On the stable rationality of certain real threefolds

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JLCT, A. Pirutka et F. Scavia, Variétés réelles semi-algébriquement connexes non stablement rationnelles, juin 2025, https ://arxiv.org/abs/2505.21477
JLCT et A. Pirutka, Certaines fibrations en surfaces quadriques réelles, juin 2024, https ://arxiv.org/abs/2406.00463

Let X be a smooth projective variety of dimension d over a field k. Assume that it is geometrically rational, i.e. birational to projective space over a finite field extension, and that it has at least one k-point. Is it (stably) k-birational to $\mathbb{P}^d_k$?

Among the birational invariants which have been used to study this problem, we find :

– The quotient of the Brauer group $\mathrm{Br}(X) = \mathrm{Br}_{nr}(k(X)/k)$ by $\mathrm{Br}(k)$, more generally unramified cohomology.

– The Chow group of zero cycles $CH_0(X)$.

One must also consider the quotients $\mathrm{Br}(X_L)/\mathrm{Br}(L)$ and $CH_0(X_L)$ over any field L and in particular $L = k(X)$.

If $CH_0(X_{k(X)}) = \mathbb{Z}$ holds, one says that X admits an integral "decomposition of the diagonal", or that X is universally CH_0-trivial.

If X stably rational, then X is universally CH_0-trivial.

In this talk, we discuss rationality questions for varieties over the reals $\mathbb{R}$, and more generally over real closed fields.

A real field is a field in which -1 is not a sum of squares. A real closed field F is a real field with no nontrivial real algebraic extension. Equivalently the field $F[t]/(t^2+1)$ is algebraically closed.

Example : the field $R = \cup_{n=1}^{\infty} \mathbb{R}((t^{1/n}))$ of real Puiseux series. We shall denote it $\mathbb{R}\{\{t\}\}$. This field is equipped with a unique order, for which $t > 0$ is smaller than any real number in $\mathbb{R}_{>0}$.

Let R be a real closed field. A semi-algebraic set in R^n is a subset belonging to the smallest family containing sets $P(x_1, \ldots, x_n) > 0$ (with P a polynomial) and stable under finite union, finite intersection, and taking complements.
This definition is extended to the set $X(R)$ of R-points of an algebraic variety X/R.
The closure of a semi-algebraic set is semi-algebraic.

A semi-algebraic set in $A \subset X(R)$ is called (semi-algebraically) connected if for each pair F_1, F_2 of semi-algebraic subsets closed in A with $A = F_1 \cup F_2$, either $F_1 = A$ or $F_2 = A$.
Any semi-algebraic set A in $X(R)$ is the union of finitely many semi-algebraically connected subsets $C_1, \ldots, C_s$ of A which are open and close in A.
[Over $R = \mathbb{R}$ the notion coincides with the usual topological notion of connectedness.]

The notion was studied by H. Delfs and M. Knebusch (1981), with later works by Bochnak, Coste, Roy (Springer Ergebnisse), and by C. Scheiderer.

For smooth, projective, algebraic varieties X over a real closed field R, the number s of s.a. connected components of $X(R)$ is a stable birational invariant. It may be computed in various ways. Assume $X(R) \neq \emptyset$.
We have $CH_0(X)/2 = (\mathbb{Z}/2)^s$ (CT-Ischebeck 1981).
We have $H^i_{nr}(R(X)/R, \mathbb{Z}/2) = (\mathbb{Z}/2)^s$ for $i > \mathrm{dim(X)}$
(CT-Parimala 1990, further work by Scheiderer)

Known results (Comessatti 1912)
If $X/\mathbb{R}$ is a smooth, projective, geometrically rational surface, it is $\mathbb{R}$-rational if and only if $X(\mathbb{R})$ is connected.

What about higher dimensional geometrically rational varieties, such as :
– smooth intersections of two quadrics in $\mathbb{P}^n_{\mathbb{R}}$, $n \geq 5$
– fibrations into quadric hypersurfaces of dimension $n \geq 2$ over $\mathbb{P}^1_{\mathbb{R}}$

Definition. Over a field k of char. zero, a "good" quadric surface fibration over $\mathbb{P}^1_k$ is a smooth connected threefold X equipped with a flat morphism $X \to \mathbb{P}^1_k$ all fibres of which are quadric surfaces, and all geometric singular fibres of which are irreducible.
(In other words, the finitely many geometric singular fibres are cones over a conic.)
Over a real closed field R, "special" good quadric surface fibration over $\mathbb{P}^1_R$ are defined by affine equations

$$x^2 + y^2 + z^2 = t.p(t)$$

with $p(t)$ positive on R.
In CT-Pirutka 2024, we gave some sufficient conditions for universal CH_0-triviality (more on this topic if I have time at the end of the talk).
Over the real Puiseux field $\mathbb{R}\{\{t\}\}$, there exist special quadric surface fibrations over $\mathbb{P}^1_{\mathbb{R}\{\{t\}\}}$ which are not universally CH_0-trivial (Benoist-Pirutka 2024).

