Complex Immersions and Arakelov Geometry

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Contents

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Abstract

In this paper we establish an arithmetic Riemann–Roch–Grothendieck Theorem for immersions. Our final formula involves the Bott–Chern currents attached to certain holomorphic complexes of Hermitian vector bundles, which were previously introduced by the authors. The functorial properties of such currents are studied. Explicit formulas are given for Koszul complexes.

In [BGS3] and in [GS3], the direct image by a submersion of an Hermitian vector bundle on an arithmetic variety was defined. One may wonder [Ma], [GS3] whether a Riemann–Roch–Grothendieck formula holds for such direct images, which would involve the characteristic classes defined in [GS2], with values in the arithmetic Chow groups of [GS1]. Such a formula would be stronger than the corresponding Riemann–Roch–Grothendieck theorem with values in Chow groups [SGA6] and the Riemann–Roch–Grothendieck theorem at the level of differential forms obtained in [BGS2,3] for submersions, using Quillen metrics [Q2].

When proving his Riemann–Roch theorem with values in Chow groups, Grothendieck used the factorization of any projective map between smooth manifolds as a regular closed immersion followed by the projection attached to a projective bundle [SGA6]. The main step of the proof is then to study the direct image by a regular immersion by blowing up the subvariety. It seems that the same reduction to the immersion case will be necessary for proving the arithmetic Riemann–Roch–Grothendieck theorem conjecture of [Ma] and [GS3].

The purpose of this paper is to prove an arithmetic Riemann–Roch–Grothendieck theorem for immersions, in which our datas are:

- an immersion $i: Y \to X$ of arithmetic varieties;
- a vector bundle η on Y, equipped with an Hermitian metric g^η;
- a resolution of the sheaf $i_*\eta$ by a complex of vector bundles (ξ, v) on X, equipped with an Hermitian metric h^ξ.

Our main result in Theorem 4.13 calculates the arithmetic Chern character of ξ [GS2] in terms of the Chern character of η, of the Todd genus of the normal bundle N of Y in X which is equipped with a metric g^N, and of a secondary invariant $T(h^\xi)$ attached to the Hermitian chain complex (ξ, v) introduced in Bismut-Gillet-Soulé [BGS4] under the name of a Bott–Chern singular current. In fact the current $T(h^\xi)$ solves the equations of currents on X

$$\frac{\bar\partial\partial}{2i\pi} T(h^\xi) = Td^{-1}(N, g^N) ch(\eta, g^\eta)\delta_Y - ch(\xi, h^\xi) \tag{0.1}$$

where in (0.1), the various characteristic classes are calculated by Chern–Weil theory using the holomorphic Hermitian connections associated with the corresponding metrics. The construction of the current $T(h^{\xi})$ given in [BGS4] uses in an essential way Quillen's superconnections [Q1], and a result proved in Bismut [B2, Theorem 3.2] on the large parameter behaviour of Quillen's superconnection Chern character forms. When $Y = \emptyset$, i.e. when the complex (ξ, v) is acyclic, the currents $T(h^{\xi})$ have already been introduced in Bismut–Gillet–Soulé [BGS1] and were shown to coincide with objects considered earlier by Bott–Chern [BoC] and Donaldson [D].

An important intermediary result, proved in Theorem 2.7, concerns the behaviour of the currents $T(h^{\xi})$ under composition of immersions. In fact any result of the type of the Riemann-Roch-Grothendieck theorem on direct images implies that the considered objects behave functorially with respect to the composition of maps. Here the functorial behaviour of the currents $T(h^{\xi})$ under composition of immersions is one of the key tools by which we prove our Riemann–Roch–Grothendieck theorem for immersions.

A second key instrument is the explicit computation of the currents $T(h^{\xi})$ associated with Koszul complexes. This computation makes use of the formalism of Mathai–Quillen [MQ] in a complex setting. We also introduce Euler–Green currents in the sense of [GS1], and we relate them to the currents $T(h^{\xi})$ of Koszul complexes. Such Euler–Green currents are obtained by a double transgression formula from the Mathai–Quillen–Thom forms [MQ].

Let us here point out that Bott–Chern singular currents can be constructed in an enormous variety of ways, and that much of our work consists in showing that two Bott–Chern currents which solve equation (0.1) differ by ∂ or $\bar{\partial}$ coboundaries, i.e. they represent the same class in Bott–Chern theory.

A third instrument is the deformation to the normal cone described in Baum–Fulton–MacPherson [BaFM], by which we show that our currents $T(h^{\xi})$ are related to the arithmetic characteristic classes of Gillet–Soulé [GS1].

A common technical feature of our proofs is that we use the properties of the wave front set of the considered currents. In fact it was shown in [BGS4], by using microlocal estimates of [B2], that the wave front set of the current $T(h^{\xi})$ is included in the conormal bundle to Y in X. By using standard arguments in [H, Chapter 8], we can in particular multiply two currents $T(h^{\xi})$ associated with transversal submanifolds in X.

Our paper is organized as follows. In Section 1, we recall the main results of [B2] and [BGS4] concerning the Quillen's Chern character currents and the Bott–Chern singular currents. In Section 2, we study various functorial properties of our Bott–Chern singular currents, including a transitivity property under composition of immersions.

In Section 3, we calculate the Bott–Chern singular current associated with a Koszul complex, and we compare this current with the Euler–Green current, which is one component of the arithmetic Euler class defined in [GS2]. Notice that this is one of the very few cases where an explicit formula can be given for a Green current. The transitivity property established in Section 2 is used here to check the multiplicativity of the Euler–Green current of an orthogonal direct sum. Finally, in Section 4, we study the deformation to the normal cone (a variant of Grothendieck's blowing-up introduced in [BaFM]) to prove our Riemann–Roch–Grothendieck theorem for immersions. Notice however that our final formulas given in Theorem 4.13 does not lie in the arithmetic Chow group of the ambiant variety, but rather is integrated down to a base on which both the variety and the subvariety project smoothly (we do not define direct images for immersions in arithmetic Chow groups and arithmetic Grothendieck groups).

The results obtained in this paper were announced in [BGS5].

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1. A Bott–Chern Singular Current

Let $i: M' \to M$ be an immersion of complex manifolds. Let η be a holomorphic vector bundle on M', let (ξ, v) be a holomorphic chain complex of vector bundles on M which is such that there is an exact sequence of sheaves

$$\mathcal{O}_M(\xi) \to i_* \mathcal{O}_{M'}(\eta) \to 0.$$

We assume that the vector bundles in ξ are equipped with Hermitian metrics.

In this section, we recall the main results of Bismut [B2] and Bismut–Gillet–Soulé [BGS4] which concern:

- The asymptotic behavior of the Quillen superconnection forms naturally associated with the Hermitian complex (ξ, v) [B2].
- The corresponding construction of singular Bott–Chern currents [BGS4].

This section is organized as follows. In a), we give our main assumptions and notations. In b), we introduce assumption (A) for the metrics on the complex (ξ, v). In c), we briefly recall elementary properties of wave front sets. In d), we review Quillen's superconnections [Q1]. In e), we recall the double transgression formulas of [BGS1]. In f), we state the results of convergence of superconnection currents which were proved in [B2]. In g), we recall our construction of Bott–Chern singular currents [BGS4].

(a) *A holomorphic chain complex.* Let M be a compact connected complex manifold of complex dimension ℓ. Let $M' = \bigcup_1^n M'_j$ be a finite union of compact connected complex submanifolds of M, such that, for $j \neq j'$, $M'_j \cap M'_{j'} = \emptyset$. Let i be the embedding $M' \to M$. Let N be the normal vector bundle to M' in M, and let N^* be its dual.

Let

$$(1.1) \qquad (\xi, v): 0 \to \xi_m \to_v \xi_{m-1} \cdots \to_v \xi_0 \to 0$$

be a holomorphic chain complex of vector bundles on M.

Let η be a holomorphic vector bundle on M'. We assume there is a holomorphic restriction map $r: \xi_{0|M'} \to \eta$ which is such that we have an exact sequence of sheaves

$$(1.2) \quad 0 \to \mathcal{O}_M(\xi_m) \to_v \mathcal{O}_M(\xi_{m-1}) \to \cdots \to_v \mathcal{O}_M(\xi_0) \to_r i_* \mathcal{O}_{M'}(\eta) \to 0.$$

In particular the complex (ξ, v) is acyclic on $M \setminus M'$.

For $x \in M'$, $0 \leq k \leq m$, let $F_{k,x}$ be the kth homology group of the complex $(\xi, v)_x$. Set $F_x = \bigoplus_0^m F_{k,x}$.

The following results are consequences of the local uniqueness of resolutions (see Serre [S, IV Appendix 1] and Eilenberg [E, Theorem 8]) and are proved in [B2, Section 1]:

- For $k = 0, \ldots, m$, $x \in M'$, the dimension of $F_{k,x}$ is constant on each M'_j, so that F_k is a holomorphic vector bundle on M'.
- For $x \in M'$, $U \in T_x M$, let $\partial_U v(x)$ be the derivative of the chain map v calculated in any given local holomorphic trivialization of (ξ, v) near x. Then $\partial_U v(x)$ acts on F_x. When acting on F_x, $\partial_U v(x)$ only depends on the image y of U in N_x. So we now write $\partial_y v(x)$ instead of $\partial_U v(x)$.
- For any $x \in M'$, $y \in N$, $(\partial_y v)^2(x) = 0$. If $y \in N$, let i_y be the interior multiplication operator by y acting on the exterior algebra $\Lambda(N^*)$. Let i_y act like $i_y \otimes 1$ on $\Lambda N^* \otimes \eta$. Then the graded holomorphic complex $(F, \partial_y v)$ on the total space of the vector bundle N is canonically isomorphic to the Koszul complex $(\Lambda N^* \otimes \eta, i_y)$.

(b) *Assumption (A) on the Hermitian metrics of a chain complex.* We now assume that $\xi_0, \ldots, \xi_m$ are equipped with smooth Hermitian metrics $h^{\xi_0}, \ldots, h^{\xi_m}$. We equip $\xi = \bigoplus_{k=0}^m \xi_k$ with the metric h^ξ which is the orthogonal sum of the metrics $h^{\xi_0}, \ldots, h^{\xi_m}$. Let v^* be the adjoint of v with respect to the metric h^ξ. Using finite dimensional Hodge theory, we get the identification of smooth vector bundles on M' for $0 \leq k \leq m$

$$(1.3) \qquad F_k \cong \{ f \in \xi_k; vf = 0, v^* f = 0 \}.$$

As a smooth subvector bundle of ξ_k, the right hand side of (1.3) inherits a Hermitian metric from the metric h^{ξ_k}. Using the identification (1.3), we find that for every $k = 0, \ldots, m$, F_k is a holomorphic Hermitian vector bundle on M'. Let h^{F_k} denote the Hermitian metric on F_k. We equip $F = \bigoplus_0^m F_k$ with the metric h^F which is the orthogonal sum of the metrics $h^{F_0}, \ldots, h^{F_m}$.

Let g^N, g^η be Hermitian metrics on the vector bundles N, η. We equip the vector bundle $\Lambda N^* \otimes \eta$ with the tensor product of the metric induced by g^N on $\Lambda(N^*)$ and of the metric g^η.

Definition 1.1. Given metrics g^N, g^η on N, η, we will say that the metrics $h^{\xi_0}, \ldots, h^{\xi_m}$ on $\xi_0, \ldots, \xi_m$ verify assumption (A) with respect to g^N, g^η if the canonical identification of holomorphic chain complexes on the total space of N

$$(F, \partial_y v) \cong (\Lambda N^* \otimes \eta, i_y) \tag{1.4}$$

also identifies the metrics.

Proposition 1.2. *Given metrics g^N, g^η, on N, η, there exist metrics $h^{\xi_0}, \ldots, h^{\xi_m}$ on $\xi_0, \ldots, \xi_m$ which verify assumption* (A) *with respect to g^N, g^η.*

Proof. This result is proved in [B2, Proposition 1.6]. □.

(c) *Wave front sets.* If γ is a current on M, we note $WF(\gamma)$ the wave front set of γ. For the definition and properties of wave front sets, we refer to Hörmander [H, Chapter VIII]. Let us just recall that $WF(\gamma)$ is a closed conic subset of $T^*_R M \setminus \{0\}$. Also if p is the projection $T^*_R M \to M$, $p(WF(\gamma))$ is exactly the singular support of γ, whose complement in M is the set of points x such that γ is C^∞ on a neighborhood of x.

Let $\mathcal{D}'_{N^*_R}$ be the set of currents γ on M which are such that $WF(\gamma) \subset N^*_R$. In particular currents in $\mathcal{D}'_{N^*_R}$ are smooth on $M \backslash M'$. By [H, Definition 8.2.2], $\mathcal{D}'_{N^*_R}$ has a natural topology which we now describe.

Let U be a small open set in M, which we identify with an open ball in $R^{2\ell}$. Over U, we identify $T^*_R M$ with $U \times R^{2\ell}$. Let Γ be a closed conic set in $R^{2\ell}$ such that if $x \in U$, $\Gamma \cap N^*_{R,x} = \emptyset$. Let φ be a smooth current on $R^{2\ell}$ with compact support included in U and let m be an integer. If γ is a current, let $\widehat{\varphi\gamma}(\xi)$ be the Fourier transform of $\varphi\gamma$ (which is here considered as a current on $R^{2\ell}$). If $\gamma \in \mathcal{D}'_{N^*_R}$, set

$$p_{U,\Gamma,\varphi,m}(\gamma) = \sup_{\xi \in \Gamma} |\xi|^m |\widehat{\varphi\gamma}(\xi)|. \tag{1.5}$$

If γ_n is a sequence of currents on $\mathcal{D}'_{N^*_R}$, we say that γ_n converges to $\gamma \in \mathcal{D}'_{N^*_R}$ if

- $\gamma_n \to \gamma$ in the sense of distributions.
- If U, Γ, φ, m are taken as before

$$p_{U,\Gamma,\varphi,m}(\gamma_n - \gamma) \to 0. \tag{1.6}$$

Definition 1.3. $P^M_{M'}$ denotes the vector space of currents ω on M which have the following two properties:

- ω is a sum of currents of type (p,p).
- The wave front set of ω is included in N^*_R.

$P^{M,0}_{M'}$ is the vector space of current $\omega \in P^M_{M'}$ which are such that there exist currents $\alpha, \beta \in \mathcal{D}'_{N^*_R}$ for which $\omega = \partial\alpha + \bar{\partial}\beta$.

We equip $P^M_{M'}$ with the topology induced by $\mathcal{D}'_{N^*_R}(M)$.

If $M' = \emptyset$, we will write P^M, $P^{M,0}$ instead of $P^M_{M'}$, $P^{M,0}_{M'}$.

(d) *Quillen's superconnections.* We now assume that $\xi_0, \ldots, \xi_m$ are equipped with Hermitian metrics $h^{\xi_0}, \ldots, h^{\xi_m}$. We otherwise use the notations of Section 1b).

Set

$$\xi_+ = \bigoplus_{k \text{ even}} \xi_k \qquad \xi_- = \bigoplus_{k \text{ odd}} \xi_k. \tag{1.7}$$

Then $\xi = \xi_+ \oplus \xi_-$ is a Z_2-graded Hermitian vector bundle. $\operatorname{End}\xi$ is naturally Z_2-graded, the even (resp. odd) elements in $\operatorname{End}\xi$ commuting (resp. anticommuting) with the operator $\tau = \pm 1$ on $\xi_\pm$ which defines the Z_2-grading.

For $0 \le k \le m$, let ∇^{ξ_k} be the holomorphic Hermitian connection on ξ_k. Then $\nabla^\xi = \bigoplus_0^m \nabla^{\xi_k}$ is the holomorphic Hermitian connection on the vector bundle ξ.

We now briefly recall the definition of a superconnection in the sense of Quillen [Q1]. The algebra $\Lambda(T^*_R M) \hat{\otimes} \operatorname{End}\xi$ is naturally Z_2-graded. Let S be a smooth section of $(\Lambda(T^*_R M) \hat{\otimes} \operatorname{End}\xi)^{\text{odd}}$. Then by definition $\nabla^\xi + S$ is a superconnection on the Z_2-graded vector bundle ξ.

In the sequel ∇^ξ will be considered as a first order differential operator acting on the set of smooth sections of $\Lambda(T^*_R M) \hat{\otimes} \operatorname{End}\xi$. The curvature $(\nabla^\xi + S)^2$ of the superconnection $\nabla^\xi + S$ is then a smooth section of $(\Lambda(T^*_R M) \hat{\otimes} \operatorname{End}\xi)^{\text{even}}$.

If $A \in \operatorname{End}\xi$, we define its supertrace $Tr_s[A] \in \mathbb{C}$ by

$$Tr_s[A] = Tr[\tau A]. \tag{1.8}$$

We extend Tr_s as a linear map from $\Lambda(T_R^*M) \hat{\otimes} \operatorname{End}\xi$ into $\Lambda(T_R^*M)$, with the convention that if $\omega \in \Lambda(T_R^*M)$, $A \in \operatorname{End}\xi$

$$Tr_s[\omega A] = \omega Tr_s[A].$$

If $B, B' \in \Lambda(T_R^*M) \hat{\otimes} \operatorname{End}\xi$, let $[B, B']$ be the supercommutator

$$[B, B'] = BB' - (-1)^{\deg B \deg B'} B'B.$$

Then by [Q1], Tr_s vanishes on supercommutators. Let φ be the homomorphism of $\Lambda^{\text{even}}(T_R^*M)$ into itself which to $\omega \in \Lambda^{2p}(T_R^*M)$ associates $(2\pi i)^{-p}\omega$.

Let S be an odd smooth section of $\Lambda(T_R^*M) \hat{\otimes} \operatorname{End}\xi$. The basic result of Quillen [Q1] asserts that the form $\varphi\big(Tr_s\big[\exp\big(-(\nabla^\xi + S)^2\big)\big]\big)$ is closed and represents in cohomology the Chern character of $\xi_0 - \xi_1 + \cdots + (-1)^m \xi_m$.

(e) *Double transgression formulas.* We make the same assumptions as in Section 1b). Set

$$V = v + v^*. \tag{1.9}$$

Then V is a smooth section of $\operatorname{End}^{\text{odd}}\xi$. For $u \geq 0$, let A_u be the superconnection on ξ

$$A_u = \nabla^\xi + \sqrt{u}\, V. \tag{1.10}$$

Then the curvature A_u^2 of A_u is a smooth section of $\big(\Lambda(T_R^*M) \hat{\otimes} \operatorname{End}\xi\big)^{\text{even}}$.

Let N_H be the number operator of the complex (ξ, v). Namely N_H acts on ξ_k $(0 \leq k \leq m)$ by multiplication by k. We now recall a result of [BGS1].

Theorem 1.4. *The forms $Tr_s[\exp(-A_u^2)]$ and $Tr_s[N_H \exp(-A_u^2)]$ lie in P^M, and depend smoothly on $u \geq 0$. Moreover, for $u > 0$, the following identities hold*

$$\begin{aligned} \frac{\partial}{\partial u} Tr_s\big[\exp(A_u^2)\big] &= -d\, Tr_s\left[\frac{V}{2\sqrt{u}} \exp(-A_u^2)\right] \\ Tr_s\left[\frac{V}{\sqrt{u}} \exp(-A_u^2)\right] &= (\bar{\partial} - \partial) Tr_s\left[\frac{N_H}{u} \exp(-A_u^2)\right] \end{aligned} \tag{1.11}$$

In particular

$$\frac{\partial}{\partial u} Tr_s\left[\exp(-A_u^2)\right] = \frac{1}{u}\bar{\partial}\partial\, Tr_s\left[N_H \exp(-A_u^2)\right]. \tag{1.12}$$

Proof. These formulas are proved in [BGS1, Theorem 1.15], or in [B2, Theorem 2.4]. Observe that signs have been changed with respect to [BGS1], since here v decreases the grading in ξ by 1, while in [BGS1], v increases the grading in ξ by 1. □

(f) *Convergence of superconnection currents.* Set

$$F_+ = \bigoplus_{k \text{ even}} F_k \qquad F_- = \bigoplus_{k \text{ odd}} F_k.$$

$F = F_+ \oplus F_-$ is a Hermitian Z_2-graded vector bundle. If $y \in N$, let $\bar{y}$ be the conjugate element of y in $\bar{N}$. Then $y \in N$ represents $Y = y + \bar{y} \in N_R$. In particular if N is equipped with a metric g^N, $|Y|^2 = 2|y|^2$.

