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Martingales, the Malliavin Calculus and Hypoellipticity Under General Hörmander's Conditions

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In a remarkable series of papers [10, 11], Malliavin has developped probabilistic techniques to prove hypoellipticity for certain second order differential operators written in Hörmander's form

$$\mathscr{L} = X_0 + \tfrac{1}{2} \sum_{i=1}^{m} X_i^2 \tag{0.1}$$

where $X_0, X_1, \ldots, X_m$ are a family of smooth vector fields on R^d. These techniques have been exploited by Stroock in a paper which also includes various applications [12].

The reasoning in [10] proceeds in the following way. First, it is possible to associate to the operator $\mathscr{L}$ the diffusion given by the stochastic differential equation

$$\begin{aligned} dx &= X_0(x)\,dt + X_i(x) \cdot dw^i \\ x(0) &= x \end{aligned} \tag{0.2}$$

where $w = (w^1 \ldots w^m)$ is a Brownian motion. The measure Q on $\mathscr{C}(R^+; R^d)$ associated to Eq. (0.2) is then the image measure of the brownian measure P on $\mathscr{C}(R^+; R^m)$ by the mapping

$$\omega = (w_{\bullet}) \to x_{\bullet}(\omega).$$

At time t, the fundamental solution $P_t(x, .)$ associated with the operator $\mathscr{L}$ is then the probability law of $x_t(\omega)$.

The key idea in [10] is then to exploit the fact that $P_t(x, .)$ is the image of the gaussian measure P on $\mathscr{C}(R^+; R^m)$ by the mapping $\omega \to x_t(\omega)$. In fact, to prove that $P_t(x, .)$ is given by a smooth density, it suffices to prove that all the differentials of $P_t(x, .)$ in distribution sense are bounded measures, or equivalently to show that there exists constants C_m such that for any function f in $\mathscr{D}$ (which is the set of C^∞ functions with compact support) then:

$$\left| E\left[\frac{\partial^m f}{\partial x^m}(x_t)\right]\right| \leqq C_m \sup_{y \in R^d} |f(y)|. \tag{0.3}$$

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To prove (0.3), an integration by parts argument is needed. In [10, 12], this uses essentially a generalized Ornstein-Uhlenbeck operator $\mathscr{A}$ which is an unbounded self-adjoint non negative operator on the Hilbert space $L_2(\Omega; P)$ and the associated Ornstein-Uhlenbeck process $s \to \omega_s \in \mathscr{C}(R^+; R^m)$ whose generator is precisely $\mathscr{A}$, and for which the brownian measure P on the state space $\mathscr{C}(R^+; R^m)$ is an invariant measure. It is then necessary to prove that all the functions $\omega \to x_t(\omega)$ are in the domain of the operator $\mathscr{A}$, and to compute the Meyer decomposition of the process $s \to x_t(\omega_s)$ (at least in a generalized sense). This in turn permits us to derive the needed integration by parts formula.

In this paper we will try to obtain this formula directly using a simpler argument. The two key points are the following:

a) If x is given by (0.2), x is a very regular function of w, in the sense that if $u \in L_1(R^+; R^m)$ and if x^u is the solution of the stochastic differential equation

$$\begin{aligned} & d x^u = X_0(x^u)\, dt + X_i(x^u) \cdot (d w^i + u^i\, dt) \\ & x^u(0) = x, \end{aligned} \tag{0.4}$$

it is possible to define a regular version of x^u in such a way that, for a.e. w, $u \to x^u(w) \in \mathscr{C}(R^+; R^d)$ is smooth. This uses essentially the theory of stochastic flows developped by Malliavin [10], Baxendale [1], Elworthy [8] and ourselves [2–4].

b) The Brownian measure P possesses a strong quasi-invariance property in the sense that if $u = (u^1, \ldots, u^m)$ is a bounded adapted process defined on $\mathscr{C}(R^+; R^m)$, then the probability law P^u of $w_t + \int_0^t u\, ds$ is equivalent to P on any σ-field $\mathscr{B}(w_s | s \leqq t)$ and the density is explicity given by the Cameron-Martin-Maruyama-Girsanov formula [13].

If g is a bounded differentiable function on the space $\mathscr{C}([0, T]; R^d)$, the combination of a) and b) permits us to establish directly a formula of integration by parts relative to $g(x_{\cdot}(\omega))$ where $x_{\cdot}(\omega)$ is given by (0.2). This formula is in fact equivalent to the martingale representation result of Haussmann [9–19] for general diffusions, which extends the result of Clark [6] for the Brownian motion. Section 2 is in fact devoted to the proof of the result of Haussmann [9–19] with a method of calculus of variations connected with [19].

One of the main interests of the paper is to show that the infinite dimensional aspect of the Malliavin calculus may be easily taken care of by the Cameron-Martin-Maruyama-Girsanov transformation, while all the differential analysis is in fact strictly finite dimensional.

Using these techniques, we are able to derive the existence of a density for the resolvent operators of (0.1) under the general conditions of Hörmander [15]. This requires the lifting techniques of Rothschild and Stein [17] (see also Hörmander and Melin [16]) and change of time. In the special case where the distribution generated by $X_1, \ldots, X_m, [X_i, X_j]_{0 \leqq i, j \leqq m} \ldots$ (i.e. all the brackets of $X_0, X_1, \ldots, X_m$ except X_0 itself) generates a foliation of R^d by immersed submanifolds, the diffusion is shown to be factored in the "product" of a deterministic motion along X_0 and transversal to the leaves of the foliation, and of a diffusion with a regular semi-group in the leaves. These results are closely related to the results of Ichihara and Kunita [20] who classified some hypoelliptic diffusions using Hörmander's theorem [15].

The paper is divided into five sections. In Sect. 1, a few results on stochastic differential equations and their flows are given. In Sect. 2, an integration by parts formula is established which leads to the explicit martingale representation of Haussmann [9–19] of random variables of the type $g(x_{\cdot}(\omega))$ where g is a differentiable function on $\mathscr{C}([0, T]; R^d)$ and $x_{\cdot}(\omega)$ is given by (0.2). This result is the key to Sect. 3, where the Malliavin formula of integration by parts is established. For the sake of completeness, Sect. 4 rederives the existence of a density of transition for the process given by (0.2) when $\mathscr{L}$ verifies restricted Hörmander's conditions, just as in Malliavin [10–11]. The integration by parts formula is given in a completely explicit form which exhibits Lie brackets of vector fields constructed via the flow associated to the stochastic differential equation.

Finally, Sect. 5 is devoted to a probabilistic proof of the existence of a density for the resolvents of (0.1) under general Hörmander's conditions.

1. A Few Results on Stochastic Flows

Ω denotes the space $\mathscr{C}(R^+; R^m)$ of continuous functions defined on R^+ with values in R^m. The trajectory of a point ω is written $w_t = (w_t^1, \ldots, w_t^m)$. The σ-field F_t is defined by $F_t = \mathscr{B}(w_s; s \leqq t)$. P is the brownian measure on Ω with the condition $P(w_0 = 0) = 1$. $\{F_t^+\}_{t \geqq 0}$ is the filtration obtained by completing $\{F_t\}_{t \geqq 0}$ with the negligible sets in F_∞ and by taking the right regularization of this new filtration [7].

$X_0, X_1, \ldots, X_m$ is a family of $m+1$ vector fields defined on R^d with values in R^d, which are C^∞, bounded with bounded differentials of all orders.

We consider the stochastic differential equation

$$\begin{aligned} dx &= X_0(x)\,dt + X_i(x) \cdot dw^i \\ x(0) &= x \end{aligned} \tag{1.1}$$

where dw^i denotes the Stratonovitch differential of w^i, or the equivalent equation

$$\begin{aligned} dx &= \left(X_0 + \frac{1}{2}\frac{\partial X_i}{\partial x} X_i\right)(x)\,dt + X_i(x) \cdot \delta w^i \\ x(0) &= x \end{aligned} \tag{1.2}$$

where δw^i is the Ito differential of w^i.

For every $x \in R^d$, (1.1) has an essentially unique a.s. continuous solution. Moreover, using an argument of Kolmogorov, it is possible to prove elementarily that it is possible to define a mapping $(\omega, t, x) \to \varphi_t(\omega, x)$ such that:

a) $\varphi_t(\omega, x)$ is measurable in the variable ω, and continuous in the variables (t, x);

b) For any $x \in R^d$, $t \to \varphi_t(\omega, x)$ is the essentially unique solution of (1.1).

Moreover the following is proved in [2] (see also Malliavin [10], Elworthy [8], Baxendale [1]):

Theorem 1.1. *A.s., for every $t \geqq 0$, $\varphi_t(\omega, x)$ is a C^∞ diffeomorphism of R^d onto R^d, and the differentials $\frac{\partial^m \varphi}{\partial x^m} t(\omega, x)$ are continuous on $R^+ \times R^d$. For any $x \in R^d$, $Z_t = \frac{\partial \varphi}{\partial x} t(\omega, x)$ and $Z'_t = \left[\frac{\partial \varphi}{\partial x} t(\omega, x)\right]^{-1}$ are the unique solutions of the stochastic differential equations*

$$\begin{aligned} &dZ = \frac{\partial X_0}{\partial x}(x_t) Z\, dt + \frac{\partial X_i}{\partial x}(x_t) Z \cdot dw^i \\ &Z(0) = I \\ &dZ' = -Z' \frac{\partial X_0}{\partial x}(x_t)\, dt - Z' \frac{\partial X_i}{\partial x}(x_t) \cdot dw^i \\ &Z'(0) = I. \end{aligned} \tag{1.3}$$

For any $T > 0$, $R > 0$, $p (1 \leqq p < +\infty)$, $m (0 \leqq |m| < +\infty)$ the random variables

$$\sup_{\substack{0 \leqq t \leqq T \\ |x| \leqq R}} \left| \frac{\partial^m \varphi}{\partial x^m} t(\omega, x) \right|, \qquad \sup_{\substack{0 \leqq t \leqq T \\ |x| \leqq R}} \left| \left(\frac{\partial \varphi}{\partial x} t(\omega, x) \right)^{-1} \right| \tag{1.4}$$

are in L_p.

If L is an adapted locally integrable process with values in R^d, if $z_0 \in R^d$ and z_t is the continuous adapted process defined by

$$z_t = z_0 + \int_0^t L\, ds$$

then $\varphi_t(\omega, z_t)$ is a continuous semi-martingale which may be written:

$$\begin{aligned} \varphi_t(\omega, z_t) = z_0 &+ \int_0^t X_0(\varphi_s(\omega, z_s))\, ds + \int_0^t X_i(\varphi_s(\omega, z_s)) \cdot dw_s^i \\ &+ \int_0^t \frac{\partial \varphi}{\partial x} s(\omega, z_s) \cdot dz_s, \end{aligned} \tag{1.5}$$

or equivalently:

$$\begin{aligned} \varphi_t(\omega, z_t) = z_0 &+ \int_0^t \left(X_0 + \frac{1}{2} \frac{\partial X_i}{\partial x} X_i \right) (\varphi_s(\omega, z_s))\, ds \\ &+ \int_0^t X_i(\varphi_s(\omega, z_s)) \cdot \delta w_s^i + \int_0^t \frac{\partial \varphi}{\partial x} s(\omega, z_s) \cdot dz_s. \end{aligned} \tag{1.6}$$

Proof. These results are contained in Theorems I.1.2, I.2.1 and I.5.1 of [2]. A proof of (1.5) is also given in [4] (see [3]). □

We now give a result concerning the integration of a class of stochastic differential equations obtained from (1.1) by a perturbation of the drift.

Let $u=(u^1,\dots,u^m)$ be a bounded adapted process defined on $\Omega\times R^+$ with values in R^m. We then consider the stochastic differential equation:

$$\begin{aligned} dx &= X_0(x)\,dt + X_i(x)\cdot[dw^i + u^i\,dt] \\ x(0) &= x \end{aligned} \tag{1.7}$$

which may be written equivalently as:

$$\begin{aligned} dx &= \left[X_0(x) + \frac{1}{2}\frac{\partial X_i}{\partial x} X_i(x) + X_i(x)\,u^i\right] dt + X_i(x)\cdot\delta w^i \\ x(0) &= x. \end{aligned} \tag{1.8}$$

(1.7) has a essentially unique solution x^u. We now have the following:

Theorem 1.2. *The differential equation:*

$$\begin{aligned} dz^u &= \left[\frac{\partial\varphi}{\partial x} t(\omega, z^u)\right]^{-1} [X_i(\varphi_t(\omega, z^u))\,u^i]\,dt \\ z^u(0) &= x \end{aligned} \tag{1.9}$$

has one unique solution defined on a stochastic interval $[0, T^u[$ *where* T^u *is a stopping time which is equal to* $+\infty$, *except on a negligible set (which may depend on u). The process* $\varphi_t(\omega, z_t^u)$ *is the essentially unique solution* x^u *of* (1.7).

Proof. The coefficients of (1.9) are C^∞ in the variable z^u. By standard results on differential equations, (1.9) is easily proved to have a unique solution on a stochastic interval $[0, T^u[$, where T^u is a stopping time such that on $(T^u < +\infty)$, $\lim_{t\to T^u} \|z_t^u\| = +\infty$. Let T_n^u be a sequence of stopping times increasing to T^u such that $(T_n^u < T^u)$ a.s. Applying (1.5), we see that on $[0, T_n^u]$:

$$\begin{aligned} \varphi_t(\omega, z_t^u) = x &+ \int_0^t X_0(\varphi_s(\omega, z_s^u))\,ds + \int_0^t X_i(\varphi_s(\omega, z_s^u))\cdot dw^i \\ &+ \int_0^t X_i(\varphi_s(\omega, z_s^u))\,u^i\,ds. \end{aligned} \tag{1.10}$$

$\varphi_t(\omega, z_t^u)$ is then equal on $[0, T_n^u]$ to the essentially unique solution x^u of (1.7), i.e. $z_t^u = \varphi_t^{-1}(\omega, x_t^u)$ on $[0, T^u[$. Since $\varphi_t^{-1}(\omega, .)$ is also a continuous family of diffeomorphisms of R^d (for the topology of C^∞ compact convergence), it follows that a.s., $\varphi_t^{-1}(\omega, x_t^u)$ is a continuous process, and consequently that $T^u = +\infty$ a.s. $\square$

Remark 1. For an extension of this result, see [2–4]. The difficulty here is that there is no honest majoration of $\left[\frac{\partial\varphi}{\partial x} t(\omega, z)\right]^{-1}$ by a linear function of z which would permit a global integration of (1.9), without considering a stopping time T^u.

To overcome this slight difficulty, we will have to make a somewhat restrictive assumption which will be dropped.

