Compositio Math. (Page 1 of 39)
doi:10.1112/S0010437X10005002

Is the function field of a reductive Lie algebra purely transcendental over the field of invariants for the adjoint action?

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с уважением и благодарностью

Abstract

Let k be a field of characteristic zero, let G be a connected reductive algebraic group over k and let $\mathfrak{g}$ be its Lie algebra. Let $k(G)$, respectively, $k(\mathfrak{g})$, be the field of k-rational functions on G, respectively, $\mathfrak{g}$. The conjugation action of G on itself induces the adjoint action of G on $\mathfrak{g}$. We investigate the question whether or not the field extensions $k(G)/k(G)^G$ and $k(\mathfrak{g})/k(\mathfrak{g})^G$ are purely transcendental. We show that the answer is the same for $k(G)/k(G)^G$ and $k(\mathfrak{g})/k(\mathfrak{g})^G$, and reduce the problem to the case where G is simple. For simple groups we show that the answer is positive if G is split of type A_n or C_n, and negative for groups of other types, except possibly G_2. A key ingredient in the proof of the negative result is a recent formula for the unramified Brauer group of a homogeneous space with connected stabilizers. As a byproduct of our investigation we give an affirmative answer to a question of Grothendieck about the existence of a rational section of the categorical quotient morphism for the conjugating action of G on itself.

Contents

Received 27 May 2009, accepted in final form 9 April 2010.
2010 Mathematics Subject Classification 14E08, 17B45, 14L30 (primary), 20C10, 20G15, 14F22 (secondary).
Keywords: algebraic group, simple Lie algebra, rationality problem, integral representation, algebraic torus, unramified Brauer group.

Introduction

A field extension E/F is called *pure* (or *purely transcendental* or *rational*) if E is generated over F by a finite collection of algebraically independent elements. A field extension E/F is called *stably pure* (or *stably rational*) if E is contained in a field L which is pure over both F and E. Finally, we shall say that E/F is *unirational* if E is contained in a field L which is pure over F. In summary

$$\begin{tikzcd} L \arrow[d, dash, "\text{pure}"'] \arrow[dd, dash, bend left, "\text{pure}"] \\ E \arrow[d, dash, "\text{stably pure}"'] \\ F \end{tikzcd} \qquad \begin{tikzcd} L \arrow[d, dash] \arrow[dd, dash, bend left, "\text{pure}"] \\ E \arrow[d, dash, "\text{unirational}"'] \\ F \end{tikzcd}$$

Let k be a field. Unless otherwise mentioned, we assume that $\mathrm{char}(k) = 0$. This is in particular a standing assumption in this section.

Let G be a connected reductive algebraic group over k. Let V be a finite dimensional k-vector space and let $G \hookrightarrow \mathrm{GL}(V)$ be an algebraic group embedding over k. Let $k(V)$ denote the field of k-rational functions on V and $k(V)^G$ the subfield of G-invariants in $k(V)$. It is natural to ask whether $k(V)/k(V)^G$ is pure (or stably pure).

This question may be viewed as a birational counterpart of the classical problem of freeness of the module of (regular) covariants, i.e., the $k[V]^G$-module $k[V]$; cf. [PV94, §§ 3 and 8]. (Here $k[V]$ is the algebra of k-regular functions on V and $k[V]^G$ is the subalgebra of its G-invariant elements.) The question of rationality of $k(V)$ over $k(V)^G$ also comes up in connection with counterexamples to the Gelfand–Kirillov conjecture; see [AOV96] and the paragraph of this introduction right after the statement of Theorem 0.4.

Recall that a connected reductive group G is called *split* if there exists a Borel subgroup B of G defined over k and a maximal torus in B is split.

If G is split and the G-action on V is *generically free*, i.e., the G-stabilizers of the points of a dense open set of V are trivial, then the following conditions are equivalent:

(i) the extension $k(V)/k(V)^G$ is pure;

(ii) the extension $k(V)/k(V)^G$ is unirational;

(iii) the group G is a 'special group'.

Over an algebraically closed field, special groups were defined by Serre [Ser58] and classified by him and Grothendieck [Gro58] in the 1950s; cf. § 1.3. The equivalence of these conditions follows from Corollary 3.6 below; see also Lemma 3.10.

The purity problem for $k(V)/k(V)^G$ is thus primarily of interest in the case where the G-action on V is faithful but not generically free. For k algebraically closed, such actions have been extensively studied and even classified, under the assumption that either the group G or the G-module V is simple; for details and further references, see [PV94, § 7.3].

For these G-modules, purity for $k(V)/k(V)^G$ is known in some special cases. For instance, one can show that this is the case if $k[V]^G$ is generated by a quadratic form. In [AOV96, Appendix A] one can find a sketch of a proof that $k(V)/k(V)^G$ is pure if $G = \mathrm{SL}_2$ and $\dim(V) = 4$ or 5.

Theorem 0.2(b) below (with G adjoint) gives the first known examples of a connected linear algebraic group G over an algebraically closed field k with a faithful but not generically free G-module V such that $k(V)$ is not pure (and not even stably pure) over $k(V)^G$.

Let $\mathfrak{g}$ be the Lie algebra of G. The homomorphism $\mathrm{Int}: G \to \mathrm{Aut}(G)$ sending $g \in G$ to the map $\mathrm{Int}(g): G \to G$, $x \mapsto gxg^{-1}$, determines the *conjugation action* of G on itself, $G \times G \to G$, sending (g, x) to $\mathrm{Int}(g)(x)$. The differential of $\mathrm{Int}(g)$ at the identity is the linear map $\mathrm{Ad}(g): \mathfrak{g} \to \mathfrak{g}$. This defines an action of G on $\mathfrak{g}$, called the *adjoint action*. As usual, we will denote the fields of k-rational functions on G, respectively, $\mathfrak{g}$, by $k(G)$, respectively, $k(\mathfrak{g})$, and the fields of invariant k-rational functions for the conjugation action, respectively, the adjoint action, by $k(G)^G$, respectively, $k(\mathfrak{g})^G$.

The purpose of this paper is to address the following purity questions.

Questions 0.1. Let G be a connected reductive group over k and let $\mathfrak{g}$ be its Lie algebra.

(a) Is the field extension $k(\mathfrak{g})/k(\mathfrak{g})^G$ pure? Stably pure?

(b) Is the field extension $k(G)/k(G)^G$ pure? Stably pure?

It is worth mentioning here two other natural purity questions arising in this situation, namely, that of the purity of $k(\mathfrak{g})^G$ and $k(G)^G$ over k. They are, however, not directly related to the questions we are asking and both have affirmative answers: for $k(\mathfrak{g})^G$ this is proved in [Kos63] and, for $k(G)^G$, in [Ste65] (for simply connected semisimple G) and [Pop] (for the general case).

The main case of interest for us is that of split groups, but some of our results hold for arbitrary reductive groups.

We shall give a nearly complete answer to Questions 0.1 for split groups, in particular, when k is algebraically closed. Our results can be summarized as follows.

(i) (Corollary 4.8) *Let G be a connected split reductive group over k. Then the field extensions $k(\mathfrak{g})/k(\mathfrak{g})^G$ and $k(G)/k(G)^G$ are unirational.*

This is closely related to Theorem 0.3 below.

(ii) (Theorem 4.10) *For a given connected reductive group G over k, the answers to Questions 0.1(a) and (b) are the same.*

(iii) (Proposition 5.1) *For a connected reductive group G over k and a central k-subgroup Z of G, the answers to Questions 0.1 for G/Z are the same as for G.*

Taking Z to be the radical of G, we thus reduce Questions 0.1 to the case where G is semisimple. We shall further reduce them to the case where G is simple as follows. Recall that a semisimple group G is called *simple* if its Lie algebra is a simple Lie algebra. Its centre is then finite but need not be trivial. In the literature such a group is sometimes referred to as an *almost simple* group.

(iv) (Proposition 5.3) *Suppose that G is connected, semisimple, and split. Denote the simple components of the simply connected cover of G by $G_1, \ldots, G_n$. Let $\mathfrak{g}_i$ denote the Lie algebra of G_i. Then the following properties are equivalent:*

(a) *$k(\mathfrak{g})/k(\mathfrak{g})^G$ is stably pure;*

(b) *$k(\mathfrak{g}_i)/k(\mathfrak{g}_i)^{G_i}$ is stably pure for every $i = 1, \ldots, n$.*

Similarly, the following properties are equivalent:

(a) *$k(G)/k(G)^G$ is stably pure;*

(b) *$k(G_i)/k(G_i)^{G_i}$ is stably pure for every $i = 1, \ldots, n$.*

If we replace 'stably pure' by 'pure', we can still show that the field extension $k(\mathfrak{g})/k(\mathfrak{g})^G$ (respectively, $k(G)/k(G)^G$) is pure if each $k(\mathfrak{g}_i)/k(\mathfrak{g}_i)^{G_i}$ (respectively, each $k(G_i)/k(G_i)^{G_i}$) is pure, but we do not know whether or not the converse holds.

Finally, in the case where G is simple our main theorem is as follows.

Theorem 0.2. *Let G be a connected, simple algebraic group over k and let $\mathfrak{g}$ be its Lie algebra. Then the field extensions $k(G)/k(G)^G$ and $k(\mathfrak{g})/k(\mathfrak{g})^G$ are:*

(a) *pure, if G is split of type A_n or C_n;*

(b) *not stably pure if G is not of type A_n, C_n, or G_2.*

To prove Theorem 0.2, we show that the two equivalent Questions 0.1(a) and (b) are equivalent to the question of (stable) $K_{\rm gen}$-rationality of the homogeneous space $G_{K_{\rm gen}}/T_{\rm gen}$, where $T_{\rm gen}$ is the generic torus of G, defined over the field $K_{\rm gen}$; see Theorem 4.10. (For the definition of $T_{\rm gen}$ and $K_{\rm gen}$ see § 4.2.) We then address this rationality problem for $G_{K_{\rm gen}}/T_{\rm gen}$ by using the main result of [CTK06], which gives a formula for the unramified Brauer group of a homogeneous space with connected stabilizers; see § 6. This allows us to prove Theorem 0.2(b) in § 7 by showing that if G is not of type A_n, C_n, or G_2, then the unramified Brauer group of $G_{K_{\rm gen}}/T_{\rm gen}$ is nontrivial over some field extension of $K_{\rm gen}$. This approach also yields Theorem 0.2(a), with 'pure' replaced by 'stably pure' (Proposition 8.2). The proof of the purity assertion in part (a) requires additional arguments, which are carried out in § 9.

A novel feature of our approach is a systematic use of the notions of (G,S)-fibration and versal (G,S)-fibration, generalizing the well known notions of G-torsor and versal G-torsor; cf., e.g., [GMS03, § 1.5]. Here S is a k-subgroup of G. For details we refer the reader to §§ 2 and 3.

As a byproduct of our investigations we obtain the following two results which are not directly related to Questions 0.1 but are, in our opinion, of independent interest. Recall that a connected reductive group over a field k is called *quasisplit* if it has a Borel subgroup defined over k.

Theorem 0.3 (Corollary 4.8(a)). *Let G be a connected quasisplit reductive group over k. Then the categorical quotient map $G \to G/\!\!/G$ for the conjugation action has a rational section.*

In the classical case $G = \mathrm{SL}_n$ such a section is given by the companion matrices. The existence of a regular section for an arbitrary connected, split, semisimple, simply connected group G is a theorem of Steinberg [Ste65, Theorems 1.4 and 1.6]. In a letter to Serre, dated January 15, 1969, Grothendieck asked whether or not a rational section exists if G is not assumed to be simply connected; see [GS01, p. 240].[1] Theorem 0.3 answers this question in the affirmative. The example specifically mentioned by Grothendieck is PGL_2, or $\mathrm{GP}(1)$, in his notation; an explicit rational section in this case is constructed in Remark 4.9. Our proof of Theorem 0.3 does not use Steinberg's result, but it uses Kostant's result on the existence of sections in the Lie algebra case [Kos63, Kot99].

If k is algebraically closed and G is not simply connected, then by [Pop] there is no regular section of the categorical quotient map $G \to G/\!\!/G$.

THEOREM 0.4 (See Propositions 7.1 and 8.1). *Let G be a connected, split, simple, simply connected algebraic group defined over k and let W be the Weyl group of G. The weight lattice $P(G)$ of G fits into an exact sequence of W-lattices*

$$0 \to P_2 \to P_1 \to P(G) \to 0,$$

with P_1 and P_2 permutation, if and only if G is of type A_n, C_n, or G_2.

Recently the results and methods of this paper have played an important part in A. Premet's (negative) solution of the Gelfand–Kirillov conjecture for finite-dimensional simple Lie algebras of every type, other than A_n, C_n, and G_2. The Gelfand–Kirillov conjecture is known to be true for Lie algebras of type A_n and is still open for the types C_n and G_2. For details, including background material on the Gelfand–Kirillov conjecture, we refer the reader to [Pre10].

This paper is dedicated to Valentin Evgen'yevich Voskresenskiĭ, who turned 80 in 2007. Professor Voskresenskiĭ's work (see [Vos77, Vos98]) was the starting point for many of the methods and ideas used in the present paper.

Some terminology

By definition, a k-variety is a separated k-scheme of finite type. If X is a k-variety, it is naturally equipped with its structure morphism $X \to \operatorname{Spec} k$. As a consequence, any Zariski open set $U \subset X$ is naturally a k-variety.

The fibre product over $\operatorname{Spec} k$ of two k-schemes X and Y is denoted $X \times_k Y$, or simply $X \times Y$ when the context is clear.

The set of k-rational points of a k-variety X is defined by $X(k) = \mathrm{Mor}_k(\operatorname{Spec} k, X)$.

An algebraic group G over k, sometimes simply called a k-group, is a k-variety equipped with a structure of algebraic group over k. In other words, there is a multiplication morphism $G \times_k G \to G$ and a neutral element in $G(k)$ satisfying the usual properties. In other terms, it is a k-group scheme of finite type.

The ring $k[X]$ is the ring of global sections of the sheaf $\mathcal{O}_X$ over the k-variety X. The group $k[X]^\times$ is the group of invertible elements of $k[X]$.

[1] The exact quote is as follows: 'Le théorème de Steinberg [...] est-il vrai uniquement pour le [groupe] simplement connexe [...] ? Que se passe-t-il par exemple pour GP(1) ? Y a-t-il une section rationnelle de G sur $I(G)$ ('invariants') dans ce cas ?'.

If X is an integral (i.e., reduced and irreducible) k-variety, we let $k(X)$ be the field of rational functions on X. This is the direct limit of the fields of fractions of the k-algebras $k[U]$ for U running through the dense open subsets U of X.

When we consider two k-varieties X and Y, a k-morphism from X to Y will sometimes simply be called a morphism or even a map.

Similarly, if H and G are algebraic groups over k, if the context is clear, a k-homomorphism of k-group schemes from H to G will sometimes simply be called a homomorphism, or even a morphism.

For any field extension K/k, we may consider the K-variety $X_K = X \times_k K$, where the latter expression is shorthand for $X \times_{\operatorname{Spec} k} \operatorname{Spec} K$. We write $K[X] = K[X_K]$. If the K-variety X_K is integral, we let $K(X)$ be the function field of X_K.

An integral k-variety X is called *stably k-rational* if its function field $k(X)$ is stably rational over k (see the first paragraph of the introduction) or, equivalently, if there exists a k-birational isomorphism between $X \times_k \mathbb{A}^n_k$ and $\mathbb{A}^m_k$ for some integers $n, m \geqslant 0$. If $n = 0$, X is called *k-rational.*

1. Preliminaries on lattices, tori, and special groups

For the details on the results of this section, see [CTS77, CTS87a, Vos98], or [Lor05].

1.1 Γ-lattices

Let Γ be a finite group. A *Γ-lattice* M is a free abelian group of finite type equipped with a homomorphism $\Gamma \to \operatorname{Aut}(M)$. When the context is clear, we shall say lattice instead of Γ-lattice.

In this subsection we recall some basic properties of such lattices.

The *dual lattice* of a lattice M is the lattice $M^0 = \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ where for $\gamma \in \Gamma$, $m \in M$ and $\varphi \in M^0$, we have $(\gamma \cdot \varphi)(m) = \varphi(\gamma^{-1} \cdot m)$.

A *permutation lattice* is a lattice which has a $\mathbb{Z}$-module basis whose elements are permuted by Γ. The dual lattice of a permutation lattice is a permutation lattice.

Any lattice M may be realized as a sublattice of a permutation lattice P with torsion-free quotient P/M [CTS87a, Lemma 0.6].

Two lattices M_1 and M_2 are called *stably equivalent* if there exist permutation lattices P_1 and P_2 and an isomorphism $M_1 \oplus P_1 \cong M_2 \oplus P_2$.

A lattice M is called a *stably permutation lattice* if there exist permutation lattices P_1 and P_2 and an isomorphism $M \oplus P_1 \cong P_2$.

A lattice M is called *invertible* if there exists a lattice N such that $M \oplus N$ is a permutation lattice.

In these definitions one may replace Γ by its image in the group of automorphisms of M. Because of this one may give the analogous definitions for Γ a profinite group with a continuous and discrete action.

Let M be a Γ-lattice. For any integer $i \geqslant 0$ one writes

$$Ш^i_\omega(\Gamma, M) = \operatorname{Ker}\left[H^i(\Gamma, M) \to \prod_{\gamma \in \Gamma} H^i(\langle\gamma\rangle, M)\right].$$

For $i = 1, 2$, this kernel only depends on the image of Γ in the group of automorphisms of M. So for these i it is natural to extend the above definition to the case where Γ is a profinite group and the action is continuous and discrete.

If Γ is the absolute Galois group of a field K, one refers to lattices as Galois lattices and one uses the notation $Ш^i_\omega(K,M)$.

