Zero-cycles (and rational points) over a global field : Per Salberger 1983 to 1993, and beyond

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Some topics to which Per Salberger contributed during the years 1983 to 1993, before he turned his attention to bounds for the number of points of bounded height.
More are discussed in the slides of my talk on Per's 60th birthday.

• Algebraic K-Theory of orders and their Severi-Brauer schemes (Ph.D, Göteborg, 1985)
• Finiteness of torsion of certain Chow groups of zero-cycles on varieties over finitely generated fields
• Over a number field, Hasse principle and weak approximation for rational points, Brauer-Manin obstruction, descent (torsors) and fibration methods
• Over a number field, analogues for zero-cycles

In this short talk, I shall restrict myself to two topics.

(i) Arithmetic of zero-cycles: the 1988 Inventiones paper and its developments.
(I refer to the slides of my talk on Salberger's 60th birthday for some ideas of proofs.)
https://www.imo.universite-paris-saclay.fr/ jean-louis.colliot-thelene/salberger60talkjuillet2017.pdf

(ii) A more specialized topic : effectivity questions for Chow classes of zero-cycles on rational surfaces (unpublished section (b) of Salberger's Ph. D. thesis, 1985).

Let X be a projective variety (of finite type) over a field k. A point P of the scheme X is closed if and only if its residue field $k(P)$ is a finite extension of k.
One associates to X the free abelian group $Z_0(X)$ on closed points. Given a proper k-morphism $f : C \to X$ from a projective, integral, smooth curve C to X and a rational function $g \in k(C)^*$, one associated the zero-cycle $f_*(div_C(g)) \in Z_0(X)$.
The Chow group of zero-cycles $CH_0(X)$ is the quotient of $Z_0(X)$ by the group spanned by all such zero-cycles when (C, f, g) vary.
There is a degree map $CH_0(X) \to \mathbb{Z}$. Its kernel is denoted $A_0(X)$.
There is a natural bilinear pairing $CH_0(X) \times \mathrm{Br}(X) \to \mathrm{Br}(k)$ sending a pair (P, α) (P closed point of X with residue field $k(P)$, α element of the Brauer group of X) to $Norm_{k(P)/k}(\alpha(P)$, where $\alpha(P) \in \mathrm{Br}(k(P)$ is the evaluation of α at P.

Let k be a number field, Ω the set of places of k, k_v the completion of k at v. Class field theory gives injections $\mathrm{Br}(k_v) \hookrightarrow \mathbb{Q}/\mathbb{Z}$ and an exact sequence (vast generalization of the law of quadratic reciprocity)

$$0 \to \mathrm{Br}(k) \to \oplus_{v\in\Omega}\mathrm{Br}(k_v) \to \mathbb{Q}/\mathbb{Z} \to 0.$$

Let X be a smooth, projective, geometrically connected variety over a number field k. One then has a natural pairing

$$\oplus_{v\in\Omega} CH_0(X_{k_v}) \times \mathrm{Br}(X) \to \mathbb{Q}/\mathbb{Z}$$

which vanishes on the diagonal image of $CH_0(X)$ in the LHS.

The following conjecture is the final form of a conjecture first stated and tested on special classes of varieties. It was initially motivated by the work of Cassels on elliptic curves at the beginning of the 60s, by the work of CT-Sansuc 1981 on rational surfaces (with numerical tests and some theoretical evidence), and by the higher class field theory of Bloch and Kato-Saito in the 80s.

Conjecture E (CT-Sansuc 81, Kato–Saito 86, Saito 89, CT 95 and 99, van Hamel 2003, Wittenberg 2012)
Let X be a smooth, projective, geometrically connected variety over a number field k. Let Ω_k be the set of places v of k and k_v be the completion. The natural complex

$$\widehat{CH_0(X)} \to \prod_v \widehat{CH'_0(X_v)} \to Hom(\mathrm{Br}(X), \mathbb{Q}/\mathbb{Z})$$

is exact.
Here $\widehat{A} = \mathrm{proj\,lim}_n A/n$.
Here $CH'_0(X_v) = CH_0(X \times_k k_v)$ for v a nonarchimedean place, and a modified version at the archimedean places.

A special case is :

Conjecture E_1 (Shuji Saito 89, CT 95)

Let X be a smooth, projective, geometrically connected variety over a number field k. If there exists a family $\{z_v\}_{v\in\Omega_k}$ of local zero-cycles of degree one orthogonal to the Brauer group of X ("there is no Brauer-Manin obstruction"), then there exists a global zero-cycle of degree one.