Theorem 1.3. *If the vector fields $X_1, \dots, X_m$ have compact support, there exists a negligible set $\mathcal{N}$ in Ω, such that for any $\omega \notin \mathcal{N}$, $x \in R^d$, if $t \to v_t$ is a Borel function defined on $[0, T]$ with values in R^m which belongs to $L_1([0, T], dt)$, then the differential equation*[1]

$$\begin{aligned} dy &= (\varphi_t^{*-1} X_i)(y_t)\, v_t^i\, dt \\ y(0) &= x \end{aligned} \tag{1.11}$$

has a unique solution $y_t^{(v)}(\omega)$; the mapping

$$v \in L_1([0, T], dt) \to y^{(v)}_{\bullet}(\omega) \in \mathscr{C}([0, T]; R^d) \tag{1.12}$$

is infinitely differentiable, and its differential at $v = 0$ is given by the linear mapping

$$v \in L_1([0, T], dt) \to \int_0^t (\varphi_s^{*-1} X_i)(x)\, v_s^i\, ds. \tag{1.13}$$

Proof. Let us first prove that, for a.e. ω, for any $T > 0$, $\left[\frac{\partial \varphi}{\partial x} t(\omega, x)\right]^{-1}$ is uniformly bounded on $[0, T] \times R^d$. Let $k = \sup_{x \in R^d} \|X_0(x)\|$. Then, it is clear that if R is taken large enough so that the ball of center 0 and radius R contains the supports of $X_1, \dots, X_m$, if $x \in R^d$ is such that $\|x\| \geqq R + kT$, for $t \leqq T$, the unique solution of (1.1) is given by the ordinary differential equation:

$$\begin{aligned} dx &= X_0(x)\, dt \\ x(0) &= x. \end{aligned} \tag{1.14}$$

From (1.3), we see that, for $x \in R^d$, $\|x\| \geqq R + kT$, $t \leqq T$, $\left[\frac{\partial \varphi}{\partial x} t(\omega, x)\right]^{-1}$ is given by the ordinary differential equation

$$\begin{aligned} dZ' &= -Z' \frac{\partial X_0}{\partial x}(x_t)\, dt \\ Z'(0) &= I. \end{aligned} \tag{1.15}$$

Since $\frac{\partial X_0}{\partial x}$ is uniformly bounded, $\left[\frac{\partial \varphi}{\partial x} t(\omega, x)\right]^{-1}$ is uniformly bounded on $\{(t, x); t \leqq T, \|x\| \geqq kT + R\}$. By a continuity argument, $\left[\frac{\partial \varphi}{\partial x} t(\omega, x)\right]^{-1}$ is uniformly bounded on $[0, T] \times R^d$. The various functions in (1.11) satisfy the usual sufficient conditions for the global existence and uniqueness of the solution of (1.11) on $[0, T]$. (1.12) is then an easy consequence of the implicit function theorem. In fact, let F be the function defined on $\mathscr{C}([0, T]; R^d) \times L_1([0, T], dt) \to \mathscr{C}([0, T]; R^d)$:

$$(y, v) \to y_t - \int_0^t (\varphi_s^{*-1} X_i)(y_s)\, v^i\, ds. \tag{1.16}$$

[1] If Y is a vector field, $(\varphi_t^{*-1} Y)(x)$ is the vector field $\left[\frac{\partial \varphi}{\partial x} t(\omega, x)\right]^{-1} Y(\varphi_t(\omega, x))$

Then, its partial differential F_y at $(y, 0)$ is the identity and the partial differential F_v at $(y, 0)$ is given by:

$$v' \in L_1([0, T], dt) \to -\int_0^t (\varphi_s^{*-1} X_i)(y_s) v'^i ds. \tag{1.17}$$

(1.13) follows then from a differentiability result associated to the implicit function theorem. □

Corollary. *Under the assumptions of Theorem 1.3, if u is a bounded adapted process defined on $\Omega \times R^+$ with values in R^m, then $\varphi_t(\omega, y_t^{u(\omega)})$ is the essentially unique solution of the stochastic differential equation:*

$$\begin{aligned} dx &= (X_0(x) + X_i(x) u^i) dt + X_i(x) \cdot dw^i \\ x(0) &= x. \end{aligned} \tag{1.18}$$

Proof. This is obvious from Theorems 1.2 and 1.3. □

Remark 2. It must be noted that this corollary gives a stronger result than Theorem 1.2 since the negligible set which is eliminated in Theorem 1.3 does not depend on u.

2. Integration by Parts and Martingale Representation

This section is essentially devoted to the proof of an integration by parts formula, which is closely related to the martingale representation of certain random variables of Haussmann [9–19].

The assumptions are the same as in Sect. 1.

T is a positive real number.

g is a bounded function defined on $\mathscr{C}([0, T]; R^d)$ with values in R which is continuous and (strongly) differentiable. For every $y \in \mathscr{C}([0, T]; R^d)$ the differential $dg(y)$ of g at y is defined by a finite measure $d\mu^y(t)$ on $[0, T]$ with values in R^d by

$$z \in \mathscr{C}([0, T]; R^d) \to \langle dg(y), z \rangle = \int_0^T \langle z_t, d\mu^y(t) \rangle. \tag{2.1}$$

From the point of view of differential geometry, we may identify $d\mu^y(t)$ to a generalized element of $T_{y_t}^*(R^d)$.

We finally assume that g is uniformly Lipschitz, or equivalently that the differentials $dg(y)$ are uniformly bounded.

We have now the following key result, which is a consequence of a result of Haussmann [9–19] which we rederive using ideas of the calculus of variations. The proof given here is related to the proof given in [19]. For another proof, see Davis [21].

Theorem 2.1. *If $u = (u^1, \ldots, u^m)$ is a bounded adapted process defined on $\Omega \times R^+$ with values in R^m, then for any x in R^d*

$$E\left[g(\varphi_{\cdot}(\omega,x))\int_0^T u^i\cdot\delta w^i\right]$$

$$=E\left[\int_0^T u^i\,ds\langle X_i(\varphi_s(\omega,x)),\int_{[s,T]}\varphi_s^*(\omega,x)\,\varphi_v^{*-1}(\omega,x)\,d\mu^{\varphi_{\cdot}(\omega,x)}(v)\rangle\right]. \quad (2.2)$$

Proof. Let us first note that the R.H.S. of (2.2) is well defined. In fact, $\frac{\partial\varphi}{\partial x}t(\omega,x)$ and $\left[\frac{\partial\varphi}{\partial x}t(\omega,x)\right]^{-1}$ are given by Z and Z' in (1.3) and we know by Theorem 1.1 that for any $p(1\leqq p<+\infty)$ $\sup_{0\leqq t\leqq T}|Z_t|$ and $\sup_{0\leqq t\leqq T}|Z'_t|$ are in L_p. It must also be noted that formula (2.2) is geometrically invariant.

We may also assume that u is a step predictable process, since (2.2) is easily extended to any u by density. We then suppose that there is $n\in N$ such that on any interval $\left[k/2^n,\frac{k+1}{2^n}\right[$, u is constant, bounded and $F_{k/2^n}$ measurable. We may even assume that on $\left[k/2^n,\frac{k+1}{2^n}\right[$, has the form

$$u_t=u^k[w_{t_1},\ldots,w_{t_{l_k}}], \qquad 0\leqq t_1<t_2<\ldots<t_{l_k}\leqq k/2^n$$

where u^k is a bounded C^∞ function with bounded differentials of all orders since such random variables are dense in $L_2^{F_{k/2^n}}$.

Let η be an element of $\mathscr{C}(R^+;R^m)$ which is null at 0. For $l\in R$, let us consider the functional equation

$$\rho_t=\eta_t-\int_0^t l\,u_s(\rho)\,ds. \quad (2.3)$$

Note that Eq. (2.3) is not considered as a stochastic differential equation, but must be solved for each η. Now (2.3) is a trivial difference equation, which may be solved recursively on each dyadic interval $\left[k/2^n,\frac{k+1}{2^n}\right[$, and its unique solution is written $\rho^l_{\cdot}(\eta)$. Obviously, for each η, $\rho^l_{\cdot}(\eta)$ is a differentiable function of l.

Let v^l_t be the adapted continuous process

$$v^l_t=w_t+\int_0^t l\,u_s(w)\,ds. \quad (2.4)$$

From what has been said previously, it follows that

$$w_{\cdot}=\rho^l_{\cdot}(v^l_{\cdot}). \quad (2.5)$$

In (2.4), $u_s(w)$ is then adapted to v^l_t, since we have

$$u_s(w)=u_s(\rho^l_{\cdot}(v^l_{\cdot})). \quad (2.6)$$

Now, by the Cameron-Martin-Maruyama-Girsanov formula [13] we know that the probability law Q^l of v^l_t on $\mathscr{C}(R^+;R^m)$ is equivalent to the brownian measure on the σ-field F_T. More precisely, if w denotes the generic trajectory in $\mathscr{C}(R^+;R^m)$, we have

$$\frac{dQ^l}{dP} F_T = \exp\left\{\int_0^T \langle l u_s(\rho_\cdot^l(w)), \delta w\rangle - \tfrac{1}{2}\int_0^T |l u_s(\rho_\cdot^l(w))|^2\, ds\right\}. \tag{2.7}$$

Let Z_T^{lu} be the density (2.7).

Let us then consider the stochastic differential equation:

$$\begin{aligned} &dx = (X_0(x) + X_i(x)\, l u^i(w))\, dt + X_i(x)\cdot dw^i \\ &x(0) = x \end{aligned} \tag{2.8}$$

which may also be written

$$\begin{aligned} &dx = X_0(x)\, dt + X_i(x)\cdot(dw^i + l u^i(w)\, dt) \\ &x(0) = x. \end{aligned} \tag{2.9}$$

From Theorem 1.2, we know that the essentially unique solution x^{lu} of (2.8) is given by

$$x_t^{lu} = \varphi_t(\omega, z_t^{lu}) \tag{2.10}$$

where z^{lu} is the solution of the differential equation

$$\begin{aligned} &dz^{lu} = (\varphi_t^{*-1} X_i)(z_t^{lu})\, l u^i(w)\, dt \\ &z^{lu}(0) = x. \end{aligned} \tag{2.11}$$

Using (2.7) and (2.9) it follows that

$$E^P[g(\varphi_\cdot(\omega, z_\cdot^{lu}(\omega)))] = E^P[g(\varphi_\cdot(\omega, x))\, Z_T^{lu}(\omega)]. \tag{2.12}$$

We will now take the differential of both sides of (2.12) at $l=0$. We will first assume that $X_1, \ldots, X_m$ have compact support. By Theorem 1.3, we know that $l \to z_\cdot^{lu}$ is differentiable for a.e. ω, and that the differential at $l=0$ is given by the continuous function

$$\int_0^t (\varphi_s^{*-1} X_i)(x)\, u^i\, ds. \tag{2.13}$$

The differential of $l \to g(\varphi_\cdot(\omega, z^{lu}))$ at $l=0$ is then given for a.e. ω by

$$\int_0^T \langle \varphi_t^*(\omega, x) \int_0^t (\varphi_s^{*-1} X_i)(x)\, u^i\, ds, d\mu^{\varphi_\cdot(\omega,x)}(t)\rangle. \tag{2.14}$$

We now must prove that the differential of the L.H.S. of (2.12) is precisely the expected value of (2.14). Using a uniform integrability argument, it suffices to prove that for $l \neq 0$:

$$\frac{1}{l^2} E\, |g(\varphi_\cdot(\omega, z^{lu})) - g(\varphi_\cdot(\omega, x))|^2 \tag{2.15}$$

is uniformly bounded. Since g is Lipschitz, (2.15) may be bounded by:

$$\frac{C}{l^2} E[\|\varphi_{\cdot}(\omega, z_{\cdot}^{lu}) - \varphi_{\cdot}(\omega, x)\|^2_{\mathscr{C}([0,T];R^d)}]. \tag{2.16}$$

Now from (2.9)–(2.10), we know that $\varphi_{\cdot}(\omega, z_{\cdot}^{lu}) = x_{\cdot}^{lu}$ is the unique solution of the stochastic differential equation

$$\begin{aligned} dx^{lu} &= \left(X_0 + \frac{1}{2}\frac{\partial X_i}{\partial x} X_i\right)(x^{lu})\,dt + X_i(x^{lu})\,\delta w^i + X_i(x^{lu})\, l u^i\, dt \\ x^{lu}(0) &= x. \end{aligned} \tag{2.17}$$

Knowing that $\varphi_{\cdot}(\omega, x) = x_{\cdot}^0$ is the solution of (1.2), using the uniform boundedness of $X_0, X_1, \ldots, X_m$ and their differentials, we have:

$$E|x_t^{lu} - x_t^0|^2 \leqq C\left\{\int_0^t E|x_s^{lu} - x_s^0|^2\, ds + l^2 t\right\}, \qquad t \leqq T. \tag{2.18}$$

From Gronwall's lemma, we find:

$$E|x_t^{lu} - x_t^0|^2 \leqq l^2(e^{Ct} - 1), \qquad t \leqq T. \tag{2.19}$$

Using Doob and Burkholder-Davis-Gundy inequalities [7], we get:

$$\begin{aligned} E[\|x^{lu} - x^0\|^2_{\mathscr{C}([0,T];R^d)}] &\leqq C'\left\{\int_0^T E|x_s^{lu} - x_s^0|^2\, ds + l^2 T\right\} \\ &\leqq C'' l^2. \end{aligned} \tag{2.20}$$

The boundedness of (2.15) is proved.

It follows immediately that the differential at $l=0$ of the L.H.S. of (2.12) is given by:

$$E\int_0^T \left\langle \varphi_t^*(\omega, x)\int_0^t \varphi_s^{*-1}(\omega, x)\, X_i(\varphi_s(\omega, x))\, u^i\, ds,\, d\mu^{\varphi_{\cdot}(\omega, x)}(t)\right\rangle. \tag{2.21}$$

We now differentiate the R.H.S. of (2.12).

First note that in (2.3), the differential $K^l(\eta)$ of $\rho^l(\eta)$ at l is such that:

$$K_t^l(\eta) = -\int_0^t u_s(\rho^l(\eta))\, ds - \int_0^t l \frac{d}{dl}[u_s(\rho^l(\eta))]\, ds \tag{2.22}$$

and of course $\frac{d}{dl}[u_s(\rho^l(\eta))]$ is given by:

$$\frac{d}{dl}[u_s(\rho^l(\eta))] = \langle du_s(\rho^l(\eta)), K^l(\eta)\rangle. \tag{2.23}$$

Now (2.22)–(2.23) is still a recursive equation. Since the various functions u^k have bounded differentials of all orders, it follows that if $|l| \leqq L$, the differential $K^l(\eta)$ is uniformly bounded in $\mathscr{C}([0,T];R^m)$ independently of η. It follows that for $|l|\leqq L$, the function

$$s \to \frac{d}{dl} u_s(\rho^l(\eta)) \tag{2.24}$$

is uniformly bounded by a constant independent of η.

