Stable rationality and quadratic forms
Part II

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Hassett-Pirutka-Tschinkel Acta Math. 2018 (specialization, Brauer group)
Surveys (Voisin, CT) in Birational Geometry of Hypersurfaces (Gargnano del Garda 2018), Springer LNUMI 26 (2019)
Schreieder1 DMJ 2019 Quadric bundles over $\mathbb{P}^n$ (specialization, unramified cohomology)
Schreieder2 ANT 2018 Quadric bundles of relative dimension 2 over $\mathbb{P}^2$ (specialization, Brauer group)
Schreieder3 JAMS 2019 Hypersurfaces with small slope (specialization, unramified cohomology)
Schreieder4 ANT 2020 Minimum degree for unirationality (specialization, unramified cohomology)

Hassett-Pirutka-Tschinkel Acta Math. 2018

Quadric bundles of type (2,2) in $\mathbb{P}^3 \times \mathbb{P}^2$.

Starts with a specific example over $\mathbb{P}^2_{\mathbb{C}}$. Configuration : a smooth conic $F = 0$ and three tangent lines.
$F(x, y, z) = x^2 + y^2 + z^2 - 2(xy + yz + zx)$ and the coordinate axes : $x = 0, y = 0, z = 0$. The conic $F = 0$ does not contain points such as $(1, 0, 0)$. The homogeneous form F induces a square on each coordinate axis.
Let $W \subset \mathbb{P}^2 \times \mathbb{P}^3$ be given by

$$yzX^2 + zxY^2 + xyZ^2 + F(x, y, z)T^2 = 0.$$

This is a family $W \to \mathbb{P}^2$ of 2-dimensional quadrics.

HPT compute an explicit resolution of singularities $\tilde{W} \to W$. They show the morphism $\tilde{W} \to W$ is universally CH_0-trivial. This part is technically demanding. As Schreieder later proved (see below), one can dispense with the explicit resolution.

For Q a smooth 2-dimensional quadric over a field F, if its discriminant is not a square then the map $\mathrm{Br}(F) \to \mathrm{Br}(F(Q))$ is injective (and its image is $\mathrm{Br}(Q)$)).

Consider the quaternion symbol $(x/z, y/z) \in \mathrm{Br}(\mathbb{C}(\mathbb{P}^2))$. One shows that the image of $(x/z, y/z)$ dans $\mathrm{Br}(\mathbb{C}(W))$ is nonzero and is unramified. This relies on the tangency properties of the given plane configuration. Same argument as for the Artin-Mumford example (in the CT-Oj version, without using the explicit desingularisation).

Using this example and the specialization method, HPT then prove :

Theorem (HPT). *There exist* **smooth** *projective families $X \to S$ with connected S over $\mathbb{C}$, whose very general fibres are not stably rational but which have some rational special fibres.*
The fibres $X_s, s \in S$, are 4-dimensional varieties which are quadric surface bundles over $\mathbb{P}^2_{\mathbb{C}}$. One uses bidegree $(2, 2)$ in $\mathbb{P}^2 \times \mathbb{P}^3$ and applies Bertini's theorem (would not work in $\mathbb{P}^2 \times \mathbb{P}^2$).
Some fibres are rational : one specializes a $(4, 4)$ symmetric matrix with coefficients homogenous polynomials $a_{i,j}$, $0 \leq i, j \leq 3$, of degree 2 en (x_0, x_1, x_2), so that $a_{0,0} = 0$ for the special fibre. The special fibre is then a quadric family $X_s \to \mathbb{P}^2$ with a section, hence X_s is rational. In fact, the points of S with rational fibres are dense for the complex topology, by an argument due to C. Voisin).

From last time, recall :

Theorem (CT-Pirutka 2014)

Let A be a discrete valuation ring, K its field of fractions, k its residue field. Assume k algebraically closed, char.(k) zero.
Let $\mathcal{X}/A$ be proper and faithfully flat with geometrically integral fibres.
Assume the special fibre $Y = \mathcal{X} \times_A k$ admits a desingularization $f : Z \to Y$ **such that f is (universally) CH_0-trivial**.
Let $\overline{K}$ be an algebraic closure of K. Let $\overline{X} := X \times_K \overline{K}$ and $W \to \overline{X}$ a desingularisation.
/...

