Zero-cycles for rational surfaces in the function field case : an introduction to recent progress

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The Spring of Rational Points
University of Bath
May 7th, 2024

Let X be a smooth projective variety over a field k. Let $X(k)$ denote the set of k-points (rational points) of X/k.
The Brauer group $\mathrm{Br}(X)$ of X may be defined in a number of ways.
There is a natural evaluation pairing

$$X(k) \times \mathrm{Br}(X) \to \mathrm{Br}(k).$$

Let k be a number field, let $k_v, v \in \Omega$, denote the completions.
Let X be a smooth projective variety over k.
Let $X(\mathbb{A}_k) = \prod_{v \in \Omega} X(k_v)$.
There are evaluation pairings $X(k_v) \times \mathrm{Br}(X) \to \mathrm{Br}(k_v) \subset \mathbb{Q}/\mathbb{Z}$ (last inclusion by local class field theory) which by summation induce a (well-defined) pairing

$$X(\mathbb{A}_k) \times \mathrm{Br}(X) \to \mathbb{Q}/\mathbb{Z}.$$

The Brauer-Manin set of X, denoted $X(\mathbb{A}_k)^{\mathrm{Br}}$, is the left kernel of this pairing.
Manin's original observation :

$$X(k) \subset X(\mathbb{A}_k)^{\mathrm{Br}} \subset X(\mathbb{A}_k).$$

Manin's original observation (1969) :

$$X(k) \subset X(\mathbb{A}_k)^{\mathrm{Br}} \subset X(\mathbb{A}_k).$$

This accounted for many explicit counterexamples to the Hasse principle : one may have $X(\mathbb{A}_k) \neq \emptyset$ but $X(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$.
In 1985 it accounted for the 1966 Cassels-Guy example over $\mathbb{Q}$

$$5x^3 + 9y^3 + 10z^3 + 12t^3 = 0.$$

(Note that any prime occurring divides two coefficients.)
A first unconditional example with $X(\mathbb{A}_k)^{\mathrm{Br}} \neq \emptyset$ and $X(k) = \emptyset$ was built by Skorobogatov (1999). Several have followed.

In 1979, Sansuc and I had asked :
For geometrically rational surfaces X, is $X(k)$ dense in $X(\mathbb{A}_k)^{\mathrm{Br}}$ (weak approximation inside the Brauer-Manin set) ?
[This includes cubic surfaces and conic bundles over the projective line.]
In 1999, I extended the question to all geometrically "rationally connected" varieties (in the sense of Kollár-Miyaoka-Mori). This includes quadric bundles over projective space, geometrically unirational varieties, homogeneous spaces of linear algebraic groups, and smooth Fano varieties (anticanonical bundle ample).

For Fano varieties, a recent statistical support for the conjecture is :
Theorem (Browning, Le Boudec, Sawin, 2023)
Fix $n \geq 4$ and $d \leq n$. The Hasse principle holds for almost all hypersurfaces in $\mathbb{P}^n_{\mathbb{Q}}$ of degree d.
(One does not know which ones.)

Almost all : the hypersurfaces are ordered by the maximal absolute value H of the coefficients of a primitive equation in $\mathbb{Z}[x_0, \ldots, x_n]$ and one counts the number of equations for which the result holds, as H grows.

A similar result for cubic surfaces ($d = 3, n = 3$). This corresponds to the fact, taught to us by experience, that although Brauer-Manin obstruction to the Hasse principle may occur for such surfaces, they very rarely happen.

Progress on the original conjecture has been achieved for several classes of such varieties, for example for a large class of conic bundles over $\mathbb{P}^1_{\mathbb{Q}}$, when the Green-Tao-Ziegler results offered a substitute for Schinzel's hypothesis.

But if one wants to study cubic surfaces, in general the best one can in general do is to fibre them by a pencil of curves of genus one. A very ingenious method was initiated by Swinnerton-Dyer (1993, 1998, 2001, with later developments) but it is conditional on finiteness of Tate-Shafarevich groups of elliptic curves.

Rather than asking for existence of rational points, one may ask for a weaker property.
Given a smooth projective variety X over a field k, one defines its index $I(X/k)$ as the g.c.d. of the degree of finite field extensions L/k with $X(L) \neq \emptyset$. One may ask whether $I(X/k) = 1$.

