

Structure of motivic cohomology groups

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This note is a short report on the divisibility properties of the *motivic cohomology groups* $H_M^i(X, \mathbb{Z}(j))$ for smooth projective varieties over separably closed fields and over finite fields. The note does not claim any originality.

1 Introduction

In the 80s, Spencer Bloch and then other authors [CTSS83, CTR85] investigated the abelian group structure of some particular K -cohomology groups, which later turned out to be motivic cohomology groups. The key rôle was played by results of Merkurjev and Suslin, and of Suslin, and by weight arguments which are consequences of Deligne's proof of the Weil conjectures. After the drastic progress in the definition and study of motivic cohomology groups by Rost, Suslin, Voevodsky, it is natural to ask what can be proven for arbitrary motivic cohomology groups $H_M^i(X, \mathbb{Z}(j))$.

The abelian group structure of the *étale motivic cohomology groups* $H_L^i(X, \mathbb{Z}(j))$ for smooth projective varieties X over a separably closed field was discussed in papers of A. Rosenschon and V. Srinivas [RS16, Prop. 3.1] (2016), of Geisser (2017) [G17, Theorem 1.1], and of B. Kahn [Ka18]. Kahn's paper discusses results for open smooth varieties over a separably closed field [Ka18, Thm. 1.3]. Geisser's paper discusses smooth projective varieties over separably closed fields, finite fields, local fields and also arithmetic schemes. Over a field of characteristic $p > 0$, he also considers the p -primary torsion of the étale motivic cohomology groups.

As pointed out by various authors, one should first compute étale motivic cohomology, for which one gets neat results in a whole range of indices, and then deduce results for motivic cohomology in the restricted range where these two theories coincide. It is however a curious fact that the proofs in [G17] and [RS16] for the groups $H_L^i(X, \mathbb{Z}(j))$ rely on the fact that the two cohomologies coincide when taken with rational coefficients, combined with the identification of the groups $H_M^i(X, \mathbb{Z}(j))$ with Bloch's higher Chow groups.

Thanks to Tamás Szamuely for pointing out the paper [G17] by Thomas Geisser on étale motivic cohomology groups.

2 Preliminaries

Let k be a field and X/k a smooth variety.

For $i, j \in \mathbb{Z}$ and $j \geq 0$, let $H_M^i(X, \mathbb{Z}(j))$, resp. $H_L^i(X, \mathbb{Z}(j))$, denote the Zariski motivic cohomology groups, resp. the étale motivic cohomology groups.

After work of Voevodsky and other authors, the following statements are known.

There is a natural map $H_M^i(X, \mathbb{Z}(j)) \rightarrow H_L^i(X, \mathbb{Z}(j))$ which is an isomorphism after tensoring with \mathbb{Q} . In particular the kernel and the cokernel of this map are torsion groups. This implies that for all primes ℓ the map $H_M^i(X, \mathbb{Z}(j)) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \rightarrow H_L^i(X, \mathbb{Z}(j)) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell$ is surjective.

Let $j \geq 0$ and $i \geq 0$. For X smooth over a field, the natural map $H_M^i(X, \mathbb{Z}(j)) \rightarrow H_L^i(X, \mathbb{Z}(j))$ is an isomorphism if $i \leq j + 1$, and it is injective if $i = j + 2$. This is referred to as the integral Beilinson–Lichtenbaum conjecture, closely related to the statement $H_L^{n+1}(F, \mathbb{Z}(n)) = 0$ for F an arbitrary field (higher Hilbert’s theorem 90). See [Vo1, Thm. 6.6], [Vo2, Thm. 6.18], [Riou, Thm. 2.2].

