Zero-cycles on del Pezzo surfaces (Variations upon a theme by Daniel Coray)

Zoom algebraic geometry seminar
September 22nd, 2020
(online)

Jean-Louis Colliot-Thélène
(CNRS et Université Paris-Saclay)

The text

Zéro-cycles sur les surfaces de del Pezzo
(Variations sur un thème de Daniel Coray)

is available on arXiv
arXiv : 2005.06876v2 (June 2020)
A version starting with a developed introduction (for the general public) was put on my webpage yesterday.
The present presentation is a reorganized version of “online” presentations in Paris (June 5) and in Moscow (June 11).

Introduction and statement of main results

Let X be an algebraic variety over a field k.
One lets $X(k)$ be the set of rational points of X.
If P is a closed point of X, we denote by $k(P)$ the residue field at P. This is a finite extension of k. Its degree $d_P := [k(P) : k]$ is called the degree of P.
A closed point P of degree 1 is a rational point.
The *index* $I(X) = I(X/k)$ is the g.c.d. of the degrees d_P for all closed points P of X.
It is also the g.c.d. of the degrees of the finite field extension K/k such that $X(K) \neq \emptyset$.
If $X(k) \neq \emptyset$, then clearly $I(X) = 1$.
Naive question : what about the converse ? One classical case : yes for quadrics (Artin 37, Springer 1952). One less classical case : yes for intersections of two quadrics (Amer 76, Coray 77, Brumer 78).

Zero-cycle on a k-variety X : finite linear combination with integral coefficients of closed points $\sum_P n_P P$, $n \in \mathbb{Z}$
Effective cycle : all $n_P \geq 0$

Degree of the zero-cycle (over k) : $\sum_P n_P [k(P) : k] \in \mathbb{Z}$

Rational equivalence on the group $Z_0(X)$ of zero-cycles :
for any proper morphism $p : C \to X$ from a normal integral k-curve and any rational function $f \in k(C)^*$, mod out by $p_*(div_C(f))$.
If X/k is proper, then induced degree map

$$CH_0(X) = Z_0(X)/\mathrm{rat} \to \mathbb{Z}$$

from the Chow group of degree zero-cycles to $\mathbb{Z}$.
The image is $\mathbb{Z}.I(X) \subset \mathbb{Z}$, where $I(X)$ is the index.
The kernel $A_0(X)$ is the reduced Chow group of zero-cycles.

Curves (zero-cycle = divisor)

Riemann's inequality $\ell(z) \geq deg_k(z) + 1 - g$ for a divisor z on a smooth, projective, geometrically connected curve C/k of genus g implies :

• Any zero-cycle of degree at least equal to g on C is rationally equivalent to an effective cycle.

• For $g > 1$, if $I(C) = 1$, then there exist effective zero-cycles of degree g and of degree $g + 1$, hence closed points of coprime degrees $\leq g + 1$.

• If $g \geq 1$ and $C(k) \neq \emptyset$, then $CH_0(C)$ is generated by closed points of degree at most g.

• For $g = 0$ and $g = 1$, if $I(C) = 1$, then $C(k) \neq \emptyset$.

Naive questions. For X/k a smooth, projective, geometrically connected variety over a field k, do we have similar results with a suitable integer in place of g ?

Over $k = \mathbb{C}$ the complex field, only the first property is relevant. The answer is well known to be NO in general. This is the famous result of Mumford 1969 on a problem of Severi, expanded by Roitman 1971, with the proof by Spencer Bloch 1979 by means of algebraic correspondances, expanded by Bloch and Srinivas and then by many other authors.

Theorem. Let $X/\mathbb{C}$ be as above. If there exists an integer $N \geq 1$ such that any zero-cycle of degree at least N is rationally equivalent to an effective zero-cycle, then $H^0(X, \Omega^i_X) = 0$ for $i \geq 2$.

УДК 513.015.7

Г-эквивалентность нульмерных циклов

А. А. Ройтман (Москва)

Введение

Пусть X — гладкое проективное многообразие над $\mathbf{C}$. Мы изучаем связь между глобальными свойствами этого многообразия (наличие на нем регулярных дифференциалов, линейчатость, рациональность) и свойством его нульмерных циклов быть между собой (Γ, x_1, x_2)-эквивалентными.