Over a number field k, one may ask for the following version of the Hasse principle :

For a given X/k, if $I(X_{k_v}) = 1$ for all v, do we have $I(X) = 1$?

The index of the variety $(x^3-2)(x^2-13)(x^2-17)(x^2-221)=0$ over $k=\mathbb{Q}$ is 1, but there is no rational point. It has $\mathbb{Q}_v$-points for all v.

For a smooth projective model X of

$$x^2+y^2-7(t^2-t-1)(t^2+t-1)(t^2-2)=0$$

over $k=\mathbb{Q}$, we have

- $X(\mathbb{A}_{\mathbb{Q}})\neq\emptyset$, hence $I(X_{\mathbb{Q}_v})=1$ for all v
- $X(\mathbb{Q})\subset X(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}}=\emptyset$, hence no rational point.
- $I(X)=1$. There exists a point in a cubic extension of $\mathbb{Q}$.

(CT-Sw.-D. 1994).

To study the index, one is led to the following definitions.
Let X/k be a smooth projective (geom. connected) variety over a field k. A closed point P of X is a reduced irreducible subscheme of dimension 0. It has a finite degree $[k(P):k]$.
A zero-cycle on X is a finite integral linear combination $\sum_P n_P P$, $n_P \in \mathbb{Z}$ of closed points of X. Its degree is

$$deg_k(\sum_P n_P P) := \sum_P n_P [k(P):k].$$

Thus $I(X/k) = 1$ iff there exists a zero-cycle of degree 1.

The zero-cycles on X form an abelian group $Z_0(X)$, equipped with the degree map $deg_k : Z_0(X) \to \mathbb{Z}$.
Given a smooth projective integral curve C/k, a rational function $\phi \in k(C)^\times$ and a k-morphism $p : C \to X$, one may build the zero-cycle on X :

$$p_*(div_C(\phi)) \in Z_0(X).$$

This zero-cycle has degree zero, because "for a rational function on a projective curve, the number of zeros is equal to the number of poles".
The subgroup of $Z_0(X)$ generated by all such $p_*(div_C(\phi))$ is called the group of zero-cycles rationally equivalent to zero. The quotient of $Z_0(X)$ by this subgroup is the Chow group $CH_0(X)$ of zero-dimensional cycles modulo rational equivalence. It inherits a degree map

$$deg_k : CH_0(X) \to \mathbb{Z}.$$

Let X be a smooth projective variety over a field k.
The pairing $X(k) \times \mathrm{Br}(X) \to \mathrm{Br}(k)$ extends to a pairing

$$CH_0(X) \times \mathrm{Br}(X) \to \mathrm{Br}(k).$$

If k is a number field, one then has a complex of abelian groups

$$CH_0(X) \to \prod_{v \in \Omega} CH_0(X_{k_v}) \to \mathrm{Hom}(\mathrm{Br}(X), \mathbb{Q}/\mathbb{Z}).$$

which extends the Manin inclusion $X(k) \subset X(\mathbb{A}_k)^{\mathrm{Br}}$.

In particular, a necessary condition for $I(X) = 1$ is that there exists a family $z_v \in Z_0(X_{k_v})$, $v \in \Omega$, of zero-cycles of degree 1 such that for all $\alpha \in \mathrm{Br}(X)$, one has

$$\sum_v < \alpha, z_v >= 0 \in \mathbb{Q}/\mathbb{Z}.$$

Conjecture (E_1)
For *any* smooth projective geom. connected variety, this necessary condition is sufficient for $I(X) = 1$.

This is a special case of :

Conjecture (E) For *any* smooth projective geom. connected variety X/k, the complex

$$limproj_n CH_0(X)/n \to \prod_{v \in \Omega} limproj_n CH_0(X_{k_v})/n \to \mathrm{Hom}(\mathrm{Br}(X), \mathbb{Q}/\mathbb{Z}).$$

is exact.

(CT/Sansuc 1981, Kato/Saito 1986, CT 1995)

For X a smooth curve, the validity of this conjecture essentially follows from the (would-be) finiteness of the Tate-Shafarevich group of the jacobian of the curve.

