Closed points on del Pezzo surfaces of degree 4 and index 2

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The purpose of this note is to sketch the proof of :

Theorem. *Let $X \subset \mathbf{P}^4_k$ be a del Pezzo surface of degree 4 over a field of characteristic zero. Suppose its index, i.e. the gcd of the degrees of closed points is 2. Then there exists a closed point on X of degree $2d$ with $d \in \{1, 3, 7\}$.*

The proof follows from the combination of Propositions 1 and 2 below. The reason why at this point it is only a sketch is that one uses general position assumptions. I am confident that the arguments recently developed in §2 of [CT] will easily show that this is legitimate for the purpose at hand, but I have not yet checked that yet. The restriction to characteristic zero presumably is not important but it is made in [CT]. One can probably deduce the positive characteristic case from the characteristic zero case.

Proposition 1 (JLCT, September 11th, 2020). *Let k be a field of characteristic zero. Let $X \subset \mathbf{P}^4_k$ be a del Pezzo surface of degree 4 over k. If there exists a closed point of degree $2d$ on X, with d odd and $d \notin \{1, 3, 5, 7, 11, 19\}$, then there exists a closed point of degree $2d'$ with d' odd and $d' < d$.*

Proof. The method of proof is due to Coray [C], with variations in [CT].

For any $n \geq 1$, the linear system $nH = -nK$ on X satisfies

$$l(nH) \geq 2n(n+1) + 1$$

(Riemann-Roch on the surface) and for any smooth projective geometrically connected curve Γ in the linear system nH,

$$p_a(\Gamma) = 2n(n-1) + 1.$$

Suppose there is no closed point of degree 2.

Suppose we have a closed point P of degree $2d$ with d odd. We also have a closed point Q of degree 4.

One chooses n minimal such that

$$2n(n+1) + 1 \geq 2d + 4 + 1.$$

This ensures that there exists a curve Γ in the linear system nH which passes through P and Q.

We have $n(n+1) - 2 \geq dn(n-1) - 2$ hence $n(n+1) - 3 \geq d \geq n(n-1) - 1$ since d is odd.

On the curve Γ we look for an effective zero-cycle of degree $2d - 4$ which is the degree of $P - Q$. This will be possible, and by Riemann-Roch on the curve, will ensure that one may strictly decrease d, if one has

$$2d - 4 \geq p_a(\Gamma) = 2n(n-1) + 1,$$

that is $d \geq n(n-1) + 5/2$, that is $d \geq n(n-1) + 3$.

The above arguments assume that one can choose the curve Γ smooth.

In the paper [C] , Coray had to go through a discussion of possible degenerate cases. In the recent paper [CT], one explains how one can avoid such a discussion.

There is a problem if $n(n+1) - 2 \geq d \geq n(n-1) - 1$ and $d < n(n-1) + 3$, i.e. since d is odd, $d \leq n(n-1) + 1$.

The bad cases are when d is of one of the following forms:

$$d = n(n-1) + 1$$

or

$$d = n(n-1) - 1.$$

If the odd d is not of one of these two special shapes, then one may produce an effective zero-cycle of degree $2d'$ with $d' < d$ odd on Γ, hence on X.

To get rid of the bad cases, Coray used the trick of searching for a curve Γ with a lower geometric genus by imposing singularities. The analogue here would be to look for a curve Γ in the linear system nH passing through P and singular at the point Q. The condition at Q imposes 12 linear conditions, and decreases the geometric genus by 4. One would then consider the desingularisation Γ_1 of a curve in the system.

One needs

$$2n(n+1)+1-12-2d \geq 1,$$

and

$$2d-4 \geq 2n(n-1)+1-4.$$

So we want

$$n(n+1)-d \geq 6$$

and

$$d \geq n(n-1)+1.$$

If $d = n(n-1)+1$, and $n(n+1)-n(n-1)-1 \geq 6$, that is $n \geq 4$ this works. So the only bad cases here are $n = 1, 2, 3$. This corresponds to $d = 1$ which me may ignore, $d = 3$ and $d = 7$.

But if $d = n(n-1)-1$ we are still in trouble. In that case we pick two points Q_1 and Q_2 of degree 4 and we look for curves in the linear system nH passing through P and singular at Q_1 and at Q_2. This imposes 24 linear conditions and decreases the genus by 8.

We then want

$$2n(n+1)+1-24-2d \geq 1$$

and

$$2d-4 \geq 2n(n-1)+1-8.$$

The first condition reads $4n \geq 22$, i.e. $n \geq 6$. So the only bad cases are for $n \leq 5$. This corresponds to $d = 5$, $d = 11$, $d = 19$.

This proves Proposition 1.

Proposition 2 (message from J. Kollár to JLCT, November 12th, 2020).
Let k be a field of characteristic zero. Let $X \subset \mathbf{P}^4_k$ be a del Pezzo surface of degree 4 over k.
(i) If there is a degree 10 closed point on X, then there is an effective zero-cycle of degree 6.
(ii) If there is a degree 22 closed point on X, then there is an effective zero-cycle of degree 14.
(iii) If there is a degree 38 closed point on X, then there is an effective zero-cycle of degree 26.

Proof.

(1) The linear system associated to $2H$ has dimension 12, so through 10 points we still get a 2-dimensional linear system. Pick 2, their intersection is 16 points, get a residual 6.

(2) The linear system associated to $3H$ has dimension 24. Pick 2 through 22 points, get a residual 14.

(3) The linear system associated to $4H$ has dimension 40. Pick 2 through 38 points, get a residual 26.

QED

Theorem. *Let $X \subset \mathbf{P}^4_k$ be a del Pezzo surface of degree 4 over a field of characteristic zero. Suppose the index of X, i.e. the gcd of the degrees of closed points on X, is 2. Then there exists a closed point on X of degree $2d$ with $d \in \{1, 3, 7\}$.*

Proof. Just combine the two propositions.

References

[C] Daniel F. Coray, Algebraic points on cubic hypersurfaces, Acta Arith. 30 (1976), no. 3, 267–296.

[CT] J.-L. Colliot-Thélène, Zéro-cycles sur les surfaces de del Pezzo (Variations sur un thème de Daniel Coray), à paraître dans L'Enseignement Mathématique.