R$.

To obtain a counterexample to the local-global principle we can thus take any of the examples with non-trivial $G(k)/R$ in Section 7.1. In particular, as in Example 7.2(b), we can choose $k = \mathbb{Q}(\sqrt{17})(x)(y)$ and $D = (-1, x) \otimes (2, y)$, so that $\mathrm{III}(F, G) \simeq G(k)/R \simeq \mathbb{Z}/2\mathbb{Z}$. Alternatively, using other cases in those examples, we get a non-trivial $\mathrm{III}(F, G)$ by taking $k = \mathbb{Q}(\sqrt{17})(x, y)$; or by taking $k = \kappa(x)(y)$ or $\kappa(x, y)$, where κ is itself a field of the form $\kappa_0(u)(v)$ or $\kappa_0(u, v)$. Thus, in particular, we obtain counterexamples to the local-global principle in the case that k is of the form $\kappa_0(u, v, x, y)$ or $\kappa_0(u)(v)(x)(y)$ for some field κ_0 .

In addition, by applying Example 7.2(d) (or Proposition 7.4) with any of these choices of k , we obtain a field extension \tilde{k}/k such that $\mathrm{III}(\tilde{F}, G)$ is infinite, where $\tilde{F} = F \otimes_{k((t))} \tilde{k}((t))$.

Our next example involves a model \mathcal{X} whose reduction graph Γ is a tree that is not monotonic.

Example 7.7. As in Examples 7.7 and 8.9(a) of [CHHKPS20], let k be a field such that $\mathrm{char}(k) \neq 2$, let $R = k[[t]]$, let c be a non-square in k , and let F be the function field of $\mathcal{X} = \mathrm{Proj}(R[x, y, z]/((y - x)(xy - cz^2) + tz^3))$. Then \mathcal{X} is a normal crossings model of F over R whose closed fiber is reduced and consists of two copies of \mathbb{P}_k^1 that meet at a single closed point P whose residue field is $k' := k(\sqrt{c})$. Thus the reduction graph is a tree, but is not a monotonic tree (and the base change of the closed fiber from k to k' is not a tree). Let U_1, U_2 be the complements of P in the two copies of \mathbb{P}_k^1 , and for $i = 1, 2$ let \wp_i be the branch at P lying on U_i . With notation as at the beginning of Section 5, we have $\kappa(U_i) = k$ and $\kappa(P) = \kappa(\wp_i) = k'$ for $i = 1, 2$. By Proposition 5.2(b), for any reductive group G over R we have a surjection

$$\begin{aligned} \mathrm{III}(F, G) &\rightarrow \prod_{U \in \mathcal{U}} (G(\kappa(U)/R) \setminus \prod_{\wp \in \mathcal{B}} (G(\kappa(\wp))/R) / \prod_{P \in \mathcal{P}} (G(\kappa(P))/R)) \\ &= (G(k)/R \times G(k)/R) \setminus (G(k')/R \times G(k')/R) / (G(k')/R). \end{aligned}$$

Here the maps $G(k)/R \times G(k)/R \rightarrow G(k')/R \times G(k')/R$ on the left hand side are the natural maps induced by the inclusion $k \subset k'$, and the map $G(k')/R \rightarrow G(k')/R \times G(k')/R$ on the right hand side is the diagonal map.

To use this to obtain an explicit counterexample to the local-global principle we can let $k = \mathbb{Q}((x))((y))$ and $c = 17$, so that $k' = \mathbb{Q}(\sqrt{17})((x))((y))$. Let D be the biquaternion algebra $(-1, x) \otimes (2, y)$ over k and let $G = \mathrm{SL}_1(D)$. By Example 7.2(b), $G(k)/R$ is trivial, while $G(k')/R = \mathbb{Z}/2\mathbb{Z}$. Thus $\mathrm{III}(F, G)$ maps surjectively onto the quotient of $(\mathbb{Z}/2\mathbb{Z})^2$ by the diagonal $\mathbb{Z}/2\mathbb{Z}$, and hence is non-trivial. As in Example 7.6, we can then find a field extension \tilde{k}/k such that $\mathrm{III}(\tilde{F}, G)$ is infinite, where $\tilde{F} = F \otimes_{k((t))} \tilde{k}((t))$.

Observe that in all our examples of non-trivial $\mathrm{III}(F, G)$ and non-trivial $G(F)/R$ for $G = \mathrm{SL}_1(D)$ and F a semi-global field over $k((t))$, we have $\mathrm{cd}(k) \geq 4$. Namely, in our examples, $k = \kappa(x, y)$ or $\kappa((x))((y))$, where κ is a number field or a field of the form $\kappa_0(u, v)$ or $\kappa_0((u))((v))$. In fact, in using our approach, this condition on the cohomological dimension is necessary for examples where Theorem 6.3 applies:

Proposition 7.8. *Let k be a field, with $\mathrm{char}(k) \neq 2$. Let D be a biquaternion division algebra over k , and let $G = \mathrm{SL}_1(D)$. Let $R = k[[t]]$, and let F be a semi-global field over R that admits a normal crossings model \mathcal{X} whose closed fiber is reduced and consists of projective k -lines meeting at k -points. If $\mathrm{cd}_2(k) \leq 3$, then $\mathrm{III}(F, G) = 1$.*

Proof. By Proposition 7.3, $\mathrm{SK}_1(D)$ is trivial. But $G(k)/R \simeq \mathrm{SK}_1(D)$ by [Vosk98, §18.2]. Hence $G(k)/R$ is also trivial. Thus $\mathrm{III}(F, G)$ is trivial, by the description given in Theorem 6.3. \square

Remark 7.9. Proposition 7.8 can be extended as follows:

- (a) If $\mathrm{char}(k) = 0$, then the conclusion of Proposition 7.8 holds even without assuming that the biquaternion algebra D is a division algebra, because Voskresenskii showed in [Vosk98, §18.2] that the isomorphism $G(k)/R \simeq \mathrm{SK}_1(D)$ remains true for general central simple algebras in that situation. Hence in that case, we additionally obtain that $G(k')/R$ is trivial for every finite extension of k . Thus we conclude that $\mathrm{III}(F, G)$ is trivial even if we assume only that the components of the closed fiber of \mathcal{X} are projective lines (not necessarily over k , or meeting at k -points).
- (b) Let $k' = \kappa((x))((y))$ and let D' be a product of two cyclic algebras as in Theorem 7.1. Assume that their degrees are not divisible by the characteristic. If $\mathrm{cd}(k') \leq 3$, then $\mathrm{cd}(\kappa) \leq 1$, hence $T(\kappa)/R$ is trivial for any torus T over κ (by [CTS77, Corollaire 5(i), p. 201]). Thus $G(k)/R \simeq \mathrm{SK}_1(D) \simeq T(\kappa)/R$ is trivial, with T as in Theorem 7.1(b). Hence $\mathrm{III}(F, G)$ is trivial, and so Proposition 7.8 extends to this setting.

The above proposition and remark provide examples in which the local-global principle for F and G holds in the situation where $\mathrm{cd}(k) \leq 3$. This suggests the question of whether $\mathrm{III}(F, G)$ is trivial for every semisimple simply connected group G over a complete discrete valuation ring R whose residue field k satisfies $\mathrm{cd}(k) \leq 3$, where F is a semi-global field over R .

On the other hand, instead of considering the local-global obstruction set $\mathrm{III}(F, G)$, which is taken with respect to all the divisorial discrete valuations on F arising from regular models \mathcal{X} over R , one could consider the *a priori* larger obstruction set $\mathrm{III}_K(F, G)$ taken with respect to the set Ω_K of discrete valuations on F that are trivial on K (or equivalently, the ones that arise from closed points on the generic fiber of $\mathcal{X} \rightarrow \mathrm{Spec}(R)$). This obstruction was considered, for example, in [CH15], [HSS15], and [HS16]. As the next example shows, the

associated local-global principle can fail for semisimple simply connected groups even in the case of semi-global fields over a p -adic field. This answers a question posed by D. Harari.