If M is a permutation lattice, then for any subgroup Γ' of Γ,

$$H^1(\Gamma',M)=0, \quad H^1(\Gamma',M^0)=0.$$

Moreover,

$$Ш^2_\omega(\Gamma',M)=0, \quad Ш^2_\omega(\Gamma',M^0)=0.$$

If there exists an exact sequence

$$0\to P_1\to P_2\to M\to 0$$

with P_1 and P_2 permutation lattices, then $Ш^1_\omega(\Gamma',M)=0$ for any subgroup Γ' of Γ.

1.2 Tori

Let K be a field, let K_{s} be a separable closure of K, and let Γ denote the Galois group of K_{s}/K. A K-torus T is an algebraic K-group which over an algebraic field extension L/K is isomorphic to a product of copies of the multiplicative group $\mathbb{G}_{m,L}$. The field L is then called *a splitting field* for T. Inside K_{s} there is a smallest splitting field for T, it is a finite Galois extension of K, called *the splitting field* of T.

To any K-torus T one may associate two Γ-lattices: its (geometric) *character group*

$$T^*=\mathrm{Hom}_{K_{\mathrm{s}}\text{-gr}}(T_{K_{\mathrm{s}}},\mathbb{G}_{m,K_{\mathrm{s}}})$$

and its (geometric) *cocharacter group*

$$T_*=\mathrm{Hom}_{K_{\mathrm{s}}\text{-gr}}(\mathbb{G}_{m,K_{\mathrm{s}}},T_{K_{\mathrm{s}}}).$$

These two Γ-lattices are dual of each other.

The association $T\mapsto T_*$ defines an equivalence between the category of K-tori and the category of Γ-lattices. The association $T\mapsto T^*$ defines a duality (anti-equivalence) between the category of K-tori and the category of Γ-lattices.

The K-torus whose character group is T_* is denoted T^0 and is called *the torus dual to T*.

A K-torus is called *quasitrivial*, respectively, *stably quasitrivial*, if its character group, or equivalently its cocharacter group, is a permutation lattice, respectively, is a stably permutation lattice. A quasitrivial torus T is K-isomorphic to a product of tori of the shape $R_{L/K}\mathbb{G}_m$, i.e., Weil restriction of scalars of the multiplicative group $\mathbb{G}_{m,L}$ from L to K, where L/K is a finite separable field extension. A quasitrivial K-torus is an open set of an affine K-space, hence is K-rational.

By a theorem of Voskresenskiĭ, a K-torus of dimension at most 2 is K-rational [Vos98, § 2.4.9, Examples 6 and 7]. This implies the following property. For any Γ-lattice M which is a direct sum of lattices of rank at most 2, there exist exact sequences

$$0\to P_2\to P_1\to M\to 0,$$

where P_1 and P_2 are permutation lattices.

1.3 Special groups

Let K be a field of characteristic zero. Recall from [Ser58] that an algebraic group G over K is called *special* if for any field extension L/K, the Galois cohomology set $H^1(L,G)$ is reduced to one point. In other words, G is special if every principal homogeneous space under G over a field containing K is trivial. Such a group is automatically linear and connected [Ser58]. An extension of a special group by a special group is a special group. A unipotent group is special. A quasitrivial torus is special. So is a direct factor of a quasitrivial torus (such a K-torus need not be stably K-rational). If K is algebraically closed and G is semisimple, then G is special if and only if it is isomorphic to a direct product

$$\mathrm{SL}_{n_1} \times \cdots \times \mathrm{SL}_{n_r} \times \mathrm{Sp}_{2m_1} \times \cdots \times \mathrm{Sp}_{2m_s}$$

for some integers $r, s, n_1, \ldots, n_r, m_1, \ldots, m_s$. That such groups are special is proved in [Ser58], that only these are is proved in [Gro58].

2. Quotients, (G, S)-fibrations, and (G, S)-varieties

We recall that k is a field of characteristic zero. Let $\overline{k}$ be an algebraic closure of k, and let G be a (not necessarily connected) linear algebraic group over k.

2.1 Geometric quotients

Let us recall some standard definitions and facts. For references, see [Bor91, § II.6], [Hum75, § 12], [Spr98, §§ 5.5, 12.2], [PV94, § 4], and [CTS07].

Let X be a k-variety endowed with an action of the k-group G. A *geometric quotient* of X by G is a pair (Y, π), where Y is a k-variety, called the *quotient space*, and $\pi: X \to Y$ is a k-morphism, called the *quotient map*, such that:

(i) π is an open *orbit map*, i.e., constant on G-orbits and induces a bijection of $X(\overline{k})/G(\overline{k})$ with $Y(\overline{k})$;

(ii) for every open subset V of Y, the natural homomorphism $\pi^*: k[V] \to k[\pi^{-1}(V)]^G$ is an isomorphism.

If such a pair (Y, π) exists, it has the *universal mapping property*, i.e., for every k-morphism $\alpha: X \to Z$ constant on the fibres of π, there is a unique k-morphism $\beta: Y \to Z$ such that $\alpha = \beta \circ \pi$. In particular, (Y, π) is unique up to a unique G-equivariant isomorphism of total spaces commuting with quotient maps. Given this, we shall denote Y by X/G.

Remark 2.1. Let X and hence X/G be geometrically integral. Being constant on G-orbits, π induces an embedding of fields $\pi^*: k(X/G) \hookrightarrow k(X)^G$. Conditions (i) and (ii) imply that, in fact, the latter is an isomorphism of fields $\pi^*: k(X/G) \xrightarrow{\cong} k(X)^G$, see, e.g., [Bor91, II, 6.5] (this property holds for the ground field k of arbitrary characteristic; for $\mathrm{char}(k) = 0$, it follows already from condition (i), see, e.g., [PV94, Lemma 2.1]).

If G acts on a reduced k-variety Z whose irreducible components are open, B is a normal k-variety, $\varrho: Z \to B$ is a k-morphism constant on G-orbits, and ϱ induces a bijection of $Z(\overline{k})/G(\overline{k})$ with $B(\overline{k})$, then (B, ϱ) is the geometric quotient of Z by G; see [Bor91, Proposition 6.6].

Example 2.2. If H is a closed k-subgroup of G, the action of H on G by right translation gives rise to a geometric quotient $\pi_{G,H}: G \to G/H$ called the quotient of G by H. The group G acts on G/H by left translation and, up to G-isomorphism, G/H is uniquely defined among the homogeneous spaces of G by the corresponding universal property, see, e.g., [Spr98, §§ 5.5, 12.2] or [Hum75, § 12].

For any reduced k-variety X endowed with a G-action, a theorem of Rosenlicht [Ros56, Ros63] (cf. also [PV94, §§ 2.1–2.4], [Spr89, Satz 2.2], [Tho86, Proposition 4.7]) ensures that there exist a G-invariant dense open subset U of X, a k-variety Y, and a smooth k-morphism $\alpha : U \to Y$ such that (Y, α) is the geometric quotient of U by G.

2.2 (G, S)-fibrations

Consider the category $\mathcal{M}_G$ whose objects are k-morphisms of k-schemes $\pi : X \to Y$ such that X is endowed with an action of G and π is constant on G-orbits, and a morphism of $\pi_1 : X_1 \to Y_1$ to $\pi_2 : X_2 \to Y_2$ is a commutative diagram,

$$\begin{tikzcd} X_1 \arrow[r, "\alpha"] \arrow[d, "\pi_1"'] & X_2 \arrow[d, "\pi_2"] \\ Y_1 \arrow[r, "\beta"] & Y_2 \end{tikzcd} \tag{2.1}$$

where α and β are k-morphisms and α is G-equivariant. The notion of composition of morphisms is clear. A morphism as in (2.1) is an isomorphism if and only if α and β are isomorphisms.

Let $\pi : X \to Y$ be an object of $\mathcal{M}_G$ and let $\mu : Z \to Y$ be a k-morphism of k-schemes. Then G acts on $X \times_Y Z$ via X, and the second projection $X \times_Y Z \to Z$ is an object of $\mathcal{M}_G$. We say that it is obtained from π by the *base change* μ.

Definition 2.3. Let F be a k-scheme endowed with an action of G and let $\pi : X \to Y$ be an object of $\mathcal{M}_G$. The morphism π is called:

(i) *trivial* (*over* Y) *with fibre* F if there exists an isomorphism between π and $\mathrm{pr}_2 : F \times_k Y \to Y$ where G acts on $F \times_k Y$ via F;

(ii) *fibration* (*over* Y) *with fibre* F if π becomes trivial with fibre F after a surjective étale base change $\mu : Y' \to Y$. In this case, we say that π *is trivialized by* μ.

Example 2.4. If $F = G$ with the G-action by left translation, then the notion of fibration over Y with fibre G coincides with that of G-*torsor* over Y.

The following definition extends the definition of a G-torsor (the latter corresponds to the case where S is the trivial subgroup $\{1\}$).

Definition 2.5. Let S be a closed k-subgroup of G. A fibration with fibre G/S, where G acts on G/S by left translation, is called (G, S)-fibration.

If X is a k-scheme endowed with an action of G and there is a (G, S)-fibration $X \to Y$, then we say that X *admits the structure of a* (G, S)*-fibration* over Y.

Remark 2.6. Replacing 'étale' by 'smooth' in Definition 2.3(ii), one obtains an equivalent definition. This follows from the fact that a surjective smooth morphism $Y' \to Y$ admits sections locally for the étale topology on Y: there exists an étale surjective morphism $Y'' \to Y$ such that $Y' \times_Y Y'' \to Y''$ has a section [Gro64, 17.16.3].

Remark 2.7. If the k-scheme X admits the structure of a (G, S)-fibration, then it admits the structure of a (G, S')-fibration for any k-subgroup S' of G such that $S_{\overline{k}}$ and $S'_{\overline{k}}$ are conjugate subgroups of $G_{\overline{k}}$. Such k-groups S and S' need not be k-isomorphic.

We list some immediate properties without proof.

Proposition 2.8. *Let $\pi\colon X \to Y$ be a (G,S)-fibration. Then the following properties hold:*

(i) *π is a smooth surjective morphism;*

(ii) *a morphism obtained from π by a base change is a (G,S)-fibration.*

Assume that X is a k-variety. Then:

(iii) *the G-stabilizers of points of $X(\overline{k})$ are conjugate to the subgroup $S_{\overline{k}}$ of $G_{\overline{k}}$;*

(iv) *(Y,π) is the geometric quotient of X by G.*

If a k-group S acts on a k-variety Z, then the functor $A \mapsto Z(A)^{S(A)}$ on commutative k-algebras is representable by a closed k-subvariety Z^S of Z (see [Fog73, SGA3bis10]).

Let Y be a k-variety and let $Z \to Y$ be an object of $\mathcal{M}_S$. If Y' is a k-variety and $Y' \to Y$ is a k-morphism, then the natural Y'-morphism $Z^S \times_Y Y' \to (Z \times_Y Y')^S$ is an isomorphism

$$Z^S \times_Y Y' \xrightarrow{\cong} (Z \times_Y Y')^S. \tag{2.2}$$

Let S be a closed k-subgroup of a k-group G and let N be the *normalizer* of S in G. Assume that X is a k-variety endowed with an action of G. Then the subvariety X^S is N-stable and, since S acts trivially on it, the action of N on X^S descends to an action of the group

$$H := N/S. \tag{2.3}$$

Let Y be a k-variety and let $\pi\colon X \to Y$ be an object of $\mathcal{M}_G$. Put

$$\pi^S := \pi|_{X^S}\colon X^S \to Y$$

and let $\pi'\colon X \times_Y X^S \to X^S$ be the morphism obtained from π by the base change π^S:

$$\begin{tikzcd} X \times_Y X^S \arrow[r] \arrow[d, "\pi'"'] & X \arrow[d, "\pi"] \\ X^S \arrow[r, "\pi^S"] & Y. \end{tikzcd}$$

Since π^S is constant on H-orbits, H acts on $X \times_Y X^S$ via X^S. The actions of G and H on $X \times_Y X^S$ commute. Therefore $X \times_Y X^S$ is endowed with an action of $G \times_k H$. The morphism π' is H-equivariant and constant on G-orbits.

The group H also acts on G/S by right multiplication. This action and the action of H on X^S determine the H-action on $(G/S) \times_k X^S$. It commutes with the G-action on $(G/S) \times_k X^S$ via left translation of G/S. Therefore $(G/S) \times_k X^S$ is endowed with an action of $G \times_k H$. The second projection $(G/S) \times_k X^S \to X^S$ is H-equivariant and constant on G-orbits.

The natural morphism $H \to G/S$ yields the basic isomorphism

$$H \xrightarrow{\cong} (G/S)^S. \tag{2.4}$$

Proposition 2.9. *For every (G,S)-fibration $\pi\colon X \to Y$ where X and Y are k-varieties, the following properties hold:*

(i) *$\pi^S\colon X^S \to Y$ is an H-torsor and every base change trivializing π trivializes π^S as well;*

(ii) *for the (G,S)-fibration $X \times_Y X^S \to X^S$ obtained from π by the base change π^S, the $(G \times_k H)$-equivariant X^S-map*

$$\varphi\colon (G/S) \times_k X^S \to X \times_Y X^S, \quad (\widetilde{g}, x) \mapsto (g \cdot x, x),$$

where $\widetilde{g} = \pi_{G,S}(g)$ (see Example 2.2), is an isomorphism;

(iii) *the morphism*

$$\mathrm{pr}_1 \circ \varphi \colon (G/S) \times_k X^S \to X, \quad (\widetilde{g}, x) \mapsto g \cdot x, \tag{2.5}$$

is an H-torsor.

Proof. (i) Suppose π is trivialized by a (surjective étale) base change $\mu \colon Y' \to Y$, i.e., there is a G-equivariant Y'-isomorphism $(G/S) \times_k Y' \to X \times_Y Y'$. By (2.2) and (2.4) we then have the H-equivariant Y'-isomorphisms $H \times_k Y' = (G/S)^S \times_k Y' \to (X \times_Y Y')^S \to X^S \times_Y Y'$. Hence, π^S is an H-torsor which is trivialized by μ. This proves (i).

(ii) The morphism φ is a Y-map with respect to the compositions of the second projections with π^S. By [Gro64, Vol. 24, Proposition 2.7.1(viii)] it is enough to prove the claim for the morphism of varieties obtained by the base change μ considered in the above proof of (i). By virtue of (i) and (2.3) this reduces the problem to proving that the map

$$(G/S) \times_k (N/S) \to (G/S) \times_k (N/S), \quad (\widetilde{g}, \widetilde{n}) \mapsto (\widetilde{gn}, \widetilde{n}),$$

is an isomorphism. But this is clear since $(\widetilde{g}, \widetilde{n}) \mapsto (\widetilde{gn^{-1}}, \widetilde{n})$ is the inverse map. This proves (ii).

(iii) By (i) and Proposition 2.8(ii) the morphism $X \times_Y X^S \to X$ obtained from π^S by the base change π is an H-torsor. Since φ is an isomorphism, this proves (iii). $\square$

Let C be an algebraic k-group. Consider a C-torsor

$$\alpha \colon P \to Y$$

over a k-variety Y. Let F be a k-variety endowed with an action of C. If every finite subset of F is contained in an open affine subset of F (for instance, if F is quasi-projective), then for the natural action of C on $F \times_k P$ determined by the C-actions on F and P, the geometric quotient exists; it is usually denoted by

$$F \times^C P.$$

Moreover, the quotient map $F \times P \to F \times^C P$ is actually a C-torsor over $F \times^C P$ (see [Flo08, Proposition 2.12] and use Proposition 2.8(iv) above; see also [PV94, § 4.8]).

Since the composition of morphisms $F \times_k P \xrightarrow{\mathrm{pr}_2} P \xrightarrow{\alpha} Y$ is constant on C-orbits, by the universal mapping property of geometric quotients this composition factors through a k-morphism

$$\alpha_F \colon F \times^C P \to Y.$$

Let $\mu \colon Y' \to Y$ be a surjective étale k-morphism such that α becomes the trivial morphism $\mathrm{pr}_2 \colon C \times_k Y' \to Y'$ after the base change μ. Then, after the same base change, α_F becomes the morphism

$$F \times^C (C \times_k Y') = F \times_k Y' \xrightarrow{\mathrm{pr}_2} Y'.$$

Hence α_F is a fibration over Y with fibre F.

Since the variety G/S is quasi-projective (see [Bor91, Theorem 6.8]), this construction is applicable for $C = H$ and $F = G/S$.

Given a k-variety Y, we now have two constructions:

- if $\pi \colon X \to Y$ is a (G, S)-fibration, then $\pi^S \colon X^S \to Y$ is an H-torsor;
- if $\alpha \colon P \to Y$ is an H-torsor, then $\alpha_{G/S} \colon (G/S) \times^H P \to Y$ is a (G, S)-fibration.

Proposition 2.10. *These two constructions are inverse to each other and they are functorial in Y.*

Proof. Since by Proposition 2.9(iii) morphism (2.5) is an H-torsor, X is the geometric quotient for the H-action on $(G/S)\times_k X^S$. Hence, by the uniqueness of geometric quotient, there is a G-equivariant isomorphism $(G/S)\times^H X^S \to X$.

Let $P\to Y$ be an H-torsor. The natural H-action on $(G/S)\times_k P$ and the G-action via left translation of G/S commute. From this and (2.4) we deduce the isomorphisms

$$((G/S)\times^H P)^S \xrightarrow{\cong} (G/S)^S \times^H P \xrightarrow{\cong} H\times^H P \xrightarrow{\cong} P.$$

Functoriality in Y is clear. □

Remark 2.11. The étale Čech cohomology set $H^1(Y,H)$ classifies H-torsors over Y. On the other hand, if $\mathrm{Aut}_G(G/S)$ is the algebraic k-group of G-equivariant automorphisms of G/S, then according to the general principle outlined at the beginning of [Ser94, § 3] (see also the references there for a more rigorous treatment), the étale Čech cohomology set $H^1(Y,\mathrm{Aut}_G(G/S))$ classifies (G/S)-fibrations over Y. Since the G-action on G/S by left translation commutes with the H-action by right multiplication, we have an injection $H\hookrightarrow \mathrm{Aut}_G(G/S)$. It is well known (and easy to prove) that, in fact, $H=\mathrm{Aut}_G(G/S)$. We thus get a bijection between $H^1(Y,H)$ and $H^1(Y,\mathrm{Aut}_G(G/S))$, i.e., between H-torsors and (G,S)-fibrations over Y. Proposition 2.10 is an explicit version of this fact.