Let ℓ be a prime invertible in k . Building upon work of Suslin and Voevodsky, Geisser and Levine [GL01, Theorem 1.5] proved that multiplication by ℓ^r on the étale complex $\mathbb{Z}(j)_{\text{ét}}$ induces an exact triangle

$$\mathbb{Z}(j)_{\text{ét}} \rightarrow \mathbb{Z}(j)_{\text{ét}} \rightarrow \mu_{\ell^r}^{\otimes j}.$$

One then gets exact sequences

$$0 \rightarrow H_L^i(X, \mathbb{Z}(j))/\ell^r \rightarrow H_{\text{ét}}^i(X, \mu_{\ell^r}^{\otimes j}) \rightarrow H_L^{i+1}(X, \mathbb{Z}(j))[\ell^r] \rightarrow 0$$

and then, after taking direct limits,

$$0 \rightarrow H_L^i(X, \mathbb{Z}(j)) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \rightarrow H_{\text{ét}}^i(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j)) \rightarrow H_L^{i+1}(X, \mathbb{Z}(j))\{\ell\} \rightarrow 0.$$

3 A lemma on abelian groups

Lemma 3.1 *Let A be an abelian group. Assume $A \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell = 0$ and assume that the ℓ -primary torsion group $A\{\ell\}$ is an extension of a group F of finite exponent by a divisible group. Then there is a natural surjective map $A \rightarrow F$, compatible with the given map $A\{\ell\} \rightarrow F$ and whose kernel is the maximal ℓ -divisible subgroup of A .*

Proof The exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}[1/\ell] \rightarrow \mathbb{Q}_\ell/\mathbb{Z}_\ell \rightarrow 0$$

induces the following exact sequence

$$0 \rightarrow A\{\ell\} \rightarrow A \rightarrow A \otimes \mathbb{Z}[1/\ell] \rightarrow A \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \rightarrow 0,$$

Under our hypothesis, this gives the exact sequence

$$0 \rightarrow A\{\ell\} \rightarrow A \rightarrow A \otimes \mathbb{Z}[1/\ell] \rightarrow 0.$$

We have a given exact sequence

$$0 \rightarrow B \rightarrow A\{\ell\} \rightarrow F \rightarrow 0,$$

with B ℓ -divisible and F a group of exponent a finite power of ℓ .

The arrow $A\{\ell\} \rightarrow F$ gives rise to the following commutative diagram of exact sequences

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
& & B & \longrightarrow & B & & \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A\{\ell\} & \longrightarrow & A & \longrightarrow & A \otimes \mathbb{Z}[1/\ell] \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & F & \longrightarrow & A' & \longrightarrow & A \otimes \mathbb{Z}[1/\ell] \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \\
& & 0 & & 0 & &
\end{array}$$

Since F is an ℓ -primary group of finite exponent, the lower sequence is split in a unique way. This produces a surjective map $A' \rightarrow F$ whose kernel is the ℓ -divisible group $A \otimes \mathbb{Z}[1/\ell]$. The kernel A'' of the composite, surjective map $A \rightarrow A' \rightarrow F$ is an extension of $A \otimes \mathbb{Z}[1/\ell]$ by B . These two groups are ℓ -divisible, hence so is A'' . QED

4 Motivic cohomology over a separably closed field

Proposition 4.1 *Let k be a separably closed field. Let X be a smooth, projective, geometrically integral variety over k . Let ℓ be a prime invertible in k . Let $j \geq 0$ be an integer. Let $i \geq 0$.*

- (a) *If $i \neq 2j$, then $H_L^i(X, \mathbb{Z}(j)) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell = 0$.*
- (b) *For $i \neq 2j + 1$ and $i \geq 1$, we have*

$$H_L^i(X, \mathbb{Z}(j))\{\ell\} \simeq H_{\text{ét}}^{i-1}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j)).$$

The group $H_L^i(X, \mathbb{Z}(j))\{\ell\}$ is an extension of the finite group $H_{\text{ét}}^i(X, \mathbb{Z}_\ell(j))_{\text{tors}}$ by the divisible group $H_{\text{ét}}^{i-1}(X, \mathbb{Q}_\ell(j))/H_{\text{ét}}^{i-1}(X, \mathbb{Z}_\ell(j))$.

Proof Let k_0 be a field of finite type and k/k_0 a separable closure. Let $G = \text{Gal}(k/k_0)$. Let X_0/k_0 be smooth, projective, geometrically integral. Let ℓ be a prime invertible in k . Let $j \geq 0$ be an integer. Let $i \geq 0$.