Example 7.10. Let p be an odd prime, and let $k = \mathbb{F}_p$, $R = \mathbb{Z}_p$, and $K = \mathbb{Q}_p$. Consider the projective R -curve $\mathcal{X} = \text{Proj}(R[u, v, w]/(uvw - p(u + v + w)^3))$, with function field F . Thus \mathcal{X} is a normal crossings model of the semi-global field F over R , whose closed fiber consists of three copies of \mathbb{P}_k^1 , forming a triangle. (This is a mixed characteristic analog of the model in Example 7.6.) Let C be the fiber of \mathcal{X} over the generic point of $\text{Spec}(R)$. Thus C is a smooth projective curve over $K = \mathbb{Q}_p$, with function field F . By [HKP21, Lemma 5.10(a),(b)], $\text{III}_K^3(F, \mathbb{Z}/2\mathbb{Z}) = \text{III}_K(F, \mathbb{Z}/2) = \mathbb{Z}/2\mathbb{Z}$. (This can also be obtained using [Kat86, Corollary 2.9]. We note that [HKP21] indirectly relied on [Kat86] via its use of [HS16].) By [PS99, Theorem 3.9], the unique non-trivial element ξ of $\text{III}_K^3(F, \mathbb{Z}/2\mathbb{Z}) \subseteq H^3(F, \mathbb{Z}/2)$ is decomposable; i.e., is represented by a symbol $(a) \cup (b) \cup (c)$, for some $a, b, c \in F^\times$ (where we identify $H^1(F, \mathbb{Z}/2\mathbb{Z})$ with $F^\times/F^{\times 2}$). Since $\xi \in \text{III}_K^3(F, \mathbb{Z}/2\mathbb{Z})$, its image $\xi_v \in H^3(F_v, \mathbb{Z}/2\mathbb{Z})$ is trivial, for all $v \in \Omega_K$.

If G is a simple F -group of type G_2 , then for field extensions L/F there is a functorial injective map of pointed sets $\iota_L : H^1(L, G) \rightarrow H^3(L, \mathbb{Z}/2)$ whose image is the set of decomposable elements (see [Ser95, Section 8.1, Théorème 9]). Since the above decomposable element $\xi \in \text{III}_K^3(F, \mathbb{Z}/2\mathbb{Z})$ is non-trivial, it is the image of some non-zero element ζ of $H^1(F, G)$. The image ζ_v of ζ in $H^1(F_v, G)$ is trivial for all $v \in \Omega_K$, since the injective map $\iota_{F_v} : H^1(F_v, G) \rightarrow H^3(F_v, \mathbb{Z}/2\mathbb{Z})$ sends ζ_v to $\xi_v = 0$. Thus ζ is a non-trivial element of $\text{III}_K(F, G)$.

Alternatively, with a, b, c as above, let D be the quaternion algebra (b, c) over F , and let $G = \text{SL}_1(D)$. The Brauer class of D is 2-torsion and lies in $H^2(F, \mathbb{Z}/2\mathbb{Z})$, where it may be identified with the cup product $(b) \cup (c)$; here $(b), (c) \in F^\times/F^{\times 2} = H^1(F, \mathbb{Z}/2\mathbb{Z})$. For each L/F , we may consider the map $H^1(L, \mathbb{Z}/2\mathbb{Z}) = L^\times/L^{\times 2} \rightarrow H^3(L, \mathbb{Z}/2\mathbb{Z})$ given by $s \mapsto s \cup D_L$. By a theorem of Merkurjev-Suslin (see [Ser95, Section 7.2, Théorème 8]), the kernel of this map consists of the square classes of elements of $\text{Nrd}(D_L^\times) \supseteq L^{\times 2}$; and so there is an induced injection $\alpha_L : H^1(L, G) = L^\times/\text{Nrd}(D_L^\times) \rightarrow H^3(L, \mathbb{Z}/2\mathbb{Z})$. The class of $a \in F^\times$ is sent by α_F to the non-zero element ξ , and the class of its image $a_v \in F_v^\times$ is sent by the injective map $\alpha_v : H^1(F_v, G) \rightarrow H^3(F_v, \mathbb{Z}/2\mathbb{Z})$ to $\xi_v = 0$ for all $v \in \Omega_K$. Hence the class of a in $H^1(F, G)$ is a non-trivial element of $\text{III}_K(F, G)$.

APPENDIX A. SPECIALIZATION IN DIMENSIONS ONE AND TWO

Let A be a regular local ring with fraction field L and residue field ℓ . Let G be a reductive group over A . In [Gil04, Théorème 0.2], under the hypothesis that A is a discrete valuation ring whose residue characteristic is not 2, P. Gille constructs a *specialization map* $G(L)/R \rightarrow G(\ell)/R$ on R -equivalence classes. This is a group homomorphism such that the composition $G(A) \rightarrow G(L)/R \rightarrow G(\ell)/R$ agrees with the restriction map $G(A) \rightarrow G(\ell)$ composed with the canonical map $G(\ell) \rightarrow G(\ell)/R$. In this appendix, we show that such a specialization map may be defined for all regular local rings A of dimension one or two, with no restriction on the residue characteristic. We do this first in the case of dimension one (i.e., A is a discrete valuation ring), in Theorem A.10, and then we use that case to treat the case of dimension two, in Theorem A.14. Before turning to this construction, we state some preliminary results concerning tori and anisotropic groups.

A.1. Preliminaries on tori and anisotropy. Below we consider a reductive group G over a regular local ring A , and we relate the split tori in G over A to those over the fraction field and residue field of A . We also discuss criteria for isotropy; i.e., for the existence of split tori.

Proposition A.1. *Let G be a reductive group over a normal domain A . Assume either that A is complete with respect to an ideal I , or that A is a Henselian local ring with maximal ideal I .*

- (a) *Every split torus in $G_{A/I}$ lifts to a split torus in G .*
- (b) *The maximal ranks of split tori in G and $G_{A/I}$ agree.*

Proof. For part (a), let \bar{T} be a split torus in $G_{A/I}$, say of rank n . Since G is smooth and since groups of the form \mathbb{G}_m^n are of multiplicative type, [SGA70b, Exp. XI, Corollaire 4.2] applies, and asserts that the functor $\underline{\mathrm{Hom}}_{A\text{-groups}}(\mathbb{G}_m^n, G)$ is representable by a smooth separated A -scheme Z . Thus \bar{T} corresponds to an A/I -point $\bar{\zeta}$ of Z .

We claim that $\bar{\zeta}$ lifts to an A -point ζ of Z . If A is I -adically complete then this follows from formal smoothness. If A is a Henselian local ring with maximal ideal I , then for some affine neighborhood $Y \subseteq Z$ of the closed point $\bar{\zeta}$, the structure morphism $Y \rightarrow \mathrm{Spec}(A)$ factors through an étale morphism $Y \rightarrow \mathbb{A}_A^N$ for some N , by smoothness; and so a lift ζ exists since A is Henselian.

The above lift ζ corresponds to a lift of \bar{T} to a split torus T in G . This proves part (a), and then part (b) is an immediate consequence. \square

Recall that a reductive group G over a ring A is *anisotropic* if it does not contain an isomorphic copy of $\mathbb{G}_{m,A}$. For an affine group scheme G over A and an A -algebra B , we write G_B for $G \times_A B$; this is an affine group scheme over B .

Proposition A.2. *Let G be a reductive group over a regular local ring A (of arbitrary dimension) with fraction field L and residue field ℓ . Consider the reductive groups G_L over L and G_ℓ over ℓ .*

- (a) *If G_ℓ is anisotropic, then G_L is anisotropic.*
- (b) *If G_L is anisotropic, then G is anisotropic.*
- (c) *If A is Henselian, then these three conditions are equivalent: G is anisotropic, G_L is anisotropic, G_ℓ is anisotropic.*

Proof. For (a), assume that G is isotropic over L . Let $x \in A$ be part of a regular system of parameters of A , and consider the completion \widehat{A}_x of the local ring of A at the corresponding height one prime \mathfrak{p} . Thus \widehat{A}_x is a discrete valuation ring, whose fraction field is the completion L_x of L with respect to the \mathfrak{p} -adic valuation. Since the group G is isotropic over L , it is also isotropic over the overfield L_x ; and since G is reductive it is then also isotropic over the discrete valuation ring \widehat{A}_x , by [Guo20, Section 3, Lemma 4]. Thus G is isotropic over the residue field of \widehat{A}_x , which is the fraction field of the regular local ring A/xA of dimension one less than that of A . By induction on the dimension of A , we conclude that G is isotropic over ℓ .

Part (b) is immediate, since if G is isotropic over A , then it is trivially isotropic over L .