There are parallel conjectures for rational points, but only for geometrically rationally connected varieties.

For the simplest nontrivial case, Châtelet surfaces, given by an affine equation

$$y^2 - az^2 = P(x),$$

with $P(x)$ of degree 4, Conjecture E was proved as a consequence of a theorem on rational points (CT, Sansuc and Swinnerton-Dyer 1984-1987).

Breakthrough :

Theorem (Salberger 1988 $+ \epsilon$)
Conjecture E on zero-cycles holds for all surfaces X with a conic bundle structure $X \to \mathbb{P}^1_k$.
These are surfaces birationally given by an affine equation

$$a(t)x^2 + b(t)y^2 + c(t) = 0$$

with $a(t), b(t), c(t) \in k[t]$, none of them zero.

Salberger 1988 if $\mathrm{Br}(X)/\mathrm{Br}(k) = 0$.
$+\epsilon$: CT-Swinnerton-Dyer 1993-1994, other proof Salberger 1993-2003.

One arithmetic argument in Salberger's paper may be viewed as a simple but successful substitute for Schinzel's hypothesis. One is given an irreducible polynomial $P(t)$ over a number field k. It is hard to find an almost integral value $\alpha \in k$ such that $P(\alpha)$ is almost a prime. However for any degree $N \geq deg(P)$ (hence in coprime degrees) one may easily produce a field extension L/k of degree N and an almost integral element $\beta \in L$ such that $P(\beta)$ is almost a prime in L.

This plays a rôle in the final argument, where one finds a conic over number fields of coprime degree which has points in all completions except possibly one. That is enough for the conics to have rational points, and for the surface to have a zero-cycle of degree 1.

Parenthesis : From zero-cycles to rational points

For conic bundles over $\mathbb{P}^1_k$ with at most 5 geometric degenerate fibres, any zero-cycle of degree 1 is rationally equivalent to a rational point (second part of this talk).
Used by Salberger to study local-global principle for rational points on arbitrary conic bundles over $\mathbb{P}^1$ with 4 degenerate fibres.
Using Salberger 1988 together with the descent method, Salberger-Skorobogatov 1991 then control weak approximation for rational points on del Pezzo surfaces of degree 4 with a rational point (they are birational to conic bundles with 5 degenerate fibres).

Back to zero-cycles

Salberger's 1988 fundamental work on zero-cycles on conic bundles over $\mathbb{P}^1$ was extended over the years to other varieties fibred over the projective line.

Contributions were made by CT, Swinnerton-Dyer, Skorobogatov, 1994-1998; Harari 1994; CT 2000 ; Frossard 2003; van Hamel 2003; Wittenberg 2012; Dasheng Wei 2014.
Until 2015, progress was hindered by the restriction that the bad reduction fibres were requested to contain a multiplicity one component split by an abelian extension (as occurs for conic bundles).

Here is the vast generalisation of Salberger's 1988 theorem. Here X is a smooth projective irreducible variety.

Theorem (Harpaz and Wittenberg 2016) *Let k be a number field and $f : X \to \mathbb{P}^1_k$ be a family of rationally connected varieties. If conjecture E holds for the smooth fibres of f over closed points, then it holds for X.*

Using work of Sansuc, Borovoi, Yonqi Liang, this gives :
Theorem (Harpaz and Wittenberg 2016) *Conjecture E holds for any $f : X \to \mathbb{P}^1_k$ such that the generic fibre is birational to a homogeneous space of a connected linear algebraic group.*

Results in the function field case

Let X be a smooth projective variety over $k = \mathbb{F}(C)$ the function field of a smooth projective curve C. Let $\mathcal{X}/C$ a projective flat model, with $\mathcal{X}/\mathbb{F}$ smooth.
It was observed by Shuji Saito and others that conjecture E for X/k is closely related to a strong integral version of the ℓ-adic Tate conjecture for 1-dimensional cycles on $\mathcal{X}/\mathbb{F}$. This was used in the following results.

Theorem (Parimala and Suresh 2016). *Conjecture E holds for surfaces with a conic bundle structure over* $\mathbb{P}^1_{\mathbb{F}(C)}$.

