Arithmetic upon intersections of two quadrics

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Let k be a number field. Let k_v run through the completions of k. Let $X \subset \mathbb{P}^n_k$, be a *smooth* complete intersection of two quadrics :

$$f(x_0, \cdots, x_n) = g(x_0, \cdots, x_n) = 0.$$

A well known conjecture asserts :
For $n \geq 5$, for any such X, the Hasse principle holds, namely

$$\prod_v X(k_v) \neq \emptyset \Longrightarrow X(k) \neq \emptyset.$$

When $X(k) \neq \emptyset$, and $n \geq 5$, one knows that $X(k) \subset \prod_v X(k_v)$ is dense.

For $n = 3$, the Hasse principle need not hold. One then has a curve of genus one, the obstruction to the Hasse principle is related to the Tate-Shafarevich group of the jacobian of the curve.

For $n = 4$, the Hasse principle need not hold (first smooth example : Birch and Swinnerton-Dyer 1975). Conjecturally, the defect is controlled by the Brauer-Manin obstruction.

Results were obtained for $n \geq 12$ by Mordell (1959) and for $n = 10$ by Swinnerton-Dyer (1964).

Assume k is totally imaginary, and $n = 12$. Assume $f(x_0, \ldots, x_{12})$ is non-degenerate. Here is Mordell's argument. The quadratic form f may be written as the direct sum of a totally hyperbolic quadratic form in 10 variables and a quadratic form in 3 variables. On a linear space of codimension $5 + 3 = 8$, that is a $\mathbb{P}^4_k$, the form f identically vanishes. The restriction of g to this $\mathbb{P}^4_k$ is given by a quadratic form in 5 variables, it has a nontrivial zero over k.

For smooth X over a formally real field, like $\mathbb{Q}$, one uses weak approximation on k at the real places and an elegant trick of Mordell over the reals : consider the behaviour of the signature of the quadratic form $af + bg$ as $a, b \in \mathbb{R}$ vary over $a^2 + b^2 = 1$. One proves the existence of quadratic forms in the pencil over $\mathbb{R}$ with 6 hyperbolics.

The Hasse principle for X **smooth** complete intersection of two quadrics in $\mathbb{P}^n_k$ is known to hold :

For $n \geq 8$ (CT–Sansuc–Swinnerton-Dyer 1987) [Note : for $n \geq 8$, $X(k_v) \neq \emptyset$ for v nonarchimedean].

For $n \geq 4$ if X contains two lines globally defined over k (the case $n = 4$ was known before 1970).

For $n \geq 5$ if X contains a conic (Salberger 1993).

For $n = 7$ (Heath-Brown 2018).

Taking two difficult conjectures (finiteness of Ш of elliptic curves and Schinzel's hypothesis) for granted, Wittenberg (2007) gave a proof of the Hasse principle for any smooth X for $n \geq 5$.

The Hasse principle was also proved for smooth projective models Y of various classes of *singular* projective, non conical, geometrically integral, complete intersections X of two quadrics.
In particular, the Hasse principle holds for smooth models Y of $X \subset \mathbb{P}^n_k$ for $n \geq 6$, when X contains a pair of conjugate singular points (CT–Sansuc–Swinnerton-Dyer 1987).
These results in the singular case also play a key rôle in the proof of the smooth case in higher dimension, via the fibration method.

Let k be a number field. Let Y/k be a smooth projective model of a possibly singular, geometrically integral, non-conical complete intersection of two quadrics in $\mathbb{P}^n_k$.
A purely algebraic computation gives : for $n \geq 6$, one has $\mathrm{Br}(Y)/\mathrm{Br}(k) = 0$.
This motivated :

Conjecture (CT-S-SD 1987) : if $n \geq 6$, then the Hasse principle holds for Y.

For $n \geq 8$, this was proven in [CT-S-SD 1987].
The case $n = 7$ is the topic of the present talk.

I shall discuss the main steps in the proof of the following theorem of A. Molyakov (2023), which completes and encompasses results of Heath-Brown (2018) and myself (2022).