Given any α dans $H_M^i(X, \mathbb{Z}(j)) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell$, up to replacing k_0 by a finite extension and X_0 by the similar base change, we may assume that α is in the image of a class in $H_M^i(X_0, \mathbb{Z}(j)) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell$. This uses the theorem that the motivic cohomology group $H_M^i(X, \mathbb{Z}(j))$ is identified with one of Bloch's higher Chow groups. The image of the composite map

$$\begin{aligned}
H_M^i(X_0, \mathbb{Z}(j)) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell &\rightarrow H_M^i(X, \mathbb{Z}(j)) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \rightarrow H_L^i(X, \mathbb{Z}(j)) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \\
&\rightarrow H_L^i(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j)) = H_{\text{ét}}^i(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j))
\end{aligned}$$

is invariant under G . From Deligne's result, we know [CTR85, Thm. 1.5]) that the group of G -invariants of $H_{\text{ét}}^i(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j))$ is finite if $i \neq 2j$. As the group $H_M^i(X_0, \mathbb{Z}(j)) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell$ is divisible, the composite map vanishes. Thus the composite map

$$H_M^i(X, \mathbb{Z}(j)) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \rightarrow H_L^i(X, \mathbb{Z}(j)) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \rightarrow H_{\text{ét}}^i(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j))$$

vanishes if $i \neq 2j$. The LHS map is onto (see preliminaries) and the RHS map is injective (see preliminaries). We conclude that the group $H_L^i(X, \mathbb{Z}(j)) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell$ vanishes if $i \neq 2j$.

From this and the exact sequence in the preliminaries we deduce that for $i \neq 2j$ there is a natural isomorphism

$$H_L^{i+1}(X, \mathbb{Z}(j))\{\ell\} \simeq H_{\text{ét}}^i(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j)).$$

With other indexing : For $i \neq 2j + 1$ and $i \geq 1$, we have a natural isomorphism

$$H_L^i(X, \mathbb{Z}(j))\{\ell\} \simeq H_{\text{ét}}^{i-1}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j)).$$

QED

Remark 4.2 The idea to use coincidence of motivic cohomology and étale motivic cohomology with rational coefficients, so that one may then use the representation of classes over some small subfield, is in [G17, Proof of Thm. 1.1] and [RS16, Prop. 3.1]. Geisser uses inverse limits. Here we used direct limits.

Upon use of Lemma 3.1 we deduce :

Proposition 4.3 *Let X/k be smooth and projective over a separably closed field k . Let ℓ be a prime invertible in k . Assume $i \geq 1$ and $i \neq 2j, 2j + 1$.*

(a) *There is a natural surjective map $H_L^i(X, \mathbb{Z}(j)) \rightarrow H_{\text{ét}}^i(X, \mathbb{Z}_\ell(j))_{\text{tors}}$ whose kernel B_ℓ is the maximal ℓ -divisible subgroup of $H_L^i(X, \mathbb{Z}(j))$.*

(b) *The ℓ -primary torsion subgroup of B_ℓ is the group $H_{\text{ét}}^{i-1}(X, \mathbb{Q}_\ell(j))/H_{\text{ét}}^{i-1}(X, \mathbb{Z}_\ell(j))$, which is the maximal divisible subgroup of $H_{\text{ét}}^{i-1}(X, \mathbb{Q}_\ell(j))$.*

(c) *If $\text{char}(k) = 0$, the group $H_L^i(X, \mathbb{Z}(j))$ is an extension of the finite group $\bigoplus_\ell H_{\text{ét}}^i(X, \mathbb{Z}_\ell(j))_{\text{tors}}$ by a divisible subgroup whose ℓ -primary torsion is $H_{\text{ét}}^{i-1}(X, \mathbb{Q}_\ell(j))/H_{\text{ét}}^{i-1}(X, \mathbb{Z}_\ell(j))$.*

(d) *If moreover $i \leq j + 1$, the isomorphism $H_M^i(X, \mathbb{Z}(j)) \simeq H_L^i(X, \mathbb{Z}(j))$ yields the same structure results for $H_M^i(X, \mathbb{Z}(j))$.*

We also have :

Proposition 4.4 *Let X/k be smooth, projective, connected of dimension d over a separably closed field k . Let ℓ be a prime invertible in k . Let $i \geq 0$ and $j \geq 2$. If one of the following hypotheses holds:*

(1) $2j < i$

(2) $i \leq j + 1$

(3) $d \leq j$ and $i \neq 2j$

then $H_M^i(X, \mathbb{Z}(j)) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell = 0$.

