

ZERO-CYCLES ON SURFACES DOMINATED BY PRODUCTS OF HYPERELLIPTIC CURVES

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ABSTRACT. Conditionally on the finiteness of the relevant Tate–Shafarevich groups, we prove a local-to-global result for zero-cycles of degree 1 on the surfaces given by $y^2 = f_1(x_1)f_2(x_2)$, where the polynomials f_1 and f_2 are algebraically general. The proof combines the fibration method, parity results for 2-Selmer groups in quadratic twist families, and a variant of a theorem of Morgan on the variation of the Cassels–Tate pairing.

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1. INTRODUCTION

Let X be a smooth, projective, geometrically integral variety over a number field k . Assume that the adelic space

$$X(\mathbf{A}_k) = \prod_v X(k_v)$$

is non-empty. It has been known for a long time that this condition does not suffice to ensure that $X(k)$ is non-empty. It does not even suffice to ensure that the index $I(X)$ of X is equal to 1, that is, that there exists a zero-cycle of degree 1 on X , as shows the example of smooth projective curves of genus 1.

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Most counterexamples known before 1970 were accounted for by Manin by means of the Brauer–Grothendieck group $\mathrm{Br}(X)$. Namely, class field theory gives a pairing $X(\mathbf{A}_k) \times \mathrm{Br}(X) \rightarrow \mathbb{Q}/\mathbb{Z}$, and one has an inclusion

$$X(k) \subset X(\mathbf{A}_k)^{\mathrm{Br}},$$

where $X(\mathbf{A}_k)^{\mathrm{Br}}$ denotes the Brauer–Manin set of adelic points orthogonal to $\mathrm{Br}(X)$.

Question 1.1. *If $X(\mathbf{A}_k)^{\mathrm{Br}}$ is non-empty, does it follow that $X(k) \neq \emptyset$?*

For arbitrary smooth projective varieties, Question 1.1 has a negative answer [Sko99]. It is conjectured that Question 1.1 has a positive answer in the following cases: for rationally connected varieties [CTS21, Conjecture 14.1.2], for K3 surfaces [CTS21, Conjecture 14.3.12], and for homogeneous spaces of abelian varieties, where Manin observed that a positive answer to Question 1.1 follows from the finiteness of the Tate–Shafarevich groups (see [CTS21, p. 379]).

Although Question 1.1 has a negative answer in general, it is still open whether the following weaker form holds for arbitrary smooth projective varieties.

Conjecture 1.2. *If $X(\mathbf{A}_k)^{\mathrm{Br}}$ is non-empty, then $I(X) = 1$.*

Remark 1.3. Conjecture 1.2 holds for a smooth, projective, geometrically integral curve C if the Tate–Shafarevich group of the Jacobian $J = \mathrm{Pic}_{C/k}^0$ is finite. Indeed, the canonical embedding $C \hookrightarrow \mathrm{Pic}_{C/k}^1$ gives an inclusion $C(\mathbf{A}_k)^{\mathrm{Br}} \subset \mathrm{Pic}_{C/k}^1(\mathbf{A}_k)^{\mathrm{Br}}$, so the last set is non-empty. By Manin, the finiteness of $\mathrm{III}(J)$ then implies that $\mathrm{Pic}_{C/k}^1$ has a k -point. A well-known argument (Lemma 6.1 below) gives $I(C) = 1$.

In this paper we extend this observation to certain surfaces of general type dominated by products of two hyperelliptic curves.

Conjecture 1.2 follows from a more precise conjecture put forward by Colliot-Thélène–Sansuc and Kato–Saito [CTS21, Conjecture 15.1.3]. Starting with work of Salberger on conic bundle surfaces [Sal88], substantial progress has been made on Conjecture 1.2 for varieties admitting a dominant morphism to $\mathbb{P}_k^1$ with rationally connected fibres of a type already under control; see, for example, work of Harpaz and Wittenberg [HW16].

In this introduction and in the main theorem of this paper we assume the finiteness of the Tate–Shafarevich groups of the relevant abelian varieties.

For a variety X with a dominant morphism to $\mathbb{P}_k^1$ whose generic fibre is a torsor under an abelian variety A_t over $k(\mathbb{P}_k^1) = k(t)$, Swinnerton-Dyer [SD95] found a method to prove, under rather stringent assumptions, that Question 1.1 has a positive answer. These assumptions are due to the technical nature of the method and are presumably too strong. In many cases where the method applies, only a very small subgroup of $\mathrm{Br}(X)$ comes into play, and the final result simply takes the form of the Hasse principle:

$$X(\mathbf{A}_k) \neq \emptyset \implies X(k) \neq \emptyset.$$

The goal is to find a value $t_0 \in k$ of the parameter t on $\mathbb{P}_k^1$ such that the fibre X_{t_0} is smooth, has points in all completions of k , and the relevant Selmer group of the abelian variety A_{t_0} is sufficiently small. Under the finiteness hypothesis for $\mathrm{III}(A_{t_0})$, this then implies that $X_{t_0}(k) \neq \emptyset$, and hence $X(k) \neq \emptyset$.

Swinnerton-Dyer’s method was elaborated in [CTSSD98] and further developed in papers by Swinnerton-Dyer, Colliot-Thélène, Swinnerton-Dyer–Skorobogatov,

and Wittenberg. In [HS16], using some ideas of Mazur and Rubin, Harpaz and Skorobogatov introduced a new, alternative method. In [Har19] Harpaz extended the method by introducing the second descent and variation of the Cassels–Tate pairing under quadratic twists.

In his recent paper [Mor25], Morgan proved the existence of quadratic twists that do not change the 2-Selmer group and give the Cassels–Tate pairing on it any form allowed by the constraints of Poonen and Stoll [PS99]. In the present paper, we use Morgan’s new method as well as some of his earlier results. For another recent application of this technique see [MS24, Section 4].

The authors of [SSD05] and [HS16] studied Question 1.1 on Kummer surfaces with affine equation

$$(1.1) \quad y^2 = f_1(x_1)f_2(x_2),$$

where the polynomials f_i have degree at most 4. The relevant family over $\mathbb{P}_k^1$ arises by considering the pair of simultaneous quadratic-twist equations

$$(1.2) \quad z_1^2 = tf_1(x_1), \quad z_2^2 = tf_2(x_2).$$

In the present paper we initiate the study of Conjecture 1.2 for smooth projective surfaces with affine equation (1.1), where f_1 and f_2 have degree greater than 4. These are surfaces of general type, for which an affirmative answer to Question 1.1 is not expected. More precisely, here is the setting that we will consider.

Situation 1.4. *Let k be a number field. For $i = 1, 2$, let $g_i \geq 2$ be an integer. Let $f_i(x) \in k[x]$ be a separable polynomial of degree $2g_i + 2$, discriminant Δ_i and splitting field L_i . Assume that $\text{Gal}(L_i/k) \simeq S_{2g_i+2}$ for $i = 1, 2$, and that L_1 and L_2 are linearly disjoint over k .*

Let S_0 be a finite set of places of k containing the archimedean places, the primes dividing 2, the primes v where at least one of $f_1(x)$ and $f_2(x)$ is not in $\mathcal{O}_v[x]$, or the leading coefficient of at least one of $f_1(x)$ and $f_2(x)$ is not in $\mathcal{O}_v^\times$, and the primes v with residue field of size $< 4\max\{g_1, g_2\}^2$. Assume the existence of primes $w_1 \neq w_2$ not in S_0 such that each $f_i(x)$ has a unique double root modulo w_i and $\text{val}_{w_i}(\Delta_i) = 2$, and $f_i(x)$ is separable modulo w_j , for $i \neq j$.

Let C_i be a smooth projective hyperelliptic curve with affine equation $y^2 = f_i(x)$. Let X be a smooth, projective, geometrically integral variety birational to the affine variety with equation

$$y^2 = f_1(x_1)f_2(x_2).$$

Let us denote by $\iota_1: C_1 \rightarrow C_1$ and $\iota_2: C_2 \rightarrow C_2$ the hyperelliptic involutions. We may take X to be the quotient of the blow-up of $C_1 \times_k C_2$ at the fixed locus of the involution (ι_1, ι_2) , by the action of the involution extending (ι_1, ι_2) . This is the minimal desingularization of the quotient of $C_1 \times C_2$ by (ι_1, ι_2) .

Our main result is the following.

Theorem 1.5. *We place ourselves in Situation 1.4. Assume that the 2-primary torsion subgroups of the Tate–Shafarevich groups of the Jacobians of all quadratic twists of C_1 and C_2 over k are finite. Then $X(\mathbf{A}_k) \neq \emptyset$ implies that X has a zero-cycle of degree 1.*

Several comments on Theorem 1.5 are in order.

- (1) The surfaces X appearing in Situation 1.4 are of general type (Lemma A.1).

- (2) The set of k -points $(a_0, \dots, a_{2g_1+2}; b_0, \dots, b_{2g_2+2})$ of $\mathbb{A}_k^{2g_1+3} \times_k \mathbb{A}_k^{2g_2+3}$ such that Situation 1.4 holds for the polynomials $f_1(x) = a_{2g_1+2}x^{2g_1+2} + \dots + a_0$ and $f_2(x) = b_{2g_2+2}x^{2g_2+2} + \dots + b_0$ satisfies weak approximation. In fact, one can even fix suitable w_1 and w_2 in advance; see Remark 6.4.
- (3) If the degree of at least one f_i is odd, then it is not difficult to show that any smooth projective model X has a k -rational point.
- (4) The assumption that the polynomials $f_i(x)$ are algebraically general (linear independence of their splitting fields and maximal Galois action) is used in the proof of Proposition 2.1 to show that the vertical Brauer group of a certain fibration over $\mathbb{P}_k^1$ attached to X comes from $\text{Br}(k)$. It is also used in the proofs of Propositions 3.4 and 3.5.
- (5) Local conditions similar to those imposed in Situation 1.4 appear also in the aforementioned papers, for example in [HS16, Theorems A, B] and [Mor25, Theorem 1.2]. It would be interesting to understand to what extent the combination of algebraic genericity of the polynomials with such local conditions implies that there is no Brauer-Manin obstruction to the existence of a zero-cycle of degree one.

Here is a brief outline of the paper. In Section 2 we use the well-known fibration theorem for the universal family of quadratic twists to prove the existence of $t \in k^\times$ such that both curves in (1.2) are everywhere locally soluble. Section 3 summarizes various known results from the literature. In Section 4 we prove that it is possible to choose $t \in k^\times$ such that both curves in (1.2) are everywhere locally soluble and, moreover, the 2-Selmer groups of their Jacobians are $\mathbb{F}_2$ -vector spaces of odd dimension. In Section 5 we apply Morgan's method to prove the existence of $t \in k^\times$ such that, in addition to the above conditions, both Cassels–Tate pairings have any given form allowed by the theory of Poonen and Stoll. The proof of Theorem 1.5 is completed in Section 6 by choosing $t \in k^\times$ so that the kernel of each of the two Cassels–Tate pairings is generated by the class of $\text{Pic}_{C_i/k}^1$. The existence of a zero-cycle of degree 1 on each twisted curve is an easy consequence. Finally, in the appendix we show that the surfaces featuring in Situation 1.4 are of general type.

Notation. For a field k , let $\bar{k}$ be a separable closure of k and $\Gamma_k := \text{Gal}(\bar{k}/k)$ the absolute Galois group of k .

For a smooth projective curve C over a field k , we denote by $\text{Pic}_{C/k}$ the Picard scheme of C over k .

For an abelian variety A over a field k we denote by $A^\vee$ the dual abelian variety. When k is a number field, we denote by $\text{III}(A)$ the Tate–Shafarevich group of A and by $\text{III}(A)_{\text{nd}}$ the quotient of $\text{III}(A)$ by its (conjecturally trivial) maximal divisible subgroup. For an integer $m \geq 1$, let $\text{Sel}^m(A)$ be the m -Selmer group of A . Let $\text{rk}_2(A)$ be the 2^∞ -Selmer rank of A , defined as the $\mathbb{Z}_2$ -rank of $\varprojlim \text{Sel}^{2^n}(A)$. If the 2-primary subgroup of $\text{III}(A)$ is finite, then $\text{rk}_2(A) = \text{rk}(A(k))$.