Два цикла a и b называются (Γ, x_1, x_2)-эквивалентными, где Γ — гладкая кривая и $x_1, x_2 \in \Gamma$, $x_1 \neq x_2$, если существует алгебраическая система циклов $T(u)$, параметризованная кривой Γ, такая, что $T(x_1) = a$, $T(x_2) = b$. Г-эквивалентность является обобщением рациональной эквивалентности и во многом на нее похожа. (Рациональная эквивалентность есть $(\mathbf{P}^1, \alpha, \beta)$-эквивалентность для любых точек $\alpha, \beta \in \mathbf{P}^1$, $\alpha \neq \beta$). Более точно (Γ, x_1, x_2)-эквивалентность определяется в § 1.

В § 2 доказывается, что множество эффективных циклов степени n, Г-эквивалентных между собой, образует счетное объединение замкнутых подмножеств в $S^n X$ — n-ой симметрической степени X. Набросок доказательства этого факта дает Мамфорд в [1]. Однако его метод проходит только для рациональной эквивалентности. В § 3 изучается связь между Г-эквивалентностью и поведением дифференциалов.

Два последних параграфа являются основными в работе. В § 4 вводится понятие геометрического инварианта многообразия. Этот инвариант показывает, как много среди циклов X Г-эквивалентных. Более точно: $\operatorname{inv}_\Gamma X = \lim\limits_{n\to\infty} \frac{\dim W_A}{n}$ для достаточно общей точки $A \in S^n X$, где $W_A \subset S^n X$; $W_A = \{B \in S^n X \mid B \underset{\Gamma}{\sim} A\}$. Мы доказываем, что последовательность $\frac{\dim W_A}{n}$ неубывающая, а ее предел — целое число. Это понятие хорошо согласуется с условием конечномерности, введенным Мамфордом в [1]. (Ему принадлежат условия 1), 2), 3) определения-леммы 8.) Доказывается, что $A_\Gamma(X)$ конечномерна в том и только в том случае, когда $\operatorname{inv}_\Gamma X = \dim X$. Основной результат этого параграфа следующий.

Теорема 5. *Пусть* $\dim H^0(X, \Omega_X^q) > 0$ *для некоторого* $q \geqslant 2$. *Тогда* $A_\Gamma(X)$ *не конечномерна.*

Перед тем, как говорить о последнем параграфе, необходимо отметить, что эта работа появилась как результат изучения уже упоминавшейся работы Мамфорда [1] и двух работ Маттука [2] и [3].

One is then led to restrict our naive questions and only consider smooth, projective geometrically connected varieties X/k such that, over (arbitrary) algebraically closed extensions Ω/k, the geometry of $Y := X \times_k \Omega$ is "reasonable". In decreasing order of generality :

(1) CH_0-representable : there exists a proper curve C and a proper morphism $p : C \to Y$ such that $p_* : CH_0(C) \to CH_0(Y)$ is onto
(2) CH_0-trivial : the degree map $CH_0(Y) \to \mathbb{Z}$ is an isomorphism.
(3) Rationally connected varieties (Kollár-Miyaoka-Mori), e.g. Fano varieties.
(4) Unirational varieties

For surfaces, classes (3) and (4) coincide with the class of rational surfaces.

One special class of varieties over k of geometric type (4) is that of smooth compactifications X of a homogeneous space E of a connected linear algebraic group.

For such X, the question whether $I(X) = 1$ implies $X(k) \neq \emptyset$ has been much investigated.
For E a principal homogeneous space, the question is open in general. There are positive results (Serre, Sansuc over number fields, Bayer–Lenstra).
For projective homogeneous spaces, we have Springer's theorem on quadrics.
But in the general case, one may have $I(X) = 1$ and $X(k) = \emptyset$ (Florence, Parimala).

In this talk, we shall concentrate on the class of geometrically rational surfaces over a field k of characteristic zero. We shall see that analogues of the properties for curves hold for such surfaces.

