Arithmetic upon intersection of two quadrics

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References

Retour sur l'arithmétique des intersections de deux quadriques, avec un appendice par A. Kuznetsov
https ://arxiv.org/abs/2208.04121v2

Also to be found on my webpage
https ://www.imo.universite-paris-saclay.fr/ jean-louis.colliot-thelene/
See also the note "Lichtenbaum's theorems."

Let k be a number field. Let k_v run through the completions of k. Let $X \subset \mathbb{P}^n_k$, be a smooth complete intersection of two quadrics :

$$f(x_0, \cdots, x_n) = g(x_0, \cdots, x_n) = 0.$$

A well known conjecture asserts :
For $n \geq 5$, for any such X, the Hasse principle holds, namely

$$\prod_v X(k_v) \neq \emptyset \Longrightarrow X(k) \neq \emptyset.$$

When $X(k) \neq \emptyset$, and $n \geq 5$, one knows that $X(k) \subset \prod_v X(k_v)$ is dense.

For $n = 3$, the Hasse principle need not hold. One then has a curve of genus one, the obstruction to the Hasse principle is related to the Tate-Shafarevich group of the jacobian of the curve.

For $n = 4$, the Hasse principle need not hold (first explicit example : Birch and Swinnerton-Dyer 1975). Conjecturally, the defect is controlled by the Brauer-Manin obstruction.

Results were obtained for $n \geq 12$ by Mordell (1959) and for $n = 10$ by Swinnerton-Dyer (1964).

Assume k is totally imaginary, and $n = 12$. Assume $f(x_0, \ldots, x_{12})$ is non-degenerate. Here is Mordell's argument. The quadratic form f may be written as the direct sum of a totally hyperbolic quadratic form in 10 variables and a quadratic form in 3 variables. On a linear space of codimension $5 + 3 = 8$, that is a $\mathbb{P}^4_k$, the form f identically vanishes. The restriction of g to this $\mathbb{P}^4_k$ is given by a quadratic form in 5 variables, it has a nontrivial zero over k.

Formally real fields are handled by an elegant trick over the reals : consider the behaviour of the signature of the quadratic form $af + bg$ as (a, b) varies over $a^2 + b^2 = 1$. One proves the existence of quadratic forms in the pencil over $\mathbb{R}$ with 6 hyperbolics.

The Hasse principle for *smooth* complete intersections of two quadrics in $\mathbb{P}^n_k$ is known to hold :

For $n \geq 8$ (CT–Sansuc–Swinnerton-Dyer 1987) [Note : for $n \geq 8$, $X(k_v) \neq \emptyset$ for v nonarchimedean].

For $n \geq 4$ if X contains two lines globally defined over k (the case $n = 4$ was known before 1970).

For $n \geq 5$ if X contains a conic (Salberger 1993, unpublished).

For $n = 7$ (Heath-Brown 2018).

Taking two difficult conjectures (finiteness of Ш of elliptic curves and Schinzel's hypothesis) for granted, Wittenberg (2007) gave a proof of the Hasse principle for $n \geq 5$.

Here is another special case for $X \subset \mathbb{P}^5_k$.

Theorem (J. Iyer and R. Parimala 2022). *Let $X \subset \mathbb{P}^5_k$ be a smooth complete intersection of two quadrics $f = g = 0$ over a number field. Assume that X contains a line over each completion k_v of k. Assume also that the curve of genus 2 defined by*

$$y^2 = -det(\lambda f + \mu g)$$

has index 1, for instance has a rational point. Then $X(k) \neq \emptyset$.

Creutz and Viray (2021-2022) have investigated the question whether a smooth complete intersection of two quadrics $X \subset \mathbb{P}^n_k$, $n \geq 4$, over a local or over a global field k has a point in an extension K/k of degree ≤ 2. They have also considered the question whether the index $I(X)$ (gcd of degrees of closed points) is 1, 2 or 4. They proved :

• *For k p-adic and $n \geq 4$, there exists a quadratic extension K/k such that $X(K) \neq \emptyset$.*
• *For k a number field and $n \geq 4$, $I(X)$ divides 2.* The proof is quite delicate. For $n = 4$ it uses the birational equivalence between $\mathrm{Sym}^2 X$ and the variety parametrizing pairs (L, Q) with Q quadric in $\mathbb{P}^4$ in the pencil of quadrics containing X and L line of $\mathbb{P}^4$ lying in the quadric Q. This variety is birational to the total space of a fibration over $\mathbb{P}^1_k$ with general fibre a Severi-Brauer variety.