Finally, if A is Henselian and G is isotropic over ℓ then G is also isotropic over A by Proposition A.1(a). So by parts (a) and (b), part (c) follows. \square

Proposition A.3. *Let A be a discrete valuation ring with fraction field L and residue field ℓ . Let G be a reductive group over A . If G_ℓ is anisotropic then $G(A) = G(L)$; and if $G(A) = G(L)$ then G and G_L are anisotropic.*

Proof. For the first assertion, note that if G_ℓ is anisotropic then so is G_L , by Proposition A.2. In the case that A is Henselian, the equality $G(A) = G(L)$ then essentially follows from a result of Bruhat-Tits-Rousseau (see [Pra82, pp. 197-198]). Namely, under the Henselian hypothesis they showed that G_L is anisotropic if and only if $G(L)$ is bounded. But by [Mac17, Theorem 1.1], the subgroup $G(A)$ is a maximal bounded subgroup of $G(L)$; hence $G(A) = G(L)$. (For another proof, see [Guo20, Section 3, Proposition 6].)

In the general case, where the discrete valuation ring A is not assumed Henselian, let \widehat{L} , \widehat{A} be the completions of L and A respectively. Since ℓ is the residue field of \widehat{A} , it follows that $G(\widehat{A}) = G(\widehat{L})$ by the previous case; and so $G(A) = G(\widehat{A}) \cap G(L) = G(\widehat{L}) \cap G(L) = G(L)$.

For the second assertion, assume that G is isotropic over A or L . (For a discrete valuation ring, those conditions are equivalent by [Guo20, Section 3, Lemma 4].) Then there is a closed immersion $\mathbb{G}_m \hookrightarrow G$ over A . But $\mathbb{G}_m(A)$ is strictly contained in $\mathbb{G}_m(L)$; and $\mathbb{G}_m(A) = G(A) \cap \mathbb{G}_m(L)$ since $\mathbb{G}_m \subseteq G$ is a closed immersion. So $G(A)$ is indeed strictly contained in $G(L)$. \square

Thus if A is a Henselian discrete valuation ring with fraction field L and G is reductive over A , then the condition $G(A) = G(L)$ is equivalent to each of the three anisotropy conditions in Proposition A.2(c).

Proposition A.4. *Let G be a reductive group over a regular local ring A with fraction field L . Let $S \subseteq G$ be a maximal split torus in G , and let $H = C_G(S)$ be its centralizer.*

- (a) *Then the group H/S is anisotropic over L .*
- (b) *If A is a Henselian discrete valuation ring, then $H(L) = H(A)S(L)$.*

Proof. Part (a) will follow from showing that a central extension of a split torus by a split torus, in the category of groups, is itself a split torus. But any such extension is in fact a group of multiplicative type, by [SGA70b, Exp. XVII, Proposition 7.1.1]; and hence it is also an extension of split tori in the category of groups of multiplicative type. Such extensions are classified by $\text{Ext}^1(\mathbb{Z}^r, \mathbb{Z}^s)$; i.e., extensions of locally constant sheaves in the étale topology on $\text{Spec}(A)$ (where r, s are the ranks of the two split tori). But this group is trivial, thereby yielding the assertion in part (a).

For part (b), observe first that H/S is anisotropic over L by part (a), and hence also over the residue field, by Proposition A.2(c). Thus $(H/S)(L) = (H/S)(A)$ by Proposition A.3. Since S is split and A is local, it follows that $H^1(A, S) = 0$; and so the map $H(A) \rightarrow (H/S)(A)$ is surjective. Thus for every $h \in H(L)$ there is some $h' \in H(A)$ whose image in $(H/S)(A) = (H/S)(L)$ agrees with that of h . That is, $h = h's$ for some $s \in S(L)$, as desired. \square

We note that in the statement of Proposition A.4, the centralizer is taken in a functorial sense, in the context of presheaves on the category of schemes (see [SGA70c, Exp. XXVI, Section 6]). As a result, the formation of centralizers commutes with base change. In particular, in the situation of the proposition, if $B = A/\mathfrak{p}$ for some prime ideal \mathfrak{p} , then H_B is the centralizer of S_B in G_B .

A.2. Specialization in dimension one. Before turning specifically to discrete valuation rings, we begin with a general lemma. This lemma applies in particular to complete regular local rings, since such rings are excellent by [Gro65, Chap IV, Scholie 7.8.3(iii)].

Lemma A.5. *Let X, Y be schemes of finite type over an excellent regular local ring A , with X regular and connected of dimension two, and with Y proper over A . Let $g : X \dashrightarrow Y$ be a rational map of A -schemes. Then there is a morphism $\pi : \tilde{X} \rightarrow X$ obtained by a sequence of blow-ups at closed points, together with a morphism $\tilde{g} : \tilde{X} \rightarrow Y$, such that $g \circ \pi = \tilde{g}$ as rational maps.*

Proof. Since A is excellent, so are X and Y , by [Gro65, Chap IV, Proposition 7.8.6(i)]. Let \tilde{X}' be the closure of the graph of g in $X \times_A Y$, with projection maps $\alpha : \tilde{X}' \rightarrow X$ and $\gamma : \tilde{X}' \rightarrow Y$. Thus $g \circ \alpha = \gamma$ as rational maps; and α is proper, since $Y \rightarrow \text{Spec}(A)$ is. By excellence, [Sta20, Section 0ADX] and [Sta20, Theorem 0BGP] apply, and assert that there exists a resolution $\beta : \tilde{X} \rightarrow \tilde{X}'$ of the singularities of \tilde{X}' (which is in particular proper). Thus $\pi = \alpha \circ \beta : \tilde{X} \rightarrow X$ is a proper birational morphism; and so by [Sta20, Lemma 0C5R] it is a composition of blowups at closed points. The composition $\tilde{g} := \gamma \circ \beta$ is a morphism $\tilde{X} \rightarrow Y$, and $g \circ \pi = g \circ \alpha \circ \beta = \gamma \circ \beta = \tilde{g}$ as rational maps. \square

In the above situation, we view the residue field ℓ of A as an A -algebra, and so we write $X(\ell)$ for $\text{Hom}_A(\text{Spec}(\ell), X)$. Note that every ℓ -point of X lies over the closed point of $\text{Spec}(A)$; i.e., $X(\ell)$ is contained in the closed fiber X_ℓ of $X \rightarrow \text{Spec}(A)$.

Lemma A.6. *Let A be an excellent regular local ring with residue field ℓ , and let X be a regular connected two-dimensional scheme of finite type over A . If $\tilde{X} \rightarrow X$ is a sequence of blow-ups at closed points, then the natural map $\tilde{X}(\ell)/\mathbb{R} \rightarrow X(\ell)/\mathbb{R}$ is a bijection.*

Proof. Let X_ℓ be the fiber of X over the closed point of $\text{Spec}(A)$. If X_ℓ is empty then the result is trivial; so we assume that X_ℓ is non-empty.

Note that any two \mathbb{R} -equivalent points of $\tilde{X}(\ell)$ map to \mathbb{R} -equivalent points of $X(\ell)$, and so there is a well defined map on \mathbb{R} -equivalence classes. To prove that this map is bijective, by induction we are reduced to the case that $\tilde{X} \rightarrow X$ is a blow-up at a single closed point P of X . If P has residue field larger than ℓ , then $\tilde{X}(\ell) = X(\ell)$ and there is nothing to prove. In the case that P is an ℓ -point, the exceptional divisor E is isomorphic to \mathbb{P}_ℓ^1 ; thus $\tilde{X}(\ell) \rightarrow X(\ell)$ is surjective, and hence so is $\tilde{X}(\ell)/\mathbb{R} \rightarrow X(\ell)/\mathbb{R}$.

For injectivity, suppose that distinct points $Q, Q' \in \tilde{X}(\ell)$ have \mathbb{R} -equivalent images in $X(\ell)$ under $\pi : \tilde{X} \rightarrow X$. If $\pi(Q) = \pi(Q') \in X(\ell)$, then that common image is P , and the points Q, Q' both lie in $E(\ell) \cong \mathbb{P}^1(\ell)$ and hence are \mathbb{R} -equivalent. So we may assume that $\pi(Q), \pi(Q') \in X(\ell)$ are distinct \mathbb{R} -equivalent points; and thus there is a chain of ℓ -points $P_0 = \pi(Q), \dots, P_n = \pi(Q')$ of \tilde{X} , with ℓ -morphisms $f_i : U_i \rightarrow X_\ell$ directly linking P_i to P_{i+1} . Here each P_i is of the form $\pi(Q_i)$ for some point $Q_i \in \tilde{X}(\ell)$, since $\tilde{X}(\ell) \rightarrow X(\ell)$ is surjective. We are thus reduced to the case of two points $Q, Q' \in \tilde{X}(\ell)$ whose images in $X(\ell)$ are distinct and directly \mathbb{R} -equivalent. In this situation there is a single morphism $U \rightarrow X$ from an open subset $U \subseteq \mathbb{P}_\ell^1$ to X , having $\pi(Q)$ and $\pi(Q')$ in its image.