The first theorem is a substitute for the would-be statement : $I(X) = 1 \Longrightarrow X(k) \neq \emptyset$. It generalizes the result for cubic surfaces obtained by Coray (1974) in his thesis (to be discussed further below).

Theorem A

Let X/k be a smooth, projective, geometrically rational surface. There exists an integer $N(X) \geq 1$, which depends only on the geometry of X, such that, if $I(X) = 1$, then there exist closed points of coprime degrees all less than $N(X)$.

The second theorem generalizes a result of Coray and mine (1979) on conic bundles over the projective line.

Theorem B

Let X/k be a smooth, projective, geometrically rational surface with a k-rational point. There exists an integer $M(X) \geq 1$, which depends only on the geometry of X, such that any zero-cycle on X of degree at least $M(X)$ is rationally equivalent to an effective cycle. In particular, the Chow group of zero-cycles is generated by closed points of degree at most $M(X)$.

Index 1 for del Pezzo surfaces of degree 3 : Coray's thesis

In his Ph.D. thesis (Cambridge, UK 1974), Daniel Coray (1947-2015) studied the question :

If a cubic hypersurface X in $\mathbb{P}^n_k$ has a rational point in a finite field extension K/k of degree prime to 3, does it have a rational point in k ?
That is, if $I(X) = 1$, do we have $X(k) \neq \emptyset$?

ACTA ARITHMETICA
XXX (1976)

Algebraic points on cubic hypersurfaces

by

D. F. CORAY* (Cambridge, Mass.)

Introduction. The following conjecture was apparently first formulated by Cassels and Swinnerton-Dyer (in the case $n = 3$), and it is closely connected with some of the problems mentioned by B. Segre in [15], p. 2:

CONJECTURE (CS). *Let $f(x_0, \ldots, x_n)$ be a cubic form with coefficients in a field k. Suppose f has a non-trivial solution in an algebraic extension K/k, of degree d prime to 3. Then f also has a non-trivial solution in the ground field k.*

Of course the crucial condition in this statement is that d should be prime to 3. The case $n \leqslant 2$ was already known to Henri Poincaré [11]; his proof will be given in § 2, since the geometrical ideas it involves are fundamental in the study of the case $n = 3$ and will be used throughout this paper. We begin with a few rather dry lemmas on the rationality of cycles on an algebraic variety (§ 1), which are necessary if we want to proceed on firm ground when using algebraic geometry over an arbitrary field. The use of these lemmas is exemplified in § 2, which therefore gives not only Poincaré's proof, but also a few other applications of the same type of argument. In § 3 we discuss some of the first attempts made at proving the conjecture when $n = 3$, including a very interesting descent argument due to Cassels (unpublished). This result implies in particular that (CS) holds when $n = 3$ and k is a local field. But the argument fails when the characteristic of the residue class field is equal to 2.

At this point, the exposition breaks into two parts: in §§ 4 and 5, we use a different method to prove the conjecture in full generality over any local field (i.e. for all n and without any restriction on the characteristic). This is done by purely arithmetic means, and the reader who is more interested in the geometrical aspect of the problem may proceed

* This paper forms the substance of a dissertation presented to the University of Cambridge [3]. I wish to express my gratitude to the Research Committee of the University of Geneva and to the Société Académique (Turettini Fund) for financial support.

There are two main theorems in this thesis.

Theorem (Coray). *If k is the field of fractions of a complete DVR with residue field κ, if $I(X) = 1 \Longrightarrow X(k) \neq \emptyset$ holds for cubic hypersurfaces over κ in any dimension, then it holds for cubic hypersurfaces over k in any dimension.*

This is proved by a delicate study of possible bad reduction of cubic hypersurfaces, extending earlier work of Demjanov, Lewis, Springer.

Corollary (Coray). *$I(X) = 1 \Longrightarrow X(k) \neq \emptyset$ holds for arbitrary cubic hypersurfaces over a p-adic field.*
Indeed, it is easy to prove $I(X) = 1 \Longrightarrow X(\kappa) \neq \emptyset$ for cubic hypersurfaces over a finite field κ.

This talk is concerned with the second main theorem in Coray's thesis, his method, and some improvements which lead to Theorems A and B. Here k is an arbitrary field.