There exist "bad" quadric surface fibration $Y \to \mathbb{P}^1_{\mathbb{R}}$, whose set of real points is connected, and which are not stably rational. Here is one (CT-P 24). In $\mathbb{P}^3 \times \mathbb{A}^1_u$ with coordinates $(x, y, z, t); u$ it is given by the following equation :

$$x^2 + (1 + u^2)y^2 - u(w^2 + z^2) = 0.$$

The geometric fibres on points other than $u = 0, \infty$ are irreducible. The fibres over $u = 0$ and $u = \infty$ break up over $\mathbb{C}$ into two conjugate planes, the intersection of which is a line. The only singular points on Y are conjugate complex points on each of these two lines. One easily checks that $Y(\mathbb{R})$ is connected. One may write a desingularisation $\tilde{Y} \to Y$ with $\tilde{Y}(\mathbb{R}) = Y(\mathbb{R})$, hence connected.

(a) The map $\mathrm{Br}(\mathbb{R}(u)) \to \mathrm{Br}(\mathbb{R}(Y))$ is injective (because the determinant of the generic quadric is not a square).

(b) The nonconstant class $(-1, u) \in \mathrm{Br}(\mathbb{R}(u))$ becomes unramified on $\tilde{Y}$.

Thus $\mathrm{Br}_{nr}(\mathbb{R}(\tilde{Y}))/\mathrm{Br}(\mathbb{R}) \neq 0$ and $\tilde{Y}$ is not CH_0-trivial.

One can also produce such examples among singular intersections of two quadrics in $\mathbb{P}^5_{\mathbb{R}}$.
An easy birational transformation shows that the variety Y above is $\mathbb{R}$-birational to the singular intersection W of two quadrics in $\mathbb{P}^5_{\mathbb{R}}$ given by the system

$$a^2 + b^2 + c^2 - cd = 0,$$

$$e^2 + f^2 - bd = 0.$$

The only singular points on W are 4 complex points. The space $W(\mathbb{R})$ is connected (birational invariance of the number of connected components). We have $\mathrm{Br}_{nr}(\mathbb{R}(W)/\mathbb{R})/\mathrm{Br}(\mathbb{R}) \neq 0$. Hence no desingularisation of W is CH_0-trivial.

The original hope : deform these examples to good, smooth projective varieties over $\mathbb{R}$ with the same properties.

Let $p : \mathcal{X} \to U \subset \mathbb{A}^1_{\mathbb{R}}$ be a flat proper morphism to an open set U, with $O \in U(\mathbb{R})$. Assume that the morphism p is smooth outside of O, and that the special fibre $\mathcal{X}_O$ has no singular real point, and that $\mathcal{X}_O(\mathbb{R})$ is connected. Ehresmann's theorem on $\mathbb{C}^\infty$ manifolds then gives that locally around O, the fibration $\mathcal{X}(\mathbb{R}) \to U(\mathbb{R})$ is a product. In particular for any $P \in U(\mathbb{R})$ close to O, $\mathcal{X}_P(\mathbb{R})$ is connected.
Suppose we know that the special fibre $\mathcal{X}_O/\mathbb{R}$ is not stably rational thanks to some unramified cohomology obstruction. Can we find points $P \in U(\mathbb{R})$ close to O such that $\mathcal{X}_P/\mathbb{R}$ is not stably rational ?

In complex geometry, starting some 10 years ago, a technique was started by C. Voisin, developed by CT-Pirutka and then further by S. Schreieder, to disprove stable rationality of "very general" varieties of many classes of varieties which a priori are close to being rational. One uses a fibration $p : \mathcal{X} \to U \subset \mathbb{A}^1_{\mathbb{C}}$, assumes that the special, singular, fibre $\mathcal{X}_O$ is not stably rational *thanks to an unramified cohomology obstruction*, then uses specialisations of zero-cycles to deduce that the generic geometric fibre is not stably rational. The method requires that *the special, singular fibre, admits a reasonable desingularisation* (CH_0-trivial resolution of singularities). One then shows that over the points of $U(\mathbb{C})$ outside a countable union of *proper algebraic subsets* of $\mathbb{A}^1_{\mathbb{C}}$, i.e. outside a countable union of points, the fibres are not CH_0-trivial, hence not stably rational.

