On the integral Tate conjecture for 1-cycles on the product of a curve and a surface over a finite field

Joint work with Federico Scavia (UBC, Vancouver)

Jean-Louis Colliot-Thélène
(CNRS et Université Paris-Sud $\subset$ Paris-Saclay)

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Let X be a smooth projective variety over a finite field $\mathbb{F}$ of char. p. Let ℓ be a prime, $\ell \neq p$. For any $i \geq 0$, there is a cycle map

$$CH^i(X) \otimes \mathbb{Z}_\ell \to H^{2i}(X, \mathbb{Z}_\ell(i))$$

from the Chow groups of codimension i cycles to the projective limit of the étale cohomology groups $H^{2i}(X, \mu_{\ell^n}^{\otimes i})$, which is a $\mathbb{Z}_\ell$-module of finite type.
For $i = 1$, this map reads

$$\mathrm{Pic}(X) \otimes \mathbb{Z}_\ell \to H^2(X, \mathbb{Z}_\ell(1))$$

and Tate conjectured that it is always surjective. This is related to the conjectured finiteness of Tate-Shafarevich groups of abelian varieties over a global field.
It is known for geometrically separably unirational varieties (easy), for abelian varieties (Tate) and for most $K3$-surfaces.

For $i > 1$, Tate conjectured that the cycle map

$$CH^i(X) \otimes \mathbb{Q}_\ell \to H^{2i}(X, \mathbb{Q}_\ell(i))$$

is surjective. Very little is known. One may give examples where the statement with integral coefficients does not hold. However, for X of dimension d, it is unknown whether the Integral Tate conjecture for 1-cycles, henceforth denoted $T_1 = T^{d-1}$

(T_1) *The cycle map* $CH^{d-1}(X) \otimes \mathbb{Z}_\ell \to H^{2d-2}(X, \mathbb{Z}_\ell(d-1))$ *is onto.*

may fail.

For $d = 2$, this is a special case of the Tate conjecture.

For arbitrary d, the integral Tate conjecture for 1-cycles holds for X of any dimension $d \geq 3$ if it holds for any X of dimension 3.

For X of dimension 3, some nontrivial cases have been established.

• X is a conic bundle over a geometrically ruled surface (Parimala and Suresh).

• X is the product of a curve of arbitrary genus and a geometrically rational surface (Pirutka).

For smooth projective varieties X over $\mathbb{C}$, there is a formally parallel conjecture for cycle maps

$$CH^i(X) \to Hdg^{2i}(X, \mathbb{Z})$$

where $Hdg^{2i}(X, \mathbb{Z}) \subset H^{2i}_{Betti}(X, \mathbb{Z})$ is the subgroup of rationally Hodge classes. The conjecture with $\mathbb{Q}$-coefficients is a famous open problem. With integral coefficients, several counterexamples were given, even with $dim(X) = 3$ and 1-cycles.
A recent counterexample involves the product $X = E \times S$ of an elliptic curve E and an Enriques surface. For fixed S, provided E is "very general", then the integral Hodge conjecture fails for X (Benoist-Ottem). The proof uses the fact that the torsion of the Picard group of such a surface is nontrivial, it is $\mathbb{Z}/2$.

It is reasonable to investigate the Tate conjecture for cycles of codimension $i \geq 2$ assuming T^1 : The conjecture is true for cycles of codimension 1 over any smooth projective variety.

Theorem (CT-Scavia 2020). *Let $\mathbb{F}$ be a finite field, $\overline{\mathbb{F}}$ a Galois closure, $G = Gal(\overline{\mathbb{F}}/\mathbb{F})$. Let $E/\mathbb{F}$ be an elliptic curve and $S/\mathbb{F}$ be an Enriques surface. Let $X = E \times_{\mathbb{F}} S$. Let ℓ be a prime different from $p = \mathrm{char.}(\mathbb{F})$. Assume T^1.*
If $\ell \neq 2$, or if $\ell = 2$ but $E(\mathbb{F})$ has no nontrivial 2-torsion, then the map $CH^2(X) \otimes \mathbb{Z}_\ell \to H^4(X, \mathbb{Z}_\ell(2))$ is onto.

We actually prove a general theorem, for the product $X = C \times S$ of a curve C and a surface S which is *geometrically CH_0-trivial*, which here means :

Over any algebraically closed field Ω of $\mathbb{F}$, the degree map $CH_0(S_\Omega) \to \mathbb{Z}$ is an isomorphism.