The superconnection formalism of Quillen can also be applied to the Z_2-graded vector bundle $F = F_+ \oplus F_-$. Let $(\partial_y v)^*$ be the adjoint of $\partial_y v$ with respect to the metric h^F on F. Then $(\partial_y v)^*$ is a antiholomorphic function of y. Set

$$\partial_Y V = \partial_y v + (\partial_y v)^*. \tag{1.13}$$

$\partial_Y V$ is an odd section of End F. If we use the canonical identification (1.4), then

$$\partial_Y V = i_y + i_y^*. \tag{1.14}$$

For $0 \leq k \leq m$, let ∇^{F_k} be the holomorphic Hermitian connection on the vector bundle F_k. Then $\nabla^F = \bigoplus_{k=0}^{m} \nabla^{F_k}$ is the holomorphic Hermitian connection on F. Let B be the superconnection on F

$$B = \nabla^F + \partial_Y V. \tag{1.15}$$

Then B^2 is the curvature of the superconnection B. B^2 is a smooth section of $\left(\Lambda(T_R^* N) \hat{\otimes} \operatorname{End} F\right)^{\text{even}}$.

N_H acts naturally on F, i.e. if $0 \leq k \leq m$, $f \in F_k$, then

$$N_H f = kf. \tag{1.16}$$

By [Q1], for any $u \geq 0$, the form $Tr_s[\exp(-A_u^2)]$ is closed and

$$\varphi(Tr_s[\exp(-A_u^2)])$$

represents in cohomology the Chern character of $\xi_0 - \xi_1 + \cdots + (-1)^m \xi_m$.

$\delta_{M'}$ denotes the current of integration on the oriented manifold M'.

Let $C^1(M)$ be the set of continuous differential forms on M which have continuous first derivatives. Les $\| \, \|_{C^1(M)}$ be a norm on $C^1(M)$ such that $\|\mu^n\|_{C^1(M)} \to 0$ if and only if μ^n tends to 0 uniformly on M together with its first derivatives.

We now recall the result of Bismut announced in [B1] and proved in [B2, Theorem 3.2]

Theorem 1.5. *As $u \to \infty$, we have the following convergence of currents on M*

$$\begin{aligned} Tr_s\left[\exp(-A_u^2)\right] &\to \left[\int_N Tr_s\left[\exp(-B^2)\right]\right]\delta_{M'} \text{ in } P_{M'}^M \\ Tr_s\left[N_H \exp(-A_u^2)\right] &\to \left[\int_N Tr_s\left[N_H \exp(-B^2)\right]\right]\delta_{M'} \text{ in } P_{M'}^M. \end{aligned} \tag{1.17}$$

There exists $C > 0$ such that if μ is a smooth differential form on M, then for $u \geq 1$

$$\begin{aligned} &\left|\int_M \mu\left\{Tr_s\left[\exp(-A_u^2)\right] - \left[\int_N Tr_s\left[\exp(-B^2)\right]\right]\delta_{M'}\right\}\right| \\ &\qquad\qquad \leq \frac{C}{\sqrt{u}}\|\mu\|_{C^1(M)}. \\ &\left|\int_M \mu\left\{Tr_s\left[N_H \exp(-A_u^2)\right] - \left[\int_N Tr_s\left[N_H \exp(-B^2)\right]\right]\delta_{M'}\right\}\right| \\ &\qquad\qquad \leq \frac{C}{\sqrt{u}}\|\mu\|_{C^1(M)}. \end{aligned} \tag{1.18}$$

If U, Γ, φ, m are taken as in Section 1c), there exists $C > 0$ such that for $u \geq 1$

$$\begin{aligned} p_{U,\Gamma,\varphi,m}\left(Tr_s\left[\exp(-A_u^2)\right] - \left[\int_N Tr_s\left[\exp(-B^2)\right]\right]\delta_{M'}\right) &\leq \frac{C}{\sqrt{u}} \\ p_{U,\Gamma,\varphi,m}\left(Tr_s\left[N_H \exp(-A_u^2)\right] - \left[\int_N Tr_s\left[N_H \exp(-B^2)\right]\right]\delta_{M'}\right) &\leq \frac{C}{\sqrt{u}}. \end{aligned} \tag{1.19}$$

Proof. (1.17), (1.18), (1.19) are proved in [B2, Theorems 3.2 and 4.3]. □

Recall that the Todd polynomial is an ad-invariant polynomial on matrices which is such that if C is a diagonal matrix with diagonal entries $x_1, \dots, x_p$, then

$$Td(C) = \prod_1^p \frac{x_i}{1 - e^{-x_i}}. \tag{1.20}$$

Let $(Td^{-1})'$ be the ad-invariant polynomial which is such that if C is taken as before then

$$(Td^{-1})'(C) = \frac{\partial}{\partial b}\left\{\frac{\prod_1^p (1 - e^{-(x_i+b)})}{x_i + b}\right\}_{b=0}. \tag{1.21}$$

Let E be a holomorphic vector bundle of dimension k on M. Let Q be an ad-invariant polynomial on (k,k) matrices. If h^E is a Hermitian metric on E, and if Ω is the curvature of the corresponding holomorphic connection, we use the notation

$$Q(h^E) = Q\left(-\frac{\Omega}{2i\pi}\right). \tag{1.22}$$

Recall that ch is the polynomial $A \to Tr[\exp(A)]$.

Our definition of φ extends to an arbitrary manifold and in particular to the total space of N.

Another result of [B2] is as follows.

Theorem 1.6. *The form on M $Tr_s[N_H \exp(-A_0^2)]$ is closed. The form on M' $\int_N Tr_s[N_H \exp(-B^2)]$ is closed.*

If the metrics $h^{\xi_0}, \dots, h^{\xi_m}$ verify assumption (A) with respect to metrics g^N, g^η on N, η, then

$$\begin{aligned} \int_N \varphi\big(Tr_s[\exp(-B^2)]\big) &= Td^{-1}(g^N)ch(g^\eta) \\ \int_N \varphi\big(Tr_s[N_H \exp(-B^2)]\big) &= -(Td^{-1})'(g^N)ch(g^\eta). \end{aligned} \tag{1.23}$$

Proof. Clearly

$$Tr_s[N_H \exp(-A_0^2)] = \sum_0^m (-1)^k k\, Tr_s\big[\exp(-(\nabla^{\xi_k})^2)\big] \tag{1.24}$$

and so the form (1.24) is closed on M. The form $\int_N Tr_s[N_H \exp(-B^2)]$ is closed by [B2, Theorem 4.3]. The first line of (1.23) is a result of Mathai-Quillen [MQ, Theorem 4.5] of which a related proof is given in [B2, Theorem 3.2]. The second line of (1.23) is proved in [B2, Theorem 4.3]. □

(g) *A singular Bott–Chern current.* We make the same assumptions as in Section 1f). We now recall our definition [BGS4] of a singular Bott–Chern current associated with the Hermitian chain complex (ξ, v).

Definition 1.7. For $0 < \mathrm{Re}(s) < \frac{1}{2}$, $1 \leq A \leq +\infty$, let $\zeta_\xi^A(s)$ be the even current on M defined by the formula

$$\zeta_\xi^A(s) = \frac{1}{\Gamma(s)} \int_0^A u^{s-1} \Big\{ Tr_s\left[N_H \exp(-A_u^2)\right] - \left[\int_N Tr_s\left[N_H \exp(-B^2)\right]\right] \delta_{M'} \Big\} du. \tag{1.25}$$

By Theorem 1.5, it is clear that the current $\zeta_\xi^A(s)$ is well-defined. Also one verifies easily in [BGS4] that $\zeta_\xi^A(s)$ extends into a current depending holomorphically on s near 0.

In particular by [BGS4, Section 2a)], the current $\zeta_\xi^{A'}(0) = \dfrac{\partial \zeta_\xi^A(0)}{\partial s}$ is given by

$$\begin{aligned} \zeta_\xi^{A'}(0) = &\int_0^1 Tr_s\left[N_H\left(\exp(-A_u^2) - \exp(-A_0^2)\right)\right] \frac{du}{u} \\ &+ \int_1^A \Big\{ Tr_s\left[N_H \exp(-A_u^2)\right] \\ &- \left[\int_N Tr_s\left[N_H \exp(-B^2)\right]\right] \delta_{M'} \Big\} \frac{du}{u} \\ &- \Gamma'(1) \Big\{ Tr_s\left[N_H \exp(-A_0^2)\right] - \left[\int_N Tr_s\left[N_H \exp(-B^2)\right]\right] \delta_{M'} \Big\}. \end{aligned} \tag{1.26}$$

Of course in the case where (ξ, v) is acyclic, i.e. if M' is empty, for $A = +\infty$, the current $\zeta_\xi^{A'}(0)$ coincides with the smooth current defined in our earlier work [BGS1, Section 1c)].

Remember that the metric h^ξ on ξ is the orthogonal sum of the metrics $h^{\xi_0}, \ldots, h^{\xi_m}$ on $\xi_0, \ldots, \xi_m$.

Note that the map φ extends to even currents in the obvious way.

Definition 1.8. For $1 \leq A \leq +\infty$, let $T^A(h^\xi)$ be the current

$$T^A(h^\xi) = \varphi(\zeta_\xi^{A'}(0)). \tag{1.27}$$

Set

$$T(h^{\xi}) = T^{\infty}(h^{\xi}). \tag{1.28}$$

We define $ch(h^{\xi})$ by the formula

$$ch(h^{\xi}) = \sum_{0}^{m}(-1)^k ch(h^{\xi_k}). \tag{1.29}$$

Theorem 1.9. *For $1 \le A \le +\infty$, the current $T^A(h^{\xi})$ lies in $P^M_{M'}$. In particular $T(h^{\xi}) \in P^M_{M'}$, so that $WF(T(h^{\xi})) \subset N^*_R$. As $A \to +\infty$*

$$T^A(h^{\xi}) \to T(h^{\xi}) \quad in\ P^M_{M'}. \tag{1.30}$$

Also the current $T(h^{\xi})$ verifies the equation of currents

$$\frac{\bar{\partial}\partial}{2i\pi}T(h^{\xi}) = \left[\int_N \varphi(Tr_s[\exp(-B^2)])\right]\delta_{M'} - ch(h^{\xi}). \tag{1.31}$$

In particular, if the metrics $h^{\xi_0}, \dots, h^{\xi_m}$ verify assumption (A) *with respect to the metrics g^N, g^{η} on N, η, then*

$$\frac{\bar{\partial}\partial}{2i\pi}T(h^{\xi}) = Td^{-1}(g^N)ch(g^{\eta})\delta_{M'} - ch(h^{\xi}). \tag{1.32}$$

Proof. Theorem 1.9 is proved in [BGS4, Theorem 2.5].

Let $\tilde{M}$ be a compact complex manifold, let $f: \tilde{M} \to M$ be a holomorphic map. We assume that f is transversal to M', i.e. if $\tilde{M}' = f^{-1}(M')$, if $x \in \tilde{M}'$, then $Im[df(x)] + T_{f(x)}M' = T_xM$. Then $\tilde{M}'$ is a finite union of complex submanifolds of M.

Let $\tilde{i}$ be the embedding $\tilde{M}' \to \tilde{M}$. We still denote by f the restriction of f to $\tilde{M}'$. Using the local uniqueness of resolutions, we find that we have the exact sequence of sheaves

$$0 \to \mathcal{O}_{\tilde{M}}(f^*\xi_m) \to_{f^*v} \cdots \to_{f^*v} \mathcal{O}_{\tilde{M}}(f^*\xi_0) \to_{f^*r} \tilde{i}_*\mathcal{O}_{\tilde{M}'}f^*\eta \to 0. \tag{1.33}$$

Let $T(f^*h^{\xi})$ be the current on $\tilde{M}$ constructed as before, which is associated with the Hermitian chain complex $(f^*(\xi, v), f^*h^{\xi})$.

Since $WF(T(h^{\xi})) \subset N^*_R$, and since f is transversal to M', by [H, Theorem 8.2.4], the pulled-back current $f^*T(h^{\xi})$ on $\tilde{M}$ is well defined.

Theorem 1.10. *The following identity holds*

$$T(f^*h^\xi) = f^*T(h^\xi). \tag{1.34}$$

Proof. This result is proved in [BGS4, Theorem 2.7]. □

2. Metric and geometric properties of Bott–Chern currents

The purpose of this section is to establish the behaviour of the singular Bott–Chern currents $T(h^\xi)$ considered in [BGS4] and in Section 1g) under natural modifications of our datas, which are here an immersion of complex manifolds, a chain complex of holomorphic vector bundles, and metrics on these vector bundles.

In particular, we establish in this section a key transitivity property, which describes the behaviour of the singular Bott–Chern currents under composition of immersions.

Note that in the whole section, the microlocal properties of the considered currents play a key role in the formulation of the results and in the proofs themselves.

This section is organized as follows. In a) we study the dependence of the current $T(h^\xi)$ on the metrics h^ξ. In b), we establish a transivity formula for Bott–Chern currents associated with a commutative diagram of immersions. In c), we assume that our vector bundles are themselves replaced by acyclic complexes, and we study the corresponding Bott–Chern currents.

In this section, we make the same assumptions as in Section 1, and we use the same notations.

(a) *The current* $T(h^\xi)$ *as a function of* h^ξ. Let E be a holomorphic vector bundle on M of dimension k. Let Q be an ad-invariant polynomial on (k,k) matrices. We now use the same notations as in (1.22). Let h_0^E, h_1^E be two Hermitian metrics on E. By [BGS1, Theorem 1.29], there is a well-defined class of smooth forms $\tilde{Q}(h_0^E, h_1^E) \in P^M/P^{M,0}$ such that

$$\frac{\bar{\partial}\partial}{2i\pi}\tilde{Q}(h_0^E, h_1^E) = Q(h_1^E) - Q(h_0^E). \tag{2.1}$$

$\tilde{Q}(h_0^E, h_1^E)$ is the axiomatically defined Bott–Chern class of [BGS1, Section 1f)] associated with the exact sequence $0 \to (E, h_0^E) \to (E, h_1^E) \to 0$.

We now make the same assumptions as in Section 1a). Let $\mathcal{M}^\xi$ (resp. $\mathcal{M}^F$) be the set of Hermitian metrics $h^\xi = (h^{\xi_0}, \dots, h^{\xi_m})$ (resp. $h^F = (h^{F_0}, \dots, h^{F_m})$) on the vector bundles $\xi_0, \dots, \xi_m$ (resp. $F_0, \dots, F_m$).

If $h_0^\xi = (h_0^{\xi_0}, \ldots, h_0^{\xi_m})$, $h_1^\xi = (h_1^{\xi_0}, \ldots, h_1^{\xi_m})$ are elements of $\mathcal{M}^\xi$, set

$$\widetilde{ch}(h_0^\xi, h_1^\xi) = \sum_0^m (-1)^k \widetilde{ch}(h_0^{\xi_k}, h_1^{\xi_k}). \tag{2.2}$$

Then by (2.1)

$$\frac{\bar{\partial}\partial}{2i\pi} \widetilde{ch}(h_0^\xi, h_1^\xi) = ch(h_1^\xi) - ch(h_0^\xi). \tag{2.3}$$

Take now $h^F = (h^{F_0}, \ldots, h^{F_m}) \in \mathcal{M}^F$. We equip F with the orthogonal sum of the metrics $(h^{F_k})_{0 \leq k \leq m}$.

Let g^N, g^η be Hermitian metrics on the vector bundles N, η. We equip $\Lambda N^* \otimes \eta$ with the tensor product of the metric induced on ΛN^* by g^N and of the metric g^η.

Definition 2.1. We will say that the metrics $h^F = (h^{F_0}, \ldots, h^{F_m})$ verify assumption (A) with respect to the metrics g^N, g^η on N, η if the identification of Z-graded complexes $(F, \partial_y v) \cong (\Lambda N^* \otimes \eta, i_y)$ also identifies the metrics.

Let ∇^F be the holomorphic Hermitian connection on F. For $y \in V$, let $(\partial_y v)^*$ be the adjoint of $\partial_y v$. Set as in (1.13)

$$\partial_Y V = \partial_y v + (\partial_y v)^*. \tag{2.4}$$

Let B be the superconnection

$$B = \nabla^F + \partial_Y V. \tag{2.5}$$

Definition 2.2. Let $\theta(h^F)$ be the smooth form on M'

$$\theta(h^F) = \int_N \varphi\big(Tr_s\big[\exp(-B^2)\big]\big).$$

The form $\theta(h^F)$ is closed. By [BGS1, Proposition 1.8], $\theta(h^F) \in P^{M'}$. Also if the metrics $h^F = (h^{F_0}, \ldots, h^{F_m})$ verify assumption (A) with respect to the metrics g^N, g^η, then by Theorem 1.6

$$\theta(h^F) = Td^{-1}(g^N) ch(g^\eta). \tag{2.6}$$

If $\ell \in R \to h_\ell^F = (h_\ell^{F_0}, \ldots, h_\ell^{F_m}) \in \mathcal{M}^F$ is a smooth map from R into $\mathcal{M}^F$, let $\ell \in R \to B_\ell$ be the corresponding family of superconnections on the graded vector bundle F. For any $\ell \in R$

$$(2.7) \qquad (h_\ell^F)^{-1}\frac{\partial h_\ell^F}{\partial \ell} = \left((h_\ell^{F_0})^{-1}\frac{\partial h_\ell^{F_0}}{\partial \ell}, \ldots, (h_\ell^{F_m})^{-1}\frac{\partial h_\ell^{F_m}}{\partial \ell}\right)$$

is a smooth section of End F.

Definition 2.3. Let χ be the smooth form

$$(2.8) \qquad \chi = -\int_0^1 d\ell \int_N \varphi\left(Tr_s\left[(h_\ell^F)^{-1}\frac{\partial h^F}{\partial \ell}\exp(-B_\ell^2)\right]\right) d\ell.$$

Theorem 2.4. *The form χ lies in $P^{M'}$ and its class in $P^{M'}/P^{M',0}$ only depends on h_0^F, h_1^F. Moreover*

$$(2.9) \qquad \frac{\bar\partial\partial}{2i\pi}\chi = \theta(h_1^F) - \theta(h_0^F).$$

If the metrics h_0^F and h_1^F verify assumption (A) *with respect to metrics (g_0^N, g_0^η) and (g_1^N, g_1^η) on N, η, then*

$$(2.10) \quad \chi = \widetilde{Td^{-1}}(g_0^N, g_1^N)ch(g_0^\eta) + Td^{-1}(g_1^N)\widetilde{ch}(g_0^\eta, g_1^\eta) \quad in\ P^{M'}/P^{M',0}.$$

Proof. The proof of (2.9) follows from the extension to superconnections of a formula of Bott and Chern [BoC, 3.28]. This extension was proved in Bismut [B2, Theorem 2.1]. Using this formula, the proof of (2.9) is strictly identical to the proof of [BGS1, Theorem 1.27]. The fact that the class of χ in $P^{M'}/P^{M',0}$ only depends on h_0^F and h_1^F follows from the analogue of the formulas in [BGS1, Theorem 1.25] for superconnections which was proved in [B2, Theorem 2.2]. Of course the formulas in [B2] must be integrated along the fibre N. We can then proceed as in [BGS1, Section 1e)].

