Arithmetic upon intersections of two quadrics

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Let k be a number field. Let k_v run through the completions of k. Let X be a smooth, projective, geometrically connected k-variety. The set $X(k)$ of rational points embeds diagonally into the adèle space $X(\mathbb{A}_k) = \prod_v X(k_v)$. Let $\mathrm{Br}(X)$ denote the Brauer group of X. There is a pairing

$$X(\mathbb{A}_k) \times \mathrm{Br}(X)/\mathrm{Br}(k) \to \mathbb{Q}/\mathbb{Z}$$

induced by the pairings $X(k_v) \times \mathrm{Br}(X) \to \mathrm{Br}(k_v) \subset \mathbb{Q}/\mathbb{Z}$. The left kernel of the pairing is denoted $X(\mathbb{A}_k)^{\mathrm{Br}}$ and is called the Brauer-Manin set of X.

We have (Manin 1970)

$$X(k) \subset X(\mathbb{A}_k)^{\mathrm{Br}} \subset X(\mathbb{A}_k).$$

This then accounted for most known examples of Hasse principle failures.

The following question is open. It was raised in 1981 for surfaces (CT-Sansuc), and was extended in 2000 to higher dimensional varieties.

X is a **rationally connected variety**, *is $X(k)$ dense in $X(\mathbb{A}_k)^{\mathrm{Br}}$?*

In particular, for such X, if $\mathrm{Br}(X)/\mathrm{Br}(k) = 0$, does the Hasse principle hold ?

In this talk, I shall discuss a (very) special class of rationally connected varieties.

Let $X \subset \mathbb{P}^n_k$, $n \geq 3$ be a smooth complete intersection of two quadrics :

$$f(x_0, \cdots, x_n) = g(x_0, \cdots, x_n) = 0.$$

The general question specializes to :

Conjecture *For $n \geq 5$, for any such X over a number field k, the Hasse principle holds, namely*

$$\prod_v X(k_v) \neq \emptyset \Longrightarrow X(k) \neq \emptyset.$$

For later use :

Along with the Hasse principle problem, often comes the question whether a given smooth projective variety over a number field k satisfies weak approximation, namely the diagonal embedding $X(k) \hookrightarrow \prod_v X(k_v)$ has dense image.

For $X \subset \mathbb{P}^n_k$ a smooth complete intersection of two quadrics over a number field, *if* X *has a* k*-point and* $n \geq 5$, then weak approximation holds (easy).

For the parallel case of global fields of positive characteristic :

Theorem (Zhiyu Tian, 2017). *Let $k = \mathbb{F}(C)$ be the function field of a smooth projective curve C over a finite field $\mathbb{F}$ of odd characteristic. Let k_v run through the completions at closed points of C. For any smooth complete intersection of two quadrics in $\mathbb{P}^n_k$, and $n \geq 5$, the Hasse principle holds.*

The proof is geometric : the hypersurface $X/\mathbb{F}(C)$ extends to a fibration $\mathcal{X} \to C$ of varieties over the finite field $\mathbb{F}$, and Tian studies the spaces of sections of this fibration, which are varieties over the finite field $\mathbb{F}$.

In the rest of the talk, k denotes a number field.

Why $n \geq 5$?

For $n = 3$, the Hasse principle need not hold. One then has a curve of genus one, the obstruction to the Hasse principle is related to the Tate-Shafarevich group of the jacobian of the curve.

For $n = 4$, the Hasse principle need not hold (first smooth example : Birch and Swinnerton-Dyer 1975). Conjecturally, the closure of $X(k)$ is $X(\mathbb{A}_k)^{\mathrm{Br}}$.
Known when $X(k) \neq \emptyset$ (Salberger–Skorobogatov 1991).

Early results for $X \subset \mathbb{P}^n_k$
$n \geq 12$ (Mordell, 1959), $n = 10$ (Swinnerton-Dyer 1964).