This is the function field analogue of Salberger's result over number fields. The proof goes via third unramified cohomology of $\mathcal{X}/\mathbb{F}$ and techniques of Saltman.

Theorem (Zhiyu Tian, JAMS 2025) *Conjecture E holds for smooth geometrically rational surfaces over* $\mathbb{F}(C)$.

The proof is a tour de force involving geometric arguments (joint work with J. Kollár) and various tools of motivic cohomology, Examples are given by smooth cubic surfaces, which in general do not admit a conic bundle structure.

Vector bundles and effectivity of zero-cycle classes

Let X be a smooth, projective, geometrically connected variety X over a field k.

(Q1, Effectivity) *Does there exist an integer $M(X) \geq 0$ depending only on the geometry of X such that any zero-cycle on X of degree at least $M(X)$ is rationally equivalent to an effective cycle, i.e. of the shape $\sum_i n_i P_i$ with all $n_i \geq 0$?*

Example :
X is a quadric, $M(X) = 0$ (Swan).

Let X be a smooth, projective, geometrically *ationally connected* variety X over an arbitrary field k. Assume that X is geometrically *rationally connected*.

(Q2, Index and rational point) *Suppose there exists a zero-cycle of degree 1 on X. Is there a rational point ? If not, and there exists a closed point of degree $r > 1$ on X, what is the smallest integer s coprime to r for which there exists a closed point of degee s ?*

Example :

X is a quadric, r odd, $s = 1$ (T.A. Springer)

Theorem (Coray 1974). *If a smooth cubic surface has a zero-cycle of degree 1, then it has a closed point of degree 1, or 4, or 10.*

Given a closed point P of degree prime to 3, one produces a hypersurface section of degree as low as possible passing through P and a point of degree 3. The cut out curve then has a zero-cycle of degree one, if the genus is low enough, one may use Riemann's inequality for line bundles on the curve to get an effective zero-cycle of degree smaller than the degree of P. This works all the way down to 10, 4, 1. The curves need not be irreducible or smooth. In 2020, I gave a device which allows to consider only smooth curves. I then addressed the questions for del Pezzo surfaces of degrees 3, 2, 1.

Theorem (Coray-CT 1979). *Let $\pi : X \to \mathbb{P}^1_k$ be a relatively minimal conic bundle. Let m be the sum of the degrees of closed points P of $\mathbb{P}^1_k$ with a singular fibre X_P. Any zero-cycle on X with nonnegative degree at least $(m-3)/2$ is rationally equivalent to an effective cycle.*

The method is essentially that of Coray 1974.

Here, already for $m = 6$, one may have a zero-cycle of degree one but no rational point.

Theorem (Salberger 1985) *Let* $\pi : X \to C$ *be a relatively minimal conic bundle over a smooth projective curve* C *of genus* g*. Let* m *be the sum of the degrees of closed points* P *of* C *with a singular fibre* X_P*. Any zero-cycle on* X *with nonnegative degree at least* $2g + (m - 3)/2$ *is rationally equivalent to an effective cycle.*

This unfortunately unpublished paper uses a different and elegant proof. Namely, given such a conic bundle, there is an associated locally free bundle Λ of rank 4 on C with a local structure of maximal, hereditary order in the quaternion algebra associated to the generic fibre. There is an associated genus $g(\Lambda) \in \mathbb{Z}$. One considers suitable fractional ideals $\mathcal{L}$ in Λ. For these, Witt 1934 proved a Riemann inequality. There is a close relation between 0-cycles on X and such ideals (Salberger), with effective cycles corresponding to "integral" ideals. Computation of $g(\Lambda)$ via a noncommutative Hurwitz formula (Van den Bergh, Van Geel; Mattson) then leads to the result.

In a recent paper, Claire Voisin (2025) exhibited vector bundles of rank 2 on del Pezzo surfaces and used sections of these bundles to produce effective zero-cycles of low degree.
She got rid of the 10 in Coray's 1974 result for cubic surfaces and improved upon the bounds produced in 2020 for effectivity of zero-cycles on del Pezzo surfaces.

Final comment : One can produce a simple function $f : \mathbb{Z} \to \mathbb{N}$ such that for any smooth projective geometrically rational surface X/k with a k-rational point, any zero-cycle on X of degree at least $f((K_X.K_X))$ is rationally equivalent to an effective zero-cycle. But the proof is case by case (del Pezzo surfaces degree by degree, conic bundles).