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1.5 under the additional assumption that g_1 and g_2 are even. The extension to the case where g_1 and g_2 have arbitrary parities was explained to us by Adam Morgan.

2. THE FIBRATION ARGUMENT

Let k be a number field. Let C_i be a smooth projective hyperelliptic curve with affine equation $y^2 = f_i(x)$. For $t \in k^\times$ we write C_i^t for the quadratic twist of C_i by t . This is the hyperelliptic curve with affine equation $y^2 = tf_i(x)$. Define $\mathfrak{S}$ as the set of $t \in k^\times$ such that $C_i^t(\mathbf{A}_k) \neq \emptyset$ for $i = 1, 2$.

Proposition 2.1. *Let k be a number field. For $i = 1, 2$, let g_i be a positive integer. Let $f_i(x) \in k[x]$ be a separable polynomial of degree $2g_i + 2$, discriminant Δ_i and splitting field L_i . Assume that $\text{Gal}(L_i/k) \simeq S_{2g_i+2}$ for $i = 1, 2$, and L_1 and L_2 are linearly disjoint over k . Let C_i be a smooth projective hyperelliptic curve with affine equation $y^2 = f_i(x)$. Let X be a smooth, projective, geometrically integral variety birational to the affine variety with equation $y^2 = f_1(x_1)f_2(x_2)$. Suppose that $X(\mathbf{A}_k) \neq \emptyset$.*

Let Σ be a finite set of places of k , and for every $v \in \Sigma$ let $t_v \in k_v^\times$ be such that $C_i^{t_v}(k_v) \neq \emptyset$ for $i = 1, 2$. Then, for every real number $\epsilon > 0$, there exists $t \in \mathfrak{S}$ such that $|t - t_v|_v < \epsilon$ in k_v for all $v \in \Sigma$. In particular, $\mathfrak{S}$ is non-empty.

Proof. The second assertion follows from the first by taking $\Sigma = \emptyset$. It remains to prove the first assertion.

The fixed locus Z of the involution (ι_1, ι_2) on $C_1 \times_k C_2$ is a finite étale closed subscheme of degree $(2g_1 + 2)(2g_2 + 2)$ over k . Define Y as the blow-up of $C_1 \times_k C_2$ at Z . Let $\mathcal{X}$ denote the quotient of $Y \times_k \mathbb{G}_{m,k}$ by the diagonal action of μ_2 (acting by the extension of (ι_1, ι_2) on the first factor, and by $x \mapsto -x$ on the second factor). The fibre of $\mathcal{X}$ over $t \in \mathbb{G}_m(k)$ is the quadratic twist of Y by t . Let $\mathcal{X} \subset \tilde{\mathcal{X}}$ be a smooth compactification fitting into a commutative diagram

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \tilde{\mathcal{X}} \\ \downarrow & & \downarrow \\ \mathbb{G}_{m,k} & \longrightarrow & \mathbb{P}_k^1 \end{array}$$

The generic fibre of the projection $\mathcal{X} \rightarrow X$ is a $\mathbb{G}_m$ -torsor, and hence in particular the threefold $\tilde{\mathcal{X}}$ is stably birational to X . It is clear that X is the minimal desingularization of $(C_1^t \times C_2^t)/(\iota_1, \iota_2)$ for any $t \in k^\times$, where, by a slight abuse of notation, we denote by $\iota_i: C_i^t \rightarrow C_i^t$ the respective hyperelliptic involutions.

For $i = 1, 2$, let $l_i := k[x]/(f_i(x))$, and fix an inclusion $l_i \subset L_i$.

Claim 2.2. The vertical Brauer group of $\tilde{\mathcal{X}}$ over $\mathbb{P}_k^1$ is the image of $\text{Br}(k)$ in $\text{Br}(\tilde{\mathcal{X}})$.

Proof of Claim 2.2. This will follow from [CTS21, Proposition 11.5.2(i)] once we have shown that the natural homomorphism $k^\times/k^{\times 2} \rightarrow (l_1 l_2)^\times / (l_1 l_2)^{\times 2}$ is injective.

Since L_1 and L_2 are linearly disjoint over k , we have

$$\text{Gal}(L_1 L_2 / k) \simeq \text{Gal}(L_1 / k) \times \text{Gal}(L_2 / k) \simeq S_{2g_1+2} \times S_{2g_2+2}.$$

The subfield $l_1 l_2 \subset L_1 L_2$ is the fixed field of $H = S_{2g_1+1} \times S_{2g_2+1}$, where S_{2g_i+1} is the stabilizer of a root of f_i . The quadratic extensions of k contained in $L_1 L_2$ are $k(\sqrt{\Delta_1})$, $k(\sqrt{\Delta_2})$ and $k(\sqrt{\Delta_1 \Delta_2})$, where $\Delta_i \in k^\times$ is the discriminant of f_i for $i = 1, 2$. It suffices to show that none of these quadratic extensions of k is contained

in $l_1 l_2$. Equivalently, it is enough to show that H is not contained in the kernel of the corresponding quadratic character of $S_{2g_1+2} \times S_{2g_2+2}$. This is indeed impossible, because $2g_i + 2 \geq 4$ and hence each S_{2g_i+1} contains odd permutations. $\square$

Since X is stably birational to $\mathcal{X}$, solubility of X everywhere locally is equivalent to the solubility of $\mathcal{X}$ everywhere locally. By Claim 2.2, we may apply the fibration theorem for the universal family of quadratic twists [CTS00, Theorem A] (see also [CTS21, Corollary 14.2.19]) to find a $t \in \mathfrak{S}$ arbitrarily close to t_v for $v \in \Sigma$. $\square$

3. PRELIMINARIES ON JACOBIANS OF HYPERELLIPTIC CURVES

3.1. Theta characteristics and the Poonen–Stoll class.

Definition 3.1. Let C be a smooth, projective, geometrically integral curve over a field k , of positive genus g . Let $J = \text{Pic}_{C/k}^0$ be the Jacobian of C , and let λ be the canonical principal polarization of J .

- (1) Let $\mathfrak{c} := [\Theta_C] \in H^1(k, J[2])$, where $\Theta_C \subset \text{Pic}^{g-1}(C_{\bar{k}})$ is the $J[2]$ -torsor defined by $\Theta_C := \{L \in \text{Pic}^{g-1}(C_{\bar{k}}) : L^{\otimes 2} \simeq \omega_C\}$; see [PR11, Definition 3.8].
- (2) Let $\mathfrak{c}_\lambda \in H^1(k, J^\vee[2])$ be the class of the $J^\vee[2]$ -torsor of symmetric line bundles on J inducing the polarization λ ; see [PR11, (17)].
- (3) Let $c_\lambda \in H^1(k, J^\vee)$ be the class associated to λ by Poonen–Stoll [PS99, § 4].
- (4) Let $c \in H^1(k, J)$ be the class of the J -torsor $\text{Pic}_{C/k}^{g-1}$.

Proposition 3.2. *Let C be a smooth, projective, geometrically integral curve over a number field k , of positive genus g . Let $J = \text{Pic}_{C/k}^0$ be the Jacobian of C , and let $\lambda: J \rightarrow J^\vee$ be the canonical principal polarization. Then*

$$c_\lambda \in \text{III}(J^\vee)[2], \quad c \in \text{III}(J)[2], \quad \mathfrak{c}_\lambda \in \text{Sel}^2(J^\vee/k), \quad \mathfrak{c} \in \text{Sel}^2(J/k).$$

Moreover, the four classes fit into the following commutative diagram:

$$(3.1) \quad \begin{array}{ccc} \text{Sel}^2(J/k) & \xrightarrow{\lambda_*} & \text{Sel}^2(J^\vee/k) \\ \downarrow & & \downarrow \\ \text{III}(J)[2] & \xrightarrow{\lambda_*} & \text{III}(J^\vee)[2], \end{array} \quad \begin{array}{ccc} \mathfrak{c} & \xrightarrow{\lambda_*} & \mathfrak{c}_\lambda \\ \downarrow & & \downarrow \\ c & \xrightarrow{\lambda_*} & c_\lambda. \end{array}$$

Proof. We have $\mathfrak{c}_\lambda = \lambda_*(\mathfrak{c})$ by [PR11, Theorem 3.9], and $c_\lambda = \lambda_*(c)$ by [PS99, Corollary 4]. By [PR11, Remark 3.3], the natural map $H^1(k, J^\vee[2]) \rightarrow H^1(k, J^\vee)$ sends $\mathfrak{c}_\lambda$ to c_λ . It follows that the natural map $H^1(k, J[2]) \rightarrow H^1(k, J)$ sends $\mathfrak{c}$ to c . Finally, by [PS99, Corollary 2] we have $c_\lambda \in \text{III}(J^\vee)[2]$, and the other assertions follow from the commutativity of (3.1). $\square$

For an abelian variety A over a number field k we have the Cassels–Tate pairing

$$\text{III}(A) \times \text{III}(A^\vee) \rightarrow \mathbb{Q}/\mathbb{Z},$$

see [Mil86, Proposition I.6.9]. When J is the Jacobian of a curve with its canonical principal polarization $\lambda: J \xrightarrow{\sim} J^\vee$, we denote by $\langle x, y \rangle_J$, for $x, y \in \text{III}(J)$, the Cassels–Tate pairing of x and $\lambda_*(y)$.

Proposition 3.3. *Let k be a number field, let $C: y^2 = f(x)$ be a hyperelliptic curve of genus $g \geq 2$, where $f(x) \in k[x]$ is separable of degree $2g + 2$, and let J be the Jacobian of C . Let $\mathcal{R}$ be the set of roots of f in $\bar{k}$. Let $\mathfrak{c} \in \text{Sel}^2(J/k)$ be the class of (3.1).*

(1) There is a short exact sequence of Γ_k -modules

$$0 \longrightarrow J[2] \longrightarrow \mathbb{F}_2^{\mathcal{R}} / \langle \mathcal{R} \rangle \longrightarrow \mathbb{F}_2 \longrightarrow 0,$$

where the third map sends a subset of $\mathcal{R}$ to its cardinality modulo 2. Let $\mathfrak{w}$ be the image of $1 \in H^0(k, \mathbb{F}_2)$ under the connecting homomorphism $H^0(k, \mathbb{F}_2) \rightarrow H^1(k, J[2])$.

- (2) If g is even, then $\mathfrak{c} = \mathfrak{w}$. If g is odd, then $\mathfrak{c} = 0$.
 (3) The splitting field of f over k is equal to $k(J[2])$.
 (4) The natural map $H^1(k, J[2]) \rightarrow H^1(k, J)$ sends $\mathfrak{w}$ to $[\text{Pic}_{C/k}^1]$.
 (5) After identifying $\frac{1}{2}\mathbb{Z}/\mathbb{Z}$ with $\mathbb{F}_2$, we have

$$\langle \mathfrak{c}, \mathfrak{c} \rangle_J \equiv \dim_{\mathbb{F}_2} \text{III}(J)_{\text{nd}}[2] \equiv \dim_{\mathbb{F}_2} \text{Sel}^2(J/k) + \text{rk}_2(J) + \dim_{\mathbb{F}_2} J(k)[2] \pmod{2}.$$

- (6) After identifying $\frac{1}{2}\mathbb{Z}/\mathbb{Z}$ with $\mathbb{F}_2$, we have $\langle \mathfrak{c}, \mathfrak{c} \rangle_J \equiv N \pmod{2}$, where N is the number of places v of k such that $\text{Pic}^{g-1}(C_{k_v}) = \emptyset$. In particular, if $C(k_v) \neq \emptyset$ for every place v of k , then $\langle \mathfrak{c}, \mathfrak{c} \rangle_J = 0$ in $\mathbb{F}_2$.
 (7) For every $t \in k^\times$, let $C^t : y^2 = tf(x)$ be the quadratic twist of C , let J^t be its Jacobian, and let $\mathfrak{c}_t \in \text{Sel}^2(J^t/k) \subset H^1(k, J^t[2])$ be the class of (3.1) for J^t . Under the canonical identification $J^t[2] \simeq J[2]$, the class $\mathfrak{c}_t$ is identified with $\mathfrak{c}$.