2.3 (G,S)-varieties

Definition 2.12. Let S be a closed k-subgroup of G and let X be a k-variety endowed with a G-action. We shall say that X is a (G,S)-*variety* if X contains a dense open G-stable subset U which admits the structure of a (G,S)-fibration $U\to Y$. Generalizing a terminology introduced in [BF03], it is convenient to call such an open subset U a *friendly open subset* of X for the action of G.

If X is geometrically integral and U is a friendly open subset of X with (G,S)-fibration $\pi: U\to Y$, then π induces an isomorphism $\pi^*: k(Y)\xrightarrow{\cong} k(X)^G$.

The following statement over an algebraically closed field has previously appeared in various guises in the literature (see [PV94, 2.7] and [Pop94, 1.7.5]).

Theorem 2.13. *Let X be a geometrically integral k-variety endowed with a G-action. Let S be a closed k-subgroup of G. Then the following properties are equivalent:*

(a) *X is a (G,S)-variety;*
(b) *X contains a dense open G-stable subset such that the $G_{\overline{k}}$-stabilizer of each of its $\overline{k}$-points is conjugate to $S_{\overline{k}}$.*

Proof. That (a) implies (b) is clear. Let us assume (b). One may replace X by dense G-stable open subsets to successively ensure that:

(i) The $G_{\overline{k}}$-stabilizer of every $\overline{k}$-point of X is conjugate to $S_{\overline{k}}$.

(ii) The k-variety X is smooth over k.

(iii) (Rosenlicht's theorem, see § 2.1.) There exist a geometrically integral k-variety Y and a k-morphism $\pi: X\to Y$ such that the pair (Y,π) is the geometric quotient of X for the action of G.

(iv) The morphism π is smooth. Indeed the generic fibre of π is regular, hence smooth since $\mathrm{char}(k)=0$. The statement on π can thus be achieved by replacing Y by an open set and X by the inverse image of this open set.

(v) There exists an open set U of the reduced variety $(X^S)_{\mathrm{red}} \subset X^S$ such that the composition of maps $U \hookrightarrow X^S \xrightarrow{\pi^S} Y$ is smooth. This follows from the surjectivity of the map $\pi^S : X^S \to Y$ on $\overline{k}$-points, itself a consequence of (i) and (iii).

(vi) If we let $U \subset (X^S)_{\mathrm{red}}$ be the maximal open set such that the map $\pi|_U : U \to Y$ is smooth, then this map is surjective. This is achieved by replacing Y in the previous statement by a dense open set contained in the image of the previous U, and replacing X by the inverse image of this open set.

We now consider the following G-equivariant k-morphism of smooth k-varieties:

$$\psi : (G/S) \times_k U \to X \times_Y U, \quad (\widetilde{g}, u) \mapsto (gu, u)$$

(see the notation in Proposition 2.9(ii)). It is a U-morphism with respect to the second projections. Since every G-equivariant morphism $G/S \to G/S$ is bijective and (Y, π) is the geometric quotient, we deduce from (i) and (iii) that ψ is bijective on $\overline{k}$-points. As $\mathrm{char}(k)=0$, we then conclude by Zariski's main theorem that ψ is an isomorphism. Thus it is proved that the morphism obtained from π by the base change $\pi|_U$ is trivial over U with fibre G/S. Since $\pi|_U$ is smooth, we now deduce from Remark 2.6 that π is a (G,S)-fibration. □

Condition (b) of Theorem 2.13 gained much attention in the literature (see [PV94, § 7] and references therein). If (b) holds, one says that, for the action of G on X, there exists a *stabilizer S in general position* or that there exists a *principal orbit type* for (G, X). There are actions, even of reductive groups, for which a stabilizer in general position does not exist (see [PV94, 7.1, 2.7]). There are results ensuring its existence under certain conditions or, equivalently (by Theorem 2.13), the existence of a structure of (G,S)-variety. Theorem 2.15 below is such a result.

Recall the following definition introduced in [Pop72].

Definition 2.14. The action of an algebraic k-group G on a k-variety X is called *stable* if there exists a dense open subset U of X such that the G-orbit of every point of $U(\overline{k})$ is closed in $X_{\overline{k}}$.

Theorem 2.15. *Let X be an affine geometrically integral k-variety with an action of a reductive k-group G such that $X(k)$ is Zariski dense in X. Assume that either of the following conditions hold:*

(i) *X is smooth; or*

(ii) *the G-action on X is stable.*

Then there is a closed k-subgroup S of G such that X is a (G,S)-variety. In case (ii) this subgroup S is reductive.

Proof. If (i) holds, then by Richardson's theorem [Ric72, Proposition 5.3] (cf. also [Lun73, Corollary 8], [PV94, Theorem 7.2]) there is a closed $\overline{k}$-subgroup R of $G_{\overline{k}}$ such that the $G_{\overline{k}}$-stabilizer of a general $\overline{k}$-point of X is conjugate to R. Since $X(k)$ is Zariski dense, R can be taken as $S_{\overline{k}}$, where S is the stabilizer of a k-point of X. Then property (b) from the statement of Theorem 2.13 holds, hence X is a (G,S)-variety.

If (ii) holds, then the above subgroup S still exists by [PV94, § 7.2, Cor.], so the same argument applies. As the general orbit is closed, it is affine, whence S is reductive by Matsushima's criterion [Mat60, Oni60], cf. [Bia63, Lun73]. □

2.4 Categorical quotients

For the definition of a categorical quotient we refer the reader to [Mum65, Definition 0.5], [Bor91, 6.16, 8.19], and [PV94, § 4.3]. In this paper we shall only work with categorical quotients for reductive group actions on affine varieties, which are constructed as follows.

Let A be a finitely generated k-algebra. Assume a reductive k-group G acts on the k-variety $X = \operatorname{Spec}(A)$ (over k). Then (cf. [Mum65, Theorem 1.1, Corollary 1.2]):

(i) the ring A^G is a finitely generated k-algebra;

(ii) the inclusion $A^G \hookrightarrow A$ induces a categorical quotient map $\pi : X \to \operatorname{Spec}(A^G) =: X/\!\!/G$;

(iii) every geometric fibre of π contains a unique closed orbit.

As G-orbits are open in their closure, the latter property implies that every geometric fibre of π containing a closed G-orbit of maximal (in this fibre) dimension, coincides with this orbit.

Proposition 2.16. *Let X be a geometrically integral affine k-variety with an action of a reductive k-group G such that $X(k)$ is Zariski dense in X. Let $\pi : X \to X/\!\!/G$ be a categorical quotient. Then the following properties are equivalent:*

(a) *the action of G on X is stable;*

(b) *there exist a reductive k-subgroup S of G and a dense open subset Y of $X/\!\!/G$ such that the restriction of π to $\pi^{-1}(Y)$ is a (G, S)-fibration $\pi^{-1}(Y) \to Y$.*

The group S in (b) may be taken as the G-stabilizer of any k-point of $\pi^{-1}(Y)$.

If (a), (b) hold, then π induces an isomorphism $\pi^* : k(X/\!\!/G) \xrightarrow{\cong} k(X)^G$.

Proof. Assume that (a) holds. By Theorem 2.15, there exist a reductive k-subgroup S of G and a G-invariant open subset U_1 of X such that U_1 admits the structure of a (G, S)-fibration $\alpha : U_1 \to Z_1$.

On the other hand, by (a) there is an open subset U_2 of X such that the G-orbit of every point of $U_2(\overline{k})$ is closed. Since there is an open subset $U_{\max}$ of X such that the G-orbit of every point of $U_{\max}(\overline{k})$ has maximal (in X) dimension (cf. [Mum65, ch. 0, § 2] or [PV94, § 1.4]), we may replace U_2 by $U_2 \cap U_{\max}$ and assume in addition that this maximality property holds for every point of $U_2(\overline{k})$. The openness of U_2 in X implies that $\pi(U_2)$ contains a smooth open subset Y_1 of $X/\!\!/G$. Put $U_3 := \pi^{-1}(Y_1)$. Then the fibre of π over every point of $Y_1(\overline{k})$ contains a closed G-orbit of maximal dimension. As we mentioned right before the statement of Proposition 2.16, this implies that this fibre is a G-orbit. In turn, as we mentioned in § 2.1, this implies that $\pi|_{U_3} : U_3 \to Y_1$ is the geometric quotient for the G-action on U_3.

Let $U = U_1 \cap U_3$. Then, since α and $\pi|_{U_3}$ are open morphisms, $Z := \alpha(U)$ and $Y := \pi(U_3)$ are open subsets of Z_1 and Y_1 respectively. The morphisms $\pi|_U : U \to Y$ and $\alpha|_U : U \to Z$ are geometric quotient maps for the G-action on U. By uniqueness of geometric quotients, there is an isomorphism $\varphi : Z \to Y$ such that the diagram

$$\begin{tikzcd} & U \arrow[dl, "\alpha|_U"'] \arrow[dr, "\pi|_U"] & \\ Z \arrow[rr, "\cong", "\varphi"'] & & Y \end{tikzcd}$$

is commutative. Hence (b) holds.

Conversely, if (b) holds, then fibres of π over points of $Y(\overline{k})$ are G-orbits. Therefore these orbits are closed; whence (a).

To prove the last assertion of the proposition, we may replace $X/\!\!/G$ by Y and thus assume that π is a (G,S)-fibration. By Proposition 2.8(iv) π is a geometric quotient map; the desired conclusion now follows from Remark 2.1. □

3. Versal actions

We recall that k is a field of characteristic zero. Let $\overline{k}$ be an algebraic closure of k, and let G be a (not necessarily connected) linear algebraic group over k.

Definition 3.1. Let S be a closed k-subgroup of G. We say that a (G,S)-fibration $\pi : V \to Y$ is *versal* if Y is geometrically integral and for every field extension L/k, every (G_L, S_L)-fibration $\varrho : X \to \mathrm{Spec}(L)$, and every dense open subset Y_0 of Y, there exists a Cartesian diagram of the form

$$\begin{tikzcd} X \arrow[rr] \arrow[d, "\varrho"'] & & V \arrow[d, "\pi"] \\ \mathrm{Spec}(L) \arrow[r] & Y_0 \arrow[r, hook] & Y. \end{tikzcd} \tag{3.1}$$

In other words, there is an L-point of Y_0 and an L-isomorphism between X and the fibre product $V \times_Y \mathrm{Spec}(L)$.

Note that if $S = \{1\}$, i.e., π is a G-torsor, this definition coincides with the usual definition of a versal torsor; see [GMS03, § I.5], [BF03].

Lemma 3.2. *Let N be the normalizer of S in G and let $H = N/S$. A (G,S)-fibration $\pi : V \to Y$ over a smooth Y is versal if and only if the associated H-torsor $\pi^S := \pi|_{V^S} : V^S \to Y$ is versal.*

Proof. By Proposition 2.10 there are mutually inverse functorial correspondences between (G,S)-fibrations and H-torsors over $\mathrm{Spec}(L)$ given by passing from a (G,S)-fibration $\varrho : X \to \mathrm{Spec}(L)$ to the H-torsor $\varrho^S : X^S \to \mathrm{Spec}(L)$ and from an H-torsor $\alpha : Z \to \mathrm{Spec}(L)$ to the (G,S)-fibration $\alpha_{G/S} : (G/S) \times^H Z \to \mathrm{Spec}(L)$. This implies that a Cartesian diagram (3.1) exists if and only if a Cartesian diagram

$$\begin{tikzcd} X^S \arrow[rr] \arrow[d, "\varrho^S"'] & & V^S \arrow[d, "\pi^S"] \\ \mathrm{Spec}(L) \arrow[r] & Y_0 \arrow[r, hook] & Y \end{tikzcd}$$

exists. This means that π is versal if and only if π^S is versal. □

We say that a (G,S)-variety X is *versal* if there is a friendly open set U of X (see Definition 2.12) such that the associated (G,S)-fibration $U \to Y$ is versal.

It is easy to see that if X is a versal (G,S)-variety, then the (G,S)-fibration $U' \to Y'$ is a versal (G,S)-fibration for *every* friendly open set U' of X.

The following proposition plays an important rôle in our paper. We are grateful to the referee for the present version of this statement which strengthens our earlier result used in the proof of Theorem 0.2. Recall that the notion of stabilizer in general position utilized in the formulation of this proposition has been defined in the previous section, just after the proof of Theorem 2.13.

Proposition 3.3. *Suppose G and H are linear algebraic groups over k and G acts (algebraically) on H by group automorphisms. Assume further that:*

(i) *the group H is connected and, for this action of G on H, there exists a stabilizer S in general position;*

(ii) *the group H^S is connected.*

Then H is a versal (G,S)-variety.

Note that by Theorem 2.15(i) condition (i) automatically holds if G is reductive.

Our proof of Proposition 3.3 will rely on the following two lemmas.

Lemma 3.4. *Let C be an algebraic group over a field K, let X be a quasi-projective K-variety endowed with a C-action, and let $\pi : P \to \operatorname{Spec}(K)$ be a C-torsor. Then the following properties are equivalent.*

(a) *There exists a C-equivariant morphism $\alpha : P \to X$ defined over K.*

(b) *$X \times^C P$ has a K-point.*

Recall that here $X \times^C P$ is the K-variety defined after the proof of Proposition 2.9.

Proof of Lemma 3.4. (a) $\Rightarrow$ (b): By the universal mapping property (see § 2.1) the C-equivariant morphism $\alpha \times \mathrm{id} : P \to X \times P$ determines a K-morphism of geometric quotients $\operatorname{Spec}(K) = P/C \to (X \times P)/C = X \times^C P$, i.e., a K-point of $X \times^C P$.

(b) $\Rightarrow$ (a): Given a K-point $\mu : \operatorname{Spec}(K) \to X \times^C P$, let $\varepsilon : E \to \operatorname{Spec}(K)$ be the C-torsor obtained from the C-torsor $X \times P \to X \times^C P$ by the base change μ. By construction there is a C-equivariant morphism $E \to X \times P$ (see § 2.2). Its composition with the projection $X \times P \to P$ is a morphism of C-torsors π and ε, hence an isomorphism. This yields a C-equivariant morphism $P \to X \times P$. Its composition with the projection $X \times P \to X$ is a C-equivariant morphism $P \to X$. □

Lemma 3.5. *Let C, X, and P be as in Lemma 3.4. If X is an algebraic group and C acts on X by group automorphisms, then $X \times^C P$ has a natural structure of an algebraic group defined over K.*

Proof of Lemma 3.5. For notational simplicity we shall write ${}^P X$ in place of $X \times^C P$.

Let X_1 and X_2 be quasi-projective K-varieties endowed with C-actions and let $\varphi : X_1 \to X_2$ be a C-equivariant morphism. By the universal mapping property the C-equivariant morphism $\varphi \times \mathrm{id} : X_1 \times P \to X_2 \times P$ determines a morphism of geometric quotients ${}^P X_1 \to {}^P X_2$ which we shall denote by ${}^P \varphi$.

For $i = 1, 2$, let $\pi_i : X_1 \times X_2 \to X_i$ be the projection. We claim that the K-morphism

$$ {}^P\pi_1 \times {}^P\pi_2 : {}^P(X_1 \times X_2) \to {}^P X_1 \times {}^P X_2 \tag{3.2} $$

is, in fact, an isomorphism. To prove this claim we pass to a finite field extension K' of K such that P splits over K', next we observe that if P splits, the claim is obvious. We conclude that (3.2) is an isomorphism over K' and hence by [Gro64, Vol. 24, Proposition 2.7.1(viii)] over K.

Using this isomorphism one easily checks that the multiplication map $X \times X \to X$ and the inverse map $X \to X$ give rise to group operations on ${}^P X$. □

Proof of Proposition 3.3. Condition (i) and Theorem 2.13 yield that, for the G-action on H, there is a friendly open subset U of H. So a geometric factor $U \to Y$ for the action of G on U exists and is a (G,S)-fibration. The action of $N_G(S)/S$ on H^S is generically free and Proposition 2.9(i) yields that $U^S = H^S \cap U$ is a friendly open subset of H^S for this action.

By Lemma 3.2 it suffices to show that $U^S \to Y$ is a versal $N_G(S)/S$-torsor or, equivalently, that H^S is a versal $N_G(S)/S$-variety. Thus, given condition (ii), after replacing G by $N_G(S)/S$ and H by H^S, we may assume that $S = \{1\}$.

Our goal now is to show that, under this assumption, the G-action on the friendly open subset U of H satisfies the conditions of Definition 3.1, i.e., for every field extension L/k, every G-torsor $P \to \mathrm{Spec}(L)$, and every open dense subset Y_0 of Y, there is a G-equivariant map $P \to U_0 := \pi^{-1}(Y_0)$. By Lemma 3.4 with $C = G_L$ this map exists if and only if $(U_0)_L \times^{G_L} P$ has an L-point. Since U_0 is a dense G-invariant subset of U (and hence of H), we see that $(U_0)_L \times^{G_L} P$ is a dense open subset of $H_L \times^{G_L} P$. Thus, it suffices to show that L-points are dense in $H_L \times^{G_L} P$.

