Hilbert's irreducibility theorem and jumps in the rank of the Mordell-Weil group

Algebra, Geometry and Physics seminar

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Jean-Louis Colliot-Thélène
(CNRS et Université Paris-Saclay)

Two intertwining topics

• Behaviour of the Mordell–Weil rank in a family of abelian varieties

• Hilbert's irreducibility theorem

Aims

• Over a number field, find elliptic curves with a higher rank

• Over a number field, find new classes of varieties over which Hilbert's irreducibility holds

Summary :

• Much of what I am going to describe goes back to the work of André Néron in the 50s

k a field, in this talk, $\mathrm{char}(k) = 0$
X a smooth, geometrically integral k-variety, $d = dim(X)$.
Each of the following properties implies the next one

- X is k-rational : k-birational to projective space $\mathbb{P}^d_k$
- There exists a dominant k-morphism $f : Y \to X$ with Y a k-rational variety and with *geometrically integral* generic fibre : in this talk we say that such an X is Hilbert-unirational
- X is k-unirational : There exists a dominant k-morphism $f : Y \to X$ with Y a k-rational variety (one may here assume $dim(Y) = dim(X)$)
- The set $X(k)$ of k-rational points of X is Zariski dense in X.

X geometrically integral k-variety, $\dim(X) \geq 1$.
Thin sets in $X(k)$ (as defined by Serre) :
Subset of $X(k)$ contained in a finite union of sets of either one of the two types :

• $Y(k) \subset X(k)$ with $Y \subset X$ proper closed subvariety
• $f(Y(k)) \subset X(k)$ for Y integral k-variety, $dim(Y) = dim(X)$ and $f : Y \to X$ dominant with degree $[k(Y) : k(X)] > 1$.
• The k-variety X is called hilbertian if $X(k)$ is not thin. Then $X(k)$ is Zariski dense in X.
• A field k is called hilbertian $\mathbb{P}^1_k$ is a hilbertian k-variety.
A k-variety which is k-rational is then hilbertian.
• A number field is hilbertian.
• Any function field in at least one variable over a field is hilbertian.
• An elliptic curve over a number field is not hilbertian (weak Mordell-Weil theorem).

Easy proposition.

Let k be a field. Let $Y \to X$ be a dominant k-morphism between geometricallly integral k-varieties. Assume that **the generic fibre is geometrically integral**.
(a) If $\mathcal{R} \subset Y(k)$ is not thin in Y, then $f(\mathcal{R}) \subset X(k)$ is not thin in X.
(b) In particular, if Y/k is hilbertian, then $f(Y(k))$ is not thin in X, hence X/k is hilbertian.

Corollary. *If k is a hilbertian field (e.g. a number field), then a k-variety which is Hilbert-unirational is hilbertian.*

Given a finitely generated abelian group M, we write here $rank(M) := dim_{\mathbb{Q}}(M \otimes_{\mathbb{Z}} \mathbb{Q})$.

Néron's specialisation theorem
The proof given by Serre in his Mordell–Weil book gives :

Theorem *Let k be a field, $\mathrm{char}(k) = 0$. Let U be a smooth geometrically integral variety, $U(k) \neq \emptyset$ and $X \to U$ an abelian scheme. Let $A = X_\eta$ be the generic fibre. This is an abelian variety over $K = k(U)$. Assume the abelian group $A(K)$ is finitely generated. The set of points $m \in U(k)$ where specialisation $A(K) = X_\eta(K) \to X_m(k)$ is not injective is thin in $U(k)$.*

If k is of finite type over $\mathbb{Q}$, the group $A(K)$ is of finite type (Mordell, Weil, Néron).

If k is arbitrary and $Tr_{K/k}(A) = 0$ (no constant part for A/K) then $A(K)$ is of finite type (Lang-Néron).

Corollary. *Suppose $A(K)$ is finitely generated and k is a hilbertian field. If the k-variety U is hilbertian, for instance k-rational, then the set of point $m \in U(k)$ such that $\mathrm{rank}(X_m(k)) \geq \mathrm{rank}(A(K))$ is Zariski dense in U.*

If U is an open set of an elliptic curve E over a number field, and $E(k)$ is infinite, then we cannot get anything from this corollary since $E(k)$ is thin in E (weak Mordell–Weil theorem).

