The Brauer–Grothendieck group. Errata and comments

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Pages 118–119, Claim

Mohamed Amine Koubaa pointed out an inaccuracy in our rendition of de Jong's proof of Gabber's theorem. One should drop the second part of condition (b) on p. 118 ("but the following composition is zero" and the formula in the next line), on p. 119 remove "$\psi_2 = 0$" in line 5, and in line 6 insert "isomorphic to" before "the direct summand".

Page 300, Proposition 12.2.1 (b) should be corrected as follows:

"(b) For each irreducible divisor $Y \subset \widetilde{X}$ which does not lie over the generic point of $\mathbb{P}^2_{\mathbb{C}}$, the restriction of β to $\mathrm{Br}(\mathbb{C}(Y))$ is zero."

Page 321, Remark 13.3.9

As remarked by O. Wittenberg, the question has an easy positive answer. For any variety X over a a number field k and any finite field extension K/k, the natural inclusion $X(\mathbb{A}_k) \subset X(\mathbb{A}_K)$ induces an inclusion $X(\mathbb{A}_k)^{\mathrm{Br}} \subset X(\mathbb{A}_K)^{\mathrm{Br}}$. In particular if the LHS is not empty, then neither is the RHS.

The inclusion follows from the functoriality of the corestriction map (Prop. 3.8.1) and the fact that for a finite field extension E/F of local fields, the corestriction map $\mathrm{Br}(E) \to \mathrm{Br}(F)$ commutes with the embedding of each group into $\mathbb{Q}/\mathbb{Z}$ given in Definition 13.1.7.

Page 400, Remark 16.1.7

As pointed out by Yanshuai Qin in his paper "On geometric Brauer groups and Tate–Shafarevich groups" (available at `https://arxiv.org/abs/2012.01681v2`), the result was proved earlier by Cadoret, Hui, and Tamagawa in their paper "$\mathbf{Q}_\ell$- versus $\mathbf{F}_\ell$-coefficients in the Grothendieck–Serre/Tate conjectures" (available at `https://webusers.imj-prg.fr/~anna.cadoret/GST.pdf`).

Page 404, Proof of Theorem 16.2.3

On line 11 the proof refers to [Zar77], Thm. 1.1, which assumes $\mathrm{char}(k) \neq 2$. The correct reference is [Zar14], Cor. 2.7, where this restriction is removed.

Page 406, Proof of Corollary 16.3.2

This statement is Theorem B of [Zar14]. The proof given here should be corrected. The relevant commutative diagram of exact sequences is:

$$\begin{array}{ccccc}
\mathrm{H}^1(k,\mathrm{Pic}(X^{\mathrm{s}}\times Y^{\mathrm{s}})) & \hookrightarrow & \mathrm{Br}(X\times Y)/\mathrm{Br}(k) & \to & \mathrm{Br}(X^{\mathrm{s}}\times Y^{\mathrm{s}})^{\Gamma} \\
\uparrow & & \uparrow & & \uparrow \\
\mathrm{H}^1(k,\mathrm{Pic}(X^{\mathrm{s}}))\oplus\mathrm{H}^1(k,\mathrm{Pic}(Y^{\mathrm{s}})) & \hookrightarrow & \mathrm{Br}(X)/\mathrm{Br}(k)\oplus\mathrm{Br}(Y)/\mathrm{Br}(k) & \to & \mathrm{Br}(X^{\mathrm{s}})^{\Gamma}\oplus\mathrm{Br}(Y^{\mathrm{s}})^{\Gamma}
\end{array}$$

Exact sequence (5.31) and finiteness of $\mathrm{H}^1(k,M)$ for a Galois lattice M imply that the cokernel of the left-hand side vertical map is finite. The cokernels of the right horizontal maps are finite by Theorem 5.4.12. Since k is finitely generated over $\mathbb{Q}$, the cokernel of the right vertical map is finite by Theorem 16.3.1. A diagram chase then shows that the cokernel of the middle vertical map is finite.