Lemma 3.5 implies that $H_L \times^{G_L} P$ is an algebraic group over L. It is connected because H is connected. Let L' be a finite field extension of L such that P splits over L'. Then the L'-groups $(H_L \times^{G_L} P) \times_L L'$ and $H_L \times_L L'$ are isomorphic. Since H is linear, this yields that $(H_L \times^{G_L} P) \times_L L'$ is linear. The standard descent result [Gro64, Vol. 24, Proposition 2.7.1(xiii)] then implies that the L-group $H_L \times^{G_L} P$ is linear. But since $\mathrm{char}(L) = 0$, by Chevalley's theorem [Bor91, Theorem 18.2(ii)] any connected linear algebraic group defined over L is unirational over L. Hence $H_L \times^{G_L} P$ is unirational over L and therefore L-points are dense in it, as claimed. □

Corollary 3.6. (a) (cf. [Rei00, Proposition 7.1] and [GMS03, Example I.5.4]) *Every finite-dimensional generically free G-module V defined over k is a versal $(G, \{1\})$-variety.*

(b) *If G is reductive and V is a finite-dimensional G-module defined over k, then V is a versal (G, S)-variety for a suitable k-subgroup S of G. There exists a dense open subset U of V such that the G-stabilizer of each point of $U(k)$ is a possible choice for S.*

Proof. In both parts view V as the unipotent k-group $\mathbf{G}_a^{\dim V}$ and apply Proposition 3.3 and, in part (b), Theorem 2.15(i). □

Lemma 3.7. *Let $\varphi : X \to Y$ be a dominant morphism of integral k-varieties. Denote the generic point of Y by η and the generic fibre of φ by X_η.*

(a) *Suppose φ has a rational section $s : Y \dashrightarrow X$. Then there exists a dense open subset Y_0 of Y defined over k such that for any morphism $Z \to Y_0$ of integral schemes, the natural projection $\varphi_Z : X_Z := X \times_Y Z \to Z$ has a section $Z \to X_Z$.*

(b) *Suppose the generic fibre X_η is connected and is rational (respectively, stably rational) over $k(Y)$. Then there exists a dense open subset Y_0 of Y defined over k such that for any extension field L/k and any point $y \in Y_0(L)$, the fibre X_y of φ over y is integral and rational (respectively, stably rational) over L.*

Proof. (a) Choose $Y_0 \subset Y$ so that s is regular on Y_0 and pull back the section s to X_Z.

Before we prove (b), let us discuss a more general situation. Let $p : X \to Y$ and $p' : X' \to Y$ be two dominant k-morphisms of geometrically integral k-varieties with geometrically integral generic fibres. Assume that the generic fibres are birationally isomorphic over $k(Y)$. Then X and X' are birationally isomorphic over Y. There thus exist two dense open sets $U \subset X$ and $U' \subset X'$ and a Y-isomorphism $U \xrightarrow{\cong} U'$. Let $Y_0 \subset p(U)$ be a Zariski dense open set and replace U and U' by their restrictions over Y_0. Then all geometric fibres of $U \to Y_0$ and $U' \to Y_0$ are nonempty. For any point $y \in Y_0$ this induces a $k(y)$-isomorphism between the nonempty fibre $U_y \subset X_y$ and the nonempty fibre $U'_y \subset X'_y$. The same therefore holds over L with $k(y) \subset L$.

To prove statement (b) in the case where X_η is rational, it suffices to apply this argument to X and $X' = \mathbf{P}^d_Y$ where d is the dimension of X_η and $\mathbf{P}^d_Y$ is the d-dimensional projective space over Y.

To prove statement (b) in the case where X_η is stably rational, it suffices to apply this argument to the pair $X \times_Y \mathbf{P}^n_Y$ and $\mathbf{P}^{n+d}_Y$, with d as above and n some positive integer. □

Definition 3.8. Given a (G, S)-variety X, we shall say that it admits a rational section if for some and hence any friendly open set $U \subset X$ the quotient map $U \to U/G$ admits a section over a dense open set of U/G.

Theorem 3.9. *Let G be a linear algebraic group over k, let S be a closed k-subgroup of G, and let V be a geometrically integral versal (G, S)-variety.*

(a) *If the (G, S)-variety V admits a rational section, then for every field extension F/k every geometrically integral (G, S)-variety X over F admits a rational section.*

(b) *Assume that the homogeneous space G/S is connected. If $k(V)/k(V)^G$ is pure (respectively, stably pure), then for every field extension F/k and every geometrically integral (G, S)-variety X over F, the field extension $F(X)/F(X)^G$ is pure (respectively, stably pure).*

Proof. Note that if V is a versal (G, S)-variety over k then V_F is a versal (G, S)-variety over F. Since the hypothesis $F(V)/F(V)^G$ pure, or stably pure, holds as soon as it does for $k(V)/k(V)^G$, it is enough to prove the theorem for $F = k$. After replacing V by a friendly open subset we may assume that we are given a (G, S)-fibration $\pi : V \to Y$. Choose a dense open subset $Y_0 \subset Y$ as in Lemma 3.7(a).

(a) After replacing X by a friendly open subset, we may assume that X is the total space of a (G, S)-fibration $\alpha : X \to Z$. Let η be the generic point of Z. Since π is versal, the (G, S)-fibration $\alpha_\eta : X_\eta \to \eta$ can be obtained by pull-back from π via a morphism Spec $k(Z) \to Y_0$. In other words, after replacing X by a smaller friendly open set, we may assume that $\alpha : X \to Z$ is the pull-back of $\pi : V \to Y$ via a morphism $Z \to Y_0 \subset Y$. The desired conclusion now follows from Lemma 3.7(a).

The proof of part (b) is exactly the same, except that we appeal to Lemma 3.7(b), rather than to Lemma 3.7(a). □

Lemma 3.10. *Let G be a connected linear algebraic group over k and let X be a geometrically integral k-variety with G-action which admits a geometric quotient $\pi : X \to Y$. The following properties are equivalent.*

(a) *$\pi : X \to Y$ admits a rational section;*
(b) *$k(X)$ is unirational over $k(X)^G$.*

Proof. We know that π induces an isomorphism $\pi^* : k(Y) \xrightarrow{\cong} k(X)^G$. Let X_η be the $k(Y)$-variety which is the generic fibre of $\pi : X \to Y$.

Assume (b). The hypothesis implies that there exists a dominant $k(Y)$-rational map φ from some projective space $\mathbb{P}^n_{k(Y)}$ to X_η. This rational map is defined on a dense open set $U \subset \mathbb{P}^n_{k(Y)}$. Since rational points are Zariski dense on projective space over an infinite field, the set $U(k(Y))$ is nonempty. The $k(Y)$-morphism $\varphi : U \to X_\eta$ sends such a point to a $k(Y)$-point of X_η, i.e., to a rational section of $X \to Y$. Thus (a) holds.

Assume (a) holds. By the definition of π, the generic fibre of π is a $k(Y)$-variety with function field $k(X)$, and it is a homogeneous space of $G_{k(Y)}$ which by (a) admits a $k(Y)$-rational point.

We thus have inclusions of fields $k(Y) \subset k(X) \subset k(Y)(G)$. By a theorem of Chevalley, over a field of characteristic zero, any connected linear algebraic group is unirational (see [Bor91, Theorem 18.2] or [DG70, Vol. II, ch. XIV, Corollary 6.10]). Thus $k(Y)(G)$ embeds into a purely transcendental extension of $k(Y) \xrightarrow{\cong} k(X)^G$. This proves (b). □

4. The conjugation action and the adjoint action

We now concentrate on the main actors. We recall that k is a field of characteristic zero. Let G be a connected reductive group over k.

4.1 Quotients by the adjoint action, versal (G, S)-varieties, and Kostant's theorem

The radical $\mathrm{Rad}(G)$ of G is a central k-torus in G. The G-stabilizer of a point $g \in G$ for the conjugation action of G on itself is the centralizer of g in G. There is a Zariski dense open set of G such that the centralizers of its points in G are maximal tori of G, see [Bor91, 12.2, 13.1, 13.17, 12.3].

Lemma 4.1. *The following properties of an element $g \in G$ are equivalent:*

(i) *the conjugacy class of g is closed in G;*

(ii) *g is semisimple.*

Proof. For semisimple groups this is proved in [Ste65, 6.13]. The general case can be reduced to that of semisimple groups in the following manner. Let (G, G) be the commutator subgroup of G. It is a closed, connected, semisimple k-subgroup of G, and $G = (G, G) \cdot \mathrm{Rad}(G)$, see [Bor91, 2.3, 14.2]. Let $g = hz$ for some $h \in (G, G)$, $z \in \mathrm{Rad}(G)$, and let $g = g_s g_u$, $h = h_s h_u$ be the Jordan decompositions, see [Bor91, 4.2]. Since $\mathrm{Rad}(G)$ is a central torus, $g_s = h_s z$, $g_u = h_u$. Hence g is semisimple if and only if h shares this property. Let $G \cdot g$ and $G \cdot h$ be respectively the G-conjugacy classes of g and h in G. Since z is central, $G \cdot g = (G \cdot h)z$. Hence closedness of $G \cdot g$ in G is equivalent to that of $G \cdot h$. But $G \cdot h$ coincides with the (G, G)-conjugacy class of h in (G, G) and, since (G, G) is semisimple, the cited result in [Ste65] shows that the latter is closed in (G, G) if and only if h is semisimple. This completes the proof. □

Corollary 4.2. *The action of G on itself by conjugation is stable.*

Analogous statements hold for the adjoint action of G on $\mathfrak{g}$, see [Kos63].

Proposition 4.3. *Let G be a connected reductive group over k and $\mathfrak{g}$ its Lie algebra. Let $S \subset G$ be a maximal k-torus. Let X be either G or $\mathfrak{g}$ and let $\pi : X \to Y := X /\!\!/ G$ be the categorical quotient for the conjugation, respectively, the adjoint action. Then:*

(a) *there exists a dense Zariski open subset V of Y with inverse image $U = \pi^{-1}(V)$ such that $\pi|_U : U \to V$ is a (G, S)-fibration;*

(b) *π^* induces an isomorphism $k(Y) \xrightarrow{\cong} k(X)^G$;*

(c) *X is a versal (G, S)-variety.*

Proof. Statements (a) and (b) follow from Corollary 4.2 and Proposition 2.16 by virtue of the identification of general G-stabilizers with maximal tori of G, which are all conjugate over $\overline{k}$.

For $X = \mathfrak{g}$, part (c) is a special case of Corollary 3.6(b).

To prove part (c) for $X = G$, note that

$$X^S = \text{centralizer of } S \text{ in } G = S$$

is connected. Hence Proposition 3.3 applies to the conjugation action of G on itself, yielding the desired conclusion. □

The following well known result plays an important rôle in the sequel.

Proposition 4.4 (Kostant). *Let G be a reductive linear algebraic group over k and $\mathfrak{g}$ be its Lie algebra. Assume that the semisimple quotient $G/\mathrm{Rad}(G)$ is quasisplit. Then the categorical quotient map $\pi : \mathfrak{g} \to \mathfrak{g}/\!\!/G$ has a (regular) section.*

Proof. For algebraically closed base field k this is a theorem of Kostant [Kos63, Theorem 0.6]. For an arbitrary base field k of characteristic 0, see [Kot99, § 4.3]. □

Proposition 4.5. *Let G be a connected reductive group over k, let S be a maximal k-torus of G, and let X be a geometrically integral (G, S)-variety over k. Assume that the semisimple group $G/\mathrm{Rad}(G)$ is quasisplit. Then:*

(a) *X admits a rational section;*
(b) *$k(X)$ is unirational over $k(X)^G$.*

Proof. Going over to a friendly open set, we may assume that we are given a (G, S)-fibration $\pi : X \to Y$. Since the radical of G lies in every conjugate of S, it acts trivially on X. Thus the G-action on X descends to the semisimple group $G/\mathrm{Rad}(G)$, and we may assume that the k-group G is quasisplit semisimple.

By Proposition 4.3(c), the Lie algebra $\mathfrak{g}$ equipped with the adjoint action is a versal (G, S)-variety. By Proposition 4.4 the map $\pi : \mathfrak{g} \to \mathfrak{g}/\!\!/G$ admits a section. Statement (a) now follows from Theorem 3.9(a). As for (b), by Lemma 3.10 it follows from (a). □

Many instances of the following immediate corollary have appeared in the literature (cf. [Kot82]).

Corollary 4.6. *Let G be a connected reductive group over a field k, let K be an overfield of k. Let X be a K-variety which is a homogeneous space of G_K. If the semisimple quotient $G/\mathrm{Rad}(G)$ is quasisplit and if the geometric stabilizers of the G_K-action on X are maximal tori in $G_{\overline{K}}$, then X has a K-rational point.*

Remark 4.7. Suppose that the base field k is algebraically closed and G is a connected simple algebraic group defined over k. Let V be a faithful simple G-module over k. Theorem 2.15 tells us that V is a (G, H)-variety for some closed subgroup H of G. The list of all pairs (G, H), with $H \neq \{1\}$, which can occur in this setting can be found in [PV94, pp. 260–262]. Then the analogue of Corollary 4.6 holds, namely every G-homogeneous space X, defined over K whose geometric stabilizers are isomorphic to $H \times_k \overline{K}$ has a K-point. The proof is similar to the one above, except that instead of Kostant's result (Proposition 4.4), one uses the existence of a regular section for the categorical quotient map $V \to V/\!\!/G$, proved in [Pop92].

Part (a) of the following corollary partially generalizes a result of Steinberg [Ste65], who constructed a regular section of π in the case where G is simply connected.

Corollary 4.8. *Let G be a connected reductive group over k and let $\mathfrak{g}$ be its Lie algebra. If the semisimple quotient $G/\mathrm{Rad}(G)$ is quasisplit, then:*

(a) *the categorical quotient map $\pi: G \to G/\!\!/G$ for the conjugation action of G on itself admits a rational section;*

(b) *$k(G)$ is unirational over $k(G)^G$;*

(c) *$k(\mathfrak{g})$ is unirational over $k(\mathfrak{g})^G$.*

Proof. By Proposition 4.3 both G and $\mathfrak{g}$ are (G, S)-varieties. All three parts now follow from Proposition 4.5. □

Remark 4.9. Recall from [LPR06] that a *Cayley map* for an algebraic group G over a field k is a G-equivariant birational isomorphism

$$\varphi: G \dashrightarrow \mathfrak{g}, \tag{4.1}$$

where $\mathfrak{g}$ is the Lie algebra of G. Here, as before, G is assumed to act on itself by conjugation and on $\mathfrak{g}$ by the adjoint action. We say that G is a *Cayley group* if it admits a Cayley map. In particular, the special orthogonal group SO_n, the symplectic group Sp_{2n}, and the projective linear group PGL_n are Cayley groups for every $n \geqslant 1$ and every base field k of characteristic zero; see [LPR06, Examples 1.11 and 1.16].

In the case where G is a Cayley group, Theorem 0.3 (or equivalently, Corollary 4.8(a)) has the following simpler proof. By Proposition 4.4 the categorical quotient map $\pi_{\mathfrak{g}}: \mathfrak{g} \to \mathfrak{g}/\!\!/G$ has a section $\sigma: \mathfrak{g}/\!\!/G \to \mathfrak{g}$. Let $\pi_G: G \to G/\!\!/G$ be the categorical quotient map. Then the Cayley map (4.1) induces a commutative diagram,

$$\begin{tikzcd} G \arrow[r, dashed, "\varphi"] \arrow[d, "\pi_G"'] & \mathfrak{g} \arrow[d, "\pi_{\mathfrak{g}}"'] \\ G/\!\!/G \arrow[r, dashed, "\varphi/\!\!/G"] & \mathfrak{g}/\!\!/G \arrow[u, bend right, "\sigma"'] \end{tikzcd}$$

where φ and $\varphi/\!\!/G$ are birational isomorphisms. The section σ pulls back to a rational section of π_G via this diagram.

In the case of $G = \mathrm{PGL}_2$ explicitly mentioned by Grothendieck (see the footnote in the introduction), we have $\mathfrak{g} = \mathfrak{sl}_2$ and:

(i) $G/\!\!/G$ is the affine line $\mathbb{A}^1$ and $\pi_{\mathrm{PGL}_2}: \mathrm{PGL}_2 \to \mathbb{A}^1$ is given by $[g] \mapsto (\mathrm{tr}\, g)^2/\det g$, where $\mathrm{GL}_2 \to \mathrm{PGL}_2$, $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto [g] := \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, is the natural projection;

(ii) $\mathfrak{g}/\!\!/G$ is the affine line $\mathbb{A}^1$ and $\pi_{\mathfrak{g}}: \mathfrak{g} \to \mathbb{A}^1$ is given by $g \mapsto \det g$.

In this case a Cayley map $\varphi: \mathrm{PGL}_2 \dashrightarrow \mathfrak{sl}_2$ is given by

$$[g] \mapsto \frac{2}{\mathrm{tr}\, g} g - I_2,$$

where I_2 is the 2×2 identity matrix (see [LPR06, Example 1.11]), the map $\varphi/\!\!/G: \mathbb{A}^1 \dashrightarrow \mathbb{A}^1$ is given by $t \mapsto -1 + 4/t$, and the section $\sigma: \mathbb{A}^1 \to \mathfrak{sl}_2$ is given by $t \mapsto \left(\begin{smallmatrix} 0 & 1 \\ -t & 0 \end{smallmatrix}\right)$. The above strategy then leads to the rational section

$$\mathbb{A}^1 \dashrightarrow \mathrm{PGL}_2, \quad t \mapsto \begin{bmatrix} 1 & 1 \\ 1 - 4/t & 1 \end{bmatrix}.$$

4.2 The generic torus

The conjugation action of a connected reductive group G on itself leads to another construction, that of the generic torus. Let S be a maximal k-torus of the connected reductive k-group G and let N be the normalizer of S in G. Consider the natural map $\varphi : S \times_k (G/S) \to G \times_k (G/N)$ given by $(s, gS) \mapsto (gsg^{-1}, gN)$. Its image $H \subset G \times_k (G/N)$ is closed (see [Hum95, p. 10]). The point $\varphi(s, g)$ defines the maximal torus gSg^{-1} and the point gsg^{-1} in that torus. The second projection $\pi : H \to G/N$ gives H the structure of a torus over G/N, and the family of fibres of this projection is the family of maximal tori in G. To be more precise, the morphism $H \hookrightarrow G \times_k (G/N)$ is a morphism of G/N-group schemes, where H is a (G/N)-torus and $G \times_k (G/N)$ is the constant (G/N)-group scheme induced by base change from $G \to \operatorname{Spec} k$. The (G/N)-torus H is thus a maximal torus in the (fibrewise connected) reductive (G/N)-group $G \times_k G/N$. The variety G/N is the 'variety of maximal tori in G'. Given any field extension L/k and any maximal L-torus S in G_L, there exists an L-point $s \in (G/N)(L)$ such that $\pi^{-1}(s) = S$. The actions of G by conjugation on itself and by left translation on G/N induce a G-action on $G \times_k G/N$ with respect to which H is stable and π is G-equivariant. Since $G(L)$ is dense in G, this implies that the set of L-points of G/N whose fibre under $H \to G/N$ is isomorphic to a given L-torus is Zariski dense in G/N. The field $k(G/N)$ is *denoted* K_{gen}. The *generic torus* T_{gen} is by definition the generic fibre of π. For the details of this construction, see [Vos98, §§ 4.1–4.2].