Main Theorem *Let k be a number field and $X \subset \mathbb{P}^7_k$ be a nonconical, geom. integral complete intersection of two quadrics. For any smooth projective model Y of X, the Hasse principle holds.*

One useful tool is the theorem : *Over any field k, if an intersection of two quadrics $X \subset \mathbb{P}^n_k$ has a rational point over an odd degree extension of k then it has a rational point.*

This is an immediate consequence of Springer's theorem (same statement for one quadric, over any field) and the theorem of Amer and of Brumer :

Let $k(t)$ be the rational function field in one variable. A sytem of two quadratic forms $f = g = 0$ over a field k has a nontrivial zero over k if and only if the quadratic form $f + tg$ over the field $k(t)$ has a nontrivial zero.

When discussing a complete intersection of two quadrics $X \subset \mathbb{P}^n_k$ over a field k (char. not 2) given by a system $f = g = 0$, one is quickly led to consider the pencil of quadrics $\lambda f + \mu g = 0$ containing X.

Ignoring subtle points with the singular forms in the pencil, there is a close relation between the following statements, where we assume $r \geq 1$:

- There exists a form $\lambda f + \mu g$ in the pencil which splits off $r + 1$ hyperbolic planes.
- There exists a quadric in the pencil which contains a linear space $\mathbb{P}^r_k \subset \mathbb{P}^n_k$.
- The variety X contains an $(r-1)$-dimensional quadric $Y \subset \mathbb{P}^r_k \subset \mathbb{P}^n_k$.

The proof of the main theorem we describe here builds upon a local result in very small dimension.

Theorem (CT 2022) *Let k be a p-adic field. Let $X \subset \mathbb{P}^3_k$ be an arbitrary intersection of two quadrics given by a system*

$$f(x_0, x_1, x_2, x_3) = 0, \; g(x_1, x_2, x_3) = 0.$$

Then there exists a quadratic extension K/k with $X(K) \neq \emptyset$.

Proof. When X is not a smooth complete intersection, this is proven by a case-by-case discussion. Assume X is a smooth complete intersection. Then X is a genus one curve.

Let $\overline{k}$ be an algebraic closure of k, and $G := \mathrm{Gal}(\overline{k}/k)$. The period of a curve X is defined as the positive generator of the image of the degree map $\mathrm{Pic}(X \times_k \overline{k})^G \to \mathbb{Z}$. The index of a k-variety X is the gcd of degrees of closed points on X.

The assumption that $g(x_1, x_2, x_3)$ involves only three variables implies that the "period" of the curve X divides 2. This one sees by using the fact any conic has period 1 and that the curve X is a double cover of the conic $g(x_1, x_2, x_3) = 0$.

For a curve of genus one, it is a theorem of Lichtenbaum (1969) that the period coincides with the index (for a review of the proof, see my webpage). Thus the index divides 2. By Riemann-Roch, this implies that there exists a field K/k of degree at most 2 with $X(K) \neq \emptyset$.

Degree 2 Local Theorem (Creutz–Viray 2021) *Let k be a p-adic field. Let $X \subset \mathbb{P}^n_k$, $n \geq 4$ be an intersection of two quadrics. There exists a field K/k of degree at most 2 with $X(K) \neq \emptyset$.*

(Alternate) proof. It is enough to handle the case $n = 4$. Any quadratic form in 5 variables over a p-adic field has a nontrivial zero. This implies that one may find a hyerplane section over k such that the induced quadratic form has rank at most 3. The result then follows from the previous theorem.

The next two results will not be used in the proof of the main theorem.

Theorem (Creutz–Viray 2021). *Let k be a number field and $X \subset \mathbb{P}^n_k$ be a smooth complete intersection of two quadrics. For $n \geq 4$, the index $I(X)$ divides 2.*

The proof is very elaborate.

Theorem (CT 2022) *Let k be a number field and $X \subset \mathbb{P}^n_k$ be a smooth complete intersection of two quadrics. For $n \geq 5$ there exists a quadratic extension K/k with $X(K) \neq \emptyset$.*

The question whether this holds for $n = 4$ remains open. Partial results are given by Creutz–Viray.