Nevertheless (Silverman 1982, first version Néron ICM 1954) :

Theorem *Let k be a number field, U a smooth integral k-curve with $U(k)$ infinite, let $K = k(U)$.*
Let $X \to U$ be an abelian scheme. Assume the K/k-trace of the generic fibre $A = X_\eta$ is trivial. The set of points $m \in U(k)$ such that

$$rank(X_m(k)) \geq rank(A(K))$$

is an infinite set (Néron, lower bound on counting function). Its complement is finite (Silverman).

Néron proves and uses a pre-Faltings theorem :
Theorem (Néron, ICM 1954) *Let E/k be an elliptic curve over a number field, $E(k)$ infinite. Let h be a (reasonable) height function on $E(k)$. If C is a smooth integral curve and $f : C \to E$ is a ramified k-morphism, then the quotient*
number of points in $f(C(k)) \subset E(k)$ with height $\leq H$
by number of points of $E(k)$ with height $\leq H$
tends to 0 when H tends to infinity.

Néron's argument would be worth deciphering. It uses the complex parametrisation and some geometry of numbers.
The theorem was reestablished by Manin (1967). To prove it, he used Mumford 1965 (A remark on Mordell's conjecture).
Today the theorem is a consequence of Mordell's conjecture, as proved by Faltings. Indeed $g(C) \geq 2$.

Over a number field k, for U an open set of $\mathbb{P}^1_k$, the following question has been investigated. :

Let k be a number field, let U be a smooth, geometrically integral k-variety and $X \to U$ an abelian scheme of relative dimension ≥ 1. Let $A = X_\eta$ be the generic fibre. What is the behaviour of the set of points $m \in U(k)$ such that $rank(X_m(k)) > rank(A(K))$? Is it not thin ? Is it Zariski dense ?

Billard 1998, Salgado, Hindry, Loughran more recently, **assuming that the total space X/k of the fibration $X \to U$ is k-rational, or at least k-unirational**.

Some version of the following nearly trivial lemma may already been found in Néron 1952.

Key Lemma. *Let F be a field and A/F an abelian variety. Let $\xi \in A$ be a point of the scheme A,* **not closed**, *and let $F(\xi)$ be the residue field at ξ. The quotient $A(F(\xi))/A(F)$ is not a torsion group.*

Proof. For any field extensions $F \subset L \subset M$ with L and M algebraically closed, $A(L)_{tors} = A(M)_{tors}$.

The basic example is when $dim(A) \geq 1$ and ξ is the generic point of A.

Jumping from ρ to $\rho + 1$

"Generic point theorem" (CT 2019)

Let k be a field, $\mathrm{char}(k) = 0$. Let U be smooth, geometrically integral k-variety with $U(k) \neq \emptyset$ and $X \to U$ an abelian scheme of relative dimension ≥ 1. Let $K = k(U)$ and A/K be the generic fibre X_η. Assume $A(K)$ is finitely generated of rank ρ.

(a) If $X(k)$ is not thin in X (for example if k is hilbertian and X is k-rational), then the set of points $m \in U(k)$ such that the Mordell-Weil rank of $X_m(k)$ is $\geq \rho + 1$ is not thin in $U(k)$ (and in particular is Zariski-dense in U).

(b) Let k be hilbertian. If the k-variety X is k-unirational, then the set of points $m \in U(k)$ such that the Mordell-Weil rank of $X_m(k)$ is $\geq \rho + 1$ is Zariski dense in U.

Proof of (a).
Set $W = X$. Let $Y = X \times_U W$ be the fibre product, both maps $X \to U$ and $W = X \to U$ being the given map $X \to U$. This defines an abelian scheme $Y \to W$. By the key lemma applied to $A = X_\eta/K$ and to the generic point $\xi \in A$, the Mordell-Weil rank of the generic fibre of $Y \to W$ is $\geq \rho + 1$. By our hypothesis the set $W(k) = X(k)$ is not thin in W. Néron's specialisation theorem then gives that the set E of points $n \in W(k)$ such that Y_n/k has Mordell-Weil rank $\geq \rho + 1$ is not thin in W. The generic fibre of $q = W \to U$ is geometrically integral. The easy proposition then gives that $q(E) \subset U(k)$ is not thin in U. For $n \in E \subset W$, the fibre of $X \to U$ en $m = q(n)$ is the fibre of $X \times_U W$ in n. QED