Assume that G is split over k and S is a split maximal torus of G. Then the K_{gen}-torus T_{gen} is split by the extension $k(G/S)$ of $k(G/N)$, which is a Galois extension with Galois group the Weyl group $W = N/S$. If, moreover, G is simple, simply connected of type R, then the character lattice of the generic torus is the weight W-lattice $P(\mathsf{R})$. For proofs of these assertions, see [Vos88].

4.3 Equivalent versions of the purity questions

We consider the purity Questions 0.1(a) and (b) from the introduction.

Theorem 4.10. *Let G be a connected reductive group over k and let S be a maximal k-torus of G. Then the following three conditions are equivalent:*

(a) *$k(G)/k(G)^G$ is pure (respectively, stably pure);*

(b) *$k(\mathfrak{g})/k(\mathfrak{g})^G$ is pure (respectively, stably pure);*

(c) *for every field extension F/k and every integral (G, S)-variety X over F, the extension $F(X)/F(X)^G$ is pure (respectively, stably pure).*

The following two conditions are equivalent and are implied by the previous conditions:

(d) *for every field extension F/k and every maximal F-torus T of G_F, the F-variety G_F/T is rational (respectively, stably rational) over F;*

(e) *the K_{gen}-variety $G_{K_{\text{gen}}}/T_{\text{gen}}$ is K_{gen}-rational (respectively, stably rational).*

If G is quasisplit, then all five conditions are equivalent.

Proof. By Proposition 4.3, both G and $\mathfrak{g}$ are versal (G, S)-varieties, where S is a maximal k-torus of G. The equivalence of (a), (b) and (c) now follows from Theorem 3.9(b).

The implication (d) $\Rightarrow$ (e) is clear. Let us prove that (e) $\Rightarrow$ (d). By assumption, the generic fibre of the quotient of the (G/N)-group scheme $G \times_k (G/N)$ by the maximal torus H over G/N is $k(G/N)$-rational.

(This quotient exists as affine (G/N)-scheme. This is a consequence of the fact that H is a (fibrewise connected) reductive (G/N)-group, the scheme $G \times_k (G/N)$ is affine over G/N, and G/N is a scheme of characteristic zero. For our purposes, it is enough to know this over a dense open subset of G/N, and that is a consequence of the existence of the quotient over the field $k(G/N)$.)

By Lemma 3.7(b) it follows that over a dense Zariski open set of G/N, all fibres are rational. The F-torus T may be represented by an F-point of this open set. Hence the result.

The implication (c) $\Rightarrow$ (d) is immediate, since G_F/T is a (G,S)-variety over F.

Let us show that (d) $\Rightarrow$ (c) if G is quasisplit. Let X be a (G,S)-variety. We may assume that X is a (G,S)-fibration. By Proposition 4.5(a) the quotient map $\pi: X \to Y = X/G$ has a rational section $s: X/G \dashrightarrow X$. Let $K = k(Y)$ be the function field of Y. Let η denote the generic point of Y and let X_η be the fibre of π over η. The K-variety X_η is a G_K-homogeneous space with a K-point, namely $s(\eta) \in X_\eta(K)$. Let $T \subset G_K$ be the stabilizer of $s(\eta)$. This is a maximal K-torus in G_K. The G_K-homogeneous space X_η is isomorphic to G_K/T. By our construction, $K(G_K/T) = K(X_\eta) = k(X)$ and $K = k(X)^G$. By (d), $K(G_K/T)$ is rational (respectively, stably rational) over K. Hence, $k(X)$ is rational (respectively, stably rational) over $k(X)^G$. □

Remark 4.11. From the equivalence between (a) and (b) we conclude that the answers to Questions 0.1(a) and (b) stated in the introduction depend only on the isogeny class of the group G.

Remark 4.12. If G_F is split and the maximal F-torus $T \subset G_F$ is also split then it easily follows from the Bruhat decomposition of G that the homogeneous space G_F/T which appears in part (d) of Theorem 4.10 is F-rational; cf. the last paragraph on page 219 in [Bor91]. For general $T \subset G_F$, we shall see in this paper that the quotient G_F/T need not be F-rational.

5. Reduction to the case where the group G is simple and simply connected

In the previous section we have seen that the answers to Questions 0.1(a) and (b) stated in the introduction are the same. Moreover, these answers remain unchanged if we replace G by an isogenous group. In this section we reduce these questions for a general connected split reductive group G to the case where G is split, simple and simply connected.

Proposition 5.1. *Let G be a connected reductive algebraic group over k, Z be a central k-subgroup of G and $\widetilde{G} = G/Z$. Denote the Lie algebras of G and $\widetilde{G}$ by $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ respectively. Then the following properties are equivalent:*

(i) $k(\mathfrak{g})/k(\mathfrak{g})^G$ *is pure (respectively, stably pure);*

(ii) $k(\tilde{\mathfrak{g}})/k(\tilde{\mathfrak{g}})^{\widetilde{G}}$ *is pure (respectively, stably pure).*

Proof. We may assume without loss of generality that Z is the centre of G.

Since Z acts trivially on $\mathfrak{g}$, the adjoint action of G on $\mathfrak{g}$ descends to a $\widetilde{G}$-action, making $\mathfrak{g}$ into a $(\widetilde{G}, \widetilde{S})$-variety, where $\widetilde{S}$ is a maximal torus in the semisimple group $\widetilde{G}$. Thus both $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ are $(\widetilde{G}, \widetilde{S})$-varieties. The action of $\widetilde{G}$ on each of $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ is linear. By Corollary 3.6 both are versal. The desired conclusion now follows from Theorem 3.9(b). □

Taking Z to be the radical (connected centre) of G, we see that Proposition 5.1 reduces Questions 0.1 to the case where G is semisimple. Proposition 5.3 below further reduces it to the case where G is simple.

Lemma 5.2. *Let $G = G_1 \times \cdots \times G_n$, where each G_i is a connected reductive k-group. Denote the Lie algebra of G_i by $\mathfrak{g}_i$ and the Lie algebra of G by $\mathfrak{g} = \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_n$.*

(a) *If $k(\mathfrak{g}_i)/k(\mathfrak{g}_i)^{G_i}$ is pure (respectively, stably pure) for every $i = 1, \ldots, n$, then $k(\mathfrak{g})/k(\mathfrak{g})^G$ is pure (respectively, stably pure).*

(b) *If each G_i is split and $k(\mathfrak{g})/k(\mathfrak{g})^G$ is stably pure, then $k(\mathfrak{g}_i)/k(\mathfrak{g}_i)^{G_i}$ is stably pure for every $i = 1, \ldots, n$.*

Proof. Denote the categorical quotient map for the adjoint G_i-action by $\pi_i : \mathfrak{g}_i \to \mathfrak{g}_i /\!\!/ G_i$. Then the categorical quotient map for the adjoint G-action is

$$\pi = \pi_1 \times \cdots \times \pi_n : \mathfrak{g} = \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_n \to (\mathfrak{g}_1 /\!\!/ G_1) \times \cdots \times (\mathfrak{g}_n /\!\!/ G_n) = \mathfrak{g} /\!\!/ G.$$

Clearly if each $\mathfrak{g}_i$ is rational (respectively, stably rational) over $\mathfrak{g}_i /\!\!/ G_i$ then $\mathfrak{g}$ is rational (respectively, stably rational) over $\mathfrak{g} /\!\!/ G$. This proves part (a).

Let us prove (b). Suppose that $k(\mathfrak{g})/k(\mathfrak{g})^G$ is stably pure. By symmetry it suffices to show that $k(\mathfrak{g}_1)/k(\mathfrak{g}_1)^{G_1}$ is stably pure.

Let S_i be a split maximal torus in G_i. Then $S = S_1 \times \cdots \times S_n$ is a maximal torus in G. Consider the (G, S)-variety $X = \mathfrak{g}_1 \times (G_2/S_2) \times \cdots (G_n/S_n)$, where the G-action on X is the direct product of the adjoint action of G_1 on $\mathfrak{g}_1$ and the left translation action G_i on G_i/S_i. Clearly $X /\!\!/ G = \mathfrak{g}_1 /\!\!/ G_1$ and the quotient map for X is the composition of the projection $\mathrm{pr}_1 : X \to \mathfrak{g}_1$ to the first component and the quotient map $\pi_1 : \mathfrak{g}_1 \to \mathfrak{g}_1 /\!\!/ G_1$:

$$X = \mathfrak{g}_1 \times (G_2/S_2) \times \cdots \times (G_n/S_n) \xrightarrow{\mathrm{pr}_1} \mathfrak{g}_1 \xrightarrow{\pi_1} \mathfrak{g}_1 /\!\!/ G_1.$$

Because G_i and S_i are both split, each G_i/S_i is rational over k; see Remark 4.12. Hence, X is rational over $\mathfrak{g}_1$.

By Proposition 4.3(c) $\mathfrak{g}$ is a versal (G, S)-variety. Since $k(\mathfrak{g})/k(\mathfrak{g})^G$ is stably pure, Theorem 3.9(b) tells us that X is stably rational over $X /\!\!/ G \cong \mathfrak{g}_1 /\!\!/ G_1$. Consequently, $\mathfrak{g}_1$ is stably rational over $\mathfrak{g}_1/G_1$, i.e., $k(\mathfrak{g}_1)$ is stably pure over $k(\mathfrak{g}_1)^{G_1}$, as desired. □

PROPOSITION 5.3. *Let G be a split semisimple group over k. Let $G_1, \ldots, G_n$ denote the simple components of the simply connected cover of G. Denote the Lie algebras of $G, G_1, \ldots, G_n$ by $\mathfrak{g}, \mathfrak{g}_1, \ldots, \mathfrak{g}_n$ respectively.*

(a) *The following properties are equivalent:*
   (i) *$k(\mathfrak{g})/k(\mathfrak{g})^G$ is stably pure;*
   (ii) *$k(\mathfrak{g}_i)/k(\mathfrak{g}_i)^{G_i}$ is stably pure for every i.*

(b) *If $k(\mathfrak{g}_i)/k(\mathfrak{g}_i)^{G_i}$ is pure for every i, then $k(\mathfrak{g})/k(\mathfrak{g})^G$ is pure.*

Proof. The fields $k(\mathfrak{g})$, $k(\mathfrak{g})^G$, $k(\mathfrak{g}_i)$ and $k(\mathfrak{g}_i)^{G_i}$ remain unchanged if we replace G by its simply connected cover. Hence we may assume $G = G_1 \times \cdots \times G_n$, and the proposition follows from Lemma 5.2. □

6. Are homogeneous spaces of the form G/T stably rational?

The rest of this paper will be devoted to proving Theorem 0.2. Using Theorem 4.10 and Proposition 5.1 we may assume without loss of generality that G is simply connected and restate Theorem 0.2 in the following equivalent form.

THEOREM 6.1. *Let G be a split, simple, simply connected algebraic group over k and let T_{gen} be the generic torus of G. (Recall that T_{gen} is defined over the field $K_{\mathrm{gen}} = k(G/N)$; see § 4.2.) Then the homogeneous space $G_{K_{\mathrm{gen}}}/T_{\mathrm{gen}}$ is:*

(a) *rational over K_{gen} if G is split of type A_n $(n \geqslant 1)$ or C_n $(n \geqslant 2)$;*

(b) *not stably rational over K_{gen} if G is not of type A_n $(n \geqslant 1)$ or C_n $(n \geqslant 2)$ or G_2.*

Remark 6.2. For k algebraically closed, the question whether or not the generic torus T_{gen} of G is itself K-rational has been studied in some detail. The (almost) simple groups whose generic torus is (stably) rational are classified in [LL00] (in type A only), [CK00] (for simply connected or adjoint G of all types) and [LPR06] (for all G). A comparison of Theorem 6.1 and [CK00, Theorem 0.1] shows that there is no obvious relation between the (stable) rationality of the homogeneous space G_K/T_{gen} and that of the generic torus T_{gen}. On the other hand, in view of the results of this section one might wonder if the stable rationality of G_K/T_{gen} is related to that of the dual torus T^0_{gen}. We shall return to this question in appendix.

Let X be a K-variety. As usual, we let $\overline{K}$ denote the algebraic closure of K and set $\overline{X} := X_{\overline{K}}$. The ring $\overline{K}[X]$ and the abelian group $\operatorname{Pic}\overline{X}$ then come equipped with the actions of the Galois group of $\overline{K}$ over K.

THEOREM 6.3 [CTK06]. *Let K be a field of arbitrary characteristic, let G be a semisimple, simply connected linear algebraic K-group and let $T \subset G$ be a K-torus. Denote the character lattice of T by T^*.*

(a) *If a K-variety X_{c} is a smooth compactification of $X = G/T$, then there is an exact sequence of Galois lattices*

$$0 \to P \to \operatorname{Pic}\overline{X}_{\mathrm{c}} \to T^* \to 0$$

with P a permutation lattice.

(b) *If G/T is stably K-rational, then there exists an exact sequence of Galois lattices*

$$0 \to P_2 \to P_1 \to T^* \to 0$$

with P_2 and P_1 permutation lattices.

(c) *If G/T is stably K-rational, then $Ш^1_\omega(K, T^*) = 0$.*

Proof. For K of characteristic zero, this is an immediate consequence of the more general result [CTK06, Theorem 5.1]. This special case is easier to prove.

We let here $\overline{K}$ denote a separable closure of K. Associated to the T-torsor $G \to G/T = X$ there is a well known exact sequence of Galois lattices (see [CTS87b, Proposition 2.1.1]):

$$0 \to \overline{K}[X]^\times/\overline{K}^\times \to \overline{K}[G]^\times/\overline{K}^\times \to T^* \to \operatorname{Pic}\overline{X} \to \operatorname{Pic}\overline{G}.$$

Since G is semisimple and simply connected, we have $\overline{K}^\times = \overline{K}[G]^\times$ and $\operatorname{Pic}\overline{G} = 0$ (see [Pop74, Proposition 1]). We thus get $\overline{K}^\times = \overline{K}[X]^\times$ and $T^* \xrightarrow{\cong} \operatorname{Pic}\overline{X}$ (this is a special case of [Pop74, Theorem 4] where $\operatorname{Pic}\overline{G/H}$ for arbitrary subgroup H is described). The open immersion of smooth K-varieties $X \subset X_{\mathrm{c}}$ gives rise to an exact sequence of Galois lattices

$$0 \to \overline{K}[X]^\times/\overline{K}^\times \to \operatorname{Div}_\infty \overline{X}_{\mathrm{c}} \to \operatorname{Pic}\overline{X}_{\mathrm{c}} \to \operatorname{Pic}\overline{X} \to 0.$$

Here $\operatorname{Div}_\infty \overline{X}_{\mathrm{c}}$ is the free abelian group on points of codimension 1 of $\overline{X}_{\mathrm{c}}$ with support in the complement of $\overline{X}$. This is a permutation lattice, call it P. All in all, we get an exact sequence of torsion-free Galois lattices

$$0 \to P \to \operatorname{Pic}\overline{X}_{\mathrm{c}} \to T^* \to 0.$$

This proves (a).

If $\operatorname{char}(K) = 0$, then G/T admits a smooth K-compactification. Statement (b) is then a consequence of (a) and the well known fact that if the smooth, proper K-variety X_{c} is stably K-rational, then the Galois lattice $\operatorname{Pic}(\overline{X}_{\mathrm{c}})$ is a stably permutation lattice (see [CTS87b, Proposition 2.A.1]). Statement (c) then follows from (b) (see § 1).

There is however no need to use a smooth compactification to prove (b) and (c). If $X = G/T$ is stably K-rational, there exist natural integers r and s and dense open sets $U \subset Y = X \times_K \mathbb{A}^r_K$ and $V \subset \mathbb{A}^s_K$ together with a K-isomorphism $U \xrightarrow{\cong} V$. The natural maps $\overline{K}^\times \to \overline{K}[\mathbb{A}^s]^\times$ and $\overline{K}[X]^\times \to \overline{K}[X \times_K \mathbb{A}^r]^\times$ are isomorphisms. By what we have seen above, $\overline{K}^\times \to \overline{K}[X]^\times$ is an isomorphism. Thus $\overline{K}^\times \xrightarrow{\cong} \overline{K}[Y]^\times$. We have $\mathrm{Pic}\, \overline{\mathbb{A}}^s = 0$, hence $\mathrm{Pic}\, \overline{V} = 0$, hence $\mathrm{Pic}\, \overline{U} = 0$. The pull-back map associated to the projection $Y = X \times_K \mathbb{A}^r \to X$ induces an isomorphism of Galois modules $\mathrm{Pic}\, \overline{X} \xrightarrow{\cong} \mathrm{Pic}\, \overline{Y}$. From what we have proved above this induces an isomorphism $T^* \simeq \mathrm{Pic}\, \overline{Y}$.