By the valuative criterion for properness, there is a lift of the morphism $U \rightarrow X$ to a morphism $U \rightarrow \tilde{X}$. If $\pi(Q), \pi(Q') \in X$ are each unequal to the blown-up point P , then over

each of these two points there is a unique point of \tilde{X} ; viz., Q and Q' respectively. Thus Q and Q' lie in the image of $U \rightarrow \tilde{X}$ and so are R-equivalent. On the other hand, if one of these two points of X , say $\pi(Q')$, is the blown-up point P , then the image of $U \rightarrow \tilde{X}$ must contain Q and an ℓ -point Q'' of the exceptional divisor $E \cong \mathbb{P}_\ell^1$ (namely, the point of the image that lies over $\pi(Q')$). Thus Q and Q'' are R-equivalent in \tilde{X} . Also, $Q', Q'' \in E(\ell) \cong \mathbb{P}^1(\ell)$ are directly R-equivalent. So Q and Q' are R-equivalent in \tilde{X}_ℓ . \square

Given a two-dimensional regular local ring \mathcal{O} , by a *regular* height one prime \wp in \mathcal{O} we mean a prime of the form (π) , where π is part of a regular system of parameters $\{\pi, \delta\}$ for \mathcal{O} at its maximal ideal $\mathfrak{m}_\mathcal{O}$. Equivalently, \wp has the property that the one-dimensional local ring \mathcal{O}/\wp is regular; i.e., a discrete valuation ring.

If A is a local ring with residue field ℓ and G is a group scheme over A , then for any $g \in G(A)$ we write $\bar{g} \in G(\ell)$ for the image of g under the reduction map $G(A) \rightarrow G(\ell)$.

Proposition A.7. *Let A be a discrete valuation ring with fraction field L and residue field ℓ . Let G be a reductive group over A such that G_ℓ is anisotropic. If $g_0, g_1 \in G(A)$ are R-equivalent as elements of $G(L)$, then their images $\bar{g}_0, \bar{g}_1 \in G(\ell)$ are also R-equivalent.*

Proof. Let \hat{A} be the completion of A , with fraction field \hat{L} . Thus A, \hat{A} both have residue field ℓ , and the reduction map $G(A) \rightarrow G(\ell)$ factors through \hat{A} . The inclusion $G(A) \hookrightarrow G(\hat{A})$ induces an inclusion $G(L) \hookrightarrow G(\hat{L})$ that takes R-equivalent points to R-equivalent points. Hence it suffices to prove the result with A replaced by \hat{A} . We are therefore reduced to the case that A is complete.

As G_L is a linear algebraic group over the field L , R-equivalent points $g_0, g_1 \in G(A) \subseteq G(L)$ are directly R-equivalent over L , via some rational map $g_L : \mathbb{A}_L^1 \dashrightarrow G_L$ taking $0_L \in \mathbb{A}^1(L)$ to g_0 and $1_L \in \mathbb{A}^1(L)$ to g_1 (see [Gil97, Lemme II.1.1(b)]). We can also view g_L as a rational map $\mathbb{A}_A^1 \dashrightarrow G$. Let \tilde{G} be the closure of $G \subseteq \mathrm{GL}_{n,A}$ in $\mathbb{P}_A^{n^2}$, and let $\pi : \tilde{X} \rightarrow X := \mathbb{A}_A^1$ and $\tilde{g} : \tilde{X} \rightarrow \tilde{G}$ be as in Lemma A.5, taking $Y = \tilde{G}$ there (that lemma applies because A is complete). Thus $\tilde{X} \rightarrow X$ is a succession of blow-ups at closed points, and the morphism $\tilde{g} : \tilde{X} \rightarrow \tilde{G}$ lifts the rational map $g_L : X \dashrightarrow G$. Hence \tilde{g} and $g_L \circ \pi$ agree as rational maps. In particular, for $i = 0, 1$, we have $\tilde{g}(i_L) = g_i \in G(L)$.

Since $\tilde{X} \rightarrow X$ is an isomorphism away from the closed fiber, we may identify \tilde{X}_L with $X_L = \mathbb{A}_L^1$. Let $\tilde{0}, \tilde{1} \in \tilde{X}(A)$ be the unique lifts of the points $0_L, 1_L \in \tilde{X}_L(L) = \mathbb{A}^1(L)$ (corresponding to their closures in \tilde{X}), and let $P_0, P_1 \in \tilde{X}(\ell)$ be their images under $\tilde{X}(A) \rightarrow \tilde{X}(\ell)$. Since any two ℓ -points of $X_\ell = \mathbb{A}_\ell^1 \subset X = \mathbb{A}_A^1$ are R-equivalent in X_ℓ , it follows from Lemma A.6 that the same holds for two ℓ -points of \tilde{X}_ℓ ; and in particular, P_0, P_1 are R-equivalent in $\tilde{X}(\ell)$.

Fix $i \in \{0, 1\}$. Then the morphism $g_i : \mathrm{Spec}(A) \rightarrow G \subset \tilde{G}$ and the composition $\mathrm{Spec}(A) \xrightarrow{\tilde{i}} \tilde{X} \xrightarrow{\tilde{g}} \tilde{G}$ agree at the generic point $\mathrm{Spec}(L)$ of $\mathrm{Spec}(A)$, because $\tilde{g}(i_L) = g_i \in G(L)$. Therefore these two morphisms agree, since A is a discrete valuation ring and \tilde{G} is separated. Hence the closed point of $\mathrm{Spec}(A)$ has the same image under these two morphisms. That is,

$$(9) \quad \tilde{g}(P_i) = \bar{g}_i \in G(\ell)$$

for $i = 0, 1$.

We claim, moreover, that for every $P \in \tilde{X}(\ell)$, the point $\tilde{g}(P) \in \tilde{G}(\ell)$ lies in $G(\ell)$. To see this, first choose a regular height one prime \wp in the local ring $\mathcal{O} := \mathcal{O}_{\tilde{X}, P}$ such that $\tilde{g}(\wp) \in G$, where we view $\wp \in \text{Spec}(\mathcal{O})$. (For example, for any regular system of parameters x, y and any $n \geq 0$, the principal ideal $(x + y^n)$ is a regular height one prime \wp of \mathcal{O} , and for all but finitely many n the point $\wp \in \text{Spec}(\mathcal{O})$ lies in the dense open subset $\tilde{g}^{-1}(G)$.) Let B be the completion of the discrete valuation ring \mathcal{O}/\wp , and let E be the fraction field of B . Thus $\tilde{g}(\wp) \in G(E)$. Since G_ℓ is anisotropic, $G(E) = G(B)$ by Proposition A.3. That is, the rational map $\text{Spec}(B) \dashrightarrow G$ extends to a morphism $\text{Spec}(B) \rightarrow G$ giving a commutative diagram:

$$(10) \quad \begin{array}{ccc} & \text{Spec}(B) & \\ \swarrow & \downarrow & \searrow \\ \tilde{X} & & G \\ \searrow & \tilde{g} \downarrow & \swarrow \\ & \tilde{G} & \end{array}$$

As the closed point of $\text{Spec}(B)$ maps to $P \in \tilde{X}(\ell)$ (in the left hand side of (10)) and maps to a point of $G(\ell) \subseteq \tilde{G}(\ell)$ (in the right hand side), it follows from commutativity of the diagram that $\tilde{g}(P) \in G(\ell)$ as claimed.