If h_0^F, h_1^F verify assumption (A) with respect to (g_0^N, g_0^η) and (g_1^N, g_1^η), we can find a smooth family of metrics $\ell \to (g_\ell^N, g_\ell^\eta)$ on N, η which interpolates between (g_0^N, g_0^η), and (g_1^N, g_1^η). If h_ℓ^F is the metric on F associated with the metric (g_ℓ^N, g_ℓ^η), the family $\ell \to h_\ell^F$ interpolates between h_0^F and h_1^F.

The operator $(g_\ell^N)^{-1}\frac{\partial g_\ell^N}{\partial \ell}$ acts naturally on the exterior algebra ΛN^*. One verifies easily that since $F = \Lambda N^* \otimes \eta$,

$$(2.11)\qquad (h_\ell^F)^{-1}\frac{\partial h_\ell^F}{\partial \ell} = (g_\ell^N)^{-1}\frac{\partial g_\ell^N}{\partial \ell} \otimes 1 + 1 \otimes (g_\ell^\eta)^{-1}\frac{\partial g_\ell^\eta}{\partial \ell}.$$

Let $(\nabla_\ell^N)^2$, $(\nabla_\ell^\eta)^2$ be the curvatures of the holomorphic Hermitian connections on the holomorphic Hermitian vector bundles (N, g_ℓ^N), (η, g_ℓ^η). By proceeding as in Mathai–Quillen [MQ, Theorem 4.5], Bismut [B2, Theorem 3.2], we find that

$$(2.12)\qquad \begin{aligned} -\int_N Tr_s\left[(h_\ell^F)\frac{\partial h_\ell^F}{\partial \ell}\exp(-B_\ell^2)\right] &= (2i\pi)^{\dim N}\frac{\partial}{\partial b}\Big\{Td^{-1}\Big(-(\nabla_\ell^N)^2 \\ &- b(g_\ell^N)^{-1}\frac{\partial g_\ell^N}{\partial \ell}\Big)Tr\left[\exp(-(\nabla_\ell^\eta)^2)\right] \\ &+ Td^{-1}(-(\nabla_\ell^N)^2)Tr\left[\exp\Big(-(\nabla_\ell^\eta)^2 - b(g_\ell^\eta)^{-1}\frac{\partial g_\ell^\eta}{\partial \ell}\Big)\right]\Big\}_{b=0}. \end{aligned}$$

Using (2.8), (2.12) and also [BGS1, Remark 1.28, Corollary 1.30 and Remark 1.31] we find that

$$(2.13)\qquad \chi = \widetilde{Td^{-1} \otimes ch}\big((g_0^N, g_0^\eta), (g_1^N, g_1^\eta)\big) \quad \text{in } P^{M'}/P^{M',0}.$$

Our theorem is proved. □

We now will write $\chi(h_0^F, h_1^F)$ instead of χ the class of forms in $P^{M'}/P^{M',0}$ defined in (2.8).

Let $h_0^\xi = (h_0^{\xi_0}, \dots, h_0^{\xi_m})$, $h_1^\xi = (h_1^{\xi_0}, \dots, h_1^{\xi_m})$ be two elements of $\mathcal{M}^\xi$. Let $h_0^F = (h_0^{F_0}, \dots, h_0^{F_m})$, $h_1^F = (h_1^{F_0}, \dots, h_1^{F_m})$ be the elements of $\mathcal{M}^F$ respectively associated with h_0^ξ, h_1^ξ by the construction of Section 1b).

Theorem 2.5. *The following identity holds*

$$(2.14)\qquad T(h_1^\xi) - T(h_0^\xi) = \chi(h_0^F, h_1^F)\delta_{M'} - \widetilde{ch}(h_0^\xi, h_1^\xi) \quad \textit{in } P_{M'}^M/P_{M'}^{M,0}.$$

Proof. Let $\mathbf{P}^1$ be the complex projective plane equipped with the meromorphic coordinate z and the two distinguished points $\{0\}$ and $\{\infty\}$.

The complex (ξ, v) lifts naturally to $M \times \mathbf{P}^1$. On $M \times \mathbf{P}^1$, we equip $(\xi^0, \dots, \xi^m)$ with Hermitian metrics $h^\xi = (h^{\xi_0}, \dots, h^{\xi_m})$ which restrict to the metrics $(h_0^{\xi_0}, \dots, h_0^{\xi_m})$ and $(h_1^{\xi_0}, \dots, h_1^{\xi_m})$ on $M \times \{0\}$ and $M \times \{\infty\}$ respectively.

The vector bundle N_R^* on M' lifts naturally to $M' \times \mathbf{P}^1$. By Theorem 1.9, the wave front set $WF(T(h^\xi))$, of the current $T(h^\xi)$ on $M \times \mathbf{P}^1$ is included in N_R^*. Also the wave front set $WF(\mathrm{Log}|z|^2)$ of the distribution $\mathrm{Log}|z|^2$ on $\mathbf{P}^1$ is equal to $(T_R^*\mathbf{P}^1 \setminus \{0\})_0 \cup (T_R^*\mathbf{P}^1 \setminus \{0\})_\infty$. The current $\mathrm{Log}|z|^2$ lifts to $M \times \mathbf{P}^1$, and its wave front set $WF(\mathrm{Log}|z|^2)$ also lifts in the obvious way.

Since $WF(T(h^\xi)) \cap (-WF(\mathrm{Log}|z|^2)) = \emptyset$, by [H, Theorem 8.2.10], the product of currents $\mathrm{Log}|z|^2 T(h^\xi)$ is well defined. Also the usual rules of differential calculus can still be used. In particular

$$\frac{\bar\partial\partial}{2i\pi}(\mathrm{Log}|z|^2)T(h^\xi) - \mathrm{Log}|z|^2\frac{\bar\partial\partial}{2i\pi}T(h^\xi) = \frac{\bar\partial}{2i\pi}(\partial(\mathrm{Log}|z|^2)T(h^\xi)) + \frac{\partial}{2i\pi}(\mathrm{Log}|z|^2\bar\partial T(h^\xi)). \tag{2.15}$$

It is well-known that

$$\frac{1}{2i\pi}\bar\partial\partial\,\mathrm{Log}|z|^2 = \delta_0 - \delta_\infty. \tag{2.16}$$

Let $h^F = (h^{F_0}, \ldots, h^{F_m})$ be the metrics on the vector bundles $F_0, \ldots, F_m$ on $M' \times \mathbf{P}^1$ induced by the metrics $h^\xi = (h^{\xi_0}, \ldots, h_0^{\xi_m})$. Using Theorem 1.9, and (2.15), (2.16) we find that

$$\begin{aligned} &T(h^\xi)\delta_{M\times\{0\}} - T(h^\xi)\delta_{M\times\{\infty\}} - \mathrm{Log}|z|^2(\theta(h^F)\delta_{M'\times\mathbf{P}^1} - ch(h^\xi)) \\ &\quad = \frac{\bar\partial}{2i\pi}(\partial(\mathrm{Log}|z|^2)T(h^\xi)) + \frac{\partial}{2i\pi}(\mathrm{Log}|z|^2\bar\partial T(h^\xi)). \end{aligned} \tag{2.17}$$

We now integrate (2.17) along the fiber of $M \times \mathbf{P}^1 \to M$. By [H, Theorem 8.2.10], the wave front sets of the currents $\partial(\mathrm{Log}|z|^2)T(h^\xi)$ and $\mathrm{Log}|z|^2\bar\partial T(h^\xi)$ are included in the sum of the duals of the real normal bundles to $M', M \times \{0\}$, $M \times \{\infty\}$ in $M \times \mathbf{P}^1$. By [H, Theorem 8.2.13], the wave front set of their integral along the fibers is included in N_R^*. Also ∂ and $\bar\partial$ commute with integration along the fiber.

From (2.17), we deduce in particular that

$$T(h_0^\xi) - T(h_1^\xi) - \left[\int_{\mathbf{P}^1} \mathrm{Log}|z|^2\theta(h^F)\right]\delta_{M'} + \int_{\mathbf{P}^1} \mathrm{Log}|z|^2 ch(h^\xi) \in P_{M'}^{M,0}. \tag{2.18}$$

By [BGS1, Theorem 1.29], we know that

$$\int_{\mathbf{P}^1} \mathrm{Log}|z|^2 ch(h^\xi) = -\widetilde{ch}(h_0^\xi, h_1^\xi) \quad \text{in } P^M/P^{M,0}. \tag{2.19}$$

The metric h_0^F lifts into a Hermitian metric on the vector bundle F on $M' \times \mathbf{P}^1$. Clearly

$$\begin{aligned}\frac{\bar\partial\partial}{2i\pi}(\mathrm{Log}|z|^2)\chi(h_0^F,h^F) \quad &- \mathrm{Log}|z|^2\frac{\bar\partial\partial}{2i\pi}\chi(h_0^F,h^F)\\ &= \frac{\bar\partial}{2i\pi}(\partial\,\mathrm{Log}|z|^2\chi(h_0^F,h^F))\\ &+ \frac{\partial}{2i\pi}(\mathrm{Log}|z|^2\bar\partial\chi(h_0^F,h^F)).\end{aligned} \tag{2.20}$$

By Theorem 2.4, we know that

$$\frac{\bar\partial\partial}{2i\pi}\chi(h_0^F,h^F) = \theta(h^F) - \theta(h_0^F). \tag{2.21}$$

Observe that since h^F coincides with h_0^F on $M \times \{0\}$, the restriction of $\chi(h_0^F,h^F)$ to $M' \times \{0\}$ vanishes in $P^{M'}/P^{M',0}$. Using (2.16), (2.21) and integrating (2.20) along the fiber of $M' \times \mathbf{P}^1 \to M'$, we get

$$\chi(h_0^F,h_1^F) + \int_{\mathbf{P}^1}\mathrm{Log}|z|^2\theta(h^F) \in P^{M',0}. \tag{2.22}$$

From (2.18)–(2.22), we deduce (2.14). □

(b) *A transitivity property of Bott–Chern singular currents.* We make the same assumptions as in Section 1a).

Let $\tilde M' = \bigcup_{j'=1}^{n'} \tilde M'_{j'}$, be another finite union of compact connected complex submanifolds of M such that if $j \neq j'$, $\tilde M'_j \cap \tilde M'_{j'} = \emptyset$. Let $\tilde i$ be the embedding $\tilde M' \to M$.

We otherwise assume that to the pair M, $\tilde M'$, we have associated the analogue of the objects which were associated with the pair M, M'. These objects will be denoted with a $\sim$. In particular:

• $(\tilde\xi,\tilde v)$ is a complex of holomorphic vector bundles on M

$$(\tilde\xi,\tilde v): 0 \to \tilde\xi_{\tilde m} \to_{\tilde v} \cdots \to_{\tilde v} \tilde\xi_0 \to 0. \tag{2.23}$$

• $\tilde\eta$ is a holomorphic vector bundle on $\tilde M'$.

• There exists a holomorphic restriction map $\tilde r: \tilde\xi_{0|\tilde M'} \to \tilde\eta$ such that we have the exact sequence of sheaves on M

$$0 \to \mathcal{O}_M(\tilde\xi_{\tilde m}) \to_{\tilde v} \cdots \to \mathcal{O}_M(\tilde\xi_0) \to_{\tilde r} \tilde i_* \mathcal{O}_{\tilde M'}(\tilde\eta) \to 0. \tag{2.24}$$

$\tilde{N}$ denotes the normal bundle to $\tilde{M}'$ in M.

We now make the fundamental assumption that M' and $\tilde{M}'$ intersect transversally, i.e. if $x \in M' \cap \tilde{M}'$, then $T_xM' + T_x\tilde{M}' = T_xM$.

Let M'' be the complex submanifold of M $M'' = M' \cap \tilde{M}'$. Then if $x \in M''$, $T_xM'' = T_xM' \cap T_x\tilde{M}'$. Let i'' be the embedding $M'' \to M$. Let j, $\tilde{j}$ be the embeddings $j: M'' \to M'$, $\tilde{j}: M'' \to \tilde{M}'$.

We will denote with a $''$ the objects naturally associated with the pair M, M''. In particular, N'' is the normal bundle to M'' in M. Observe that since M' and $\tilde{M}'$ are transversal, the vector bundle N'' splits holomorphically into

$$N'' = N_{|M''} \oplus \tilde{N}_{|M''}. \tag{2.25}$$

Also $N_{|M''}$ (resp. $\tilde{N}_{|M''}$) is exactly the normal bundle to M'' in $\tilde{M}'$ (resp. in M').

Let (ξ'', v'') be the double complex

$$(\xi'', v'') = \left(\xi \,\hat{\otimes}\, \tilde{\xi}, v \,\hat{\otimes}\, 1 + 1 \,\hat{\otimes}\, \tilde{v}\right).$$

Then

- ξ'' is the graded tensor product of the graded vector bundles ξ and $\tilde{\xi}$, and so for $0 \leq k \leq m + \tilde{m}$

$$\xi''_k = \bigoplus_{j+j'=k} (\xi_j \,\hat{\otimes}\, \tilde{\xi}_{j'}) \tag{2.26}$$

- $v \,\hat{\otimes}\, 1$ acts on $\xi \,\hat{\otimes}\, \tilde{\xi}$ like $v \otimes 1$, and moreover, if $f \in \xi_j$, $\tilde{f} \in \tilde{\xi}_{j'}$, then

$$(1 \,\hat{\otimes}\, \tilde{v})(f \,\hat{\otimes}\, f) = (-1)^{\deg f} f \,\hat{\otimes}\, \tilde{v}\tilde{f}. \tag{2.27}$$

One easily verifies that $v''^2 = 0$. To simplify our notations, we will consider v and $\tilde{v}$ as odd anticommuting elements in $\operatorname{End}\xi''$, acting on ξ'' like $v \,\hat{\otimes}\, 1$ and $1 \,\hat{\otimes}\, \tilde{v}$ so that $v'' = v + \tilde{v}$.

Let $\tilde{\eta}$ be the holomorphic vector bundle on M'' $\eta'' = \eta_{|M''} \otimes \tilde{\eta}_{|M''}$. In the sequel, it will be convenient to consider η, $\tilde{\eta}$ as even vector bundles, so that $\eta'' = \eta_{|M''} \,\hat{\otimes}\, \tilde{\eta}_{|M''}$.

Let r'' be the restriction map

$$r'': \xi''_{0|M''} = \left(\xi_0 \,\hat{\otimes}\, \tilde{\xi}_0\right)\big|_{M''} \xrightarrow[r \hat{\otimes} r']{} \eta''. \tag{2.28}$$

Using again the local uniqueness of resolutions (see Serre [S, Chapter IV, Appendix 1], Eilenberg [E, Theorem 8]) one verifies easily that we have the exact sequence of sheaves

$$\begin{aligned} &0 \to \mathcal{O}_M(\xi'', v'') \to_{r''} i''_* \mathcal{O}_{M''}(\eta'') \to 0 \\ &0 \to \mathcal{O}_{\tilde{M}'}((\xi, v)_{|\tilde{M}'}) \to_r \tilde{j}_* \mathcal{O}_{M''}(\eta_{|M''}) \to 0 \\ &0 \to \mathcal{O}_{M'}((\tilde{\xi}, \tilde{v})_{|M'}) \to_{r'} j_* \mathcal{O}_{M''}(\eta'_{|M''}) \to 0. \end{aligned} \tag{2.29}$$

By [B2, Section 1b)], we know that if F (resp. $\tilde{F}$, F'') is the direct sum of the homology groups of the complex $(\xi, v)_{|M'}$ (resp. $(\tilde{\xi}, \tilde{v})_{|\tilde{M}'}$, $(\xi'', v'')_{|M''}$), then F (resp. $\tilde{F}$, F'') is a Z-graded holomorphic vector bundle. Moreover we have the identification of holomorphic vector bundles

$$F'' = F_{|M''} \hat{\otimes} \tilde{F}_{|M''}. \tag{2.30}$$

In the sequel, we always equip the direct sum of Hermitian vector bundles with the orthogonal sum of the considered metrics, and tensor products of vector bundles with the tensor product of the Hermitian metrics.

Let now $h^{\xi} = (h^{\xi_0}, \dots, h^{\xi_m}) \in \mathcal{M}^{\xi}$, $h^{\tilde{\xi}} = \left(h^{\tilde{\xi}_0}, \dots, h^{\tilde{\xi}_{\tilde{m}}}\right) \in \mathcal{M}^{\tilde{\xi}}$. Let $h^{\xi''} = \left(h^{\xi''_0}, \dots, h^{\xi''_{m+\tilde{m}}}\right) \in \mathcal{M}^{\xi''}$ be the corresponding family of Hermitian metrics on $\xi''_0, \dots, \xi''_{m+\tilde{m}}$. Let h^F (resp. $h^{\tilde{F}}$, $h^{F''}$) be the family of metrics on F (resp. $\tilde{F}$, F'') induced by h^{ξ} (resp. $h^{\tilde{\xi}}$, $h^{\xi''}$). Then (2.30) is an identification of holomorphic Hermitian vector bundles.

Note that if the metrics h^{ξ} and $h^{\tilde{\xi}}$ verify assumption (A) with respect to metrics (g^N, g^{η}) and $(g^{\tilde{N}}, g^{\tilde{\eta}})$ on N, η and $\tilde{N}, \tilde{\eta}$, then $h^{\xi''}$ verifies assumption (A) with respect to the metrics $\left(g^N \oplus g^{\tilde{N}}, g^{\eta} \otimes g^{\tilde{\eta}}\right)$ on (N'', η'').

We now construct the currents $T(h^{\xi})$, $T(h^{\tilde{\xi}})$, $T(h^{\xi''})$ associated with the holomorphic Hermitian chain complexes (ξ, v), $(\tilde{\xi}, \tilde{v})$, (ξ'', v''). By [BGS4, Theorem 2.5] or by Theorem 1.9, we know that

$$WF(T(h^{\xi})) \subset N^*_R; WF(T(h^{\tilde{\xi}})) \subset \tilde{N}^*_R; WF(T(h^{\xi''})) \subset N''^*_R. \tag{2.31}$$

Recall that M' and $\tilde{M}'$ are transversal. Using (2.31) and [H, Theorem 8.2.4], we know that the pulled back currents $i^* T(h^{\tilde{\xi}})$ and $\tilde{i}^* T(h^{\xi})$ on M' and $\tilde{M}'$ are well-defined.

By (2.29), the complex $(\xi, v)_{|\tilde{M}}$ provides a resolution on $\tilde{M}'$ of the sheaf of holomorphic sections of $\eta_{|M''}$. Therefore, if we denote by $\tilde{i}^* h^{\xi}$ the family of Hermitian metrics induced by h^{ξ} on $\xi_{|\tilde{M}'}$, we can define the current $T(\tilde{i}^* h^{\xi})$ on $\tilde{M}'$. Similarly, we also consider the current $T(i^* h^{\tilde{\xi}})$ on M'.