The open immersion $V \subset \mathbb{A}^s_K$ induces an isomorphism of Galois modules between the permutation module on irreducible divisors of $\mathbb{A}^s_{\overline{K}}$ with support in the complement of $\overline{V}$ and the Galois module $\overline{K}[V]^\times/\overline{K}^\times$, hence the Galois module $\overline{K}[U]^\times/\overline{K}^\times$ is a permutation module. The open immersion $U \subset Y$ induces an exact sequence

$$0 \to \overline{K}[U]^\times/\overline{K}^\times \to \Delta \to \mathrm{Pic}\, \overline{Y} \to 0,$$

where Δ is the permutation module on irreducible divisors of $\overline{Y}$ with support in the complement of $\overline{V}$. This completes the proof of (b), hence also of (c). □

Remark 6.4. In certain circles, the (unramified) Brauer group is a well known K-birational invariant of smooth, projective, geometrically integral K-varieties. For $X \subset X_{\mathrm{c}}$ as above, this is the group $\mathrm{Br}\, X_{\mathrm{c}}$. The connection with the above proposition is given by an isomorphism $\mathrm{Br}\, X_{\mathrm{c}}/\mathrm{Br}\, K \xrightarrow{\cong} H^1(K, \mathrm{Pic}\, \overline{X}_{\mathrm{c}})$, which one combines with an isomorphism $H^1(K, \mathrm{Pic}\, \overline{X}_{\mathrm{c}}) \xrightarrow{\cong} Ш^1_\omega(K, T^*)$ deduced from statement (a) to produce an isomorphism

$$\mathrm{Br}\, X_{\mathrm{c}}/\mathrm{Br}\, K \xrightarrow{\cong} Ш^1_\omega(K, T^*).$$

The interested reader is referred to [CTK06] for details.

Under a strong assumption on the group G, we shall now establish a converse to statement (b) in Theorem 6.3. We first prove a lemma.

Lemma 6.5. *Let K be a field of characteristic* 0 *and let H be a special linear algebraic K-group. If X is a geometrically integral K-variety with a generically free action of H, then X admits a dense H-stable open set U which is H-isomorphic to $H \times_K Y$ for a K-variety Y whose function field $K(Y)$ is K-isomorphic to $K(X)^H$. Here H acts on $H \times_K Y$ via left translation on the first factor.*

Proof. After replacing X by a friendly open set, we may assume that X is the total space of an H-torsor $X \to Y = X/H$. Since H is special, this torsor splits over a dense open set of Y. If we replace Y by this open set, we have $X = H \times_K Y$ as varieties with an H-action. □

Proposition 6.6. *Let K be a field of characteristic zero, and let G be a special, K-rational K-group. Let $T \subset G$ be a K-torus. If there exists an exact sequence of Galois modules*

$$0 \to P_2 \to P_1 \to T^* \to 0$$

with P_2 and P_1 permutation modules, then G/T is stably K-rational.

Proof. By assumption we have an exact sequence of K-tori

$$1 \to T \to T_1 \to T_2 \to 1$$

with T_1 and T_2 quasitrivial. We now identify T with the diagonal subgroup T of $G \times T_1$, and G with the subgroup $G \times \{1\}$ of $G \times T_1$. The first projection $G \times T_1 \to G$ induces a map $(G \times T_1)/T \to G/T$ which makes $(G \times T_1)/T$ into a right T_1-torsor over G/T. The second projection $G \times T_1 \to T_1$ induces a map $(G \times T_1)/T \to T_1/T$ which makes $(G \times T_1)/T$ into a left G-torsor over $T_1/T = T_2$. In summary, we have the following diagram.

$$\begin{tikzcd} & (G \times T_1)/T \arrow[dl, "\text{right } T_1\text{-torsor}"'] \arrow[dr, "\text{left } G\text{-torsor}"] & \\ G/T & & T_1/T = T_2 \end{tikzcd}$$

Since T_1 is a quasitrivial torus hence a special K-group, $(G \times T_1)/T$ is K-birationally isomorphic to $T_1 \times (G/T)$; see Lemma 6.5. Similarly, since G is special, $(G \times T_1)/T$ is K-birationally isomorphic to $G \times T_2$. Thus $T_1 \times (G/T)$ is K-birationally isomorphic to $G \times T_2$. Since T_1, T_2 and G are K-rational varieties, we conclude that G/T is stably K-rational. □

We now specialize Theorem 6.3 and Proposition 6.6 to the setting of Theorem 6.1.

The weight lattice $P(R)$ and the root lattice $Q(R)$ of a root system R of type R are equipped with the natural actions of the Weyl group W. We also denote these lattices by $P(\mathsf{R})$ and $Q(\mathsf{R})$ respectively.

Corollary 6.7. *Let G be a split, simple, simply connected algebraic group of type R, defined over k and T_{gen} be the generic torus of G. Recall that T_{gen} is defined over the field $K_{\text{gen}} = k(G/N)$; cf. § 4.2.*

(a) *If the homogeneous space $G_{K_{\text{gen}}}/T_{\text{gen}}$ is stably rational over K_{gen}, then there exists an exact sequence*

$$0 \to P_2 \to P_1 \to P(\mathsf{R}) \to 0 \tag{6.1}$$

of W-lattices, where P_1 and P_2 are permutation.

(b) *Suppose G is special. Then the converse to part (a) holds. That is, if there exists an exact sequence (6.1) of W-lattices with P_1 and P_2 permutation lattices, then $G_{K_{\text{gen}}}/T_{\text{gen}}$ is stably rational over K_{gen}.*

Proof. As recalled in § 4.2, the K_{gen}-torus T_{gen} splits over a Galois extension of K_{gen} with Galois group the Weyl group W, and the character lattice of T_{gen} is isomorphic to the weight lattice $P(\mathrm{R})$ with its natural W-action. Note also that since G is split, G is rational over k and hence $G_{K_{\text{gen}}}$ is rational over K_{gen}. Applying Theorem 6.3(b) and Proposition 6.6 to $G_{K_{\text{gen}}}$ and T_{gen} and using the correspondence between tori and lattices (§ 1) we get (a) and (b). □

7. Nonrationality

In this section we shall prove Theorem 0.2(b) or equivalently, Theorem 6.1(b). To prove the latter, one may assume that k is algebraically closed. In view of Corollary 6.7(a), it suffices to establish the following proposition.

Proposition 7.1. *Let R be a reduced, irreducible root system in a real vector space V that it spans. Let $P(R)$ be the weight lattice equipped with the action of the Weyl group $W = W(R)$. If R is not of type A_n, C_n or G_2, then there exists a subgroup $H \cong (\mathbb{Z}/2\mathbb{Z})^2$ in W such that $Ш^1_\omega(H, P(R)) \neq 0$. In particular, there does not exist an exact sequence of W-lattices*

$$0 \to P_2 \to P_1 \to P(R) \to 0$$

with P_1 and P_2 permutation lattices.

Proof. Let B be a basis of R. Let $W = W(R)$ be the Weyl group. The abelian group $Q(R) \subset V$ spanned by R is the root lattice, its rank is $l := \dim(V)$. There is an inclusion $Q(R) \subset P(R) \subset V$, where $P(R)$ is the weight lattice. See [Bou68, VI. 1.9]. Both $Q(R)$ and $P(R)$ are W-lattices.

Let $B' \subset B$ be a subset of B of cardinality l', let $V' \subset V$ be the vector space spanned by B' and let $R' = R \cap V'$. This R' is a root system in V', B' is a basis of R', and $Q(R') = Q(R) \cap V'$. See [Bou68, VI.1.7, Corollary 4, p. 162].

This implies that $Q(R')$ is a direct factor of $Q(R)$ (as an abelian group). Moreover, since $W(R')$ is generated by the reflections in the hyperplanes orthogonal to the roots $\alpha \in R'$, $W(R')$ can naturally be viewed as a subgroup of $W(R)$. From the formula $s_\alpha(\beta) = \beta - n_{\beta,\alpha}\alpha$ we see that $W(R')$ acts trivially on V/V' and hence on $Q(R)/Q(R')$. We write $Q(R)/Q(R') = \mathbb{Z}^{l-l'}$, the trivial $W(R')$-lattice. In other words, there is a short exact sequence of $W(R')$-lattices

$$0 \to Q(R') \to Q(R) \to \mathbb{Z}^{l-l'} \to 0. \tag{7.1}$$

Our proof of Proposition 7.1 will rely on the following claim.

Claim 7.2. *For a root system R dual (inverse in the terminology of* [Bou68, VI 1.1]*) to one of the root systems occurring in the statement of the proposition, there exist a subsystem $R' \subset R$ (as above), a subgroup $H \cong (\mathbb{Z}/2\mathbb{Z})^2$ in $W(R')$ and a direct factor J_H of the H-lattice $Q(R')$, where J_H is the cokernel of the map $\mathbb{Z} \to \mathbb{Z}[H]$ given by the norm.*

Indeed, assume Claim 7.2 is established. Consider the exact sequences

$$0 \to \mathbb{Z} \to \mathbb{Z}[H] \to J_H \to 0,$$
$$0 \to I_H \to \mathbb{Z}[H] \to \mathbb{Z} \to 0,$$

where the map $\mathbb{Z}[H] \to \mathbb{Z}$ is augmentation. The latter sequence yields $Ш^1_\omega(H, I_H) \cong \mathbb{Z}/2\mathbb{Z}$.

The Weyl groups of a root system R and of its dual $R^\vee$ are identical. Exact sequence (7.1) induces an exact sequence of $W(R')$-lattices (the last two are weight lattices)

$$0 \to \mathbb{Z}^{l-l'} \to P(R^\vee) \to P(R'^\vee) \to 0, \tag{7.2}$$

which we view as an exact sequence of H-lattices. Here $R^\vee$ is a root system as occurring in the proposition to be proved, i.e., a root system not of type A_n, C_n or G_2. (Recall that by the assumption G is simply connected, hence the W-lattice given by the character group of a maximal torus is the weight lattice.)

Using Claim 7.2 we conclude that the H-lattice $P(R'^\vee)$ (dual to $Q(R')$) contains the H-lattice I_H (dual to J_H) as a direct factor, hence $Ш^1_\omega(H, P(R'^\vee)) \neq 0$. From the exact sequence (7.2) we get, by a standard computation,

$$Ш^1_\omega(H, P(R^\vee)) \cong Ш^1_\omega(H, P(R'^\vee))$$

hence

$$Ш^1_\omega(H, P(R^\vee)) \neq 0.$$

To complete the proof of Proposition 7.1 it remains to establish Claim 7.2.

Proof of Claim 7.2. The root systems R dual to those considered in the proposition are those of types C_n $(n \geqslant 3)$, D_n $(n \geqslant 4)$, E_r $(r = 6, 7, 8)$ and F_4.

Any Dynkin diagram of type $\mathsf{C}_n (n \geqslant 3)$ contains a subdiagram of type C_3. All the other ones in the list above, except F_4, contain a subdiagram of type D_4. The case of F_4 can be reduced to D_4 because the weight lattices $P(\mathsf{F}_4)$ and $P(\mathsf{D}_4)$ coincide (as abelian groups in $\mathbb{R}^4$) and $W(\mathsf{D}_4) \subset W(\mathsf{F}_4) \subset \mathrm{GL}_4(\mathbb{R})$ (compare Planche IV and Planche VIII in [Bou68]).

For the reader's convenience, we reproduce some of the calculations from [CK00].

Recall that $W(\mathsf{C}_n)$ is a semidirect product $(\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$. We denote by $c_1, \ldots, c_n$ the natural generators of $(\mathbb{Z}/2\mathbb{Z})^n$.

Let us first discuss the case where R is of type C_3. We choose $H = \langle c_1c_3, c_2(13)\rangle = \langle a, b\rangle \subset W(\mathsf{C}_3)$. In the basis $\alpha_1 = \varepsilon_1 - \varepsilon_2$, $\alpha_2 = \varepsilon_2 - \varepsilon_3$, $\alpha_3 = \varepsilon_2 + \varepsilon_3$, the group H acts on $M = Q(\mathsf{C}_3)$ as follows:

$$a : \begin{cases} \alpha_1 \mapsto -\alpha_1 - \alpha_2 - \alpha_3, \\ \alpha_2 \mapsto \alpha_3, \\ \alpha_3 \mapsto \alpha_2, \end{cases} \qquad b : \begin{cases} \alpha_1 \mapsto \alpha_3, \\ \alpha_2 \mapsto -\alpha_1 - \alpha_2 - \alpha_3, \\ \alpha_3 \mapsto \alpha_1. \end{cases}$$

This coincides with the standard formulas for J_H.

Let us now discuss the case where R is of type D_4. In $\mathbb{R}^4$ equipped with the standard basis $\varepsilon_1, \ldots, \varepsilon_4$, we consider $M = Q(\mathsf{D}_4)$ with $\mathbb{Z}$-basis $\alpha_1 = \varepsilon_1 - \varepsilon_2$, $\alpha_2 = \varepsilon_2 - \varepsilon_3$, $\alpha_3 = \varepsilon_3 - \varepsilon_4$, $\alpha_4 = \varepsilon_3 + \varepsilon_4$. The Weyl group $W(\mathsf{D}_4)$ can be identified with the subgroup of $W(\mathsf{C}_4)$ consisting of the elements with even numbers of c_i's. We choose $H = \langle c_3c_4, c_1c_2(34)\rangle$. The group H acting on M respects $V' = \langle \varepsilon_2, \varepsilon_3, \varepsilon_4\rangle = \langle \alpha_2, \alpha_3, \alpha_4\rangle$, and $R' = R \cap V'$ is of type D_3. Moreover, H respects the one-dimensional $\mathbb{Z}$-module generated by α_1: indeed, c_3c_4 fixes α_1 and $c_1c_2(34)$ sends α_1 to $-\alpha_1$. Therefore the H-lattice M decomposes into a direct sum of a one-dimensional lattice and a three-dimensional lattice. It remains to note that the latter three-dimensional lattice J is isomorphic to J_H. To see that, we observe that the action of $c_1c_2(34)$ on J coincides with the action of $c_2(34) \in W(\mathsf{C}_3)$ on $Q(\mathsf{C}_3) = Q(\mathsf{D}_3)$, and we are led (up to permutation of indices) to the former case.

This completes the proof of Claim 7.2, hence of Proposition 7.1, hence of Theorems 6.1(b) and 0.2(b). □

Remark 7.3. Our proof of Theorem 6.1(b) actually establishes the following stronger assertion.

Proposition 7.4. *Let G be a simple simply connected linear algebraic group over k which is not of type A_n, C_n or G_2. Let T_{gen} be the generic torus of G. Recall that T_{gen} is defined over the field $K_{\mathrm{gen}} = k(G/N)$; cf. § 4.2. Then $(G_{K_{\mathrm{gen}}}/T_{\mathrm{gen}}) \times_{K_{\mathrm{gen}}} Y$ is not rational over K_{gen} for any K_{gen}-variety Y.*

Indeed, let X_{c} be a smooth K_{gen}-compactification of $X = G_{K_{\mathrm{gen}}}/T_{\mathrm{gen}}$. Combining Theorem 6.3, Remark 6.4 and Proposition 7.1 we find that there exists a finite field extension M/K_{gen} such that $\mathrm{Br}(X_{\mathrm{c}})_M/\,\mathrm{Br}\, M \neq 0$. On the other hand, if there exists a K_{gen}-variety Y such that $X \times_{K_{\mathrm{gen}}} Y$ is K_{gen}-birationally isomorphic to projective space, then $\mathrm{Br}(X_{\mathrm{c}})_M/\mathrm{Br}\, M = 0$ for any field extension M/K_{gen}. As a matter of fact, the nonvanishing of $\mathrm{Br}(X_{\mathrm{c}})_M/\mathrm{Br}\, M$ implies that the K_{gen}-variety $X = G_{K_{\mathrm{gen}}}/T_{\mathrm{gen}}$ is not even retract rational (a concept due to D. Saltman); see [CTS07, § 1, Proposition 5.7 and Remark 5.8]. □

8. Weight lattices for root systems of types A_n, C_n, and G_2

In this section we shall prove the following converse to Proposition 7.1.

Proposition 8.1. *Let R be a reduced, irreducible root system and $P(R)$ be the weight lattice equipped with the action of the Weyl group $W = W(R)$. If R is of type A_n, C_n or G_2, then there* *exists an exact sequence of W-lattices*

$$0 \to P_2 \to P_1 \to P(R) \to 0$$

with P_1 and P_2 permutation lattices.

Proof. Suppose R is of type G_2. Here the W-lattice $M = P(\mathsf{G}_2)$ is of rank two. Thus, as we pointed out at the end of § 1, M fits into an exact sequence

$$0 \to P_2 \to P_1 \to M \to 0,$$

with P_1 and P_2 permutation.

Now suppose R is of type A_n. Then $W = S_{n+1}$. From the Bourbaki tables [Bou68] we get the exact sequence of S_{n+1}-lattices

$$0 \to Q(\mathsf{A}_n) \to \oplus_{i=1}^{n+1} \mathbb{Z}\varepsilon_i \to \mathbb{Z} \to 0,$$

where the action of S_{n+1} on the middle term is by permutation, the action on the right-hand side $\mathbb{Z}$ is trivial and the right-hand side map to $\mathbb{Z}$ is augmentation, i.e., summation of coefficients.

If one dualizes this sequence one gets an exact sequence of S_{n+1}-lattices

$$0 \to \mathbb{Z} \to \oplus_{i=1}^{n+1} \mathbb{Z}\varepsilon_i \to P(\mathsf{A}_n) \to 0, \tag{8.1}$$

where the action of S_{n+1} on the middle term is by permutation, the action on the left-hand side $\mathbb{Z}$ is trivial and the map with source $\mathbb{Z}$ sends 1 to the sum of the ε_i.