Theorem (Coray 1974)
Let $X \subset \mathbb{P}^3_k$ be a smooth cubic surface. If it has a rational point in a finite field extension of k of degree prime to 3, then it has a rational point in an extension of degree 1, or of degree 4, or of degree 10.
("or " not exclusive)

I shall describe the main points of Coray's proof. It uses curves of low genus lying on the surface. The basic tools are classical : Riemann-Roch for line bundles on a surface and the formula for the arithmetic genus of a curve on a surface. One does not know whether the curves one produces are smooth or even irreducible.

One must then envision possible degeneracy cases. I shall then explain a general method to dodge this part of the argument, and, with the added flexibility, produce new results without too much pain.
From now on, to be on the safe side, I assume $\mathrm{char}(k) = 0$.

We assume that the smooth cubic surface $X \subset \mathbb{P}^3_k$ has a closed point of degree d prime to 3. Let d be the least such integer.
If $d = 1$, there is nothing to do. If $d = 2$, then taking the line through a quadratic point and its conjugate we get a rational point, thus in fact $d = 1$.
Let us thus assume d prime to 3 and $d \geq 4$. Let $P \in X$ be a closed point of degree d.
On the surface X we find a closed point Q of degree 3 by intersecting with a line $\mathbb{P}^1_k \subset \mathbb{P}^3_k$.

Let $n \geq 1$ be the smallest integer such that there exists a surface $\Sigma \subset \mathbb{P}^3_k$ of degree n cutting out a curve $\Gamma \subset X$ which contains both P and Q.
On the surface X we easily compute

$$h^0(X, O_X(n)) \geq 3n(n+1)/2 + 1.$$

(actually, equal)
Assume that the surface Σ of degree n cuts out a curve $\Gamma = D \subset X$ which is geometrically irreducible and smooth. On this curve there is a zero-cycle of degree 1. One computes the genus

$$g = p_a(D) = 3n(n-1)/2 + 1.$$

If on the one hand $3n(n+1)/2 - 3 \geq d$ then

$$3n(n+1)/2 + 1 \geq d + 3 + 1$$

and one may find a surface of degree n cutting out a curve Γ (assumed to be smooth) passing through the closed points P (of degree d) and Q (of degree 3).
If on the other hand $d \geq 3n(n-1)/2 + 4$, then

$$d - 3 \geq 3n(n-1)/2 + 1 = g(\Gamma),$$

thus on the smooth curve Γ, the zero-cycle $P - Q$ is rationally equivalent to an effective zero-cycle of degree $d - 3 < d$. Thus there exists a closed point of degree prime to 3 and smaller than d, contradiction.

This argument works for any integer d prime to 3 which lies in an interval

$$3n(n+1)/2 - 3 \geq d \geq 3n(n-1)/2 + 4.$$

For other values of d, a complementary argument is needed.
In particular, for integers of the shape $d = 3n(n-1)/2 + 1$, one uses a curve Γ which is the normalisation of a curve $\Gamma_0 \subset X$ cut out by a surface of degree n passing through P and having a double point at the point Q of degree 3. The genus of the curve drops down by 3, and the dimension of the linear system of interest drops down by 9.

For $d = 3n(n-1)/2 + 1$ with $n \geq 4$, there is enough room. But there is not enough room in the case $n = 2, d = 4$ and in the case $n = 3, d = 10$.

CONCLUSION (up to good position argument)

On a smooth cubic surface X/k with a closed point of degree d prime to 3, the least such d lies in $\{1, 4, 10\}$.

45 years old question : Can one eliminate 10, 4, both ?

The above argument for cubic surfaces assumes that the curves Γ found in the linear system are geometrically irreducible and smooth. In his paper, Coray then discusses the possible singular and even reducible curves which may turn up, and manages to go down to 1, 4 or 10 also in these cases.

It is clear that such cases may occur : consider the simpler question of finding a smooth plane conic through a closed point of degree 3 in $\mathbb{P}^2_k$. If the closed point happens to lie on a $\mathbb{P}^1_k \subset \mathbb{P}^2_k$, this is not possible.

Making the method flexible

I now explain how to avoid such a discussion of degenerate cases.