By [BGS4, Theorem 2.7] or by Theorem 1.10, we know that

$$
\begin{aligned}
T(\tilde{i}^* h^{\xi}) &= \tilde{i}^* T(h^{\xi}) \\
T(i^* h^{\tilde{\xi}}) &= i^* T(h^{\tilde{\xi}}).
\end{aligned}
\tag{2.32}
$$

The push-forward of a current by a smooth map is always well defined. Then using the fact that M' and $\tilde{M}'$ intersect transversally, (2.31) and [H, Theorem 8.2.10], we find that the product currents $T(h^{\xi})\delta_{M'}$, and $T(h^{\tilde{\xi}})\delta_{\tilde{M}'}$ are well defined. One easily verifies that

$$
\begin{aligned}
T(h^{\tilde{\xi}})\delta_{M'} &= i_*\left(i^* T(h^{\tilde{\xi}})\right) \\
T(h^{\xi})\delta_{\tilde{M}'} &= \tilde{i}_*\left(\tilde{i}^* T(h^{\xi})\right).
\end{aligned}
\tag{2.33}
$$

Note the commutative diagram of immersions

$$
\begin{array}{ccc}
M'' & \xrightarrow{\;j\;} & M' \\
\Big\downarrow{\scriptstyle \tilde{j}} & \searrow^{i''} & \Big\downarrow{\scriptstyle i} \\
\tilde{M}' & \xrightarrow[\;\tilde{i}\;]{} & M
\end{array}
\tag{2.34}
$$

We now will prove that to (2.34), we can associate a corresponding transitivity result for singular Bott–Chern currents.

If $x \in M' \cup \tilde{M}'$, set

$$
\begin{aligned}
(N_R^* + \tilde{N}_R^*)_x &= N_R^* \ \text{ if } x \in M' \setminus M'' \\
&= \tilde{N}_R^* \ \text{ if } x \in \tilde{M}' \setminus M'' \\
&= \tilde{N}_R''^{*} \ \text{ if } x \in M''.
\end{aligned}
\tag{2.35}
$$

Definition 2.6. $\mathcal{D}'_{N_R^* + \tilde{N}_R^*}(M)$ denotes the set of currents on M whose wave front set is included in $N_R^* + \tilde{N}_R^*$. $P^M_{M' \cup \tilde{M}'}$ is the set of currents ω on M which have the following two properties.

- ω is a sum of currents of type (p,p).
- The wave front set of ω is included in $N_R^* + \tilde{N}_R^*$.

$P^{M,0}_{M' \cup \tilde{M}'}$ is the set of currents $\omega \in P^M_{M' \cup \tilde{M}'}$, which are such that there exist currents $\alpha, \beta \in \mathcal{D}'_{N_R^* + \tilde{N}_R^*}(M)$ for which $\omega = \partial\alpha + \bar{\partial}\beta$.

Theorem 2.7. *The following identities hold*

$$(2.36)\quad \begin{aligned} T(h^{\xi''}) &= ch(h^{\tilde{\xi}})T(h^{\xi}) + i_*\big(\theta(h^F)T(i^*h^{\tilde{\xi}})\big) \text{ in } P^M_{M'\cup\tilde{M}'}/P^{M,0}_{M'\cup\tilde{M}'} \\ T(h^{\xi''}) &= ch(h^{\xi})T(h^{\tilde{\xi}}) + \tilde{i}_*\big(\theta(h^{\tilde{F}})T(\tilde{i}^*h^{\xi})\big) \text{ in } P^M_{M'\cup\tilde{M}'}/P^{M,0}_{M'\cup\tilde{M}'}. \end{aligned}$$

Proof. We first show that the equalities in (2.36) make sense. By [BGS4, Theorem 2.5] or by Theorem 1.8, we know that the wave front set of the current $i^*T(h^{\tilde{\xi}}) = T(i^*h^{\tilde{\xi}})$ is included in the real conormal bundle $\tilde{N}^*_R$ to M'' in M'. The current $\theta(h^F)$ on M' is smooth. By [H, Theorem 8.2.13], we find that the wave front set of the current $i_*\big(\theta(h^F)i^*T(h^{\tilde{\xi}})\big)$ is included in $N^*_R + \tilde{N}^*_R$. So the wave front set of the current $ch(h^{\tilde{\xi}})T(h^{\xi}) + i_*\big(\theta(h^F)i^*T(h^{\tilde{\xi}})\big)$ is included in $N^*_R + \tilde{N}^*_R$. Therefore this current lies in $P^M_{M'\cup\tilde{M}'}$. The first line in (2.36) is then an identity of currents in $P^M_{M'\cup\tilde{M}'}/P^{M,0}_{M'\cup\tilde{M}'}$. The same is true for the second line in (2.36).

We now prove (2.36). To the Hermitian chain complexes (ξ, v), $(\tilde{\xi}, \tilde{v})$, (ξ'', v''), we associate linear maps V, $\tilde{V}$, V'' as in (1.9). For $u \geq 0$, let A_u, $\tilde{A}_u$, A''_u be the superconnections on the vector bundles ξ, $\tilde{\xi}$, ξ''

$$(2.37)\quad \begin{aligned} A_u &= \nabla^{\xi} + \sqrt{u}\,V \\ \tilde{A}_u &= \nabla^{\tilde{\xi}} + \sqrt{u}\,\tilde{V} \\ A''_u &= \nabla^{\xi''} + \sqrt{u}\,V''. \end{aligned}$$

Let N_H, $\tilde{N}_H$, N''_H be the operators defining the Z-grading of the complexes (ξ, v), $(\tilde{\xi}, \tilde{v})$, (ξ'', v''). Clearly

$$(2.38)\quad N''_H = N_H \hat{\otimes} 1 + 1 \hat{\otimes} \tilde{N}_H.$$

For $0 \leq u < +\infty$, set

$$(2.39)\quad \begin{aligned} \alpha_u &= Tr_s\big[\exp(-A_u^2)\big] \quad ; \quad \beta_u = Tr_s\big[N_H \exp(-A_u^2)\big] \\ \tilde{\alpha}_u &= Tr_s\big[\exp(-\tilde{A}_u^2)\big] \quad ; \quad \tilde{\beta}_u = Tr_s\big[\tilde{N}_H \exp(-\tilde{A}_u^2)\big] \\ \alpha''_u &= Tr_s\big[\exp(-A''^2_u)\big] \quad ; \quad \beta''_u = Tr_s\big[N''_H \exp(-A''^2_u)\big]. \end{aligned}$$

By the construction of the metrics $h^{\xi''} \in \mathcal{M}^{\xi''}$ and by (2.38), one finds easily that

$$(2.40)\quad \begin{aligned} \alpha''_u &= \alpha_u \tilde{\alpha}_u \\ \beta''_u &= \alpha_u \tilde{\beta}_u + \tilde{\alpha}_u \beta_u. \end{aligned}$$

By [B2, Theorems 3.2 and 4.3], or by Theorem 1.5, we know that as $u \to +\infty$, the currents α_u, β_u, $\tilde{\alpha}_u$, $\tilde{\beta}_u$, α''_u, β''_u have limits α_∞, β_∞, $\tilde{\alpha}_\infty$, $\tilde{\beta}_\infty$, α''_∞, β''_∞ which are explicitly known. By [B2, Theorem 4.3], or by Theorem 1.6, we know that the currents β_0, β_∞, $\tilde{\beta}_0$, $\tilde{\beta}_\infty$, β''_0, β''_∞, are closed. Set

$$\text{(2.41)}\quad \begin{aligned} &\bullet \text{ for } u \leq 1 \quad \tilde{\eta}_u = \int_0^u (\tilde{\beta}_v - \tilde{\beta}_0)\frac{dv}{v} \\ &\bullet \text{ for } 1 \leq u \leq +\infty \quad \tilde{\eta}_u = \int_0^1 (\tilde{\beta}_v - \tilde{\beta}_0)\frac{dv}{v} + \int_1^u (\tilde{\beta}_v - \tilde{\beta}_\infty)\frac{dv}{v}. \end{aligned}$$

By Theorem 1.9, for any $u \in [0, +\infty]$, $\tilde{\eta}_u \in \mathcal{D}'_{\tilde{N}^{\bullet}_R}(M)$ and the map $u \in [0, +\infty] \to \tilde{\eta}_u \in \mathcal{D}'_{\tilde{N}^{\bullet}_R}(M)$ is continuous.

By [BGS1, Theorem 1.15], [B2, Theorem 2.4] or by Theorem 1.4, we know that

$$\text{(2.42)}\quad \frac{\partial}{\partial u}\tilde{\alpha}_u = \frac{1}{u}\bar{\partial}\partial\,\tilde{\beta}_u.$$

Since β_0 and β_∞ are closed currents, we deduce from (2.42) that for $0 \leq u < +\infty$

$$\text{(2.43)}\quad \tilde{\alpha}_u = \tilde{\alpha}_0 + \bar{\partial}\partial\,\tilde{\eta}_u.$$

Also by (2.40) we have

$$\text{(2.44)}\quad \beta''_u - \beta''_0 = \tilde{\alpha}_u\beta_u - \tilde{\alpha}_0\beta_0 + \alpha_u\tilde{\beta}_u - \alpha_0\tilde{\beta}_0.$$

Using (2.43), (2.44), we find that

$$\text{(2.45)}\quad \begin{aligned} \beta''_u - \beta''_0 &= \tilde{\alpha}_0(\beta_u - \beta_0) + \alpha_u\tilde{\beta}_u - \alpha_0\tilde{\beta}_0 \\ &\quad + \left(u\frac{\partial}{\partial u}\alpha_u\right)\tilde{\eta}_u - \bar{\partial}(\partial(\beta_u)\tilde{\eta}_u) - \partial(\beta_u\bar{\partial}\tilde{\eta}_u). \end{aligned}$$

Therefore from (2.45), we deduce that

$$\text{(2.46)}\quad \begin{aligned} \int_0^1 (\beta''_u - \beta''_0)\frac{du}{u} &= \tilde{\alpha}_0\int_0^1 (\beta_u - \beta_0)\frac{du}{u} + \alpha_1\tilde{\eta}_1 \\ &\quad + \tilde{\beta}_0\int_0^1 (\alpha_u - \alpha_0)\frac{du}{u} \\ &\quad - \bar{\partial}\int_0^1 \partial(\beta_u)\tilde{\eta}_u\frac{du}{u} \\ &\quad - \partial\int_0^1 \beta_u\bar{\partial}\,\tilde{\eta}_u\frac{du}{u}. \end{aligned}$$

Clearly, the integrals in (2.46) define smooth currents on M.

We can define currents η_u associated to the family of currents β_u by a formula similar to (2.41). In particular the analogue of (2.43) is

$$\alpha_u = \alpha_0 + \bar{\partial}\partial\, \eta_u. \tag{2.47}$$

Since $\tilde{\beta}_0$ is closed, we deduce from (2.46), (2.47), that

(2.48)
$$\int_0^1 (\beta''_u - \beta''_0)\frac{du}{u} = \tilde{\alpha}_0 \int_0^1 (\beta_u - \beta_0)\frac{du}{u} + \alpha_1\tilde{\eta}_1 + \bar{\partial}\partial\left(\tilde{\beta}_0 \int_0^1 \eta_u \frac{du}{u}\right)$$
$$- \bar{\partial}\int_0^1 \partial(\beta_u)\tilde{\eta}_u\frac{du}{u} - \partial\int_0^1 \beta_u\bar{\partial}\,\tilde{\eta}_u\frac{du}{u}.$$

Remember that by [H, Theorem 8.2.10], if $\omega \in \mathcal{D}'_{N^*_R}(M)$, $\tilde{\omega} \in \mathcal{D}'_{\tilde{N}^*_R}(M)$, since $N^*_{R|M''} \cap \tilde{N}^*_{R|M''} = \{0\}$, we can form the product $\omega\tilde{\omega} \in \mathcal{D}'_{N^*_R+\tilde{N}^*_R}(M)$, and the map $(\omega,\tilde{\omega}) \in \mathcal{D}'_{N^*_R}(M) \times \mathcal{D}'_{\tilde{N}^*_R}(M) \to \omega\tilde{\omega} \in \mathcal{D}'_{N^*_R+\tilde{N}^*_R}(M)$ is continuous. Also the usual rules of differential calculus can be used on such products.

In view of (2.30), it is clear that

$$\begin{aligned} \alpha''_\infty &= \alpha_\infty\tilde{\alpha}_\infty \\ \beta''_\infty &= \alpha_\infty\tilde{\beta}_\infty + \tilde{\alpha}_\infty\beta_\infty. \end{aligned} \tag{2.49}$$

Remember that by Theorem 1.5, as $u \to +\infty$, α_u, β_u converge to α_∞, β_∞ in $\mathcal{D}'_{N^*_R}(M)$, $\tilde{\alpha}_u$, $\tilde{\beta}_u$ converge to $\tilde{\alpha}_\infty$, $\tilde{\beta}_\infty$ in $\mathcal{D}'_{\tilde{N}^*_R}(M)$. So (2.49) can be considered as a consequence of (2.40) and of the previous considerations on the products of currents.

By proceeding formally as in (2.44) and using (2.43) and (2.47), we find that

$$\begin{aligned} \beta''_u - \beta''_\infty &= \tilde{\alpha}_0(\beta_u - \beta_\infty) + \alpha_u\tilde{\beta}_u - \alpha_\infty\tilde{\beta}_\infty \\ &+ u\left(\frac{\partial}{\partial u}\alpha_u\right)\tilde{\eta}_u + (\tilde{\alpha}_0 - \tilde{\alpha}_\infty)\beta_\infty \\ &- \bar{\partial}\big((\partial\beta_u)\tilde{\eta}_u\big) - \partial\big(\beta_u\bar{\partial}\,\tilde{\eta}_u\big). \end{aligned} \tag{2.50}$$

Remember that by Theorem 1.6, the current β_∞ is closed, so that using (2.43)

$$\begin{aligned} (\tilde{\alpha}_0 - \tilde{\alpha}_\infty)\beta_\infty &= -(\bar{\partial}\partial\,\tilde{\eta}_\infty)\beta_\infty = \partial((\bar{\partial}\,\tilde{\eta}_\infty)\beta_\infty) \\ \bar{\partial}\big((\partial\beta_u)\tilde{\eta}_u\big) &= \bar{\partial}\big(\partial(\beta_u - \beta_\infty)\tilde{\eta}_u\big). \end{aligned} \tag{2.51}$$

From (2.50), (2.51), we deduce that for any $A \in [1,+\infty[$

$$\begin{aligned}\int_1^A (\beta''_u - \beta''_\infty)\frac{du}{u} &= \tilde{\alpha}_0 \int_1^A (\beta_u - \beta_\infty)\frac{du}{u} + \tilde{\alpha}_A\tilde{\eta}_A - \tilde{\alpha}_1\tilde{\eta}_1 \\ &+ \int_1^A (\alpha_u - \alpha_\infty)\tilde{\beta}_\infty\frac{du}{u} \\ &- \partial \int_1^A (\beta_u\bar{\partial}\tilde{\eta}_u - \beta_\infty\bar{\partial}\tilde{\eta}_\infty)\frac{du}{u} \\ &- \bar{\partial}\int_1^A \partial(\beta_u - \beta_\infty)\tilde{\eta}_u\frac{du}{u}.\end{aligned} \tag{2.52}$$

Using (2.47) and the fact that the current $\tilde{\beta}_\infty$ is closed, we find that if $A \in [1,+\infty[$

$$\int_1^A (\alpha_u - \alpha_\infty)\tilde{\beta}_\infty\frac{du}{u} = \bar{\partial}\partial\int_1^A (\eta_u - \eta_\infty)\tilde{\beta}_\infty\frac{du}{u}. \tag{2.53}$$

We make $A \to +\infty$ in (2.52). By Theorem 1.9, we have

$$\begin{aligned}\int_1^A (\beta''_u - \beta''_\infty)\frac{du}{u} &\to \int_1^{+\infty} (\beta''_u - \beta''_\infty)\frac{du}{u} \text{ in } \mathcal{D}'_{N''^{\bullet}_R}(M). \\ \tilde{\alpha}_0\int_1^A (\beta_u - \beta_\infty)\frac{du}{u} &\to \tilde{\alpha}_0\int_1^{+\infty} (\beta_u - \beta_\infty)\frac{du}{u} \text{ in } \mathcal{D}'_{N^{\bullet}_R}(M).\end{aligned} \tag{2.54}$$

Also, again using Theorem 1.9 and the properties of the product of currents explained after equation (2.48), we find that as $A \to +\infty$

$$\begin{aligned}\alpha_A\tilde{\eta}_A &\to \alpha_\infty\tilde{\eta}_\infty \text{ in } \mathcal{D}'_{N^{\bullet}_R+\tilde{N}^{\bullet}_R}(M). \\ \int_1^A (\alpha_u - \alpha_\infty)\tilde{\beta}_\infty\frac{du}{u} &\to \int_1^{+\infty} (\alpha_u - \alpha_\infty)\tilde{\beta}_\infty\frac{du}{u} \text{ in } \mathcal{D}'_{N^{\bullet}_R+\tilde{N}^{\bullet}_R}(M).\end{aligned} \tag{2.55}$$

In the sequel, the constant C may vary from line to line.

Using (1.18) and (2.41), we find that if μ is a smooth differential form on M, then if $u \geq 1$

$$\left|\int_M \mu(\eta_u - \eta_\infty)\right| \leq C\|\mu\|_{C^1(M)}\int_u^{+\infty}\frac{dv}{v^{3/2}}$$

or equivalently that if $u \geq 1$

$$\left|\int_M \mu(\eta_u - \eta_\infty)\right| \leq C\frac{\|\mu\|_{C^1(M)}}{\sqrt{u}}. \tag{2.56}$$

From (2.56), we get

$$\int_1^A (\eta_u - \eta_\infty)\frac{du}{u} \to \int_1^{+\infty} (\eta_u - \eta_\infty)\frac{du}{u} \text{ in } \mathcal{D}'(M). \tag{2.57}$$

Also if U, Γ, φ, m are taken as in Section 1c) with respect to the submanifold M', using (1.19), we find that

$$p_{U,\Gamma,\varphi,m}(\eta_u - \eta_\infty) \leq C \int_u^{+\infty} \frac{dv}{v^{3/2}}$$

or equivalently

$$p_{U,\Gamma,\varphi,m}(\eta_u - \eta_\infty) \leq \frac{C}{\sqrt{u}}. \tag{2.58}$$

From (2.57), (2.58), we deduce that

$$\int_1^A (\eta_u - \eta_\infty)\frac{du}{u} \to \int_1^{+\infty} (\eta_u - \eta_\infty)\frac{du}{u} \text{ in } \mathcal{D}'_{N^*_R}(M). \tag{2.59}$$

Since M' and $\tilde{M}'$ are transversal, we deduce from (2.59) that

$$\int_1^A (\eta_u - \eta_\infty)\tilde{\beta}_\infty\frac{du}{u} \to \int_1^{+\infty} (\eta_u - \eta_\infty)\tilde{\beta}_\infty\frac{du}{u} \text{ in } \mathcal{D}'_{N^*_R+\tilde{N}^*_R}(M). \tag{2.60}$$

From (2.60), we get

$$\bar{\partial}\partial\int_1^A (\eta_u - \eta_\infty)\tilde{\beta}_\infty\frac{du}{u} \to \bar{\partial}\partial\int_1^{+\infty} (\eta_u - \eta_\infty)\tilde{\beta}_\infty\frac{du}{u} \text{ in } \mathcal{D}'_{N^*_R+\tilde{N}^*_R}(M). \tag{2.61}$$

Clearly, since β_∞ is closed

$$\beta_u\bar{\partial}\,\tilde{\eta}_u - \beta_\infty\bar{\partial}\,\tilde{\eta}_\infty = (\beta_u - \beta_\infty)\bar{\partial}\tilde{\eta}_u + \bar{\partial}\big(\beta_\infty(\tilde{\eta}_u - \tilde{\eta}_\infty)\big). \tag{2.62}$$

The same arguments as in (2.60) show that

$$\int_1^A \beta_\infty(\tilde{\eta}_u - \tilde{\eta}_\infty)\frac{du}{u} \to \int_1^{+\infty} \beta_\infty(\tilde{\eta}_u - \tilde{\eta}_\infty)\frac{du}{u} \text{ in } \mathcal{D}'_{N^*_R+\tilde{N}^*_R}(M). \tag{2.63}$$

and so

$$\bar{\partial}\partial\int_1^A \beta_\infty(\tilde{\eta}_u - \tilde{\eta}_\infty)\frac{du}{u} \to \bar{\partial}\partial\int_1^{+\infty} \beta_\infty(\tilde{\eta}_u - \tilde{\eta}_\infty)\frac{du}{u} \text{ in } \mathcal{D}'_{N^*_R+\tilde{N}^*_R}(M). \tag{2.64}$$

We now study the behaviour of $A \to +\infty$ of the current

$$\int_1^A (\beta_u - \beta_\infty)\bar\partial\,\tilde\eta_u \frac{du}{u}. \tag{2.65}$$

Here the situation is slightly more involved because the current $\bar\partial\,\tilde\eta_u$ depends on u. As $u \to \infty$, the current $\tilde\eta_u$ converges as a smooth current on $M \setminus \tilde M'$. Using the estimates in Theorem 1.5, it is clear that difficulties in the convergence of the currents (2.65) may occur only near M''.