Finally suppose R is of type C_n ($n \geqslant 2$). Then W is the semidirect product of S_n by $(\mathbb{Z}/2\mathbb{Z})^n$. Let us first look at the B_n-table in [Bou68]. There is an exact sequence of $W(\mathsf{B}_n)$-lattices

$$0 \to Q(\mathsf{B}_n) \to \oplus_{i=1}^{n} (\mathbb{Z}a_i \oplus \mathbb{Z}b_i) \to \oplus_{i=1}^{n} \mathbb{Z}c_i \to 0,$$

where a_i and b_i are mapped to c_i, and the action of W is as follows. On the right-hand lattice $\oplus_{i=1}^{n} \mathbb{Z}c_i$, the action is the permutation action of the quotient S_n. On the middle lattice, S_n acts by naturally permuting the a_i and the b_i. An element $(\alpha_1, \ldots, \alpha_n) \in (\mathbb{Z}/2\mathbb{Z})^n$ fixes a_i and b_i if $\alpha_i = 0$ and it permutes them if $\alpha_i = 1$. In Bourbaki's notation for B_n, we have $\varepsilon_i = a_i - b_i$. If one dualizes the above sequence, one gets the exact sequence of $W(\mathsf{C}_n)$-lattices

$$0 \to \oplus_{i=1}^{n} \mathbb{Z}\gamma_i \to \oplus_{i=1}^{n} (\mathbb{Z}\alpha_i \oplus \mathbb{Z}\beta_i) \to P(\mathsf{C}_n) \to 0, \tag{8.2}$$

where the two left lattices are permutation lattices. □

Proposition 8.2. *Let G be a split, simply connected simple group of type A_n or C_n (i.e., $G = \mathrm{SL}_n$ or Sp_{2n}). Then the field extensions $k(G)/k(G)^G$ and $k(\mathfrak{g})/k(\mathfrak{g})^G$ are stably pure.*

Proof. Since these groups are special, Propositions 6.6 and 8.1 imply that G_K/T_{gen} is stably rational over K_{gen} (or equivalently, $k(\mathfrak{g})$ is stably rational over $k(\mathfrak{g})^G$). The statement then follows from Theorem 4.10. □

Remark 8.3. This is weaker than the rationality assertion of Theorem 6.1(a) (or equivalently, of Theorem 0.2(a)), which will be proved in the next section. In the meantime, we remark that the same argument cannot be used to show that G_K/T_{gen} is stably rational (or equivalently, $k(\mathfrak{g})$ is stably rational over $k(\mathfrak{g})^G$) for the split G_2, because this group is not special and Proposition 6.6 does not apply to it. In fact, in this case we do not know whether or not G_K/T_{gen} is stably rational over K_{gen}.

9. Rationality

We now turn to the proof of Theorem 6.1(a) (or equivalently, of Theorem 0.2(a)). Our argument will be based on the rationality criterion of Lemma 9.1(c) below. For groups of type A_n, Theorem 6.1(a) will be an easy consequence of this criterion. For groups of type C_n the proof proceeds along the same lines but requires a more elaborate argument.

Let K be a field of characteristic zero, let G be a linear algebraic group over K, and let H_1, H_2 be closed K-subgroups. The actions of H_1 and H_2 on G by respectively left and right translation commute, thus giving rise to an $(H_1 \times H_2)$-action on G. The action of H_1 on G defines an H_1-torsor $\pi_1 : G \to H_1\backslash G$, and the action of H_2 defines an H_2-torsor $\pi_2 : G \to G/H_2$.

Using Rosenlicht's theorem (see § 2.1), one may find an $(H_1 \times H_2)$-stable dense open subset U of G such that the action of $H_1 \times H_2$ on U mods out to a *geometric quotient* $U \to V$ which factorizes through $U \to U_1 \to V$ and $U \to U_2 \to V$, where $U_1 \subset H_1\backslash G$ and $U_2 \subset G/H_2$ are open sets and $U \to U_1$, respectively, $U \to U_2$ is an H_1-torsor, respectively, an H_2-torsor.

In this section we shall indulge in the following notation. We shall adopt the double coset notation $H_1\backslash G/H_2$ for *some* V as above. In particular, we have

$$K(H_1\backslash G/H_2) \cong K(G)^{H_1 \times H_2}.$$

We have a commutative diagram of rational maps

$$\begin{tikzcd} & G \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[dd, dashed, "\pi"] & \\ H_1\backslash G \arrow[dr, dashed] & & G/H_2, \arrow[dl, dashed] \\ & H_1\backslash G/H_2 & \end{tikzcd}$$

where, as usual, solid arrows denote regular maps and dotted arrows denote rational maps. Let $Z(G)$ denote the centre of G.

LEMMA 9.1. *Let K be a field of characteristic zero and let G be a simple K-group. Suppose G has subgroups H and T, where T is a torus and $H \cap Z(G) = \{1\}$. Then:*

(a) *the H-action on G/T is generically free;*

(b) $\dim(H\backslash G/T) = \dim(G) - \dim(H) - \dim(T)$*;*

(c) *if H is special and both H, $H\backslash G/T$ are K-rational, then G/T is also K-rational.*

Proof. To prove (a), we may pass to an algebraic closure $\overline{K}$ of K and thus assume, without loss of generality, that K is algebraically closed.

Note that conditions:

(a) the left H-action on G/T is generically free;

(a′) the right T-action on $H\backslash G$ is generically free

are equivalent. Indeed, (a) says that $H \cap gTg^{-1} = \{1\}$ for $g \in G$ in general position and (a′) says that $T \cap gHg^{-1} = \{1\}$ for $g \in G$ in general position. Thus (a) and (a′) are equivalent, and it suffices to prove (a′).

Assume to the contrary that the T-action on $H\backslash G$ is not generically free. By a result of Sumihiro [Sum74] there is an affine T-stable dense open set U of $H\backslash G$. By the embedding theorem [PV94, Theorem 1.5], U is a G-stable closed irreducible subvariety of a finite-dimensional T-module V not contained in a proper T-submodule of V. Hence U intersects the complement to the union of weight spaces of T. But the stabilizer of a point in this complement coincides with the kernel of the action of T on V. Thus the action of T on U, hence on $H\backslash G$ has a nontrivial kernel $\Gamma \subset T$, cf. [PV94, § 7.2, Proposition]. Then Γ is contained in $N = \bigcap_{g\in G} gHg^{-1}$, which is a normal subgroup of G. Since $N \subsetneq G$ and we are assuming that G is simple, we conclude that $N \subset Z(G)$. Thus $\{1\} \subsetneq \Gamma \subset N = N \cap Z(G) \subset H \cap Z(G)$, contradicting our assumption that $H \cap Z(G) = \{1\}$. This contradiction proves part (a).

By Theorem 2.13 (case $S = \{1\}$) and the standard formula for the dimension of a variety fibred over another variety, (b) follows from (a).

Let us now prove (c). Part (a) allows us to apply Lemma 6.5 to the left H-action on G/T. By Lemma 6.5, G/T is K-birationally isomorphic to $H \times (H\backslash G/T)$. Since we are assuming that both H and $H\backslash G/T$ are K-rational, so is G/T. □

We now proceed with the proof of Theorem 6.1(a). The group G is a split simply connected simple group of type A_n or C_n over the field k. In the sequel we will set $K = K_{\mathrm{gen}}$ and will work with the generic K-torus $T = T_{\mathrm{gen}} \subset G_K$, as in § 4.2.

Type A_n. Let H denote the stabilizer of a nonzero element for the natural action of $G = \mathrm{SL}_{n+1}$ on $\mathbb{A}_k^{n+1}$. It is easy to see that H is isomorphic to a semidirect product $U \rtimes \mathrm{SL}_n$, for some unipotent group U and that $H \cap Z(\mathrm{SL}_{n+1}) = 1$. By Lemma 9.1(c), it suffices to show that (i) H is special, (ii) H is k-rational, and (iii) the 'double coset space' $H_K\backslash G_K/T_{\mathrm{gen}}$ (whose definition is explained above) is K-rational. We now proceed to prove (i), (ii) and (iii).

(i) For any field extension F/k, the natural map

$$H^1(F, H) \to H^1(F, \mathrm{SL}_n)$$

is an isomorphism; see, e.g., [San81, Lemme 1.13]. Since SL_n is a special group, we conclude that $H^1(F, H) \cong H^1(F, \mathrm{SL}_n) = \{1\}$, i.e., H is special.

(ii) In characteristic zero, any unipotent group is special (see [Ser94, Proposition III.2.1.6]) and rational (see, e.g., [LPR06, Example 1.21]). Viewing the natural projection $H \to \mathrm{SL}_n$ as a U-torsor over SL_n, we see that H is k-birationally isomorphic to $U \times \mathrm{SL}_n$. This shows that H is k-rational.

(iii) $H_K\backslash G_K/T_{\mathrm{gen}}$ is a one-dimensional K-variety; see Lemma 9.1(b). It is clearly unirational over K (it is covered by G_K). By Lüroth's theorem it is thus K-rational.

This completes the proof of Theorem 6.1 for groups of type A_n.

Type C_n. Let $G := \mathrm{Sp}_{2n}$. Once again, we let H be the Sp_{2n}-stabilizer of a nonzero vector $v \in k^{2n}$ for the natural action of G on $\mathbb{A}^{2n}$. It is well known that H is k-isomorphic to a semidirect product of $U \rtimes \mathrm{Sp}_{2n-2}$, where U is a unipotent group defined over k; see, e.g., [Wei65, pp. 35–36] or [Igu73, p. 384]. Once again, by Lemma 9.1(c), it suffices to show that:

(i) H is special;

(ii) H is k-rational;

(iii) $H_K\backslash G_K/T_{\mathrm{gen}}$ is K-rational.

The proofs of (i) and (ii) are exactly the same as for type A. In order to complete the proof of Theorem 6.1(a) (or equivalently, of Theorem 0.2(a)), it thus remains to establish (iii), which we now restate as a proposition.

Proposition 9.2. *$H_K\backslash G_K/T_{\mathrm{gen}}$ is K-rational.*

To prove the proposition, we first note that $H\backslash G$ is, by definition, the Sp_{2n}-orbit of a nonzero element v in $\mathbb{A}^{2n}$. By Witt's extension theorem [Art57, Theorem 3.9], this orbit is $\mathbb{A}^{2n}\backslash\{0\}$. Thus the group Sp_{2n} acts on $H_K\backslash G_K$ (on the right) via its natural, linear, action on $\mathbb{A}^{2n}_K$. Restricting this action to T_{gen}, we reduce our problem to showing that $\mathbb{A}^{2n}/T_{\mathrm{gen}}$ is K-rational.

The K-torus T_{gen} is split by a W-Galois extension and the character lattice of T_{gen} is the weight lattice $P(\mathsf{C}_n)$ with its W-action (see § 4.2). Here $W=(\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$ is the Weyl group of $G=\mathrm{Sp}_{2n}$. Over L we can diagonalize the T_{gen}-action on $\mathbb{A}^{2n}$ in some L-basis $f_1,\ldots,f_{2n}$. Let $\chi_1,\ldots,\chi_{2n}$ be the associated characters of $T_{\mathrm{gen}}\times_K L$. We have $t\cdot f_i=\chi_i(t)f_i$ for every $t\in T_{\mathrm{gen}}(L)$. These characters are permuted by W; denote the associated W-permutation lattice of rank $2n$ by P_1. That is, W permutes a set of generators $a_1,\ldots,a_{2n}$ of P_1; sending a_i to χ_i, we obtain a morphism of W-lattices

$$\tau : P_1 \to P(\mathsf{C}_n).$$

Since $G=\mathrm{Sp}_{2n}$ acts faithfully on $\mathbb{A}^{2n}$, so does $T_{\mathrm{gen}}\subset G_K$; hence τ is surjective and we obtain a sequence of W-lattices

$$0\to P_2\to P_1\to P(\mathsf{C}_n)\to 0$$

and the dual sequence

$$1\to T_{\mathrm{gen}}\to T_1\to T_2\to 1$$

of K-tori. The torus T_1 has a dense open orbit in $\mathbb{A}^{2n}$; identifying T_1 with this orbit, we obtain the following birational isomorphisms of K-varieties:

$$\mathbb{A}^{2n}_K/T_{\mathrm{gen}}\cong T_1/T_{\mathrm{gen}}\cong T_2.$$

It thus remains to show that the n-dimensional torus T_2 is rational over K. Since every torus of dimension $\leqslant 2$ is rational, we may assume without loss of generality that $n\geqslant 3$. We have thus reduced Proposition 9.2 to the following lemma.

Lemma 9.3. *Let $n\geqslant 3$, let $W=W(\mathsf{C}_n)=(\mathbb{Z}/2\mathbb{Z})^n\rtimes S_n$, let P be a permutation W-lattice of rank $2n$ and let*

$$0\to M\to P\xrightarrow{\varphi} P(\mathsf{C}_n)\to 0$$

be an exact sequence of W-lattices. Then there exists a W-isomorphism between this sequence and sequence (8.2). In particular, M is a permutation lattice.

Recall that $P(\mathsf{C}_n)$ has a basis $h_1,\ldots,h_n$ such that $c=(c_1,\ldots,c_n)\in(\mathbb{Z}/2\mathbb{Z})^n$ acts on $P(\mathsf{C}_n)$ by $h_i\mapsto(-1)^{c_i}h_i$ and S_n permutes $h_1,\ldots,h_n$ in the natural way.

Proof of Lemma 9.3. Denote the normal subgroup $(\mathbb{Z}/2\mathbb{Z})^n$ of W by A. We shall identify the dual group $A^*=\mathrm{Hom}(A,\mathbb{Z}/2\mathbb{Z})$ with $(\mathbb{Z}/2\mathbb{Z})^n$ in the usual way. That is, $(b_1,\ldots,b_n)\in(\mathbb{Z}/2\mathbb{Z})^n$ will denote the additive character $\chi : A\to\mathbb{Z}/2\mathbb{Z}$ taking $(a_1,\ldots,a_n)\in A$ to $b_1a_1+\cdots+b_na_n\in\mathbb{Z}/2\mathbb{Z}$.

We now observe that $P(\mathsf{C}_n)\otimes\mathbb{Q}$ decomposes as the direct sum of n one-dimensional A-invariant $\mathbb{Q}$-subspaces, upon which A acts by the characters

$$\chi_1=(1,0,\ldots,0),\ldots,\chi_n=(0,\ldots,0,1).$$

Since the exact sequence

$$0\to M\otimes\mathbb{Q}\to P\otimes\mathbb{Q}\to P(\mathsf{C}_n)\otimes\mathbb{Q}\to 0$$

of A-modules over $\mathbb{Q}$ splits, all of these characters will be present in the irreducible decomposition of $P\otimes\mathbb{Q}$ (as an A-module over $\mathbb{Q}$). Since P is a permutation W-module, the trivial character will also occur in this decomposition; denote its multiplicity by d, where $1 \leqslant d \leqslant n$. The set Λ of the remaining $n-d$ characters is permuted by S_n. Since the orbit of a character $(b_1, \dots, b_n) \in A^*$ has $\binom{n}{i}$ elements, where i is the number of times 1 occurs among $b_1, \dots, b_n \in \mathbb{Z}/2\mathbb{Z}$, and $\binom{n}{i} \geqslant n$ for any $1 \leqslant i \leqslant n-1$, we conclude that $(b_1, \dots, b_n)$ can be in Λ only if $i = n$, i.e., $b_1 = \cdots = b_n = 1$. In summary, $P \otimes \mathbb{Q}$, viewed as an A-module over $\mathbb{Q}$, is the direct sum of the following characters:

$$\begin{aligned} &(0, \dots, 0), \text{ with multiplicity } d,\\ &(1, \dots, 1), \text{ with multiplicity } n-d,\\ &\chi_1 = (1, 0, \dots, 0), \dots, \chi_n = (0, \dots, 0, 1), \text{ each with multiplicity } 1, \end{aligned} \tag{9.1}$$

where $d \geqslant 1$ is an integer.

In order to gain greater insight into the A-action on P and in particular to determine the exact value of d, we shall now compute the irreducible decomposition of $P \otimes \mathbb{Q}$ (as an A-module over $\mathbb{Q}$) in a different way. Recall that P is a permutation W-lattice. In particular, we may write

$$P \cong \mathbb{Z}[A/A_1] \oplus \cdots \oplus \mathbb{Z}[A/A_r],$$

where $\cong$ denotes an isomorphism of A-lattices. Now observe that, as an A-module over $\mathbb{Q}$, $\mathbb{Q}[A/A_i]$ is the sum of all those characters of A which vanish on A_i. Denote the set of all such characters by $\Lambda_i \subset (\mathbb{Z}/2\mathbb{Z})^n$. Thus the A-module $P \otimes \mathbb{Q}$, which we know is the direct sum of the $2n$ characters listed in (9.1), can also be written as the direct sum of the characters in $\Lambda_1, \dots, \Lambda_r$. Here some of the characters may appear with multiplicity $\geqslant 2$; note that $|\Lambda_1| + \cdots + |\Lambda_r| = 2n$ (here $|\Lambda_i|$ denotes the order of Λ_i). In other words, the $2n$ characters listed in (9.1) can be partitioned into r subsets $\Lambda_1, \dots, \Lambda_r$.

Note that each Λ_i is clearly a subgroup of $(\mathbb{Z}/2\mathbb{Z})^n$. On the other hand, since $\chi_l + \chi_m$ is not on our list (9.1) for any $l \neq m$ (here we are using the assumption that $n \geqslant 3$), we see that no two of the characters $\chi_1, \dots, \chi_n$ can be contained in the same Λ_i. This implies $r \geqslant n$. After possibly relabeling the subgroups $A_1, \dots, A_r$ and $\Lambda_1, \dots, \Lambda_r$, we may assume that $\chi_i \in \Lambda_i$ for $i = 1, \dots, n$. Since Λ_i is a subgroup of $(\mathbb{Z}/2\mathbb{Z})^n$, each Λ_i should also contain the trivial character. This shows that the irreducible decomposition of $P \otimes \mathbb{Q}$ (as an A-module) contains at least n copies of the trivial character $(0, \dots, 0)$. We conclude that $n - d = 0$ in (9.1), $r = n$, $\Lambda_i = \{(0, \dots, 0), \chi_i\}$, and

$$A_i = \operatorname{Ker}(\chi_i) = (\mathbb{Z}/2\mathbb{Z}) \times \cdots \times (\mathbb{Z}/2\mathbb{Z}) \times \{1\} \times (\mathbb{Z}/2\mathbb{Z}) \times \cdots \times (\mathbb{Z}/2\mathbb{Z}),$$

where $\{1\}$ occurs in the ith position.