Proof. (1) See [Mor25, (8.3)].

(2) See [Mor25, Proposition 8.5].

(3) The sequence of (1) shows that $k(J[2])$ is contained in the splitting field of f . Conversely, since $2g+2 \geq 6$, the natural action of the symmetric group $\text{Sym}(\mathcal{R})$ on $\text{Ker}(\mathbb{F}_2^{\mathcal{R}} / \langle \mathcal{R} \rangle \rightarrow \mathbb{F}_2)$ is faithful. Thus, by (1), any element of Γ_k acting trivially on $J[2]$ acts trivially on $\mathcal{R}$. Therefore $k(J[2])$ contains the splitting field of f .

(4) Let $\eta \in \text{Pic}_{C/k}^2(k)$ be the class of a fibre of the hyperelliptic map $C \rightarrow \mathbb{P}_k^1$. Then $\mathfrak{w}$ is the class of the torsor under $J[2]$ which is the preimage of η under multiplication by 2 on $\text{Pic}_{C/k}$. Since $\deg(\eta) = 2$, this torsor is naturally a closed subset of $\text{Pic}_{C/k}^1$, and this embedding is compatible with the natural injection $J[2] \rightarrow J$. Thus the image of $\mathfrak{w}$ in $H^1(k, J)$ is $[\text{Pic}_{C/k}^1]$.

(5) By [PS99, Theorem 8], we have $\langle \mathfrak{c}, \mathfrak{c} \rangle_J \equiv \dim_{\mathbb{F}_2} \text{III}(J)_{\text{nd}}[2]$, where $\text{III}(J)_{\text{nd}}$ is the quotient of $\text{III}(J)$ by its maximal divisible subgroup. The conclusion follows from the exact sequence

$$0 \longrightarrow J(k)/2J(k) \longrightarrow \text{Sel}^2(J/k) \longrightarrow \text{III}(J)[2] \longrightarrow 0$$

and the definition of $\text{rk}_2(J)$.

(6) See [PS99, Corollary 12].

(7) The set of roots of $tf(x)$ coincides with the set of roots $\mathcal{R}$ of $f(x)$. We obtain a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & J[2] & \longrightarrow & \mathbb{F}_2^{\mathcal{R}} / \langle \mathcal{R} \rangle & \longrightarrow & \mathbb{F}_2 \longrightarrow 0 \\ & & \downarrow \wr & & \parallel & & \parallel \\ 0 & \longrightarrow & J^t[2] & \longrightarrow & \mathbb{F}_2^{\mathcal{R}} / \langle \mathcal{R} \rangle & \longrightarrow & \mathbb{F}_2 \longrightarrow 0 \end{array}$$

where the rows are the short exact sequences of (1) for J and J^t , and where the left vertical arrow is the unique map making the left square commute. The fact that the isomorphism $H^1(k, J[2]) \rightarrow H^1(k, J^t[2])$ sends $\mathfrak{c}$ to $\mathfrak{c}_t$ now follows from (2). $\square$

3.2. The case of maximal Galois image.

Proposition 3.4. *Let $C : y^2 = f(x)$ be a hyperelliptic curve over a number field k of genus $g \geq 2$, where $f \in k[x]$ is separable of degree $2g + 2$, and let J be the Jacobian of C . Suppose that $G := \text{Gal}(f) \simeq S_{2g+2}$.*

- (1) *We have $J(k)[2] = 0$.*
- (2) *The G -module $J[2]$ is simple, and $\text{End}_G(J[2]) = \mathbb{F}_2$.*
- (3) *We have $H^1(G, J[2]) \simeq \mathbb{F}_2$, and the inflation map $H^1(G, J[2]) \rightarrow H^1(k, J[2])$ sends the non-zero element to the class $\mathfrak{w}$ of Proposition 3.3(1). Consequently, if $C(\mathbf{A}_k) \neq \emptyset$, then*

$$\text{Sel}^2(J/k) \cap H^1(G, J[2]) = \langle \mathfrak{w} \rangle \quad \text{inside } H^1(k, J[2]).$$

Proof. (1) Immediate from Proposition 3.3(1).

(2) See [Mor25, Lemma 8.2].

(3) The first statement is [Mor25, Proposition 8.4]. By Proposition 3.3(4), if $C(\mathbf{A}_k) \neq \emptyset$, then $\mathfrak{w} \in \text{Sel}^2(J/k)$. $\square$

Proposition 3.5. *Let $C : y^2 = f(x)$ be a hyperelliptic curve over a number field k of genus $g \geq 2$, where $f \in k[x]$ is separable of degree $2g + 2$, and let J be the Jacobian of C . Suppose that $G := \text{Gal}(f) \simeq S_{2g+2}$.*

- (1) *For every cycle $h \in G \simeq S_{2g+2}$ of length $2g + 1$, we have $J[2]^h = 0$.*
- (2) *If v is a prime of good reduction for C , unramified in the splitting field of f , and if Frob_v acts as a cycle of length $2g + 1$ on the roots of f , then f has a root in k_v , and consequently every quadratic twist of C has a k_v -point.*

Proof. (1) We use the identification of $J[2]$ given in Proposition 3.3(1). The h -fixed subspace $(\mathbb{F}_2^{\mathcal{R}})^h$ has basis e_1, e_2 , where e_1 is the fixed point and e_2 is the sum of the elements in the cycle of length $2g + 1$. The coordinate sum of $a_1 e_1 + a_2 e_2$ is $a_1 + (2g + 1)a_2 = a_1 + a_2$. Hence the intersection of $(\mathbb{F}_2^{\mathcal{R}})^h$ and the kernel of the augmentation map $\mathbb{F}_2^{\mathcal{R}} \rightarrow \mathbb{F}_2$ is generated by $e_1 + e_2 = \mathcal{R}$. Since $2g + 1$ is odd, we have $H^1(\mathbb{Z}/(2g + 1), \mathbb{Z}/2) = 0$, which implies that $J[2]^h = 0$.

(2) Since Frob_v acts as a cycle of length $2g + 1$, it fixes one root of the reduction of f modulo v . Since v is a place of good reduction, this root is simple. By Hensel's lemma, we may lift this root to a root $\alpha \in k_v$ of f over k_v . For any quadratic twist $C^a : y^2 = af(x)$, the point $(\alpha, 0)$ belongs to $C^a(k_v)$. $\square$

Corollary 3.6. *Let k be a number field, let $C : y^2 = f(x)$ be a hyperelliptic curve of genus $g \geq 2$, where $f(x) \in k[x]$ is separable of degree $2g + 2$, and let J be the Jacobian of C . Suppose that $\text{Gal}(f) \simeq S_{2g+2}$. Let $\mathfrak{c} \in \text{Sel}^2(J/k)$ be the class of (3.1). For every $t \in k^\times$, viewing $\mathfrak{c}$ as an element of $\text{Sel}^2(J^t/k)$ via the canonical identification $J^t[2] \simeq J[2]$, we have*

$$\langle \mathfrak{c}, \mathfrak{c} \rangle_{J^t} \equiv \dim_{\mathbb{F}_2} \text{Sel}^2(J^t/k) + \text{rk}_2(J^t) \pmod{2}.$$

In particular, if $C^t(k_v) \neq \emptyset$ for every place v of k , then

$$\dim_{\mathbb{F}_2} \text{Sel}^2(J^t/k) + \text{rk}_2(J^t) \equiv 0 \pmod{2}.$$

Proof. By Proposition 3.3(7), the class $\mathfrak{c}$ is identified with the Poonen–Stoll class of J^t . Applying Proposition 3.3(5) to C^t , we obtain

$$\langle \mathfrak{c}, \mathfrak{c} \rangle_{J^t} \equiv \dim_{\mathbb{F}_2} \text{Sel}^2(J^t/k) + \text{rk}_2(J^t) + \dim_{\mathbb{F}_2} J^t(k)[2] \pmod{2}.$$

By Proposition 3.4(1), $J^t(k)[2] = 0$, and the desired congruence follows.

If $C^t(k_v) \neq \emptyset$ for every place v , then $\text{Pic}^{g-1}(C_{k_v}^t) \neq \emptyset$ for every v . Hence $\langle \mathfrak{c}, \mathfrak{c} \rangle_{J^t} = 0$ by Proposition 3.3(6), giving the final assertion. $\square$

3.3. The minimal regular model.

Lemma 3.7. *Let K be a non-archimedean local field of odd residue characteristic, and let C/K be the hyperelliptic curve $y^2 = f(x)$, where $f \in \mathcal{O}_K[x]$ is separable of degree $2g + 2$ and has leading coefficient in $\mathcal{O}_K^\times$. Suppose that the reduction $\bar{f}$ has exactly one double root and no other multiple root, and that $\text{val}(\Delta(f)) = 2$. Then the minimal proper regular model of C over $\mathcal{O}_K$ has special fibre consisting of two smooth geometrically integral irreducible components geometrically meeting in two ordinary double points.*

Proof. This is a special case of results of [Liu96]; we give a direct proof. Let $\mathbb{F}$ be the residue field of K , and let π be a uniformizer. Let

$$F(u) = u^{2g+2}f(1/u) \in \mathcal{O}_K[u].$$

Consider the flat proper model $\mathcal{C}_0$ of C over $\mathcal{O}_K$ obtained by gluing the two affine schemes

$$U_0 := \text{Spec } \mathcal{O}_K[x, y]/(y^2 - f(x)), \quad U_\infty := \text{Spec } \mathcal{O}_K[u, z]/(z^2 - F(u)).$$

On U_0 , the special fibre is singular exactly at the multiple roots of $\bar{f}$. By assumption, there is only one such root, and it is a double root. On U_∞ , the points at infinity are given by the equations $u = 0$, $z^2 = \bar{a}_{2g+2}$, where a_{2g+2} is the leading coefficient of f . Since $a_{2g+2} \in \mathcal{O}_K^\times$ and $\text{char}(\mathbb{F})$ is odd, these points are smooth, because the partial derivative with respect to z is nonzero.

Let $\alpha \in \mathbb{F}$ be the unique double root of $\bar{f}$, and let P be the corresponding singular point of $\mathcal{C}_{0, \mathbb{F}}$. After translating x , we may assume that $\alpha = 0$, so that $P = (x, y, \pi)$. We have $\widehat{\mathcal{O}}_{\mathcal{C}_0, P} = \mathcal{O}_K[[x, y]]/(y^2 - f(x))$.

