Unramified cohomology (survey talk)

Jean-Louis Colliot-Thélène

C.N.R.S., Université Paris-Sud

Ramification in algebra and in geometry,
Emory University, 16-20 May 2011

I was required to give a survey talk on a given topic. Of course I shall obey, but I shall defend myself with a celebrated quote by Jean Cocteau.

Puisque ces mystères me dépassent,
feignons d'en être l'organisateur.

Since these mysteries pass my understanding,
let us pretend I am in charge of them.

Rationality versus unirationality

1. Over any field, a unirational curve is rational (Lüroth)
2. Over an algebraically closed field of char. zero, a unirational surface is rational (Castelnuovo)
3. Let G be a finite group. If the field $(\mathbf{Q}(x_g)_{g\in G})^G$ is purely transcendental over $\mathbf{Q}$, then G is a Galois group over $\mathbf{Q}$ (Hilbert, E. Noether)
4. There are unirational surfaces over $\mathbf{R}$ which are not $\mathbf{R}$-rational (B. Segre).

Retract rationality

Theorem (Saltman 1984) *Let k be a field and X an integral k-variety.*
Equivalent :
(i) There exists a non-empty open $U \subset X$, an open set $W \subset \mathbb{A}^n_k$, and a factorization $U \to W \to U$ of identity on U.
(ii) There exists a non-empty open $V \subset X$ such that for any local k-algebra A, with residue field κ, the map $V(A) \to V(\kappa)$ is onto.

Uses it to show that GL_n/PGL_p, p prime, is retract rational.

direct factor of k-rational variety $\Longrightarrow$ retract k-rational; converse ?

retract k-rational $\Longrightarrow$ for X/k smooth and proper, X is (universally) R-trivial; converse ?
Analogous statement and question for the Chow group of zero-cycles.

retract k-rational $\Longrightarrow$ k-unirational

Let X/k be a smooth, connected, projective variety and $k(X)$ be its function field.
How may one show that X is not k-birational to projective space over k ?
Produce birational invariants, "trivial" on projective space.

retract k-rational $\Longrightarrow$ many birational invariants are trivial.

If k is not algebraically closed, there is a subtler invariant ; the stable class of the Galois module $Pic(\overline{X})$ up to addition of a permutation lattice (Manin, Voskresenskiĭ). Accounts for Swan's negative answer to Noether's problem for $G = \mathbf{Z}/47$ over $\mathbf{Q}$.

Let $k = \mathbf{C}$. To any smooth, projective, connected variety $X/\mathbf{C}$ one may associate its fundamental group $\pi_1(X)$.

Theorem (Serre 1959) If $X/\mathbf{C}$ is unirational, then $\pi_1(X) = 0$.

The Brauer group

Serre 's result shows that we cannot use $H^1_{\text{ét}}(X, G)$ with G finite to detect nonrational unirational varieties.

In 1972, three completely independent methods were devised to produce nonrational unirational varieties over $\mathbf{C}$:
Clemens-Griffiths, Iskovskikh-Manin, Artin-Mumford.
Artin and Mumford used some version of the Brauer group.

For A a dvr with fraction field K and with residue field κ, and $n \in \kappa^*$, residue map

$$\partial_A : H^2(K, \mu_n) \to H^1(\kappa, \mathbf{Z}/n).$$

For any $j \in \mathbf{Z}$, let $\mathbf{Q}/\mathbf{Z}(j) = lim_n \mu_n^{\otimes j}$.

For $char(k) = 0$ and X/k smooth, quasiprojective, connected, equivalent definitions of the Brauer group $Br(X)$ of X :

(1) Azumaya Brauer group $Br_{Az}(X)$

(2) Étale Brauer group $H^2_{\text{ét}}(X, \mathbb{G}_m) \hookrightarrow H^2_{\text{ét}}(k(X), \mathbb{G}_m)$

(3) Image of $H^2_{\text{ét}}(X, \mathbf{Q}/\mathbf{Z}(1))$ in $H^2_{\text{ét}}(k(X), \mathbf{Q}/\mathbf{Z}(1))$