If one tries to mimic the argument in the real context, one only gets that the set of real points in $U(\mathbb{R})$ which one must a priori eliminate is a countable union of *proper semi-algebraic subsets* of $U(\mathbb{R})$. Typically, one would have to find a real point different from $t = 0$ in the intersection of the intervals $[-1/m, 1/m]$ for all integers $m > 0$.
Well, if we accept to pass from $\mathbb{R}$ to the Puiseux series field $\mathbb{R}\{\{t\}\}$ we do find a nonzero element, the infinitely small ellement t, in this intersection !
(This is not the formal argument.)

The substitute for Ehresmann's theorem.

Theorem A (on connexity)

Let $p : \mathcal{X} \to U \subset \mathbb{A}^1_{\mathbb{R}} = Spec(\mathbb{R}[t])$ *be a flat proper morphism, with* $O \in U(\mathbb{R})$ *given by* $t = 0$*. Let* $\mathcal{X}_\eta = \mathcal{X} \times_U \mathbb{R}\{\{t\}\}$*. Assume that the morphism* p *is smooth outside of* O*, and that the special fibre* $\mathcal{X}_O/\mathbb{R}$ *has no singular real point. Then the number of semi-algebraic connected components of* $\mathcal{X}_\eta(\mathbb{R}\{\{t\}\})$ *is the same as the number of connected components of* $\mathcal{X}_O(\mathbb{R})$*.*

In the paper, we give two proofs, none of which is simple-minded. The first one uses a Nash-triviality theorem of Coste and Shiota (1992). (Nash functions are analytic functions which are algebraic over algebraic functions.)

The other one uses Claus Scheiderer's *Real and étale cohomology* (Springer, 1994) which develops analogues of the smooth and proper base change theorems (SGA4) in the real context.

In the real context, we have to develop a version of the specialisation method as elaborated in (CT-Pirutka 2015).

The new point in this context is described in the next two slides. The hypothesis (i) is some kind of a substitute for the notion of CH_0-trivial resolution of singularities.

Proposition *Let* $p : Z \to Y$ *be a proper birational* $\mathbb{R}$*-morphism of geometrically integral projective* $\mathbb{R}$*-varieties. Assume that* $Z/\mathbb{R}$ *is smooth. Let* $S \neq \emptyset$ *be the singular locus of* Y*. Let* $U \subset Y$ *be the smooth locus. Assume* $p : p^{-1}(U) \to U$ *is an isomorphism, and there exists* $b \in U(\mathbb{R})$*. Assume :*

(i) *The morphism* $S \to \mathrm{Spec}\,(\mathbb{R})$ *factorizes through* $\mathrm{Spec}\,(\mathbb{C}) \to \mathrm{Spec}\,(\mathbb{R})$*. [If* S *is finite, this means* $S(\mathbb{R}) = \emptyset$*.]*

(ii) *The map*

$$\mathrm{Br}(\mathbb{R}) \to \mathrm{Ker}[\mathrm{Br}(Z) \to \mathrm{Br}(Z_{\mathbb{C}})]$$

is not surjective.

Then the difference between the generic point of Y *and the point* $b_{\mathbb{R}(Y)}$ *in* $CH_0(Y_{\mathbb{R}(Y)})$ *does not vanish.*

Proof. Let $T = p^{-1}(S)$. One compares the localization sequences for Chow groups of zero-cycles when passing over from Y to U and from Z to $p^{-1}(U)$, over the fields $\mathbb{R}(Y)$ and $\mathbb{C}(Y)$. Let $N_{\mathbb{C}/\mathbb{R}}$ denote $\mathrm{Norm}_{\mathbb{C}/\mathbb{R}}$. Hypothesis (i) gives that the projection map

$$\Phi : CH_0(Z_{\mathbb{R}(Y)})/N_{\mathbb{C}/\mathbb{R}}(CH_0(Z_{\mathbb{C}(Y)})) \to CH_0(Y_{\mathbb{R}(Y)})/N_{\mathbb{C}/\mathbb{R}}(CH_0(Y_{\mathbb{C}(Y)}))$$