By Hensel's lemma, we may write $f(x) = q(x)h(x)$, where $q(x) \in \mathcal{O}_K[x]$ is a monic polynomial of degree 2 with reduction x^2 , and $h(x) \in \mathcal{O}_K[x]$ satisfies $h(0) \in \mathcal{O}_K^\times$. The assumption $\text{val}(\Delta(f)) = 2$ implies $\text{val}(\Delta(q)) = 2$. Since $2 \in \mathcal{O}_K^\times$, again by Hensel's lemma we may write $h(x) = h(0)s(x)^2$ for some $s(x) \in \mathcal{O}_K[[x]]$ such that $s(0) \in \mathcal{O}_K^\times$. Thus $\widehat{\mathcal{O}}_{\mathcal{C}_0, P} = \mathcal{O}_K[[x, y']]/((y')^2 - h(0)q(x))$, where $y' = y/s(x)$. Write $q(x) = x^2 + bx + c$. Since $2 \in \mathcal{O}_K^\times$, we have

$$\widehat{\mathcal{O}}_{\mathcal{C}_0, P} = \mathcal{O}_K[[x', y']]/((y')^2 - h(0)(x')^2 + \mu\pi^2),$$

where $x' = x + b/2$, and where $\mu \in \mathcal{O}_K^\times$ because $\text{val}(\Delta(q)) = 2$. Therefore, after passing to an étale cover given by taking the square root of $h(0)$, the completion of $\mathcal{C}_0$ at P is given by $\mathcal{O}_K[[x'', y'']]/(x''y'' - \pi^2)$. A simple calculation now shows that the blowup $\mathcal{C}$ of $\mathcal{C}_0$ at P is regular. The special fibre of $\mathcal{C}$ is the union of the strict transform of the special fibre of $\mathcal{C}_0$ and the exceptional divisor, which meet in the points corresponding to the two tangent directions of the node (defined over the quadratic extension of $\mathbb{F}$ given by the square root of $\bar{h}(0)$). Thus $\mathcal{C}$ is the required regular minimal model. $\square$

3.4. A Weil estimate.

Lemma 3.8. *Let $g \geq 1$, let q be an odd prime power, and let $F(x) \in \mathbb{F}_q[x]$ have degree $2g + 2$. Let $a \in \mathbb{F}_q^\times$, and let Y_a be the projective curve over $\mathbb{F}_q$ with affine*

equation $y^2 = aF(x)$. Assume that F is separable, or that F has exactly one double root and no other multiple root. If $q \geq 4g^2$, then Y_a has a smooth $\mathbb{F}_q$ -point.

Proof. If F is separable, then Y_a is a smooth curve of genus g . By the Weil bound,

$$|Y_a(\mathbb{F}_q)| \geq q + 1 - 2g\sqrt{q} \geq 1.$$

Suppose now that $F(x) = (x - \alpha)^2 G(x)$ has exactly one double root α and no other multiple root. Since this double root is unique, we have $\alpha \in \mathbb{F}_q$. The normalization of Y_a is the smooth projective curve $\tilde{Y}_a$ with affine model $z^2 = aG(x)$, and it has genus $g - 1$ because $\deg G = 2g$. By the Weil bound,

$$|\tilde{Y}_a(\mathbb{F}_q)| \geq q + 1 - 2(g - 1)\sqrt{q}.$$

The normalization map is an isomorphism away from the singular point of Y_a , and the inverse image of this singular point contains at most two $\mathbb{F}_q$ -points. Thus it is enough to show that

$$q + 1 - 2(g - 1)\sqrt{q} > 2.$$

Since $q \geq 4g^2$, we have $\sqrt{q} \geq 2g$, and hence

$$q + 1 - 2(g - 1)\sqrt{q} \geq 2g\sqrt{q} + 1 - 2(g - 1)\sqrt{q} = 1 + 2\sqrt{q} > 2. \quad \square$$

4. PARITY OF SELMER GROUPS IN TWO FAMILIES

Lemma 4.1. *Let K be a non-archimedean local field of odd residue characteristic, with ring of integers $\mathcal{O}_K$. Let K'/K be the unramified quadratic extension. Let $C : y^2 = f(x)$ be a hyperelliptic curve of genus $g \geq 2$ over K , where $f \in \mathcal{O}_K[x]$ is separable of degree $2g + 2$ with leading coefficient in $\mathcal{O}_K^\times$. Let J be the Jacobian of C .*

(1) *Suppose that J has good reduction over K . Then we have*

$$J(K)/N_{K'/K}J(K') = 0,$$

where $N_{K'/K} : J(K') \rightarrow J(K)$ is the norm map.

(2) *Suppose that the reduction of f modulo the maximal ideal of $\mathcal{O}_K$ has exactly one double root and no other multiple root, and that $\text{val}(\Delta(f)) = 2$. Then we have*

$$J(K)/N_{K'/K}J(K') \simeq \mathbb{Z}/2.$$

Proof. Let $\mathbb{F}$ and $\mathbb{F}'$ be the residue fields of K and K' , respectively. Let $\mathcal{J}$ be the Néron model of J over $\mathcal{O}_K$, let $\mathcal{J}^0$ be its identity component, and let Φ be the group scheme of connected components of the special fibre.

By [Maz72, Propositions 4.2 and 4.3], we have

$$(4.1) \quad J(K)/N_{K'/K}J(K') \simeq H^1(\text{Gal}(K'/K), J(K')) \simeq H^1(\text{Gal}(\mathbb{F}'/\mathbb{F}), \Phi(\mathbb{F}')).$$

(1) If J has good reduction, then its Néron model is an abelian scheme over $\mathcal{O}_K$. In particular, the special fibre is connected, so $\Phi = 0$, and we conclude by (4.1).

(2) By Lemma 3.7, the special fibre of the minimal proper regular model of C consists of two irreducible components meeting in two ordinary double points. Thus its dual graph has two vertices joined by two edges. Therefore, by [BLR90, Chapter 9, §6, Proposition 10] we have $\Phi_{\mathbb{F}} \simeq \mathbb{Z}/2$, and hence $\Phi \simeq \mathbb{Z}/2$. Thus $\Phi(\mathbb{F}') \simeq \mathbb{Z}/2$ and we conclude again by (4.1). $\square$

For $i = 1, 2$ and $t \in k^\times$, let $J_i^t = \text{Pic}_{C_i^t/k}^0$ be the Jacobian of C_i^t . We use the canonical identification $J_i^t[2] \simeq J_i[2]$ of Galois modules.

Lemma 4.2. *We place ourselves in Situation 1.4. Let $i, j \in \{1, 2\}$ with $i \neq j$.*

(1) *Let $t \in k_{w_j}^\times$ be a unit. Then*

$$J_i(k_{w_j})/NJ_i(k_{w_j}(\sqrt{t})) = 0.$$

(2) *Let $t \in k_{w_i}^\times$ be a unit. Then*

$$\dim_{\mathbb{F}_2}(J_i(k_{w_i})/NJ_i(k_{w_i}(\sqrt{t}))) = \begin{cases} 0, & \text{if } t \in k_{w_i}^{\times 2}, \\ 1, & \text{if } t \notin k_{w_i}^{\times 2}. \end{cases}$$

Proof. (1) Let $K = k_{w_j}$. The assumptions imply that C_i has good reduction over K , and therefore J_i has good reduction over K .

If $t \in K^{\times 2}$, then $K(\sqrt{t}) = K$ and the norm quotient is zero. If $t \notin K^{\times 2}$, then since the residue characteristic of K is odd, $K(\sqrt{t})/K$ is the unramified quadratic extension, and the conclusion follows from Lemma 4.1(1).

(2) Let $K = k_{w_i}$. If $t \in K^{\times 2}$, then $K(\sqrt{t}) = K$, so the norm map is the identity map on $J_i(K)$. Hence the quotient is zero. Suppose now that $t \notin K^{\times 2}$. By assumption, the residue characteristic of K is odd, so that $K(\sqrt{t})/K$ is the unramified quadratic extension. The conclusion follows from Lemma 4.1(2). $\square$

Recall that $\mathfrak{S}$ is the set of $t \in k^\times$ such that $C_i^t(\mathbf{A}_k) \neq \emptyset$ for $i = 1, 2$.

Proposition 4.3. *We place ourselves in Situation 1.4. Assume that $X(\mathbf{A}_k) \neq \emptyset$. Then there exists $t \in \mathfrak{S}$ such that $\dim_{\mathbb{F}_2} \text{Sel}^2(J_1^t)$ and $\dim_{\mathbb{F}_2} \text{Sel}^2(J_2^t)$ are odd.*

Proof. Let $t \in k^\times$, and let $L = k(\sqrt{t})$. By [Mor19, Theorem 10.12] we have

$$(4.2) \quad \dim_{\mathbb{F}_2} \text{Sel}^2(J_i^t) \equiv \dim_{\mathbb{F}_2} \text{Sel}^2(J_i) + \sum_v \dim_{\mathbb{F}_2} (J_i(k_v)/N(J_i(L_w))) \pmod{2},$$

where the sum is over all places of k , and w is a place of L over v . (If v is split in L , then the summand is zero.)

Let S be the union of S_0 and the primes dividing $\Delta_1 \Delta_2$.

Assume $v \notin S$. Then J_i has good reduction at v , so the local root number is $w(J_i/L_w) = 1$. By [Mor23, Theorem 1.8], for such v , we thus have

$$(\Delta_i, t)_{k_v} = (-1)^{\varepsilon(C_i/k_v) + \varepsilon(C_i^t/k_v) + \dim_{\mathbb{F}_2} (J_i(k_v)/N(J_i(L_w)))},$$

where $\varepsilon(C_i/k_v) = 0$ if C_i has a k_v -rational divisor of degree $g - 1$, otherwise $\varepsilon(C_i/k_v) = 1$. Since $v \notin S$, the curve C_i has good reduction at v . Then the special fibre of C_i has an $\mathbb{F}_v$ -rational divisor of degree 1. By Hensel's lemma, it lifts to a k_v -rational divisor of degree 1. Thus C_i has a k_v -rational divisor of degree $g - 1$, so $\varepsilon(C_i/k_v) = 0$. For $v \notin S$ and any $t \in k^\times$, we thus have

$$(\Delta_i, t)_{k_v} = (-1)^{\varepsilon(C_i^t/k_v) + \dim_{\mathbb{F}_2} (J_i(k_v)/N(J_i(L_w)))}.$$

By the product formula $\prod_v (\Delta_i, t)_{k_v} = 1$, the parity of $\dim_{\mathbb{F}_2} \text{Sel}^2(J_i^t) - \dim_{\mathbb{F}_2} \text{Sel}^2(J_i)$ is determined by

$$\prod_{v \notin S} (-1)^{\varepsilon(C_i^t/k_v)} \cdot \prod_{v \in S} (\Delta_i, t)_{k_v} \cdot \prod_{v \in S} (-1)^{\dim_{\mathbb{F}_2} (J_i(k_v)/N(J_i(L_w)))} \in \{\pm 1\}.$$

If $t \in \mathfrak{S}$, then C_i^t is everywhere locally soluble, so the first product is 1. If, moreover, t is a unit at w_1 and w_2 , then since $\text{val}_{w_i}(\Delta_j)$ is even by assumption, we

have $(\Delta_i, t)_{k_{w_1}} = (\Delta_i, t)_{k_{w_2}} = 1$, hence for $i = 1, 2$ the parity of $\dim_{\mathbb{F}_2} \text{Sel}^2(J_i^t) - \dim_{\mathbb{F}_2} \text{Sel}^2(J_i)$ is determined by

$$(4.3) \quad \prod_{v \in S \setminus \{w_1, w_2\}} (\Delta_i, t)_{k_v} \cdot \prod_{v \in S} (-1)^{\dim_{\mathbb{F}_2} (J_i(k_v)/N(J_i(L_w)))} \in \{\pm 1\}.$$

This holds for any $t \in \mathfrak{S}$ that is a unit at w_1 and w_2 .