Take $x_0 \in M''$. Let U be a small open neighborhood of x_0, let Γ, φ, m as in Section 1c) with respect to the submanifold M'. Similarly, given U, and replacing M' by $\tilde M'$, we choose $\tilde\Gamma$, $\tilde\varphi$, $\tilde m$ as in Section 1c).

Then by Theorem 1.5, we know that if $u \geq 1$

$$\begin{gathered}
\left|\widehat{\varphi(\beta_u - \beta_\infty)}(\xi)\right| \leq \frac{C}{\sqrt u}(1 + |\xi|) \\
p_{U,\Gamma,\varphi,m}(\beta_u - \beta_\infty) \leq \frac{C}{\sqrt u} \\
\left|\widehat{\tilde\varphi\,\bar\partial\,\tilde\eta_u}(\xi)\right| \leq C(1 + |\xi|^2) \\
p_{U,\tilde\Gamma,\tilde\varphi,\tilde m}(\tilde\eta_u) \leq C.
\end{gathered} \tag{2.66}$$

By proceeding as in Hörmander [H, Theorem 8.2.4] and using the transversality assumption $N^*_{R|M'} \cap \tilde N^*_{R|M'} = \{0\}$, we can easily prove the following estimates.

- There exists $k \in N$ such that for $u \geq 1$

$$\left|\widehat{\varphi(\beta_u - \beta_\infty)\tilde\varphi\,\bar\partial\,\tilde\eta_u}(\xi)\right| \leq \frac{C}{\sqrt u}(1 + |\xi|)^k. \tag{2.67}$$

From (2.67), we immediately deduce that as $A \to +\infty$, the currents (2.65) converge, i.e.

$$\int_1^A (\beta_u - \beta_\infty)\bar\partial\,\tilde\eta_u \frac{du}{u} \to \int_1^{+\infty} (\beta_u - \beta_\infty)\bar\partial\,\tilde\eta_u \frac{du}{u} \text{ in } \mathcal{D}'(M). \tag{2.68}$$

- If θ is a smooth current with compact support in U, if Δ is a closed cone in $R^{2\ell}$ such that $\Delta \cap (N^*_R + \tilde N^*_R) = \{0\}$ on $(M' \cup \tilde M') \cap U$, if m'' is an integer, then for $u \geq 1$

$$p_{U,\Delta,\theta,m''}\big((\beta_u - \beta_\infty)\bar\partial\,\tilde\eta_u\big) \leq \frac{C}{\sqrt u}. \tag{2.69}$$

To prove (2.69), we break a convolution integral in the ξ variable into three pieces, which are separately estimated using (2.66). Here again it is crucial that $N^*_{R|M''} \cap \tilde{N}^*_{R|M''} = \{0\}$.

From (2.68), (2.69), it is now clear that

$$(2.70) \quad \int_1^A (\beta_u - \beta_\infty)\bar{\partial}\tilde{\eta}_u \frac{du}{u} \to \int_1^{+\infty} (\beta_u - \beta_\infty)\bar{\partial}\tilde{\eta}_u \frac{du}{u} \text{ in } \mathcal{D}'_{N^*_R + \tilde{N}^*_R}(M)$$

and so

(2.71)

$$\partial \int_1^A (\beta_u - \beta_\infty)\bar{\partial}\tilde{\eta}_u \frac{du}{u} \to \partial \int_1^{+\infty} (\beta_u - \beta_\infty)\bar{\partial}\tilde{\eta}_u \frac{du}{u} \text{ in } \mathcal{D}'_{N^*_R + \tilde{N}^*_R}(M).$$

We can prove in the same way that as $A \to +\infty$

(2.72)

$$\int_1^A \partial(\beta_u - \beta_\infty)\tilde{\eta}_u \frac{du}{u} \to \int_1^{+\infty} \partial(\beta_u - \beta_\infty)\tilde{\eta}_u \frac{du}{u} \text{ in } \mathcal{D}'_{N^*_R + \tilde{N}^*_R}(M).$$

$$\bar{\partial} \int_1^A \partial(\beta_u - \beta_\infty)\tilde{\eta}_u \frac{du}{u} \to \bar{\partial} \int_1^{+\infty} \partial(\beta_u - \beta_\infty)\tilde{\eta}_u \frac{du}{u} \text{ in } \mathcal{D}'_{N^*_R + \tilde{N}^*_R}(M).$$

From (2.53), (2.54), (2.55), (2.61), (2.71), (2.72) we deduce that we can take the obvious limits in (2.52).

Now note that the currents which appear in (2.46) after the operators ∂, $\bar{\partial}$ or $\bar{\partial}\partial$ are smooth. Similarly as we saw in (2.60), (2.63), (2.70), (2.72), we find in particular that the wave front sets of the currents which appear after operators ∂, $\bar{\partial}$ or $\bar{\partial}\partial$ in (2.52) for $A = +\infty$ are included in $N^*_R + \tilde{N}^*_R$. From (2.46), (2.52), we get

(2.73)

$$\begin{aligned}\int_0^1 (\beta''_u - \beta''_0)\frac{du}{u} &+ \int_1^{+\infty} (\beta''_u - \beta''_\infty)\frac{du}{u} \\ &= \tilde{\alpha}_0 \left\{ \int_0^1 (\beta_u - \beta_0)\frac{du}{u} + \int_1^{+\infty} (\beta_u - \beta_\infty)\frac{du}{u} \right\} \\ &\quad + \alpha_\infty \left\{ \int_0^1 (\tilde{\beta}_u - \tilde{\beta}_0)\frac{du}{u} + \int_1^{+\infty} (\tilde{\beta}_u - \tilde{\beta}_\infty)\frac{du}{u} \right\} \\ &\quad \text{in } P^M_{M' \cup \tilde{M}'}/P^{M,0}_{M' \cup \tilde{M}'}.\end{aligned}$$

Also using (2.40), (2.49), we know that

$$(2.74) \quad \beta''_\infty - \beta''_0 = \tilde{\alpha}_0(\beta_\infty - \beta_0) + \alpha_\infty(\tilde{\beta}_\infty - \tilde{\beta}_0) + \beta_\infty(\tilde{\alpha}_\infty - \tilde{\alpha}_0) + \tilde{\beta}_0(\alpha_\infty - \alpha_0).$$

Remember that the currents $\tilde{\beta}_0$ and β_∞ are closed. Therefore using (2.42), (2.47), we find that

$$\beta_\infty(\tilde{\alpha}_\infty - \tilde{\alpha}_0) = \bar{\partial}\partial(\beta_\infty \tilde{\eta}_\infty). \tag{2.75}$$

The currents $\beta_\infty \tilde{\eta}_\infty$ and $\tilde{\beta}_0 \eta_\infty$ lie in $\mathcal{D}'_{N^\bullet_R + \tilde{N}^\bullet_R}(M)$. From (2.74), (2.75), we find that

$$\beta''_\infty - \beta''_0 = \tilde{\alpha}_0(\beta_\infty - \beta_0) + \alpha_\infty(\tilde{\beta}_\infty - \tilde{\beta}_0) \text{ in } P^M_{M' \cup \tilde{M}'}/P^{M,0}_{M' \cup \tilde{M}'}. \tag{2.76}$$

Using now formula (1.26), together with (2.73), (2.76), we deduce the first line of (2.36). By interchanging the roles of M' and $\tilde{M}'$, we also obtain the second line of (2.36). □

Remark 2.8. It is much easier to prove directly that the right hand sides of both lines of (2.36) coincide in $P^M_{M' \cup \tilde{M}'}/P^{M,0}_{M' \cup \tilde{M}'}$. In fact

$$\begin{aligned} &\bar{\partial}\partial(T(h^\xi))T(h^{\tilde{\xi}}) - T(h^\xi)\bar{\partial}\partial T(h^{\tilde{\xi}}) = \\ &\bar{\partial}(\partial T(h^\xi))T(h^{\tilde{\xi}}) + \partial(T(h^\xi)\bar{\partial}T(h^{\tilde{\xi}})). \end{aligned} \tag{2.77}$$

Using now equation (1.31) for $\frac{\bar{\partial}\partial}{2i\pi}T(h^\xi)$ and its analogue for $\frac{\bar{\partial}\partial}{2i\pi}T(h^{\tilde{\xi}})$, we get from (2.77) that

$$\begin{aligned} &T(h^{\tilde{\xi}})\theta(h^F)\delta_{M'} - ch(h^\xi)T(h^{\tilde{\xi}}) - T(h^\xi)\theta(h^{\tilde{F}})\delta_{\tilde{M}'} \\ &\qquad + ch(h^{\tilde{\xi}})T(h^\xi) \in P^{M,0}_{M' \cup \tilde{M}'}, \end{aligned} \tag{2.78}$$

which is equivalent to the equality of the right-hand sides of (2.36). Note that in (2.78), only equation (1.31) and the wave front set properties of the currents $T(h^\xi)$ and $T(h^{\tilde{\xi}})$ have been used. The identities (2.36) are much deeper, since they involve the explicit form of the currents $T(h^\xi)$, $T(h^{\tilde{\xi}})$, $T(h^{\xi''})$.

Theorem 2.7 plays a fundamental role in the sequel, in particular in our construction of singular currents associated with Koszul complexes. Also as pointed out in the introduction, it should be deeply related to a version with metrics of the Theorem of Grothendieck–Riemann–Roch.

(c) *Bott–Chern currents and double complexes.* Let now $(\xi^j, v)_{1 \le j \le p}$ be p holomorphic chain complexes of vector bundles $\xi^j_0, \dots, \xi^j_m$ on M, which provide resolutions of the sheafs $\mathcal{O}_{M'}(\eta^0), \dots, \mathcal{O}_{M'}(\eta^p)$, where $\eta^0, \dots, \eta^p$

are holomorphic vector bundles on M'. Let r denote the restriction map $\xi^j_{0|M'} \to \eta^j$ $(1 \le j \le p)$.

We also assume that there are holomorphic chain maps $\tilde{v}: \xi^j \to \xi^{j-1}$, $\eta^j \to \eta^{j-1}$ $(1 \le j \le p)$, which are such that the following diagram commutes

$$
(2.79) \qquad
\begin{array}{ccccccccccc}
 & & 0 & & & & 0 & & 0 & & \\
 & & \uparrow & & & & \uparrow & & \uparrow & & \\
0 & \to & \xi^0_m & \to_v & \cdots & \to_v & \xi^0_0 & \to_r & \eta^0 & \to & 0 \\
 & & \uparrow \tilde{v} & & & & \uparrow \tilde{v} & & \uparrow \tilde{v} & & \\
0 & \to & \xi^1_m & \to_v & \cdots & \to_v & \xi^1_0 & \to_r & \eta^1 & \to & 0 \\
 & & \uparrow & & & & & & \uparrow & & \\
 & & \vdots & & & & & & \vdots & & \\
 & & \uparrow \tilde{v} & & & & & & \uparrow \tilde{v} & & \\
0 & \to & \xi^p_m & \to_v & \cdots & \to_v & \xi^p_0 & \to_r & \eta^p & \to & 0 \\
 & & \uparrow & & & & \uparrow & & \uparrow & & \\
 & & 0 & & & & 0 & & 0 & &
\end{array}
$$

Let $(\xi_m, \tilde{v}), \dots, (\xi_0, \tilde{v})$, $(\eta, \tilde{v})$ denote the vertical holomorphic complexes of vector bundles in (2.79).

We now make the fundamental assumption that the complexes $(\xi_m, \tilde{v}), \dots, (\xi_0, \tilde{v})$, $(\eta, \tilde{v})$ are acyclic.

We equip the $(\xi^j_i)_{\substack{1 \le i \le m \\ 1 \le j \le p}}$, and the $(\eta^j)_{1 \le j \le p}$ with Hermitian metrics $(h^{\xi^j_i})_{\substack{1 \le i \le m \\ 1 \le j \le p}}$, and g^{η_j} $(1 \le j \le p)$, and we equip the normal bundle N with a Hermitian metric g^N. Let h^{ξ_i}, h^{ξ^j} be the induced Hermitian metric on ξ_i, ξ^j, and let g^η be the induced Hermitian metric on η.

We also make the assumption that for every $j(1 \le j \le p)$, the metrics h^{ξ^j} verify assumption (A) with respect to the metrics g^N, g^{η^j}.

For any $j(1 \le j \le p)$, we can define the currents $T(h^{\xi^j})$, whose wave front sets are included in $N^*_{\mathbf{R}}$. Similarly, since the complexes $(\xi_i, \tilde{v})_{0 \le i \le m}$ and $(\eta, \tilde{v})$ are acyclic, we can construct the associated smooth currents $T(h^{\xi_i})$ and $T(g^\eta)$ on M and M' respectively. Note that such smooth currents were already constructed in [BGS1, Section 1c)].

Theorem 2.9. *The following equality holds*

$$
(2.80) \qquad \sum_{j=0}^{p} (-1)^j T(h^{\xi^j}) = \sum_{i=0}^{m} (-1)^i T(h^{\xi_i}) - i_*(Td^{-1}(g^N) T(g^\eta))
$$

in $P^M_{M'}/P^{M,0}_{M'}$.

Proof. Let $\mathbf{P}^1$ be the one dimensional projective plane with distinguished points 0 and ∞, and let z be the canonical meromorphic coordinate on $\mathbf{P}^1$. By [BGS1, Section 1f)] or by the Grassman graph construction of [BaFM, Chapter II] which will be explained in more detail in Section 4, one finds easily that there exist a double complex on $M \times \mathbf{P}^1$ which will be noted as in (2.79), having the following three properties.

- The restriction of the new complex to $M \times \{0\}$ coincides with the complex (2.79).

- The rows in the double complex provide resolution of sheaves of holomorphic sections of vector bundles on $M' \times \mathbf{P}^1$, and the columns are acyclic.

- On $M' \times \{\infty\}$, the new double complex splits vertically. In particular if $p = 2$, on $M' \times \{\infty\}$, there is an identification of holomorphic chain complexes

$$(\xi^1, \eta^1) = (\xi^0, \eta^0) \oplus (\xi^2, \eta^2)$$

and $\tilde{v}$ acts as the obvious map. If $p \geq 2$, the complex breaks vertically into short exact sequences of complexes which are split vertically.

Let $\left(\bar{h}^{\xi_i^j}\right)_{\substack{0 \leq i \leq m \\ 0 \leq j \leq p}}$ be Hermitian metrics on the $\left(\xi_i^j\right)_{\substack{0 \leq i \leq m \\ 0 \leq j \leq p}}$ on $M \times \mathbf{P}^1$ which have the following two properties:

- They restrict to the given metrics $\left(h^{\xi_i^j}\right)$ on $M \times \{0\}$.

- On $M' \times \{\infty\}$, the Hermitian chain complexes $(\xi_i, \tilde{v})_{0 \leq i \leq m}$ are split in the sense of [BGS1, Section 1f)]. If $p = 2$, this means that $\xi_i^1 = \xi_i^0 \oplus \xi_i^2$ $(0 \leq i \leq m)$, and that the previous splittings are orthogonal.

For $0 \leq j \leq p$, let $T\left(\bar{h}^{\xi^j}\right)$ be the current on $M \times \mathbf{P}^1$ associated with the Hermitian chain complex ξ^j equipped with the metrics $\left(\bar{h}^{\xi^j}\right)$.

The wave front set of the current $\operatorname{Log}|z|^2$ was calculated in the proof of Theorem 2.5. By Theorem 1.9, the wave front set of the current $T\left(\bar{h}^{\xi^j}\right)$ is included in the normal bundle to $M' \times \mathbf{P}^1$ in $M \times \mathbf{P}^1$, which is the natural lift of N to $M' \times \mathbf{P}^1$. As in the proof of Theorem 2.5, we deduce that the product of currents $\operatorname{Log}|z|^2 T\left(\bar{h}^{\xi^j}\right)$ is well defined. Then we have the analogue of (2.15) i.e.

$$\begin{aligned} &\frac{\bar{\partial}\partial}{2i\pi}(\operatorname{Log}|z|^2)T(\bar{h}^{\xi^j}) - \operatorname{Log}|z|^2 \frac{\bar{\partial}\partial}{2i\pi}T(\bar{h}^{\xi^j}) \\ (2.81) \qquad &= \frac{\bar{\partial}}{2i\pi}(\partial(\operatorname{Log}|z|^2)T(\bar{h}^{\xi^j})) \\ &+ \frac{\partial}{2i\pi}(\operatorname{Log}|z|^2 \bar{\partial} T(\bar{h}^{\xi^j})). \end{aligned}$$

For $0 \leq j \leq p$, let $\bar{h}^{F^j}$ be the metric induced on the direct sum F^j of the homology groups of $(\xi^j, v)_{|M'}$ by the metrics $\bar{h}^{\xi^j}$

Using Theorem 1.9 and (2.81), we get

$$
\begin{aligned}
&\mathrm{Log}|z|^2 \sum_{j=0}^{p}(-1)^j \theta(\bar{h}^{F^j})\delta_{M'\times \mathbf{P}_1} - \mathrm{Log}|z|^2 \sum_{j=0}^{p}(-1)^j ch(\bar{h}^{\xi^j}) \\
&+ \sum_{j=0}^{p}(-1)^j T(\bar{h}^{\xi^j})\delta_{M\times\{\infty\}} - \sum_{j=0}^{p}(-1)^j T(\bar{h}^{\xi^j})\delta_{M\times\{0\}} \\
(2.82) \qquad &= -\frac{\bar{\partial}}{2i\pi}\left(\partial(\mathrm{Log}|z|^2)\left(\sum_{j=0}^{p}(-1)^j T(\bar{h}^{\xi^j})\right)\right) \\
&- \frac{\partial}{2i\pi}\left((\mathrm{Log}|z|^2)\bar{\partial}\left(\sum_{j=0}^{p}(-1)^j T(\bar{h}^{\xi^j})\right)\right)
\end{aligned}
$$

We claim that

$$
(2.83) \qquad \sum_{j=0}^{P}(-1)^j T(\bar{h}^{\xi^j})\delta_{M\times\{\infty\}} = 0.
$$

In fact by the analogue of (2.32), the current $T(\bar{h}^{\xi^j})\delta_{M\times\{\infty\}}$ is the image of the restriction to $M \times \{\infty\}$ of $T(\bar{h}^{\xi^j})$ by the embedding $M \times \{\infty\} \to M \times \mathbf{P}^1$. By [BGS4, Theorem 2.7] or by Theorem 1.10, the restriction of $T(\bar{h}^{\xi^j})$ to $M \times \{\infty\}$ is the Bott–Chern current associated with the chain complex $(\xi^j, v)_{|M\times\{\infty\}}$. Since on $M \times \{\infty\}$, for $1 \leq i \leq m$, the Hermitian chain complex $(\xi_i, \tilde{v})$ splits, it is elementary to verify that for any $u \geq 0$, if $A_u^{\xi^j}$ is the superconnection (1.10) associated with the Hermitian chain complex (ξ^j, v) and if $N_H^{\xi^j}$ is the corresponding number operator, then the restriction to $M \times \{\infty\}$ of the form

$$
\sum_{j=0}^{p}(-1)^j Tr_s[N_H^{\xi^j}\exp(-A_u^{\xi^j})^2]
$$

vanishes identically. (2.83) is now obvious using formula (1.26).