Each of the following statements implies the next one :
(i) The $\overline{K}$-variety $\overline{X}$ is stably rational.
(ii) The smooth $\overline{K}$-variety W is (universally) CH_0-trivial.
(iii) The (smooth) k-variety Z is (universally) CH_0-trivial.
This last property implies :
(a) For any $i \geq 0$ and any $n > 0$, for any overfield L of k, the map $H^i(L, \mathbb{Z}/n) \to H^i_{nr}(L(Z)/L, \mathbb{Z}/n)$ is an isomorphism.
(b) For any overfield L of k, the natural map $\mathrm{Br}(L) \to \mathrm{Br}(Z_L)$ is an isomorphism.
(c) $\mathrm{Br}(Z) = 0$.

Schreieder's methods and results

• *Avoids explicit CH_0-trivial resolution of singularities of the special fibre of a quadric fibration $\mathcal{X}/A$.*
• *Finds non-stably rational r-quadric bundles over $\mathbb{P}^n_{\mathbb{C}}$ with smooth total space for all r with $2^{n-1} - 1 \leq r \leq 2^n - 2$, by specialisation to special families of Pfister quadrics or closely related quadrics*
• *Of course still needs elaborate configurations of hypersurfaces in the bottom $\mathbb{P}^n$ to ensure the auxiliary ramified cohomology classes on $\mathbb{C}(\mathbb{P}^n)$ become unramified in the total space of the special fibre $Z \to \mathbb{P}^n$.*
• *New trick to ensure the cohomology class does not vanish over the total space (of the special fibre $Z \to \mathbb{P}^n$), trick which avoids the use of Witt, Arason, Voevodsky type of results : auxiliary specialisation to a situation with a section.*

- *Handles positive characteristic via alterations, thanks to the method avoiding CH_0-trivial resolution of singularities*
- *Degeneration of hypersurfaces of degree d to hypersurfaces of degree d which contain a suitable linear subspace of multiplicity $d-2$, so that these hypersurfaces are birational to quadric bundles over projective space handled by the previous method.*
- *Use of "Pfister-Fermat hypersurfaces" instead of Pfister quadric hypersurfaces to handle char. 2, and gets strong lower bounds for unirationality degree.*

Avoiding explicit CH_0-trivial resolution of singularities of the special fibre

Theorem A. *Let R be a discrete valuation ring, K its field of fractions, k its residue field. Assume $k = \mathbb{C}$. Let $\overline{K}$ be an algebraic closure of K. Let $\mathcal{X}/R$ be a flat integral projective scheme over R with geometrically integral fibres. Let X/k be the generic fibre Z/k be the special fibre. Let $f : \tilde{Z} \to Z$ be a desingularisation. Let $i \geq 1$, $m \geq 2$. Let $\alpha \in H^i_{nr}(k(Z), \mathbb{Z}/m)$.*

Assume

(i) At any schematic point $y \in \tilde{Z}$ of codimension 1 which lies over the singular locus of Z (there are finitely many), the restriction of α in $H^i(k(y), \mathbb{Z}/m)$ vanishes.

(ii) The $\overline{K}$-variety $X_{\overline{K}}$ admits a smooth projective model which is CH_0-trivial, for instance $X_{\overline{K}}$ is stably rational.

Then $\alpha = 0 \in H^i(k(Z), \mathbb{Z}/m)$.

The proof is rather formal. It uses

– The functoriality of CH_0 under proper morphisms

– the (obvious) localization sequence for Chow groups of zero-cycles for restriction to an open set

– The fact that unramified cohomology of a smooth projective integral variety W/F admits a pairing with the Chow group of zero-cycles

$$CH_0(W) \times H^i_{nr}(F(W), \mathbb{Z}/m) \to H^i(F, \mathbb{Z}/m)$$

– henselisation, Hensel's lemma, finite extension of K, base change from k to $k(Z)$.