In that case $\mathrm{Pic}(S_\Omega)$ is a finitely generated abelian group.

For $\overline{\mathbb{F}}$ a Galois closure of $\mathbb{F}$, $G = \mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F})$, and J the jacobian of C, still assuming T^1, we prove that $CH^2(X) \otimes \mathbb{Z}_\ell \to H^4(X, \mathbb{Z}_\ell(2))$ is onto under the condition $\mathrm{Hom}_G(\mathrm{Pic}(\mathrm{S}_{\overline{\mathbb{F}}})_{\mathrm{tors}}, \mathrm{J}(\overline{\mathbb{F}})) = 0$.

We do not know whether this condition is necessary.

In the rest of the talk, I shall sketch some ingredients of the proof.

Let M be a finite Galois-module over a field k. Given a smooth, projective, integral variety X/k with function field $k(X)$, and $i \geq 1$ an integer, one lets

$$H^i_{nr}(k(X), M) := \mathrm{Ker}[H^i(k(X), M) \to \oplus_{x \in X^{(1)}} H^{i-1}(k(x), M(-1))]$$

Here $k(x)$ is the residue field at a codimension 1 point $x \in X$, the cohomology is Galois cohomology of field, and the maps on the right hand side are "residue maps".
One is interested in $M = \mu_{\ell^n}^{\otimes j}$, for which $M(-1) = \mu_{\ell^n}^{\otimes (j-1)}$ and in the direct limit $\mathbb{Q}_\ell/\mathbb{Z}_\ell(j) = \lim_j \mu_{\ell^n}^{\otimes j}$, for which the cohomology groups are the limit of the cohomology groups.

Theorem (Kahn, CT-Kahn) *For $X/\mathbb{F}$ smooth, projective of arbitrary dimension, the torsion subroup of the (conjecturally finite) group*

$$\mathrm{Coker}[CH^2(X) \otimes \mathbb{Z}_\ell \to H^4(X, \mathbb{Z}_\ell(2))]$$

is isomorphic to the quotient of $H^3_{\mathrm{nr}}(\mathbb{F}(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$ by its maximal divisible subgroup.

A **basic exact sequence** (Kahn, CT-Kahn). Let $\overline{\mathbb{F}}$ be an algebraic closure of $\mathbb{F}$, let $\overline{X} = X \times_{\mathbb{F}} \overline{\mathbb{F}}$ and $G = \mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F})$.

For $X/\mathbb{F}$ a smooth, projective, geometrically connected variety over a finite field, long exact sequence

$$0 \to \mathrm{Ker}[CH^2(X)\{\ell\} \to CH^2(\overline{X})\{\ell\}] \to H^1(\mathbb{F}, H^2(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)))$$

$$\to \mathrm{Ker}[H^3_{\mathrm{nr}}(\mathbb{F}(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) \to H^3_{\mathrm{nr}}(\overline{\mathbb{F}}(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))]$$

$$\to \mathrm{Coker}[CH^2(X) \to CH^2(\overline{X})^G]\{\ell\} \to 0.$$

The proof relies on work of Bloch and on the Merkurjev-Suslin theorem. Via Deligne's theorem on the Weil conjectures, one has

$$H^1(\mathbb{F}, H^2(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))) = H^1(\mathbb{F}, H^3(\overline{X}, \mathbb{Z}_\ell(2))_{\mathrm{tors}})$$

and this is finite.

For X a curve, all groups are zero.
For X a surface, $H^3(\overline{\mathbb{F}}(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = 0$.
For $X/\mathbb{F}$ a surface, one also has

$$H^3_{\mathrm{nr}}(\mathbb{F}(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = 0.$$

This vanishing was remarked in the early stages of higher class field theory (CT-Sansuc-Soulé, K. Kato, in the 80s). It uses a theorem of S. Lang, which relies on Tchebotarev's theorem.

For our 3-folds $X = C \times S$, S as above, we have an isomorphism of finite groups

$$\mathrm{Coker}[CH^2(X) \otimes \mathbb{Z}_\ell \to H^4(X, \mathbb{Z}_\ell(2))] \simeq H^3_{\mathrm{nr}}(\mathbb{F}(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)),$$

and, under the assumption T^1 for all surfaces over a finite field, a theorem of Chad Schoen implies $H^3_{\mathrm{nr}}(\overline{\mathbb{F}}(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = 0$.