Actual progress on these conjectures was achieved for some classes of varieties X equipped with a fibration $\pi : X \to \mathbb{P}^1_k$, when one knows that the conjecture holds for the smooth fibres $X_m, m \in \mathbb{P}^1(k)$, of the fibration.
The case of conic bundles given by a system

$$a(t)X^2 + b(t)Y^2 + c(t)Z^2 = 0$$

in $\mathbb{P}^2 \times \mathbb{A}^1_t$ was the testing ground. A crucial new method is due to Salberger 1988, the widest developments are in work of Harpaz and Wittenberg 2016. But these results are not enough to control cubic surfaces, which one can in general fibre only into curves of genus one.

When things get too hard over a number field, one should try to understand what happens over a global field $\mathbb{F}(C)$, namely the function field of a curve C over a finite field $\mathbb{F}$ of char. $p > 0$. The p-torsion phenomena are often hard to handle. All statements in the present talk should be understood as "for $\ell \neq p$". For instance we only investigate whether the index is prime to ℓ for $\ell \neq p$.

In 2016, Parimala and Suresh proved the vanishing of the third unramified cohomology group $H^3_{nr}(\mathbb{F}(W), \mathbb{Z}/2)$ for $W/\mathbb{F}$ with a conic bundle structure over a surface over a finite field. This implies :

Theorem. *Assume* $\mathrm{char}(\mathbb{F}) \neq 2$. *For* $X/\mathbb{F}(C)$ *a surface equipped with a conic bundle structure over* $\mathbb{P}^1_{\mathbb{F}(C)}$*, conjecture (E) holds.*

Theorem (Zhiyu Tian 2023). *Let* $k = \mathbb{F}(C)$. *Assume* $\mathrm{char}(\mathbb{F}) \neq 3$. *Let* X/k *be a smooth cubic surface. If* $X(\mathbb{A}_k)^{\mathrm{Br}} \neq \emptyset$, *then* $I(X) = 1$.

Theorem (Zhiyu Tian 2023). *Let* $k = \mathbb{F}(C)$. *Assume* $\mathrm{char}(\mathbb{F}) \neq 2$. *Let* X/k *be a del Pezzo surface of degree 4, i.e. a smooth complete intersection of two quadrics in* $\mathbb{P}^4_k$. *If* $X(\mathbb{A}_k)^{\mathrm{Br}} \neq \emptyset$, *then* $X(k) \neq \emptyset$.

What Zhiyu Tian actually proves is conjecture (E) for zero-cycles on these surfaces.

The first step is to translate the conjectures on zero-cycles on $X/\mathbb{F}(C)$ into conjectures on 1-cycles (linear combinations of curves) on a model $\mathcal{X}/C/\mathbb{F}$.

Quite generally, for a projective variety W over a field k, and an integer i, one considers the free abelian group $Z^i(W)$ on reduced irreducible closed subvarieties of codimension i, and define on it

- rational equivalence, with quotient the Chow group $CH^i(W)$
- algebraic equivalence, with an exact sequence

$$0 \to CH^i(W)_{alg} \to CH^i(W) \to CH^i(W)/alg \to 0.$$

Proposition (S. Saito 1989, CT 1999) *Let $\mathbb{F}$ be a finite field, let $C/\mathbb{F}$ be a smooth, projective, geometrically connected curve over $\mathbb{F}$. Let $\mathbb{F}(C)$ be its function field. Let $\mathcal{X}$ be a smooth, projective, geometrically integral variety over $\mathbb{F}$ of dimension d, equipped with a flat morphism $\mathcal{X} \to C$ whose generic fibre $X = \mathcal{X}_\eta/\mathbb{F}(C)$ is smooth and geometrically integral. Let ℓ be a prime number, $\ell \neq \mathrm{char}(\mathbb{F})$.*

a) If for all $\ell \neq p$ the étale cycle class map $\mathrm{CH}^{d-1}(\mathcal{X}) \otimes \mathbb{Z}_\ell \to H^{2d-2}_{et}(\mathcal{X}, \mathbb{Z}_\ell(d))$ is onto then Conjecture (E) holds for $X/\mathbb{F}(C)$.

b) If for all $\ell \neq p$ the étale cycle class map $\mathrm{CH}^{d-1}(\mathcal{X}) \otimes \mathbb{Z}_\ell \to H^{2d-2}_{et}(\mathcal{X}, \mathbb{Z}_\ell(d))$ is onto modulo torsion, then Conjecture (E_1) holds for $X/\mathbb{F}(C)$.