Note that Hypothesis (i) is automatically satisfied if the closures of the y's are all rational. Indeed, in that case $H^i_{nr}(k(y)/k, \mathbb{Z}/m) = 0$. Vanishing may also occur for Galois cohomological dimension reasons, in situations where the closures of the y are far from being rational, as may occur after alterations.

In Schr1, Schr 2 (HPT case), and the first part of Schr3, the hypothesis

(i) the restriction of $\alpha \in H^i_{nr}(k(Z), \mathbb{Z}/m)$ to each y vanishes

in Theorem A was checked in ad hoc manners for the schemes $\mathcal{X}/R$ and Z and the relevant integer i under consideration (in the HPT case it is actually quite easy to do it). *The varieties Z/k considered there are all equipped with a proper surjective morphism $p : Z \to \mathbb{P}^n_{\mathbb{C}}$ for some $n \geq 1$, with generic fibre a smooth quadric.* For such $p : Z \to \mathbb{P}^n_{\mathbb{C}}$, by consideration of the possible bad reduction for an arbitrary smooth quadric over the field of fractions of a DVR, Schreieder (Schr3) establishes the general result :

Theorem B. *Let $Z \to \mathbb{P}^n_{\mathbb{C}}$ be as above. Suppose $\alpha \in H^n_{nr}(k(Z), \mathbb{Z}/2)$ is of the shape $\alpha = p^*(\beta)$ for some $\beta \in H^n(\mathbb{C}(\mathbb{P}^n), \mathbb{Z}/2)$ (same n). Then condition (i) in Theorem A is automatically fulfilled.*

This gives the :

Theorem C. *Let R be a discrete valuation ring, K its field of fractions, k its residue field. Assume $k = \mathbb{C}$. Let $\overline{K}$ be an algebraic closure of K. Let $\mathcal{X}/R$ be a flat integral projective scheme over R with geometrically integral fibres.*
Assume the special fibre Z/k admits a proper morphism $p : Z \to \mathbb{P}^n_k$ whose generic fibre is a smooth quadric of dimension at least 1. If there exists a cohomology class $\beta \in H^n(k(\mathbb{P}^n), \mathbb{Z}/2)$ (same n) whose image $\alpha \in H^n(k(Z), \mathbb{Z}/2)$ is **unramified** *and* **nonzero**, *then no smooth projective model of $\mathcal{X} \times_R \overline{K}$ is CH_0-trivial. In particular $\mathcal{X} \times_R \overline{K}$ is not stably rational.*

We now look for a complex rational variety S of dimension n with a dominant fibration $Z \to S$ whose generic fibre is a smooth quadric and could be used in the above theorem C. For $r > 2^n - 2$, Tsen-Lang gives the existence of a rational section, hence the function field of Z is purely transcendental.
In Schr1, Schreieder produces examples in all smaller relative dimensions. Let us just consider the case of maximal relative dimension.
Theorem D. *Let $r = 2^n - 2$. A very general hypersurface in $\mathbb{P}^{r+1} \times \mathbb{P}^n$ with bidegree $(2, d)$ and $d \geq n + 2r + 1$ is not stably rational.*

Construction of "special" quadrics Z over $S = \mathbb{P}^n$. The generic fibre over $\mathbb{C}(S)$ is a Pfister form $<< a_1, \ldots, a_{n-1}, b_1.b_2 >>$ and the a_i's and b_j's enjoy valuative properties similar to those in CTOj89. This follows from strong tangency properties for the zeros and poles of the a_i's and b_j's.
The element $\beta = (a_1) \cup \ldots (a_{n-1}) \cup (b_1)$ in $H^n(\mathbb{C}(S), \mathbb{Z}/2)$ has nonzero image α in $H^n(\mathbb{C}(Z), \mathbb{Z}/2)$ *does not vanish* and is *unramified*.
$\alpha \neq 0 \in H^n(\mathbb{C}(Z), \mathbb{Z}/2)$: Analogue of the Witt, Arason argument in CTOj89 : Prove that α does not vanish by studying the ramification of β on $\mathbb{P}^n_{\mathbb{C}}$ and invoking Orlov-Vishik-Voevodsky (OVV 2007) : for the function field F of a Pfister neighbour of $<< a_1, \ldots, a_n >>$ over a field k, the kernel of $H^n(k, \mathbb{Z}/2) \to H^n(F, \mathbb{Z}/2)$ is spanned by the cup-product $(a_1) \cup \cdots \cup (a_n)$ where $(b) \in k^*/k^{*2} = H^1(k, \mathbb{Z}/2)$ is the class of $b \in k^*$.