(4) $Ker[H^2_{\text{ét}}(k(X), \mathbf{Q}/\mathbf{Z}(1)) \to \oplus_{x \in X^{(1)}} H^1_{\text{ét}}(k(x), \mathbf{Q}/\mathbf{Z})]$

If X is moreover projective

(5) Unramified $Br_{nr}(k(X)/k) = H^2_{nr}(k(X), \mathbf{Q}/\mathbf{Z}(1))$: For Ω the set of all rank one discrete valuations on $k(X)$, trivial on k, $Ker[H^2_{\text{ét}}(k(X), \mathbf{Q}/\mathbf{Z}(1)) \to \prod_{v \in \Omega} H^1_{\text{ét}}(k(v), \mathbf{Q}/\mathbf{Z})]$

Étale cohomology definition gives functoriality under arbitrary morphisms. Also enables use of the Kummer sequence (Grothendieck, 1968).

For X smooth and projective over $\mathbf{C}$, exact sequence

$$0 \to NS(X) \otimes \mathbf{Q}/\mathbf{Z} \to H^2_{\text{ét}}(X, \mathbf{Q}/\mathbf{Z}(1)) \to Br(X) \to 0$$

which gives

$$0 \to (\mathbf{Q}/\mathbf{Z})^{(b_2-\rho)} \to Br(X) \to H^3(X(\mathbf{C}), \mathbf{Z})\{tors\} \to 0.$$

For X unirational, $b_2 - \rho = 0$.

For X retract rational, $Br(X) = 0$.

Artin and Mumford produced a smooth projective X with a conic bundle structure over $\mathbb{P}^2_{\mathbf{C}}$ for which they compute $H^3(X(\mathbf{C}), \mathbf{Z})\{tors\} \neq 0$.

Hard to exhibit smooth projective models of function fields in high dimension, hence hard to compute $H^3(X(\mathbf{C}), \mathbf{Z})\{tors\}$ of such a model.

Saltman 1984 : First example of a finite group G with a faithful linear action on a f.d. complex vector space V such that $\mathbf{C}(V)^G$ is not rational. Does not compute a smooth projective model !
Uses the unramified definition of the Brauer group:

$$Br_{nr}(k(X)) = Ker[H^2_{\text{ét}}(k(X), \mathbf{Q}/\mathbf{Z}(1)) \to \prod_{v \in \Omega} H^1_{\text{ét}}(k(v), \mathbf{Q}/\mathbf{Z})]$$

Proof slightly devious :
Produces a field $L/\mathbf{C}$, product of function fields of Severi-Brauer varieties over $\mathbf{C}(a,b,c,d)$, with $Br_{nr}(L/\mathbf{C}) \neq 0$, thus $L/\mathbf{C}$ not retract rational. This uses knowledge of $Ker[Br(F) \to Br(F(W))]$ for W/F Severi-Brauer (Witt, Châtelet, Amitsur).
Then uses the lifting characterisation of retract rationality to show that $L/\mathbf{C}$ is retract rational if and only if $\mathbf{C}(V)^G/\mathbf{C}$ is, for G a suitable p-group of class 2.

Further work on Noether's problem : Bogomolov (1987, 1989).

Theorem.
G finite group of automorphisms of a function field $L/\mathbf{C}$
Then

$$Br_{nr}(L^G) = \{\alpha \in Br(L^G),\ \forall H \subset G \text{ bicyclic},\ \alpha \in Br_{nr}(L^H)\}$$

Idea : a nontrivial residue is a class in $H^1(\kappa(v), \mathbf{Q}/\mathbf{Z})$ hence is detected on a cyclic group, and one is reduced to considering a central extension of such a cyclic group by an inertia group, cyclic, hence this extension is a bicyclic group.

Application to the Noether problem. Using Fisher's theorem, Bogomolov 1987 then proves :

Theorem *Let G finite act linearly and faithfully on a finite dimensional vector space V. Then*

$$Br_{nr}(\mathbf{C}(V)^G) \simeq ker[H^2(G, \mathbf{Q}/\mathbf{Z}) \to \prod_{A \ bicyclic} H^2(A, \mathbf{Q}/\mathbf{Z})].$$

(May here replace "bicyclic" by "abelian".)