Proof of (b).
We are given $f : X \to U$ and a dominant k-morphism $W \to X$ with W k-rational. Let $g : W \to X \to U$ be the composed map. Let $Y = X \times_U W$ be the fibre product of f and g.
This map admits the section $W \to X \times_U W = Y$ given by $W \to X$, and the composite map $W \to Y \to X$ is the given dominant morphism. By the key lemma, the Mordell–Weil rank of the generic fibre of $Y \to W$ is $\sigma > \rho = rank(A(K))$.
Néron's specialisation theorem, when applied to $Y \to W$, gives that the set $\mathcal{R}$ of points $n \in W(k)$ with $rank(Y_n(k)) \geq \sigma$ is not thin in W. Since k is hilbertian and W k-rational, this set is Zariski dense in W. Its image $f(\mathcal{R})$ under $f : W \to U$ is thus Zariski dense in U. For $n \in W(k)$, we have $X_{f(n)} = Y_n$, with $rank(X_{f(n)}) \geq \rho + 1$. QED

Note. Let k be a number field and $X \to U$ a family of elliptic curves over k. Under the mere assumption that $X(k)$ is Zariski dense in X, Merel's theorem on uniform torsion implies that the set of points $m \in U(k)$ whose fibre X_m has Mordell-Weil rank at least 1 is Zariski-dense in U.

Families of elliptic curves whose total space is k-rational

$y^2 = x^3 + rx + s$ with $(r, s) \in U \subset \mathbb{A}_k^2$
$y^2 = x^3 + ax + s$ with fixed $a \in k$ and $s \in U \subset \mathbb{A}_k^1$
$y^2 = x^3 + rx$ with $r \in U \subset \mathbb{A}_k^1$
$ry^2 = x^3 + ax + b$, with fixed $a \in k$, $b \in k^*$, and $r \in U \subset \mathbb{A}_k^1$
(quadratic twists).

Jumping from ρ to $\rho + 2$
(Salgado 2012, Loughran–Salgado 2019)

Let k be a number field. Let $f : X \to \mathbb{P}^1_k$ be a family of elliptic curves. Assume that $X \times_k k^{\mathrm{alg}}$ is k^{alg}-rational.
[Example : the fibration into elliptic curves associated to a del Pezzo surface of degree 1.]
Assume there exists $C \subset X$ with $C \simeq \mathbb{P}^1_k$ such that the composite map $C \to \mathcal{X} \to \mathbb{P}^1_k$ is of degree 2 (thus a bisection of the fibration). Assume that the generic fibre of f is not constant after a quadratic extension of $k(\mathbb{P}^1)$.

Theorem. *Under the above hypotheses – in particular k is a number field – if the generic rank is ρ, then the set of points $m \in \mathbb{P}^1(k)$ with $X_m(k)$ of rank $\geq \rho + 2$ is not thin.*

Proof (sketch).
The linear system $H^0(X, \mathcal{O}_X(C))$ makes X into a conic fibration over $\mathbb{P}^1_k$. One then shows X is k-unirational. One then has many bisections $\mathbb{P}^1_k \simeq C_\alpha \to \mathbb{P}^1_k$ for $f : X \to \mathbb{P}^1_k$. One proves the ramification of the maps $C_\alpha \to \mathbb{P}^1_k$ varies. Using a variant of the "generic point theorem" one shows there are many $D := D_{\alpha,\beta} := C_\alpha \times_{\mathbb{P}^1_k} C_\beta$ such that the generic fibre of $X \times_{\mathbb{P}^1_k} D$ has rank at least $\rho + 2$. In general, such a D has genus 1. With care, one may arrange that $D(k)$ is infinite. One then applies the Néron–Silverman theorem *over an elliptic curve* to specialize at points of $D(k)$ while keeping $\rho + 2$ for the Mordell-Weil rank. Letting the ramification of the C_α's and C_β's vary one shows that the set of points $m \in \mathbb{P}^1(k)$ whose fibre X_m has MW rank at least $\rho + 2$ is not thin in $\mathbb{P}^1_k$.