Now since u_s is a step function, which is constant on the dyadic intervals $\left[k/2^n, \frac{k+1}{2^n}\right[$

$$\int_0^T \langle l u_s(\rho^l(w)), \delta w\rangle = \Sigma\, l \langle u_{k/2^n}(\rho^l(w)), w_{T\wedge\frac{k+1}{2^n}} - w_{T\wedge k/2^n}\rangle. \tag{2.25}$$

From (2.25), it follows clearly that, a.e. Z_T^{lu} is a differentiable function of l. Moreover since the differential of $l \to l u_s(\rho^l(\eta))$ at $l=0$ is clearly given by

$$u_s(\eta) \tag{2.26}$$

because in particular $\rho^0(\eta)=\eta$, we see that

$$\frac{dZ_T^{lu}}{dl}_{l=0} = \int_0^T \langle u_s(w), \delta w\rangle. \tag{2.27}$$

The differential at $l=0$ of $l \to g(\varphi_{\cdot}(\omega, x))Z_T^{lu}$ is then equal to

$$g(\varphi_{\cdot}(\omega, .))\int_0^T \langle u_s(w), \delta w\rangle. \tag{2.28}$$

From (2.25), we see that

$$\begin{aligned}\frac{d}{dl}\int_0^T \langle l u_s(\rho^l(w)), \delta w\rangle &= \Sigma \langle u_{k/2^n}(\rho^l(w)), w_{T\wedge\frac{k+1}{2^n}} - w_{T\wedge k/2^n}\rangle \\ &\quad + \Sigma\, l \left\langle \frac{d}{dl}(u_{k/2^n}(\rho^l(w)), w_{T\wedge\frac{k+1}{2^n}} - w_{T\wedge\frac{k+1}{2^n}}\right\rangle.\end{aligned} \tag{2.29}$$

It follows from what has been previously said that if $|l|\leqq L$, there is a constant C_L such that:

$$\left|\frac{d}{dl}\int_0^T \langle l u_s(\rho^l(w)), \delta w\rangle\right| \leqq C_L(\Sigma |w_{T\wedge\frac{k+1}{2^n}} - w_{T\wedge k/2^n}|). \tag{2.30}$$

For the same reason, for $l\leqq L$:

$$\left|\frac{d}{dl}\int_0^T |l u_s(\rho^l(w))|^2\, ds\right| \tag{2.31}$$

is uniformly bounded. Moreover, trivially, from (2.7) and (2.25), we see that for $|l|\leqq L$

$$Z_T^{lu} \leqq \exp\{D_L(\Sigma |w_{T\wedge\frac{k+1}{2^n}} - w_{T\wedge k/2^n}|)\}. \tag{2.32}$$

From (2.31)-(2.32), we see that for $|l|\leqq L$

$$\left|\frac{d}{dl} Z_T^{lu}\right| \leqq [C_L(\Sigma |w_{T\wedge\frac{k+1}{2^n}} - w_{T\wedge\frac{k+1}{2^n}}|) + C'_L] \exp\{D_L(\Sigma |w_{T\wedge\frac{k+1}{2^n}} - w_{T\wedge\frac{k+1}{2^n}}|)\}. \tag{2.33}$$

The R.H.S. of (2.33) is trivially integrable. From (2.27), it follows that the differential at $l=0$ of $l\to E(g(\varphi_{\bullet}(\omega,x))Z_T^{lu})$ is equal to:

$$E\left[g(\varphi_{\bullet}(\omega,.))\int_0^T u^i \cdot \delta w^i\right]. \tag{2.34}$$

Equating (2.21) and (2.34), we obtain formula (2.2), at least when $X_1,\dots,X_m$ have compact support. We now extend (2.2) to the general case. Let $\{r_n\}$ be a family of C^∞ functions defined on R^d with values in R such that:

a) For any n, $0\leqq r_n\leqq 1$, and the differentials of r_n are bounded uniformly in n;

b) r_n is equal to 1 on the ball of center 0 and radius n and is 0 out of the ball of center 0 and radius $n+1$.

Let $\varphi^n_{\bullet}(\omega,.)$ be the flows corresponding to the vector fields $(X_0, r_nX_1, r_nX_2,\dots,r_nX_m)$. It is trivially proved that for any x, a.e. $\varphi^n_{\bullet}(\omega,x)$, $\frac{\partial\varphi^n}{\partial x}\cdot(\omega,x)$, $\left[\frac{\partial\varphi^n}{\partial x}\cdot(\omega,x)\right]^{-1}$ converge uniformly to $\varphi_{\bullet}(\omega,x)$, $\frac{\partial\varphi}{\partial x}\cdot(\omega,x)$, $\left[\frac{\partial\varphi}{\partial x}\right]^{-1}\cdot(\omega,x)$ on $[0,T]$, and even coincide with these processes for n large enough.

Trivially

$$E\left[g(\varphi^n_{\bullet}(\omega,x))\left[\int_0^T u^i\cdot\delta w^i\right]\right]\to E\left[g(\varphi_{\bullet}(\omega,x))\left[\int_0^T u^i\cdot\delta w^i\right]\right]. \tag{2.35}$$

Moreover, noting that for n large enough, $\varphi^n_{\bullet}(\omega,x)$ and $\varphi_{\bullet}(\omega,x)$ coincide, it follows that:

$$\begin{aligned}&\int_0^T u^i\,ds\langle X_i(\varphi^n_s(\omega,x)), \int_{[s,T]} \varphi^{n*}_s(\omega,x)\,\varphi^{n*-1}_v(\omega,x)\,d\mu^{\varphi^n_{\bullet}(\omega,x)}(v)\rangle\\ &\qquad\to\int_0^T u^i\,ds\langle X_i(\varphi_s(\omega,x)), \int_{[s,T]} \varphi^*_s(\omega,x)\,\varphi^{*-1}_v(\omega,x)\,d\mu^{\varphi_{\bullet}(\omega,x)}(v)\rangle.\end{aligned} \tag{2.36}$$

To prove that the expectations also converge, we need only to prove that the random variables in (2.36) are uniformly bounded in L_2. From the uniform boundedness of dg, and a uniform bound for the L_4 norms of

$$\sup_{0\leqq t\leqq T}\left|\frac{\partial\varphi^n}{\partial x}t(\omega,x)\right| \qquad \sup_{0\leqq t\leqq T}\left|\left(\frac{\partial\varphi^n}{\partial x}\right)^{-1}t(\omega,x)\right| \tag{2.37}$$

which is trivial, we have proved the convergence of expectations in (2.36). From equality (2.2) already proved for $\varphi^n_{\bullet}(\omega,.)$, we obtain equality (2.2) for a general $\varphi_{\bullet}(\omega,.)$. □

Remark 1. The key point in the proof is formula (2.12). The proof that both sides are differentiable in l is merely technical, but cannot be avoided. In particular, it is absolutely necessary to choose first a process u which has a simple form, in order that the discussion starting in (2.3) proceeds smoothly. The denomination of "integrations by parts formula" for (2.2) comes from the analogy of (2.2) with the finite dimensional gaussian case.

Formula (2.2) is directly connected with the martingale representation of $g(\varphi_{\bullet}(\omega,x))$. We have in fact the fundamental result which is equivalent to Theorem 2.1 and which was first proved by Haussmann [9-19] extending Clark [6].

Theorem 2.2. *Under the assumptions of Theorem 2.1, let M_t be the continuous martingale*

$$M_t = E^{F_t^+}\, g(\varphi_{\bullet}(\omega,x)). \tag{2.38}$$

Then, if H_t is the predictable projection of the process

$$U_t = \int_{[t,T]} \varphi_t^*(\omega,x)\, \varphi_v^{*-1}(\omega,x)\, d\mu^{\varphi_{\cdot}(\omega,x)}(v). \tag{2.39}$$

M_t has the representation

$$M_t = \sum_{i=1}^{m} \int_0^t \langle H_s, X_i(\varphi_s(\omega,x))\rangle\, \delta w^i + E[g(\varphi_{\bullet}(\omega,x))]. \tag{2.40}$$

In particular,

$$g(\varphi_{\bullet}(\omega,x)) = \sum_{i=1}^{m} \int_0^T \langle H_s,\ X_i(\varphi_s(\omega,x))\rangle\, \delta w^i + E[g(\varphi_{\bullet}(\omega,x))]. \tag{2.41}$$

Proof. From a fundamental result of Ito [13], we know that any square integrable martingale for the canonical filtration of the brownian motion w may be written as

$$\int_0^t H_i\, \delta w^i + a. \tag{2.42}$$

Identifying (2.42) to M_t and applying the usual rules of Ito calculus, we find that for any adapted bounded process u with values in R^m

$$E\left[g(\varphi_{\bullet}(\omega,x)) \int_0^T u^i \cdot \delta w^i\right] = E\int_0^T H_i u^i\, dt. \tag{2.43}$$

Identifying the R.H.S. of (2.43) and the R.H.S. of (2.2), (2.41) follows. □

Remark 2. It must be noted that since U is left-continuous with right hand limits, by Theorems VI-48 49 in [7], H is also left-continuous with right hand limits. Moreover by [2-4], the flow $\varphi_{\bullet}(\omega,.)$ possesses a strong Markov multiplicative property, in the sense that if θ_t is the canonical translation operator on $\mathscr{C}(R^+; R^m)$ defined by: $\theta_t(w_s) = (w_{s+t} - w_s)$, if S is a stopping time relative to $\{F_t^+\}_{t\geq 0}$, then on $(S < +\infty)$ a.s. for any $(t,x) \in R^+ \times R^d$:

$$\varphi_{t+S}(\omega,x) = \varphi_t(\theta_S\,\omega, \varphi_S(\omega,x)).$$

If S is a constant time $\leqq T$

$$\int_{[S,T]} \varphi_S^*(\omega,x)\,\varphi_v^{*-1}(\omega,x)\,d\mu^{\varphi.(\omega,x)}(v)=\int_{[S,T]} \varphi_{v-S}^{*-1}(\theta_S\omega,\varphi_S(\omega,x))\,d\mu^{\varphi.(\omega,x)}(v). \tag{2.44}$$

Since $\varphi_{\bullet}(\omega,x)$ is given by $\varphi_t(\omega,x)$ for $t\leqq S$, and by $\varphi_{t-S}(\theta_S\omega,\varphi_S(\omega,x))$ for $t\geqq S$, noting that F_S and $\theta_{S^{-1}}F_\infty$ are independent, the conditional expectation of (2.44) with respect to F_S is then a function of $\{\varphi_t(\omega,x);\, t\leqq S\}$. Let then $\{G_t^+\}_{t\geqq 0}$ be the family of σ-fields generated by the process $\{\varphi_{\bullet}(\omega,x)\}$. By the previous argument, the predictable projections of U with respect to $\{F_t^+\}_{t\geqq 0}$ and to $\{G_t^+\}_{t\geqq 0}$ coincide at constant times. Since they are left-continuous, they coincide a.s. for every time.

This is in agreement with the general results concerning the representation of martingales for well-posed martingale problems of Dellacherie, Jacod and Yor [14]. In fact we also have:

$$M_t=E^{G_t^+}\,g(\varphi_{\bullet}(\omega,x)) \tag{2.45}$$

and (2.40) may be written

$$M_t=\int_0^t \left\langle H_s,\delta x_s-\left(X_0+\frac{1}{2}\frac{\partial X_i}{\partial x}X_i\right)(x_s)\,ds\right\rangle+E[g(x_{\bullet}(\omega))] \tag{2.46}$$

where $x_t(\omega)$ denotes the Markov process $\varphi_t(\omega,x)$. (2.44) involves only the process x itself and not the Brownian motion w which may contain more information than x.

Remark 3. A question of interest is to know if the process H_t may be defined by functions $H_t(x.)$, where H_t is smooth on $\mathscr{C}([0,T];R^d)$. By (2.44), this question may have an explicit answer.

3. The Malliavin Formula of Integration by Parts

We will now apply Theorem 2.1 to obtain the Malliavin formula of integration by parts [10–12].

Let f be a bounded C^∞ function defined on R^d with values in R, whose all differentials are uniformly bounded.

Let h be a continuous function defined on $\mathscr{C}([0,T];R^d)$ with values in the cotangent plane $T_x^*(R^d)$, which is bounded, (strongly) differentiable and uniformly Lipschitz, i.e. its differentials $dh(y)$ are uniformly bounded. If $y\in\mathscr{C}([0,T]);R^d)$, the differential $dh(y)$ may be identified to a finite measure $d\upsilon^y(t)$ on $[0,T]$ with values in $R^d\otimes R^d$.

$$z\in\mathscr{C}([0,T];R^d)\to dh(y)[z]=\int_0^T d\upsilon^y(t)(z_t).$$

From the point of view of differential geometry, $d\upsilon^y(t)$ may be identified to a generalized linear mapping from $T_{y_t}(R^d)$ in $T_x^*(R^d)$.

In particular, if $l \in T_x(R^d)$, we define the action of $\varphi_t^{*-1} d\upsilon^y(t)$ on l by:

$$\int_0^T [\varphi_t^{*-1} d\upsilon^y(t)](l) = \int_0^T d\upsilon^y(t)\, \varphi_t^*(l) \in T_x^*(R^d). \tag{3.1}$$

We then have the fundamental

Theorem 3.1. *The following equality holds*

$$\begin{aligned} &E\left(f(\varphi_T(\omega,x))\left\langle h(\varphi_{\bullet}(\omega,x)), \int_0^T (\varphi_s^{*-1} X_i)(x)\, \delta w^i\right\rangle\right) \\ &\quad = E\left\langle h(\varphi_{\bullet}(\omega,x)), \int_0^T (\varphi_s^{*-1} X_i)(x)\right\rangle \langle (\varphi_s^{*-1} X_i)(x)\, ds, (\varphi_T^{*-1} df)(x)\rangle \\ &\quad + E\left[f(\varphi_T(\omega,x)) \int_0^T \left\langle (\varphi_s^{*-1} X_i)(x), \int_{[s,T]} [\varphi_v^{*-1}(\omega,x)\, d\upsilon^{\varphi_{\cdot}(\omega,x)}(v)](\varphi_s^{*-1} X_i)(x)\right\rangle ds\right]. \end{aligned} \tag{3.2}$$

Proof. Let us first note that (3.2) is coordinate invariant. Let $(l_1, \ldots, l_d)$ be a basis in $T_x(R^d)$, $(l^1, \ldots, l^d)$ the dual basis in $T_x^*(R^d)$. We may write

$$\begin{aligned} h(\cdot) &= \sum_1^d h_k(\cdot)\, l^k \\ \varphi_s^{*-1} X_i(x) &= \sum_1^d (\varphi_s^{*-1} X_i)^k(x)\, l_k. \end{aligned} \tag{3.3}$$

A density argument proves that in (2.2) we may choose u adapted and such that

$$E \int_0^T |u|^2\, ds < +\infty. \tag{3.4}$$

For a given k, we take in (2.2)

$$g(y) = h_k(y) f(y_T), \qquad u_s^i = (\varphi_s^{*-1} X_i)^k(x). \tag{3.5}$$

Using (2.2) and summing in k, we get (3.2). □

Equality (3.2) is the basis of the Malliavin formula of integration by parts.