α **is unamified** : Here comes a combinatorial but technically involved part : to define the a_i's and b_j's , one produce configurations of hyperplanes and quadrics in $\mathbb{P}^n$, with “tangency conditions” which will play for $Z/\mathbb{P}^n$ the rôle of configurations of lines in $\mathbb{P}^2$ and planes in $\mathbb{P}^3$ in CTOj89. In the article Schr1, rather complicated solution, coefficients of rather high degree. In Schr3, simpler solution.

The varieties $Z/\mathbb{P}^n_{\mathbb{C}}$ which one produces are in general singular. One wants to deform them into a flat projective family W/T where $W_0 = Z$ and the general fibre W_t is smooth and equiped with a flat projective map $W_t \to \mathbb{P}^n_{\mathbb{C}}$ all fibres of which are quadrics.
To achieve this, one needs to choose good "types" of fibrations and apply Bertini theorems for linear systems without base point.
This imposes further restrictions on the degrees such as the one in Theorem D.

Schreieder 2 ANT 2018 Quadric surface bundles $X \to \mathbb{P}^2$
Gets rid of CH_0-trivial resolution of singularities of the special fibre given by

$$yzX^2 + zxY^2 + xyZ^2 + F(x, y, z)T^2 = 0.$$

Method avoiding explicit resolution of singularities particularly simple in this case. Rest of the argument as in HPT.
Schr2 produces many more types of quadric surface bundles. For example :

- A very general hypersurface of bidegree $(2, d)$ in $\mathbb{P}^3 \times \mathbb{P}^2$ is not stably rational as soon as $d \geq 2$.
- Inspiration for Schreiderer 3 : use of some quadratic forms which slightly differ from Pfister forms.

Schreieder 3 JAMS 2019, Hypersurfaces with small slopes

Theorem. *A very general smooth hypersurface* $X \subset \mathbb{P}^{N+1}_{\mathbb{C}}$ *with dimension* $N \geq 3$ *and degree* $d \geq log_2 N + 2$ *is not stably rational.*
Recall : Kollár, Totaro had obtained $d \geq 2N/3$ (roughly).

Let $N \geq 3$. There is a unique way to write $N = n + r$ with $n \geq 2$ and $r \geq 1$ and $2^{n-1} - 2 \leq r \leq 2^n - 2$. The integer n thus determined is of the order of $log_2 N$. Precise statement :
Therorem (Schreieder) *Fix N and let n be as above. A very general hypersurface* $X \subset \mathbb{P}^{N+1}_{\mathbb{C}}$ *with degree* $d \geq n + 2$ *is not stably rational.*

That powers of 2 appear has to do with the proof : one uses Pfister forms, whose rank is a power of 2.

Method : **Use Theorem C, with special fibre defined by new type of diagonal quadric fibrations over $\mathbb{P}^n$, generalizing those of HPT and Schr2.**

Let $x_i = X_i/X_0$. Let $F = \mathbb{C}(\mathbb{P}^n) = \mathbb{C}(x_1, \ldots, x_n)$. One uses an even degree homogeneous form $G(X_0, \ldots, X_n)$ which induces a square on each coordinate axis $X_i = 0$ and which does not pass through $(1, 0, \ldots, 0)$. Let $g = G(1, x_1, \ldots, x_n)$. Take a quadratric form

$$q =< g, c_1, \ldots, c_{r+1} >$$

over F, with $r \leq 2^n - 2$ and each $c_i \in F^\times, i = 1, \ldots, r$ a product of x_j's pour $j = 1, \ldots, n$. This defines a smooth quadric Q over $\mathbb{C}(\mathbb{P}^n)$. The form q is slightly deformed with respect to the form $< 1, c_1, \ldots, c_{r+1} >$, which itself is a subform of the Pfister form $<< x_1, \ldots, x_n >>$.