By Proposition 2.1, the assumption $X(\mathbf{A}_k) \neq \emptyset$ implies that $\mathfrak{S} \neq \emptyset$. For every $v \in S \setminus \{w_1, w_2\}$, choose $t_v \in k_v^\times$ such that $C_i^{t_v}(k_v) \neq \emptyset$ for $i = 1, 2$. Then, using Lemma 4.2, choose units $t_{w_1} \in k_{w_1}^\times$ and $t_{w_2} \in k_{w_2}^\times$ such that

$$(4.4) \quad \prod_{v \in S \setminus \{w_1, w_2\}} (\Delta_i, t_v)_{k_v} \cdot \prod_{v \in S} (-1)^{\dim_{\mathbb{F}_2} (J_i(k_v)/N(J_i(k_v(\sqrt{t_v}))))}$$

takes any given value in $\{\pm 1\}$, for $i = 1, 2$. We note that $C_i^{t_{w_j}}(k_{w_j}) \neq \emptyset$ for all $i, j = 1, 2$ by Lemma 3.8 and Hensel's lemma. By Proposition 2.1, there exists $t \in \mathfrak{S}$ sufficiently close to t_v for all $v \in S$. Then the value of (4.3) equals the value of (4.4), which we can choose to be such that $\dim_{\mathbb{F}_2} \text{Sel}^2(J_i^t)$ is odd for $i = 1, 2$. $\square$

5. SIMULTANEOUS VARIATION OF THE CASSELS–TATE PAIRING

Definition 5.1. Let k be a number field, let J be the Jacobian of a smooth projective hyperelliptic curve over k , let $L = k(J[2])$, and set $G = \text{Gal}(L/k)$. Let $\mathfrak{w}$ be the class defined in Proposition 3.3(1), and suppose that

$$(5.1) \quad \text{Sel}^2(J) \cap H^1(G, J[2]) = \langle \mathfrak{w} \rangle \quad \text{as subgroups of } H^1(k, J[2]).$$

Let $m := \dim_{\mathbb{F}_2} \langle \mathfrak{w} \rangle \in \{0, 1\}$ and $n := \dim_{\mathbb{F}_2} \text{Sel}^2(J)/\langle \mathfrak{w} \rangle$. Thus we have $n + m = \dim_{\mathbb{F}_2} \text{Sel}^2(J)$. Let

$$(5.2) \quad \begin{aligned} \mathcal{S} &= \mathcal{S}_0 \sqcup \mathcal{S}_1, \\ \mathcal{S}_0 &= \begin{cases} \emptyset, & \text{if } m = 0, \\ \{n+1\} \times \{1, \dots, n\}, & \text{if } m = 1, \end{cases} \quad \mathcal{S}_1 = \{(r, q) : 1 \leq r < q \leq n\}. \end{aligned}$$

By definition, a *Morgan datum* for J is a tuple

$$(5.3) \quad (T = \{\mathbf{a}_i\}_{i=1}^n, T' = \{\mathbf{a}_i\}_{i=n+1}^{n+m}, \{a_i\}_{i=1}^{n+m}, \{\gamma_\nu\}_\nu)$$

with the following properties:

- (1) $T \sqcup T' = \{\mathbf{a}_i\}_{i=1}^{n+m}$ is a basis of $\text{Sel}^2(J)$,
- (2) the image of $T = \{\mathbf{a}_i\}_{i=1}^n$ in $\text{Sel}^2(J)/\langle \mathfrak{w} \rangle$ is a basis,
- (3) $T' = \{\mathbf{a}_i\}_{i=n+1}^{n+m}$ is a basis of $\langle \mathfrak{w} \rangle \subset \text{Sel}^2(J)$,
- (4) for each $i \in \{1, \dots, n+m\}$, the map $a_i : \Gamma_k \rightarrow J[2]$ is a cocycle representing $\mathbf{a}_i$, which we require to be the inflation of a cocycle $\tilde{a}_i : G \rightarrow J[2]$ when $i > n$,
- (5) for each $\nu = (r, q) \in \mathcal{S}$, the map $\gamma_\nu : \Gamma_k \rightarrow \mu_2$ is a 1-cochain such that $d\gamma_\nu = a_r \cup a_q$, where the cup-product is formed using the Weil pairing on $J[2]$; such a cochain γ_ν exists because $\mathbf{a}_r \cup \mathbf{a}_q = 0$ in $H^2(k, \mu_2)$ by [Mor25, Remark 3.4].

Construction 5.2. Let k be a number field, let J be the Jacobian of a smooth projective hyperelliptic curve over k , let $L = k(J[2])$, and set $G = \text{Gal}(L/k)$. Let $\mathfrak{w}$ be the class defined in Proposition 3.3(1), and suppose that (5.1) holds. Fix a

Morgan datum (5.3) for J . We attach to this datum a finite Galois extension M/k and functions ψ_ν for $\nu \in \mathcal{S}$.

For every $\tau \in \Gamma_k$, let $\bar{\tau}$ denote its image in $G = \text{Gal}(k(J[2])/k)$. The cocycles $a_1, \dots, a_n$ define a homomorphism

$$\varphi_T: \Gamma_k \longrightarrow J[2]^n \rtimes G, \quad \tau \longmapsto ((a_i(\tau))_{1 \leq i \leq n}, \bar{\tau}),$$

where G acts diagonally on $J[2]^n$. Set

$$\Gamma := J[2]^n \rtimes G.$$

For $1 \leq i \leq n$, let f_i be the i th coordinate cocycle on Γ . For $n < i \leq n+m$, let $f_i: \Gamma \rightarrow J[2]$ be the inflation of the cocycle $\tilde{a}_i: G \rightarrow J[2]$ of Definition 5.1(4) along the projection $\Gamma \rightarrow G$. For every $1 \leq i \leq n+m$, we have $a_i = f_i \circ \varphi_T$.

Following [Mor25, Section 4.3 and Notation 4.12], let E be the central extension of Γ by μ_2^S associated with the 2-cocycle $(f_r \cup f_q)_{\nu=(r,q) \in \mathcal{S}}$. Since $d\gamma_\nu = a_r \cup a_q$ for every $\nu = (r, q) \in \mathcal{S}$, the cochains γ_ν give a lift of φ_T to a homomorphism

$$\varphi_{T,T'}: \Gamma_k \longrightarrow E.$$

Explicitly, this lift is characterized by the requirement that its projection to Γ is φ_T , and that its ν -th μ_2 -coordinate is γ_ν .

We define $M = k_{T,T'}$ to be the fixed field of $\ker(\varphi_{T,T'})$. Thus $\varphi_{T,T'}$ identifies $\text{Gal}(M/k)$ with a subgroup of E . In what follows, for every $\sigma \in \text{Gal}(M/k)$ we write $a_i(\sigma)$ and $\gamma_\nu(\sigma)$ for $a_i(\tilde{\sigma})$ and $\gamma_\nu(\tilde{\sigma})$, where $\tilde{\sigma} \in \Gamma_k$ is a lift of σ . These values are independent of the choice of $\tilde{\sigma}$, since $\ker(\varphi_{T,T'})$ is contained in the kernels of all the maps a_i and γ_ν .

Let $\sigma \in \text{Gal}(M/k)$, and suppose that the image of σ in G acts on $J[2]$ without non-zero fixed points. It follows that the endomorphism $\sigma - 1$ of $J[2]$ is invertible and hence, for each $1 \leq i \leq n+m$, there is a unique element $P_{i,\sigma} \in J[2]$ such that

$$a_i(\sigma) = \sigma P_{i,\sigma} - P_{i,\sigma}.$$

We write μ_2 additively and identify it with $\mathbb{F}_2$. For $\nu = (r, q) \in \mathcal{S}$, define

$$\psi_\nu(\sigma) = e_2(P_{r,\sigma}, a_q(\sigma)) + \gamma_\nu(\sigma) \in \mathbb{F}_2,$$

where $e_2: J[2] \times J[2] \rightarrow \mu_2$ is the Weil pairing associated with the chosen principal polarization.

Lemma 5.3. *Let k be a number field, let J be the Jacobian of a smooth projective hyperelliptic curve over k , let $L = k(J[2])$, and set $G = \text{Gal}(L/k)$. Let $\mathfrak{w}$ be the class defined in Proposition 3.3(1). Suppose that (5.1) holds, that is,*

$$\text{Sel}^2(J) \cap H^1(G, J[2]) = \langle \mathfrak{w} \rangle \quad \text{as subgroups of } H^1(k, J[2]).$$

Fix the sets $\mathcal{S} = \mathcal{S}_0 \sqcup \mathcal{S}_1$ of (5.2), fix a Morgan datum for J (in the sense of Definition 5.1), and let $M = k_{T,T'}$ and $\{\psi_\nu\}_{\nu \in \mathcal{S}}$ be as in Construction 5.2. Assume, moreover, that J satisfies [Mor25, Assumption 1.3(A),(B)]:

- (A) $J[2]$ is a simple $\mathbb{F}_2[G]$ -module and $\text{End}_G(J[2]) = \mathbb{F}_2$;
- (B) the set $\{g \in G : J[2]^g = 0\}$ is not empty.

Then the following hold.

- (a) *The field extension M/L is a 2-extension, that is, a finite Galois extension whose Galois group is a 2-group. Moreover, every quadratic character $\text{Gal}(M/k) \rightarrow \mu_2$ factors through G .*

- (b) Let $g \in G$ satisfy $J[2]^g = 0$. For every choice of values $\epsilon_\nu \in \mathbb{F}_2$, where $\nu \in \mathcal{S}$, there exists $\sigma \in \text{Gal}(M/k)$ whose image in G is g , and such that $\psi_\nu(\sigma) = \epsilon_\nu$ for every $\nu \in \mathcal{S}$.
- (c) Let Σ_0 be a finite set of places of k containing all places above 2, all archimedean places, all places of bad reduction for J , and all places ramified in M/k . Let $\chi: \Gamma_k \rightarrow \mu_2$ be a quadratic character such that χ_v is trivial for every $v \in \Sigma_0$, and such that, for every $v \notin \Sigma_0$ at which χ is ramified, one has $J[2]^{\text{Frob}_v} = 0$. Then

$$\text{Sel}^2(J^\chi/k) = \text{Sel}^2(J/k) \quad \text{as subgroups of } H^1(k, J[2]),$$

and

$$\text{rk}_2(J^\chi/k) \equiv \text{rk}_2(J/k) \pmod{2}.$$

- (d) Under the hypotheses of (c), for all $\nu = (r, q) \in \mathcal{S}$, one has

$$\langle \mathbf{a}_r, \mathbf{a}_q \rangle_{J^\chi} = \langle \mathbf{a}_r, \mathbf{a}_q \rangle_J + \sum_{v \in \text{Ram}(\chi) \setminus \Sigma_0} \psi_\nu(\text{Frob}_v).$$

Proof. Part (a) follows from [Mor25, Proposition 4.14]. Part (b) is [Mor25, Corollary 4.19]. Part (c) is [Mor25, Lemma 2.8]. Part (d) is [Mor25, Proposition 3.3]. $\square$

Lemma 5.4. *Let M/k be a finite Galois extension, and let Σ be a finite set of places of k , containing all archimedean places, all places above 2, and all places ramified in M/k . Let $\mathcal{C} \subset \text{Gal}(M/k)$ be a nonempty union of conjugacy classes. For every $v \in \Sigma$, let $\chi_v: G_{k_v} \rightarrow \mu_2$ be a quadratic character. Suppose that, for every quadratic character $\beta: \text{Gal}(M/k) \rightarrow \mu_2$ which is trivial on $\mathcal{C}$, one has*

$$(5.4) \quad \sum_{v \in \Sigma} \text{inv}_v(\chi_v \cup \text{res}_v \beta) = 0.$$

Then there exists a global quadratic character $\chi: \Gamma_k \rightarrow \mu_2$ such that $\text{res}_v(\chi) = \chi_v$ for every $v \in \Sigma$ and such that every place $v \notin \Sigma$ at which χ is ramified satisfies $\text{Frob}_v \in \mathcal{C}$.

Proof. This is [Mor25, Proposition 5.2]. $\square$

Lemma 5.5. *For $i = 1, 2$, let $C_i: y^2 = f_i(x)$ be hyperelliptic curves over k , with f_i separable of degree $2g_i + 2 \geq 6$, and let J_i be their Jacobians. Let L_i be the splitting field of f_i , and assume that $G_i := \text{Gal}(L_i/k) \simeq S_{2g_i+2}$. Assume that L_1 and L_2 are linearly disjoint over k .*

Let $t \in k^\times$. For $i = 1, 2$, we have the canonical identification of Γ_k -modules $J_i^t[2] \simeq J_i[2]$; in particular $L_i = k(J_i^t[2])$. For $i = 1, 2$, assume that (5.1) holds for J_i^t , fix a Morgan datum for J_i^t , and let M_i be the finite Galois extension of k attached to this datum by Construction 5.2. Then M_1 and M_2 are linearly disjoint over k . Moreover, every quadratic character $\text{Gal}(M_1 M_2/k) \rightarrow \mu_2$ factors through $\text{Gal}(L_1 L_2/k) \simeq G_1 \times G_2$.