Néron's classical example (1951)

Let k be a field, $\mathrm{car}(k) = 0$. Choose 8 rational points in general position in $\mathbb{P}^2_k$. Then all curves in the pencil of cubic curves through these 8 points are geometrically integral. There is a 9th point $P \in \mathbb{P}^2(k)$ which they all contain. One blows up the 9 points, this gives a k-rational surface with a morphism $X \to \mathbb{P}^1_k$ all of which fibres are geometrically integral. Restriction to the generic fibre X_η produces an exact sequence $0 \to \mathbb{Z} \to Pic(X) \to Pic(X_\eta) \to 0$. Let $K = k(\mathbb{P}^1)$. The rank of $Pic(X)$ is 10. Each blown-up point defines a section of $X \to \mathbb{P}^1_k$. Fixing one such point P gives X_η the structure of an elliptic curve E/K of Mordell-Weil rank 8.

(0) Néron's specialisation theorem gives : the set of points $m \in \mathbb{P}^1(k)$ with $rank(X_m(k)) \geq 8$ is not thin in $\mathbb{P}^1_k$.

(1) Since the total space X is k-rational, the "generic point theorem" gives that the set of points $m \in \mathbb{P}^1(k)$ with $rank(X_m(k)) \geq 9$ is not thin in $\mathbb{P}^1_k$.

(2) The Loughran–Salgado hypotheses hold for $X \to \mathbb{P}^1_k$ because the lines of $\mathbb{P}^2_k$ through P define a bisection of the pencil. If k is a number field, their result gives that the set of points $m \in \mathbb{P}^1(k)$ with $rank(X_m(k)) \geq 10$ is not thin in $\mathbb{P}^1_k$.

Over any number field, beyond the rank ≥ 8 case, Néron proved that there are elliptic curves of rank ≥ 9 and ≥ 10.
Serre in his Mordell-Weil book (Chapter 11) describes Néron's technique for rank ≥ 9 and ≥ 10.
Over an arbitrary hilbertian field k, one may use the proofs and the "generic point theorem" to improve upon these lower bounds by 1.

Let us discuss the improvement from $\rho \geq 9$ to $\rho \geq 10$.
Start from Serre, §11.2. Given 9 general points in $\mathbb{P}^2_k$, there is a unique cubic curve which contains them. The graph of the relation "9 points of $\mathbb{P}^2_k$ and a cubic of $\mathbb{P}^2_k$ through these 9 points" gives rise to a smooth fibration $X \to U$, with U open set in the $\mathbb{P}^9_k$ parametrizing cubics, and X k-birational to $(\mathbb{P}^2_k)^9$. The generic fibre is an elliptic curve $E/k(U)$ with $rank(E(k(U)) \geq 9$ (use the 9 points and the trace of a line in $\mathbb{P}^2_k$).
Néron's specialisation theorem over $U \subset \mathbb{P}^9_k$ gives elliptic curves E_m/k of rank ≥ 9. *As the total space X is k-rational*, the "generic point theorem" gives elliptic curves E_m/k with MW rank ≥ 10. In each case the set $m \in U(k)$ of such points is not thin hence nonempty if k is hilbertian.

Project : Check that Serre's §11.3 enables one to get MW rank ≥ 11.

Hilbert's irreducibility theorem holds over some classes of varieties

Methods to prove that certain varieties are hilbertian.

- Weak-weak approximation
- From above : Hilbertian unirationality
- From below : fibrations

Weak weak approximation

Let k be a number field and X a smooth connected k-variety, $X(k) \neq \emptyset$.

• Weak weak approximation holds for X with $X(k) \neq \emptyset$ if there exists a finite set S_0 of places of k such that for any finite set S of places of k with $S \cap S_0 = \emptyset$, $X(k)$ is dense in $\prod_{v \in S} X(k_v)$.

This property is conjectured (CT-Sansuc, CT) for k-varieties X which are geometrically rationally connected and satisfy $X(k) \neq \emptyset$, hence in particular for k-unirational varieties. Only known for some classes of varieties.

Theorem. *Let X be a smooth, projective, geometrically integral variety over a number field k. If weak weak approximation with respect to some S_0 as above holds, then for any thin set $E \subset X(k)$ and any S as above, $S \cap S_0 = \emptyset$, the closure of $X(k) \setminus E$ in $\prod_{v \in S} X(k_v)$ coincides with the closure of $X(k)$.*
[Tools : Tschebotarev and Lang-Weil estimates for varieties over a finite field.]

Proposition (Ekedahl, CT). *Weak weak approximation for X/k implies that X is hilbertian : $X(k)$ is not thin.*

Reference : Serre, Topics in Galois Theory, Chap. 3.
Consequence : If weak-weak approximation holds for unirational k-varieties, then any finite group is a Galois group over $\mathbb{Q}$.