The aim of the talk is to present alternate proofs and slight improvements of the recent results (2018-2022) listed above, except for the last statement in the case $n = 4$.

A general remark is that there are good reasons to try to get results also for arbitrary, possibly singular, intersections of two quadrics.

One useful tool is the theorem : *Over any field, if an intersection of two quadrics* $X \subset \mathbb{P}^n_k$ *has a rational point over an odd degree extension of* k *then it has a rational point.*

Thisi is an immediate consequence of Springer's theorem (same statement for one quadric, over any field) and the theorem of Amer and of Brumer :

Let $k(t)$ *be the rational function field in one variable. A sytem of two quadratic forms* $f = g = 0$ *over a field* k *has a nontrivial zero if and only if the quadratic form* $f + tg$ *over the field* $k(t)$ *has a nontrivial zero.*

When discussing a complete intersection of two quadrics $X \subset \mathbb{P}^n_k$ over a field k (char. not 2) given by a system $f = g = 0$, one is quickly led to consider the pencil of quadrics $\lambda f + \mu g = 0$ containing X.
Ignoring subtle points with the singular forms in the pencil, there is a close relation between the following statements, where we assume $r \geq 1$:

- There exists a form $\lambda f + \mu g$ in the pencil which splits off $r + 1$ hyperbolic planes.
- There exists a quadric in the pencil which contains a linear space $\mathbb{P}^r_k \subset \mathbb{P}^n_k$.
- The variety X contains an $r - 1$-dimensional quadric $Y \subset \mathbb{P}^r_k \subset \mathbb{P}^n_k$.

In this talk I shall igore the "subtle points". They are addressed in my typescript.

Theorem 1 (CT 2022) *Let k be a p-adic field. Let $X \subset \mathbb{P}^3_k$ be an intersection of two quadrics given by a system*

$$f(x_0, x_1, x_2, x_3) = 0, \; g(x_1, x_2, x_3) = 0.$$

Then there exists a quadratic extension K/k with $X(K) \neq \emptyset$.

Proof.
When X is not a smooth complete intersection, this is proven by a case-by-case discussion. Assume X is a smooth complete intersection. Then X is a genus one curve.

Let $\overline{k}$ be an algebraic closure of k, and $G := \mathrm{Gal}(\overline{k}/k)$. The period of a curve X is defined as the positive generator of the image of the degree map $\mathrm{Pic}(X \times_k \overline{k})^G \to \mathbb{Z}$.

The assumption that $g(x_1, x_2, x_3)$ involves only three variables implies that the "period" of the curve X divides 2. This one sees by using the fact any conic has period 1 and that tthe curve X is a double cover of the conic $g(x_1, x_2, x_3) = 0$.

For a curve of genus one, it is a theorem of Lichtenbaum (1969) that the period coincides with the index. Thus the index divides 2. By Riemann-Roch, this implies that there exists a field K/k of degree at most 2 with $X(K) \neq \emptyset$.

Theorem 2 (Creutz–Viray 2022) *Let k be a p-adic field. Let $X \subset \mathbb{P}^n_k$, $n \geq 4$ be an intersection of two quadrics. There exists a field K/k of degree at most 2 with $X(K) \neq \emptyset$.*

(Alternate) proof. It is enough to handle the case $n = 4$. Singular cases are handled by a case by case analysis. Assume X is a smooth complete intersection. It is then given by a system

$$h(x_0, x_1, x_2) + x_3 x_4 = 0 = g(x_0, \cdots, x_4).$$

The section by $x_4 = 0$ is an intersection of two quadrics in $\mathbb{P}^3_k$ as in the previous theorem. QED

Theorem (Creutz–Viray 2021). *Let k be a number field and $X \subset \mathbb{P}^n_k$ be a smooth complete intersection of two quadrics. For $n \geq 4$, the index $I(X)$ divides 2.*

The proof is very elaborate.

Theorem 3 (CT 2022) *Let k be a number field and $X \subset \mathbb{P}^n_k$ be a smooth complete intersection of two quadrics. For $n \geq 5$ there exists a quadratic extension K/k with $X(K) \neq \emptyset$.*

The question whether this holds for $n = 4$ remains open. Partial results are given by Creutz–Viray.