Integral Tate "conjecture"

For a smooth projective variety $\mathcal{X}/\mathbb{F}$ of dimension d, and $\ell \neq p$, we have étale cycle class maps

$$CH^i(\mathcal{X}) \otimes \mathbb{Z}_\ell \to H^{2i}_{et}(\mathcal{X}, \mathbb{Z}_\ell(i)).$$

For $i = 1$, Tate conjectured this map to surjective.
Fo $i > 1$, Tate conjectured the cokernel to be finite. This is still an open problem.
There are examples where this integral cycle map is not onto.
For $i = d$, it is surjective.
For $i = d - 1$ (i.e. 1-cycles) surjectivity is an important open problem.

Let now $\mathcal{X}/\mathbb{F}$ be a threefold. To study the surjectivity of the cycle map

$$\mathrm{CH}^2(\mathcal{X}) \otimes \mathbb{Z}_\ell \to H^4_{et}(\mathcal{X}, \mathbb{Z}_\ell(2))$$

one may use the exact sequence (Hochschild-Serre)

$$0 \to H^1(\mathbb{F}, H^3_{et}(\overline{\mathcal{X}}, \mathbb{Z}_\ell(2))) \to H^4_{et}(\mathcal{X}, \mathbb{Z}_\ell(2)) \to H^4_{et}(\overline{\mathcal{X}}, \mathbb{Z}_\ell(2))^g \to 0$$

where $H^1(\mathbb{F}, H^3_{et}(\overline{\mathcal{X}}, \mathbb{Z}_\ell(2)))$ is finite (Deligne), and try to show that the composite map $\mathrm{CH}^2(\mathcal{X}) \otimes \mathbb{Z}_\ell \to H^4_{et}(\overline{\mathcal{X}}, \mathbb{Z}_\ell(2))^g$ is onto, which would already give conjecture (E_1) for the surface $X/\mathbb{F}(C)$, then try to cover the subgroup $H^1(\mathbb{F}, H^3_{et}(\overline{\mathcal{X}}, \mathbb{Z}_\ell(2)))$ by images of cycle classes with trivial image in $H^4_{et}(\overline{\mathcal{X}}, \mathbb{Z}_\ell(2))$, which would complete the proof of conjecture (E).

Recent work in complex algebraic geometry (Benoist–Ottem 2021) has attracted attention on subtleties regarding the filtration $N^c H^i(X, \mathbb{Z})$ of integral cohomology groups by codimension c of support. Let X be smooth and projective over $\mathbb{C}$ of dimension d. Already in codimension $c = 1$, the case of interest today, given a class in $H^i(X, \mathbb{Z})$ one may ask whether it vanishes by restriction to some nonempty Zariski open set, a property which defines the classical coniveau subgroup $N^1 H^i(X, \mathbb{Z})$, or whether it belongs to the a priori smaller subgroup spanned by the images of all $f_* : H^{i-2}(T, \mathbb{Z}) \to H^i(X, \mathbb{Z})$ for $f : T \to X$ morphism, T *smooth* proper, $dim(T) = d - 1$.
One denotes this subgroup $\tilde{N}^1 H^i(X, \mathbb{Z}) \subset N^1 H^i(X, \mathbb{Z})$. It may be smaller (Benoist–Ottem). There are similar questions and examples in the étale cohomology context (Scavia–Suzuki 2023).

Theorem SS23 (Scavia-Suzuki 2023) *Let $\mathcal{X}/\mathbb{F}$ be smooth and projective of arbitrary dimension. Let $\overline{\mathcal{X}} := \mathcal{X} \times_{\mathbb{F}} \mathbb{F}^{sep}$. Assume*

$$\tilde{N}^1 H^3_{et}(\overline{\mathcal{X}}, \mathbb{Z}_\ell(2)) = N^1 H^3_{et}(\overline{\mathcal{X}}, \mathbb{Z}_\ell(2)).$$

Then the ℓ-adic Abel-Jacobi map on homologically trivial cycles in $\mathrm{CH}^2(\mathcal{X}) \otimes \mathbb{Z}_\ell$, with values in $H^1(\mathbb{F}, H^3_{et}(\overline{\mathcal{X}}, \mathbb{Z}_\ell(2)))$, induces an isomorphism

$$\mathrm{CH}^2(\mathcal{X})_{alg} \otimes \mathbb{Z}_\ell \simeq H^1(\mathbb{F}, N^1 H^3_{et}(\overline{\mathcal{X}}, \mathbb{Z}_\ell(2)))$$

Kollár and Tian 2023 using elaborate deformation arguments prove :

Theorem KT23. *Let $\mathcal{X} \to C$ be a dominant morphism of smooth projective varieties over $\mathbb{F}$, with C a curve and with generic fibre a smooth geometrically separably rationally connected variety over $\mathbb{F}(C)$ (e.g. a geometrically rational variety). The map*

$$\mathrm{CH}_1(\mathcal{X})/alg \to (\mathrm{CH}_1(\overline{\mathcal{X}})/alg)^g$$

is an isomorphism.