Under T^1 for all surfaces, for our threefolds $X = C \times S$ with S geometrically CH_0-trivial, we thus have an exact sequence of finite groups

$$0 \to \mathrm{Ker}[CH^2(X)\{\ell\} \to CH^2(\overline{X})\{\ell\}] \to H^1(\mathbb{F}, H^2(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)))$$

$$\xrightarrow{\theta_X} H^3_{\mathrm{nr}}(\mathbb{F}(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) \to \mathrm{Coker}[CH^2(X) \to CH^2(\overline{X})^G]\{\ell\} \to 0.$$

Under these hypotheses, the surjectivity of

$$CH^2(X) \otimes \mathbb{Z}_\ell \to H^4(X, \mathbb{Z}_\ell(2))$$

(integral Tate conjecture) is therefore equivalent to the combination of two hypotheses :

Hypothesis 1
The composite map

$$\rho_X : H^1(\mathbb{F}, H^2(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))) \to H^3(\mathbb{F}(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$$

of θ_X *and* $H^3_{\mathrm{nr}}(\mathbb{F}(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) \subset H^3(\mathbb{F}(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$ *vanishes.*

Hypothesis 2 $\mathrm{Coker}[CH^2(X) \to CH^2(\overline{X})^G]\{\ell\} = 0.$

(Optional slide)

Hypothesis 1 is equivalent to :

Hypothesis 1a. The (injective) map from

$$\mathrm{Ker}[CH^2(X)\{\ell\} \to CH^2(\overline{X})\{\ell\}]$$

to the (finite) group

$$H^1(\mathbb{F}, H^2(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))) \simeq H^1(\mathbb{F}, H^3(\overline{X}, \mathbb{Z}_\ell(2))_{tors})$$

is onto.

In dimension > 2 , we do not see how to establish the validity of this hypothesis directly – unless of course when the finite group $H^3(\overline{X}, \mathbb{Z}_\ell(2))_{tors}$ vanishes. The group $H^3(\overline{X}, \mathbb{Z}_\ell(1))_{tors}$ is the nondivisible part of the ℓ-primary Brauer group of $\overline{X}$.

Discussion of Hypothesis 1 :
The map $\rho_X : H^1(\mathbb{F}, H^2(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))) \to H^3(\mathbb{F}(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$ vanishes.
This map is the composite of the Hochschild-Serre map

$$H^1(\mathbb{F}, H^2(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))) \to H^3(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)))$$

with the restriction to the generic point of X.

We prove :
Theorem. *Let Y and Z be two smooth, projective geometrically connected varieties over a finite field $\mathbb{F}$. Let $X = Y \times_{\mathbb{F}} Z$. Assume that the Néron-Severi group of Z is free with trivial Galois action. If the maps ρ_Y and ρ_Z vanish, then so does the map ρ_X.*

Corollary. For the product X of a surface and arbitrary many curves, the map ρ_X vanishes.

On must study $H^1(\mathbb{F}, H^2(\overline{X}, \mu_{\ell^n}^{\otimes 2}))$ under restriction from X to its generic point.
As may be expected, the proof uses a Künneth formula, along with standard properties of Galois cohomology of a finite field.
As a matter of fact, it is an unusual Künneth formula, with coefficients $\mathbb{Z}/\ell^n$, $n > 1$. That it holds for H^2 of the product of two smooth, projective varieties over an algebraically closed field, is a recent result of Skorobogatov and Zarhin, who used it in an other context (the Brauer-Manin set of a product).

Discussion of Hypothesis 2 :

$\mathrm{Coker}[CH^2(X) \to CH^2(\overline{X})^G]\{\ell\} = 0.$
Here we restrict to the special situation : C is a curve and S is geometrically CH_0-trivial surface.
One lets $K = \mathbb{F}(C)$ and $L = \overline{\mathbb{F}}(C)$.
On considers the projection $X = C \times S \to C$, with generic fibre the K-surface S_K. Restriction to the generic fibre gives a natural map from

$$\mathrm{Coker}[CH^2(X) \to CH^2(\overline{X})^G]\{\ell\}$$

to

$$\mathrm{Coker}[CH^2(S_K) \to CH^2(S_L)^G]\{\ell\}.$$

Using the hypothesis that S is geometrically CH_0-trivial, which implies $b_1 = 0$ and $b_2 - \rho = 0$ (Betti number b_i, rank ρ of Néron-Severi group), one proves :

Theorem. *The natural, exact localisation sequence*

$$\mathrm{Pic}(\overline{C}) \otimes \mathrm{Pic}(\overline{S}) \to CH^2(\overline{X}) \to CH^2(S_L) \to 0.$$

may be extended on the left with a finite p-group.