is an *isomorphism*. This map sends the generic point of the smooth variety Z to the generic point of the singular variety Y.
Fix $\alpha \in \mathrm{Ker}[\mathrm{Br}(Z) \to \mathrm{Br}(Z_{\mathbb{C}})]$ nonzero, with $\alpha(b) = 0$. We have $\mathrm{Br}(Z) \subset \mathrm{Br}(\mathbb{R}(Z))$. Let η be the generic point of Z. There is a pairing

$$CH_0(Z_{\mathbb{R}(Y)})/N_{\mathbb{C}/\mathbb{R}}(CH_0(Z_{\mathbb{C}(Y)})) \times \mathrm{Ker}[\mathrm{Br}(Z) \to \mathrm{Br}(Z_{\mathbb{C}})] \to \mathrm{Br}(\mathbb{R}(Y))$$

with $< \eta - b_{\mathbb{R}(Y)}, \alpha >= \alpha_{\mathbb{R}(Y)} \neq 0 \in \mathrm{Br}(\mathbb{R}(Y)) = \mathrm{Br}(\mathbb{R}(Z))$.
Via the isomorphism Φ above, one concludes that *the difference between the generic point of Y and the constant point $b_{\mathbb{R}(Y)}$ is a nontrivial class in $CH_0(Y_{\mathbb{R}(Y)})/N(CH_0(Y_{\mathbb{C}(Y)}))$. In particular the difference is not zero in $CH_0(Y_{\mathbb{R}(Y)})$.*

Theorem B (on stable rationality) *Let $U \subset \mathbb{A}^1_{\mathbb{R}}$ be an open set containing $t = 0$. Let $\mathcal{X} \to U$ be a proper map with smooth generic fibre. Assume that the special fibre $Y = \mathcal{X}_0/\mathbb{R}$ satisfies the conditions of the previous theorem. Then the $\mathbb{R}\{\{t\}\}$-variety $\mathcal{X} \times_U \mathbb{R}\{\{t\}\}$ is not universally CH_0-trivial, in particular it is not stably rational.*

Sketch of proof. If it is universally CH_0-trivial, then over some $\mathbb{R}((t^{1/n}))$ one uses specialisation of zero-cycles from $\mathcal{X} \times_U \mathbb{R}((t^{1/n}))$ to zero-cycles on $Y/\mathbb{R}$ and deduces that the generic point of the (singular) variety Y is rationally equivalent over $\mathbb{R}(Y)$ to a point coming from $Y(\mathbb{R})$. Contradiction with the above proposition.

Starting from the singular examples earlier described and putting Theorems A and B together we get :

Theorem (CT-Pirutka-Scavia 2025)

Over the field $\mathbb{R}\{\{t\}\}$ of real Puiseux series, among each of the following classes of varieties $X/\mathbb{R}\{\{t\}\}$

– Smooth intersections of two quadrics in $\mathbb{P}^5_{\mathbb{R}\{\{t\}\}}$.

– Good, nonspecial, quadric surface fibrations over $\mathbb{P}^1_{\mathbb{R}\{\{t\}\}}$

there exist examples such that

(i) $X(\mathbb{R}\{\{t\}\})$ is s.a. connected,

and

(ii) $X_{\mathbb{R}\{\{t\}\}}$ is not universally CH_0-trivial, and in particular not stably rational.

A refined version of the technique using $H^3_{nr}(\mathbb{R}(Y)/\mathbb{R}, \mathbb{Z}/2)$ instead of the Brauer group $H^2_{nr}(\mathbb{R}(Y)/\mathbb{R}, \mathbb{Z}/2)$ leads to :

Theorem (CT-Pirutka-Scavia 2025)
Over the field $\mathbb{R}\{\{t\}\}$, among each of the following classes of varieties $X/\mathbb{R}\{\{t\}\}$
– Smooth intersections of two quadrics in $\mathbb{P}^9_{\mathbb{R}\{\{t\}\}}$.
– Good quadric hypersurface fibrations of relative dimension 6 over $\mathbb{P}^1_{\mathbb{R}\{\{t\}\}}$
there exist examples such that
(i) $X(\mathbb{R}\{\{t\}\})$ is s.a. connected,
and
(ii) $X_{\mathbb{R}\{\{t\}\}}$ is not universally CH_0-trivial, and in particular not stably rational.

For smooth intersections of two quadrics $X \subset \mathbb{P}^n_R$ over a real closed field, with $X(R)$ s. a. connected, we have :

For $n = 4$ (Comessatti) and $n = 6$ (Hassett, Kollár, Tschinkel 2022), X is rational. For $n = 8$, HKT also have some positive results on rationality.