Ideas :

• When available, use results of the type : if there is a k-rational point on a k-variety X of the type under study, then the k-rational points are Zariski dense.

• use the Bertini theorems (not very original !)

• replace k by the "large" field $F = k((t))$, so that there are many F-points on whichever smooth variety appears in the process (the original variety, or some parameter space) as soon as there is at least one F-point.

• For each of the problems under consideration here, to prove a result for a k-variety X, it is enough to solve it positively for the $k((t))$-variety $X \times_k k((t))$.

Theorem (a variation on the Bertini theorems, as found in Jouanolou's book)

Let X be a smooth, projective, geom. connected k-variety. Let $f : X \to \mathbb{P}^n_k$ be a k-morphism. Assume its image has dimension at least 2 and generates $\mathbb{P}^n_k$.
Let $r \leq n$ be an integer. There exists a nonempty open set $U \subset X^r$ such that, for any field L containing k and any L-point $(P_1, \ldots, P_r) \in U(L)$, there exists a hyperplane $h \subset \mathbb{P}^n_L$ whose inverse image $f^{-1}(h) \subset X_L$ is a smooth, geometrically integral L-variety which contains the points $\{P_1, \ldots, P_r\}$.

Here we just say : " If there is a point in $U(L)$, then ...". But for a given L, $U(L)$ could be empty.

Let X be a smooth k-variety and $m > 0$ be an integer. Consider the open set W of X^m consisting of m-tuples $(x_1, \ldots, x_m)$ with $x_i \neq x_j$ for $i \neq j$.

The symmetric group $\mathfrak{S}_m$ acts on W, the quotient is a smooth k-variety $Sym^m_{sep}X$. It parametrizes effective zero-cycles of degree m which are "separable".

Theorem (zero-cycles version of previous theorem)

Let X be a smooth, projective, geom. connected k-variety. Let $f : X \to \mathbb{P}^n_k$ be a k-morphism. Assume its image has dimension at least 2 and generates $\mathbb{P}^n_k$. Let $s_1, \dots, s_t$ be natural integers such that $\sum_i s_i \leq n$. There exists a nonempty open set U of the product $\mathrm{Sym}^{s_1}_{sep}X \times \cdots \times \mathrm{Sym}^{s_t}_{sep}X$ such that, for any field L containing k and any L-point of U, corresponding to a family $\{z_i\}$ of separable effective zero-cycles of respective degrees s_i, there exists a hyperplane $h \subset \mathbb{P}^n_L$ whose inverse image $X_h = f^{-1}(h) \subset X_L$ is a smooth, geometrically integral L-variety which contains the points of the supports of the cycle $\sum_i z_i$.

Same comment as before on $U(L)$ being possibly empty.
Note : Let $s = s_1 + \cdots + s_t$. For the proofs of Theorems A,B,C, we use $\mathrm{Sym}^{s_1}_{sep}X \times \cdots \times \mathrm{Sym}^{s_t}_{sep}X$ and not only $\mathrm{Sym}^{s}_{sep}X$.

Let k be a field, $\mathrm{char}(k) = 0$. Let X be a smooth, projective, geom. connected k-variety.

In this talk, we say that X has the *density property* if it satisfies : for any finite field extension L/k with $X(L) \neq \emptyset$, the set $X(L)$ is Zariski dense in X_L.

R-equivalence on $X(k)$ is the equivalence relation generated by the elementary relation : $A, B \in X(k)$ both lie in the image of $\mathbb{P}^1(k)$ under a k-morphism $\mathbb{P}^1_k \to X$.

In this talk, we say that X has the *R-density property* if it satisfies : for any finite field extension L/k and $P \in X(L)$, the set of points of $X(L)$ which are R-equivalent to P on X_L is Zariski dense on X_L.

Smooth cubic hypersurfaces in $\mathbb{P}^n_k$, $n \geq 3$, satisfy both properties.

Theorem (Bertini for varieties with density properties)

Let k be a field, $\mathrm{char}(k) = 0$. Let X be a smooth, projective, geom. connected k-variety. Let $f : X \to \mathbb{P}^n_k$ be a k-morphism. Assume its image has dimension at least 2 and generates $\mathbb{P}^n_k$. Let $P_1, \ldots, P_t$ be closed points of X of respective degrees $s_1, \ldots, s_t$ such that $\sum_i s_i \leq n$.