Proof. By Bertini it is enough to prove the case $n = 5$. In this case the variety $F_1(X)$ of lines on X is geometrically integral – it is actually a principal homogeneous space under an abelian variety. Hence there exists a finite set S of places of k such that $F_1(X)(k_v) \neq \emptyset$ for $v \notin S$. Thus for almost all v, any $\lambda f + \mu g$ splits off 2 hyperbolics over k_v.

For any place v, Theorem 2 gives a point of X in an extension of k_v of degree 2, hence there exists a $\lambda_v f + \mu_v g$ in the pencil over k_v which splits off two hyperbolics.

Using weak approximation, we find $(\lambda, \mu) \in \mathbb{P}^1(k)$ such that $\lambda f + \mu g$ splits off 2 hyperbolics over each k_v. By a result of Hasse (1924) it splits off 2 hyperbolics over k. Thus X contains a point over a quadratic extension of k.

Theorem 4 (Salberger, CT, 1988/89) *Let k be a number field and $X \subset \mathbb{P}^4_k$ be a smooth complete intersection of two quadrics which contains a conic. Then $X(k)$ is dense in the Brauer-Manin set $X(\mathbb{A}_k)^{\mathrm{Br}(X)} \subset X(\mathbb{A}_k)$.*

Because X contains a conic, it admits a fibration into conics $X \to \mathbb{P}^1_k$ with 4 geometric degenerate fibres. Salberger's proof uses his work on zero-cycles. He proves $I(X) = 1$ and on such X this implies $X(k) \neq \emptyset$. My proof uses universal torsors and results from CT–Sansuc–Swinnerton-Dyer 1987.

Theorem 5 (Salberger 1993, Harari 1994) *Let k be a number field and $X \subset \mathbb{P}^n_k$, $n \geq 5$, be a smooth complete intersection of two quadrics which contains a conic. Then the Hasse principle holds and $X(k)$ is dense in $\prod_v X(k_v)$.*

The proof uses the fibration method to reduce to Theorem 4. Salberger also discusses singular intersections.

Theorem 6 (Iyer and Parimala 2022) *Let k be a number field and $X \subset \mathbb{P}^5_k$ be a smooth complete intersection of two quadrics given by a system $f = g = 0$. Let C be the double cover of $\mathbb{P}^1_k$ given by the equation $y^2 = -det(\lambda f + \mu g)$. This is a curve of genus two. Assume $I(C) = 1$. If X contains a line over each k_v, then $X(k) \neq \emptyset$.*

Simplified proof (CT 2022)
Let t be a variable. The quadratic form $f + tg$ over the field $k(t)$ splits off two hyperbolics over each $k_v(t)$, that is we have an isomorphism
$f + tg \simeq < 1, -1 > \perp < 1, -1 > \perp \rho_v < 1, det(\lambda f + \mu g >$
for some $\rho_v \in k_v(t)^*$. Going over to the function field $k(C)$, one gets that $f + tg$ is isomorphic to the sum of there hyperbolics over each $k_v(C)$. One now uses the assumption $I(C) = 1$. Standard reductions reduce to the case where there is a k-point of C in good position with respect to $C \to \mathbb{P}^1_k$. The image of this point gives a $t_0 \in k$ with the property that the quadratic $f + t_0 g$ over k splits off 3 hyperbolics over each k_v. By Hasse's result, it splits off 3 hyperbolics over k, hence $f + t_0 g = 0$ contains a $\mathbb{P}^2_k$, hence X contains a conic. Since $X(k_v) \neq$ for each v, Theorem 5 gives $X(k) \neq \emptyset$.

Theorem 7 (Heath-Brown 2018) *Let k be a local field. Let $X \subset \mathbb{P}^7_k$ be a smooth complete intersection of two quadrics given by $f = g = 0$. If $X(k) \neq \emptyset$, then there exists a nondegenerate form $\lambda f + \mu g$ in the pencil which splits off three hyperbolics.*

Proof (CT 2022) Let $P \in X(k)$. The intersection C of X with the tangent $\mathbb{P}^5_k$ at P is a cone with vertex P over an intersection of two quadrics $Y \subset \mathbb{P}^4_k$. By Theorem 2 (Creutz–Viray) there exists a point on Y in a quadratic extension K/k. This defines a line over K on C passing through the vertex P of the cone. One thus gets a pair of lines in $C \subset X$ passing through P and globally defined over k. Fix a k-point Q in the plane $\mathbb{P}^2_k$ defined by these two lines, outside of the two lines. The form $\lambda f + \mu g$ vanishing at Q vanishes on the plane $\mathbb{P}^2_k$ spanned by the two lines. If nondegenerate, this form splits off 3 hyperbolics. There is a simple way to handle the case where the form is of rank 7.