One considers the family $Z/\mathbb{P}^n$ of quadrics over $\mathbb{P}^n$ defined by q over $F = \mathbb{C}(\mathbb{P}^n)$ and the cohomology class
$\beta = (x_1) \cup \cdots \cup (x_n) \in H^n(\mathbb{C}(\mathbb{P}^n), \mathbb{Z}/2)$
(where $(x_i) \in F^\times/F^{\times 2} = H^1(F, \mathbb{Z}/2)$).
Uses the techniques from Schr1 (avoiding CH_0-trivial resolution of singularities), one shows :
(a) the image $\alpha \in H^n(\mathbb{C}(Z), \mathbb{Z}/2)$ of β in $H^n(\mathbb{C}(\mathbb{P}^n), \mathbb{Z}/2)$ is unramified.
(b) The class α vanishes on smooth components of a desingularisation which do not lie over the generic point of $\mathbb{P}^n$ (direct computation may be avoided by Theorem B on quadric fibrations).

One also wants to show $\alpha \neq 0 \in H^n(\mathbb{C}(Z), \mathbb{Z}/2)$. If $r = 2^n - 2$ one may use OVV. This would not work for all r.
New trick, avoiding the recourse to Witt, Arason, ..., Voevodsky. One puts a variable t in the coefficients of g, one specializes to a relative quadric $Q_{t=0}$ which has a rational point, at this level we have injection, so the class β specializes at $t = 0$ to something nonzero, hence is nonzero.

How to disprove stable rationality for very general smooth hypersurfaces of small slope in $\mathbb{P}^{N+1}$?

For suitable r, d, N, a (very general) smooth hypersurface X_0 of degree d in $\mathbb{P}^{N+1}_{\mathbb{C}}$ may be specialized to a given hypersurface $X_1 \subset \mathbb{P}^{N+1}$ of degree d with multiplicity $d-2$ along an r-plane $L \subset X_1$. Blowing up the r-plane L in $\mathbb{P}^{N+1}$, the proper transform of X_1 is a quadric bundle $Z \to \mathbb{P}^n$ of relative dimension r. One may arrange that Z is exactly as in the above construction. Let $\tilde{Z} \to Z$ be a desingularisation. One then has the composite map $\pi : \tilde{Z} \to Z \to X_1$. One produces a nonzero element $\alpha \in H^n_{nr}(\mathbb{C}(\tilde{Z}), \mathbb{Z}/2) = H^n_{nr}(\mathbb{C}(Z_{1)}, \mathbb{Z}/2) = H^n_{nr}(\mathbb{C}(X_1), \mathbb{Z}/2$ which vanishes on each codimension 1 point of $\tilde{Z}$ which lies over a singular point of X_1 [A specific argument is required for the points lying over $L \subset X_1$.] Theorem C then gives that the hypersurface X_0 is not CH_0-trivial, and in particular is not stably rational.

Here are some details. Set $N = n + r$ with $n \geq 2$ and $r \geq 1$ and $2^{n-1} - 2 \leq r \leq 2^n - 2$. Here we only discuss the case n odd.
One looks for hypersurfaces of even degree $d \geq n + 2$ in $\mathbb{P}^{N+1}$.
Homogeneous coordinates $(X_0, \dots, X_n, Y_1, \dots, Y_{r+1})$ for $\mathbb{P}^{N+1}$.
Let t be a variable and $H = \sum_{i=0}^n X_i^{(n+1)/2}$and
$G = t^2 H^2 - \prod_{i=0}^n X_i$. Let $g = G/X_0^{n+1}$.
For $\varepsilon : \{1, \dots, n\} \to \{0, 1\}$, let $C_\varepsilon := \prod_{i=1}^n X_i^{\varepsilon(i)}$. Let
$c_\varepsilon(x_1, \dots, x_n) = C_\varepsilon(1, x_1, \dots, x_n)$.
Renumber the C_ε as $C_0, \dots, C_{2^n-1}$. We have the quadratic form
$< g, c_1, \dots, c_{r+1} >$ and the n-Pfister neighbour
$\psi =< 1, c_1, \dots, c_r >$ (recall : $r \geq 2^{n-1} - 2$). One arranges that
$c_1 = x_1 \cdot \ldots \cdot x_r$.