The "generic point theorem" admits a variant in the framework of weak weak approximation.

Theorem. *Let $X \to U$ be a family of abelian varieties of relative dimension ≥ 1, with $U(k) \neq \emptyset$. Let $\mathcal{R} \subset U(k)$ be the set of points $m \in U(k)$ such that $rank(X_m(k)) > rank(X_\eta(k(U)))$. If weak weak approximation holds for the total space X, then there exists a finite set S of places of k such that $\mathcal{R}$ is dense in $\prod_{v \notin S} U(k_v)$.*

Weak weak approximation is known for the following classes of smooth varieties X over a number field k with $X(k) \neq \emptyset$, which are thus hilbertian.

• Conic bundle fibrations over $\mathbb{P}^1_k$ with a k-point and at most 5 geometric singular fibres.

Example. Châtelet surfaces. Equation $y^2 - az^2 = p(x)$ with $p(x)$ separable of degree 3.
Mais also be written $d(t)y^2 = p(x)$ with $d(t)$ separable of degree 2 and $p(x)$ separable of degree 3. This is a family of quadratic twists of the elliptic curve $y^2 = p(x)$. Using the "generic point theorem" and CT-Sansuc-Swinnerton-Dyer 1987, one gets a generalisation of a result of Rohrlich 1993 : over $k = \mathbb{Q}$, the set of $t \in \mathbb{Q}$ whose fibre has MW rank at least 1 is dense in $\mathbb{R}$.

• del Pezzo surfaces of degree 4 (Salberger–Skorobogatov).
• Smooth cubic hypersurfaces (Swinnerton-Dyer).
• Conic bundle fibrations $X \to \mathbb{P}^1_{\mathbb{Q}}$ whose geometric singular fibres X_m occur only at $\mathbb{Q}$-points m (Browning, Matthiesen et Skorobogatov 2014, building upon results in additive combinatorics). For example

$$y^2 - az^2 = \prod_{i=1}^{n} (x - e_i).$$

• Granting the Dickson-Bouniakowsky-Schinzel hypothesis : arbitrary conic bundles over $\mathbb{P}^1_k$, k number field.

More general results for some classes of rationally connected varieties : Harpaz and Wittenberg 2016, 2020.

From above : Hilbertian unirationality

Recall : *If k (char. zero) is a hilbertian field (e.g. a number field), then a k-variety which is Hilbert-unirational is hilbertian.*

For each of the following types of k-variety X, if $X(k) \neq \emptyset$ then X is Hilbert-unirational.

• $X = G/H$ with $H \subset G$ connected, linear algebraic groups (CT-Sansuc 1987)

• X conic bundle over $\mathbb{P}^1_k$ with 4 geometric degenerate fibres (CT-Skorobogatov 1987)

Whether the Hilbert-unirationality property holds for del Pezzo dP_n with $n = 4, 3, 2$ is an open question.
Even more so for dP_1's and general conic bundles over $\mathbb{P}^1_k$, for which k-unirationality and even Zariski density of k-points is an open question.

From below : fibrations

Theorem (BSFP, Bary-Soroker, Fehm, Petersen) *Let k be a hilbertian field. Let $f : X \to S$ be a morphism of geometrically integral k-varieties with geometrically integral generic fibre. Suppose that the set of points $m \in S(k)$ such that the fibre X_m is Hilbertian is not a thin subset of $S(k)$. Then X is hilbertian.*

Proof (Sketch). Let $Y \to X$ be a generically finite generic morphism of degree > 1, with Y/k geometrically integral. If for the Stein factorisation $Y \to T \to S$ of the composite map $Y \to X \to S$ the map $T \to S$ is not birational, then the image of $T(k)$ to $S(k)$ is thin. Otherwise for $m \in S(k)$ general the map $Y_m \to X_m$ is a generically finite map of degree > 1 between geometrically integral varieties.

The result follows.