Bogomolov also produced a precise formula in the case G is a central extension of an abelian p-group by an abelian p-group. This led to many examples with $Br_{nr}(\mathbf{C}(V)^G) \neq 0$.

Theorem (Kunyavskiĭ 2010) *For a finite simple group G, $Br_{nr}(\mathbf{C}(V)^G) = 0$.*

Saltman 1987 establishes a connexion between $\mathbf{C}(GL_n/H)$ for $H \subset GL_n$ semisimple and (possibly twisted) multiplicative invariants under the Weyl group of H.
Earlier result : Formanek, Procesi. Motivated by the case $H = PGL_r$ ("the centre of the ring of generic matrices").

Saltman 1987, 1990 : computation of $Br_{nr}(\mathbf{C}(M)^G)$ for multiplicative invariants (M a faithful G-lattice),

$$Br_{nr}(\mathbf{C}(M)^G) = Ker[H^2(G, \mathbf{C}^* \oplus M) \to \prod_{A \ bicyclic} H^2(A, \mathbf{C}^* \oplus M)].$$

and for twisted multiplicative invariants.

Saltman 1985 : Over any field k,

$$Br(k) = Br_{nr}(k(GL_n/PGL_r)) = Br_{nr}(k(SL_n/PGL_r))$$

Theorem (Bogomolov 1987, 1989) : Over $\mathbf{C}$, connected reductive groups $H \subset G$, if G semisimple and simply connected, then $Br_{nr}(\mathbf{C}(G/H)) = 0$.

Open question : For such $H \subset G$ over $\mathbf{C}$, is G/H rational ?

Higher unramified cohomology

For A a dvr with fraction field K and with residue field κ, $n \in \kappa^*$, any $i > 0$ and any $j \in \mathbf{Z}$, there is a residue map

$$\partial_A : H^i(K, \mu_n^{\otimes j}) \to H^{i-1}(\kappa, \mu_n^{\otimes (j-1)})).$$

For X/k a smooth connected variety, $n \in k^*$,
Ojanguren and I (1989) defined

$$H^i_{nr}(X, \mu_n^{\otimes j}) = Ker[H^i(k(X), \mu_n^{\otimes j}) \to \prod_{x \in X^{(1)}} H^{i-1}(k(x), \mu_n^{\otimes (j-1)})].$$

In a different guise, these groups are already in Bloch-Ogus (1974).

The Gersten conjecture (Bloch-Ogus 1974) ensures that for any $x \in X$ any class in $H^i_{nr}(X, \mu_n^{\otimes j})$ comes from $H^i(O_{X,x}, \mu_n^{\otimes j})$. This implies that for X/k smooth projective

$$H^i_{nr}(X, \mu_n^{\otimes j}) \subset H^i(k(X), \mu_n^{\otimes j})$$

is a birational invariant. It may also be defined purely in terms of valuations

$$H^i_{nr}(k(X), \mu_n^{\otimes j}) = Ker \ [H^i(k(X), \mu_n^{\otimes j}) \to \prod_{v \in \Omega} H^{i-1}(k(v), \mu_n^{\otimes (j-1)})],$$

where Ω is the set of rank one dvr's on $k(X)$, trivial on k.

The valuation theoretic definition in CT/Ojanguren 1989 was inspired by Saltman's unramified version of the Brauer group.

The group $H^3_{nr}(\mathbf{C}(X), \mathbf{Z}/2)$ was then used to give examples of nonrational unirational varieties for which the previously known methods may not be used to detect nonrationality.
Unramified classes are obtained by the method "ramification eats up ramification".
Nonvanishing of the classes uses Arason's theorem (1974) : control of kernel $H^3(F, \mathbf{Z}/2) \to H^3(F(Y), \mathbf{Z}/2)$ for Y a 3-fold Pfister quadric (Arason's result is a forerunner of later breakthroughs in algebraic K-theory).

For $i \leq 2$, the map $H^i_{\text{ét}}(X, \mu_n^{\otimes j}) \to H^i_{nr}(X, \mu_n^{\otimes j})$ is onto, but this need not be so for $i \geq 3$.