Proof. By Bertini it is enough to prove the case $n = 5$. In this case the variety $F_1(X)$ of lines on X is geometrically integral – it is actually a principal homogeneous space under an abelian variety. Hence there exists a finite set S of places of k such that $F_1(X)(k_v) \neq \emptyset$ for $v \notin S$. Thus for almost all v, any $\lambda f + \mu g$ splits off 2 hyperbolics over k_v.
For any place v, the Degree 2 Local Theorem gives a point of X in an extension of k_v of degree 2, hence there exists a $\lambda_v f + \mu_v g$ in the pencil over k_v which splits off two hyperbolics.
Using weak approximation, we find $(\lambda, \mu) \in \mathbb{P}^1(k)$ such that $\lambda f + \mu g$ splits off 2 hyperbolics over each k_v. By a result of Hasse (1924) it splits off 2 hyperbolics over k. Thus X contains a point over a quadratic extension of k.

Theorem (Salberger 1993 + ε) *Let k be a number field and $X \subset \mathbb{P}^n_k$, $n \geq 4$, be a geometrically integral, nonconical, complete intersection of two quadrics, and let Y/k be a smooth projective model of X.* **Assume that X contains a conic $C \subset \mathbb{P}^2_k \subset \mathbb{P}^n_k$.** *Then*
(a) The set $Y(k)$ is dense in the Brauer-Manin set $Y(\mathbb{A}_k)^{\mathrm{Br}(Y)} \subset Y(\mathbb{A}_k)$.
(b) For $n \geq 6$, the Hasse principle and weak approximation hold for Y.
(c) For $n = 5$ and X smooth, the Hasse principle and weak approximation hold for X.

The proof of the theorem relies in part on several works (CTSaSD 87, Coray-Tsfasman 88). Salberger's proof of the case $n = 4$ builds upon his seminal work on zero-cycles.

Salberger's result is used in various proofs of the following theorem (not needed for this talk.)

Theorem (J. Iyer and R. Parimala 2022) *Let k be a number field and $X \subset \mathbb{P}^5_k$ be a smooth complete intersection of two quadrics given by $f = g = 0$. Assume :*
(a) The genus 2 curve given by $y^2 = -det(\lambda f + \mu g)$ has a zero-cycle of degree one.
(b) Over each completion k_v of k, X contains a line defined over k_v
Then $X(k) \neq \emptyset$.

Local Theorem (Heath-Brown 2018) *Let k be a local field. Let $X \subset \mathbb{P}^7_k$ be a smooth complete intersection of two quadrics given by $f = g = 0$. If $X(k) \neq \emptyset$, then there exists a nondegenerate form $\lambda f + \mu g$ in the pencil which splits off three hyperbolics.*

Proof (CT 2022) Let $P \in X(k)$. The intersection C of X with the tangent space $\mathbb{P}^5_k$ at P is a cone with vertex P over an intersection of two quadrics $Y \subset \mathbb{P}^4_k$. By the Degree 2 Local Theorem (Creutz–Viray) there exists a point on Y in a quadratic extension K/k. This defines a line over K on C passing through the vertex P of the cone. One thus gets a pair of lines in $C \subset X$ passing through P and globally defined over k. Fix a k-point Q in the plane $\mathbb{P}^2_k$ defined by these two lines, outside of the two lines. The form $\lambda f + \mu g$ vanishing at Q vanishes on the plane $\mathbb{P}^2_k$ spanned by the two lines. If nondegenerate, this form splits off 3 hyperbolics. (There is a simple way to handle the case where the form is of rank 7.)