(a) If X satisfies the density property, then there exists a hyperplane $h \subset \mathbb{P}^n_k$ defined over k such that $X_h = f^{-1}(h) \subset X$ is smooth, geom. integral and contains effective zero-cycles $z_1, \ldots, z_t$ of respective degrees $s_1, \ldots, s_t$.

(b) If X is satisfies the R-density property, then one may moreover ensure that, for each i, the zero-cycle z_i is rationally equivalent to the zero-cycle P_i.

Definition (F. Pop)
A field F is said to be a *large field* (in French, *corps fertile*) if, for any smooth geometrically connected variety X over F, if $X(F) \neq \emptyset$ then the set $X(F)$ of F-rational points is Zariski dense in X.

If a field F is large, then any finite field extension of F is large.

Thus any smooth geom. connected variety over a large field satisfies the density property.

The formal power series field $F = k((t))$ over any field k is a large field.

Theorem (Bertini over a large field)

Let F be a large field, $\mathrm{char}(F) = 0$. Let X be a smooth, projective, geom. connected F-variety. Let $f : X \to \mathbb{P}^n_F$ be an F-morphism. Assume its image has dimension at least 2 and generates $\mathbb{P}^n_F$. Let $P_1, \dots, P_t$ be closed points of X of respective degrees $s_1, \dots, s_t$ such that $\sum_i s_i \leq n$.

(a) There exists a hyperplane $h \subset \mathbb{P}^n_F$ defined over F such that $X_h = f^{-1}(h) \subset X$ is smooth, geom. integral and contains effective zero-cycles $z_1, \dots, z_t$ of respective degrees $s_1, \dots, s_t$.

(b) If X is geometrically rationally connected, then one may moreover ensure that, for each i, the zero-cycle z_i is rationally equivalent to the zero-cycle P_i.

For the proof of (a) :
The family $P_1, \ldots, P_t$ defines an F-point of the smooth, connected k-variety $\mathrm{Sym}^{s_1}_{sep} X \times \cdots \times \mathrm{Sym}^{s_t}_{sep} X$. Since F is large, any nonempty Zariski open set of that k-variety contains an F-point.

For the proof of (b), one moreover uses a result due to Kollár (1999) (deformation method) : for any F-point P on a smooth, projective geometrically (separably) rationally connected variety X over a large field F, the set of F-points which are R-equivalent to P, hence in particular are rationally equivalent to P, is Zariski dense in X.

(Easy) Proposition

Let k be a field and $F = k((t))$. Let X be a proper k-variety.

(a) The gcd of degrees of closed points coincides for X/k and X_F/F.

(b) For any integer $r \geq 1$, the smallest degree of a closed point of degree prime to r, which is also the smallest degree of an effective zero-cycle of degree prime to r, coincides for X/k and X_F/F.

(c) Let I be a set of natural integers. If the Chow group of zero-cycles on X_F may be generated by the classes of effective cycles of degree $d \in I$, then the same holds for X.

(d) Let $d \geq 0$ be an integer. If every zero-cycle on X_F of degree at least d is rationally equivalent to an effective cycle, then the same holds for X.

One may then run Coray's proof using only smooth projective curves in the linear systems of interest. There are two ways to do this.

One may use the density property of smooth cubic surfaces and apply Bertini's theorem (a) for varieties with this property.

Or one may reduce to the case of large fields F via replacing k by $k((t))$, use Bertini theorem (a) for large fields, and then use the fact that the statement of the theorem for $X_{k((t))}$ over $k((t))$ implies it for X over k.

In any case, an important point has been to be able to move the effective zero-cycles through which one wants curves of a given linear system to pass and simultaneously be smooth.

The gained flexibility enables one to prove the next theorems by Coray's method without too much effort.

Index 1 for del Pezzo surfaces of degree 2

“Bertini over a large field” (a) is enough to prove :

Theorem

Let X be a del Pezzo surface of degree 2, i.e. a double cover of $\mathbb{P}^2_k$ ramified along a smooth quartic. If there exists a closed point of odd degree on X, then there exists a closed point of degre 1, or 3, or 7.