Proof. Since $G_i \simeq S_{2g_i+2}$, with $2g_i + 2 \geq 6$, the hypotheses (A) and (B) of Lemma 5.3 hold for J_i^t . Thus Lemma 5.3(a), applied to J_i^t , shows that M_i/L_i is a 2-extension, and that every quadratic subextension of M_i/k is contained in L_i .

We first show that $L_1 \cap M_2 = k$ as subfields of $\bar{k}$. Set $F = L_1 \cap M_2$; since L_1/k and M_2/k are Galois, F/k is also Galois. Since L_1 and L_2 are linearly disjoint over k and $F \subset L_1$, we have $F \cap L_2 = k$. The compositum FL_2 is contained in M_2 , and M_2/L_2 is a 2-extension. Hence FL_2/L_2 is a 2-extension. Since $F \cap L_2 = k$

and F/k is Galois, it follows that F/k is a 2-extension. On the other hand, $F \subset L_1$ and $\text{Gal}(L_1/k) \simeq S_{2g_1+2}$. Since $2g_1 + 2 \geq 6$, the only nontrivial 2-group quotient of S_{2g_1+2} is its quotient of order 2, corresponding to the sign character. Therefore, if $F \neq k$, then by Galois theory F contains the quadratic subfield of L_1 . But $F \subset M_2$, and every quadratic subextension of M_2/k is contained in L_2 , contradicting $L_1 \cap L_2 = k$. Hence $L_1 \cap M_2 = k$. The same argument gives $L_2 \cap M_1 = k$.

Now set $F' = M_1 \cap M_2$. Since $F' \cap L_1 \subset M_2 \cap L_1 = k$, the fields F' and L_1 are linearly disjoint over k . Moreover, F'/k is Galois, since both M_1/k and M_2/k are Galois. The compositum $F'L_1$ is contained in M_1 , and M_1/L_1 is a finite Galois extension whose Galois group is a 2-group. Since $F' \cap L_1 = k$, restriction gives an isomorphism $\text{Gal}(F'L_1/L_1) \simeq \text{Gal}(F'/k)$. Hence F'/k is a 2-extension.

If $F' \neq k$, then F'/k has a quadratic subextension K/k . Since $K \subset M_1$ and $K \subset M_2$, Lemma 5.3(a) gives $K \subset L_1$ and $K \subset L_2$, contradicting $L_1 \cap L_2 = k$. Therefore $M_1 \cap M_2 = k$. This shows that the Galois extensions M_1/k and M_2/k are linearly disjoint over k . In particular, $\text{Gal}(M_1M_2/k) \simeq \text{Gal}(M_1/k) \times \text{Gal}(M_2/k)$. By Lemma 5.3(a) every quadratic character of $\text{Gal}(M_i/k)$ factors through G_i , for $i = 1, 2$. It follows that every quadratic character of $\text{Gal}(M_1M_2/k)$ factors through $G_1 \times G_2 \simeq \text{Gal}(L_1L_2/k)$, as desired. $\square$

Definition 5.6. Let A be a principally polarized abelian variety over k , and let $\mathbf{c} \in \text{Sel}^2(A)$ be the Poonen–Stoll class attached to the chosen principal polarization. We identify $\frac{1}{2}\mathbb{Z}/\mathbb{Z}$ with $\mathbb{F}_2$. Following Morgan [Mor25, § 1.3], a symmetric bilinear form

$$P: \text{Sel}^2(A) \times \text{Sel}^2(A) \longrightarrow \mathbb{F}_2$$

is called *admissible* if $P(\mathbf{c}, x) = P(x, x)$ for all $x \in \text{Sel}^2(A)$, and

$$P(\mathbf{c}, \mathbf{c}) \equiv \dim_{\mathbb{F}_2} \text{Sel}^2(A) + \text{rk}_2(A) + \dim_{\mathbb{F}_2} A(k)[2] \pmod{2}.$$

The following elementary observation is implicit in Morgan’s proof of [Mor25, Theorem 1.14].

Lemma 5.7. *Let k be a number field, let J/k and $\mathcal{S} = \mathcal{S}_0 \sqcup \mathcal{S}_1$ be as in Definition 5.1, and fix a Morgan datum (5.3) for J . Let $P, Q: \text{Sel}^2(J) \times \text{Sel}^2(J) \rightarrow \mathbb{F}_2$ be admissible pairings. Suppose that $P(\mathbf{a}_r, \mathbf{a}_q) = Q(\mathbf{a}_r, \mathbf{a}_q)$ for every $\nu = (r, q) \in \mathcal{S}$. Then $P = Q$.*

Proof. It is enough to show that the values of P on the basis $T \sqcup T' = \{\mathbf{a}_i\}_{i=1}^{n+m}$ are determined by its values on the pairs indexed by $\mathcal{S}$.

Consider the case where g is even. Then by Proposition 3.3(2) we have $\mathbf{w} = \mathbf{c}$.

Suppose first that $\mathbf{w} = \mathbf{c} = 0$. Then $m = 0$, so $T' = \emptyset$ and $T = \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ is a basis of $\text{Sel}^2(J)$. The values on pairs $\mathbf{a}_r, \mathbf{a}_q$ with $1 \leq r < q \leq n$ are precisely the values indexed by $\mathcal{S} = \mathcal{S}_1$. Moreover, admissibility gives $P(x, x) = P(\mathbf{c}, x) = 0$ for every $x \in \text{Sel}^2(J)$, so all diagonal values are determined. Thus P is determined by its values on $\mathcal{S}$ in this case.

Suppose now that $\mathbf{w} = \mathbf{c} \neq 0$. Then $m = 1$ and, with our notation, $\mathbf{a}_{n+1} = \mathbf{c}$. The values on pairs $\mathbf{a}_r, \mathbf{a}_q$ with $1 \leq r < q \leq n$ are precisely the values indexed by $\mathcal{S}_1$. The values $P(\mathbf{c}, \mathbf{a}_q)$, for $1 \leq q \leq n$, are precisely the values indexed by $\mathcal{S}_0$. These determine the diagonal values $P(\mathbf{a}_q, \mathbf{a}_q)$, since $P(\mathbf{a}_q, \mathbf{a}_q) = P(\mathbf{c}, \mathbf{a}_q)$ by the admissibility of P . Finally

$$P(\mathbf{c}, \mathbf{c}) \equiv \dim_{\mathbb{F}_2} \text{Sel}^2(A) + \text{rk}_2(A) \pmod{2}$$

is also fixed by the admissibility of P . Thus P is determined by its values on $\mathcal{S}$ also in this case.

Now consider the case where g is odd. By Proposition 3.3(2) we have $\mathfrak{c} = 0$, so any admissible pairing is alternating. In this case the proof is straightforward. $\square$

The following proposition is a variant of [Mor25, Theorem 1.14].

Proposition 5.8. *Let k be a number field. For $i = 1, 2$, let $C_i : y^2 = f_i(x)$ be hyperelliptic curves with f_i separable of degree $2g_i + 2 \geq 6$, and let J_i be their Jacobians. Assume that $G_i := \text{Gal}(f_i) \simeq S_{2g_i+2}$, and that the splitting fields L_1 and L_2 of f_1 and f_2 are linearly disjoint over k . Suppose given $t \in \mathfrak{S}$. For $i = 1, 2$, let*

$$P_i : \text{Sel}^2(J_i^t/k) \times \text{Sel}^2(J_i^t/k) \longrightarrow \mathbb{F}_2$$

be an admissible pairing (as in Definition 5.6). Then there exists $s \in \mathfrak{S}$ such that, for $i = 1, 2$, one has $\text{Sel}^2(J_i^s/k) = \text{Sel}^2(J_i^t/k)$ inside $H^1(k, J_i[2])$, one has $\text{rk}_2(J_i^s/k) \equiv \text{rk}_2(J_i^t/k) \pmod{2}$, and the Cassels–Tate pairing on $\text{Sel}^2(J_i^s/k)$ is equal to P_i .

Proof of Proposition 5.8. We note that, for $i = 1, 2$, property (5.1) holds:

$$\text{Sel}^2(J_i^t/k) \cap H^1(G_i, J_i[2]) = \langle \mathfrak{w}_i \rangle.$$

This follows from $t \in \mathfrak{S}$ and Proposition 3.4(3).

Observe that the principally polarized abelian varieties J_1^t and J_2^t satisfy the assumptions of Lemma 5.3. Indeed, for $i = 1, 2$, we have $J_i^t[2] \simeq J_i[2]$, so that $k(J_i^t[2]) = L_i$, and hence $G_i = \text{Gal}(L_i/k) \simeq S_{2g_i+2}$. Thus Assumptions (A) and (B) of Lemma 5.3 hold for J_i^t , while (5.1) holds by hypothesis. For each $i = 1, 2$, choose a Morgan datum $(T_i, T'_i, \{a_{i,r}\}_r, \{\gamma_{i,\nu}\}_{\nu \in \mathcal{S}_i})$ for J_i^t , in the sense of Definition 5.1, where $\mathcal{S}_i = (\mathcal{S}_i)_0 \sqcup (\mathcal{S}_i)_1$ is the corresponding finite indexing set. Applying Construction 5.2 we obtain, for $i = 1, 2$, a finite Galois extension $M_i = k_{i,T_i,T'_i}/k$ and functions

$$\psi_{i,\nu} : \{\sigma \in \text{Gal}(M_i/k) : J_i[2]^{\bar{\sigma}} = 0\} \longrightarrow \mathbb{F}_2, \quad \nu \in \mathcal{S}_i,$$

where for every $\sigma \in \text{Gal}(M_i/k)$ we denote by $\bar{\sigma}$ the image of σ in G_i .

Let $M = M_1 M_2$. By Lemma 5.5, we have $\text{Gal}(M/k) \simeq \text{Gal}(M_1/k) \times \text{Gal}(M_2/k)$. We shall write an element of $\text{Gal}(M/k)$ as a pair (σ_1, σ_2) , with $\sigma_i \in \text{Gal}(M_i/k)$.

For $i = 1, 2$, choose a cycle $h_i \in G_i \simeq S_{2g_i+2}$ of length $2g_i + 1$. By Proposition 3.5(1), one has $J_i[2]^{h_i} = 0$. For $\nu = (r, q) \in \mathcal{S}_i$, define

$$\delta_{i,\nu} := \langle \mathfrak{a}_{i,r}, \mathfrak{a}_{i,q} \rangle_{J_i^t} + P_i(\mathfrak{a}_{i,r}, \mathfrak{a}_{i,q}) \in \mathbb{F}_2.$$

By Lemma 5.3(b), for each $i = 1, 2$ we can choose elements $\sigma_i^*, \sigma_i^0 \in \text{Gal}(M_i/k)$, both mapping to $h_i \in G_i$, such that $\psi_{i,\nu}(\sigma_i^*) = \delta_{i,\nu}$ and $\psi_{i,\nu}(\sigma_i^0) = 0$ for all $\nu \in \mathcal{S}_i$. Define $\tau_1 := (\sigma_1^*, \sigma_2^0)$ and $\tau_2 := (\sigma_1^0, \sigma_2^*)$, viewed as elements of $\text{Gal}(M/k)$.

Choose a finite set of places Σ_0 containing

- (i) all archimedean places and all places above 2,
- (ii) all places ramified in M/k ,
- (iii) all places of bad reduction for C_1 or C_2 ,
- (iv) every place v such that $t \notin \mathcal{O}_v^\times$, and
- (v) all places v for which the residue field is too small to guarantee a smooth $\mathbb{F}_v$ -point on every smooth projective curve over $\mathbb{F}_v$ of the form $y^2 = af_i(x)$, with $a \in \mathbb{F}_v^\times$ and $i = 1, 2$.

The last condition excludes only finitely many places, by Lemma 3.8.

By the Chebotarev density theorem, we may choose distinct primes $\mathfrak{p}_1, \mathfrak{p}_2 \notin \Sigma_0$ such that $\text{Frob}_{\mathfrak{p}_i}$ is conjugate to τ_i in $\text{Gal}(M/k)$, for $i = 1, 2$. We define $\Sigma := \Sigma_0 \cup \{\mathfrak{p}_1, \mathfrak{p}_2\}$.