Since P_0, P_1 are R-equivalent in $\tilde{X}(\ell)$, and since $\tilde{g}(P_i) = \bar{g}_i \in G(\ell)$ for $i = 0, 1$ (by display (9)), in order to conclude the proof it suffices to show that R-equivalent points in $\tilde{X}(\ell)$ have R-equivalent images in $G(\ell)$ under \tilde{g} . For this, it suffices to consider points $P, P' \in \tilde{X}(\ell)$ that are directly R-equivalent; i.e., such that there is a rational map $\phi : \mathbb{A}_\ell^1 \dashrightarrow \tilde{X}$ connecting P and P' . By the previous paragraph, $P, P' \in \tilde{g}^{-1}(G)(\ell)$; and so the generic point of \mathbb{P}_ℓ^1 is also mapped by ϕ into $\tilde{g}^{-1}(G)$. We thus obtain a rational map $g \circ \phi : \mathbb{A}_\ell^1 \dashrightarrow G$, showing that $\tilde{g}(x), \tilde{g}(x') \in G(\ell)$ are (directly) R-equivalent. \square

Proposition A.8 (Decomposition of elements in reductive groups). *Let A be a regular local ring of dimension at most 2 and let G be a reductive group over A . Let $S \subseteq G$ be a maximal split torus, and let $H = C_G(S)$. Then H is reductive, and there are parabolic subgroups of G with unipotent radicals $\mathcal{U}, \mathcal{U}'$, respectively, such that the map*

$$H \times_A \mathcal{U} \times_A \mathcal{U}' \rightarrow G$$

induced by multiplication is an open immersion that gives an isomorphism of $H \times_A \mathcal{U} \times_A \mathcal{U}'$ with the dense open subset $\mathcal{C} = H\mathcal{U}\mathcal{U}'$ of G .

Proof. Choose a minimal parabolic subgroup P of G containing S , and let \mathcal{U} be the unipotent radical of P . The group H is a reductive Levi subgroup of P ; the group P is a semi-direct product of H with the normal subgroup \mathcal{U} ; and $H\mathcal{U} \cong H \times_A \mathcal{U}$ as schemes via the multiplication map (see [SGA70c, Exp. XXVI, Proposition 1.6, Proposition 6.8, Remark 6.18]). By [SGA70c, Exp. XXVI, Theorem 4.3.2], there is a unique parabolic subgroup P' of G such that $P \cap P' = H$, and its unipotent radical \mathcal{U}' has the property that the multiplication map $P \times_A \mathcal{U}' \rightarrow G$ is an open immersion. (Here we apply the inversion map on G to reverse the order of the factors from the one given in part (vi) of that theorem.) We conclude that the composition $H \times_A \mathcal{U} \times_A \mathcal{U}' \rightarrow P \times_A \mathcal{U}' \rightarrow G$ is an open immersion with image $\mathcal{C} = H\mathcal{U}\mathcal{U}'$. That is, $H \times_A \mathcal{U} \times_A \mathcal{U}' \rightarrow P \times_A \mathcal{U}' \rightarrow \mathcal{C}$ is an isomorphism and $\mathcal{C} \subseteq G$ is a dense open subset. \square

Corollary A.9. *With notation as in Proposition A.8, let L be the fraction field of A .*

- (a) *The isomorphism $H \times \mathcal{U} \times \mathcal{U}' \xrightarrow{\sim} \mathcal{C} \subseteq G$ and the map $H \times \mathcal{U} \times \mathcal{U}' \rightarrow H \rightarrow H/S$ defined by projection induce bijections $G(L)/R \leftarrow \mathcal{C}(L)/R \rightarrow H(L)/R \rightarrow (H/S)(L)/R$ that together define a group isomorphism $G(L)/R \xrightarrow{\sim} (H/S)(L)/R$.*
- (b) *If, in addition, A is a Henselian discrete valuation ring, then every element of $G(L)$ is R -equivalent to an element of $H(A)$.*

Proof. If L is a finite field (and so is equal to A), then the set of R -equivalence classes of L -points in any linear algebraic group over L consists of just one element, by [CTS77, Corollaire 6, p. 202]. In particular, this applies to the groups $G, H, S, \mathcal{U}, \mathcal{U}'$; and also to $\mathcal{C}(L)/R$ since $\mathcal{C} = H\mathcal{U}\mathcal{U}'$. So the corollary is trivial in that case.

Thus we may assume that L is infinite. Since G_L is reductive over L , and since $\mathcal{C} \subseteq G$ is a dense open subset, [CTS77, Proposition 11, p. 196] then says that the natural map $\mathcal{C}(L)/R \rightarrow G(L)/R$ is a bijection. For part (a) we next show that we have bijections $\mathcal{C}(L)/R \rightarrow H(L)/R \rightarrow (H/S)(L)/R$.

By [SGA70c, Exp. XXVI, §2, Corollaire 2.5], the unipotent radicals \mathcal{U} and \mathcal{U}' are isomorphic to affine spaces \mathbb{A}_A^n for some values of n . Thus the morphism $\mathcal{C} \xrightarrow{\sim} H \times \mathcal{U} \times \mathcal{U}' \rightarrow H$ induces a bijection $\mathcal{C}(L)/R \rightarrow H(L)/R$ on R -equivalence classes. Since S is a split torus, it is isomorphic to some power of $\mathbb{G}_{m,A}$; so each fiber of the S -torsor $H \rightarrow H/S$ over an L -point of H/S contains L -points, all of which are R -equivalent. Thus $H(L)/R \rightarrow (H/S)(L)/R$ is surjective. For injectivity, if two L -points P, Q of H/S are R -equivalent, then they are connected by a smooth rational affine L -curve $C \subseteq H/S$; and the restriction of the S -torsor $H \rightarrow H/S$ to C is trivial, since C has trivial Picard group. Thus C lifts to a smooth rational affine curve $\tilde{C} \subseteq H$, and \tilde{C} meets the fibers of $H \rightarrow H/S$ over P and Q at L -points of H . So all the L -points of H lying over P are R -equivalent to those over Q , proving the bijectivity of $H(L)/R \rightarrow (H/S)(L)/R$.

Next, observe that the above bijection $H(L)/R \rightarrow G(L)/R$ carries the R -equivalence class of $h \in H(L)$ to the R -equivalence class of h in $G(L)$; and so it agrees with the homomorphism induced by $H \rightarrow G$ (which is then an isomorphism). Since the bijection $H(L)/R \rightarrow (H/S)(L)/R$ is the map induced by $H \rightarrow H/S$, it is also an isomorphism, and hence so is $G(L)/R \xrightarrow{\sim} (H/S)(L)/R$, completing part (a).

Since $H(L)/R \rightarrow G(L)/R$ is an isomorphism, every element of $G(L)$ is R -equivalent to an element of $H(L)$. In the case that A is a Henselian discrete valuation ring, Proposition A.4(b) says that $H(L) = H(A)S(L)$. Since S_L is a power of $\mathbb{G}_{m,L}$, any two L -points of S are R -equivalent, and it follows that every element of $H(L)$ is R -equivalent to an element of $H(A)$. This proves part (b). \square

Below we introduce our specialization map for a reductive group G over a discrete valuation ring A with fraction field L and residue field ℓ . Given another discrete valuation ring A' , we will consider injective local homomorphisms between them; i.e., injective homomorphisms $\phi : A \rightarrow A'$ of discrete valuation rings such that $\phi(\mathfrak{m}_A) \subseteq \mathfrak{m}_{A'}$. Such maps ϕ induce inclusions $L \rightarrow L'$ and $\ell \rightarrow \ell'$, where L', ℓ' are the fraction field and residue field of A' , respectively.

Here and below, we write $[\]_R$ to denote the R -equivalence class of an element. For a reductive group G over a domain A with fraction field L , by $G(A)/R$ we will mean the subset of $G(L)/R$ consisting of R -equivalence classes that contain an element of $G(A)$.

Theorem A.10. *Let A be a discrete valuation ring with fraction field L and residue field ℓ , and let G be a reductive group over A .*

(a) *There is a unique homomorphism*

$$\mathrm{sp} = \mathrm{sp}_A = \mathrm{sp}_{A,G} : G(L)/\mathbb{R} \rightarrow G(\ell)/\mathbb{R}$$

such that $\mathrm{sp}([g]_{\mathbb{R}}) = [\bar{g}]_{\mathbb{R}}$ for all $g \in G(A) \subseteq G(L)$, and which is functorial in G and in A (in the latter case with respect to injective local homomorphisms of discrete valuation rings).

(b) *If A is Henselian, then $\mathrm{sp} : G(L)/\mathbb{R} \rightarrow G(\ell)/\mathbb{R}$ is surjective.*

Proof. Since $G(\ell)/\mathbb{R}$ is trivial if ℓ is finite (by [CTS77, Corollaire 6, p. 202]), we may assume that ℓ is infinite.