In the proof, just as for cubic surfaces, in certain cases, one needs to blow up points on X. To apply the Bertini types of results, one needs to know that certain invertible sheaves are very ample. Here one may use Reider's criteria (1988).

For del Pezzo surfaces of degree 2 with a k-rational point not in a very special situation, k-unirationality is known. But the trick with large fields enables us to handle our problem without using k-unirationality.

Remark (Kollár-Mella 2017). *There exist examples of del Pezzo surfaces X of degree 2 with a closed point of degree 3, hence $I(X) = 1$, but with no rational point.*

Suppose k is a field with a quadratic field extension $k(\sqrt{a})/k$, a cubic field extension and a quintic field extension.
Let $C \subset \mathbb{P}^2_k$ a conic with a smooth k-point.
Let $Q \subset \mathbb{P}^2_k$ be a smooth quartic with $Q \cap C = \{A, B\}$, with A closed point of degree 3 and B closed point of degree 5.
Let $F = k(t)$. Let X/F be the smooth del Pezzo surface of degree 2 defined by the equation

$$z^2 - aC(u, v, w)^2 + tQ(u, v, w) = 0.$$

It has obvious points of degree 3 and 5.
However congruences modulo powers of t show it has no F-point.

Effectivity results for del Pezzo surfaces

Using either “Bertini over a large field” (b) or “Bertini for varieties with the R-density property” (b), one proves :

Theorem

Let $X \subset \mathbb{P}^3_k$ be a smooth cubic surface. Suppose $X(k) \neq \emptyset$

(i) Every zero-cycle of degree zero on X is rationally equivalent to the difference of two effective cycles of degree 3.

(ii) Every zero-cycle on X of degree ≥ 10 is rationally equivalent to an effective zero-cycle.

The discussion of closed points of degree $d = 3n(n-1)/2 + 1$, resp. $d = d = 3n(n-1)/2$, requires the use of curves with one, resp. two double rational points.

Theorem

Let X/k be a del Pezzo surface of degree 2. Assume $X(k) \neq \emptyset$.

(i) Every zero-cycle of degree zero is rationally equivalent to the difference of two effective zero-cycles of degree 6.

(ii) Every zero-cycle of degree at least 43 is rationally equivalent to an effective cycle.

The discussion of closed points of degree $d = n^2 - n + 1$, resp. $d = n^2 - n$, requires the use of curves with one, resp. two double rational points.

Since we do not know the R-density property for del Pezzo surfaces of degree 2, the proof here relies on Bertini over a large field (b), the combination of the reduction trick from k to $k((t))$ and Kollár's result on R-density for geometrically rationally connected varieties (proved using deformation theory).

Theorem

Let X/k be a del Pezzo surface of degree 1. It has a k-point, the fixed point of the anticanonical system.

(i) Every zero-cycle of degree zero is rationally equivalent to the difference of effective zero-cycles of degree 21.

(ii) Every zero-cycle of degree at least 904 is rationally equivalent to an effective cycle.

The discussion of closed points of degree $d = n(n-1)/2 + 1$, resp. $d = n(n-1)/2$, requires the use of curves with one, resp. two double rational points.

Since we do not know the density property and even less the R-density property for del Pezzo surfaces of degree 1, the proof here relies on Bertini over a large field (b), the combination of the reduction trick from k to $k((t))$ and Kollár's result on R-density for geometrically rationally connected varieties.

With some effort, one should be able to give analogous theorems without assuming $X(k) \neq \emptyset$.
Here is one example (full details have not been written down).

Let $X \subset \mathbb{P}^4_k$ be a del Pezzo surface of degree 4. Assume $I(X) = 2$.
Over number fields, Creutz and Viray have recently discussed whether this implies the existence of a closed point of degree 2.
Over an arbitary field of char. zero, using the technique described in this talk, one may prove :
For X/k as above, if $I(X) = 2$, then there exists a closed point of degree $2d$ with d odd in $\{1, 3, 5, 7, 11\}$.