Let $E_0(X_0,\ldots,X_n)=(X_0+X_1)^{d-(n+1)}G(X_0,\ldots,X_n)$. This is a form of degree d.
For $i=1,\ldots,r+1$ define $E_i(X_0,\ldots,X_n)=X_0^{d-2-d_i}C_i$, where $d_i=deg(C_i)$. These are forms of degree $d-2$.
Define the degree d form in $N+2$ variables :

$$F(X_0,\ldots,X_n,Y_1,\ldots,Y_{r+1})=E_0(X_0,\ldots,X_n)+\sum_{i=1}^{r+1}E_i(X_0,\ldots,X_n).Y_i^2.$$

The hypersurface $F=0$, of degree d, contains the linear space $L\subset\mathbb{P}^{N+1}$ defined by $X_0=\cdots=X_n=0$, and this space is $(d-2)$-multiple on the hypersurface.
One blows up L in $\mathbb{P}^{N+1}$ and considers the proper transform of $F=0$. This gives a quadric fibration $Z\to\mathbb{P}^n$ whose generic fibre is given by $<g,c_1,\ldots,c_{r+1}>=0$.

On shows that the class $\beta = (x_1) \cup \cdots \cup (x_n) \in H^n(\mathbb{C}(\mathbb{P}^n), \mathbb{Z}/2)$ has an image $\alpha \in H^n(\mathbb{C}(Z), \mathbb{Z}/2)$ which satisfies

(a) α **is unramified**. Here one reduces to a "small" ground field and one uses an element t transcendental over that field. One uses the special form $G = t^2H^2 - \prod_{i=0}^n X_i$ and the fact that the cohomology class $(x_1) \cup \cdots \cup (x_n)$ vanishes in the field of functions of a Pfister neighbour of $\psi =<< x_1, \ldots, x_n >>$.

(b) If $\phi : \tilde{Z} \to Z$ is a resolution of singularities inducing $\phi^{-1}(U) \simeq U$ and $Z \setminus \phi^{-1}(U)$ is a union of smooth components none of them lying over the generic point of $\mathbb{P}^n$, then α **vanishes on these components**. This follows from (a) since $Z \to \mathbb{P}^n$ is a quadric bundle fibration.

(c) $\alpha \neq 0$: to show this, one specializes t in $t = 0$: one gets a fibration $Z_0 \to \mathbb{P}^n$ with a section, and α specializes to α_0 which is the image of $\beta_0 = \beta \neq 0 \in H^n(\mathbb{C}(\mathbb{P}^n), \mathbb{Z}/2)$. The proof is elementary but a bit more elaborate. This gets rid of the OVV argument, which here works only if $r = 2^{n-2}$.

As hinted above, an extra argument is required to establish the vanishing of α at codimension 1 points of $\tilde{Z}$ lying over L. This extra argument reduces to the fact that for a Pfister neighbour q of a Pfister form $<< a_1, \ldots, a_n >>$ over a field F, the cohomology class $(a_1) \cup \cdots \cup (a_n) \in H^n(F, \mathbb{Z}/2)$ vanishes in $H^n(F(q), \mathbb{Z}/2)$ (again no OVV, only the easy part).

One then applies Theorem C.

Thus :

Therorem (Schreieder) *Let $N = n + r$ with $n \geq 2$ and $r \geq 1$ and $2^{n-1} - 2 \leq r \leq 2^n - 2$. . A very general hypersurface $X \subset \mathbb{P}_{\mathbb{C}}^{N+1}$ with degree $d \geq n + 2$ is not stably rational.*

The paper Schr3 actually handles the situation over uncountable algebraically closed fields of arbitrary characteristic, except $p = 2$ – as one might expect because of the use of Pfister forms.

In positive characteristic, use is made of the alteration results of de Jong and Gabber.

