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PROBABILITY THEORY METHODS IN ZERO-SUM STOCHASTIC GAMES*

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Abstract. The purpose of this paper is to apply the methods of optimal stochastic control introduced by the author to a class of zero-sum stochastic games.

1. Introduction. The purpose of this paper is to apply the methods used by the author in his previous work on the optimal control of diffusions [1] to a wide class of stochastic games.

Let us consider two compact metrizable spaces U and U'.

f is a bounded function defined on $R^+ \times R^d \times U \times U'$ with values in R^d, Borel on $R^+ \times R^d$ for a given $(u, u') \in U \times U'$, and continuous on $U \times U'$ for $(t, x) \in R^+ \times R^d$. We consider the diffusion process

$$\begin{aligned} &dx = f(t, x_t, u(t, x_t), u'(t, x_t))\, dt + \sigma(t, x_t) \cdot d\beta \\ &x_s = x, \end{aligned} \tag{1.1}$$

where u and u' are Borel functions defined on $R^+ \times R^d$ with values in U and U'.

A is a Borel set in $R^+ \times R^d$.

T_A is the stopping time:

$$T_A = \inf \{t > s\,;\, (t, x_t) \in A\}. \tag{1.2}$$

p is a constant >0.

We consider the criterion

$$e^{ps} E \int_s^{T_A} e^{-pt} K(t, x_t, u(t, x_t), u'(t, x_t))\, dt, \tag{1.3}$$

where K is a bounded function satisfying the same assumptions as f.

The purpose of this paper is to prove the existence of a minimax couple of strategies where the minimum corresponds to u and the maximum to u'.

This problem has been considered by Friedman in [5], where partial differential equation techniques are used.

Friedman assumes that A^c may be written as $Q_T =]0, T[\times \Omega$ where Ω is an open domain of R^d, whose boundary is sufficiently regular. Differentiability assumptions are also done in [5] on the matrix $a = \sigma\sigma^*$.

Then Friedman proves in Theorem 3 of [5] that if H, X_1, X_2 are defined by

$$\begin{aligned} &H(t, x, u, u', p) = K(t, x, u, u') + \langle p, f(t, x, u, u')\rangle, \\ &X_1(t, x, p) = \inf_{u \in U} \sup_{u' \in U'} H(t, x, u, u', p), \\ &X_2(t, x, p) = \sup_{u' \in U'} \inf_{u \in U} H(t, x, u, u', p), \end{aligned} \tag{1.4}$$

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then under Isaac's condition $h=-X_1=-X_2$, the partial differential equation

$$(1.5)\qquad \begin{aligned}&\frac{\partial V}{\partial t}+\frac{1}{2}\sum_{i,j} a_{ij}(t,x)V_{x_ix_j}=h\left(t,x,V,\frac{\partial V}{\partial x}\right)\quad \text{on } Q_T,\\ &V=0 \quad \text{on } (]0,T[\times\partial\Omega)\cup(\{T\}\times\bar{\Omega})\end{aligned}$$

has one unique solution, which is the cost function of the game. V is then proved to be continuous on $\bar{Q}_T$, and $\nabla_x V$ is proved to be Hölder continuous on Q_T. Moreover, Theorem 4 of [5] shows that a solution of the game exists.

In this paper we consider the case where f and K may be written

$$(1.6)\qquad \begin{aligned}f(t,x,u,u')&=b(t,x,u)+b'(t,x,u'),\\ K(t,x,u,u')&=L(t,x,u)+L'(t,x,u').\end{aligned}$$

We do not have any differentiability assumptions on a, and we accept A to be any Borel set in $R^+\times R^d$. The partial differential equation (1.5) is then not well defined.

A solution of the game is proved to exist by using the results of [1]. The main interest of the method is the use of the deep convex structure of some problems of stochastic control. Moreover, the method is extendable to games on processes other than diffusions [2].

