The Brauer–Grothendieck group. Errata and comments

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Pages 118–119, Claim

Mohamed Amine Koubaa pointed out an inaccuracy in our rendition of de Jong's proof of Gabber's theorem. One should drop the second part of condition (b) on p. 118 ("but the following composition is zero" and the formula in the next line), on p. 119 remove "$\psi_2 = 0$" in line 5, and in line 6 insert "isomorphic to" before "the direct summand".

Page 127, Theorem 5.2.5

As pointed out by Otto Overkamp, the assumption "X is geometrically integral and geometrically connected" in this theorem can be replaced by a weaker assumption that $\mathrm{H}^0(X, \mathcal{O}_X) = k$. Indeed, then $\mathcal{O}_k \to p_*\mathcal{O}_X$ is an isomorphism, hence $\mathbb{G}_{m,k} \to p_*\mathbb{G}_{m,X}$ is an isomorphism.

In the proof of this theorem (page 128, lines 14-15) the inclusion

$$\mathrm{H}^0(k, R^2p_*\mathbb{G}_{m,X}) \hookrightarrow \mathrm{H}^0(\bar{k}, R^2p_*\mathbb{G}_{m,X})$$

is not automatic since $\mathrm{Spec}(\bar{k}) \to \mathrm{Spec}(k)$ is not in general an fppf-cover as it may not be of finite presentation. For a proof of this inclusion see [D'Ad, Lemma 3.2]. Alternatively, note that $\mathrm{Br}(\overline{X})$ is the inductive limit of $\mathrm{Br}(X \times_k L)$, where $L \subset \bar{k}$ is a finite field extension of k, see Section 2.2.2 (ii). Since $\mathrm{Spec}(L) \to \mathrm{Spec}(k)$ is an fppf-cover, we have $\mathrm{H}^0(k, R^2p_*\mathbb{G}_{m,X}) \hookrightarrow \mathrm{H}^0(L, R^2p_*\mathbb{G}_{m,X})$. The same proof works with $\bar{k}$ replaced by L.

Page 149, proof of Theorem 5.6.1 (iv)

As observed by Otto Overkamp, after replacing k by $K = \mathrm{H}^0(D, \mathcal{O}_D)$ the structure morphism $D \to \mathrm{Spec}(k)$ factors through $D \to \mathrm{Spec}(K)$, but in general there is no reason why D should be geometrically integral over K (as claimed in line 23). However, the more general version of Theorem 5.2.5 given above can be applied, which allows one to complete the proof.

Page 273, Definition 11.3.1

In the last line of the definition, replace "smooth over k" with "regular".

Page 300, Proposition 12.2.1 (b) should be corrected as follows:

"(b) For each irreducible divisor $Y \subset \widetilde{X}$ which does not lie over the generic point of $\mathbb{P}^2_{\mathbb{C}}$, the restriction of β to $\mathrm{Br}(\mathbb{C}(Y))$ is zero."

Page 321, Remark 13.3.9

Olivier Wittenberg pointed out that this question has a simple positive answer. Indeed, for any variety X over a number field k and any finite field extension K/k, the natural inclusion $X(\mathbb{A}_k) \subset X(\mathbb{A}_K)$ induces an inclusion $X(\mathbb{A}_k)^{\mathrm{Br}} \subset X(\mathbb{A}_K)^{\mathrm{Br}}$, hence $X(\mathbb{A}_k)^{\mathrm{Br}} \neq \emptyset$ implies $X(\mathbb{A}_K)^{\mathrm{Br}} \neq \emptyset$. This follows from the functoriality of the corestriction map (Proposition 3.8.1) and the fact that for a finite extension of local fields E/F, the corestriction map cores: $\mathrm{Br}(E) \to \mathrm{Br}(F)$ satisfies

$$\mathrm{inv}_F \circ \mathrm{cores}_{E/F} = \mathrm{inv}_F,$$

where inv_F is defined in Definition 13.1.7, cf. Proposition 1.4.7.

Page 400, Remark 16.1.7

As pointed out by Yanshuai Qin in his paper "On geometric Brauer groups and Tate–Shafarevich groups" (available at `https://arxiv.org/abs/2012.01681v2`), the result was proved earlier by Cadoret, Hui, and Tamagawa in their paper "$\mathbf{Q}_\ell$- versus $\mathbf{F}_\ell$-coefficients in the Grothendieck–Serre/Tate conjectures" (available at `https://webusers.imj-prg.fr/~anna.cadoret/GST.pdf`).

Page 404, Proof of Theorem 16.2.3

On line 11 the proof refers to [Zar77], Thm. 1.1, which assumes $\mathrm{char}(k) \neq 2$. The correct reference is [Zar14, Cor. 2.7], where this restriction is removed.

References

[D'Ad] M. D'Adezzio. Boundedness of the p-primary torsion of the Brauer group of an abelian variety. arXiv:2201.07526

[Zar14] Yu.G. Zarhin. Abelian varieties over fields of finite characteristic. *Cent. Eur. J. Math.* **12** (2014) 659–674.