Assume k is totally imaginary. Then any quadratic form in at least 5 variables has a nontrivial zero. Let $n = 12$. Assume $f(x_0, \ldots, x_{12})$ is non-degenerate. The quadratic form f may be written as the direct sum of a totally hyperbolic quadratic form in 10 variables and a quadratic form in 3 variables. On a linear space of codimension $5 + 3 = 8$, that is a $\mathbb{P}^4_k$, the form f identically vanishes. The restriction of g to this $\mathbb{P}^4_k$ is given by a quadratic form in 5 variables, it has a nontrivial zero over k.

For smooth X over a real number field, one uses weak approximation on k at the real places and an elegant trick of Mordell over the reals : consider the behaviour of the signature of the quadratic form $af + bg$ as $a, b \in \mathbb{R}$ vary over $a^2 + b^2 = 1$. One proves the existence of quadratic forms in the pencil over $\mathbb{R}$ with 6 hyperbolics.

The Hasse principle for X **smooth** complete intersection of two quadrics in $\mathbb{P}^n_k$ is known to hold :

For $n \geq 8$ (CT–Sansuc–Swinnerton-Dyer 1987) [Note : for $n \geq 8$, one always has $X(k_v) \neq \emptyset$ for v nonarchimedean].

For $n \geq 4$ if X contains two lines globally defined over k (ibidem)
The case $n = 4$ was known before 1970, and is used in the proof.

For $n \geq 5$ if X contains a conic (Salberger 1993).

For $n = 7$ (Heath-Brown 2018).

Conditional result : Taking two difficult conjectures (finiteness of Ш of elliptic curves and Schinzel's hypothesis) for granted, by a method initiated by Swinnerton-Dyer, Wittenberg (2007) gave a proof of the Hasse principle for any smooth X for $n \geq 5$.

The Hasse principle was also proved for smooth projective models Y of some *singular* projective, non conical, geometrically integral, complete intersections X of two quadrics.
In particular, the Hasse principle holds for smooth models Y of $X \subset \mathbb{P}^n_k$ for $n \geq 6$, when X contains a pair of conjugate singular points (CT–Sansuc–Swinnerton-Dyer 1987).
These results in the singular case also play a key rôle in the proof of the smooth case in higher dimension, via the fibration method.

Let k be a number field. Let X be a possibly singular, geometrically integral, non-conical complete intersection of two quadrics in $\mathbb{P}^n_k$. Let Y/k be a smooth projective model of X. An algebraic computation gives : for $n \geq 6$, one has $\mathrm{Br}(Y)/\mathrm{Br}(k) = 0$ (this may fail for X singular and $n = 5$). This motivated :

Conjecture (CT-S-SD 1987) : if $n \geq 6$, then the Hasse principle holds for Y.

For $n \geq 8$, this was proven in [CT-S-SD 1987].
The case $n = 7$ is the topic of the present talk.

I shall discuss the main steps in the proof of the following theorem. In the smooth case, this was proved by R. Heath-Brown (2018) and revisited by me in 2022. The possibly singular case was proved by A. Molyakov (2023).

Main Theorem (of the lecture)
Let k be a number field and let $X \subset \mathbb{P}^7_k$ be a nonconical, geometrically integral, complete intersection of two quadrics. For any smooth projective model Y of X, the Hasse principle holds.

One useful tool is the theorem : *Over any field k, if an intersection of two quadrics $X \subset \mathbb{P}^n_k$ has a rational point over an odd degree extension of k then it has a rational point.*

This is an immediate consequence of Springer's theorem (same statement for one quadric, over any field) and the theorem of Amer and of Brumer :

Let $k(t)$ be the rational function field in one variable. A sytem of two quadratic forms $f = g = 0$ over a field k has a nontrivial zero over k if and only if the quadratic form $f + tg$ over the field $k(t)$ has a nontrivial zero.

When discussing a complete intersection of two quadrics $X \subset \mathbb{P}^n_k$ over a field k (char. not 2) given by a system $f = g = 0$, one is quickly led to consider the pencil of quadrics $\lambda f + \mu g = 0$ containing X.