Global Theorem (Heath-Brown, 2018) *Let k be a number field. Let $X \subset \mathbb{P}^7_k$ be a smooth complete intersection of two quadrics given by $f = g = 0$. The Hasse principle holds for X.*

Proof of Global Theorem (CT 2022, some ingredients from HB's proof).
The variety $F_2(X)$ of planes $\mathbb{P}^2_k \subset X \subset \mathbb{P}^7_k$ is a geometrically integral variety – it is actually a principal homogeneous spaces under an abelian variety. Hence there exists a finite set S of places of k such that $F_2(X)(k_v) \neq \emptyset$ for $v \notin S$. Thus for each $v \notin S$, any nondegenerate $\lambda f + \mu g$ splits off 3 hyperbolics over k_v. By the Local Theorem, for each $v \in S$ the assumption $X(k_v) \neq \emptyset$ implies that there exists a point $(\lambda_v, \mu_v) \in \mathbb{P}^1(k_v)$ such that $\lambda_v f + \mu_v g$ is nondegenerate and contains 3 hyperbolics. By weak approximation on $\mathbb{P}^1_k$, there exists $(\lambda, \mu) \in \mathbb{P}^1(k)$ such that $\lambda f + \mu g$ is nondegenerate and contains 3 hyperbolics over each k_v. By Hasse 1924 it contains 3 hyperbolics over k. Thus X contains a conic. Salberger's theorem and the hypothesis $\prod_v X(k_v) \neq \emptyset$ then give $X(k) \neq \emptyset$.

What about singular complete intersections of two quadrics ?

Let k be a number field and $X \subset \mathbb{P}^n_k$ a possibly singular complete intersection of two quadrics. Assume it is geometrically integral and not a cone. One is interested in the Hasse principle for a smooth projective model Y of X.
In CT-Sansuc-Swinnerton-Dyer 1987, we proved the Hasse principle for Y under the assumption $n \geq 8$. We proposed :

Conjecture. *For $n = 6$ and $n = 7$, the Hasse principle holds for Y.*

For such n, one has $\mathrm{Br}(Y)/\mathrm{Br}(k) = 0$ so there is no Brauer-Manin obstruction. Under various additional hypotheses on X, the conjecture is proved in CTSaSD 1987. As we saw, Salberger 1993 proves it when X contains a conic.
A. Molyakov recently proved the above conjecture for $n = 7$.

Theorem (Molyakov 2023) *Let k be a number field. Let $X \subset \mathbb{P}^7_k$ be a non-conical geom. integral complete intersection of two quadrics. Then the Hasse principle holds for any smooth projective model of X.*

In the rest of this talk, I sketch the main steps of Molyakov's proof.

A local result

Theorem *Let k be a local field. Let $X \subset \mathbb{P}^7_k$ be a nonconical geom. integral complete intersection of two quadrics given by $f = g = 0$. If $X_{smooth}(k) \neq \emptyset$, and* **there is no form of rank** ≤ 5 *in the geometric pencil $\lambda f + \mu g$, then there exists a nondegenerate form $\lambda f + \mu g$ in the pencil over k which splits off three hyperbolics.*

The proof is similar to the proof in the smooth case but is geometrically more involved. Namely, one finds a smooth k-point $P \in X(k)$ such that the intersection of the tangent space T_P at X in the point P is a cone over a reasonable intersection of two quadrics $Y \subset \mathbb{P}^4$. Then there exists a quadratic point on Y over the p-adic field, which leads to a (degenerate conic) lying in $T_P \cap X$. A quadric in the pencil containing a conic is defined by a quadratic form which splits off three hyperbolics.

Global result, the "regular" case

Theorem *Let k be a number field. Let $X \subset \mathbb{P}^7_k$ be a nondegenerate geom. integral complete intersection of two quadrics given by $f = g = 0$. Assume* **there is no form of rank** ≤ 6 *in the geometric pencil $\lambda f + \mu g$. Then the Hasse principle holds for any smooth projective model of X*

Proof. Under the geometric hypothesis one knows that the variety parametrizing the planes $\mathbb{P}^2 \subset X$ is a generalized jacobian (X. Wang 2018) and in particular is **geometrically integral**.
Via Lang-Weil and Hensel this shows there is a finite set S of places such that for $v \notin S$, there exists a $\mathbb{P}^2_{k_v} \subset X_{k_v}$. Thus any form $\lambda f + \mu g$ contains 3 hyperbolics over k_v for $v \notin S$.
The previous local result together with weak approximation then produce a $\lambda f + \mu g$ over k with 3 hyperbolics over each k_v hence over k by Hasse, hence we have a conic lying on X and may conclude by Salberger's theorem.