Moreover it must be noted that from Theorem 1.1 and especially from (1.3), we know that for any $x \in R^d$, $Z_t = \frac{\partial \varphi}{\partial x} t(\omega, x)$ and $Z_t' = \left[\frac{\partial \varphi}{\partial x} t(\omega, x)\right]^{-1}$ are the solutions of the stochastic differential Eqs. (1.3). We may add to Eqs. (1.1) and (1.3) the new stochastic differential equation:

$$\begin{aligned} dR &= Z_t' X_i(x_t) \cdot \delta w^i \\ R(0) &= 0. \end{aligned} \tag{3.6}$$

The left hand side of (3.2) may then be written

$$E[f(\varphi_T(\omega,x)) \langle h(\varphi_{\bullet}(\omega,x)), R_T \rangle]. \tag{3.7}$$

The function: $(x_., Z_., Z'_., R_.) \to f(x_T)\langle h(x_.), R_T\rangle$ is still a differentiable function on the set $\mathscr{C}([0,T]; R^d \times (R^d \otimes R^d)^2 \times R^d)$ of continuous functions $(x_., Z_., Z'_., R_.)$ defined on R_+ with values in $R^d \times (R^d \otimes R^d)^2 \times R^d$. This explains that formula (3.2) may be applied to this new function. Similarly, if function h has two differentials, the same can be said for the R.H.S. of (3.2). This possibility of iterating (3.2) will be the key of the estimates obtained in Sect. 4, where h itself will be a function of $(x_., Z_., Z'_.)$.

4. Application: Properties of the Semi-Groups under Restricted Hörmander's Conditions

For the sake of completeness, we now rederive the argument of Malliavin [10] to prove the existence of a density of transition under restricted Hörmander's conditions. We will also have something to say about hypoellipticity.

We make the following assumption:

H_1: For any $x \in R^d$, the space in $T_x(R^d)$ spanned by $X_1, X_2, \ldots, X_m$, $[X_i, X_j]_{0 \leq i,j \leq m}$, $[X_i, [X_j, X_k]]_{0 \leq i,j,k \leq m} \ldots$ is equal to $T_x(R^d)$.

Assumption H_1 is exactly assumption (E) in the work of Ichihara and Kunita [20].

Proposition 4.1. *For any x, a.s., for any $T > 0$,*

$$\int_0^T (\varphi_s^{*-1} X_i)(x)\rangle\langle(\varphi_s^{*-1} X_i)(x)\, ds \tag{4.1}$$

is a positive definite form on $T_x^*(R^d)$.

Proof. Let U_s be the vector space in $T_x(R^d)$ spanned by $\varphi_s^{*-1} X_i(x)_{1 \leq i \leq m}$ and V_t be the vector space spanned by

$$\bigcup_{s \leq t} (U_s). \tag{4.2}$$

We define V_t^+ by

$$V_t^+ = \bigcap_{s > t} V_s. \tag{4.3}$$

By the zero-one law, we know that a.s. V_0^+ is a fixed space not depending on ω. Let us assume that $V_0^+ \neq T_x(R^d)$.

Then, if S is the stopping time

$$S = \inf\{t > 0;\ V_t \neq V_0^+\}; \tag{4.4}$$

then a.s. S is > 0. Let f be a non-zero element in $T_x^*(R^d)$ orthogonal to V_0^+. Then

$$\langle f, (\varphi_s^{*-1} X_i)(x)\rangle = 0 \qquad s \leq S. \tag{4.5}$$

Now it is easily proved that

$$\langle f, (\varphi_t^{*-1} X_i)(x)\rangle = \langle f, X_i(x)\rangle + \int_0^t \langle f, (\varphi_s^{*-1}[X_0, X_i])(x)\rangle\, ds$$
$$+ \int_0^t \langle f, (\varphi_s^{*-1}[X_j, X_i])(x)\rangle \,.\, dw^j; \tag{4.6}$$

or equivalently

$$\langle f, (\varphi_t^{*-1} X_i)(x)\rangle$$
$$= \langle f, X_i(x)\rangle + \int_0^t \langle f, (\varphi_s^{*-1}([X_0, X_i] + \tfrac{1}{2}[X_j, [X_j, X_i]]))(x)\rangle\, ds$$
$$+ \int_0^t \langle f, (\varphi_s^{*-1}[X_j, X_i])(x)\rangle \,.\, \delta w^j. \tag{4.7}$$

Canceling the martingale term in (4.7), we see from (4.7) that

$$\langle f, \varphi_s^{*-1}[X_j, X_i](x)\rangle = 0, \quad s \leqq S,\ 1 \leqq i,\ j \leqq m. \tag{4.8}$$

Iterating (4.7) on (4.8), this implies in particular that

$$\langle f, \varphi_s^{*-1}[X_j, [X_j, X_i]](x)\rangle = 0, \quad s \leqq S,\ 1 \leqq i,\ j \leqq m. \tag{4.9}$$

Canceling now the bounded variation process in (4.7), we have also that

$$\langle f, \varphi_s^{*-1}[X_0, X_i](x)\rangle = 0, \quad s \leqq S,\ 1 \leqq i \leqq m. \tag{4.10}$$

By iteration, we see that if $X_{[I]}$ is any of the brackets defined in H_1, then a.s.

$$\langle f, \varphi_s^{*-1} X_{[I]}(x)\rangle = 0 \quad s \leqq S. \tag{4.11}$$

In particular, by taking $s=0$ in (4.11), we see that f is orthogonal to the vector spaces spanned by these brackets at x. This is a contradiction of the fact that this space is exactly $T_x(R^d)$.

This implies that, a.s. $V_0^+ = T_x(R^d)$. The Proposition is proved. □

Let now ψ be a C^∞ function defined on $R^d \otimes R^d$ with values in R, which has bounded differentials of all orders and which is such that:

a) $0 \leqq \psi \leqq 1$;

b) ψ is equal to one on the set of invertible elements A in $R^d \otimes R^d$ such that $\|A^{-1}\| \leqq l$ $(l>0)$, and zero on the set of invertible elements A in $R^d \otimes R^d$ such that $\|A^{-1}\| \geqq 2l$ and on the set of non invertible elements.

T is a fixed real number >0.

We then have the following

Theorem 4.2. *Let f be a bounded C^∞ function defined on R^d with values in R which has bounded derivatives, and let Y be a bounded C^∞ vector field on R^d which has bounded derivatives. Then the following equality holds*

$$
\begin{aligned}
&E\{\psi(C(\omega))(Yf)(\varphi_T(\omega,x))\}\\
&\quad=E\left\{\psi(C(\omega))f(\varphi_T(\omega,x))\left\langle\varphi_T^{*-1}Y,\,C^{-1}(\omega)\int_0^T\varphi_s^{*-1}X_i\,\delta w^i\right\rangle\right\}\\
&\quad-E\left\{\psi(C(\omega))f(\varphi_T(\omega,x))\int_0^T\langle C^{-1}(\omega)[\varphi_s^{*-1}X_i,\varphi_T^{*-1}Y],\varphi_s^{*-1}X_i\rangle\,ds\right\}\\
&\quad+E\left\{\psi(C(\omega))f(\varphi_T(\omega,x))\int_0^T ds\,\langle C^{-1}(\omega)\,\varphi_s^{*-1}X_i,\varphi_T^{*-1}Y\rangle\right.\\
&\quad\left.\cdot\int_0^s\langle C^{-1}(\omega)[\varphi_v^{*-1}X_j,\varphi_s^{*-1}X_i],\varphi_v^{*-1}X_j\rangle\,dv\right\}\\
&\quad+E\left\{\psi(C(\omega))f(\varphi_T(\omega,x))\int_0^T ds\left\langle C^{-1}(\omega)\,\varphi_s^{*-1}X_i,\right.\right.\\
&\quad\left.\left.\int_0^s\langle[\varphi_v^{*-1}X_j,\varphi_s^{*-1}X_i],\;C^{-1}(\omega)\,\varphi_T^{*-1}Y\rangle\,\varphi_v^{*-1}X_j\right\rangle dv\right\}\\
&\quad-E\left[f(\varphi_T(\omega,x))\left(\frac{\partial\psi}{\partial x}(C(\omega)),\;\int_0^T ds\,\varphi_s^{*-1}X_i\left\langle\int_0^s[\varphi_v^{*-1}X_j,\varphi_s^{*-1}X_i],\right.\right.\right.\\
&\quad(\langle C^{-1}(\omega)\,\varphi_T^{*-1}Y,\varphi_v^{*-1}X_j\rangle)\,dv\\
&\quad\left.\left.+\int_0^T ds\int_0^s\langle C^{-1}(\omega)\,\varphi_T^{*-1}Y,\varphi_v^{*-1}X_j\rangle[\varphi_v^{*-1}X_j,\varphi_s^{*-1}X_i]\,dv<\varphi_s^{*-1}X_i,.\right)\right],
\end{aligned}\tag{4.14}
$$

where $C(\omega)$ is the linear mapping from $T_x^*(R^d)$ in $T_x(R^d)$

$$C(\omega)=\int_0^T(\varphi_s^{*-1}X_i)(x))\rangle\langle((\varphi_s^{*-1}X_i)(x)\,ds. \tag{4.15}$$

Proof. We will show that (4.14) is a consequence of Theorem 3.1. We consider the flow $\Psi.(\omega,.)$ associated to the stochastic differential equation

$$
\begin{aligned}
&dx=X_0(x)\,dt+X_i(x)\cdot dw^i\\
&x(0)=x\\
&dZ=\frac{\partial X_0}{\partial x}(x)Z\,dt+\frac{\partial X_i}{\partial x}(x)Z\cdot dw^i\\
&Z(0)=Z_0\\
&dZ'=-Z'\frac{\partial X_0}{\partial x}(x)\,dt-Z'\frac{\partial X_i}{\partial x}(x)\cdot dw^i\\
&Z'(0)=Z'_0,
\end{aligned}\tag{4.16}
$$

which takes its values in $R^d\times(R^d\otimes R^d)^2$. By Theorem 1.1, we know that if $Z_0=Z'_0=I$, then:

$$Z_t=\frac{\partial\varphi}{\partial x}t(\omega,x),\qquad Z'_t=\left[\frac{\partial\varphi}{\partial x}t(\omega,x)\right]^{-1}.$$

To make the computations easier, instead of applying Theorem 3.1 directly, we will do some supplementary intrinsic computations in order to obtain an intrinsic formula.

Let us temporarily take the assumptions of Theorem 1.3. By Theorem 1.3, we know that for a given $v \in L_\infty(R^+; R^m)$, if y^v is the solution of the differential equation

$$\begin{aligned} d y^v &= (\varphi_t^{*-1} X_i)(y^v) v^i dt \\ y^v(0) &= x; \end{aligned} \tag{4.17}$$

then, $\varphi_t(\omega, y_t^v)$ is the unique solution of the stochastic differential equation

$$\begin{aligned} d x^v &= (X_0(x^v) + X_i(x^v) v^i) dt + X_i(x^v) dw^i \\ x^v(0) &= x. \end{aligned} \tag{4.18}$$

Let $\rho_t^v(\omega, x)$ be the flow associated to (4.17). Then the differential of the mapping $x \to x_t^v = \varphi_t^v(\omega, x)$ is given by

$$\varphi_t^{v*}(\omega, x) = \varphi_t^*(\omega, \rho_t^v(\omega, x)) \rho_t^{v*}(\omega, x) \tag{4.19}$$

where $Z_t^v = \rho_t^{v*}(\omega, x)$ is the solution of the differential equation

$$\begin{aligned} dZ^v &= \frac{\partial}{\partial x} (\varphi_t^{*-1} X_i)(y^v) Z^v v^i dt \\ Z^v(0) &= I. \end{aligned} \tag{4.20}$$

Let $\overline{T}$ be a C^∞ vector field on R^d. Let us compute the differential at $s=0$ of the mapping

$$s \to [\varphi_t^{sv*}(\omega, x)]^{-1} \overline{T}(\varphi_t^{sv}(\omega, x)). \tag{4.21}$$

The R.H.S. of (4.21) is equal to

$$[\rho_t^{sv*}(\omega, .)^{-1} [\varphi_t^*(\omega, .)^{-1} \overline{T}]](x). \tag{4.22}$$

Trivially, from (4.20), we see that:

$$\frac{\partial}{\partial s} [\rho^{sv*} t(\omega, x)]_{s=0} = \int_0^t \frac{\partial}{\partial x} (\varphi_s^{*-1} X_i)(x) v^i ds, \tag{4.23}$$

since in particular for $v=0$, $\rho_{\cdot}^v(\omega, x) = x$ and $\rho_{\cdot}^{v*}(\omega, x) = I$. Similarly from (4.17) we get:

$$\left[\frac{\partial}{\partial s} \rho_t^{sv}(\omega, x)\right]_{s=0} = \int_0^t (\varphi_s^{*-1} X_i)(x) v^i ds \tag{4.24}$$

and

$$\begin{aligned} &\frac{\partial}{\partial s} ([\varphi_t^{sv*}(\omega, x)]^{-1} \overline{T}(\varphi_t^{sv}(\omega, x)))_{s=0} \\ &= -\left[\int_0^t \frac{\partial}{\partial x} (\varphi_s^{*-1} X_i)(x) v^i ds\right] [\varphi_t^{*-1} \overline{T}](x) \\ &\quad + \frac{\partial}{\partial x} (\varphi_t^{*-1} \overline{T})(x) \int_0^t (\varphi_s^{*-1} X_i)(x) v^i ds. \end{aligned} \tag{4.25}$$

The R.H.S. of (4.25) may be expressed via Lie brackets; we get

$$\frac{\partial}{\partial s}([\varphi_t^{sv*}(\omega,x)]^{-1}\,\overline{T}(\varphi_t^{sv}(\omega,x))_{s=0}=\left[\int_0^t(\varphi_s^{*-1}X_i)(x)\,v^i\,ds,(\varphi_t^{*-1}\overline{T})(x)\right]. \quad (4.26)$$