Here algebraic equivalence is over $\mathbb{F}$, resp. over $\overline{\mathbb{F}}$.

By a delicate combination of geometric theorems of Kollár-Tian 2023 and of motivic tools (Bloch's higher Chow groups, Merkurjev-Suslin, Suslin-Voevodsky) Tian 2023 shows :

Theorem T23. *Let C/k be a curve over an algebraically closed field. For a threefold W equipped with a morphism $W \to C$ whose generic fibre is a geometrically rational surface, the Scavia-Suzuki condition is fulfilled :*

$$\tilde{N}^1 H^3_{et}(W, \mathbb{Z}_\ell(2)) = N^1 H^3_{et}(W, \mathbb{Z}_\ell(2)).$$

For a 3-fold $W/C/k$ with k algebraically closed and with geometric generic fibre a rational surface, earlier work (including consequences of the Merkurjev-Suslin theorem) gives

$$\mathrm{CH}^2(W)/alg \otimes \mathbb{Z}_\ell \simeq H^4_{et}(W, \mathbb{Z}_\ell(2))$$

$$N^1 H^3_{et}(W, \mathbb{Z}_\ell(2)) = H^3_{et}(W, \mathbb{Z}_\ell(2)).$$

Putting everything together gives the final theorem :

Theorem (Z. Tian 2023) *Let* $\mathcal{X} \to C$ *be a dominant morphism of smooth projective varieties over a finite field* $\mathbb{F}$, *with* $\dim(\mathcal{X}) = 3$ *and* C *a curve, and with generic fibre a smooth geometrically rational surface over* $\mathbb{F}(C)$. *Then the integral cycle class map*

$$\mathrm{CH}^2(\mathcal{X}) \otimes \mathbb{Z}_\ell \to H^4_{et}(\mathcal{X}, \mathbb{Z}_\ell(2))$$

is surjective.

As explained earlier, this implies Conjectures (E) and (E_1) for the surface $X = \mathcal{X} \times_C F(C)$, hence the announced theorems on index and rational points :

Theorem (Tian). Let $k = \mathbb{F}(C)$. Assume $\mathrm{char}(\mathbb{F}) \neq 3$. Let X/k be a smooth cubic surface. If $X(\mathbb{A}_k)^{\mathrm{Br}} \neq \emptyset$, then $I(X) = 1$.
[By Coray's 1974 result there then exists a field extension K/k of degree 1, 4 or 10 such that $X(K) \neq \emptyset$.]

Theorem (Tian). Let $k = \mathbb{F}(C)$. Assume $\mathrm{char}(\mathbb{F}) \neq 2$. Let X/k be a del Pezzo surface of degree 4, i.e. a smooth complete intersection of two quadrics in $\mathbb{P}^4_k$. If $X(\mathbb{A}_k)^{\mathrm{Br}} \neq \emptyset$, then $X(k) \neq \emptyset$.
[The proof gives $I(X) = 1$. One then uses the Amer-Brumer theorem to conclude $X(k) \neq \emptyset$.]

In earlier work, by exclusively geometric methods, Tian had proven the Hasse principle for

• Smooth intersection of two quadrics in $\mathbb{P}^n_{\mathbb{F}(C)}$ for $n \geq 5$, if $char(\mathbb{F}) \geq 2$.

• Smooth cubic hypersurfaces in $\mathbb{P}^n_{\mathbb{F}(C)}$, $n \geq 5$, over $\mathbb{F}(C)$ if $char(\mathbb{F}) \geq 7$.

For a cubic hypersurfaces in $X \subset \mathbb{P}^4_{\mathbb{F}(C)}$, the above result in $\mathbb{P}^3_{\mathbb{F}(C)}$ and a standard technique should give : if $X(\mathbb{A}_k) \neq \emptyset$, then there exists a point of X in an extension of degree 1, 4 or 10.