Define a subset $\mathcal{C} \subset \text{Gal}(M/k)$ as follows. An element $\tau = (\rho_1, \rho_2)$ belongs to $\mathcal{C}$ if and only if, for $i = 1, 2$, the image of ρ_i in G_i is conjugate to h_i , and $\psi_{i,\nu}(\rho_i) = 0$ for every $\nu \in \mathcal{S}_i$. The set $\mathcal{C}$ is nonempty, since it contains (σ_1^0, σ_2^0) . It is a union of conjugacy classes, because conjugation preserves the conjugacy class of the image in G_i , and the functions $\psi_{i,\nu}$ are constant on conjugacy classes; see [Mor25, Remark 3.2].

We now prescribe local quadratic characters at the places of Σ . Set $\chi_v = 1$ for every $v \in \Sigma_0$. For $j = 1, 2$, choose a nontrivial ramified quadratic character $\chi_{\mathfrak{p}_j} : G_{k_{\mathfrak{p}_j}} \rightarrow \mu_2$. We verify condition (5.4) in Lemma 5.4. Let $\beta : \text{Gal}(M/k) \rightarrow \mu_2$ be a quadratic character which is trivial on $\mathcal{C}$. By Lemma 5.5, the character β factors through $G_1 \times G_2$. Since $\sigma_i^*, \sigma_i^0 \in \text{Gal}(M_i/k)$ map to h_i , for each $i = 1, 2$, the images of τ_1 and τ_2 in $G_1 \times G_2$ are both equal to (h_1, h_2) . Since $\mathcal{C}$ contains an element with image (h_1, h_2) , and since β is trivial on $\mathcal{C}$, we get $\beta(\tau_1) = \beta(\tau_2) = 1 \in \mu_2$. Equivalently, $\text{res}_{\mathfrak{p}_j} \beta$ is an unramified quadratic character taking the value 1 $\in \mu_2$ on $\text{Frob}_{\mathfrak{p}_j}$ for $j = 1, 2$. Hence $\text{res}_{\mathfrak{p}_j} \beta$ is trivial, and therefore $\text{inv}_{\mathfrak{p}_j}(\chi_{\mathfrak{p}_j} \cup \text{res}_{\mathfrak{p}_j} \beta) = 0$ for $j = 1, 2$. At all places of Σ_0 , the prescribed local character is trivial, so the corresponding local invariants are zero. Hence condition (5.4) holds:

$$\sum_{v \in \Sigma} \text{inv}_v(\chi_v \cup \text{res}_v \beta) = 0.$$

By Lemma 5.4, there exists a global quadratic character $\chi : \Gamma_k \rightarrow \mu_2$ such that $\text{res}_v(\chi) = 1$ for all $v \in \Sigma_0$, such that $\text{res}_{\mathfrak{p}_j}(\chi) = \chi_{\mathfrak{p}_j}$ for $j = 1, 2$, and such that every place $v \notin \Sigma$ at which χ is ramified satisfies $\text{Frob}_v \in \mathcal{C}$. Let $u \in k^\times / k^{\times 2}$ be the square-class corresponding to χ , and set $s = tu$.

We first prove the assertions about Selmer groups and 2^∞ -Selmer ranks. Fix $i \in \{1, 2\}$. The character χ is trivial at every place of Σ_0 . If $v \notin \Sigma_0$ and χ is ramified at v , then either $v = \mathfrak{p}_1$, or $v = \mathfrak{p}_2$, or $v \notin \Sigma$ and $\text{Frob}_v \in \mathcal{C}$. In all cases, the image of Frob_v in G_i is conjugate to h_i , and therefore $J_i[2]^{\text{Frob}_v} = 0$ by Proposition 3.5(1). Applying Lemma 5.3(c) to $A = J_i^t$, for $i = 1, 2$, we obtain $\text{Sel}^2(J_i^s/k) = \text{Sel}^2(J_i^t/k)$ inside $H^1(k, J_i[2])$, and $\text{rk}_2(J_i^s/k) \equiv \text{rk}_2(J_i^t/k) \pmod{2}$.

We now compute the Cassels–Tate pairings. Fix $i \in \{1, 2\}$ and $\nu = (r, q) \in \mathcal{S}_i$. By Lemma 5.3(d), we have

$$\langle \mathbf{a}_{i,r}, \mathbf{a}_{i,q} \rangle_{J_i^s} = \langle \mathbf{a}_{i,r}, \mathbf{a}_{i,q} \rangle_{J_i^t} + \sum_{v \in \text{Ram}(\chi) \setminus \Sigma_0} \psi_{i,\nu}(\text{Frob}_v).$$

Suppose first that $i = 1$. The contribution at $\mathfrak{p}_1$ is $\psi_{1,\nu}(\tau_1) = \delta_{1,\nu}$, while the contribution at $\mathfrak{p}_2$ is $\psi_{1,\nu}(\tau_2) = 0$. Every further ramified prime outside Σ has Frobenius in $\mathcal{C}$, so its $\psi_{1,\nu}$ -value is zero. Therefore $\langle \mathbf{a}_{1,r}, \mathbf{a}_{1,q} \rangle_{J_1^s} = P_1(\mathbf{a}_{1,r}, \mathbf{a}_{1,q})$. The same argument for $i = 2$, with $\mathfrak{p}_1$ and $\mathfrak{p}_2$ interchanged, gives $\langle \mathbf{a}_{2,r}, \mathbf{a}_{2,q} \rangle_{J_2^s} = P_2(\mathbf{a}_{2,r}, \mathbf{a}_{2,q})$ for every $\nu = (r, q) \in \mathcal{S}_2$.

Both the Cassels–Tate pairing on $\text{Sel}^2(J_i^s/k)$ and P_i are admissible. By Lemma 5.7 an admissible pairing is determined by its values on the pairs indexed by $\mathcal{S}_i$, hence the Cassels–Tate pairing on $\text{Sel}^2(J_i^s/k)$ is equal to P_i .

It remains to prove that $s \in \mathfrak{S}$. Equivalently, we must show that $C_i^s(k_v) \neq \emptyset$ for every place v of k and for $i = 1, 2$.

Let $v \in \Sigma_0$. Since χ_v is trivial, u is a square in $k_v^\times$. Hence $C_i^s \simeq C_i^t$ over k_v . Since $t \in \mathfrak{S}$, we have $C_i^t(k_v) \neq \emptyset$, and therefore $C_i^s(k_v) \neq \emptyset$.

Now let $v \notin \Sigma_0$ be a place where χ is ramified. Then either $v = \mathfrak{p}_1$, or $v = \mathfrak{p}_2$, or $v \notin \Sigma$ and $\text{Frob}_v \in \mathcal{C}$. In all cases, the image of Frob_v in $G_i \simeq S_{2g_i+2}$ is a cycle of length $2g_i + 1$. By Proposition 3.5(2), f_i has a root $\alpha \in k_v$. Thus the point $(\alpha, 0)$ lies on $C_i^s : y^2 = sf_i(x)$, so $C_i^s(k_v) \neq \emptyset$.

Finally suppose that $v \notin \Sigma_0$ and that χ is unramified at v . Since $v \notin \Sigma_0$, the curve C_i has good reduction at v , the leading coefficient of f_i is a unit at v , and $t \in \mathcal{O}_v^\times$. Since χ is unramified, u is represented by a unit in $k_v^\times$. Thus $s = tu$ is represented by a unit at v , and C_i^s has good reduction at v . By the choice of Σ_0 , the special fibre of C_i^s has a smooth $\mathbb{F}_v$ -point. Hensel's lemma lifts this point to a k_v -point on C_i^s .

We have shown that $C_i^s(k_v) \neq \emptyset$ for every v and for $i = 1, 2$. Hence $s \in \mathfrak{S}$, as desired. $\square$

6. PROOF OF THEOREM 1.5

Lemma 6.1. *Let C be a smooth, projective, geometrically integral curve over a number field k . If $\text{Pic}^1(C_{k_v}) \neq \emptyset$ for all places v of k and $\text{Pic}_{C/k}^1(k) \neq \emptyset$, then $\text{Pic}^1(C) \neq \emptyset$, that is, C has a zero-cycle of degree 1.*

Proof. We have $\text{Pic}_{C/k}(k) = \text{Pic}(C_k)^{\Gamma_k}$. The standard Hochschild–Serre spectral sequence gives rise to an exact sequence

$$0 \rightarrow \text{Pic}(C) \rightarrow \text{Pic}(C_k)^{\Gamma_k} \rightarrow \text{Br}(k) \rightarrow \text{Br}(C).$$

By the Albert–Brauer–Hasse–Noether theorem, the natural map $\text{Br}(k) \rightarrow \bigoplus_v \text{Br}(k_v)$ is injective. We obtain that the map $\text{Br}(k) \rightarrow \text{Br}(C)$ is injective when C has a zero-cycle of degree 1 over all completions of k . This implies that every element of $\text{Pic}(C_k)^{\Gamma_k}$ of degree 1 lifts to an element of $\text{Pic}(C)$ of degree 1. $\square$

Theorem 6.2. *We place ourselves in Situation 1.4. Assume that $X(\mathbf{A}_k) \neq \emptyset$ and that $\text{III}(J_i^u)[2^\infty]$ is finite for every $u \in \mathfrak{S}$ and $i = 1, 2$. Then there exists $s \in \mathfrak{S}$ such that $(\mathfrak{w}_1, \mathfrak{w}_2)$ maps to zero in $\text{III}(J_1^s \times_k J_2^s)$. In particular, $C_1^s \times_k C_2^s$ has a zero-cycle of degree 1.*

Proof. By Proposition 4.3, there exists $t \in \mathfrak{S}$ such that both $\dim_{\mathbb{F}_2} \text{Sel}^2(J_1^t)$ and $\dim_{\mathbb{F}_2} \text{Sel}^2(J_2^t)$ are odd. Since $t \in \mathfrak{S}$, we have $\langle \mathfrak{c}_i, \mathfrak{c}_i \rangle_{J_i^t} = 0$ by [PS99, Corollary 12]. Hence, by Corollary 3.6,

$$\dim_{\mathbb{F}_2} \text{Sel}^2(J_i^t) + \text{rk}_2(J_i^t) \equiv 0 \pmod{2}.$$

Thus, in this situation, a symmetric bilinear form $P : \text{Sel}^2(J_i^t) \times \text{Sel}^2(J_i^t) \rightarrow \mathbb{F}_2$ is admissible (in the sense of Definition 5.6) if and only if $P(\mathfrak{c}_i, x) = P(x, x)$ for all $x \in \text{Sel}^2(J_i^t)$, and $P(\mathfrak{c}_i, \mathfrak{c}_i) = 0$.

Since $t \in \mathfrak{S}$, we have $\mathfrak{w}_i \in \text{Sel}^2(J_i^t)$ for $i = 1, 2$ by Proposition 3.3(4). Moreover, for $i = 1, 2$ we have $\mathfrak{w}_i \neq 0$ by Proposition 3.4(3).

Let us construct an admissible pairing on $\text{Sel}^2(J_i^t)$. Since $\dim \text{Sel}^2(J_i^t)$ is odd, there is an even-dimensional $\mathbb{F}_2$ -vector subspace $V \subset \text{Sel}^2(J_i^t)$ such that $\text{Sel}^2(J_i^t) = \mathbb{F}_2 \mathfrak{w}_i \oplus V$. Choose an alternating non-degenerate pairing on V and extend it to a pairing $P_i(x, y)$ on $\text{Sel}^2(J_i^t)$ with kernel $\mathbb{F}_2 \mathfrak{w}_i$. The pairing $P_i(x, y)$ is admissible, because $\mathfrak{c}_i = \mathfrak{w}_i$ if g_i is even, or $\mathfrak{c}_i = 0$ if g_i is odd.