(4.14) is then exactly formula (3.2) computed on the flow $\Psi_{\cdot}(\omega,.)$ where h is given by:

$$\begin{aligned}(x_\cdot,Z_\cdot,Z'_\cdot)\in\mathscr{C}([0,T];R^d\times(R^d\otimes R^d)^2)\to\psi(\int_0^T Z'_sX_i(x_s)\rangle\langle Z'_sX_i(x_s)\,ds)\\ \cdot(\int_0^T Z'_sX_i(x_s)\rangle\langle Z'_sX_i(x_s)\,ds)^{-1}\,Z'_T\,Y(x_T)\end{aligned} \quad (4.27)$$

and initial point (x,I,I). It is easier to see this by using (4.26) and starting again the computations leading to (2.2). Taking u as in Theorem 2.1, we have that:

$$\begin{aligned}&E\left[f(\varphi_T^{lu}(\omega,x))(\int_0^T\varphi_s^{lu*-1}X_i(x)\rangle\langle\varphi_s^{lu*-1}X_i(x)\,ds)^{-1}(\varphi_T^{lu*-1}Y)(x)\right.\\ &\left.\cdot\psi(\int_0^T\cdot\varphi_s^{lu*-1}X_i(x)\rangle\langle\varphi_s^{lu*-1}X_i(x)\cdot ds)\right]=E\left[f(\varphi_T(\omega,x))(\int_0^T\varphi_s^{*-1}X_i(x)\rangle\right.\\ &\left.\cdot\langle\varphi_s^{*-1}X_i(x)\,ds)^{-1}(\varphi_T^{*-1}Y)(x)\,\psi(\int_0^T\cdot\varphi_s^{*-1}X_i(x)\rangle\langle\varphi_s^{*-1}X_i(x)\cdot ds)\,Z_T^{lu}(\omega)\right].\end{aligned} \quad (4.28)$$

Taking the differential at $l=0$ of the L.H.S. of (4.28) and using (4.26), we get:

$$\begin{aligned}&E\left(\left\langle\frac{\partial f}{\partial x}(\varphi_T(\omega,x)),\varphi_T^*\int_0^T(\varphi_s^{*-1}X_i)\,u^i\,ds\right\rangle C(\omega)^{-1}(\varphi_T^{*-1}Y)(x)\,\psi(C(\omega))\right)\\ &\quad-E\left(f(\varphi_T(\omega,x))\,C(\omega)^{-1}\int_0^T ds\left(\left[\int_0^s\varphi_v^{*-1}X_ju^j,\varphi_s^{*-1}X_i\right]dv\,\langle\varphi_s^{*-1}X_i,.\right.\right.\\ &\quad\left.\left.+\varphi_s^{*-1}X_i\langle\left[\int_0^s\varphi_v^{*-1}X_ju^i,\varphi_s^{*-1}X_i\right]dv,C^{-1}(\omega)(\varphi_T^{*-1}Y(x)\rangle\,\psi(C(\omega))\right)\right)\\ &\quad+E\left(f(\varphi_T(\omega,x))\,C^{-1}(\omega)\left[\int_0^T\varphi_s^{*-1}X_ju^j\,ds,\varphi_T^{*-1}Y\right]\psi(C(\omega))\right)+\{\ \}\end{aligned} \quad (4.29)$$

where in (4.29) the differential of ψ appears in a trivial form left to the reader.

Similarly the differential at $l=0$ of the R.H.S. of (4.28) is

$$E\left[f(\varphi_T(\omega,x))\,C^{-1}(\omega)(\varphi_T^{*-1}Y)(x)\,\psi(C(\omega))\int_0^T u^i\cdot\delta w^i\right]. \quad (4.30)$$

Extending the equality of (4.29) and (4.30) to any square integrable u, taking

$$u^i=(\varphi_s^{*-1}X_i)^k$$

and summing in k the values of the k-components of both sides of the equality, we obtain (4.14).

When $X_1 \dots X_m$ do not have compact supports, we may proceed as in Theorem 2.1. □

As noted by Malliavin in [11], (4.14) is enough to see that the probability law of $\varphi_T(\omega, x)$ has a density. In fact (4.14) shows in particular that

$$|E[\psi(C(\omega))(Yf)(\varphi_T(\omega, x))]| \leqq C \sup_{x \in R^d} |f(x)|. \tag{4.31}$$

From a result in harmonic analysis, this implies that the image of the measure $\psi(C(\omega))\,dP$ by the mapping $\omega \to \varphi_T(\omega, x))$ has a density. Since by Proposition 3.1 $C(\omega)$ is a.s. invertible, the same result is true for the image of measure P by the mapping $\omega \to \varphi_T(\omega, x)$.

But more may be said. Namely, let Y' be another C^∞ vector field on R^d which is bounded with bounded differentials. Formula (4.14) may be applied to function $Y'f$, and the R.H.S. of (4.14) will make function $Y'f$ appear. Now the same procedure as the previous one may be used to evaluate this R.H.S. in a formula where f appears explicitly as such, by noting that all the objects in the R.H.S. may be submitted to the calculus of variations. The only term for which this is not entirely clear is the stochastic integral $\int_0^t (\varphi_s^{*-1} X_i)(x) . \delta w^i$, but this is in fact trivial since to the system (4.16) a new variable may be added which is given precisely by

$$\begin{aligned} dR &= Z'_s X_i(x_t)\, \delta w^i \\ R(0) &= 0 \end{aligned} \tag{4.32}$$

(incidently note that the Ito integrals are in fact Stratonovitch integrals).

By iteration, we find that if Q is any differential operator whose coefficients are functions with bounded differentials, then

$$|E[\psi(C(\omega))(Qf)(\varphi_T(\omega, x))]| \leqq C_Q \sup_{x \in R^d} |f(x)|. \tag{4.33}$$

By a classical result on Fourier transform, this implies that the image of measure $\psi(C(\omega))\,dP$ by $\omega \to \varphi_T(\omega, x)$ is given by a C^∞ density.

As noted by Malliavin [11], if $[C(\omega)]^{-1}$ belongs to all the L_p $(1 \leqq p < +\infty)$, then formula (4.14) is still true by taking $\psi = 1$, since this will follow from a trivial limit argument and from (1.4). The same can be said for the iterated formulas of (4.14) taken with $\psi = 1$, since they will make appear higher powers of C^{-1}, which may be easily taken care of, since C^{-1} is in all the L_p spaces. (4.33) is true with $\psi = 1$, which implies that the probability law of $\varphi_t(\omega, x)$ has a C^∞ density. For a non trivial example in the non elliptic case, we refer to Stroock [12]. The complete proof when the vector space spanned by $X_1, \dots, X_m$ and their brackets is full appears in Malliavin [11] and Ikeda and Watanabe [22].

Remark 1. As noted by Stroock [12] all these results are of local nature. Since we will use extensively this fact in Sect. 5, we repeat the argument of [12]. If

$x_0 \in R^d$ and if assumption H_1 is verified on a neighborhood $B(x_0, R)$, then if $x \in R^d$, $P_t(x, .)$ restricted to $B(x_0, R)$ has a density. To see this note first that if $x \in B(x_0, R)$, then $P_t(x, .)$ has a density by what has been previously said.

Take $0<\rho<R$, $x \in B(x_0, R)$ and let $\xi(x)$ be the stopping time

$$\xi(x)=\inf\{t \geqq 0;\ |\varphi_t(\omega, x)-x_0| \geqq \rho\}. \tag{4.34}$$

Since for any Borel set Γ

$$E(1_{\varphi_t(\omega,x)\in\Gamma}(\xi(x) \geqq t)) \leqq E(1_{\varphi_t(\omega,x)\in\Gamma})=P_t(x, \Gamma). \tag{4.35}$$

the L.H.S. defines a measure $\hat{P}_t(x,.)$ which has a density. If γ is taken such that $0<\gamma<\rho$, for $x \in R^d$ define $\tau^n(x)$ to be the sequence of stopping times defined by induction by

$$\begin{aligned} \tau_0(x)&=\inf\{t \geqq 0;\ |\varphi_t(\omega, x)-x_0| \leqq \gamma\} \\ \tau_{2n}(x)&=\inf\{t \geqq \tau_{2n-1};\ |\varphi_t(\omega, x)-x_0| \leqq \gamma\} \\ \tau_{2n+1}(x)&=\inf\{t \geqq \tau_{2n};\ |\varphi_t(\omega, x)-x_0| \geqq \rho\}. \end{aligned} \tag{4.36}$$

Clearly $\tau_n(x) \to +\infty$ a.s. Now for $\Gamma \in B(x_0, \gamma)$, by the Markov property of the diffusion $\varphi_t(\omega, .)$, we have:

$$\begin{aligned} P_t(x, \Gamma)&=\sum_n E(1_{\varphi_t(\omega,x)\in\Gamma}(\tau_{2n}(x) \leqq t<\tau_{2n+1}(x))) \\ &=\sum_n E[\hat{P}_{t-\tau_{2n}(x)}(\varphi_{\tau_{2n}(x)}(\omega, x)), \Gamma). \end{aligned} \tag{4.37}$$

Since each of the terms in (4.37) has a density, the R.H.S. of (4.37) has a density.

To work on a manifold, this procedure of Stroock may then be used in the same way. If N is a metrizable connected C^∞ manifold, assume that H_1 is verified on N, i.e. at each $x \in N$, the vector space spanned in $T_x(R^d)$ by $X_1, \ldots, X_m$, $[X_0, X_1]$, $[X_0, X_m] \ldots$ is full. Take $x_0 \in N$ and assume that $\mathscr{V}(x_0)$ is a neighborhood of x_0, which is diffeomorphic to $B(0, 2R)$ in R^d. Extending $X_0, \ldots, X_m$ as C^∞ bounded vector fields on R^d out of $B(0, R)$, which is diffeomorphic to $\mathscr{V}'(x_0)$, we see that if $\eta(x)$ is defined by

$$\eta(x)=\inf\{t \geqq 0;\ \varphi_t(\omega, x) \notin \mathscr{V}'(x_0)\} \tag{4.38}$$

the measure

$$\Gamma \in \mathscr{V}'(x_0) \to E(1_{\varphi_t(\omega,x)\in\Gamma}\, \eta(x) \geqq t) \tag{4.39}$$

has a density. Repeating (4.37), we see that for each $x \in N$, $P_t(x, .)$ has a density on $\mathscr{V}'(x_0)$.

Since N is locally compact and countable at infinity, for any $x \in N$, $P_t(x, .)$ has a density.

5. Application: Hypoellipticity under General Hörmander's Conditions

In this section we will apply the techniques of the previous parts to study the hypoellipticity of a second order differential operator under general

Hörmander's conditions [15] i.e. when, at each $x \in R^d$, the vector space generated by $X_0, X_1, \ldots, X_m$, $[X_i, X_j]_{0 \leqq i, j \leqq m} \ldots$ is full. The difference with Sect. 4 is that X_0 comes in explicity as a generating member of the family, and not only via its brackets with $X_1, \ldots, X_m$.

A typical example is given by the generator of the non homogeneous heat equation

$$\frac{\partial}{\partial t} + \tfrac{1}{2} \Sigma \frac{\partial^2}{\partial x^{i2}}. \tag{5.1}$$

In this case, the semi-group $P_t(x, .)$ is not given by densities, but the fact of integrating in dt to obtain the resolvent makes that the potentials or (5.1) have a density.

The main results contained in this section are the following:

a) If T'_x is the vector space generated in $T_x(R^d)$ by $X_1(x), X_2(x), \ldots, X_m(x)$, $[X_i, X_j]^{(x)}_{0 \leqq i, j \leqq m} \ldots$, then if T' is nowhere full, the diffusion may be factored in the product of a diffusion along the leaves which are the integral submanifolds of T', and a deterministic process transversal to the leaves. In each leaf, the analysis done in Sect. 4 may be used. Integration in t explains why hypoellipticity holds.

b) If T'_x is full, then the methods of Sect. 4 may be used.

c) In the general case, where T' is full on an open set A the methods of Sect. 4 may be used when x is in the fine closure $\bar{A}^f$ of A. On the complementary set ${}^c\bar{A}^f$, it is not clear that the methods of a) may be used, since T' does not necessarily generate a foliation. An alternative method is then used. In fact, the process is lifted to a larger vector space and a time change is done on the lifted process, which makes its distribution $\tilde{T}'$ full at least locally, so that the methods of Sect. 4 apply on the lifted process. Going down to the initial process, hypoellipticity is shown to hold. A consequence of this is to verify that the distribution T' is essential to understanding the semi-group $P_t(x, .)$, but is (fortunately) very unstable through time changes.

In subsection 5.1, a detailed analysis is done of the process of operators

$$\int_0^t \varphi_s^{*-1} X_i(x)\rangle \langle \varphi_s^{*-1} X_i(x)\, ds. \tag{5.3}$$

The importance of the fine closure $\bar{A}^f$ of A is explicited. In subsection 5.2 the complete analysis of the case when the distribution T' fibrates is done. The factorization theorem is proved, whose consequence is hypoellipticity in this special case. In subsection 5.3, hypoellipticity is proved in the general case using a lifting procedure.

The analysis done in Sects. 5.1 and 5.2 is closely related to the results of Ichihara and Kunita [20].

5.1 Sweeping the Tangent Space

In the whole section, we make the general assumption of Hörmander [15]:

H_2: At each $x \in R^d$, the vector space spanned by $X_0, X_1, \ldots, X_m$, $[X_i, X_j]_{0 \leqq i, j \leqq m} \ldots$ is equal to $T_x(R^d)$.

Definition 5.1. At $x \in R^d$, T'_x is the vector space spanned by $X_1, \dots, X_m$, $[X_i, X_j]_{0 \leqq i, j \leqq m} \dots$ in $T_x(R^d)$.

By H_2, the codimension of T'_x is at most 1.

We have seen in Sect. 4 that it is essential to know if $\int_0^t \varphi_s^{*-1} X_i \rangle \langle \varphi_s^{*-1} X_i\, ds$ is invertible. Under H_2, this will not necessarily be the case. It is however of key importance to know what the image of this operator is. We have:

Theorem 5.2. *For any $x \in R^d$, if $H_T(\omega, x)$ is the subspace in $T_x(R^d)$ which is the image of $T_x^*(R^d)$ by the linear mapping $C_T(\omega, x)$*

$$p \in T_x^*(R^d) \to \int_0^T \langle p, (\varphi_s^{*-1} X_i)(x) \rangle (\varphi_s^{*-1} X_i)(x)\, ds, \tag{5.4}$$

then a.s., for any $T > 0$, the vector space in $T_x(R^d)$ spanned by

$$\bigcup_{s \leqq T} \varphi_s^{*-1}(\omega, x)\, T'_{\varphi_s(\omega, x)}$$

is equal to $H_T(\omega, x)$.