Finally, we come back to the more general system (1.1)–(1.3), and derive conditions for existence of solutions of this game. The reader is referred to [5] for a precise comparison of theses results with the results of Friedman.

In a different framework, Duncan and Varaiya have examined in [3] the problem of existence in the general nonanticipating case, where f and K depend on the entire trajectory of x and where (u,u') are taken as nonanticipating functions of x. They also assume that K does not depend on (u,u'), and that the first equality holds in (1.6) with $b_ib'_i=0 (i=1,\cdots,d)$. They also have a convexity condition on $\{b(t,x,u)|u\in U\}$ and $\{b'(t,x,u')|u'\in U'\}$. In [4] Elliott has extended the results of Duncan and Varaiya and obtains an existence result under Isaac's condition. However, the techniques used in [3] and [4] are rather different from the method used here which applies more directly to Markov processes.

2. Definition of the problem. a is a continuous function defined on $R^+\times R^d$ with values in $R^d\otimes R^d$ such that:

a is bounded,

a is positive definite.

b is a Borel bounded function defined on $R^+\times R^d$ with values in R^d.

$Q^b_{(s,x)}$ is the unique measure on the space of continuous functions defined on R^+ with values in R^d, which is a solution of the martingale problem defined in [7], with (s,x) as the starting point.

$E^b_{(s,x)}$ is the expectation operator for $Q^b_{(s,x)}$.

σ is the positive square root of a.

p is a strictly positive constant.

A is a Borel set in $R^+\times R^d$.

L is a real-valued bounded Borel function defined on $R^+\times R^d$.

For $c=(b, L)$, we define the function V_c as

$$V_c(s, x) = e^{ps} E^b_{(s,x)} \int_s^{T_A} e^{-pt} L(t, x_t)\, dt. \tag{2.1}$$

K and K' are two bounded Borel set-valued mappings defined on $R^+ \times R^d$ with nonempty compact values in $R^d \times R$.

$\mathscr{L}$ (resp. $\mathscr{L}'$) is the set of Lebesgue equivalence classes of the Borel selections of K (resp. K').

$\mathscr{L}$ (resp. $\mathscr{L}'$) has the topology $\sigma(L_\infty(R^+ \times R^d), L_1(R^+ \times R^d))$, which is metrizable.

DEFINITION 2.1. Problem R is defined as the search of

$$c_0 = (b_0, L_0) \in \mathscr{L}$$

and

$$c'_0 = (b'_0, L'_0) \in \mathscr{L}'$$

such that for $(c, c') \in \mathscr{L} \times \mathscr{L}'$,

$$V_{c_0+c'} \leqq V_{c_0+c'_0} \leqq V_{c+c'_0} \tag{2.2}$$

The relation between this formulation of problem R and the formulation given in § 1 is derived in the same way as in [1, Chap. IV, Part 1].

THEOREM 2.1. *Problem* R *has a solution.*

Proof. The next parts are devoted to the proof of the theorem.

3. The convex case. We define first two measures μ and ν on $R^+ \times R^d$.

μ is a probability measure on $R^+ \times R^d$ mutually absolutely continuous with the Lebesgue measure.

ν is the measure on $R^+ \times R^d$ defined by

$$\nu(\varphi) = E^0_\mu e^{ps} \int_s^{T_A} e^{-pt} \varphi(t, x_t)\, dt. \tag{3.1}$$

ν is then absolutely continuous relative to the Lebesgue measure of $R^+ \times R^d$, by Theorem 8.1 of [7].

We assume in this part that K and K' have convex values.