Ignoring subtle points with the singular forms in the pencil, there is a close relation between the following statements, where we assume $r \geq 1$:

• There exists a form $\lambda f + \mu g$ in the pencil of quadratic forms which splits off $r + 1$ hyperbolic planes.

• There exists a quadric in the pencil which contains a linear space $\mathbb{P}^r_k \subset \mathbb{P}^n_k$.

• The variety X contains an $(r-1)$-dimensional quadric $Y \subset \mathbb{P}^r_k \subset \mathbb{P}^n_k$.

The proof of the main theorem I describe here builds upon a local result in very small dimension.

Theorem (CT 2022) *Let k be a p-adic field. Let $X \subset \mathbb{P}^3_k$ be an intersection of two quadrics given by a system*

$$f(x_0, x_1, x_2, x_3) = 0, \ g(x_1, x_2, x_3) = 0.$$

Then there exists a quadratic extension K/k with $X(K) \neq \emptyset$.

Proof. When X is not a smooth complete intersection, this is proven by a (tedious) case-by-case discussion. Assume X is a smooth complete intersection. Then X is a genus one curve.

Let $\overline{k}$ be an algebraic closure of k, and $G := \mathrm{Gal}(\overline{k}/k)$. The period of a curve X is defined as the positive generator of the image of the degree map $\mathrm{Pic}(X \times_k \overline{k})^G \to \mathbb{Z}$. The index of a k-variety X is the gcd of degrees of closed points on X.

The assumption that $g(x_1, x_2, x_3)$ involves only three variables implies that the "period" of the curve X divides 2. This one sees by using the fact any conic has period 1 and that the curve X is a double cover of the conic $g(x_1, x_2, x_3) = 0$.

For a curve of genus one, it is a theorem of Lichtenbaum (1969) (building upon Tate's duality theorem for abelian varieties over a local field) that the period coincides with the index (for a review of the proof, see my webpage). Thus the index divides 2. By Riemann-Roch, this implies that there exists a field K/k of degree at most 2 with $X(K) \neq \emptyset$.

Degree 2 Local Theorem (Creutz–Viray 2021) *Let k be a p-adic field. Let $X \subset \mathbb{P}^n_k$, $n \geq 4$ be an intersection of two quadrics. There exists a field K/k of degree at most 2 with $X(K) \neq \emptyset$.*

(Alternate) proof. It is enough to handle the case $n = 4$.
Any quadratic form in 5 variables over a p-adic field has a nontrivial zero. This implies that one may find a hyerplane section over k such that the induced quadratic form has rank at most 3.
The result then follows from the previous theorem.

The next two results will not be used in the proof of the main theorem.

Theorem (Creutz–Viray 2021). *Let k be a number field and $X \subset \mathbb{P}^n_k$ be a smooth complete intersection of two quadrics. For $n \geq 4$, the index $I(X)$ divides 2.*

The proof is very elaborate.

Theorem (CT 2022) *Let k be a number field and $X \subset \mathbb{P}^n_k$ be a smooth complete intersection of two quadrics. For $n \geq 5$ there exists a quadratic extension K/k with $X(K) \neq \emptyset$.*

The question whether this holds for $n = 4$ remains open. Partial results are given by Creutz–Viray.

Proof. By Bertini it is enough to prove the case $n = 5$. In this case the variety $F_1(X)$ of lines on X is geometrically integral – it is actually a principal homogeneous space under an abelian variety. Hence there exists a finite set S of places of k such that $F_1(X)(k_v) \neq \emptyset$ for $v \notin S$. Thus, for $v \notin S$, any nondegenerate $\lambda f + \mu g$ splits off 2 hyperbolics over k_v.
For any place v, the Degree 2 Local Theorem gives a point of X in an extension of k_v of degree 2, hence there exists a $\lambda_v f + \mu_v g$ in the pencil over k_v which splits off two hyperbolics.
Using weak approximation, we find $(\lambda, \mu) \in \mathbb{P}^1(k)$ such that $\lambda f + \mu g$ is nondegenerate and splits off 2 hyperbolics over each k_v. By a result of Hasse (1924) it splits off 2 hyperbolics over k. Thus X contains a point over a quadratic extension of k.