Theorem 8 (Heath-Brown, 2018) *Let k be a number field. Let $X \subset \mathbb{P}^7_k$ be a smooth complete intersection of two quadrics given by $f = g = 0$. The Hasse principle holds for X.*

Hasse principle for $X \subset \mathbb{P}^7_k$

Proof (CT 2022, some ingredients from HB's proof). The variety $F_2(X)$ of planes $\mathbb{P}^2_k \subset X \subset \mathbb{P}^7_k$ is a geometrically integral variety – it is actually a principal homogeneous spaces under an abelian variety. Hence there exists a finite set S of places of k such that $F_2(X)(k_v) \neq \emptyset$ for $v \notin S$. Thus each $v \notin S$, any nondegenerate $\lambda f + \mu g$ splits off 3 hyperbolics over k_v. By Theorem 7, for each $v \in S$ the assumption $X(k_v) \neq \emptyset$ implies that there exists a point $(\lambda_v, \mu_v) \in \mathbb{P}^1(k_v)$ such that $\lambda_v f + \mu_v g$ is nondegenerate and contains 3 hyperbolics. By weak approximation on $\mathbb{P}^1_k$, there exists $(\lambda, \mu) \in \mathbb{P}^1(k)$ such that $\lambda f + \mu g$ is nondegenerate and contains 3 hyperbolics over each k_v. By Hasse 1924 it contains 3 hyperbolics over k. Thus X contains a conic. Theorem 5 (Salberger) and the hypothesis $\prod_v X(k_v) \neq \emptyset$ then give $X(k) \neq \emptyset$.

What about singular complete intersections of two quadrics ?

Let k be a number field and $X \subset \mathbb{P}^n_k$ a possibly singular complete intersection of two quadrics. Assume it is geometrically integral and not a cone. One is interested in the Hasse principle for a smooth projective model Y of X.

In CT-Sansuc-Swinnerton-Dyer 1987, we proved the Hasse principle for Y under the assumption $n \geq 8$. We proposed :

Conjecture. *For $n = 6$ or $n = 7$ the Hasse principle holds for Y.*

For such n, one has $\mathrm{Br}(Y)/\mathrm{Br}(k) = 0$ so there is no Brauer-Manin obstruction. Under various additional hypotheses on Y, the conjecture is proved in CT–S–SD 1987. Salberger 1993 proves it when X contains a smooth conic.

The present proof of Theorem 7 (Heath-Brown's local theorem) could open the way to a proof of the above conjecture for $n = 7$.

More on $X \subset \mathbb{P}^5_k$

Let k be a field and $X \subset \mathbb{P}^5_k$ be a smooth complete interesection of two quadrics $f = g = 0$. The following construction appears in M. Reid's thesis (1972). It has been recently considered by Hassett and Tschinkel.
Let C be the double cover of $\mathbb{P}^1_k$ defined by $y^2 = -det(\lambda f + \mu g)$.
Let $G_2(X)$ be the variety of pairs (H, Q) where Q is a quadric of $\mathbb{P}^5_k$ containing X and $H \subset Q \subset \mathbb{P}^5_k$ is a linear space $\mathbb{P}^2_k \subset \mathbb{P}^5_k$. We have the Stein factorisation

$$G_2(X) \to C \to \mathbb{P}^1_k$$

of the map sending (H, Q) to (λ, μ) with Q defined by $\lambda f + \mu g = 0$.

The map $G_2(X) \to C$ defines a Severi-Brauer scheme with associated class $\alpha \in \mathrm{Br}(C)[2]$ of index 4 in $\mathrm{Br}(k(C)$.
The image of α in $\mathrm{Br}(k(C))$ is the class of the Clifford algebra associated to the quadratic form $f + tg$ over $k(C)$, where $k(\mathbb{P}^1) = k(t)$.

Theorem. *If there is a point $m \in C(k)$ unramified over C and such that $\alpha(m) = 0$, then X contains a conic.*

Theorem. *The variety X contains a $\mathbb{P}^1_k \subset \mathbb{P}^5_k$ if and only if $\alpha = 0 \in \mathrm{Br}(C)$.*

One may use these results to give a variant proof of the Iyer-Parimala theorem.