For part (a), we first consider the case of complete discrete valuation rings A . Given such an A , choose a maximal split torus $S \subset G$, and let $H = C_G(S)$. We have a natural bijection $G(L)/\mathbb{R} \xrightarrow{\sim} (H/S)(L)/\mathbb{R}$, by Corollary A.9(a). Here H/S is anisotropic over L by Proposition A.4(a), and over ℓ by Proposition A.2(c) since A is complete. Thus $(H/S)(L) = (H/S)(A)$ by Proposition A.3, and the reduction map $(H/S)(A) \rightarrow (H/S)(\ell)$ induces a homomorphism $(H/S)(A)/\mathbb{R} \rightarrow (H/S)(\ell)/\mathbb{R}$ by Proposition A.7. Moreover S_{ℓ} is a maximal split torus in G_{ℓ} by Proposition A.1, and $H_{\ell} = C_{G_{\ell}}(S_{\ell})$ (as noted after Proposition A.4); and so we have an isomorphism $(H/S)(\ell)/\mathbb{R} \xrightarrow{\sim} G(\ell)/\mathbb{R}$ by Corollary A.9(a). We define the map sp_A as the composition

$$G(L)/\mathbb{R} \xrightarrow{\sim} (H/S)(L)/\mathbb{R} = (H/S)(A)/\mathbb{R} \rightarrow (H/S)(\ell)/\mathbb{R} \xrightarrow{\sim} G(\ell)/\mathbb{R}.$$

This composition is a homomorphism since each of the factors is.

To show that $\mathrm{sp}([g]_{\mathbb{R}}) = [\bar{g}]_{\mathbb{R}}$ for all $g \in G(A)$ (still under the assumption that A is complete), we first suppose that $g \in \mathcal{C}(A)$, with \mathcal{C} as in Proposition A.8. Then, with $\mathcal{U}, \mathcal{U}'$ as in that proposition, $\mathrm{sp}([g]_{\mathbb{R}})$ is the image of g under the composition $\mathcal{C}(A) = \mathcal{U}'(A) \times \mathcal{U}(A) \times H(A) \rightarrow H(A) \rightarrow (H/S)(A) \rightarrow (H/S)(A)/\mathbb{R} \rightarrow (H/S)(\ell)/\mathbb{R} \rightarrow G(\ell)/\mathbb{R}$. It is thus equal to $[\bar{g}]_{\mathbb{R}}$, for $g \in \mathcal{C}(A)$.

To prove that $\mathrm{sp}([g]_{\mathbb{R}}) = [\bar{g}]_{\mathbb{R}}$ for an arbitrary element $g \in G(A)$ (i.e., not necessarily in $\mathcal{C}(A)$), note that the dense open subset $\mathcal{C} \subseteq G$ meets the closed fiber G_{ℓ} non-trivially (e.g., at the identity), as does its translate $g\mathcal{C}$. So the intersection $\mathcal{C} \cap g\mathcal{C}$ is open and dense, and also meets G_{ℓ} non-trivially. But G_{ℓ} is unirational, by [Bor91, Chap. V, Theorem 18.2]. Since ℓ is infinite, it follows that $\mathcal{C} \cap g\mathcal{C} \cap G_{\ell}$ contains a ℓ -point. That ℓ -point lifts to a point $c \in G(A)$ by the smoothness of G and the completeness of A ; and c lies in $\mathcal{C}(A) \cap g\mathcal{C}(A)$ since its reduction lies in the open set $\mathcal{C} \cap g\mathcal{C}$. Thus $c = gc'$ for some $c' \in \mathcal{C}(A)$. So $g = c(c')^{-1} \in G(A)$. Since $c, c' \in \mathcal{C}(A)$ and since sp is a homomorphism, the above special case yields $\mathrm{sp}([g]_{\mathbb{R}}) = \mathrm{sp}([c]_{\mathbb{R}}) \mathrm{sp}([(c')^{-1}]_{\mathbb{R}}) = [\bar{c}]_{\mathbb{R}} [(\bar{c}')^{-1}]_{\mathbb{R}} = [\bar{g}]_{\mathbb{R}}$, as needed.

The functoriality condition on sp with respect to G and A asserts the commutativity of the diagrams

$$\begin{array}{ccc} G(L)/\mathbb{R} & \xrightarrow{\mathrm{sp}_A} & G(\ell)/\mathbb{R} \\ \downarrow & & \downarrow \\ G'(L)/\mathbb{R} & \xrightarrow{\mathrm{sp}_A} & G'(\ell)/\mathbb{R} \end{array} \qquad \begin{array}{ccc} G(L)/\mathbb{R} & \xrightarrow{\mathrm{sp}_A} & G(\ell)/\mathbb{R} \\ \downarrow & & \downarrow \\ G(L')/\mathbb{R} & \xrightarrow{\mathrm{sp}_{A'}} & G(\ell')/\mathbb{R}, \end{array}$$

where $G \rightarrow G'$ is a morphism of reductive groups over A , and where $A \rightarrow A'$ is an injective local homomorphism of discrete valuation rings with L' the fraction field of A' and $\ell \rightarrow \ell'$ the induced map on residue fields (which exists by the local homomorphism hypothesis). In the current situation in which the discrete valuation rings that we consider are assumed complete, the commutativity of these diagrams follows from the facts that the vertical maps are well defined (on R-equivalence classes); that every R-equivalence class in $G(L)/R$ contains an element of $H(A) \subseteq G(A)$ (by Corollary A.9(b)); and that sp is given by reduction modulo the maximal ideal for elements of $G(A)$.

For uniqueness, suppose that sp'_A is another such functorial map defined for complete discrete valuation rings A . As above, by Corollary A.9(b), each $g \in G(L)$ is R-equivalent to some $h \in H(A) \subseteq G(A) \subseteq G(L)$. Thus $\text{sp}'([g]_R) = \text{sp}'([h]_R) = [\bar{h}]_R = \text{sp}([h]_R) = \text{sp}([g]_R)$. This shows that $\text{sp}' = \text{sp}$, proving uniqueness and completing the proof of (a) in the case of complete rings A .

For a more general discrete valuation ring A , say with completion \widehat{A} , define $\text{sp}_A = \text{sp}_{\widehat{A}} \circ \iota_A$, where $\iota_A : A \rightarrow \widehat{A}$ is the natural injection. This is still a well defined homomorphism. It follows immediately from this definition and from the complete case that $\text{sp}([g]_R) = [\bar{g}]_R$ for all $g \in G(A)$. Functoriality also follows from the complete case, because a local homomorphism of discrete valuation rings induces such a homomorphism between their completions, and because $\text{sp}_A = \text{sp}_{\widehat{A}} \circ \iota_A$. Uniqueness similarly follows from the complete case, since if sp'_A is another such functorial map, then sp'_A factors through $\text{sp}'_{\widehat{A}}$ by functoriality and thus $\text{sp}'_A = \text{sp}_A$. This completes the proof of part (a) in the general case.

Finally, if A is Henselian, then $G(A) \rightarrow G(\ell)$ is surjective by formal smoothness of G . Thus $G(A)/R \rightarrow G(\ell)/R$ is surjective, and hence so is $\text{sp} : G(L)/R \rightarrow G(\ell)/R$, proving part (b). \square

We call the map sp in Theorem A.10 the *specialization map* for G over the discrete valuation ring A .

Remark A.11. (a) As a consequence of the uniqueness assertion in Theorem A.10, the above map sp agrees with the map in [Gil04, Théorème 0.2], under the hypothesis that $\text{char}(\ell) \neq 2$.

(b) Given a discrete valuation ring A with fraction field L and residue field ℓ , let \widehat{A} be the completion of A , say with fraction field \widehat{L} (and again residue field ℓ). By the construction in the above proof (or by functoriality), the map $\text{sp} = \text{sp}_A : G(L)/R \rightarrow G(\ell)/R$ then factors through $\text{sp}_{\widehat{A}} : G(\widehat{L})/R \rightarrow G(\ell)/R$. As a result, the study of the specialization map on discrete valuation rings can be reduced to the complete case.