Schreieder 4 ANT 2020 Torsion order of Fano hypersurfaces
Uses higher degree analogues of (quadratic) Pfister forms.
A Pfister form $<< a_1, \ldots, a_n >>$ over a field k is a diagonal quadratic form in 2^n variables x_i. Schreieder defines Fermat-Pfister forms $\Phi_{n,m}$. These are forms of degree m in 2^n-variables $x_1, \ldots, x_{2^n}$ obtained by taking the Pfister form $<< a_1, \ldots, a_n >>$ and replacing each x_i^2 by x_i^m. Such Fermat-Pfister forms had already been used by Krashen-Matzri for a different purpose (comparing Galois cohomological and Lang's properties C_i).
There is the following analogue of a basic fact for Pfister forms :
Proposition. *Assume* $\mathrm{char}(k)$ *does not divide* m*. Let* F *be the function field of the projective hypersurface* $\Phi_{n,m} = 0$*. Then the symbol* $(a_1) \cup \cdots \cup (a_n) \in K_n^M(k)/m$ *has trivial image in* $K_n^M(F)/m$*.*

Over a field whose characteristic does not divide m, Schr4 establishes stable irrationality results for the total space Z of special families of Fermat-Pfister hypersurfaces of degree m over a projective space $\mathbb{P}^r$, using $H^r_{nr}(\bullet, \mathbb{Z}/m)$. Gets a cohomology class of order m.
(In particular, gets result for char. 2).
Tangency conditions ensuring class coming from $\mathbb{P}^r$ becomes unramified : one starts with a form of degree m on $\mathbb{P}^r$ which induces m-th powers on the coordinate axes (cf. HPT, Schr1, Schr2, Schr3).
One does not have a general Theorem B as for quadric families, but a direct proof as had been first done for Pfister quadric families works in this higher degree situation.

Using a specialisation technique to a situation which a section as we had in Schr3 (no need of a would-be Pfister-Fermat analogue of OVV) to conclude that one gets a nontrivial cohomology class. Then degeneration of smooth hypersurfaces to hypersurfaces containing a suitable multiple linear space, which are birational to the total space Z of families $Z \to \mathbb{P}^n$ of special Fermat-Pfister hypersurfaces.

For a smooth, projective rationally connected variety $X/\mathbb{C}$, there exists an integer $M > 0$ such that for any field F containing $\mathbb{C}$, the integer M annihilates $A_0(X_F)$. The smallest M is called the torsion order (Roitman, Levine–Chatzistamatiou). It gives a lower bound for the degree of unirationality of X.

By keeping track of the order of the cohomology classes involved in the above argument, decomposing the fixed degree d as a sum of two integers in various ways, and using a variant with reducible special fibre of the specialisation technique, one gets :

Theorem (Schr4). *The torsion order of a very general Fano hypersurface $X_d \subset \mathbb{P}^{N+1}$ of degree $d \geq 4$ is divisible by any integer $m \leq d - log_2 N$.*

Example : For $X_{100} \subset \mathbb{P}^{100}$, the torsion order is divisible by

$$2^5.3^3.5^2.7. \prod_{p \leq 89} p.$$

$(> 7x10^{38})$.

Some further reading

Specialisation of (stable) rationality – and obstruction to (stable) rationality.

Motivic volume, tropical degenerations ...

Larsen–Lunts 2003, Nicaise–Shinder 2019, Kontsevich–Tschinkel 2019, Nicaise–Ottem 2020, 2021

In some cases, get a situation where the special fibre is a divisor with normal crossings, $D = \sum E_i$, and the geometric generic fibre is not rational because some (lower-dimensional!) intersection of the E_i's is not rational (this last fact often proven by a Brauer (Artin-Mumford) argument).

Birational Geometry of Hypersurfaces (Gargnano del Garda 2018), Springer LN UMI 26 (2019) Ed. Hochenegger, Lehn, Stellari.

Rationality of Varieties (Schiermonnikoog 2019) Birkhäuser Progress in Mathematics 342 (2021). Ed. Farkas, van der Geer, Shen, Taelman