Statement (1) is a general result on motivic cohomology. Indeed, for $i > 2j$, one has $H_M^i(X, \mathbb{Z}(j)) = 0$. Statement (2) is a consequence of (a) and (d) in Proposition 4.3. For any smooth, connected, quasiprojective X over k , for $d \leq j$ and $i \leq 2j$, Suslin [Su99, Cor. 3, p. 254] proved that the maps

$$H_M^i(X, \mathbb{Z}/\ell^r(j)) \rightarrow H_{\text{ét}}^i(X, \mu_{\ell^r}^{\otimes j})$$

are isomorphisms. For $d \leq j$, and any $i \geq 0$, one then gets injections

$$H_M^i(X, \mathbb{Z}(j)) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \hookrightarrow H_{\text{ét}}^i(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j)).$$

For X smooth, connected, and projective, and $i \neq 2j$, a weight argument gives that this map is zero.

5 Motivic cohomology over a finite field

Let X/\mathbb{F} be a smooth, projective, geometrically integral variety over a finite field. Let ℓ be a prime invertible in \mathbb{F} . Let $i \geq 0$. We have the exact sequence

$$0 \rightarrow H_L^i(X, \mathbb{Z}(j)) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \rightarrow H_{\text{ét}}^i(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j)) \rightarrow H_L^{i+1}(X, \mathbb{Z}(j))\{\ell\} \rightarrow 0.$$

Assume $i \neq 2j, 2j+1$. By Deligne's results on the Weil conjectures, the group $H_{\text{ét}}^i(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j))$ is then finite [CTSS83, Thm. 2 p. 780]. We conclude that

$$H_L^i(X, \mathbb{Z}(j)) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell = 0$$

and there is an isomorphism of finite groups

$$H_L^{i+1}(X, \mathbb{Z}(j))\{\ell\} \simeq H_{\text{ét}}^{i+1}(X, \mathbb{Z}_\ell(j))_{\text{tors}}.$$

If $i \neq 2j+1, 2j+2$, we thus have an isomorphism of finite groups

$$H_L^i(X, \mathbb{Z}(j))\{\ell\} \simeq H_{\text{ét}}^i(X, \mathbb{Z}_\ell(j))_{\text{tors}}.$$

Upon use of Lemma 3.1 we deduce :

Proposition 5.1 *Let X/\mathbb{F} be a smooth, projective, geometrically integral variety over a finite field. Let ℓ be a prime invertible in \mathbb{F} . If $i \neq 2j, 2j+1, 2j+2$, the group $H_L^i(X, \mathbb{Z}(j))$ is an extension of the finite group $H_{\text{ét}}^i(X, \mathbb{Z}_\ell(j))_{\text{tors}}$ by the maximal ℓ -divisible subgroup of $H_L^i(X, \mathbb{Z}(j))$, and that subgroup is uniquely ℓ -divisible.*

If moreover $i \leq j+1$, then the isomorphism $H_M^i(X, \mathbb{Z}(j)) \simeq H_L^i(X, \mathbb{Z}(j))$ yields the same structure results for $H_M^i(X, \mathbb{Z}(j))$.

We also have :

Proposition 5.2 *Let X/\mathbb{F} be smooth, projective, connected of dimension d over a finite field \mathbb{F} . Let ℓ be a prime invertible in \mathbb{F} . Let $i \geq 0$ and $j \geq 2$. If one of the following hypotheses holds:*

- (1) $2j < i$
 - (2) $i \leq j+1$
 - (3) $d \leq j$ and $i \neq 2j$
- then $H_M^i(X, \mathbb{Z}(j)) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell = 0$.

Statement (1) is a general result on motivic cohomology. Indeed, for $i > 2j$, one has $H_M^i(X, \mathbb{Z}(j)) = 0$.

Statement (2) is a consequence of (a) and (d) in Proposition 5.1.

For Statement (3), one uses results of Kerz and Saito [KS12].

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