Indeed for $k = \mathbf{C}$, all $H^i_{\text{ét}}(X, \mathbf{Z}/n)$ are finite but for $i \geq 3$, the groups $H^i_{nr}(\mathbf{C}(X), \mathbf{Z}/n)$ need not be finite (C. Schoen).

However for $X/\mathbf{C}$ unirational, $H^3_{nr}(\mathbf{C}(X), \mathbf{Z}/n)$ is finite.

E. Peyre's thesis, 1993 : examples of unirational varieties X with $H^3_{nr}(\mathbf{C}(X), \mathbf{Z}/p) \neq 0$ and examples with $H^4_{nr}(\mathbf{C}(X), \mathbf{Z}/2) \neq 0$, while the lower invariants are zero.
Use of techniques à la Bogomolov to decide if certain cohomology classes are unramified.
Control of kernel of restriction maps
$H^3(F, \mathbf{Z}/p) \to H^3(F(Y), \mathbf{Z}/p)$ for certain norm varieties (Suslin) and $H^n(F, \mathbf{Z}/2) \to H^n(F(Y), \mathbf{Z}/2)$ for anisotropic n-fold Pfister quadrics Y (case $n = 4$, Jacob-Rost).
One can now go further, Orlov-Vishik-Voevodsky, see Asok 2010.

Natural question

For the higher H^i_{nr}'s, are there analogues of the results of Bogomolov and Saltman for H^2_{nr} for the function fields of GL_n/G when G is reductive (finite or connected) ?

Similar question for the fields $\mathbf{C}(M)^G$, where G is a finite group acting on a lattice M.

Theorem (Saltman 1995/97) : $H^3_{nr}(\mathbf{C}(GL_{n,\mathbf{C}}/PGL_r), \mathbf{Q}/\mathbf{Z}) = 0$.

The proof involves a study of $H^3_{nr}(\mathbf{C}(M)^G, \mathbf{Q}/\mathbf{Z})$.
The 1995 paper has inspired further work by Peyre, but the two 1997 J. Algebra papers would deserve further reading.

In one of these papers there is an analysis of residue maps on the image of the composite map

$$H^3(G, M) \to H^3(G, \mathbf{C}(M)^*) \to H^3_{\text{ét}}(\mathbf{C}(M)^G), \mathbb{G}_m)$$

$$= H^3_{\text{ét}}(\mathbf{C}(M)^G, \mathbf{Q}/\mathbf{Z}(1))$$

Rost, Totaro, Serre, Merkurjev : Relation between cohomological invariants of $H^1(., G)$ with values in $H^d(., M)$ (M finite Galois module) and $H^d_{nr}(SL_n/G, M)$.

Used by Merkurjev 2002 to compute $H^3_{nr}(k(SL_{n,k}/G), \mathbf{Q}/\mathbf{Z}(2)))$ for G semisimple simply connected (classical) and k arbitrary.
Examples where $H^3_{nr}(k(SL_{n,k}/G), \mathbf{Q}/\mathbf{Z}(2)) \neq H^3(k, \mathbf{Q}/\mathbf{Z}(2))$, hence $SL_{n,k}/G$ not retract rational.
Exceptional groups handled by Garibaldi 2006

Let G be a finite group, V a faithful finite dimensional complex linear representation of G. Fix $\mathbf{Q}/\mathbf{Z} \simeq \mathbf{Q}/\mathbf{Z}(1)$.
For any $i > 1$, there is a natural composite map

$$H^i(G, \mathbf{Q}/\mathbf{Z}) \to H^i(G, \mathbf{C}(V)^*) \to H^i(\mathbf{C}(V)^G, \mathbf{Q}/\mathbf{Z}).$$

What is the kernel of this map ?

Is $H^i_{nr}(\mathbf{C}(V)^G, \mathbf{Q}/\mathbf{Z})$ in the image of $H^i(G, \mathbf{Q}/\mathbf{Z})$?