There exist $X \subset \mathbb{P}^5_{\mathbb{R}}$ with $X(\mathbb{R})$ connected and X not rational (IJT obstruction, Hassett-Tschinkel, Benoist-Wittenberg, 2020).

For $n = 5$ and $n = 9$, and $R = \mathbb{R}\{\{t\}\}$, there exist X/R with $X(\mathbb{R})$ connected and X not stably rational (above, CT-Pirutka-Scavia 2025).

[In the study of smooth complete intersection of two quadrics in $\mathbb{P}^n$, there often is a strong difference between the case n even and the case n odd.]

Special quadric surface fibrations X over $\mathbb{P}^1_{\mathbb{R}}$ (CT-Pirutka 2024)

Let X over $\mathbb{R}$ be given by affine equations

$$x^2 + y^2 + z^2 = u.p(u)$$

with $p(u)$ strictly positive on $\mathbb{R}$. Projection to $u \in \mathbb{A}^1$ extends to a projection $\mathcal{X} \to \mathbb{P}^1$. Let $\Delta/\mathbb{R}$ be the curve given by the affine equation $w^2 = v.p(-v)$. Let $W = X \times_{\mathbb{R}} \Delta$. Note : $u + v$ is a sum of 6 squares in $\mathbb{R}(W)$.

Theorem *The following conditions are equivalent :*
(i) The variety X is universally CH_0-trivial.
(ii) $u + v$ is a sum of 4 squares in $\mathbb{R}(W)$.
(iii) The cup-product $(u + v, -1, -1) \in H^3(\mathbb{R}(W), \mathbb{Z}/2)$ vanishes.
(iv) $H^3(F, \mathbb{Z}/2) = H^3_{nr}(F(X)/F, \mathbb{Z}/2)$ for $F = \mathbb{R}(\Delta)$.

The proof uses an injective cycle class map for zero-cycles into some H^3-cohomology introduced in CT/Skorobogatov 1993.

Let us consider the case $p(u) = u^2 + au + b$. We have $b > a^2/4$. The j-invariant of the elliptic curve $z^2 = u.p(u)$ satisfies $j \in [-\infty, 1728]$.

Recall X is given by $x^2 + y^2 + z^2 = u.p(u)$.

Theorem A. *If $b \geq a^2/3$, or equivalently if $j(\Delta) \geq 0$, then X is universally CH_0-trivial.*

Theorem B. *If the curve $z^2 = u.p(u)$ has odd complex multiplication, i.e. $End_{\mathbb{C}}(E) = \mathbb{Z}[\omega]$ with $\omega^2 - d\omega + c = 0$, $c, d \in \mathbb{Z}$ and d odd, then X is universally CH_0-trivial.*

Zarhin showed : the (countable) set of j-invariants of such real curves is dense in $[-\infty, 1728]$.

Proof of Theorem A. We have $x^2 + y^2 + z^2 = u.p(u)$ and $w^2 = v.p(-v)$. In $\mathbb{R}(u, v)$, we have

$$up(u) + vp(-v) = (u + v)(u^2 - uv + v^2 + au - av + b) =$$
$$= (u + v)\left(\left(u + \frac{a - v}{2}\right)^2 + \frac{3}{4}\left(v - \frac{a}{3}\right)^2 + b - \frac{a^2}{3}\right).$$

We assumed $b - \frac{a^2}{3} \geq 0$. Thus

$$r(u, v) = \frac{up(u) + vp(-v)}{u + v} = \left(u + \frac{a - v}{2}\right)^2 + \frac{3}{4}\left(v - \frac{a}{3}\right)^2 + b - \frac{a^2}{3}$$

is a sum of 4 squares in $\mathbb{R}(u, v)$. In $\mathbb{R}(W)$, we have

$$x^2 + y^2 + z^2 + w^2 = up(u) + vp(-v) = (u + v).r(u, v).$$

Since nonzero sums of 4 squares are stable under multiplication and division (Euler) we get that $u + v$ is a sum of 4 squares in $\mathbb{R}(W)$. Apply the general theorem.

Among real varieties with equation

$$x^2 + y^2 + z^2 = u.p(u)$$

with $p(u)$ of degree 2 strictly positive on $\mathbb{R}$,
– no example is known with $X/\mathbb{R}$ rational, or already stably rational
– no example is known with $X/\mathbb{R}$ not stably rational, or already not rational