Global result, the irregular case

We now allow the existence of a form of rank ≤ 6 in the geometric pencil. In this case the variety parametrizing the $\mathbb{P}^2 \subset X \subset \mathbb{P}^7$ need not be geometrically connected.
There is an interesting case by case discussion. Some cases were handled in [CT/Sa/SD], for example the case where there exists one form in the pencil over k which has rank 6, which corresponds to X having two conjugate singular points. This case is handled by the fibration method for rational points, ultimately reducing to the case of Châtelet surfaces.

But two cases
– The geometric pencil contains two conjugate forms of rank 6.
– The geometric pencil exactly contains 4 forms of rank 6.
require a new, specific argument.
One uses the **fibration method for zero-cycles** (Harpaz-Wittenberg 2016), which is more flexible than the fibration method for rational points.

When the geometric pencil contains two conjugate forms f, g of rank 6

(General case)

The vertex of the cone $f = 0$ is a line of singular points which intersects $g = 0$ in 2 points. Similarly for $g = 0$. The 4 points are globally defined over k, are singular on X, and span $H \simeq \mathbb{P}^3 \subset \mathbb{P}^7_k$. The family of $\mathbb{P}^4$'s containing H defines a rational map h from X to $\mathbb{P}^3_k$. The fibres X_m are intersections of two quadrics in $\mathbb{P}^4_k$ with 4 singular points, they all contain a skew quadrilateral on the 4 points.

By Coray-Tsfasman 1988, these fibres over k are birational to k-forms of $\mathbb{P}^1 \times \mathbb{P}^1$, hence over a number field satisfy (smooth) Hasse principle and weak approximation.

After a resolution of singularities $Y \to X$ one gets a fibration $f : Y \to \mathbb{P}^3_k$ with smooth general fibres. One is then in a good position to apply the fibration method. However one cannot apply the fibration method for rational points since one does not control the ramification locus of f.

The smooth fibres satisfy the Hasse principle and weak approximation for rational points, and are geometrically simple enough. A general result of Y. Liang 2013 then ensures that conjecture (E) on zero-cycles (an analogue for zero-cycles of the hypothesis that the Brauer group controls the local-global obstruction for rational points) holds for these fibres.

Since $n = 7$, we have $\mathrm{Br}(Y)/\mathrm{Br}(k) = 0$.
The fibration theorem for zero-cycles (Harpaz-Wittenberg 2016 – there is no restriction on the ramification locus of $Y \to \mathbb{P}^3_k$) then ensures that the k-variety Y satisfies conjecture (E) hence in particular has a zero-cycle of degree one as soon as $Y(\mathbb{A}_k) \neq \emptyset$.
This implies that X has a zero-cycle of degree one, Hence by Amer-Brumer X has a rational point. If such a k-point is smooth then $Y(k) \neq \emptyset$. If one is given a singular k-point on X then X is k-birational to a quadric. QED

When the geometric pencil contains exactly 4 forms of rank 6
Here one reduces to the case where X is k-birational to a fibration $Y \to \mathbb{P}^1$, Y/k smooth, whose generic fibre is a smooth compactification of a principal homogeneous space under a torus. It is a classical result (Sansuc 1981) that the Brauer-Manin obstruction fro rational points is the only obstruction for smooth compactification of a principal homogeneous space under a torus over a number field. Liang's theorem then gives the result for zero-cycles : conjecture (E) holds for the smooth fibres. By Harpaz-Wittenberg one gets Conjecture (E) for the total space. From $Y(\mathbb{A}_k) \neq \emptyset$ and $\mathrm{Br}(Y)/\mathrm{Br}(k) = 0$, one gets that Y has a zero-cycle of degree one, and then X has a zero-cycle of degree one, hence a k-rational point by Amer-Brumer.

This completes the proof.