Putting everything together

Analogues of these theorems for conic bundles over the projective line, the other class of geometrically rational surfaces, analogues of the above theorems were proved by Coray and me in 1979 – the tedious way, discussing possible degenerations. I have not investigated whether one could simplify the argument by using the large field trick.
The case of del Pezzo surfaces of degree $5 \leq d \leq 9$, as usual, is easily handled (the case $d = 6$ being a little more difficult).
For del Pezzo surfaces of degree 4, one has $I(X) = 1 \Longrightarrow X(k) \neq \emptyset$ (Amer 1976, Coray 1977, Brumer 1978). If $X(k) \neq \emptyset$, then blowing up a suitable k-point leads to a conic bundle over $\mathbb{P}^1_k$.
Using the k-birational classification of geometrically rational surfaces, one gets Theorems A and B since they do not depend on the k-birational equivalence class.

Theorem A

Let X/k be a smooth, projective, geometrically rational surface. There exists an integer $N(X)$, which depends only on the geometry of X, such that, if $I(X) = 1$, then there exist closed points of coprime degrees less than $N(X)$.

Theorem B

Let X/k be a smooth, projective, geometrically rational surface with a k-rational point. There exists an integer $M(X)$, which depends only on the geometry of X, such that any zero-cycle on X of degree at least $M(X)$ is rationally equivalent to an effective cycle. In particular, the Chow group of zero-cycles is generated by closed points of degree at most $M(X)$.

These results are k-birational. They are proved by a case-by-case analysis relying on the k-birational classification of geometrically rational surfaces (Enriques, Manin, Iskovskikh, Mori). This raises questions.

Going home with problems

(i) Can one give a proof of Theorems A and B avoiding the birational k-classification ?

(ii) Do these results extend to higher dimensional geometrically rationally connected varieties ?
Favourite 3-folds : smooth cubic hypersurfaces in $\mathbb{P}^4_k$, conic bundles over $\mathbb{P}^2_k$, quadric bundles over $\mathbb{P}^1_k$?

(iii) What about geometrically CH_0-trivial varieties ? Enriques surfaces ?

(iv) What about geometrically CH_0-representable varieties ?
Salberger (unpublished, 1985) proved such a theorem for conic bundles over curves of arbitrary genus.

Going home with exercises for the class

Springer's theorem

Let $q(x_0, \ldots, x_n)$*, be a quadratic form over a field* k*. If it has a nontrivial zero over a field extension* K/k *of odd degree, then it has a nontrivial zero over* k*.*

Proof. May assume K/k simple, $K = k[t]/P(t)$, minimal, odd degree d. Assume q does not represent zero. Then may write

$$q(R_0(t), \ldots, R_n(t)) = P(t)Q(t)$$

with $deg(R_i) < deg(P)$, not all zero. LHS even degree at most $2d - 2$. Then $deg(Q) < deg(P)$ and of odd degree. Then pick up an irreducible factor of Q of odd degree and reduce modulo this factor. Contradiction.

Theorem A for the total space of a "constant" 1-parameter family of quadrics of arbitrary dimension.

Let $n \geq 1$ et $q(x_0, \ldots, x_n)$ be a nondegenerate quadratic form over a field k. Let $r(y) \in k[y]$ be a polynomial of degree 2δ.
Assume

$$r(y) - q(x_0, \ldots, x_n) = 0 \qquad (1)$$

has a solution in an odd degree extension K/k.
Then
– Either the leading coefficients of r is represented by q over k (which essentially gives a rational point above $y = \infty$)
– Or equation (1) has a solution in a field extension K/k of odd degree $d \leq \delta$.

Proof. Start with a solution

$$r(\eta) - q(\xi_0, \ldots, \xi_n) = 0$$

in an extension K/k of odd degree. One has the tower $k \subset k(\eta) \subset K$. By the previous result there exists a solution of

$$r(\eta) - q(\xi_0, \ldots, \xi_n) = 0$$

with the $\xi_i \in k(\eta) := k[t]/P(t)$. Then consider the identity

$$r(t) - q(x_0(t), \ldots, x_n(t)) = P(t)Q(t)$$

where $\deg(x_i(t)) < \deg(P(t))$ and argue with parity of degrees.