(Optional slide)

To prove this, we use correspondences on the product $C \times S$, over $\overline{\mathbb{F}}$.
We use various pull-back maps, push-forward maps, intersection maps of cycle classes :

$$\mathrm{Pic}(C) \otimes \mathrm{Pic}(S) \to \mathrm{Pic}(X) \otimes \mathrm{Pic}(X) \to CH^2(X)$$

$$CH^2(X) \otimes \mathrm{Pic}(S) \to CH^2(X) \otimes \mathrm{Pic}(X) \to CH^3(X) = CH_0(X) \to CH_0(C)$$

$$\mathrm{Pic}(C) \otimes \mathrm{Pic}(S) \to CH^2(X) = CH_1(X) \to CH_1(S) = \mathrm{Pic}(S)$$

Not completely standard properties of G-lattices for $G = \mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F})$ applied to the (up to p-torsion) exact sequence of G-modules

$$0 \to \mathrm{Pic}(\overline{C}) \otimes \mathrm{Pic}(\overline{S}) \to CH^2(\overline{X}) \to CH^2(S_L) \to 0$$

then lead to :

Theorem. *The natural map from* $\mathrm{Coker}[CH^2(X) \to CH^2(\overline{X})^G]\{\ell\}$ *to* $\mathrm{Coker}[CH^2(S_K) \to CH^2(S_L)^G]\{\ell\}$ *is an isomorphism.*

One is thus left with controlling this group. Under the CH_0-triviality hypothesis for S, it coincides with

$$\mathrm{Coker}[CH^2(S_K)\{\ell\} \to CH^2(S_L)\{\ell\}^G].$$

At this point, for a geometrically CH_0-trivial surface over $L = \overline{\mathbb{F}}(C)$, which is a field of cohomological dimension 1, using the K-theoretic mechanism, one may produce an exact sequence parallel to the basic exact sequence over $\mathbb{F}$ which we saw at the beginning. In the particular case of the constant surface $S_L = S \times_{\mathbb{F}} L$, the left hand side of this sequence gives an injection

$$0 \to A_0(S_L)\{\ell\} \to H^1(L, H^3(\overline{S}, \mathbb{Z}_\ell(2)\{\ell\})$$

where $A_0(S_L) \subset CH^2(S_L)$ is the subgroup of classes of zero-cycles of degree zero on the L-surface S_L.

Study of this situation over completions of $\overline{\mathbb{F}}(C)$ (Raskind) and a good reduction argument in the weak Mordell-Weil style, plus a further identification of torsion groups in cohomology of surfaces over an algebraically closed field then yield a Galois embedding

$$A_0(S_L)\{\ell\} \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Pic}(\overline{S})\{\ell\}, J(C)(\overline{\mathbb{F}})),$$

hence an embedding

$$A_0(S_L)\{\ell\}^G \hookrightarrow \mathrm{Hom}_G(\mathrm{Pic}(\overline{S})\{\ell\}, J(C)(\overline{\mathbb{F}})).$$

If this group $\mathrm{Hom}_G(\mathrm{Pic}(\overline{S})\{\ell\}, J(C)(\overline{\mathbb{F}}))$ vanishes, then

$$\mathrm{Coker}[CH^2(S_K)\{\ell\} \to CH^2(S_L)\{\ell\}^G] = 0$$

hence

$$\mathrm{Coker}[CH^2(X) \to CH^2(\overline{X})^G]\{\ell\} = 0,$$

which is Hypothesis 2, and completes the proof of the theorem :

Theorem (CT/Scavia) *Let* $\mathbb{F}$ *be a finite field,* $G = \mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F})$*. Let* ℓ *be a prime,* $\ell \neq \mathrm{char.}(\mathbb{F})$*. Let* C *be a smooth projective curve over* $\mathbb{F}$*, let* $J/\mathbb{F}$ *be its jacobian, and let* $S/\mathbb{F}$ *be a smooth, projective, geometrically* CH_0*-trivial surface.*
Assume the usual Tate conjecture for codimension 1 cycles on varieties over a finite field.
Under the assumption

$$\mathrm{Hom}_G(\mathrm{Pic}(S_{\overline{\mathbb{F}}})\{\ell\}, J(\overline{\mathbb{F}})) = 0,$$

the cycle class map $CH^2(X) \otimes \mathbb{Z}_\ell \to H^4_{et}(X, \mathbb{Z}_\ell(2))$ *is onto.*