Theorem (Salberger 1993 + ε) *Let k be a number field and $X \subset \mathbb{P}^n_k$, $n \geq 4$, be a geometrically integral, nonconical, complete intersection of two quadrics, and let Y/k be a smooth projective model of X.* **Assume that X contains a conic $C \subset \mathbb{P}^2_k \subset \mathbb{P}^n_k$.** *Then*

(a) The set $Y(k)$ is dense in the Brauer-Manin set $Y(\mathbb{A}_k)^{\mathrm{Br}(Y)} \subset Y(\mathbb{A}_k)$.

(b) For $n \geq 6$, the Hasse principle and weak approximation hold for Y.

(c) For $n = 5$ and X smooth, the Hasse principle and weak approximation hold for X.

The proof of the theorem relies in part on several works (CTSaSD 87, Coray-Tsfasman 88). Salberger's proof of the case $n = 4$ builds upon his seminal work on zero-cycles. Another proof builds on CTSaSD87.

Salberger's result is used in various proofs of the following theorem (not neeed for this talk.)

Theorem (J. Iyer and R. Parimala 2022) *Let k be a number field and $X \subset \mathbb{P}^5_k$ be a smooth complete intersection of two quadrics given by $f = g = 0$. Assume :*
(a) The genus 2 curve given by $y^2 = -det(\lambda f + \mu g)$ has a zero-cycle of degree one.
(b) Over each completion k_v of k, X contains a line defined over k_v
Then $X(k) \neq \emptyset$.

Proof of the main theorem in the smooth case

Local Theorem (Heath-Brown 2018) *Let k be a local field. Let $X \subset \mathbb{P}^7_k$ be a smooth complete intersection of two quadrics given by $f = g = 0$. If $X(k) \neq \emptyset$, then there exists a nondegenerate form $\lambda f + \mu g$ in the pencil which splits off three hyperbolics.*

Proof (CT 2022) Let $P \in X(k)$. The intersection C of X with the tangent space $\mathbb{P}^5_k$ at P is a cone with vertex P over an intersection of two quadrics $Y \subset \mathbb{P}^4_k$. By the Degree 2 Local Theorem (Creutz–Viray) there exists a point on Y in a quadratic extension K/k. This defines a line over K on C passing through the vertex P of the cone. One thus gets a pair of lines in $C \subset X$ passing through P and globally defined over k. Fix a k-point Q in the plane $\mathbb{P}^2_k$ defined by these two lines, outside of the two lines. The form $\lambda f + \mu g$ vanishing at Q vanishes on the plane $\mathbb{P}^2_k$ spanned by the two lines. If nondegenerate, this form splits off 3 hyperbolics. (There is a simple way to handle the case where the form is of rank 7.)

Global Theorem (Heath-Brown, 2018) *Let k be a number field. Let $X \subset \mathbb{P}^7_k$ be a smooth complete intersection of two quadrics given by $f = g = 0$. The Hasse principle holds for X.*