We now integrate (2.82) along the fiber of the projection map $M \times \mathbf{P}^1 \to M$. Taking (2.83) into account, we get

$$\begin{aligned}\sum_{j=0}^{p}(-1)^j T(h^{\xi^j}) = &-\int_{\mathbf{P}^1} \operatorname{Log}|z|^2 \sum_{j=0}^{p}(-1)^j ch(\bar{h}^{\xi^j}) \\ &+ \int_{\mathbf{P}^1} \operatorname{Log}|z|^2 \sum_{j=0}^{p}(-1)^j \theta(\bar{h}^{F^j})\delta_{M'\times\mathbf{P}^1} \\ &+ \frac{\bar{\partial}}{2i\pi}\int_{\mathbf{P}^1} \partial(\operatorname{Log}|z|^2)\sum_{j=0}^{p}(-1)^j T(\bar{h}^{\xi^j}) \\ &+ \frac{\partial}{2i\pi}\int_{\mathbf{P}^1} \operatorname{Log}|z|^2 \bar{\partial}\left(\sum_{j=0}^{p}(-1)^j T(\bar{h}^{\xi^j})\right).\end{aligned} \tag{2.84}$$

By [BGS1, proof of Theorem 1.29], we know that

$$\sum_{i=0}^{m}(-1)^i T(h^{\xi_i}) = -\int_{\mathbf{P}^1} \operatorname{Log}|z|^2 \sum_{i=0}^{m}(-1)^i ch(\bar{h}^{\xi_i}) \quad \text{in } P^M/P^{M,0}. \tag{2.85}$$

Note the trivial relation

$$\sum_{j=0}^{p}(-1)^j ch(\bar{h}^{\xi^j}) = \sum_{i=0}^{m}(-1)^i ch(\bar{h}^{\xi_i}). \tag{2.86}$$

Let $(\tilde{g}^{\eta^j})_{0\le j\le p}$ be Hermitian metrics on the $(\eta^j)_{0\le j\le p}$ on $M' \times \mathbf{P}^1$ which have the following properties:

- They coincide with the metrics $(g^{\eta^j})_{0\le j\le p}$ on $M' \times \{0\}$.
- On $M' \times \{\infty\}$, the Hermitian chain complex $(\eta, \tilde{v})$ splits.

For $0 \le j \le p$, let $(\tilde{h}^{F^j})_{0\le j\le p}$ be the metric on $F^j = \Lambda N^* \otimes \eta^j$ associated with the given metric g^N on N and with the metric $\tilde{g}^{\eta^j}$. Note that

$$\tilde{h}^{F^j} = h^{F^j} \quad \text{on} \quad M' \times \{0\}. \tag{2.87}$$

In Theorem 2.4, a class of forms $\chi(\tilde{h}^{F^j}, \bar{h}^{F^j}) \in P^{M'\times\mathbf{P}^1}/P^{M'\times\mathbf{P}^1,0}$ was constructed such that

$$\frac{\bar{\partial}\partial}{2i\pi}\chi(\tilde{h}^{F^j}, \bar{h}^{F^j}) = \theta(\bar{h}^{F^j}) - \theta(\tilde{h}^{F^j}). \tag{2.88}$$

Using (2.87), we find from Definition 2.3 that $\chi(\tilde{h}^{F^j}, \bar{h}^{F^j})$ restricts to the zero class of forms on $M' \times \{0\}$. For $0 \le j \le p$, let $\chi_\infty(\tilde{h}^{F^j}, \bar{h}^{F^j})$ be the restriction of $\chi(\tilde{h}^{F^j}, \bar{h}^{F^j})$ to $M' \times \{\infty\}$ which we identify with M', so that $\chi_\infty(\tilde{h}^{F^j}, \bar{h}^{F^j})$ is now a form on M'. By proceeding as in (2.15)–(2.19), we get

$$(2.89)\quad \begin{aligned} \int_{\mathbf{P}^1} \mathrm{Log}|z|^2 \Big(\sum_{j=0}^{p}(-1)^j \theta(\bar{h}^{F^j})\Big) &= \int_{\mathbf{P}^1} \mathrm{Log}|z|^2 \Big(\sum_{j=0}^{p}(-1)^j \theta(\tilde{h}^{F^j})\Big) \\ &\quad - \sum_{j=0}^{p}(-1)^j \chi_\infty(\tilde{h}^{F^j}, \bar{h}^{F^j}) \text{ in } P^{M'}/P^{M',0}. \end{aligned}$$

By (2.6), we know that

$$(2.90)\quad \theta(\tilde{h}^{F^j}) = Td^{-1}(g^N) ch(\tilde{g}^{\eta^j}).$$

Using Theorem 1.9, (2.16), (2.90), we find that

$$(2.91)\quad \int_{\mathbf{P}^1} \mathrm{Log}|z|^2 \left(\sum_{j=0}^{p}(-1)^j \theta(\tilde{h}^{F^j})\right) = -Td^{-1}(g^N)T(g^\eta) \text{ in } P^{M'}/P^{M',0}.$$

Using (2.84), (2.85), (2.86), (2.89), (2.91) and proceeding as in the proof of Theorem 2.5, we get

$$(2.92)\quad \begin{aligned} \sum_{j=0}^{p}(-1)^j T(h^{\xi^j}) &= \sum_{i=0}^{m}(-1)^i T(h^{\xi_i}) - i_* Td^{-1}(g^N)T(g^\eta) \\ &\quad - \sum_{j=0}^{p}(-1)^j i_* \chi_\infty(\tilde{h}^{F^j}, \bar{h}^{F^j}) \text{ in } P^M_{M'}/\tilde{P}^{M,0}_{M'}. \end{aligned}$$

To prove Theorem 2.9, we only need to show that

$$(2.93)\quad \sum_{j=0}^{p}(-1)^j \chi_\infty(\tilde{h}^{F^j}, \bar{h}^{F^j}) = 0 \quad \text{in } P^{M'}/P^{M',0}.$$

We now use the notations of Section 1a). Note that since $v\tilde{v} + \tilde{v}v = 0$, if $y \in TM$, then

$$(2.94)\quad \partial_y v\tilde{v} + \tilde{v}\partial_y v + v\partial_y \tilde{v} + \partial_y \tilde{v} v = 0.$$

Clearly $\tilde{v}$ acts on the F^j's and maps F^j into F^{j-1} $(0 \leq j \leq p)$. Using (2.94), we find that if $y \in N$, we have the identity of operators acting on the F^j's

(2.95) $$\partial_y v \tilde{v} + \tilde{v} \partial_y v = 0.$$

Also remember that for any j $(0 \leq j \leq p)$, we have a canonical isomorphism of Z-graded complexes

(2.96) $$(F^j, \partial_y v) \cong (\Lambda N^* \otimes \eta^j, i_y).$$

In particular r provides the canonical isomorphism $F_0^j \cong \eta^j$. Tautologically, the action of $\tilde{v}$ on the F_0^j's coincides with the action of $\tilde{v}$ on the η^j's, i.e.

(2.97) $$(F_0, \tilde{v}) \cong (\eta, \tilde{v}).$$

Using the canonical isomorphism (2.96), (2.95) can be rewritten in the form

(2.98) $$i_y \tilde{v} + \tilde{v} i_y = 0.$$

From (2.96) and (2.98), we deduce easily that $\tilde{v}$ acts on $\Lambda N^* \hat{\otimes} \eta$ like $1 \hat{\otimes} \tilde{v}$.

Let $\bar{h}^F, \tilde{h}^F$ be the metrics on $F = \bigoplus_{j=0}^{p} F^j$ which are the orthogonal sums of the metrics $\bar{h}^{F^j}$, $\tilde{h}^{F^j}$ respectively. Remember that on $M' \times \{\infty\}$, the holomorphic Hermitian complex (2.79) splits vertically. Therefore on $M' \times \{\infty\}$, when equipped with the metric $\bar{h}^F$, the complex $(\Lambda N^* \hat{\otimes} \eta, i_y + \tilde{v})$ also splits vertically. On the other hand, since on $M' \times \{\infty\}$ the Hermitian complex $(\tilde{\eta}, \tilde{v})$ is split, on $M' \times \{\infty\}$ the complex $(\Lambda N^*, i_y + \tilde{v})$ equipped with the metric $\tilde{h}^F$ is also split. For $0 \leq \ell \leq 1$, set

(2.99) $$\tilde{h}_\ell^F = (1-\ell)\bar{h}^F + \ell \tilde{h}^F.$$

Then the complex $(\Lambda N^* \otimes \eta, i_y + \tilde{v})$ still splits vertically as a Hermitian complex when equipped with the metric $\tilde{h}_\ell^F$. Using the notations in Section 2a), we thus find that for $\ell \in [0,1]$

(2.100) $$\sum_{j=0}^{p} (-1)^j Tr_s \left[(\tilde{h}_\ell^{F^j})^{-1} \frac{\partial \tilde{h}_\ell^{F^j}}{\partial \ell} \exp(-(B_\ell^j)^2) \right] = 0.$$

and so from (2.8) and (2.100) we get (2.93).

Our Theorem is proved. □

3. Bott–Chern currents, Euler–Green currents and Koszul complexes

Let M be a complex manifold, let E be a holomorphic Hermitian vector bundle on M. If M^E is the total space of M, if i is the embedding $M \to M^E$, the Koszul complex on $M^E(\Lambda E^*, i_y)$ provides a resolution of the sheaf $i_*\mathcal{O}_M$.

If g^E is the metric of E, and if h^E is the metric induced by g^E on ΛE^*, we first calculate the Bott–Chern current $\tilde{T}(g^E) = T(h^E)$ on M^E. If $e(g^E)$ is the Chern–Weil representative of the highest Chern class of E, then

$$\frac{\bar{\partial}\partial}{2i\pi}\tilde{T}(g^E) = Td^{-1}(g^E)\big(\delta_M - e(g^E)\big). \tag{3.1}$$

A simpler equation of currents on M^E is given by

$$\frac{\bar{\partial}\partial}{2i\pi}\tilde{e}(g^E) = \delta_M - e(g^E), \tag{3.2}$$

so that $-\tilde{e}(g^E)$ is a Green current in the sense of [GS1].

By extending the formalism of Mathai–Quillen [MQ] in a complex geometric framework, we exhibit an explicit locally integrable current $\tilde{e}(g^E)$ on M^E which solves equations (3.2). If $\dim E = 1$, $\tilde{e}(g^E)$ coincides with the obvious solution $\mathrm{Log}|y|^2$.

This section is organized as follows. In a), we recall results of [BGS4] which express the currents $T(h^\xi)$ as a principal part of a nonintegrable current. In b), we calculate the Bott–Chern current of a Koszul complex. In c), we briefly develop a differential geometric setting for the de Rham theory of M^E. In d), we give a double transgression formula for the canonical Thom form of (E, g^E) constructed in the Mathai–Quillen [MQ]. In e), we study the asymptotic behaviour of Mathai–Quillen currents, which here depend on a parameter $u \geq 0$. In f), we explicitly construct a Green current $-\tilde{e}(g^E)$. In g) we compare $\tilde{T}(g^E)$ with $\tilde{e}(g^E)$. In h), we describe $\tilde{e}(g^E)$ as a function of g^E. Finally in i), we study the behaviour of $\tilde{e}(g^E)$ in direct sums.

This section relies on the formalism of Mathai–Quillen [MQ], to which the reader is referred when necessary.

(a) *The current $T(h^\xi)$ as a finite part.* Our assumptions are the same as in Section 1g).

Let $\omega(h^\xi)$ be the restriction of the current $T(h^\xi)$ to the open set $M\backslash M'$. Then by [BGS4, Theorem 2.5], we know that $\omega(h^\xi)$ is a smooth current on $M \setminus M'$. By [BGS4, Theorem 3.3], we know that, in general, $\omega(h^\xi)$ is nonintegrable on $M \setminus M'$. We now briefly recall a result of [BGS4].

Let g^M be a Hermitian metric on TM. We identify N with the vector bundle orthogonal to TM' in TM. Then g^M induces on N a metric g^N. Let g^η be a Hermitian metric on η.

For $\varepsilon > 0$, let M_ε be the set of points of M whose Riemannian distance to M' is larger than ε.

Theorem 3.1. *Let μ be a smooth differential form on M. Then*

(3.3)
$$\begin{aligned}\int_M \mu T(h^\xi) = \lim_{\varepsilon\to 0}\Big\{\int_{M_\varepsilon} &\mu\omega(h^\xi)\\ &+ 2\operatorname{Log}\varepsilon \int_{M'} i^*\mu \int_N \varphi(Tr_s[N_H \exp(-B^2)])\Big\}\\ &- \int_{M'} i^*\mu \int_N (2\operatorname{Log}(|Y|_{g^N}) - \Gamma'(1))\varphi(Tr_s[N_H\exp(-B^2)])\end{aligned}$$

If the metrics $h^\xi = (h^{\xi_0}, \dots, h^{\xi_m})$ verify assumption (A) with respect to the metrics g^N, g^η, then

$$\int_N \varphi(Tr_s[N_H\exp(-B^2)]) = -(Td^{-1})'(g^N)ch(g^\eta)$$

(3.4)
$$\begin{aligned}&\int_N (2\operatorname{Log}(|Y|_{g^N}) - \Gamma'(1))\varphi(Tr_s[N_H\exp(-B^2)])\\ &\quad = -\left(\sum_{k=1}^{\dim N-1}\frac{1}{k} + \operatorname{Log}2\right)(Td^{-1})'(g^N)ch(g^\eta).\end{aligned}$$

Proof. This result was proved in [BGS4, Theorem 3.4]. □

(b) *The singular Bott-Chern current of a Koszul complex.* Let now M be a complex compact connected manifold of dimension ℓ. Let E be a holomorphic vector bundle on M of dimension k. Let M^E be the total space of E. Let p be the projection $M^E \to M$. We embed M in M^E as the zero section of M^E. Let i be this embedding.

Let E^* be the dual of E, and let $\Lambda E^* = \bigoplus_{j=0}^{k} \Lambda^j E^*$ be the exterior algebra of E^*. ΛE^* is a Z-graded vector bundle on M, which lifts to M^E.

If $y \in E$, the interior multiplication operator i_y acts naturally on $(\Lambda E^*)_{py}$. Then the Koszul complex $(\Lambda E^*, i_y)$ provides a resolution of the sheaf $i_* \mathcal{O}_M$, i.e. we have the exact sequence of sheaves

$$0 \to \mathcal{O}_{M^E}(\Lambda^k E^*) \to_{i_y} \cdots \to_{i_y} \mathcal{O}_{M^E} \to_r i_* \mathcal{O}_M \to 0.$$

Assume now that g^E is a Hermitian metric on E. Then g^E induces metrics $h^E = \left(h^{\Lambda^0 E^*}, \dots, h^{\Lambda^k E^*}\right)$ on $\Lambda^0 E^*, \dots, \Lambda^k E^*$. For $y \in E$, the adjoint i_y^* of i_y is given by $i_y^* = \bar{y}\wedge$.

The normal bundle N to M in M^E is exactly E. Note that since E is equipped with the metric g^E, assumption (A) of Section 1b) is verified.

The manifold M^E is noncompact. Still currents on M^E are paired with smooth compactly supported forms on M^E. It is then quite easy to verify that all the results of [B2], [BGS4] and of Sections 1 and 2 are still valid on M^E.

In particular, let $T(h^E)$ be the current on M^E associated with the holomorphic Hermitian chain complex $(\Lambda E^*, i_y)$. To note that $T(h^E)$ depends only on g^E, we will use the notation $\tilde{T}(g^E) = T(h^E)$.

Let e be the ad invariant polynomial on (k,k) matrices $e: A \to \det A$. Then using the notations of Section 1f), $e(g^E)$, $Td(g^E)$ are smooth forms on M. When lifting such forms to M^E, we omit the notation p^*.

Theorem 3.2. *The current $\tilde{T}(g^E)$ lies in $P_M^{M^E}$. Also*

$$\frac{\bar{\partial}\partial}{2i\pi}\tilde{T}(g^E) = Td^{-1}(g^E)\big(\delta_M - e(g^E)\big). \tag{3.5}$$

Proof. Theorem 3.2 is a consequence of Theorem 1.9 and of the classical relation

$$ch(h^E) = e(g^E) Td^{-1}(g^E).$$ □

Let now $\omega(g^E)$ be the restriction of $\tilde{T}(g^E)$ to M^E/M. Then $\omega(g^E)$ is smooth on M^E/M. Also by Theorem 3.1, $\omega(g^E)$ entirely determines $\tilde{T}(g^E)$.

Let ∇^E be the holomorphic Hermitian connection on (E, g^E), and let $\Omega^E = (\nabla^E)^2$ be its curvature.

Before we proceed, we will explain the notations and conventions of Mathai–Quillen [MQ], which will be used in the sequel. First observe that the connection ∇^E defines a horizontal subspace $T^H M^E$ in TM^E so that

$$TM^E = T^H M^E \oplus E. \tag{3.6}$$

If $Y \in T_R E$, let Y^V be its component in E_R with respect to the splitting (3.6). If A is an antisymmetric tensor in $\operatorname{End} E_R$, we identify A with the two form on $T_R M^E$

$$Y, Z \in T_R M^E \to \langle Y^V, AZ^V \rangle. \tag{3.7}$$

The two form (3.7) will still be noted A, and its exterior powers $A^2, \ldots, A^k$. Note that we omit the wedge product sign.

We now follow Mathai–Quillen [MQ]. Namely assume temporarily that A is invertible. Then we can define the forms A^{-1}, $(A^{-1})^2, \ldots, (A^{-1})^k$ which are the powers in $\Lambda(E_R^*)$ of the two form A^{-1}. If $Pf(A)$ is the Pfaffian of A, the forms $Pf(A)A^{-1}, \ldots, Pf(A)(A^{-1})^k$ are in fact rational functions of A, which can be extended by continuity to an arbitrary A. We will still note them this way, even if A is noninvertible.

If $A \in \operatorname{End} E$, A induces the $(1,1)$ form on M^E, $(Y,Z) \in T^{(0,1)}M^E \times T^{(1,0)}M^E \to \langle Y^V, AZ^V \rangle$ which we still note A. Therefore, we can extend to any such A the previous considerations. In particular, if I_E is the identity map of E, we identify I_E with the $(1,1)$ form on M^E defined by $(Y,Z) \in T^{(0,1)}M^E \times T^{(1,0)}M^E \to \langle Y^V, Z^V \rangle$.

Let J_E be the complex structure of E_R. Then, with the previous conventions $J_E = \sqrt{-1} I_E$.

If like in usual Chern–Weil's theory, we replace formally A by the $(1,1)$ form Ω^E on M which takes values in $\operatorname{End} E$, we get a form on M with values in forms on M^E. By antisymmetrization in all the indices, we finally get a differential form on the manifold M^E.