Can one describe the inverse image of $H^i_{nr}(\mathbf{C}(V)^G, \mathbf{Q}/\mathbf{Z})$ in $H^i(G, \mathbf{Q}/\mathbf{Z})$?

Using a suitable limit formalism to define BG (Bogomolov, Totaro) one may ask similar questions for any reductive G.

Peyre 1998, 1999, 2008 : For G finite, $k = \mathbf{C}$, exact sequences

$$0 \to CH^2_G(\mathbf{C}) \to H^3(G, \mathbf{Q}/\mathbf{Z}) \to H^3(\mathbf{C}(V)^G, \mathbf{Q}/\mathbf{Z})$$

$$0 \to CH^2_G(\mathbf{C}) \to H^3(G, \mathbf{Q}/\mathbf{Z}) \to H^3_{nr}(BG, \mathbf{Q}/\mathbf{Z}) \to 0$$

(more on this sequence later on)

$$0 \to CH^2_G(\mathbf{C}) \to H^3_{NR}(G, \mathbf{Q}/\mathbf{Z}) \to H^3_{nr}(\mathbf{C}(V)^G, \mathbf{Q}/\mathbf{Z}) \to 0$$

for $H^3_{NR}(G, \mathbf{Q}/\mathbf{Z}) \subset H^3(G, \mathbf{Q}/\mathbf{Z})$ a subgroup defined group-theoretically.

Used to produce systematic examples, à la Bogomolov, of groups G of order p^{12} with $Br_{nr}(\mathbf{C}(V)^G) = 0$ but $H^3_{nr}(\mathbf{C}(V)^G, \mathbf{Q}/\mathbf{Z}) \neq 0$.

Some important aspects :

1) Analysis of $Ker[H^3(G,\mathbf{Q}/\mathbf{Z}) \to H^3(\mathbf{C}(V)^G,\mathbf{Q}/\mathbf{Z})]$
Serre's geometrically negligible classes, Saltman's permutation negligible classes.
Description of the equivariant Chow group $CH^2_G(\mathbf{C})$ (involves work of many people).

2) For each pair (g, D), $g \in G$, $D \subset G$ with $g \in Z_G(D)$, definition of a residue map

$$\partial_{g,D} : H^3(G,\mathbf{Q}/\mathbf{Z}) \to H^2(D,\mathbf{Q}/\mathbf{Z}).$$

$H^3_{NR}(G,\mathbf{Q}/\mathbf{Z}) :=$ kernel of all these maps
(It would be interesting to compare these residue maps with those defined by Saltman for lattice invariants)

2010 Thesis by Nguyen Thi Kim Ngan, some ideas of Bruno Kahn.
Definition (Bruno Kahn) of residues in a quite general context.
In particular for each of the two families of functors F^i on smooth varieties : $H^i(X, \mathbf{Q}/\mathbf{Z}(i-1))$ and $H^i_{nr}(X, \mathbf{Q}/\mathbf{Z}(i-1))$, for G finite, for each pair (g, D), $g \in G$, $D \subset G$ with $g \in Z_G(D)$, definition of a residue map

$$\partial_{g,D} : F^i(BG) \to F^{i-1}(BD)$$

Define $F^i_{NR}(BG)$ as the intersection of all kernels of such maps.

Theorem (Nguyen Thi Kim Ngan)
For $F^i(X) = H^i_{nr}(X, \mathbf{Q}/\mathbf{Z}(i-1))$, the group $F^i_{NR}(BG)$ coincides with the group $H^i_{nr}(\mathbf{C}(V)^G, \mathbf{Q}/\mathbf{Z}(i-1))$.

The Chow group of zero-cycles

A known birational invariant of a smooth projective variety X is the Chow group $CH_0(X)$ of zero-cycles modulo rational equivalence. If X is rational, then $deg : CH_0(X) \simeq \mathbf{Z}$.

Pairings

$$CH_0(X) \times H^i_{nr}(X, \mu_n^{\otimes j}) \to H^i(k, \mu_n^{\otimes j})$$

provide a link between the two types of invariants.

The birational invariance of unramified cohomology over smooth projective varieties extends in the context of Rost's cycle modules, and so do the above pairings.