A.3. Specialization in dimension two. We now turn to the case of a two-dimensional regular local ring A , with fraction field L and residue field ℓ . To define the specialization map in this situation, we can use Theorem A.10 twice in succession to obtain a specialization map for reductive groups over A . Namely, we pick a regular height one prime \wp in A (see the discussion just before Proposition A.7). The localization A_\wp is a discrete valuation ring with fraction field L , and its residue field $k_\wp := \text{frac}(A/\wp)$ is itself the fraction field of a discrete valuation ring that has residue field ℓ . So for any reductive group G over A we can take the

composition

$$(11) \quad \mathrm{sp}_\varphi = \mathrm{sp}_{A,\varphi} = \mathrm{sp}_{A,\varphi,G} : G(L)/\mathbb{R} \xrightarrow{\mathrm{sp}_{A,\varphi}} G(k_\varphi)/\mathbb{R} \xrightarrow{\mathrm{sp}_{A/\varphi}} G(\ell)/\mathbb{R}$$

of specialization maps associated to discrete valuation rings.

Proposition A.12. *Let A be a two-dimensional regular local ring with fraction field L and residue field ℓ , let φ be a regular height one prime of A , and let G be a connected reductive group over A . Then the following hold:*

- (a) *The map $\mathrm{sp}_{A,\varphi} : G(L)/\mathbb{R} \rightarrow G(\ell)/\mathbb{R}$ is a group homomorphism.*
- (b) *For all $g \in G(A)$, we have $\mathrm{sp}_{A,\varphi}([g]_{\mathbb{R}}) = [\bar{g}]_{\mathbb{R}}$.*
- (c) *The map $\mathrm{sp}_{A,\varphi}$ is surjective if A is complete.*
- (d) *The map $\mathrm{sp}_{A,\varphi}$ factors through $\mathrm{sp}_{\widehat{A},\widehat{\varphi}}$, where \widehat{A} is the completion of A and $\widehat{\varphi}$ is the unique height one prime of \widehat{A} lying over φ .*

Proof. This is immediate from the definition of $\mathrm{sp}_{A,\varphi}$ in display (11) together with Theorem A.10, where in part (d) we use the functoriality of sp for discrete valuation rings. \square

Above, if we write L_φ for the fraction field of the completion \widehat{A}_φ of the discrete valuation ring A_φ , then the first map $\mathrm{sp}_{A,\varphi} : G(L)/\mathbb{R} \rightarrow G(k_\varphi)/\mathbb{R}$ in (11) is the same as the composition

$$(12) \quad G(L)/\mathbb{R} \rightarrow G(L_\varphi)/\mathbb{R} \xrightarrow{\mathrm{sp}_{\widehat{A}_\varphi}} G(k_\varphi)/\mathbb{R},$$

by Remark A.11(b).

In the case that A is complete and has fraction field L and residue field ℓ , we can also describe the map sp_φ in terms of a maximal split torus $S \subseteq G$ and its centralizer $H = C_G(S)$. Namely, by Corollary A.9(a), $G(L)/\mathbb{R} \xrightarrow{\sim} (H/S)(L)/\mathbb{R}$. The group $(H/S)_L$ is anisotropic by Proposition A.4(a); and then so are $(H/S)_\ell$ and hence $(H/S)_{A/\varphi}$ and $(H/S)_{\widehat{A}_\varphi}$ by successive applications of Proposition A.2. Together with Proposition A.3, we can then reformulate $\mathrm{sp}_{A,\varphi,G}$ as the following composition:

$$(13) \quad \begin{aligned} G(L)/\mathbb{R} &\xrightarrow{\sim} (H/S)(L)/\mathbb{R} \rightarrow (H/S)(L_\varphi)/\mathbb{R} = (H/S)(\widehat{A}_\varphi)/\mathbb{R} \\ &\rightarrow (H/S)(k_\varphi)/\mathbb{R} = (H/S)(A/\varphi)/\mathbb{R} \\ &\rightarrow (H/S)(\ell)/\mathbb{R} \xrightarrow{\sim} G(\ell)/\mathbb{R}, \end{aligned}$$

or equivalently as

$$(14) \quad G(L)/\mathbb{R} \xrightarrow{\sim} (H/S)(L)/\mathbb{R} \xrightarrow{\mathrm{sp}_{A,\varphi,H/S}} (H/S)(\ell)/\mathbb{R} \xrightarrow{\sim} G(\ell)/\mathbb{R}.$$

In order to prove the analog of Theorem A.10, we will show that this map is independent of the choice of regular height one prime φ , with the help of the next proposition.

Proposition A.13. *Let A be a two-dimensional regular local ring with fraction field L and residue field ℓ . Let $G \subseteq \mathrm{GL}_{n,A}$ be a reductive group over A such that G_ℓ is anisotropic. Let $g \in G(L)$, viewed as a rational map $X := \mathrm{Spec}(A) \dashrightarrow G$. Then there is a morphism $\widetilde{X} \rightarrow X$, obtained by a sequence of blow-ups at closed points, such that*

- (i) *the rational map $g : X \dashrightarrow G$ lifts to a unique morphism $\widetilde{g} : \widetilde{X} \rightarrow \widetilde{G}$, where \widetilde{G} is the closure of G in $\mathbb{P}_A^{n^2}$; and*

(ii) for each regular height one prime \wp of A , there is a point $P \in \tilde{X}(\ell)$ with $\tilde{g}(P) \in G(\ell)$, such that $\text{sp}_{\wp}([g]_{\mathbb{R}}) \in G(\ell)/\mathbb{R}$ is the \mathbb{R} -equivalence class of $\tilde{g}(P)$.

Proof. We claim that it suffices to prove this in the case of complete rings. Namely, given A, L, ℓ as above, let \hat{A} be the completion of A and let \hat{L} be the fraction field of \hat{A} . Then a regular height one prime \wp of A extends to a regular height one prime $\hat{\wp}$ of \hat{A} , and the diagram

$$(15) \quad \begin{array}{ccc} G(L)/\mathbb{R} & \xrightarrow{\text{sp}_{A, \wp}} & G(\ell)/\mathbb{R} \\ \downarrow & & \parallel \\ G(\hat{L})/\mathbb{R} & \xrightarrow{\text{sp}_{\hat{A}, \hat{\wp}}} & G(\ell)/\mathbb{R} \end{array}$$

commutes by Proposition A.12(d). Moreover every succession of blow-ups of $\text{Spec}(\hat{A})$ at closed points is induced by a succession of blow-ups \tilde{X} of $\text{Spec}(A)$ at closed points, since pulling back from $\text{Spec}(A)$ to $\text{Spec}(\hat{A})$ preserves the closed fiber. Now the locus of indeterminacy of a rational map $\tilde{X} \dashrightarrow \tilde{G}$ consists just of closed points P , and a rational map is defined at P if and only if it is defined on the complete local ring at P . Since the pullback from $\text{Spec}(A)$ to $\text{Spec}(\hat{A})$ preserves complete local rings at closed points, it preserves whether a given rational map is a morphism. Thus we may replace the given ring A by \hat{A} ; and so we will assume that A is complete.

Applying Lemma A.5 to the rational map $g : X \dashrightarrow G \subseteq \tilde{G}$ (thus taking $Y = \tilde{G}$ in that lemma), we obtain a succession of blow-ups $\tilde{X} \rightarrow X$ at closed points satisfying assertion (i) of the proposition (where the uniqueness of the lift $\tilde{g} : \tilde{X} \rightarrow \tilde{G}$ follows from the fact that regular morphisms from an integral scheme to a separated scheme are equal if they agree as rational maps). It remains to show that assertion (ii) holds.

Let \wp be a regular height one prime of A , viewed as a codimension one point of X . Since $\tilde{X} \rightarrow X$ is an isomorphism away from the closed point of X , there is a unique codimension one point $\tilde{\wp}$ of \tilde{X} that lies over \wp , such that \wp and $\tilde{\wp}$ have the same local rings and residue fields. Thus there is a section of $\tilde{X} \rightarrow X$ over the point \wp , taking \wp to $\tilde{\wp}$. Since \wp is regular, A/\wp is a discrete valuation ring; and by the valuative criterion for properness, the section over \wp extends to a section σ over $\text{Spec}(A/\wp)$. Let $P \in \tilde{X}(\ell)$ be the image of the closed point of X under σ ; write $\tilde{A} = \mathcal{O}_{\tilde{X}, P}$; and let $Z \subset \tilde{X}$ be the (isomorphic) image of $\text{Spec}(A/\wp)$ in \tilde{X} under σ . (Here we identify the codimension one point $\tilde{\wp} \in \tilde{X}$ with the corresponding height one prime ideal of \tilde{A} .) Thus $L = \text{frac}(\tilde{A})$. Also, $Z = \text{Spec}(B)$, where $B := \tilde{A}/\tilde{\wp}$ is a discrete valuation ring isomorphic to A/\wp , whose fraction field $k_{\tilde{\wp}}$ is isomorphic to $k_{\wp} = A_{\wp}/\wp = \text{frac}(A/\wp)$.

