More or less well known facts about quadratic forms in 4 variables, or quadric surfaces.

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Let k be a field of characteristic different from 2.

To a nondegenerate quadratic form q in 4 variables and the smooth quadrics $Q \subset \mathbf{P}^3_k$ it defines, one associates the discriminant $\delta \in k^*/k^{*2}$ and a quaternion class α in the Brauer group $\mathrm{Br}(K)$, where K/k is the trivial or quadratic extension defined by δ. As a matter of fact, if C/K is the conic associated to α, if $\delta = 1$ hence $K = k$ then $Q \simeq C \times_k C$ and if $\delta \neq 1$ then $Q = R_{K/k}(C)$ (Weil restriction of scalars).

The quadric Q has a rational point ($Q(k) \neq \emptyset$) if and only if it is k-rational, i.e. k-birational to projective space.

The quadric surface Q has a rational point if and only if Q has a K-rational point.

Let $F := k(Q)$ denote the function field of Q. The kernel of the restriction map $\mathrm{Br}((k) \to \mathrm{Br}((F)$ is zero, unless $Q(k) = \emptyset$, and $\delta = 1$. In that case, the kernel is $Z/2$, spanned by the class $\alpha \in \mathrm{Br}((k)$, which is nonzero.

Proposition. *If two smooth quadric surfaces Q_1 and Q_2 without a k-rational point are k-birational to each other then one of the following statements holds:*

(i) $\mathrm{Ker}[\mathrm{Br}((k) \to \mathrm{Br}((F_1)] = \mathrm{Ker}[\mathrm{Br}((k) \to \mathrm{Br}((F_2)]$*, and this group is nonzero.*

(ii) Each of $\mathrm{Ker}[\mathrm{Br}((k) \to \mathrm{Br}((F_1)]$ *and* $\mathrm{Ker}[\mathrm{Br}((k) \to \mathrm{Br}((F_2)]$ *is zero, and* $\delta_1 = \delta_2 \in k^*/k^{*2}$.

Les Q_1 and Q_2 be two smooth quadric surfaces which are k-birational to each other.

If one of the quadrics has a k-rational point, then so has the other, each of them is k-birational to $\mathbf{P}^2_k$, they are k-birational to each other.

Suppose now that the sets of rational points $Q_1(k)$ and $Q_2(k)$ are empty.

Since Q_1 and Q_2 are k-birational to each other,

$$\mathrm{Ker}[\mathrm{Br}((k) \to \mathrm{Br}((F_1)] = \mathrm{Ker}[\mathrm{Br}((k) \to \mathrm{Br}((F_2)].$$

Let us call this group B.

If $B \neq 0$, then $\delta_1 = 1$ and $\delta_2 = 1$ and the group B is spanned by the class α attached to a quaternion algebra, defining a conic C over k, then $Q_1 \simeq C \times_k C \simeq Q_2$.

If $B = 0$, then $\delta_1 \neq 1$ and $\delta_2 \neq 1$.

Suppose $\delta_1 \neq \delta_2 \in k^*/k^{*2}$. Let us go over to the field K_1. We have $Q_1(K_1) = \emptyset$ hence $Q_2(K_1) = \emptyset$. The kernel of $\mathrm{Br}((K_1) \to \mathrm{Br}((K_1(Q_1))$ is $Z/2$. The kernel of $\mathrm{Br}((K_1) \to \mathrm{Br}((K_1(Q_2))$ is 0 since ${\delta_2}_{K_1} \neq 1$. Since Q_1 and Q_2 are birational over k hence over K_1, this is not possible.

Suppose $B = 0$ and $\delta_1 = \delta_2 \in k^*/k^{*2}$ and this element is not 1. Let K/k be the corresponding quadratic exension. Since Q_1 is k-birational to Q_2, this also holds over K, and we have $\alpha_1 = \alpha_2 \in \mathrm{Br}((K)$, thus $C_1 \simeq C_2$ over$K$, thus $Q_1 \simeq R_{K/k}(C_1) \simeq R_{K/k}(C_2) \simeq Q_2$.

Conclusion :

Proposition. *Two smooth quadric surfaces Q_1 and Q_2 are k-birational to each other if and only if either*

(a) Q_1 and Q_2 both have a rational point. An then there are k-birational to $\mathbf{P}^2_k$.

or

(b) none has a rational point, $disc(Q_1) = disc(Q_2) = d \in k^*/k^{*2}$ and in $\mathrm{Br}((k(\sqrt{d}))$ *the associated Brauer classes α_1 and α_2 coincide. And then $Q_1 \simeq Q_2$.*

References

Théorème 2.5 in:

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