Theorem (Merkurjev, 2007) *Let X/F be a smooth proper, geometrically connected variety. The following conditions are equivalent*
(i) For every cycle module M over F, we have $M(F) = M(F(X))_{nr}$.
(ii) For every field extension L/F, the degree map $deg_L : CH_0(X_L) \to \mathbf{Z}$ *is an isomorphism.*

$H^3_{nr}(X, \mathbf{Q}/\mathbf{Z}(2))$ and the Chow group $CH^2(X)$

One key tool in Peyre's computation of $H^3_{nr}(\mathbf{C}(V)^G, \mathbf{Q}/\mathbf{Z})$ is the exact sequence

$$0 \to CH^2_G(\mathbf{C}) \to H^3(G, \mathbf{Q}/\mathbf{Z}) \to H^3_{nr}(BG, \mathbf{Q}/\mathbf{Z}) \to 0$$

This is a special case ($X = BG$) of a basic exact sequence for smooth k-varieties (Lichtenbaum 1990, Kahn 1996) provided by motivic cohomology (Zariski and étale).

$$0 \to CH^2(X) \to \mathbb{H}^4_{\text{ét}}(X, \mathbf{Z}_{\text{ét}}(2)) \to H^3_{nr}(X, \mathbf{Q}/\mathbf{Z}(2)) \to 0.$$

Here $\mathbf{Z}_{\text{ét}}(2)$ is the étale version of the Zariski complex $\mathbf{Z}(2)$ defined by Lichtenbaum and Voevodsky. The proof uses the Merkurjev–Suslin theorems.

In étale motivic cohomology there is a Kummer exact exact triangle (n invertible on X)

$$\mathbf{Z}_{\text{ét}}(i) \xrightarrow{\times n} \mathbf{Z}_{\text{ét}}(i) \to \mu_n^{\otimes i} \to \mathbf{Z}_{\text{ét}}(i)[1]$$

If one uses this for $i = 2$ and the snake lemma for multiplication by an integer $n > 0$ on the "basic exact sequence" one recovers a long long exact sequence of Bloch-Ogus 1974, one term of which is $CH^2(X)/n$.

The "basic exact sequence" and the Kummer triangle may be used to prove :

"Basic theorem" *Suppose F has 'finite Galois cohomology' (e.g. F algebraically closed, real closed, finite, p-adic, higher local). The following groups are finite and isomorphic :*
(i) The quotient of $H^3_{nr}(X, \mathbf{Q}_l/\mathbf{Z}(2))$ *by its maximal divisible subgroup*
(ii) The torsion subgroup of the cokernel of the cycle map

$$CH^2(X) \otimes \mathbf{Z}_l \to H^4_{\text{ét}}(X, \mathbf{Z}_l(2)).$$

Earlier Betti version over $F = \mathbf{C}$, CT-Voisin 2010
l-adic version Kahn 2011, CT-Kahn 2011

The "basic theorem" has been applied both ways.

Over $F = \mathbf{C}$, CT-Voisin 2010

1) use the CT-Ojanguren examples to produce examples of unirational 6-folds for which the integral Hodge conjecture fails for cycles of codimension 2.

2) use complex algebraic results of Voisin to show $H^3_{nr}(X, \mathbf{Q}/\mathbf{Z}) = 0$ for any uniruled threefold X.

Theorem (CT-Kahn 2011). *Let* $\mathbf{F}$ *be a finite field of characteristic* p, C *a smooth projective curve over* $\mathbf{F}$ *with function field* $K = \mathbf{F}(C)$, $\mathcal{X}$ *a smooth projective threefold over* $\mathbf{F}$, *and* $f : \mathcal{X} \to C$ *a dominant morphism with smooth generic fibre* X/K.
(i) Tate's conjecture holds for divisors on $\mathcal{X}$
(ii) $H^3_{nr}(\mathcal{X}, \mathbf{Q}/\mathbf{Z}(2)) = 0$
(iii) The Brauer-Manin set of the K*-surface* X *is not empty.*
Then there exists a zero-cycle of degree a power of p *on the surface* X.