Observe that the formal forms $(\Omega^E)^{-1}, \ldots, ((\Omega^E)^{-1})^k$ should be considered as forms of total degree 0.

Similarly if $A \in \operatorname{End} E_R$, and $Y \in E_R$, we identify AY with the one form $Z \in T_R M^E \to \langle AY, Z^V \rangle$. Then the form $Pf(\Omega^E)(\Omega^E)^{-1}Y$ is well defined. Note that the formal degree of the form $(\Omega^E)^{-1}Y$ is -1.

In the previous constructions, for any $b \in \mathbf{C}$, we can replace Ω^E by $\Omega^E + bJ_E$, and we still get a meaningful form on M^E.

For instance if $\det(B)$ is the determinant of $B \in \operatorname{End} E$, then

$$\det\left(\frac{\Omega^E}{2i\pi} + bI_E\right)\left(\frac{\Omega^E}{2\pi} + bJ_E\right)^{-j} \quad (0 \le j \le k)$$

is a well defined series of forms on M^E.

If $y \in E$, y represents $Y = y + \bar{y} \in E_R$, so that $|Y|^2 = 2|y|^2$.

Let φ be the homomorphism from $\Lambda^{\text{even}}(T_R^* M_E)$ into itself which to $\alpha \in \Lambda^{2j}(T_R^* M_E)$ associates $(2\pi i)^{-j}\alpha$.

Theorem 3.3. *On $M^E \setminus M$, we have the following identity of differential forms*

$$\omega(g^E) = -\frac{\partial}{db}\Big[\det\left(I_E - \exp\left(\frac{\Omega^E}{2i\pi} + bI_E\right)\right) \cdot \mathrm{Log}\left(\frac{|Y|^2}{2} + \left(2\pi\left(\frac{\Omega^E}{2\pi} + bJ_E\right)\right)^{-1}\right)\Big]_{b=0}. \tag{3.8}$$

Equivalently

$$\omega(g^E) = -\frac{\partial}{db}\Big[\det\left(I_E - \exp\left(\frac{\Omega^E}{2i\pi} + bI_E\right)\right) \left\{\mathrm{Log}\left(\frac{|Y|^2}{2}\right) + \sum_{j=1}^{\dim E} \frac{2^j}{j|Y|^{2j}} \left(\left(-2\pi\left(\frac{\Omega^E}{2\pi} + bJ_E\right)\right)^{-1}\right)^j\right\}\Big]_{b=0}. \tag{3.9}$$

Let $\gamma(g^E)$ be the current on M^E

$$\gamma(g^E) = \frac{1}{2}\left(\frac{\bar\partial - \partial}{2i\pi}\right)\tilde{T}(g^E). \tag{3.10}$$

Then $\gamma(g^E)$ is a locally integrable current on M^E given by the formula

$$\gamma(g^E) = \frac{1}{2}\det\left(I_E - \exp\left(\frac{\Omega^E}{2i\pi}\right)\right)(-\Omega^E)^{-1}Y\left(\frac{|Y|^2}{2} + (\Omega^E)^{-1}\right)^{-1} \tag{3.11}$$

or equivalently

$$\gamma(g^E) = \frac{1}{2}\det\left(I_E - \exp\left(\frac{\Omega^E}{2i\pi}\right)\right)(-\Omega^E)^{-1}Y\sum_{j=1}^{\dim E}\frac{2^j}{|Y|^{2j}}((-\Omega^E)^{-1})^{j-1}. \tag{3.12}$$

Also

$$d\gamma(g^E) = Td^{-1}(g^E)\big(e(g^E) - \delta_M\big). \tag{3.13}$$

Proof. Let N_H be the operator acting on $\Lambda^j(E^*)$ $(0 \le j \le k)$ by multiplication by j. We still denote by ∇^E the holomorphic Hermitian connection on ΛE^*. Set $V = i_y + \bar{y}\Lambda$. For $u > 0$, set $A_u = \nabla^E + \sqrt{u}V$. By [B2, equations (3.142), (4.21) and (4.22)], we know that

$$
\begin{aligned}
Tr_s\left[N_H \exp(-A_u^2)\right] = \frac{\partial}{\partial b}\Big[&\det\left(I - \exp(\Omega^E + bI_E)\right) \\
&\cdot \exp\left\{-u\left(\frac{|Y|^2}{2} + (\Omega^E + bI_E)^{-1}\right)\right\}\Big]_{b=0}.
\end{aligned}
\tag{3.14}
$$

From (3.14), we deduce that on $M^E \setminus M$, if $s \in \mathbb{C}$, $\mathrm{Re}(s) > 0$

$$
\begin{aligned}
&\frac{1}{\Gamma(s)}\int_0^{+\infty} u^{s-1} Tr_s\left[N_H \exp(-A_u^2)\right] du \\
&\quad = \frac{\partial}{\partial b}\left[\det\left(I - \exp(\Omega^E + bI_E)\right)\left\{\frac{|Y|^2}{2} + (\Omega^E + bI_E)^{-1}\right\}^{-s}\right]_{b=0}.
\end{aligned}
\tag{3.15}
$$

and so

$$
\begin{aligned}
&\left[\frac{1}{\Gamma(s)}\int_0^{+\infty} u^{s-1} Tr_s\left[N_H \exp(-A_u^2)\right] du\right]'(0) \\
&\quad = -\frac{\partial}{\partial b}\left[\det\left(I - \exp(\Omega^E + bI_E)\right)\mathrm{Log}\left(\frac{|Y|^2}{2} + (\Omega^E + bI_E)^{-1}\right)\right]_{b=0}.
\end{aligned}
\tag{3.16}
$$

Using (3.16) we obtain (3.8). Note here that as a two form, Ω^E gets rescaled by a factor $\frac{1}{2i\pi}$, and that as a two form of partial vertical degree 2, $(\Omega^E + bI_E)^{-1}$ is also rescaled by the same factor. By expanding the Log in (3.8), we obtain (3.9). Note that the expansion terminates at $k = \dim E$, since this corresponds to the highest possible power of a two form on E_R.

By [BGS4, Theorem 3.3], we know that the current $\gamma(g^E)$ is locally integrable. We temporarily fix an arbitrary square root of $i = \sqrt{-1}$. We extend the map φ to the whole $\Lambda(T_R^* M^E)$ so that if $\alpha \in \Lambda^k(T_R^* M^E)$, then $\varphi(\alpha) = (2\pi i)^{-k/2}\alpha$. Set

$$
\eta = \int_0^{+\infty} Tr_s\left[\sqrt{u}\, V \exp(-A_u^2)\right] \frac{du}{2u}.
\tag{3.17}
$$

Then by [BGS4, Theorem 2.5], η defines a locally integrable current M^E, and moreover

$$
\frac{1}{2}(\bar{\partial} - \partial)\tilde{T}(g^E) = (2i\pi)^{1/2}\varphi(\eta).
\tag{3.18}
$$

By [B2, equations (4.9) and (4.11)], we know that

$$Tr_s\left[\sqrt{u}\, V \exp(-A_u^2)\right] = -i_Y Tr_s\left[\exp(-A_u^2)\right].$$

Using [MQ, Theorem 4.5], or [B2, equations (3.138), (3.139)] we find that

$$\begin{aligned} i_Y Tr_s\left[\exp(-A_u^2)\right] &= i_Y \det\left(I - \exp(\Omega^E)\right) \\ &\quad \exp\left\{-u\left(\frac{|Y|^2}{2} + (\Omega^E)^{-1}\right)\right\}. \end{aligned} \tag{3.19}$$

Clearly, $i_Y(\Omega^E)^{-1} = -(\Omega^E)^{-1}Y$, and so from (3.19), we find that

$$\begin{aligned} i_Y Tr_s\left[\exp(-A_u^2)\right] &= u \det\left(I - \exp(\Omega^E)\right)(\Omega^E)^{-1}Y \\ &\quad \cdot \exp\left\{-u\left(\frac{|Y|^2}{2} + (\Omega^E)^{-1}\right)\right\}. \end{aligned} \tag{3.20}$$

Using (3.17) and (3.20) we find that

$$\eta = -\frac{1}{2}\det\left(I - \exp(\Omega^E)\right)(\Omega^E)^{-1}Y\left(\frac{|Y|^2}{2} + (\Omega^E)^{-1}\right)^{-1}.$$

Since the form $(\Omega^E)^{-1}Y$ is of formal degree -1, we then get

$$\begin{aligned} \varphi(\eta) = -\frac{1}{2}\det\left(I - \exp\left(\frac{\Omega^E}{2i\pi}\right)\right)(2i\pi)^{1/2}(\Omega^E)^{-1}Y \\ \cdot\left(\frac{|Y|^2}{2} + (\Omega^E)^{-1}\right)^{-1}. \end{aligned} \tag{3.21}$$

From (3.18), (3.21) we obtain (3.11). By expanding (3.11), we get (3.12). (3.13) now follows from (3.5) and (3.10).

Remark 3.4. By formula (3.9), it is clear that in general the current $\omega(g^E)$ is not locally integrable, since its singularity near M is of the form $|Y|^{-2\dim E}$. The singularity of $\gamma(g^E)$ near M^E is controlled by $|Y|^{-2\dim E+1}$ which is integrable near 0.

Remark 3.5. Let s be a holomorphic section of E on M which is transversal to M in M^E. Namely we assume that if $x \in M$ is such that $s(x) = 0$, if $ds(x)$ is the differential of s at x, then $\mathrm{Im}[ds(x)] = E$. Set

$$M' = \{x \in M; s(x) = 0\}. \tag{3.22}$$

Then on M', ds identifies $E_{|M'}$ with the normal bundle N to M'. Let i be the embedding $M' \to M$.

The chain complex $(\Lambda E^*, i_s) = s^*(\Lambda E^*, i_y)$ provides a resolution of the sheaf $i_* \mathcal{O}_{M'}$, i.e. we have the exact sequence of sheaves

$$\mathcal{O}_M(\Lambda E^*, i_s) \to_r i_* \mathcal{O}_{M'} \to 0.$$

Let h^E be the metric induced by g^E on ΛE^*. Let $\tilde{T}(g^E, s) = T(h^E, s)$ be the singular Bott–Chern current on M associated with the complex $(\Lambda E^*, i_s)$. Remember that $WF(\tilde{T}(g^E)) \subset E_R$. By [H, Theorem 8.2.4], the pulled-back current $s^* \tilde{T}(g^E)$ on M is well-defined. It follows from [BGS4, Theorem 2.7] or from Theorem 2.10 that

$$\tilde{T}(g^E, s) = s^* \tilde{T}(g^E). \tag{3.23}$$

(c) *Equivariant cohomology and differential forms.* We make the same assumptions as in Section 3b), and we use the same notations.

Clearly if p is the projection $M^E \to M$, then $T^H M^E \cong p^* TM$. In the sequel, we will generally omit the notation p^*. From (3.6), we deduce an isomorphism of smooth vector bundles

$$\Lambda(T_R^* M^E) \cong \Lambda(T_R^* M) \hat{\otimes} \Lambda(E_R^*). \tag{3.24}$$

If β is a smooth section of $\Lambda^j(E_R^*)$ on M^E, $\nabla^E \beta$ is a smooth section of $\Lambda^1(T_R^* M^E) \hat{\otimes} \Lambda^j(E_R^*)$. We denote by ${}^a\nabla^E \beta$ the corresponding $j+1$ form on M^E which is obtained by using (3.24) and antisymmetrization in the indices of $\nabla^E \beta$.

If α and β are smooth sections on M^E of $\Lambda(T_R^* M)$ and $\Lambda(E_R^*)$ respectively, set

$${}^a\nabla^E(\alpha\beta) = (d\alpha)\beta + (-1)^{\deg \alpha} \alpha({}^a\nabla^E \beta). \tag{3.25}$$

From (3.24), we deduce that ${}^a\nabla^E$ acts on the smooth sections on M^E of $\Lambda(T_R^* M^E)$.

$\Omega^E Y$ is a two form on M with values in vectors in $E_R \subset T_R M_E$. Therefore the operator $i_{\Omega^E Y}$ acts on $\Lambda(T_R^* M^E)$ and increases the total degree in $\Lambda(T_R^* M^E)$ by 1.

In the sequel d denotes exterior differentiation acting on smooth sections of $\Lambda(T_R^* M^E)$ on M^E.

Proposition 3.6. *The following identity holds*

$$d = {}^a\nabla^E + i_{\Omega^E Y}. \tag{3.26}$$

Proof. Let ∇^M be any torsion free connection on $T_R M$. ∇^M lifts into a connection on $T^H M^E$, which we still note ∇^M. Using the splitting (3.6) of TM^E, the connection $\nabla = \nabla^M \oplus \nabla^E$ defines a connection on $T_R M^E$. Let T be the torsion of ∇. One verifies easily that if $Y \in M^E$, $U, U' \in (T_R^* M^E)_Y$, then

$$T_Y(U, U') = \Omega^E(p_* U, p_* U')Y. \tag{3.27}$$

The connection ∇ maps smooth sections of $\Lambda^j(T_R^* M^E)$ into smooth sections of $\Lambda^1(T_R^* M^E) \hat{\otimes} \Lambda^j(T_R^* M^E)$. By antisymmetrization in all the indices, we obtain an operator ${}^a\nabla$ which map smooth sections of $\Lambda^j(T_R^* M^E)$ into smooth sections of $\Lambda^{j+1}(T_R^* M^E)$. Classically

$$d = {}^a\nabla + i_T. \tag{3.28}$$

Using (3.27), we get

$$d = {}^a\nabla + i_{\Omega^E Y}. \tag{3.29}$$

We now take α, β as in (3.25). Since $\Omega^E Y$ takes its values in E_R, $i_{\Omega^E Y}\alpha = 0$ and so

$${}^a\nabla\alpha = d\alpha. \tag{3.30}$$

Similarly using (3.29), we get

$$({}^a\nabla + i_{\Omega^E Y})\beta = d\beta. \tag{3.31}$$

From (3.30), (3.31), we obtain (3.26). □

Remark 3.7. Equation (3.26) is the differential geometric counterpart to the algebraic considerations in [MQ, Section 5].

(d) *Double transgression formulas for the Mathai–Quillen Thom form.* We now recall the basic result of Mathai–Quillen [MQ, Theorem 4.10] in the context of complex geometry.

Theorem 3.8. *For any $u > 0$, let a_u be the form on M^E*

$$a_u = \det\left(\frac{-\Omega^E}{2i\pi}\right) \exp\left\{-u\left(\frac{|Y|^2}{2} + (\Omega^E)^{-1}\right)\right\}. \tag{3.32}$$

Then for any $u > 0$, the form a_u is closed and lies in P^{M^E}. It is integrable on M^E and represents the Thom class of E_R.

Proof. As a two form on M with values in End E, Ω^E is of type $(1,1)$. Also since Ω^E is a complex endomorphism of E_R, for any $U, V \in T_R M$, $\Omega^E(U,V)$ induces a two form on E_R of type $(1,1)$. It is now clear that a_u is a form of type (k,k).

By Bianchi's identities, ${}^a\nabla^E \Omega^E = 0$. We now use the rotations $(\Omega^E)^{-1}$ with the conventions described before. We find that ${}^a\nabla(\Omega^E)^{-1} = 0$. $(\Omega^E)^{-1}Y$ is a form of degree -1 taking values in E. Using Proposition 3.6, we find that

$$d(\Omega^E)^{-1}Y = -2(\Omega^E)^{-1} - |Y|^2 \tag{3.33}$$

and so

$$d\left(\frac{|Y|^2}{2} + (\Omega^E)^{-1}\right) = 0. \tag{3.34}$$

We thus find that $da_u = 0$. By proceeding as in [MQ, Theorem 4.10], we find that the integral along the fiber $\int_{E_R} a_u$ is equal to 1. Therefore a_u represents the Thom class of E_R. □

In [MQ, Theorem 7.6], Mathai–Quillen gave a formula for the Euler form of a real oriented Euclidean vector bundle, by transgressing their formula (3.32) for a_u. We will here reprove this result, and besides obtain a double transgression formula.

Definition 3.9. For $u \geq 0$, let b_u and c_u be the forms on M^E

$$\begin{aligned} b_u &= \frac{1}{2}\det\left(\frac{-\Omega^E}{2i\pi}\right)(-\Omega^E)^{-1}Y\exp\left\{-u\left(\frac{|Y|^2}{2} + (\Omega^E)^{-1}\right)\right\}. \\ c_u &= \frac{\partial}{\partial b}\left[\det\left(-\left(\frac{\Omega^E}{2i\pi} + bI_E\right)\right)\right. \\ &\quad \left.\cdot\exp\left\{-u\left(\frac{|Y|^2}{2} + \left(2\pi\left(\frac{\Omega^E}{2\pi} + bJ_E\right)\right)^{-1}\right)\right\}\right]_{b=0}. \end{aligned} \tag{3.35}$$

Theorem 3.10. *The form b_u is of total degree $2\dim E - 1$, and the form c_u is of complex type $(\dim E - 1, \dim E - 1)$. For any $u > 0$*

$$\begin{aligned} \frac{\partial}{\partial u}a_u &= -db_u \\ b_u &= \frac{1}{2u}\left(\frac{\bar\partial - \partial}{2i\pi}\right)c_u. \end{aligned} \tag{3.36}$$

In particular for $u > 0$

$$\frac{\partial}{\partial u} a_u = \frac{\bar{\partial}\partial}{2i\pi}\left(\frac{c_u}{u}\right). \tag{3.37}$$

Proof. Equation (3.36) is proved in Mathai–Quillen [MQ, Section 7]. In fact using (3.33), we have

$$a_u = \det\left(-\frac{\Omega^E}{2i\pi}\right)\exp\left(\frac{ud}{2}(\Omega^E)^{-1}Y\right) \tag{3.38}$$

and so

$$\begin{aligned}\frac{\partial}{\partial u} a_u &= \frac{1}{2}\det\left(-\frac{\Omega^E}{2i\pi}\right) d(\Omega^E)^{-1}Y \exp\left\{-u\left(\frac{|Y|^2}{2} + (\Omega^E)^{-1}\right)\right\} \\ &= d\left(\frac{1}{2}(\Omega^E)^{-1}Y a_u\right).\end{aligned} \tag{3.39}$$

Let θ be the Kähler form of E_R. If $X, Y \in E_R$, then $\theta(X,Y) = \langle X, J_E Y\rangle$. If $X \in E_R$, the element of E_R^* which corresponds to X by the metric is given by

$$\sqrt{-1}(-i_{X^{(0,1)}} + i_{X^{(1,0)}})\theta.$$

Let A be an invertible skew-adjoint matrix in $\operatorname{End} E$. Let θ^A be the $(1,1)$ form on E

$$\begin{aligned}(U,V) \to \theta^A(U,V) = \theta(A^{-1}U, A^{-1}V) &= -\theta(A^{-2}U, V) \\ &= -\theta(U, A^{-2}V).\end{aligned} \tag{3.40}$$

From (3.40), we deduce that if $Y \in E_R$, if $A^{-1}Y \in E_R$ is identified with the corresponding element in E_R^*, then

$$A^{-1}Y = \sqrt{-1}(i_{AY^{(0,1)}} - i_{AY^{(1,0)}})\theta^A. \tag{3.41}$$

If $d^E = \partial^E + \bar{\partial}^E$ is the exterior differential on one fiber E, set

$$\partial_A^E = \partial^E + i_{AY^{(0,1)}};\ \bar{\partial}_A^E = \bar{\partial}^E + i_{AY^{(1,0)}}. \tag{3.42}$$