For $c \in \mathscr{L}$ (resp. $c' \in \mathscr{L}'$), we define Γ'_c (resp. $\Gamma_{c'}$) by

$$\Gamma'_c = \{c' \in \mathscr{L}';\ \forall (s, x) \in R^+ \times R^d,\ V_{c+c'}(s, x) = \sup_{\bar{c}' \in \mathscr{L}'} V_{c+\bar{c}'}(s, x)\} \tag{3.2}$$

(resp.

$$\Gamma_{c'} = \{c \in \mathscr{L};\ \forall (s, x) \in R^+ \times R^d,\ V_{c+c'}(s, x) = \inf_{\bar{c} \in \mathscr{L}} V_{\bar{c}+c'}(s, x)\}). \tag{3.2'}$$

PROPOSITION 3.1. Γ'_c *(resp.* $\Gamma_{c'}$*) has nonempty compact convex values.*

Proof. The nonemptiness and the compactness of Γ'_c (resp. $\Gamma_{c'}$) follow from [1, Thm. V-1]. Moreover, Theorems IV-5 and IV-8 of [1] applied to Part V of [1], prove that one can find a Borel function H_c (resp. $H_{c'}$) such that a necessary and sufficient condition for $c' = (b', L')$ to be in Γ'_c (resp. $c = (b, L)$ to be in $\Gamma_{c'}$) is that,

ν-a.e., the following relation holds:

$$L'(t,x)+\langle H_c(t,x),\sigma^{-1}(t,x)b'(t,x)\rangle=\max_{(\tilde{b}',\tilde{L}')\in K'(t,x)}\tilde{L}'+\langle H_c(t,x),\sigma^{-1}(t,x)\tilde{b}'\rangle \tag{3.3}$$

(resp.

$$L(t,x)+\langle H_{c'}(t,x),\sigma^{-1}(t,x)b(t,x)\rangle=\min_{(\tilde{b},\tilde{L})\in K(t,x)}\tilde{L}+\langle H_{c'}(t,x),\sigma^{-1}(t,x)\tilde{b}\rangle) \tag{3.4}$$

K and K' having convex values, the result follows. □

PROPOSITION 3.2. *The set-valued mapping defined on* $\mathscr{L}\times\mathscr{L}'$ *with values in* $\mathscr{L}\times\mathscr{L}'$

$$(c,c')\to\Gamma_{c'}\times\Gamma'_c \tag{3.5}$$

is upper semicontinuous.

Proof. We have to prove that if

$$(c_n,c'_n)\to(c,c')$$

and if

$$(\tilde{c}_n,\tilde{c}'_n)\in\Gamma_{c'_n}\times\Gamma'_{c_n}\to(\tilde{c},\tilde{c}'),$$

then

$$(\tilde{c},\tilde{c}')\in\Gamma_{c'}\times\Gamma'_c.$$

We know that, for any $\gamma'\in\mathscr{L}'$,

$$V_{c_n+\tilde{c}'_n}\geqq V_{c_n+\gamma'}. \tag{3.6}$$

By Theorem V-1 of [1], which proves the continuity of $c\to V_c$, we find that, for $\gamma'\in\mathscr{L}'$,

$$V_{c+\tilde{c}'}\geqq V_{c+\gamma'} \tag{3.7}$$

and $\tilde{c}'\in\Gamma'_c$. Similarly, we find that $\tilde{c}\in\Gamma_{c'}$, and then

$$(\tilde{c},\tilde{c}')\in\Gamma_{c'}\times\Gamma'_c.\quad □$$

We then have:

THEOREM 3.1. *Problem* R *has a solution.*

Proof. The set-valued mapping

$$(c,c')\to\Gamma'_{c'}\times\Gamma'_c$$

has nonempty compact convex values and is upper semicontinuous. By Kakutani's theorem, it has a fixed point (c_0,c'_0). This point has the property that

$$\text{if } \gamma'\in\mathscr{L}',\quad V_{c_0+c'_0}\geqq V_{c_0+\gamma'},$$
$$\text{if } \gamma\in\mathscr{L},\quad V_{c_0+c'_0}\leqq V_{\gamma+c'_0}.$$

It is then a solution of problem R. □

If (c_0,c'_0) is a solution of problem R, it is easy to check that, for $(s,x)\in R^+\times R^d$,

$$V_{c_0+c'_0}(s,x)=\inf_{c\in\mathscr{L}}\sup_{c'\in\mathscr{L}'}V_{c+c'}(s,x)=\sup_{c'\in\mathscr{L}'}\inf_{c\in\mathscr{L}}V_{c+c'}(s,x). \tag{3.8}$$

The function $V_{c_0+c'_0}$ does not depend then on the particular solution of problem R which is considered.