Since G_{ℓ} is anisotropic, and since ℓ is the residue field of the complete discrete valuation ring A/\wp , whose fraction field is k_{\wp} , it follows from Proposition A.2 that $G_{k_{\wp}}$ is anisotropic. But k_{\wp} is also the residue field of the discrete valuation ring A_{\wp} , whose fraction field is L . So it follows from Proposition A.3 that the element $g \in G(L)$ lies in $G(A_{\wp})$, and thus restricts to an element $\bar{g} \in G(k_{\wp})$. Since \bar{g} lifts g , it follows that $\tilde{g} : \tilde{X} \rightarrow \tilde{G}$ carries the point $\tilde{\wp}$ to a point of G . Now $\tilde{\wp}$ is the generic point of $Z = \text{Spec}(B)$, and B is a complete discrete valuation ring having fraction field $k_{\tilde{\wp}}$ and residue field $\ell = \kappa(P)$. So $G(k_{\tilde{\wp}}) = G(B)$ by

Proposition A.3; and thus the restriction of \tilde{g} to $\tilde{\wp}$ extends to a morphism $h : Z \rightarrow G \subset \tilde{G}$. But the restriction $\tilde{g}|_Z$ of \tilde{g} to Z is a morphism $Z \rightarrow \tilde{G}$ that agrees with h at the point $\tilde{\wp}$. Since \tilde{G} is separated, it follows that $h = \tilde{g}|_Z$; and hence $\tilde{g}(P) = h(P) \in G(\ell)$.

By displays (11) and (12), the specialization map $\text{sp}_{A,\wp} : G(L)/\mathbb{R} \rightarrow G(\ell)/\mathbb{R}$ factors through the natural map $G(L)/\mathbb{R} \rightarrow G(L_\wp)/\mathbb{R}$, where L_\wp is the fraction field of \hat{A}_\wp . Similarly, $\text{sp}_{\tilde{A},\tilde{\wp}}$ factors through $G(L)/\mathbb{R} \rightarrow G(L_{\tilde{\wp}})/\mathbb{R}$, where $L_{\tilde{\wp}}$ is the fraction field of the completion of $\tilde{A}_{\tilde{\wp}}$. But the natural map $G(L_\wp) \rightarrow G(L_{\tilde{\wp}})$ is an isomorphism, since $A_\wp \rightarrow \tilde{A}_{\tilde{\wp}}$ is. With respect to this isomorphism, the one-dimensional specialization maps sp_{A_\wp} and $\text{sp}_{\tilde{A}_{\tilde{\wp}}}$ in display (11) are identified with the corresponding maps $\text{sp}_{\tilde{A}_{\tilde{\wp}}}$ and $\text{sp}_{\tilde{A}/\tilde{\wp}}$. Hence $\text{sp}_{A,\wp}([g]_{\mathbb{R}}) = \text{sp}_{\tilde{A},\tilde{\wp}}([g]_{\mathbb{R}}) = \text{sp}_{\tilde{A},\tilde{\wp}}([\tilde{g}]_{\mathbb{R}}) = [\tilde{g}(P)]_{\mathbb{R}}$ by Proposition A.12, since $\tilde{g} \in G(\tilde{A})$. \square

Using the above, we now show that the map sp_\wp is independent of \wp , yielding a *two-dimensional specialization map* sp that depends just on A and G . We also show other key properties of the map sp .

Theorem A.14. *Let A be a regular local ring of dimension two, with fraction field L and residue field ℓ . Let G be a reductive group over A . Then*

- (a) *There is a unique group homomorphism $\text{sp} = \text{sp}_A = \text{sp}_{A,G} : G(L)/\mathbb{R} \rightarrow G(\ell)/\mathbb{R}$ such that for every regular height one prime \wp of A , $\text{sp} = \text{sp}_\wp$.*
- (b) *If A is complete, then sp is surjective.*
- (c) *For $g \in G(A)$, $\text{sp}_A([g]_{\mathbb{R}}) = [\bar{g}]_{\mathbb{R}}$, where \bar{g} is the image of g under the homomorphism $G(A) \rightarrow G(\ell)$ induced by the reduction map $A \rightarrow \ell$.*
- (d) *The map sp is functorial in G and in A (in the latter case with respect to injective local homomorphisms of two-dimensional regular local rings).*
- (e) *The group homomorphisms $\text{sp}_A : G(L)/\mathbb{R} \rightarrow G(\ell)/\mathbb{R}$, as A varies over regular local rings of dimension 2, are uniquely characterized by properties (c) and (d).*

Proof. We begin by showing that the map sp_\wp is independent of the choice of the choice of the regular height one prime \wp in A . By Proposition A.12(d), we may assume that A is complete. In that situation, first consider the case where G_ℓ is anisotropic. With notation as in Proposition A.13, for each $g \in G(L)$ we have a morphism $\tilde{X} \rightarrow X$, a lifting $\tilde{g} : \tilde{X} \rightarrow \tilde{G}$ of $g : X \dashrightarrow G$, and a point $P \in \tilde{X}(\ell)$ such that $\tilde{g}(P) \in G(\ell)$ and $\text{sp}_\wp([g]_{\mathbb{R}}) = [\tilde{g}(P)]_{\mathbb{R}}$. Since every point in $\tilde{X}(\ell)$ lies over the closed point of X , it follows from Lemma A.6 that any two such points are \mathbb{R} -equivalent, and hence so are their images under \tilde{g} . Thus sp_\wp is independent of the choice of \wp , if G_ℓ is anisotropic. For a more general reductive group G over the complete ring A , display (14) says that the specialization map sp_\wp for G factors through the corresponding map for the group H/S , which is anisotropic over ℓ . So the assertion follows from the previous case.

Using this, parts (a), (b), and (c) follow from Proposition A.12, by taking $\text{sp} = \text{sp}_\wp$ for any choice of \wp . Part (d) follows from display (11) together with the functoriality in Theorem A.10(a).

Finally, for part (e), suppose that for each regular local ring A of dimension 2 and each reductive group G over A , we have a homomorphism $s_A : G(L)/\mathbb{R} \rightarrow G(\ell)/\mathbb{R}$, with L, ℓ denoting the fraction field and residue field of A , such that $s_A([g]_{\mathbb{R}}) = [\bar{g}]_{\mathbb{R}}$ for all $g \in G(A)$,

and such that the maps s_A are functorial in A and G . We wish to show that $s_A = \text{sp}_A$ for all A and G .

First consider the case that G_ℓ is anisotropic, where A, L, ℓ, G are as above. Let $g \in G(L)$. As in Proposition A.13, we obtain $\tilde{X}, \tilde{G}, \tilde{g}, P$ such that $\text{sp}_\varphi([g]_R) = [\tilde{g}(P)] \in G(\ell)/R$. Here $\tilde{g} : \tilde{X} \rightarrow \tilde{G}$ is a morphism, and $\tilde{g}(P) \in G(\ell)$; so $\tilde{g} \in G(A')$, where A' is the complete local ring of \tilde{X} at P , say with fraction field L' . The inclusion $A \rightarrow A'$ is a local homomorphism of complete regular local rings of dimension two, inducing an isomorphism on residue fields. So, by functoriality, sp_A is the composition of the natural inclusion $G(L) \rightarrow G(L')$ with the map $\text{sp}_{A'}$, and $\text{sp}_A(g) = \text{sp}_{A'}(g) = \text{sp}_{A'}(\tilde{g})$. Since s_A is also functorial, we similarly have $s_A(g) = s_{A'}(\tilde{g})$. But since $\tilde{g} \in G(A')$, it follows from property (c) that $\text{sp}_{A'}(\tilde{g}) = [\tilde{g}(P)]_R = s_{A'}(\tilde{g})$. Thus $\text{sp}_A(g) = s_A(g)$, as desired.

For a more general reductive group G , the same conclusion follows from the above special case by the functoriality condition and by the functorial isomorphisms $G(L)/R \rightarrow (H/S)(L)/R$ as in Corollary A.9, since $(H/S)_\ell$ is anisotropic. \square

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