Proof. The transpose of operator $C_T(\omega, x)$ is equal to itself. Moreover $C_T(\omega, x)$ defines a positive symmetric form on $T_x^*(R^d)$. Since, for $T \leqq t$

$$\langle p, C_T(\omega, x) p \rangle \leqq \langle p, C_t(\omega, x) p \rangle, \tag{5.5}$$

the kernels $\operatorname{Ker} C_T(\omega, x)$ are decreasing with T, and moreover $T \to \operatorname{Ker} C_T(\omega, x)$ is left-continuous, i.e. $\operatorname{Ker} C_T(\omega, x) = \bigcap_{T > T'} \operatorname{Ker} C_{T'}(\omega, x)$. By orthogonality, this implies that the images $H_T(\omega, x)$ are increasing with T. The result stated in the Theorem is clearly equivalent to proving that a.s., for every $T > 0$

$$\operatorname{Ker} C_T(\omega, x) = \bigcap_{s \leqq T} [(\varphi_s^{*-1}(\omega, x)\, T'_{\varphi_s(\omega, x)})^{\perp}]. \tag{5.6}$$

Since, if $X_{[I]}$ is one of the brackets generating T', $(\varphi_s^{*-1} X_{[I]})(x)$ is a continuous process, it is equivalent to prove that for any $T > 0$

$$\operatorname{Ker} C_T(\omega, x) = \bigcap_{s \leqq T} [(\varphi_s^{*-1}(\omega, x)\, T'_{\varphi_s(\omega, x)})^{\perp}] \quad \text{a.s.} \tag{5.7}$$

For p to be in $\operatorname{Ker} C_T(\omega, x)$, it is necessary and sufficient that:

$$\langle p, (\varphi_s^{*-1} X_i)(x) \rangle = 0 \qquad s \leqq T. \tag{5.8}$$

(5.8) shows that since $X_1, \dots, X_m$ are in the distribution T',

$$\operatorname{Ker} C_T(\omega, x) \supset \bigcap_{s \leqq T} [(\varphi_s^{*-1}(\omega, x)\, T'_{\varphi_s(\omega, x)})^{\perp}]. \tag{5.9}$$

We now prove the reverse inclusion. Let $t_1, \dots, t_n$ be an equipartition of $[0, T]$ such that

$$t_{j+1} - t_j = T/n \qquad t_0 = 0. \tag{5.10}$$

Let $l_1, \ldots, l_d$ be a fixed basis in $T_x(R^d)$. In the sequel, we decompose the vectors in $T_x(R^d)$ on this basis. Clearly, if $Y_1, \ldots, Y_m$ are C^∞ vector fields on R^d, for any b, b' $(1 \leqq b, b' \leqq d)$ when $n \to +\infty$

$$\sum_k \int_{t_k}^{t_{k+1}} (\varphi_s^{*-1} Y_i(x))^b \, \delta w^i \int_{t_k}^{t_{k+1}} (\varphi_s^{*-1} Y_j(x))^{b'} \delta w^j$$
$$\to \int_0^T (\varphi_s^{*-1} Y_i(x))^b (\varphi_s^{*-1} Y_i(x))^{b'} \, ds \quad \text{in probability.} \tag{5.11}$$

By a measurable selection theorem, it is possible a family of bounded random variables $f_1, \ldots, f_d$ with values in $T_x^*(R^d)$ so that for a.e. ω, $f_1(\omega), \ldots, f_d(\omega)$ generate $\operatorname{Ker} C_T(\omega, x)$ (they are not necessarily free). Now using (4.6), we have that for any i $(0 \leqq i \leqq m)$

$$\begin{aligned}\varphi_t^{*-1} X_i(x) &= X_i(x) + \int_0^t \varphi_s^{*-1} [X_0, X_i](x) \, ds + \int_0^t \varphi_s^{*-1} [X_j, X_i] \cdot dw^j \\ &= X_i(x) + \int_0^t \varphi_s^{*-1} ([X_0, X_i] + \tfrac{1}{2}[X_j, [X_j, X_i]])(x) \, ds \\ &\quad + \int_0^t \varphi_s^{*-1} [X_j, X_i](x) \, \delta w^j.\end{aligned} \tag{5.12}$$

Using (5.11) and (5.12) it is then clear that for any l $(1 \leqq l \leqq d)$, when $n \to +\infty$

$$\sum_k |\langle f_l(\omega), \varphi_{t_{k+1}}^{*-1} X_i(x) - \varphi_{t_k}^{*-1} X_i(x) \rangle|^2$$
$$\to \int_0^T |\langle f_l(\omega), \varphi_s^{*-1} [X_j, X_i](x) \rangle|^2 \, ds \quad \text{in probability} \tag{5.13}$$

((5.13) does not follow from stochastic calculus on $\langle f_l(\omega), \varphi_s^{*-1} X_i(x) \rangle$!). Using (5.13), we see that:

$$\int_0^t |\langle f_l(\omega), \varphi_s^{*-1} [X_j, X_i](x) \rangle|^2 \, ds = 0, \quad t \leqq T, \ 1 \leqq i, \ j \leqq m \quad \text{a.s.} \tag{5.14}$$

By continuity, we get:

$$\langle f_l(\omega), \varphi_s^{*-1} [X_j, X_i](x) \rangle = 0, \quad s \leqq T, \ 1 \leqq i, \ j \leqq m \quad \text{a.s.} \tag{5.15}$$

Applying the same method to (5.15), we see that in particular:

$$\langle f_l(\omega), \varphi_s^{*-1} [X_j, [X_j, X_i]](x) \rangle = 0, \quad s \leqq T, \ 1 \leqq i, \ j \leqq m \quad \text{a.s.} \tag{5.16}$$

Now we know that $\int_0^t \varphi_s^{*-1} [X_j, X_i](x) \, \delta w^j$ is the uniform limit in probability of

$$\sum \varphi_{t_k}^{*-1} [X_j, X_i](x) (w_{t_{k+1} \wedge t}^j - w_{t_k \wedge t}^j). \tag{5.17}$$

From (5.15), we get:

$$\left\langle f_l(\omega), \int_0^t \varphi_s^{*-1}[X_j, X_i](x)\,\delta w^j \right\rangle = 0, \qquad t \leqq T,\ 1 \leqq i,\ j \leqq m \quad \text{a.s.} \tag{5.18}$$

From (5.12), (5.15) and (5.18), we have

$$\left\langle f_l(\omega), \int_0^t \varphi_s^{*-1}[X_0, X_i](x)\,ds \right\rangle = 0, \qquad t \leqq T,\ 1 \leqq i \leqq m \quad \text{a.s.} \tag{5.19}$$

or equivalently

$$\langle f_l(\omega), \varphi_s^{*-1}[X_0, X_i](x)\rangle = 0, \qquad s \leqq T,\ 1 \leqq i \leqq m \quad \text{a.s.} \tag{5.20}$$

Now (5.14)-(5.20) are equivalent to

$$\langle f_l(\omega), \varphi_s^{*-1}[X_j, X_i](x)\rangle = 0, \qquad s \leqq T,\ 0 \leqq j \leqq m,\ 1 \leqq i \leqq m \quad \text{a.s.} \tag{5.21}$$

Iterating the procedure, we find that if $X_{[I]}$ is any bracket in the distribution T', then

$$\langle f_l(\omega), \varphi_s^{*-1} X_{[I]}(x)\rangle = 0, \qquad s \leqq T \quad \text{a.s.} \tag{5.22}$$

(5.22) shows that:

$$\operatorname{Ker} C_T(\omega, x) \subset \bigcap_{s \leqq T} [(\varphi_s^{*-1}(\omega, x)\, T'_{\varphi_s(\omega, x)})^{\perp}]. \tag{5.23}$$

The Theorem is proved. □

Another connected result is the following

Theorem 5.3. *For any $x \in R^d$, a.s., for any $s, T, s \leqq T$, then*

$$\varphi_s^{*-1} X_0(x) - X_0(x) \in H_T(\omega, x). \tag{5.24}$$

Proof. Since $H_T(\omega, x)$ increases with T, it suffices to prove the stated result for $s = T$. Using the left continuity of $T \to \operatorname{Ker} C_T(\omega, x)$ and the continuity of the process $\varphi_t^{*-1} X_0(x)$, this is equivalent to proving that for any T, a.s. $\varphi_T^{*-1} X_0(x) - X_0(x)$ is orthogonal to $\operatorname{Ker} C_T(\omega, x)$. Now from (5.12), we have

$$\begin{aligned}\varphi_t^{*-1} X_0(x) &= X_0(x) + \int_0^t \varphi_s^{*-1}[X_j, X_0](x) \cdot dw^j \\ &= X_0(x) + \int_0^t \tfrac{1}{2} \varphi_s^{*-1}[X_j, [X_j, X_0]](x)\,ds + \int_0^t \varphi_s^{*-1} X_j(x)\,\delta w^j. \end{aligned} \tag{5.25}$$

Using the notations of Theorem 5.2 and proceeding as in this Theorem, it is clear that any $f_l(\omega)$ is a.s. orthogonal to $\varphi_T^{*-1} X_0(x) - X_0(x)$. The result is proved. □

Let $T_{(\omega, x)}$ be defined by

$$T_{(\omega, x)} = \inf\{t \geqq 0;\ C_t(\omega, x) \text{ is invertible}\}. \tag{5.26}$$

For every $x \in R^d$, $T_{(\omega,x)}$ is a $\{F_t^+\}_{t \geqq 0}$ stopping time. By the zero-one law either $T_{(\omega,x)}=0$ a.s. or $T_{(\omega,x)}>0$ a.s. Let F be the set

$$F=\{x \in R^d;\ T_{(\omega,x)}=0 \text{ a.s.}\}. \tag{5.27}$$

Before identifying the set F, we give some of its properties.

Theorem 5.4. *F is a G_δ set in R^d which is finely closed for the fine topology of R^d induced by the strong Markov process* (1.1). *Moreover, for every $x \in R^d$ such that $T'_x \neq T_x(R^d)$, the following identifies hold a.s.*

$$\begin{aligned} T_{(\omega,x)} &= \inf\{t \geqq 0;\ \varphi_t(\omega,x) \in F\} \\ &= \inf\{t \geqq 0;\ \varphi_t^{*-1} X_i(x) \notin T'_x \text{ for one of the } i\colon 1 \leqq i \leqq m\} \\ &= \inf\{t \geqq 0;\ \varphi_t^{*-1}(\omega,x)\, T'_{\varphi_t(\omega,x)} \neq T'_x\}. \end{aligned} \tag{5.28}$$

Proof. Clearly

$$P(T_{(\omega,x)}=0)=P\Big(\bigcap_{t\downarrow\downarrow 0}\{C_t(\omega,x) \text{ is invertible}\}\Big). \tag{5.29}$$

For x to be in F, it is necessary and sufficient that, for any $t>0$

$$P(C_t(\omega,x) \text{ is invertible})=1. \tag{5.30}$$

From Theorem 5.2, we know that a.s., for every $t>0$, $H_t(\omega,x)$ contains T'_x. Since for $t \leqq T_{(\omega,x)}$, $C_t(x,\omega)$ is not invertible, since the codimension of T'_x is 1 when $T'_x \neq T_x(R^d)$, necessarily, for $t \leqq T_{(\omega,x)}$, $t>0$, we have:

$$H_t(\omega,x)=T'_x. \tag{5.31}$$

Using Theorem 5.2 again, we see that for $t \leqq T_{(\omega,x)}$, $\varphi_t^{*-1}(\omega,x)\,T'_{\varphi_t(\omega,x)}$ is included in T'_x. Since the dimension of $T'_{\varphi_t(\omega,x)}$ is at least $d-1$, this is possible only if equality holds.

Note then that if $\varphi_t^{*-1}(\omega,x)\,T'_{\varphi_t(\omega,x)} \neq T'_x$, since the codimension of T'_x is one, the union of these two spaces spans the whole space $T_x(R^d)$. The third equality in (5.28) is then an easy consequence of Theorem 5.2. Since from (5.8), $H_t(\omega,x)$ is the space spanned by $\varphi_s^{*-1} X_i(x)\,(s \leqq t, 1 \leqq i \leqq m)$, the second equality in (5.28) is also trivial.

From the strong multiplicative Markov property of $\varphi_{\cdot}(\omega,.)$ ([2, 3]), we know that if θ_t is the canonical translation operator on $\Omega=\mathscr{C}(R^+;R^m)$

$$\theta_t(w_s)=(w_{s+t}-w_t) \tag{5.32}$$

if S is a $\{F_t^+\}_{t \geqq 0}$ stopping time, then, a.s. on $(S<+\infty)$, for any (t,x) in $R^+ \times R^d$

$$\varphi_{t+S}(\omega,x)=\varphi_t(\theta_S\omega, \varphi_S(\omega,x)). \tag{5.33}$$

We apply (5.33) with $S=T_{(\omega,x)}$. In the sequel, we write S instead of $T_{(\omega,x)}$. Knowing that on $(S<+\infty)$, $\varphi_S^{*-1}\,T'_{\varphi_S(\omega,x)}=T'_x$, using (5.33), we have on $(S<+\infty)$:

$$\inf\{t \geqq 0; \varphi_t^{*-1}(\theta_S\omega, \varphi_S(\omega,x))\,T'_{\varphi_t(\theta_S\omega,\varphi_S(\omega,x))} \neq T'_{\varphi_S(\omega,x)}\}=0. \tag{5.34}$$

Since on $(S<+\infty)$, $\theta_S^{-1}F_\infty$ and F_S are independent and $\theta_S\omega$ is still a Brownian motion, necessarily on $(S<+\infty)$, $\varphi_S(\omega,x)\in F$. If D_F is the stopping time defined by

$$D_F=\inf\{t\geqq 0;\varphi_t(\omega,x)\in F\} \tag{5.35}$$

we have proven that $D_F\leqq T_{(\omega,x)}$ a.s. Now, on $(D_F<+\infty)$, using (5.33) again with $S=D_F$, we see that a.s., on $(D_F<+\infty)$

$$\inf\{t\geqq D_F;\varphi_t^{*-1}(\omega,x)T'_{\varphi_t(\omega,x)}\neq\varphi_{D_F}^{*-1}(\omega,x)T'_{\varphi_{D_F}(\omega,x)}\}=D_F. \tag{5.36}$$

For t taken as in (5.36), $\varphi_t^{*-1}(\omega,x)T'_{\varphi_t(\omega,x)}$ and $\varphi_{D_F}^{*-1}(\omega,x)T'_{\varphi_{D_F}(\omega,x)}$ generate the whole $T_x(R^d)$. Using Theorem 5.2, we see that $T_{(\omega,x)}\leqq D_F$ a.s. The first equality in (5.28) is proved.