By Proposition 3.4(3), applied to C_i^t , and using Proposition 3.3(7), we have

$$\mathrm{Sel}^2(J_i^t/k) \cap H^1(G_i, J_i[2]) = \langle \mathfrak{w}_i \rangle.$$

Thus the hypotheses of Proposition 5.8 are satisfied for the two admissible pairings P_i just constructed. Therefore there exists a single $s \in \mathfrak{S}$ such that, for $i = 1, 2$,

$$\mathrm{Sel}^2(J_i^s/k) = \mathrm{Sel}^2(J_i^t/k) \quad \text{inside } H^1(k, J_i[2]),$$

and the Cassels–Tate pairing on $\mathrm{Sel}^2(J_i^s/k)$ is equal to P_i . Thus $\dim_{\mathbb{F}_2} \mathrm{Sel}^2(J_i^s/k)$ is odd for $i = 1, 2$.

Let $q_i: \mathrm{Sel}^2(J_i^s) \rightarrow \mathrm{III}(J_i^s)[2]$ be the natural surjection. Since the Cassels–Tate pairing on $\mathrm{Sel}^2(J_i^s)$ is the pullback of the Cassels–Tate pairing on $\mathrm{III}(J_i^s)$, we have

$$\ker(q_i) \subset \mathrm{rad}(P_i) = \mathbb{F}_2 \mathfrak{w}_i.$$

Thus $\ker(q_i)$ is either 0 or $\mathbb{F}_2 \mathfrak{w}_i$.

Assume $\ker(q_i) = 0$. Then

$$\dim_{\mathbb{F}_2} \mathrm{III}(J_i^s)[2] = \dim_{\mathbb{F}_2} \mathrm{Sel}^2(J_i^s),$$

which is odd. Under the hypothesis that $\mathrm{III}(J_i^u)[2^\infty]$ is finite for every $u \in \mathfrak{S}$, this implies that $\dim_{\mathbb{F}_2} \mathrm{III}(J_i^s)_{\mathrm{nd}}[2]$ is odd. By [PS99, Theorem 8, (1) $\iff$ (3)] (see Proposition 3.3(5) above), $\dim_{\mathbb{F}_2} \mathrm{III}(J_i^s)_{\mathrm{nd}}[2]$ is odd if and only if $\langle \mathfrak{c}_i, \mathfrak{c}_i \rangle_{J_i^s} \neq 0$. This is a contradiction, since for $s \in \mathfrak{S}$, by Proposition 3.3(6), we have $\langle \mathfrak{c}_i, \mathfrak{c}_i \rangle_{J_i^s} = 0$.

Therefore $\ker(q_i) = \mathbb{F}_2 \mathfrak{w}_i$, and hence $\mathfrak{w}_i$ maps to zero in $\mathrm{III}(J_i^s)$. Since $\mathfrak{w}_i$ maps to $[\mathrm{Pic}_{C_i^s/k}^1]$ in $H^1(k, J_i)$, this implies that $\mathrm{Pic}_{C_i^s/k}^1(k) \neq \emptyset$. We conclude by Lemma 6.1. $\square$

Remark 6.3. In contrast to [Mor25], we do not need to use a second descent argument. Indeed, $s \in \mathfrak{S}$ implies $\langle c_i, c_i \rangle_{J_i^s} = 0$ (see Proposition 3.3(6)), and hence $\dim_{\mathbb{F}_2} \mathrm{III}_{\mathrm{nd}}(J_i^s)[2]$ is even (see Proposition 3.3(5)).

Proof of Theorem 1.5. Since $X(\mathbf{A}_k) \neq \emptyset$, by Theorem 6.2 there exists $s \in \mathfrak{S}$ such that $C_1^s \times_k C_2^s$ has a zero-cycle of degree 1. Letting Y be the blow-up of $C_1^s \times_k C_2^s$ at the fixed locus of the involution, we deduce that Y has a zero-cycle of degree 1. Since X is birational to the quotient of Y by the induced involution, we conclude that X has a zero-cycle of degree 1, as desired. $\square$

Remark 6.4. The set of k -points $(a_0, \dots, a_{2g_1+2}; b_0, \dots, b_{2g_2+2})$ of $\mathbb{A}_k^{2g_1+3} \times_k \mathbb{A}_k^{2g_2+3}$ such that Situation 1.4 holds for the polynomials $f_1(x) = a_{2g_1+2}x^{2g_1+2} + \dots + a_0$ and $f_2(x) = b_{2g_2+2}x^{2g_2+2} + \dots + b_0$ satisfies weak approximation.

Indeed, let S_0 be a set of places of k containing the 2-adic places, the archimedean places, and the non-archimedean places whose residue field has size $< 4 \max\{g_1, g_2\}^2$. Fix two distinct places w_1 and w_2 not in S_0 , let $\mathfrak{p}_1$ and $\mathfrak{p}_2$ be the corresponding prime ideals, and let $\pi_1 \in \mathfrak{p}_1$ and $\pi_2 \in \mathfrak{p}_2$ be uniformizers.

Let $h_1 \in \mathcal{O}_k[x_1]$ and $h_2 \in \mathcal{O}_k[x_2]$ be such that, for $i = 1, 2$, the leading coefficient of h_i is a local unit at w_1 and w_2 , and $h_i(x_i) \equiv (x_i^2 - \pi_i^2)q_i(x_i) \pmod{\mathfrak{p}_i^3}$, for some polynomial $q_i(x_i) \in \mathcal{O}_k[x_i]$ whose reduction modulo $\mathfrak{p}_i$ is separable and non-zero at $x_i = 0$, and the reduction of $h_i(x_i)$ modulo $\mathfrak{p}_j$ is separable for $j \neq i$.

By a theorem of Ekedahl [Eke90, Theorem 1.3], the set of separable polynomials $f_1(x_1)$ and $f_2(x_2)$ with linearly disjoint splitting fields and such that, for $i = 1, 2$, the polynomial $f_i(x_i)$ has degree $2g_i + 2$ and Galois group S_{2g_i+2} , and is sufficiently close to $h_i(x_i)$ at w_1 and w_2 , satisfies weak approximation.

APPENDIX A. THE SURFACES IN SITUATION 1.4 ARE OF GENERAL TYPE

In this appendix we prove the following statement.

Lemma A.1. *Let k be a field of characteristic zero. For $i = 1, 2$, let $f_i(x) \in k[x]$ be a separable polynomial of degree $2g_i + 2 \geq 6$. Let X be a smooth, projective, geometrically integral variety birational to the affine surface*

$$y^2 = f_1(x_1)f_2(x_2).$$

Then X is of general type.

Proof. We may assume that k is algebraically closed. Let $B_1 \subset \mathbb{P}^1$ and $B_2 \subset \mathbb{P}^1$ be the reduced divisors defined by f_1 and f_2 , respectively. For $i = 1, 2$, since $\deg(f_i) = 2g_i + 2$ and f_i is separable, B_i has degree $2g_i + 2$. Let $Y := \mathbb{P}_k^1 \times_k \mathbb{P}_k^1$, and for $i = 1, 2$ let $\text{pr}_i: Y \rightarrow \mathbb{P}_k^1$ be the i th projection. Then $D := \text{pr}_1^* B_1 + \text{pr}_2^* B_2$ is a reduced divisor of bidegree $(2g_1 + 2, 2g_2 + 2)$ on Y . In particular

$$\mathcal{O}_Y(D) \simeq \mathcal{O}_Y(2g_1 + 2, 2g_2 + 2) \simeq \mathcal{O}_Y(g_1 + 1, g_2 + 1)^{\otimes 2}.$$

Let $L = \mathcal{O}_Y(g_1 + 1, g_2 + 1)$, so that $\mathcal{O}_Y(D) \simeq L^{\otimes 2}$. The equation $y^2 = f_1(x_1)f_2(x_2)$ defines, after compactification and normalization, the double cover $\pi: Z \rightarrow Y$ associated with the line bundle L and the branch divisor D . Thus X is birational to a resolution of singularities of Z .

By construction $Z = \text{Spec}_Y(\mathcal{O}_Y \oplus L^{-1})$, where the multiplication on $\mathcal{O}_Y \oplus L^{-1}$ is determined by a section $s \in H^0(Y, L^{\otimes 2})$ with zero locus D . It follows that Z is, Zariski-locally on Y , a hypersurface inside the smooth variety $\mathbb{A}_k^1 \times_k Y$. In particular, Z is Gorenstein, and hence the canonical sheaf ω_Z is invertible. By the canonical bundle formula for double covers, we have

$$\omega_Z \simeq \pi^*(\omega_Y \otimes L);$$

see [BHPV04, Chapter I, Lemma 17.1(iii)]. Since $\omega_Y \simeq \mathcal{O}_Y(-2, -2)$, we get

$$\omega_Z \simeq \pi^*\mathcal{O}_Y(g_1 - 1, g_2 - 1).$$

Since $g_1, g_2 \geq 2$, the line bundle $\mathcal{O}_Y(g_1 - 1, g_2 - 1)$ is ample. It follows that ω_Z is ample.

The divisor D is a union of vertical and horizontal lines. Away from the crossing points of D , the double cover π is smooth. Near a crossing point, one may choose local parameters s_1, s_2 on Y such that D is given by $s_1 s_2 = 0$, and so Z is locally given by $z^2 = s_1 s_2$.

The minimal resolution of the singularity $z^2 = st$ is obtained by blowing up the origin. Let $A = \mathbb{A}_{s,t,z}^3$ and let $W \subset A$ be the hypersurface $z^2 = st$. If $b: \tilde{A} \rightarrow A$ is the blow-up of the origin, E is the exceptional divisor, and $\tilde{W}$ is the strict transform of W , then $\omega_{\tilde{A}} \simeq b^*\omega_A \otimes \mathcal{O}_{\tilde{A}}(2E)$, see [Har77, Chapter II, Exercise 8.5(b)]. On the other hand, an explicit computation shows $\mathcal{O}_{\tilde{A}}(b^*W) \simeq \mathcal{O}_{\tilde{A}}(\tilde{W} + 2E)$. By the adjunction formula we have

$$\omega_W \simeq (\omega_A \otimes \mathcal{O}_A(W))|_W, \quad \omega_{\tilde{W}} \simeq (\omega_{\tilde{A}} \otimes \mathcal{O}_{\tilde{A}}(\tilde{W}))|_{\tilde{W}}.$$

Therefore

$$\begin{aligned}
\omega_{\widetilde{W}} &\simeq (b^*\omega_A \otimes \mathcal{O}_{\widetilde{A}}(2E) \otimes \mathcal{O}_{\widetilde{A}}(\widetilde{W}))|_{\widetilde{W}} \\
&\simeq (b^*\omega_A \otimes \mathcal{O}_{\widetilde{A}}(2E) \otimes \mathcal{O}_{\widetilde{A}}(b^*W - 2E))|_{\widetilde{W}} \\
&\simeq (b^*(\omega_A \otimes \mathcal{O}_A(W)))|_{\widetilde{W}} \\
&\simeq (b|_{\widetilde{W}})^*\omega_W.
\end{aligned}$$

Now let $\rho: \widetilde{Z} \rightarrow Z$ be the minimal resolution. Since Z is Gorenstein and $\widetilde{Z}$ is smooth, the invertible sheaf $\omega_{\widetilde{Z}} \otimes \rho^*\omega_Z^{-1}$ is of the form $\mathcal{O}_{\widetilde{Z}}(R)$ for a Cartier divisor R supported on the exceptional locus of ρ . Writing

$$R = \sum_P a_P E_P,$$

where P runs through the singular points of Z and E_P is the exceptional curve over the A_1 -singularity P , the local calculation above gives $a_P = 0$ for every P . Hence $R = 0$, and therefore $\omega_{\widetilde{Z}} \simeq \rho^*\omega_Z$. Since ω_Z is ample, its pullback $\rho^*\omega_Z$ is nef and big. Hence $\omega_{\widetilde{Z}}$ is nef and big. Therefore $\widetilde{Z}$ has Kodaira dimension 2. Since X is birational to $\widetilde{Z}$, the surface X is of general type. $\square$

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