Proof of Global Theorem (CT 2022, some ingredients from HB's proof).
The variety $F_2(X)$ of planes $\mathbb{P}^2_k \subset X \subset \mathbb{P}^7_k$ is a geometrically integral variety – it is actually a principal homogeneous spaces under an abelian variety. Hence there exists a finite set S of places of k such that $F_2(X)(k_v) \neq \emptyset$ for $v \notin S$. Thus for each $v \notin S$, any nondegenerate $\lambda f + \mu g$ splits off 3 hyperbolics over k_v. By the Local Theorem, for each $v \in S$ the assumption $X(k_v) \neq \emptyset$ implies that there exists a point $(\lambda_v, \mu_v) \in \mathbb{P}^1(k_v)$ such that $\lambda_v f + \mu_v g$ is nondegenerate and contains 3 hyperbolics. By weak approximation on $\mathbb{P}^1_k$, there exists $(\lambda, \mu) \in \mathbb{P}^1(k)$ such that $\lambda f + \mu g$ is nondegenerate and contains 3 hyperbolics over each k_v. By Hasse 1924 it contains 3 hyperbolics over k. Thus X contains a conic. Salberger's theorem and the hypothesis $\prod_v X(k_v) \neq \emptyset$ then give $X(k) \neq \emptyset$.

What about singular complete intersections of two quadrics ?

Let k be a number field and $X \subset \mathbb{P}^n_k$ a possibly singular complete intersection of two quadrics. Assume it is geometrically integral and not a cone. One is interested in the Hasse principle for a smooth projective model Y of X.
As mentioned earlier on, CT-Sansuc-Swinnerton-Dyer 1987 proposed the conjecture ;

Conjecture *For $n = 6$ and $n = 7$, the Hasse principle holds for Y.*

Under various additional hypotheses on X, the conjecture is proved in CTSaSD 1987. As we saw, Salberger 1993 proves it when X contains a conic. And we have seen a proof in the case where X is smooth.
A. Molyakov (preprint on arXiv, 2023) proved the general case of the above conjecture for $n = 7$.

Theorem (Molyakov 2023) *Let k be a number field. Let $X \subset \mathbb{P}^7_k$ be a non-conical geom. integral complete intersection of two quadrics. Then the Hasse principle holds for any smooth projective model of X.*

I sketch the main steps of the proof.

Recall : A complete intersection of two quadrics $f = g = 0$ in $\mathbb{P}^n$ is smooth if and only if the determinant $P(\lambda, \mu) = det(\lambda f + \mu g)$ is a nonzero separable polynomial. In that case all forms in the pencil have rank at least n.

A local result

Theorem *Let k be a local field. Let $X \subset \mathbb{P}^7_k$ be a nonconical geom. integral complete intersection of two quadrics given by $f = g = 0$. If $X_{smooth}(k) \neq \emptyset$, and* **there is no form of rank** ≤ 5 *in the geometric pencil $\lambda f + \mu g$, then there exists a nondegenerate form $\lambda f + \mu g$ in the pencil over k which splits off three hyperbolics.*

The proof is similar to the proof in the smooth case but is geometrically more involved. Namely, one finds a smooth k-point $P \in X(k)$ such that the intersection of the tangent space T_P at X in the point P is a cone over a reasonable intersection of two quadrics $Y \subset \mathbb{P}^4$. Then there exists a quadratic point on Y over the p-adic field, which leads to a (degenerate conic) lying in $T_P \cap X$. A quadric in the pencil containing a conic is defined by a quadratic form which splits off three hyperbolics.

Global result, the "regular" case

Theorem *Let k be a number field. Let $X \subset \mathbb{P}^7_k$ be a nondegenerate geom. integral complete intersection of two quadrics given by $f = g = 0$. Assume* **there is no form of rank** ≤ 6 *in the geometric pencil $\lambda f + \mu g$. Then the Hasse principle holds for any smooth projective model of X*

Proof. Under the hypothesis on ranks of forms in the pencil one knows that the variety parametrizing the planes $\mathbb{P}^2 \subset X$ is a generalized jacobian (Xiaoheng Wang 2018) and in particular is **geometrically integral**.

Via Lang-Weil and Hensel this shows there is a finite set S of places such that for $v \notin S$, there exists a $\mathbb{P}^2_{k_v} \subset X_{k_v}$. Thus any form $\lambda f + \mu g$ contains 3 hyperbolics over k_v for $v \notin S$.

The previous local result together with weak approximation then produce a $\lambda f + \mu g$ over k with 3 hyperbolics over each k_v hence over k by Hasse, hence we have a conic lying on X and may conclude by Salberger's theorem.