For a.e. ω, $x\to 1_{C_t(\omega,x)\,\text{invertible}}$ is l.s.c., since $C_t(\omega,x)$ depends continuously on x. This implies that the function g_t defined by

$$x\to P(C_t(\omega,x)\text{ is invertible}) \tag{5.37}$$

is l.s.c. Since $g_t\leqq 1$, we have

$$\{x\in R^d;g_t(x)=1\}=\bigcap_n\left\{x\in R^d;g_t(x)>1-\frac{1}{n}\right\} \tag{5.38}$$

and the R.H.S. of (5.38) is clearly a G_δ set. From (5.30), we see that F is a countable intersection of G_δ i.e. still a G_δ set. F is trivially finely closed. □

For the moment, F apparently depends on $X_0,X_1,\dots,X_m$ and not only on the generator $\mathscr{L}=X_0+\frac{1}{2}\sum_1^m X_i^2$, i.e. on the Markov process defined by (1.1). We now show this independence.

Let A be the open set:

$$A=\{x\in R^d;T'_x=T_x(R^d)\}.$$

It is easily shown that A depends only on $\mathscr{L}$. We now have

Theorem 5.5. *F is equal to the fine closure $\bar{A}^f$ of A. For every $x\in R^d$, we have:*

$$\begin{aligned}T_{(\omega,x)}&=\inf\{t\geqq 0;X_0(\varphi_t(\omega,x))\in T'_{\varphi_t(\omega,x)}\}\\&=\inf\{t\geqq 0;X_0(x)\in\varphi_t^{*-1}(\omega,x)T'_{\varphi_t(\omega,x)}\}.\end{aligned} \tag{5.39}$$

Proof. We first prove that if x belongs to the fine interior of ${}^cA$, then $x\notin F$. Let S be the stopping time

$$S=\inf\{t\geqq 0;\varphi_t(\omega,x)\in A\}. \tag{5.40}$$

Clearly $S>0$ a.s. Let $Y_1,\dots,Y_{d-1}$ be a family of brackets of the type $X_1,\dots,X_m$, $[X_0,X_i]\dots$ in the distribution T' which are free at x and generate T'_x. Let H be a C^∞ $d-1$ differential form on R^d, such that at x, we have $H(Y_1,\dots,Y_{d-1})=\varepsilon>0$. S' is the stopping time

$$S'=\inf\{t\geqq 0;H(Y_1,\dots,Y_{d-1})(\varphi_t(\omega,x))=\varepsilon/2\}\wedge S. \tag{5.41}$$

Clearly, for $t \leqq S'$, the $Y_i(\varphi_t(\omega,x))$ are free. Since $T'_{\varphi_t(\omega,x)}$ is of dimension $d-1$, they still generate $T'_{\varphi_t(\omega,x)}$. It follows that any bracket $[X_j, Y_k](\varphi_t(\omega,x))$, $[X_j,[X_j,Y_k]](\varphi_t(\omega,x))$ may be written

$$\begin{aligned}[X_j, Y_k](\varphi_t(\omega,x)) &= \sum c^n_{jk}(t)\, Y_n(\varphi_t(\omega,x))\\ [X_j,[X_j,Y_k]](\varphi_t(\omega,x)) &= \sum d^n_{jk}(t)\, Y_n(\varphi_t(\omega,x))\end{aligned} \tag{5.42}$$

where c^n_{jk}, d^n_{jk} are continuous adapted processes. This is true in particular because

$$\begin{aligned}&[X_j, Y_k] \in T' \qquad (0 \leqq j \leqq m)\\ &[X_j,[X_j,Y_k]] \in T' \quad (0 \leqq j \leqq m).\end{aligned} \tag{5.43}$$

We know that:

$$\varphi_t^{*-1} Y_k(x) = Y_k(x) + \int_0^t \varphi_s^{*-1}([X_0, Y_k] + \tfrac{1}{2}[X_j,[X_j,Y_k]])\, ds + \int_0^t \varphi_s^{*-1}[X_j, Y_k]\, \delta w^j. \tag{5.44}$$

Using (5.42), we get:

$$\varphi_t^{*-1} Y_k(x) = Y_k(x) + \int_0^t \left(c^n_{0k} + \frac{d^n_{jk}}{2}\right) \varphi_s^{*-1} Y_n\, ds + \int_0^t c^n_{jk}\, \varphi_s^{*-1} Y_n\, \delta w^j. \tag{5.45}$$

Since $T'_x \neq T_x(R^d)$, there is $p \in T_x^*(R^d) \neq 0$ such that $\langle p, Y_k(x)\rangle = 0\ (1 \leqq k \leqq d-1)$. If r_k is the process defined by: $r_k(t) = \langle p, \varphi_t^{*-1} Y_k(x)\rangle$, then the r_k verify the system

$$r_k(t) = \int_0^t \left(c^n_{0k} + \frac{d^n_{jk}}{2}\right) r_n\, ds + \int_0^t c^n_{jk}\, r_n\, \delta w^j. \tag{5.46}$$

Clearly (5.46) has a unique solution which is $r_k = 0$. This shows that for $t \leqq S'$, $\varphi_t^{*-1} T'_{\varphi_t(\omega,x)} = T'_x$. Equivalently this means exactly that $T_{(\omega,x)} \geqq S'$. Since S' is a.s. >0, we have $T_{(\omega,x)} > 0$ a.s., i.e. $x \notin F$.

Moreover, if $x \in A$, T'_x is full and $T_{(\omega,x)} = 0$ a.s., i.e. $x \in F$. This shows that the fine closure $\tilde{A}^f$ of A is also included in F. We have previously seen that $F \subset \tilde{A}^f$, and so $F = \tilde{A}^f$.

From Theorem 5.4, we know that $T_{(\omega,x)}$ is the hitting time of F. Since F is the fine closure of A, the first equality in (5.39) is proved. Since on A, T' is d-dimensional, it is clear that $T_{(\omega,x)}$ is $\geqq \inf\{t \geqq 0;\ X_0(x) \in \varphi_t^{*-1}(\omega,x)\, T'_{\varphi_t(\omega,x)}\}$. If $x \in A$, equality clearly holds. Moreover if $x \notin A$, since $X_0(x) \notin T'_x$, using the last equality in (5.28), it is easy to see that $T_{(\omega,x)} \leqq \inf\{t \geqq 0;\ X_0(x) \in \varphi_t^{*-1}(\omega,x)\, T'_{\varphi_t(\omega,x)}\}$. The last equality in (5.39) is proved. □

Remark 1. The space R^d has been divided in two regions:

a) If $x \in \tilde{A}^f$, then for every t, $P_t(x,.)$ has a density. This may be seen by the techniques of Sect. 4.

b) If $x \notin \tilde{A}^f$, then a control of the differentials of $P_t(x,.)$ may be obtained only in the directions of T'. There is however a tricky bifurcation problem when $\varphi_t(\omega,x)$ goes from ${}^c\bar{A}^f$ to $\bar{A}^f$, which makes the inversion of $C_t(\omega,x)$ difficult, just after reaching $\bar{A}^f$ since the norm of the inverse may be arbitrary.

A natural idea would then to obtain a control in the X_0 direction by using Ito's formula, which allows us to express the action of X_0 by means of the action of the $X_i\,(1\leqq i\leqq m)$ up to order 2. In subsections 5.2 and 5.3, we will use a much cleaner type of argument to overcome the bifurcation problem.

5.2. The Semi-Group P_t in the Degenerate Case

If x is the topological interior of ${}^cA$, we know [18] that since T' is an involutive distribution – i.e. stable by Lie brackets – which is locally of dimension $d-1$, T' possesses integral submanifolds of dimension $d-1$. To simplify the analysis, we will make in this subsection, besides assumption H_2, the following assumptions:

H_3: A is empty, i.e. for any $x\in R^d$, T'_x, is of dimension $d-1$ (i.e. $X_0\notin T'$).

This assumption is assumption (P) in Ishihara and Kunita [20].

X_0 will then be transverse to T' at every point.

Note that by the Frobenius theorem in its general form [18], I.2, the involutive distribution T' determines a foliation of R^d by connected disjoints maximal leaves L_α which are integral submanifolds of T', i.e.

a) R^d is the union of the L_α. For each $x\in L_\alpha$, there exists a system of coordinates $(u, y^1 \dots y^{d-1})$ such that locally the foliation is given by $\{u=cst\}$.

b) For $x\in L_\alpha$, considering L_α as an immersed submanifold, $T_x(L_\alpha)$ is exactly T'_x.

Clearly if $x\in R^d$, if $Y_1,\dots,Y_{d-1}$ are a family of brackets in T' which generate T'_x, an adequate system of coordinates describing the foliation at x is given by the local diffeomorphism

$$(u,y)\in R\times R^{d-1}\to(\exp u X_0)\circ\exp\left(\sum_1^{d-1} y^i Y_i\right). \tag{5.47}$$

However note that the L_α are generally not embedded as submanifolds of R^d, i.e. the natural topology on the leaves L_α is stronger than the induced topology (see [18]).

For $x\in R^d$, L_x is the unique leaf containing x. We consider the differential equation:

$$\begin{aligned} dy&=X_0(y)\,dt\\ y(0)&=y \end{aligned} \tag{5.48}$$

and the associated flow of diffeomorphisms of $R^d\,\psi_t$: $y\to y_t$. We then have the following elementary result:

Proposition 5.6. *For each $t\geqq 0$, the foliation L_α is stable by ψ_t, i.e. the image of a leaf by ψ_t is still a leaf. In particular for each $t\geqq 0$, $x\in R^d$*

$$\psi_t^{*-1}\,T'_{\psi_t(x)}=T'_x. \tag{5.49}$$

Proof. Let $Y_1, \ldots, Y_{d-1}$ be a family of brackets in T' which generate T'_x. They are necessarily free at x, and then free on a neighborhood, i.e. they still generate T' on a neighborhood. We have:

$$\psi_t^{*-1} Y_i(x) = Y_i(x) + \int_0^t \psi_s^{*-1} [X_0, Y_i](x)\, ds. \tag{5.50}$$

Since $[X_0, Y_i] \in T'$, for t small enough, we may write

$$[X_0, Y_i](\psi_t(x)) = \sum c_{it}^j Y_j(\psi_t(x)) \tag{5.51}$$

where c_i^j are continuous functions. Let $p \in (T'_x)^\perp$, $p \neq 0$. Then

$$\langle p, \psi_t^{*-1} Y_i(x)\rangle = \int_0^t c_{is}^j \langle p, \psi_s^{*-1} Y_j(x)\rangle\, ds. \tag{5.52}$$

From (5.52), we see that for t small enough $\langle p, \psi_t^{*-1} Y_i(x)\rangle = 0$, i.e. $T'_x = \psi_t^{*-1} T'_{\psi_t(x)}$. Define S by

$$S = \inf\{t \geqq 0, \psi_t^{*-1} T'_{\psi_t(x)} \neq T'_x\}. \tag{5.53}$$

Clearly, if $S < +\infty$, $\psi_S^{*-1} T'_{\psi_S(x)} = T'_x$. Now reasoning as in the proof of Theorem 5.4 and using the semi-group property of ψ_t, we see that if $t - S > 0$, for $t - S$ small enough, we still have $\psi_t^{*-1} T'_{\psi_t(x)} = T'_x$, which contradicts the definition of S. Hence $S = +\infty$.

Take $x \in R^d$ and $x' \in L_x$. Since L_x is connected, there is a C^∞ curve $s \in [0,1] \to x_s \in L_x$ such that

$$x_0 = x, \qquad x_1 = x'. \tag{5.54}$$

Clearly $\frac{dx}{ds} \in T'_{x_s}$, which implies that $\psi_t^* \frac{dx}{ds} \in T'_{\psi_t(x_s)} \cdot s \to \psi_t(x_s)$ is then an integral curve of the distribution T'. Necessarily it belongs to the same leaf which contains $\psi_t(x)$ and $\psi_t(x')$. The Theorem is proved. □

Note that at least locally we may take X_0 to be the vector $\frac{\partial}{\partial u}$ and T' to be the distribution

$$T' = \left\{\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^{d-1}}\right\} \tag{5.55}$$

$X_1, \ldots, X_m$ may be written

$$X_i = c_i^j(u, y) \frac{\partial}{\partial y^j} \qquad (1 \leqq i \leqq m). \tag{5.56}$$

The operator $\mathscr{L}$ has then the form

$$\mathscr{L} = \frac{\partial}{\partial u} + \frac{1}{2} \sum_1^m X_i^2. \tag{5.57}$$

We will now construct explicitly the solution of (1.1). When t is bounded by a constant T, the vector fields $\psi_t^{*-1} X_i(x)$ are clearly bounded with bounded differentials in (t, x) of any order. We then have the key result:

Theorem 5.7. *Let us consider the stochastic differential equation*

$$\begin{aligned} dz &= (\psi_t^{*-1} X_i)(z_t) \cdot dw^i \\ z(0) &= z \end{aligned} \tag{5.58}$$

and the associated flow τ_t: $z \to \tau_t(\omega, z) = z_t$. *Then a.s., for any* $z \in R^d$, $t \geqq 0$, $\tau_t(\omega, z) \in L_z$. *Moreover, a.s., if* L_α *is a leaf endowed with its natural topology, the mapping* $(t, z) \in R^+ \times L_\alpha \to \tau_t(\omega, z) \in L_\alpha$ *is continuous;* $\tau_t(\omega, .)$ *is a family of diffeomorphisms of* L_α *depending continuously on* $t \in R^+$ *(for the topology of uniform convergence over the compact sets of* L_α*).*

Proof. First note that the fact that (5.56) is not time homogeneous is no obstacle to the application of the results of Sect. 1, since the system can be made homogeneous by adding a new variable. By [3, 4, 10], we know that considered as a mapping from $\Omega \times R^+ \times R^d$ into R^d, the flow $\tau_.(\omega, .)$ is the uniform limit in probability over the compact sets in $R^+ \times R^d$ of the flows $\tau_.^n(\omega, .)$ associated to the differential equations

$$\begin{aligned} dz^n &= (\psi_t^{*-1} X_i)(z^n)\, w^{i,n}\, dt \\ z^n(0) &= z \end{aligned} \tag{5.59}$$

where

$$w^{i,n} = 2^n [w^i(([2^n t] + 1)/2^n) - w^i([2^n t]/2^n)].$$

Since z^n given by (5.59) is an integral curve of the distribution T', it is clear that for any $t \in R^+$, $z_t^n \in L_z$. Since L_z is not necessarily closed in R^d, we cannot immediately conclude that $\tau_t(\omega, z) \in L_z$.