The proof combines the "basic theorem" and a theorem of Shuji Saito (1989) on the cycle map $CH^2(\mathcal{X}) \otimes \mathbf{Z}_l \to H^4_{\text{ét}}(\mathcal{X}, \mathbf{Z}_l(2))$.

It is I believe an important question (CT-Sansuc, Kato-Saito) whether the theorem holds without assuming (i) and (ii).

Tate's conjecture for divisors is classical and holds for instance if $\mathcal{X}$ is geometrically rationally dominated by the product of a curve and a projective space.

It is an open question whether $H^3_{nr}(\mathcal{X}, \mathbf{Q}/\mathbf{Z}(2)) = 0$ for any smooth threefold over a finite field. A. Pirutka has shown this need not hold for varieties of dimension at least 5.

For the time being, there are two applications of the theorem.

(CT/Swinnerton-Dyer 2009)
$K = \mathbf{F}(t)$ and X is a surface in $\mathbb{P}^3_{\mathbf{F}(t)}$ given by an equation $f + tg = 0$, with f and g two forms over $\mathbf{F}$ of the same degree d.

Here $\mathcal{X}$ is $\mathbf{F}$-rational and the Tate and H^3 conditions are nearly obvious.

For $d \geq 5$, the surface X is of general type.

(Parimala-Suresh 2010)
The generic fibre of $f : \mathcal{X} \to C$ is a rational surface with a conic bundle structure over $\mathbb{P}^1$.
Parimala and Suresh actually prove the general result that for any smooth projective threefold $\mathcal{X}/\mathbf{F}$ with a conic bundle structure over a surface, $H^3_{nr}(\mathcal{X}, \mathbf{Q}/\mathbf{Z}(2)) = 0$ (up to p-torsion).

Their proof combines
1) Suslin's computation (1982) of $H^3_{nr}(\Gamma, \mathbf{Q}/\mathbf{Z}(2))$ for a conic Γ over an arbitrary field.
2) Vanishing of $H^3_{nr}(S, \mathbf{Q}/\mathbf{Z}(2))$ for a surface over a finite field (1983, higher class field theory)
3) Many of the ideas in the 2006/2008 papers by Saltman on central simple algebras over surfaces.

One can consider the “basic exact sequence” over a separable closure and do Galois cohomology (following Bloch, CT-Raskind, Kahn). Using many earlier results, in particular the Weil conjectures (Deligne), one gets :

Theorem (CT/Voisin 2010 , CT/Kahn 2011) (up to p-torsion)
Let $X/\mathbf{F}$ be a smooth projective variety over a finite field. Assume that the Brauer group of $\overline{X} = X \times_{\mathbf{F}} \overline{\mathbf{F}}$ is trivial.
Then (up to p-torsion) there is a natural exact sequence

$$0 \to CH^2(X) \to CH^2(\overline{X})^G \to H^3_{nr}(X, \mathbf{Q}/\mathbf{Z}(2)) \to H^3_{nr}(\overline{X}, \mathbf{Q}/\mathbf{Z}(2))$$

In particular, if $\overline{X}$ is rational, there is an exact sequence

$$0 \to CH^2(X) \to CH^2(\overline{X})^G \to H^3_{nr}(X, \mathbf{Q}/\mathbf{Z}(2)) \to 0.$$

This has been used by A. Pirutka (see her talk).

CONCLUSION

Much work has been done on computing the Brauer group $Br(X) = H^2_{nr}(X, \mathbf{Q}/\mathbf{Z}(1))$ for various varieties.

Some work has been done to compute $H^3_{nr}(X, \mathbf{Q}/\mathbf{Z}(2))$ for (compactifications of) homogeneous spaces of connected linear algebraic groups. There is also work of Rost, Kahn, Sujatha on higher unramified cohomology of quadrics.

Here is my version of "*Carthago delenda est*" :

What about cubic surfaces ?

This survey stretched over fifty years. Let me end with another quote by Jean Cocteau.

De notre naissance à notre mort, nous sommes un cortège d'autres qui sont reliés par un fil ténu.

From our birth to our death, we are a procession of other ones whom a fine thread connects.