Global result, the irregular case

We now allow the existence of a form of rank ≤ 6 in the geometric pencil. In this case the variety parametrizing the $\mathbb{P}^2 \subset X \subset \mathbb{P}^7$ need not be geometrically connected.
There is an interesting case by case discussion. Some cases were handled in [CT/Sa/SD], for example the case where there exists one form in the pencil over k which has rank 6, which corresponds to X having two conjugate singular points. This case is handled by the fibration method for rational points, ultimately reducing to the case of Châtelet surfaces.

But two cases

– The geometric pencil contains two conjugate forms of rank 6.

– The geometric pencil exactly contains 4 forms of rank 6.

require a new, specific argument.

One uses the **fibration method for zero-cycles** (Harpaz-Wittenberg 2016), which is more flexible than the fibration method for rational points.

When the geometric pencil contains two conjugate forms f, g of rank 6

(General case)

The vertex of the cone $f = 0$ is a line of singular points which intersects $g = 0$ in 2 points. Similarly for $g = 0$. The 4 points are globally defined over k, are singular on X, and span $H \simeq \mathbb{P}^3 \subset \mathbb{P}^7_k$. The family of $\mathbb{P}^4$'s containing H defines a rational map h from X to $\mathbb{P}^3_k$. The fibres X_m are intersections of two quadrics in $\mathbb{P}^4_k$ with 4 singular points, they all contain a skew quadrilateral on the 4 points.

By Coray-Tsfasman 1988, these fibres over k are birational to k-forms of $\mathbb{P}^1 \times \mathbb{P}^1$, hence over a number field satisfy (smooth) Hasse principle and weak approximation.

After a resolution of singularities $Y \to X$ one gets a fibration $f : Y \to \mathbb{P}^3_k$ with smooth general fibres. One is then in a good position to apply the fibration method. However one cannot apply the fibration method for rational points since one does not control the ramification locus of f.

The smooth fibres satisfy the Hasse principle and weak approximation for rational points, and are geometrically simple enough. A general result of Y. Liang 2013 then ensures that conjecture (E) on zero-cycles (an analogue for zero-cycles of the hypothesis that the Brauer group controls the local-global obstruction for rational points) holds for these fibres.

Since $n = 7$, we have $\mathrm{Br}(Y)/\mathrm{Br}(k) = 0$.
The fibration theorem for zero-cycles (Harpaz-Wittenberg 2016 – there is no restriction on the ramification locus of $Y \to \mathbb{P}^3_k$) then ensures that the k-variety Y satisfies conjecture (E) hence in particular has a zero-cycle of degree one as soon as $Y(\mathbb{A}_k) \neq \emptyset$. This implies that X has a zero-cycle of degree one. Hence by Amer-Brumer X has a rational point. If such a k-point is smooth then $Y(k) \neq \emptyset$. If one is given a singular k-point on X then X is k-birational to a quadric, for which the Hasse principle holds. QED

When the geometric pencil contains exactly 4 forms of rank 6

Here one reduces to the case where X is k-birational to a fibration $Y \to \mathbb{P}^1$, Y/k smooth whose generic fibre is a smooth compactification of a principal homogeneous space under a torus. It is a classical result (Sansuc 1981) that the Brauer-Manin obstruction for rational points is the only obstruction for smooth compactifications of a principal homogeneous space under a torus over a number field. Liang's theorem then gives the result for zero-cycles : conjecture (E) holds for the smooth fibres. By Harpaz-Wittenberg one gets Conjecture (E) for the total space. From $Y(\mathbb{A}_k) \neq \emptyset$ and $\mathrm{Br}(Y)/\mathrm{Br}(k) = 0$, one gets that Y has a zero-cycle of degree one, and then X has a zero-cycle of degree one, hence a k-rational point by Amer-Brumer. And we conclude as above.

This completes the proof of the main theorem in the possibly singular case.

Some references

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