Let k be a number field. Let X/k be a smooth, projective, geometrically integral surface and $f : X \to \mathbb{P}^1_k$ be a morphism with geom. integral generic fibre. For $m \in \mathbb{P}^1(k)$, let $X_m := f^{-1}(m)$. Let Y be a normal, geometrically integral, projective k-surface and $\phi : Y \to X$ be a generically finite proper morphism. Assume
(*) *The generic cover $Y_\eta \to X_\eta$ is ramified.*
Let $Y \to C \to \mathbb{P}^1_k$ be the Stein factorization of the composite map $Y \to X \to \mathbb{P}^1_k$. If $C \to \mathbb{P}^1_k$ is not an isomorphism, then the image of $C(k) \to \mathbb{P}^1(k)$ is thin. Assume that $C \to \mathbb{P}^1_k$ is an isomorphism. Then almost all fibres of $Y \to \mathbb{P}^1_k$ are geometrically integral. For all but finitely many points $m \in \mathbb{P}^1(k)$, the map $Y \to X$ induces a cover $Y_m \to X_m$ of geometrically integral curves, and this cover is ramified because of (*). Hence if $X_m(k)$ is infinite, it is not covered by $Y_m(k)$ (easy if fibres X_m are of genus zero, Néron's argument or Faltings if they are of genus one).

The same argument applies to a finite family $\phi_i : Y_i \to X$ of generically finite covers satisfying
(**) *Each generic cover $Y_{i,\eta} \to X_\eta$ is ramified.*

Theorem. *With notation as above, under (**), if the set of points $m \in \mathbb{P}^1(k)$ such that $X_m(k)$ is infinite is not thin in $\mathbb{P}^1_k$, then $X(k)$ is not covered by the union of the $\phi_i(Y_i(k))$ and the k-points of finitely many curves.*

The whole argument is due to Manin (Manin 1967 ; Кубические Формы 1972, VI.6.1, " Один отрицателныи результат"). It was rediscovered by Demeio 2018 (at some point one may replace the use of Néron's theorem by Merel's theorem on torsion if one wishes).
It was motivated by :

Corollary *Let k be a number field. Every smooth cubic hypersurface $Y \subset \mathbb{P}^n_k$, $n \geq 3$ with a k-point is hilbertian.*

Proof. Consider the case of a surface. The set $Y(k)$ is Zariski dense. Let $\phi_i : Y_i \to Y$ be finitely many generically finite covers, with Y_i/k geom integral. Since Y is geometrically simply connected, each $Y_i \to Y$ is ramified. There are many ways in which, after blow up of three colinear k-points, one may fibre Y into a family $Y' \to \mathbb{P}^1_k$ of elliptic curves of positive rank, hence *one may do it in such a way that the associated generic covers $Y'_{i,\eta} \to Y_\eta$ are ramified*, i.e. (**) holds. Using Néron's specialisation theorem for the fibration $Y' \to \mathbb{P}^1_k$, one then applies the theorem. The case of cubic hypersurfaces follows by fibration and induction on dimension from the BSFP theorem.

Double conic fibration

Theorem (S. Streeter 2019) *Let k be a hilbertian field, $\mathrm{char}(k) = 0$. If X/k is a dP_4, or a smooth cubic surface with a k-line, or a dP_2 or a dP_1 which is simultaneously a conic bundle fibration over $\mathbb{P}^1_k$, and $X(k) \neq \emptyset$, then X is hilbertian.*

Proof (sketch). Under the above hypotheses, X is k-unirational (Kollár-Mella) and X is k-birational to a Y with two distinct conic bundle fibrations $p, q : Y \to \mathbb{P}^1_k$. If a smooth Y_m at a k-point m has a k-rational point, then it is isomorphic to $\mathbb{P}^1_k$. The properties $Y_m(k) \neq \emptyset$ and Y_m hilbertian are thus equivalent.
If we can show that the set E of points $m \in \mathbb{P}^1(k)$ with $p^{-1}(m)(k) \neq \emptyset$ is not thin, the BSFP theorem will give that X is Hilbertian.