DEFINITION 3.1. q is the function defined by

$$q(s,x)=\inf_{c\in\mathscr{L}}\sup_{c'\in\mathscr{L}'} V_{c+c'}(s,x)=\sup_{c'\in\mathscr{L}'}\inf_{c\in\mathscr{L}} V_{c+c'}(s,x). \tag{3.9}$$

THEOREM 3.2. *It is possible to find a Borel function H such that for*

$$(c_0, c'_0)=((b_0, L_0), (b'_0, L'_0))$$

to be a solution of problem R, *it is necessary and sufficient that, ν—a.e., the following relations hold*:

$$L_0(t,x)+\langle H(t,x), \sigma^{-1}(t,x)b_0(t,x)\rangle=\min_{(\tilde b,\tilde L)\in K(t,x)} \tilde L+\langle H(t,x), \sigma^{-1}(t,x)\tilde b\rangle, \tag{3.10}$$

$$L'_0(t,x)+\langle H(t,x), \sigma^{-1}(t,x)b'_0(t,x)\rangle=\max_{(b',L')\in K'(t,x)} \tilde L'+\langle H(t,x), \sigma^{-1}(t,x)b'\rangle. \tag{3.11}$$

Moreover, a choice of (c_0, c'_0) verifying (3.10)–(3.11) *ν-a.e., is possible.*

Proof. The method is the same as in [1, Thm. IV-5, Cor. of Thm. IV-7 and Thm. IV-8].

With the notations of [1], we know from [1, (5.28)], that for $c=(b,L)\in L_\infty(R^+\times R^d)$, we can find a Borel function H_c and an additive functional A^c such that:

$$V_c(t,x_t)=V_c(s,x_s)+\int_s^t (pV-L)(u,x_u)\,du+\int_s^t H_c(u,x_u)\cdot d\beta_u^b+\int_s^t dA_u^c \tag{3.12}$$

(by using the results of Annex 1 in [1] we cancel the term M_t-M_s in [1, (5.28)]).

If (c_0, c'_0) is a solution of Problem R,

$$V_{c_0+c'_0}=q.$$

We can then write

$$\begin{aligned} q(t,x_t)&=q(s,x_s)+\int_s^t (pq-(L_0+L'_0))(u,x_u)\,du \\ &\quad+\int_s^t H_{c_0+c'_0}(u,x_u)\cdot d\beta_u^{b_0+b'_0}+\int_u^t dA_u^{c_0+c'_0} \end{aligned} \tag{3.13}$$

By (3.13), $H_{c_0+c'_0}$ does not depend on a particular solution of the game (c_0, c'_0).

By reasoning as in [1, Cor. of Thm. IV-7], we take for H the fixed function $H_{c_0+c'_0}$, where (c_0, c'_0) is a solution of Problem R.

The result follows from [1, Thm. IV-5 and Thm. IV-8]. □

4. The general case. We now prove Theorem 1.1.

Proof. Let $\hat K(t,x)$ and $\hat K'(t,x)$ be the closed convex hulls of $K(t,x)$ and $K'(t,x)$.

$\hat K$ and $\hat K'$ are bounded Borel set-valued functions by Corollary 3.3 of [6]. They satisfy the assumptions of § 2.

Problem $\hat R$ associated to $\hat K$ and $\hat K'$ has a solution. Let H be the function defined in Theorem 3.2 associated with $\hat R$.

Then, as in [1, Chap. IV-5], it is possible to find Borel selections c of K and c' of K' such that (3.10) and (3.11) hold ν-a.e., because $K(t, x)$ and $\hat{K}(t, x)$ (resp. $K'(t, x)$ and $\hat{K}'(t, x)$) have the same extremal points.