Let n_k be a subsequence of N so that the convergence previously defined takes place a.s., i.e. out of a negligible set $\mathcal{N}$. By Proposition IX-2 in [23], the leaves L_α are second countable (for their own topology). Fix $(t, z) \in R^+ \times R^d$ and $\omega \notin \mathcal{N}$. There is an open neighborhood V of $\tau_t(\omega, z)$ in R^d and a coordinate system $(u, y^1 \dots y^{d-1})$ on V such that the foliation in V is given by the slices $(u = cst)$ (two different slices may belong to the same leaf!).

Since the leaves L_α are second countable, by [23] IX, we know that the connected components for the topology of R^d of $L_\alpha \cap V$ are given by slices $(u = cst)$, at least if $L_\alpha \cap V \neq \emptyset$. For $|t' - t|$ small enough, and n_k large enough, we know that $\tau_{t'}^{n_k}(\omega, z) \in V$. Using the continuity of $t' \to \tau_{t'}^{n_k}(\omega, z) \in R^d$ and connexity, it is clear that for $|t' - t|$ small enough, $\tau_{t'}^{n_k}(\omega, z)$ remains in a slice (which may depend on n_k). It is then clear that for $|t' - t|$ small enough, $\tau_{t'}(\omega, z)$ remains in a slice, and then in a given leaf of the foliation. Now take $T > 0$ and cover $[0, T]$ with a finite number of intervals such that on each interval, $\tau_t(\omega, z)$ remains in a given leaf L_α. Since $\tau_0(\omega, z) = z$, for any $t \in [0, T]$, $\tau_t(\omega, z)$ remains in the same leaf L_z. Moreover $(t, z) \in R^+ \times L_\alpha \to \tau_t(\omega, z) \in R^d$ is clearly continuous. Using Proposition IX-1 in [23], it follows that $(t, z) \in R^+ \times L_\alpha \to \tau_t(\omega, z) \in L_\alpha$ is continuous. The same argument applies to the differentials of $\tau_t(\omega, .)$. □

We now have the factorization result, which is essential in this section.

Theorem 5.8. *A.s., for any* $(t,x)\in R^+\times R^d$, *we have*

$$\varphi_t(\omega,x)=\psi_t(\tau_t(\omega,x)). \tag{5.60}$$

Proof. From the Ito-Stratonovitch formula, we know that for every $x\in R^d$, then

$$\psi_t\circ\tau_t(\omega,x)=x+\int_0^t X_0(\psi_s\circ\tau_s(\omega,x))\,ds+\int_0^t X_i(\psi_s\circ\tau_s(\omega,x))\cdot dw^i \tag{5.61}$$

i.e. $\psi_t\circ\tau_t(x)$ is a solution of (1.1), which is equal to $\varphi_t(\omega,x)$. Since $\psi_t\circ\tau_t(\omega,x)$ is obviously jointly continuous in (t,x), the theorem follows. □

The diffusion (1.1) has then been factored as the "product" of a diffusion in a fixed leaf L_x and a deterministic motion in the direction X_0. This result is an improvement of the result of Ichihara and Kunita [20], which is essentially of a local nature.

For a given leaf L_x, we will consider τ_t as a flow on this leaf, and T' to be its tangent bundle.

We have now the key result to study $P_t(x,.)$.

Theorem 5.9. *For every* $x\in R^d$, *a.s., for every* $T>0$, *the quadratic form on* $T_x^*(L_x)$

$$\int_0^T \tau_t^{*-1}\psi_t^{*-1}X_i(x)\rangle\langle\tau_t^{*-1}\psi_t^{*-1}X_i(x)\,dt \tag{5.62}$$

is positive definite.

Proof. We will do the same reasoning as in the proof of Proposition 4.1, with the reservation that $\psi_t^{*-1}X_i$ is depending explicitly on t. For $f\in T_x^*(L_x)$ to be in the kernel of (5.62), it is necessary and sufficient that

$$\langle f,\tau_t^{*-1}\psi_t^{*-1}X_i(x)\rangle=0,\qquad t\leqq T. \tag{5.63}$$

Now using (5.50), we have:

$$\begin{aligned}\tau_t^{*-1}\psi_t^{*-1}X_i(x)&=X_i(x)+\int_0^t\tau_s^{*-1}\psi_s^{*-1}[X_0,X_i](x)\,ds\\&\quad+\int_0^t\tau_s^{*-1}\psi_s^{*-1}[X_j,X_i](x)\,dw^j\\&=X_i(x)+\int_0^t\tau_s^{*-1}\psi_s^{*-1}([X_0,X_i]+\tfrac{1}{2}[X_j,[X_j,X_i]])(x)\,ds\\&\quad+\int_0^t\tau_s^{*-1}\psi_s^{*-1}[X_j,X_i]\,\delta w^j.\end{aligned} \tag{5.64}$$

Reasoning as in the proofs of Proposition 4.1 and of Theorem 5.2, we see that:

$$\langle f,\tau_s^{*-1}\psi_s^{*-1}[X_j,X_i]\rangle=0,\qquad 0\leqq j\leqq m,\ 1\leqq j\leqq m,\ s\leqq t\ \text{a.s.} \tag{5.65}$$

Iterating (5.65), we get in particular that f is orthogonal to all the brackets $X_{[I]}$ in T_x', which implies f is zero in $T_x^*(L_x)$. □

Remark 2. The key fact is that X_0 has appeared only through the derivation in t of the time dependent vector fields $\psi_t^{*-1} X_i$.

From Theorem 5.9 and Remark 4.1, we get immediately:

Theorem 5.10. *For each $x \in L_x$, the probability law of $\tau_t(\omega, x)$ is given by a density on the leaf L_x.*

From Theorem 5.8 and Theorem 5.10, we get:

Theorem 5.11. *For $x \in R^d$, the probability law $P_t(x,.)$ of $\varphi_t(\omega, x)$ is given by a density on the leaf $L_{\psi_t(x)}$.*

Proof. ψ_t is a diffeomorphism of the leaf L_x on the leaf $L_{\psi_t(x)}$. From this, we get the Theorem. □

Theorem 5.12. *For any $x \in R^d$, $T>0$, the measure defined by*

$$f \to E \int_0^T f(\varphi_t(\omega, x))\, dt \tag{5.66}$$

has a density.

Proof. Given a leaf L_x, since X_0 is transversal to T', the mapping

$$(t, y) \in R \times L_x \to \psi_t(y) \in R^d \tag{5.67}$$

is non singular. Let $\tilde{P}_t(x,.)$ be the probability law of $\tau_t(\omega, x)$ on the leaf L_x. Since (5.66) is the image of the measure $1_{t \leqq T}\, dt \otimes d\tilde{P}_t(x,.)$ by the mapping (5.67), it has a density. □

5.3. *The General Case*

We will now attack the general case where only H_2 is verified. We still want to prove that the statement in Theorem 5.12 is true. If $x \in A$, then clearly $P_t(x,.)$ has a density and the result is trivial.

We will then assume that $x \notin A$ i.e. T'_x is not full. By a result of Rothschild and Stein [17] (see also Hörmander and Melin [16]), we know that if the brackets of $X_0, \ldots, X_m$ of length $\leqq r$ span $T_x(R^d)$, there exists p and C^∞ vector fields $\tilde{X}_0, \ldots, \tilde{X}_m$ on R^{d+p} such that:

a) If π is the canonical projection of R^{d+p} on R^d, then $\pi^* \tilde{X}_i = X_i$.

b) The brackets $\tilde{X}_{[I]}$ of $\tilde{X}_0, \ldots, \tilde{X}_m$ of length $\leqq r$ (i.e. $|I| \leqq r$) span $T_{(x,0)}(R^{d+p})$ and moreover they are free of order r at $(x,0)$, i.e. the only algebraic dependences between the $\tilde{X}_{[I]}(x,0)$ $(|I| \leqq r)$ are the algebraic relations which exist in a general abstract Lie algebra, like the Jacobi identities.

We may of course assume that $\tilde{X}_0, \ldots, \tilde{X}_m$ are still bounded with bounded differentials of any order.

Since T'_x is not full, $\tilde{T}'_{(x,0)}$ is not full. We now claim

Proposition 5.13. *There exists a uniformly positive bounded C^∞ function u on R^{d+p} such that if $\tilde{T}'^u$ is the distribution associated to the vector fields $(u\tilde{X}_0, \tilde{X}_1, \ldots, \tilde{X}_m)$, $\tilde{T}'^u$ is full at $(x,0)$.*

Proof. From Proposition 2 in [16], we know there exists $u\,C^\infty$ bounded such that if, for a sequence of indices $I=(i_1,\dots,i_n)$, $\tilde{X}_I$ is the differential operator $\tilde{X}_{i_1}\dots\tilde{X}_{i_n}$, then:

$$\begin{aligned}&(\tilde{X}_I u)(x,0)=0, \qquad |I|\leqq r-1\\ &(\tilde{X}_1^r u)(x,0)=1.\end{aligned} \tag{5.68}$$

By adding a constant, we may assume that u is uniformly positive. For any I, it may be easily checked that if $\tilde{X}^u_{[I]}$ are the brackets corresponding to $(u\tilde{X}_0,\tilde{X}_1,\dots,\tilde{X}_m)$, then:

$$\tilde{X}^u_{[I]}=u^{\alpha_I}\tilde{X}_{[I]}+\sum_{|J|<|I|}\beta^u_{JI}\tilde{X}_{[J]} \tag{5.69}$$

where the β^u_{JI} are linear combination of functions containing $\tilde{X}_K u$ for $|K|<I$. Since $u(x,0)$ is non zero, from (5.68), we see that for $|I|\leqq r$, the $\tilde{X}^u_{[I]}(x,0)$ are non zero multiples of the $\tilde{X}_{[I]}(x,0)$. We now compute explicitly the bracket $[\tilde{X}_1,[\tilde{X}_1,\dots,[\tilde{X}_1,u\tilde{X}_0]]]$ where $\tilde{X}_1$ is repeated r times. A trivial computation shows it is equal to

$$\sum_0^r C_r^j(\tilde{X}_1^j u)\,\tilde{Y}_{r-j} \tag{5.70}$$

where $\tilde{Y}_{r-j}$ is the bracket calculated with $u=1$ which contains $r-j\tilde{X}_1$. Using (5.68), we get that at $(x,0)$, it is equal to $u\tilde{Y}_r(x,0)+\tilde{X}_0(x,0)$. Now since the vectors $\tilde{X}_0,\dots,\tilde{X}_m$ are free of order r at $(x,0)$, $\tilde{X}_0(x,0)$ does not belong to the vector space spanned by the other brackets $\tilde{X}_{[I]}(x,0)$ with $|I|\leqq r$, $I\neq 0$, which is necessarily of codimension 1. Since $\tilde{T}^r(x,0)$ is not full, it is exactly equal to this space. In particular $\tilde{Y}_r(x,0)$ may be written as a linear combination of $\tilde{X}_{[I]}(x,0)$ with $|I|\leqq r$, $I\neq 0$. This shows in particular that $\tilde{X}_0(x,0)$ is a linear combination of $\tilde{X}^u_{[I]}(x,0)\,(|I|\leqq r+1, I\neq 0)$. $\tilde{T}'^u(x,0)$ is then necessarily full. □

We now give the result of existence of a density for the potentials of process (1.1).

Theorem 5.14. *Under H_2, for $x\in R^d$, $T\geqq 0$, the measure*

$$f\to E\int_0^T f(\varphi_t(\omega,x))\,dt$$

has a density.

Proof. We may directly assume $x\notin A$. Take $\tilde{X}_0,\dots,\tilde{X}_m$, u as in Proposition 5.13. Now $\tilde{T}'^u_{(x,0)}$ is full. Let $v\geqq 0$ be such that $v^2=u$. Let then $\tilde{\varphi}_.(\omega,.)$ be the flow in R^{d+p} associated to $(\tilde{X}_0,\dots,\tilde{X}_m)$. β_t is the change of time defined by

$$\beta_t=\inf\left\{\beta\geqq 0;\int_0^\beta\frac{ds}{v^2(\tilde{\varphi}_s(\omega,(x,0)))}=t\right\}. \tag{5.70}$$

Now if w'^i_t is the process defined by

$$w'^i_t=\int_0^{\beta_t}\frac{\delta w^i}{v(\tilde{\varphi}_s(\omega,(x,0)))} \tag{5.71}$$

$(w'^1, \dots, w'^m)$ is trivially a brownian motion. Moreover $y_t = \tilde{\varphi}_{\beta_t}(\omega, (x,0))$ is the solution of the stochastic differential equation

$$\begin{aligned} dy &= v^2(y)\left(\tilde{X}_0(y) + \frac{1}{2}\frac{\partial \tilde{X}_i}{\partial x}\tilde{X}_i(y)\right)dt + (v\tilde{X}_i)(y)\,\delta w'^i \\ y(0) &= (x,0) \end{aligned} \tag{5.72}$$

which may also be written

$$\begin{aligned} dy &= (v^2\tilde{X}_0 - \tfrac{1}{2}v(\tilde{X}_i v)\tilde{X}_i)(y)\,dt + (v\tilde{X}_i)(y)\cdot dw'^i \\ y(0) &= (x,0) \end{aligned} \tag{5.73}$$

y_t is then a new diffusion calculated with

$$\begin{aligned} X_0^* &= v^2\tilde{X}_0 - \tfrac{1}{2}v(\tilde{X}_i v)\tilde{X}_i \\ X_i^* &= v\tilde{X}_i. \end{aligned} \tag{5.74}$$

Since v is uniformly positive, it is easily checked that the distribution T'^* corresponding to $(X_0^*, \dots, X_m^*)$ is the same as the distribution $\tilde{T}'^u$ calculated for $(u\tilde{X}_0, \dots, \tilde{X}_m)$. The distribution T'^* is necessarily full at $(x,0)$. From the results of Sect. 4, we know that for any $T'>0$, the measure

$$g \to E\int_0^{T'} g(y_t)\,dt \qquad \text{has a density.} \tag{5.75}$$

Now clearly, we have:

$$E\int_0^T g(\tilde{\varphi}_t(\omega,(x,0)))\,dt = E\int_0^{+\infty} 1_{\beta_t \leqq T}\, g(y_t)\, v^2(y_t)\,dt \tag{5.76}$$

which implies that the measure defined by the L.H.S. in (5.76) has a density.

Since $\pi\tilde{\varphi}_t(\omega,(x,0)) = \varphi_t(\omega,x)$, the result is proved. □

Remark 3. The idea is of course to accelerate the process slightly so that the random variables $\varphi_{\beta_t}(\omega,x)$ have densities. Apparently, this accelaration involves the building of the extra process $\tilde{\varphi}_t(\omega,x)$.

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