If E is thin in $\mathbb{P}^1_k$, then we have finitely many smooth projective geometrically integral curves C_i and morphisms $\phi_i : C_i \to \mathbb{P}^1_k$ of degree > 1 such that up to finitely many points, $E = \bigcup_i \phi_i(C_i(k))$. Let $R \subset \mathbb{P}^1_k$ be the union of the ramification loci of the ϕ_i. Let $\Delta \subset Y$ be the reduced divisor $p^{-1}(R) \subset Y$. In the pencil defined by q, one may find a k-point $n \in \mathbb{P}^1(k)$ such that $D = q^{-1}(n)$ is smooth, isomorphic to $\mathbb{P}^1_k$ and transverse to Δ. Let $D_i = D \times_{\mathbb{P}^1} C_i$ be the fibre product of $p : D \to \mathbb{P}^1_k$ and $\phi_i : C_i \to \mathbb{P}^1_k$. The assumption on E implies that $D(k)$ up to finitely many points is the union of the images of $D_i(k) \to D(k)$. Because $D \simeq \mathbb{P}^1_k$, this is possible only if one of the $D_i \to D$ admits a section. Then there is a dominant map $D \to C_i$ over $\mathbb{P}^1_k$. This is not possible, because $D \to \mathbb{P}^1_k$ is étale above the ramification points of $C_i \to \mathbb{P}^1_k$.

Using several elliptic fibrations

Let k be a number field. Arguments in the spirit of the cubic surface theorem, involving several fibrations into curves of genus one, and using precise study of ramification loci, have been used by Corvaja, Zannier, and Demeio to prove that some varieties, though not rationally connected, are hilbertian. An open question (Corvaja, Zannier) is whether any *algebraically simply connected*, smooth, projective geom. connected variety X/k with $X(k)$ Zariski dense is hilbertian. This should be compared with the open question (Skorobogatov) whether weak-weak approximation holds for smooth $K3$-surfaces with a k-point.

The following results have been obtained.

• Quartic surface $X \subset \mathbb{P}^3_k$ given by homogeneous equation $x^4 + y^4 = z^4 + t^4$ (Corvaja-Zannier 2016) and more generally (Demeio 2018) $ax^4 + by^4 + cz^4 + dt^4 = 0$ **with** $abcd \in k^{*2}$.

Theorem. *If the rational points are Zariski dense, then the surface is hilbertian.*

• Quotient X of a product of elliptic curves $E_1 \times E_2$ by the map $u \mapsto -u$. This is given by an equation $z^2 = f(x)g(y)$ with f and g separable of degree 3. There are two obvious elliptic fibrations (and a third one). Was studied by Kuwata-Wang 1993, Rohrlich 1993 ("root numbers"), Zhizhong Huang 2020.

Theorem (Demeio 2018). *If $E_1(k)$ and $E_2(k)$ are infinite, then X is hilbertian.*

In the opposite direction, Néron's specialisation theorem easily gives :

Theorem (Holmes–Pannekoek 2015)
Let $r \geq 1$. If E is an elliptic curve over a number field k. If the Kummer variety X quotient of E^r by the involution $u \mapsto -u$ is a hilbertian variety, then there exists a quadratic twist of E over k whose Mordell-Weil rank is at least r.
Proof. Consider the fibration $E \times_k E^r \to E^r$, with generic fibre $E_\eta = E_{k(E^r)})/k(E^r)$, with Mordell-Weil rank $\geq r + rank(E(k))$. Then consider the Weil descent of E_η for the quadratic extension $k(E^r)/k(X)$. Then apply Néron's specialisation theorem on X. This gives k-points $m \in X(k)$ with a quadratic extension $K = K(m)$ with $rank(E(K) \geq r + rank(E(k))$. The quotient $R_{K/k}(E)/E$ is then an elliptic curve over k of rank $\geq r$.

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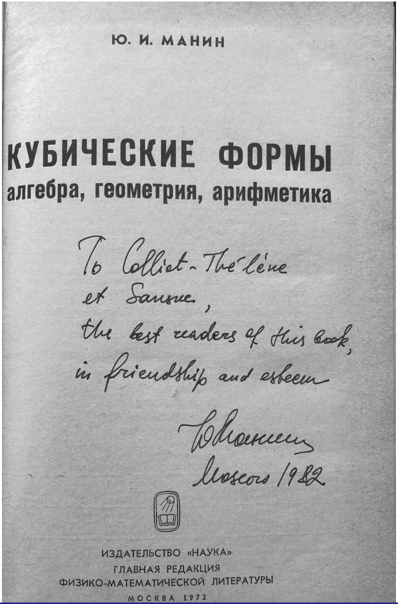
Ю.И. МАНИН



Ю. И. МАНИН

КУБИЧЕСКИЕ ФОРМЫ
алгебра, геометрия, арифметика

To Colliot-Thélène
et Sansuc,
the best readers of this book,
in friendship and esteem

Ю. Манин
Moscow / 1982

ИЗДАТЕЛЬСТВО «НАУКА»
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