We now return to the permutation W-lattice P. Let $e_1, f_1, \dots, e_n, f_n$ be a $\mathbb{Z}$-basis of P permuted by W. As we saw above, the permutation action of A on this basis is isomorphic to that on $A/A_1 \cup \cdots \cup A/A_n$. After suitably relabeling the basis elements, we may thus assume that e_i and f_i are the two elements of our basis fixed by A_i. Clearly A permutes $\{e_i, f_i\}$. On the other hand, since conjugation by S_n naturally permutes the subgroups $A_1, \dots, A_n$ of A, it also naturally permutes the (unordered) pairs $\{e_i, f_i\}$.

Since A_i fixes e_i and f_i, the elements $\varphi(e_i)$ and $\varphi(f_i)$ lie in $P(\mathsf{C}_n)^{A_i} = \mathbb{Z}h_i$. In other words, $\varphi(e_i) = m_i h_i$ and $\varphi(f_i) = -m_i h_i$ for some $m_1, \dots, m_n \in \mathbb{Z}$. Since φ is surjective, $m_i = \pm 1$ for each i. After interchanging e_i and f_i if necessary, we may assume that $m_1 = \cdots = m_n = 1$. Now S_n permutes both $\{e_1, \dots, e_n\}$ and $\{f_1, \dots, f_n\}$ in the natural way. Identifying $e_i \in P$ with α_i, f_i with β_i, we see that the exact sequence

$$0 \to M \to P \xrightarrow{\varphi} P(\mathsf{C}_n) \to 0$$

is W-equivariantly isomorphic to the sequence (8.2). $\square$

This completes the proof of Proposition 9.2, hence of Theorem 0.2(a).

10. Appendix: G/T versus T^0

Let G be a semisimple simply connected group defined over a field K and let $T \subset G$ be a K-torus. As we mentioned in Remark 6.2, there is no obvious connection between the (stable) K-rationality of G/T and that of T.

However, using the results on tori recalled in § 1.2, Theorem 6.3(d) may be rephrased in the following manner: if G/T is stably K-rational, then the dual torus T^0 is stably K-rational. Similar, Proposition 6.6 can be rephrased as follows: suppose G is a special, split K-group and $T \subset G$ is a K-torus. If the dual torus T^0 is stably K-rational, then G/T is stably K-rational.

One might then wonder if, when G is split (but not necessarily special), the (stable) K-rationality of the dual torus T^0 implies that of G/T. This is an open question; a positive answer would yield the stable rationality in the missing case G_2 in our main Theorem 0.2 or (equivalently, in Theorem 6.1).

One may go even further and ask whether or not G/T and T^0 are always stably K-birationally isomorphic (assuming G is split). The purpose of this appendix is to show that this stronger assertion is false.

Proposition 10.1. *There exist a K-torus T and a split semisimple simply connected group G with $T \subset G$ such that G/T is not stably K-birationally isomorphic to the dual torus T^0.*

Proof. Let K be a field of characteristic zero and let L/K be a finite Galois extension of fields with group Γ. The augmentation map $\mathbb{Z}[\Gamma] \to \mathbb{Z}$ gives rise to the exact sequence of Γ-lattices

$$0 \to I_\Gamma \to \mathbb{Z}[\Gamma] \to \mathbb{Z} \to 0.$$

The norm map $1 \to N_\Gamma = \sum_{g\in\Gamma} g \in \mathbb{Z}[\Gamma]$ gives rise to the exact sequence of Γ-lattices

$$0 \to \mathbb{Z} \to \mathbb{Z}[\Gamma] \to J_\Gamma \to 0$$

which is dual to the previous sequence.

Let T/K be the torus with character group $T^* = I_\Gamma$. The character group of T^0 is then $(T^0)^* = J_\Gamma$.

The torus T is the K-torus $R^1_{L/K}\mathbb{G}_m$ of norm 1 elements in L. For $d = [L : K]$ this torus is a maximal torus in $G = \mathrm{SL}_d$.

The unramified Brauer group of the K-torus T^0 (modulo $\mathrm{Br}\,K$) is $Ш^2_\omega(\Gamma, (T^0)^*)$; see [CTS87a]. We have

$$Ш^2_\omega(\Gamma, (T^0)^*) = Ш^3(\Gamma, \mathbb{Z}) = H^3(\Gamma, \mathbb{Z})$$

(recall that the cohomology of a cyclic group has period 2 and that $H^1(H, \mathbb{Z}) = 0$ for any finite group H).

The unramified Brauer group of G/T (modulo $\mathrm{Br}\,K$) is $Ш^1_\omega(\Gamma, T^*)$; see [CTK06]. We have

$$Ш^1_\omega(\Gamma, T^*) = \mathrm{Ker}\Big[\hat{H}^0(\Gamma, \mathbb{Z}) \to \prod_{g\in\Gamma} \hat{H}^0(g, \mathbb{Z})\Big].$$

This group is

$$\mathrm{Ker}\Big[\mathbb{Z}/n_\Gamma \to \prod_{g\in\Gamma} \mathbb{Z}/n_g\Big]$$

where the projection is the natural map, n_Γ is the order of Γ and n_g the exponent of $g \in \Gamma$.

Let us now take $\Gamma = (\mathbb{Z}/p)^r$, $r \geqslant 2$. The Künneth formula shows that $H^3(\Gamma, \mathbb{Z})$ is killed by p; see [Spa81, p. 247]. Thus the group $Ш^2_\omega(\Gamma, (T^0)^*)$ is killed by p.

The group

$$\operatorname{Ker}\left[\mathbb{Z}/n_\Gamma \to \prod_{g\in\Gamma} \mathbb{Z}/n_g\right]$$

is

$$p\mathbb{Z}/p^r\mathbb{Z} \subset \mathbb{Z}/p^r\mathbb{Z}.$$

Thus for any $r \geqslant 3$, the group $Ш^1_\omega(\Gamma, T^*)$ is not killed by p.

We conclude that the unramified Brauer group of G/T (modulo $\operatorname{Br} K$) is not isomorphic to the unramified Brauer group of T^0 (modulo $\operatorname{Br} K$). Therefore G/T and T^0 are not stably K-birationally isomorphic, as claimed. □

Acknowledgements

The authors are grateful to the referee for contributing Proposition 3.3 and for other helpful suggestions. Colliot-Thélène is a researcher at C.N.R.S. (France). Kunyavskiĭ was supported in part by the Minerva Foundation through the Emmy Noether Research Institute of Mathematics. Popov was supported in part by Russian grants РФФИ 08–01–00095, НШ–1987.2008.1, a (granting) program of the Mathematics Branch of the Russian Academy of Sciences, by ETH Zürich, and TU München. He also thanks the Department of Mathematics of UBC for its hospitality. Reichstein was supported in part by an NSERC Discovery grant and Accelerator supplement. Colliot-Thélène, Kunyavskiĭ and Reichstein thank the organizers of the Oberwolfach meeting 'Quadratic Forms and Linear Algebraic Groups' (June 2006). Colliot-Thélène and Reichstein also acknowledge the support of Emory University.

References

AOV96 J. Alev, A. Ooms and M. Van den Bergh, *A class of counterexamples to the Gel'fand–Kirillov conjecture*, Trans. Amer. Math. Soc. **348** (1996), 1709–1716.

Art57 E. Artin, *Geometric algebra* (Interscience Publishers Inc., New York, 1957).

BF03 G. Berhuy and G. Favi, *Essential dimension: a functorial point of view (after A. Merkurjev)*, Doc. Math. **8** (2003), 279–330.

Bia63 A. Białynicki-Birula, *On homogeneous affine spaces of linear algebraic groups*, Amer. J. Math. **85** (1963), 577–582.

Bor91 A. Borel, *Linear algebraic groups*, second enlarged edition, Graduate Texts in Mathematics, vol. 126 (Springer, Berlin, 1991).

Bou68 N. Bourbaki, *Groupes et algèbres de Lie* (Hermann, Paris, 1968), Chapters IV, V, VI.

CTK06 J.-L. Colliot-Thélène and B. È Kunyavskiĭ, *Groupe de Brauer et groupe de Picard des compactifications lisses d'espaces homogènes*, J. Algebraic Geom. **15** (2006), 733–752.

CTS77 J.-L. Colliot–Thélène and J.-J. Sansuc, *La R-équivalence sur les tores*, Ann. Sci. École Norm. Sup. (4) **10** (1977), 175–229.

CTS87a J.-L. Colliot–Thélène and J.-J. Sansuc, *Principal homogeneous spaces under flasque tori: applications*, J. Algebra **106** (1987), 148–205.

CTS87b J.-L. Colliot–Thélène and J.-J. Sansuc, *La descente sur les variétés rationnelles, II*, Duke Math. J. **54** (1987), 375–492.

CTS07 J.-L. Colliot–Thélène and J.-J. Sansuc, *The rationality problem for fields of invariants under linear algebraic groups (with special regards to the Brauer group)*, in *Proceedings of the international colloquium on algebraic groups and homogeneous spaces,* Mumbai, 2004, ed. V. Mehta (Narosa Publishing House, TIFR Mumbai, 2007), 113–186.

CK00 A. Cortella and B. Kunyavskiĭ, *Rationality problem for generic tori in simple groups*, J. Algebra **225** (2000), 771–793.

DG70 M. Demazure and A. Grothendieck, *Schémas en Groupes I, II, III*, *Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3), Dirigé par M. Demazure et A. Grothendieck*, Lecture Notes in Mathematics, vols. 151, 152, 153 (Springer, Berlin, 1970).

Flo08 M. Florence, *On the essential dimension of cyclic p-groups*, Invent. math. **171** (2008), 175–189.

Fog73 J. Fogarty, *Fixed points schemes*, Amer. J. Math. **95** (1973), 35–51.

GMS03 S. R. Garibaldi, A. Merkurjev and J.-P. Serre, *Cohomological invariants in Galois cohomology*, University Lecture Series, vol. 28 (American Mathematical Society, Providence, RI, 2003).

Gro58 A. Grothendieck, *Torsion homologique et sections rationnelles, Anneaux de Chow et Applications*, Séminaire C. Chevalley (1958), exposé 5.

Gro64 A. Grothendieck, avec la collaboration de J. Dieudonné, *Éléments de géométrie algébrique, IV, Étude locale des schémas et des morphismes de schémas*, Publ. Math. Inst. Hautes Études Sci. **20** (1964), **24** (1965), **28** (1966), **32** (1967).

GS01 *Correspondance Grothendieck–Serre*, éd. par P. Colmez et J.-P. Serre, Documents mathématiques, vol. 2 (Soc. Math. France, 2001). Reprinted and translated into English in *Grothendieck–Serre Correspondence*, Bilingual Edition (P. Colmez, J.-P. Serre, eds) (Amer. Math. Soc., Soc. Math. France, 2004).

Hum75 J. E. Humphreys, *Linear algebraic groups*, Graduate Texts in Mathematics, vol. 21 (Springer, New York, 1975).

Hum95 J. E. Humphreys, *Conjugacy classes in semisimple algebraic groups*, Mathematical Surveys and Monographs, vol. 43 (American Mathematical Society, Providence, RI, 1995).

Igu73 J.-i. Igusa, *Geometry of absolutely admissible representations*, in *Number theory, algebraic geometry and commutative algebra* (Kinokuniya, Tokyo, 1973), 373–452, in honor of Yasuo Akizuki.

Kos63 B. Kostant, *Lie group representations on polynomial rings*, Amer. J. Math. **85** (1963), 327–404.

Kot82 R. E. Kottwitz, *Rational conjugacy classes in reductive groups*, Duke Math. J. **49** (1982), 785–806.

Kot99 R. E. Kottwitz, *Transfer factors for Lie algebras*, Represent. Theory **3** (1999), 127–138.

LL00 N. Lemire and M. Lorenz, *On certain lattices associated with generic division algebras*, J. Group Theory **3** (2000), 385–405.

LPR06 N. Lemire, V. L. Popov and Z. Reichstein, *Cayley groups*, J. Amer. Math. Soc. **19** (2006), 921–967.

Lor05 M. Lorenz, *Multiplicative invariant theory*, in *Invariant theory and algebraic transformation groups, VI*, Encyclopaedia of Mathematical Sciences, vol. 135 (Springer, Berlin, 2005).

Lun73 D. Luna, *Slices étales*, Mém. Soc. Math. Fr. **33** (1973), 81–105.

Mat60 Y. Matsushima, *Espaces homogènes de Stein des groupes de Lie complexes*, Nagoya Math. J. **16** (1960), 205–218.

Mum65 D. Mumford, *Geometric invariant theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Neue Folge, Bd. 34 (Springer, Berlin, 1965), third enlarged edition; D. Mumford, J. Fogarty and F. Kirwan, *Geometric invariant theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Neue Folge, Bd. 34 (Springer, Berlin, 1994).

Oni60 A. L. Onishchik, *Complex hulls of compact homogeneous spaces*, Dokl. Akad. Nauk SSSR **130** (1960), 726–729; Engl. transl. Sov. Math. Dokl. **1** (1960), 88–91.

Pop72 V. L. Popov, *On the stability of the action of an algebraic group on an algebraic variety*, Izv. Akad. Nauk SSSR Ser. Mat. **36** (1972), 371–385; Engl. transl. Math. USSR Izv. **6** (1972), 367–379.

Pop74 V. L. Popov, *Picard groups of homogeneous spaces of linear algebraic groups and one-dimensional homogeneous vector bundles*, Izv. Acad. Nauk SSSR Ser. Mat. **38** (1974), 294–322; Engl. transl. Math. USSR Izv. **8** (1974), 301–327.

Pop92 V. L. Popov, *Groups, generators, syzygies, and orbits in invariant theory*, Translations of Mathematical Monographs, vol. 100 (American Mathematical Society, Providence, RI, 1992).

Pop94 V. L. Popov, *Sections in invariant theory*, in *The Sophus Lie memorial conference*, Oslo, 1992 (Scand. University Press, Oslo, 1994), 315–361.

Pop V. L. Popov, *Cross-sections, quotients, and representation rings of semisimple algebraic groups*, available at arXiv:0908.0826 and
http://www.math.uni-bielefeld.de/LAG/man/351.pdf.

PV94 V. L. Popov and E. B. Vinberg, *Invariant theory*, in *Algebraic geometry IV*, Encyclopaedia of Mathematical Sciences, vol. 55 (Springer, Berlin, 1994), 123–284.

Pre10 A. Premet, *Modular Lie algebras and the Gelfand–Kirillov conjecture*, Invent. math. **181** (2010), 395–420.

SGA3bis10 *Réédition de SGA*3, available at http://people.math.jussieu.fr/∼polo/SGA3/.

Rei00 Z. Reichstein, *On the notion of essential dimension for algebraic groups*, Transform. Groups **5** (2000), 265–304.

Ric72 R. W. Richardson Jr, *Principal orbit types for algebraic transformation spaces in characteristic zero*, Invent. math. **16** (1972), 6–14.

Ros56 M. Rosenlicht, *Some basic theorems on algebraic groups*, Amer. J. Math. **78** (1956), 401–443.

Ros63 M. Rosenlicht, *A remark on quotient spaces*, An. Acad. Brasil. Ciênc. **35** (1963), 487–489.

San81 J.-J. Sansuc, *Groupe de Brauer et arithmétique des groupes algébriques linéaires sur un corps de nombres*, J. reine angew. Math. **327** (1981), 12–80.

Ser58 J.-P. Serre, *Espaces fibrés algébriques*, in *Anneaux de Chow et applications*, Séminaire C. Chevalley, 1958, exposé 1. Reprinted in J.-P. Serre, *Exposés de séminaires* 1950–1999, deuxième édition, augmentée, Documents mathématiques, vol. 1 (Soc. Math. France, 2008), 107–140.

Ser94 J.-P. Serre, *Cohomologie Galoisienne*, Lecture Notes in Mathematics, vol. 5, 5ième édition (Springer, Berlin, 1994).

Spa81 E. H. Spanier, *Algebraic topology* (Springer, New York, 1981).

Spr89 T. A. Springer, *Aktionen reduktiver Gruppen auf Varietäten*, in *Algebraische Transformationsgruppen und Invariantentheorie*, DMV Seminar, Bd. 13, Herausgeb. H. Kraft, P. Slodowy and T. A. Springer (Birkhäuser, Basel, 1989), 3–39.

Spr98 T. A. Springer, *Linear algebraic groups*, Progress in Mathematics, vol. 9, second edition (Birkhäuser, Boston, 1998).

Ste65 R. Steinberg, *Regular elements of semi-simple groups*, Publ. Math. Inst. Hautes Études Sci. **25** (1965), 49–80.

Sum74 H. Sumihiro, *Equivariant completion, I*, J. Math. Kyoto Univ. **14** (1974), 1–28.

Tho86 R. W. Thomason, *Comparison of equivariant algebraic and topological K-theory*, Duke Math. J. **55** (1986), 795–825.

Vos77 V. E. Voskresenskiĭ, *Algebraicheskie tory* (in Russian) [*Algebraic tori*] (Izdat. ‘Nauka’, Moscow, 1977).

Vos88 V. E. Voskresenskiĭ, *Maximal tori without affect in semisimple algebraic groups*, Mat. Zametki **44** (1988), 309–318; English transl. Math. Notes **44** (1988), 651–655.

Vos98 V. E. Voskresenskiĭ, *Algebraic groups and their birational invariants* (American Mathematical Society, Providence, RI, 1998).

Wei65 A. Weil, *Sur la formule de Siegel dans la théorie des groupes classiques*, Acta Math. **113** (1965), 1–87.

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