By Theorem 3.2, (c, c') is a solution to problem $\hat{R}$. It is then seen immediately that it is also a solution to problem R. □

5. Extensions. By using the methods of [1], the previous results can be extended to criteria of the type

$$E^{b+b'}_{(s,x)} \int_s^{T_A} \exp -\left\{\int_s^t (m+m')(\sigma, x_\sigma)\, d\sigma\right\}(L+L')(t, x_t)\, dt, \tag{5.1}$$

where:

we ask (b, L, m) and (b', L', m') to be Borel selections of K and K' which are bounded Borel compact-valued functions from $R^+ \times R^d$ in $R^d \times R \times R^+$.

we can find $p > 0$ such that if $(b, L, m) \in K(t, x)$, $m > p$.

The results can be also extended to diffusions with boundary conditions [8] with the same methods.

Finally, in the time-homogeneous case, the solutions can also be taken to be time-homogeneous.

Another point of interest is to know if the previous methods apply to the more general systems (1.1)–(1.3).

DEFINITION 5.1. If V is a bounded finely continuous Borel function on $R^+ \times R^d$, and if h and H are Borel functions on $R^+ \times R^d$, we say that

$$\begin{aligned} \mathscr{L}^0 V &= h(t, x), \\ \partial V &= H \end{aligned} \tag{5.2}$$

if:

(a) $V(s, x) = 0$ if (s, x) is regular for A (i.e., $Q^0_{(s,x)}(T_A = 0) = 1$).

(b) For any $(s, x) \in R^+ \times R^d$,

$$l_{t<T_A} V(t, x_t) - l_{s<T_A} V(s, x_s) - \int_{s\wedge T_A}^{t\wedge T_A} (h + pV)(u, x_u)\, du \tag{5.3}$$

is a local martingale for $Q^0_{(s,x)}$.

(c) There is a predictable additive functional A such that

$$V(t, x_t) - V(s, x_s) = \int_s^t (h + pV)(u, x_u)\, du + \int_s^t dA_u + \int_s^t H(u, x_u) \cdot d\beta^0_u. \tag{5.4}$$

Let φ, ψ, h be defined by

$$\begin{aligned} \varphi(t, x, v) &= -\inf_{u\in U} \{L(t, x, u) + \langle v, b(t, x, u)\rangle\}, \\ \psi(t, x, v) &= \sup_{u'\in U'} \{L'(t, x, u') + \langle v, b'(t, x, u')\rangle\}, \\ h(t, x, v) &= \varphi(t, x, v) - \psi(t, x, v). \end{aligned} \tag{5.5}$$

Then Theorem 3.2 implies that for a bounded Borel finely continuous function V to be the cost function associated to the problem (1.6), it is necessary and sufficient that

$$\mathscr{L}^0 V = h(t, x, \sigma^{-1}\,\partial V(t, x)). \tag{5.6}$$

Equation (5.6) is then obviously the weak extension of (1.5). For each $(t, x) \in R^+ \times R^d$, $h(t, x, v)$ is the difference of two bounded uniformly Lipschitz convex functions. As a function of v, $h(t, x, \cdot)$ has a very general form, because the difference of Lipschitz convex functions is dense in the space of continuous functions for the uniform convergence on compact sets of R^d.

Let us assume that in (1.4), $h = -X_1 = -X_2$. Then if h can be written as in (5.5), equation (5.6) has one unique bounded Borel finely continuous solution, by the results of §§ 1–4.

By proceeding as Friedman in [5], we know then that for each (t, x) in $R^+ \times R^d$

$$(u, u') \to L(t, x, u, u') + \langle \partial V(t, x), \sigma^{-1}(t, x) b(t, x, u, u')\rangle$$

has a saddle point.

By using a measurable selection theorem, it is then possible to derive an existence result for the nonseparable case.

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