Clearly

$$(\partial_A^E)^2 = 0;\ (\bar{\partial}_A^E)^2 = 0. \tag{3.43}$$

Also the Lie derivative with respect to AY is given by

$$L_{AY} = (d^E + i_{AY})^2. \tag{3.44}$$

Using (3.43), (3.44), we get

$$L_{AY} = \bar{\partial}_A^E \partial_A^E + \partial_A^E \bar{\partial}_A^E. \tag{3.45}$$

Clearly $L_{AY}\theta^A = 0$, and so

$$\bar{\partial}_A^E \partial_A^E \theta^A = -\partial_A^E \bar{\partial}_A^E \theta^A. \tag{3.46}$$

Since the form θ^A is ∂^E and $\bar{\partial}^E$ closed, we obtain from (3.41) that

$$\sqrt{-1}(\partial_A^E - \bar{\partial}_A^E)\theta^A = A^{-1}Y. \tag{3.47}$$

Also

$$\frac{|Y|^2}{2} + A^{-1} = -(\partial_A^E + \bar{\partial}_A^E)\frac{A^{-1}Y}{2}. \tag{3.48}$$

From (3.46), (3.48), we get

$$\frac{|Y|^2}{2} + A^{-1} = -\sqrt{-1}\,\bar{\partial}_A^E \partial_A^E \theta^A. \tag{3.49}$$

In particular

$$\partial_A^E \left(\frac{|Y|^2}{2} + A^{-1}\right) = \bar{\partial}_A^E \left(\frac{|Y|^2}{2} + A^{-1}\right) = 0. \tag{3.50}$$

We deduce from (3.47), (3.50) that

$$\begin{aligned} &i \det\left(\frac{-A}{2i\pi}\right)(-A)^{-1}Y \exp\left\{-u\left(\frac{|Y|^2}{2} + A^{-1}\right)\right\} \\ &\quad = (\partial_A^E - \bar{\partial}_A^E)\left[\det\left(\frac{-A}{2i\pi}\right)\theta^A \exp\left\{-u\left(\frac{|Y|^2}{2} + A^{-1}\right)\right\}\right]. \end{aligned} \tag{3.51}$$

In general, the form $\det\left(\frac{-A}{2i\pi}\right)\theta^A \exp\left\{-u\left(\frac{|Y|^2}{2} + A^{-1}\right)\right\}$ cannot be extended by continuity into a well-defined form when A is noninvertible. On the contrary the form

(3.52)

$$\begin{aligned} &\frac{\partial}{\partial b}\left[\det\left(-\left(\frac{A}{2i\pi} + bI_E\right)\right)\exp\left\{-u\left(\frac{|Y|^2}{2} + \left(2\pi\left(\frac{A}{2\pi} + bJ_E\right)\right)^{-1}\right)\right\}\right]_{b=0} \\ &\quad = \frac{\partial}{\partial b}\left[\det\left(-\left(\frac{A}{2i\pi} + bI_E\right)\right)\right]_{b=0} \exp\left\{-u\left(\frac{|Y|^2}{2} + A^{-1}\right)\right\} \\ &\qquad - u \det\left(\frac{-A}{2i\pi}\right)2\pi\theta^A \exp\left\{-u\left(\frac{|Y|^2}{2} + A^{-1}\right)\right\} \end{aligned}$$

extends by continuity to arbitrary A. Also it is clear that the first form in the right-hand side of (3.52) is ∂_A^E and $\bar\partial_A^E$ closed. From (3.51), (3.52), we find that

$$\begin{aligned}(3.53)\qquad &\det\left(\frac{-A}{2i\pi}\right)(-A)^{-1}Y\exp\left\{-u\left(\frac{|Y|^2}{2}+A^{-1}\right)\right\}\\ &=\frac{1}{u}\left(\frac{\bar\partial_A^E-\partial_A^E}{2i\pi}\right)\frac{\partial}{\partial b}\left[\det\left(-\left(\frac{A}{2i\pi}+bI_E\right)\right)\exp\left\{-u\left(\frac{|Y|^2}{2}\right.\right.\right.\\ &\left.\left.\left.+\left(2\pi\left(\frac{A}{2\pi}+bJ_E\right)^{-1}\right)^{-1}\right)\right\}\right]_{b=0}\end{aligned}$$

and both sides of (3.53) extend to arbitrary A. Using Proposition 3.6 and (3.53), we obtain (3.36). (3.37) follows from (3.36).

Since $(-\Omega^E)^{-1}Y$ is of degree -1, the form b_u is of degree $2\dim E-1$. If we give the degree two to the variable b, the form

$$\begin{aligned}(3.54)\qquad d_u(b)=\det&\left(-\left(\frac{\Omega^E}{2i\pi}+bI_E\right)\right)\exp\left(-u\left(\left(\frac{|Y|^2}{2}\right.\right.\right.\\ &\left.\left.\left.+2\pi\left(\frac{\Omega^E}{2\pi}+bJ_E\right)^{-1}\right)\right)\right)\end{aligned}$$

has total degree $2\dim E$. When differentiating this form with respect to b at $b=0$, we get a form of total degree $2\dim E-2$. Since $d_u(b)$ is a sum of forms of type (p,p), c_u is of type $(\dim E-1,\dim E-1)$. Our theorem is proved. □

Remark 3.11. (3.36), (3.37) can also be derived from Theorem 1.4 and from equalities (3.14), (3.17), (3.20). Also it is no accident that the Kähler form θ appears in (3.40), (3.51). For the role of Kähler forms in double transgression formulas, we refer to [BGS2] and to [B3]. In particular the computations in (3.42)–(3.51) are related to complex equivariant cohomology and appear in another form in [B3], in relation with Quillen metrics and analytic torsion.

(e) *Convergence of Mathai–Quillen currents.* We make the same assumptions as in Section 3d), and we also use the same notations.

Theorem 3.12. *Take $n\in\mathbb{N}$. Then there exists a constant $C>0$ such that if μ is a smooth differential form on M^E with compact support* *included in* $B_n = \{Y \in M^E;\ |Y| \leq n\}$, *for* $u \geq 1$

$$\left|\int_{M^E} \mu(a_u - \delta_M)\right| \leq \frac{C}{\sqrt{u}}\|\mu\|_{C^1(M^E)}$$

$$\left|u\int_{M^E} \mu b_u\right| \leq \frac{C}{\sqrt{u}}\|\mu\|_{C^1(M^E)} \tag{3.55}$$

$$\left|\int_{M^E} \mu c_u\right| \leq \frac{C}{\sqrt{u}}\|\mu\|_{C^1(M^E)}.$$

Moreover, if U, Γ, φ, m *are taken as in Section* 1c) *with respect to the embedding* $i\colon M \to M^E$, *there exists* $C' > 0$ *such that for* $u \geq 1$

$$p_{U,\Gamma,\varphi,m}(a_u - \delta_M) \leq \frac{C}{\sqrt{u}}$$

$$p_{U,\Gamma,\varphi,m}(ub_u) \leq \frac{C}{\sqrt{u}} \tag{3.56}$$

$$p_{U,\Gamma,\varphi,m}(c_u) \leq \frac{C}{\sqrt{u}}.$$

Proof. Let τ_u be the map $Y \to \sqrt{u}Y$. Then $a_u = \tau_u^* a_1$, $ub_u = \tau_u^* b_1$, $c_u = \tau_u^* c_1$. If $\sigma_u = \tau_u^{-1}$ then

$$\int_{M^E} \mu\, a_u = \int_{M^E} (\sigma_u^* \mu) a_1$$

$$\int_{M^E} \mu\, ub_u = \int_{M^E} (\sigma_u^* \mu) b_1$$

$$\int_{M^E} \mu\, c_u = \int_{M^E} (\sigma_u^* \mu) c_1.$$

We thus deduce easily that as $u \to \infty$

$$\int_{M^E} \mu\, a_u \to \int_M (i^*\mu) \int_E a_1$$

$$\int_{M^E} \mu\, ub_u \to \int_M (i^*\mu) \int_E b_1 \tag{3.57}$$

$$\int_{M^E} \mu\, c_u \to \int_M (i^*\mu) \int_E c_1.$$

Moreover, it is clear from (3.32) that $\int_E a_1 = 1$. Also since $i_Y(\Omega^E)^{-1} =$

$-(\Omega^E)^{-1}Y$, we find that $b_1 = -\frac{1}{2} i_Y a_1$ and so $\int_E b_1 = 0$. Finally

$$\int_E c_1 = (2\pi)^{-\dim E} \frac{\partial}{\partial b}\left[\det\left(-\left(\frac{\Omega^E}{2i\pi} + bI_E\right)\right) \cdot \det\left(-\left(\frac{\Omega^E}{2i\pi} + bI_E\right)\right)^{-1}\right]_{b=0} = 0. \tag{3.58}$$

The estimates in (3.55) obviously follow from (3.57). Finally the estimates in (3.56) follow easily from the methods of [B2, proof of Theorem 3.3]. □

(f) *An Euler–Green current.* Note that the form c_0 is clearly closed.

Definition 3.13. For $s \in \mathbf{C}$, $0 < \mathrm{Re}(s) < \frac{1}{2}$, let $\rho(s)$ be the current on M^E

$$\rho(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} u^{s-1} c_u du. \tag{3.59}$$

By (3.55), one verifies that the current $\rho(s)$ is well defined. Also $\rho(s)$ extends into a meromorphic function of $s \in \mathbf{C}$ which is holomorphic at $s = 0$.

Let $\tilde{e}(g^E)$ be the current

$$\tilde{e}(g^E) = \rho'(0). \tag{3.60}$$

Equivalently

$$\tilde{e}(g^E) = \int_0^1 (c_u - c_0)\frac{du}{u} + \int_1^{+\infty} c_u \frac{du}{u} - \Gamma'(1)c_0. \tag{3.61}$$

Finally, let $\psi(g^E)$ be the current

$$\psi(g^E) = \int_0^{+\infty} b_u du. \tag{3.62}$$

By (3.55), the current $\psi(g^E)$ is also well-defined.

Remember that E is the normal bundle to M in M^E.

Theorem 3.14. *The total degree of the current* $\psi(g^E)$ *is* $2\dim E - 1$. *The current* $\tilde{e}(g^E)$ *is of complex type* $(\dim E - 1, \dim E - 1)$. *The wave* *front sets of the currents $\tilde{e}(g^E)$ and $\psi(g^E)$ are included in $E^*_{\mathbf{R}}$. In particular $\tilde{e}(g^E) \in P_M^{M^E}$. The following equations of currents hold on M^E*

$$\begin{aligned} \psi(g^E) &= \frac{1}{2}\left(\frac{\bar{\partial}-\partial}{2i\pi}\right)\tilde{e}(g^E) \\ d\psi(g^E) &= e(g^E) - \delta_M. \end{aligned} \tag{3.63}$$

In particular

$$\frac{\bar{\partial}\partial}{2i\pi}\tilde{e}(g^E) = \delta_M - e(g^E). \tag{3.64}$$

Proof. The degree of the currents $\psi(g^E)$ and $\tilde{e}(g^E)$ can be calculated by using Theorem 3.10. The wave front set properties of our currents follow easily from Theorem 3.12. The first line of (3.63) follows from (3.36), (3.61), (3.62) and from the fact that the form c_0 is closed. The second line of (3.63) also follows from the same equations. □

Theorem 3.15. *The currents $\tilde{e}(g^E)$ and $\psi(g^E)$ are locally integrable. Also we have the formulas*

$$\begin{aligned} \tilde{e}(g^E) &= -\frac{\partial}{\partial b}\Bigg[\det\left(-\left(\frac{\Omega^E}{2i\pi}+bI_E\right)\right)\mathrm{Log}\Bigg(\frac{|Y|^2}{2} \\ &\qquad + \left(2\pi\left(\frac{\Omega^E}{2\pi}+bJ_E\right)\right)^{-1}\Bigg)\Bigg]_{b=0} \\ \psi(g^E) &= \frac{1}{2}\det\left(-\frac{\Omega^E}{2i\pi}\right)(-\Omega^E)^{-1}Y\left(\frac{|Y|^2}{2}+(\Omega^E)^{-1}\right)^{-1}. \end{aligned} \tag{3.65}$$

Equivalently

(3.66)

$$\begin{aligned} \tilde{e}(g^E) &= -\frac{\partial}{\partial b}\Bigg[\det\left(-\left(\frac{\Omega^E}{2i\pi}+bI_E\right)\right) \\ &\quad \cdot\left\{\mathrm{Log}\left(\frac{|Y|^2}{2}\right)+\sum_{j=1}^{\dim E-1}\frac{2^j}{j|Y|^{2j}}\left(\left(-2\pi\left(\frac{\Omega^E}{2\pi}+bJ_E\right)\right)^{-1}\right)^j\right\}\Bigg]_{b=0} \\ \psi(g^E) &= \frac{1}{2}\det\left(-\frac{\Omega^E}{2i\pi}\right)(-\Omega^E)^{-1}Y\sum_{j=1}^{\dim E}\frac{2^j}{|Y|^{2j}}((-\Omega^E)^{-1})^{j-1}. \end{aligned}$$

In particular, if $\dim E = 1$, *then*

$$\tilde{e}(g^E) = \mathrm{Log}(|y|^2). \tag{3.67}$$

Proof. Using formula (3.61), it is clear that in order to prove that the current $\tilde{e}(g^E)$ is locally integrable, we only need to show that the current $\int_1^{+\infty} c_u \frac{du}{u}$ is locally integrable. From (3.35), we get

$$c_u = \exp\left(-\frac{u|Y|^2}{2}\right)\frac{\partial}{\partial b}\left[\det\left(-\left(\frac{\Omega^E}{2i\pi} + bI_E\right)\right)\frac{\sum_{j=0}^{\dim E}\left(\left(-2\pi\left((\Omega^E/2\pi)+bJ_E\right)\right)^{-1}\right)^j u^j}{j!}\right]_{b=0}. \tag{3.68}$$

Let us now again make the key observation (which was already made in another form in (3.58)) that in (3.68), the last index in the sum is in fact $\dim E - 1$ and not $\dim E$. Also for $k \geq 0$

$$\begin{aligned}\int_1^{+\infty} \exp\left(-\frac{u|Y|^2}{2}\right) u^{k-1} du &\leq C\left(1 + \operatorname{Log}\frac{1}{|Y|}\right) \quad \text{if } k = 0 \\ &\leq |Y|^{-2k} \quad \text{if } k > 0.\end{aligned} \tag{3.69}$$

Now the function $\operatorname{Log}|Y|$ is locally integrable on M^E. Similarly for $1 \leq k \leq \dim E - 1$, the function $|Y|^{-2k}$ is locally integrable. From (3.69), we deduce that the current $\tilde{e}(g^E)$ is locally integrable on M^E. The explicit calculation of $\tilde{e}(g^E)$ can then be done as in (3.16). The properties of the current $\psi(g^E)$ can be proved in the same way.

Remember that $|Y|^2 = 2|y|^2$. If $\dim E = 1$, again using (3.58), we find that

$$\tilde{e}(g^E) = \left[\frac{\partial}{\partial b}\left(\frac{\Omega^E}{2i\pi} + b\right)\right]_{b=0} \operatorname{Log}|y|^2 = \operatorname{Log}|y|^2.$$

Our theorem is proved. □

Remark 3.16. We make the same assumptions as in Remark 3.5. Then for essentially the same reasons as in Remark 3.5, the current $\tilde{e}(g^E)$ can be pulled back by the section s. $s^*\tilde{e}(g^E)$ is now a current on M which lies in $P_{M'}^M$, and also

$$\frac{\bar{\partial}\partial}{2i\pi} s^*\tilde{e}(g^E) = \delta_{M'} - e(g^E).$$

(g) *Comparison of the currents $\tilde{T}(g^E)$ and $\tilde{e}(g^E)$.*

Theorem 3.17. *The following equation holds*

$$\tilde{T}(g^E) - Td^{-1}(g^E)\tilde{e}(g^E) \in P_M^{M^E,0}. \tag{3.70}$$

Proof. We use the notations of the proof of Theorem 3.3. By (3.14), we know that

$$\varphi(Tr_s[N_H\exp(-A_u^2)]) = \frac{\partial}{\partial b}\Big[Td^{-1}\Big(-\Big(\frac{\Omega^E}{2i\pi}+bI_E\Big)\Big) \cdot \det\Big(-\Big(\frac{\Omega^E}{2i\pi}+bI_E\Big)\Big)\exp\Big\{-u\Big(\frac{|Y|^2}{2} + \Big(2\pi\Big(\frac{\Omega^E}{2\pi}+bJ_E\Big)^{-1}\Big)\Big)\Big\}\Big]_{b=0}. \tag{3.71}$$

Therefore

$$\varphi(Tr_s[N_H\exp(-A_u^2)]) - Td^{-1}(g^E)c_u = -(Td^{-1})'(g^E)a_u. \tag{3.72}$$

For $s\in\mathbf{C}$, $0<\mathrm{Re}(s)<\frac{1}{2}$, let $\delta(s)$ be the current

$$\delta(s) = \frac{1}{\Gamma(s)}\int_0^{+\infty} u^{s-1}\{\varphi(Tr_s[N_H\exp(-A_u^2)]) + (Td^{-1})'(g^E)\delta_M - Td^{-1}(g^E)c_u\}du. \tag{3.73}$$

By Section 1g) and by Theorem 3.12, we know that $\delta(s)$ is a well-defined current on M^E, which extends into a current which is a meromorphic function of $s\in\mathbf{C}$. This function is holomorphic near $s=0$. In particular

$$\delta'(0) = \tilde{T}(g^E) - Td^{-1}(g^E)\tilde{e}(g^E). \tag{3.74}$$

On the other hand, by (3.72), we find that if $s\in\mathbf{C}$, $0<\mathrm{Re}(s)<\frac{1}{2}$, then

$$\delta(s) = -(Td^{-1})'(g^E)\frac{1}{\Gamma(s)}\int_0^{+\infty} u^{s-1}(a_u-\delta_M)du. \tag{3.75}$$

By Theorem 3.10, we know that

$$a_u - \delta_M = -\frac{1}{2i\pi}\bar\partial\partial\int_u^{+\infty} c_v\frac{dv}{v}. \tag{3.76}$$

Using (3.75), (3.76), we find that if $0<\mathrm{Re}(s)<\frac{1}{2}$, then

$$\delta(s) = (Td^{-1})'(g^E)\frac{\bar\partial\partial}{2i\pi}\left[\frac{1}{\Gamma(s+1)}\int_0^{+\infty} u^{s-1}c_u du\right]. \tag{3.77}$$

Clearly

$$\frac{1}{\Gamma(s+1)}\int_0^{+\infty} u^{s-1}c_u du = \frac{\rho(s)}{s}. \tag{3.78}$$

On the other hand, $\rho(0) = c_0$ and c_0 is a closed form. So from (3.77), (3.78), we deduce that

$$\delta(s) = (Td^{-1})'(g^E)\frac{\bar\partial\partial}{2i\pi}\left(\frac{\rho(s)-\rho(0)}{s}\right). \tag{3.79}$$

From (3.79), we get

$$\delta'(0) = \frac{\bar\partial\partial}{2i\pi}\left\{(Td^{-1})'(g^E)\frac{\rho''(0)}{2}\right\}. \tag{3.80}$$

Using (3.74), (3.80), we find that

$$\tilde T(g^E) - Td^{-1}(g^E)\tilde e(g^E) = \frac{\bar\partial\partial}{2i\pi}\left\{(Td^{-1})'(g^E)\frac{\rho''(0)}{2}\right\}. \tag{3.81}$$

Also, it is easy to verify that the wave front set of the current $\rho''(0)$ is included in E. From (3.81), we deduce (3.70). □

Remark 3.18. It should be pointed out that although the current $\tilde T(g^E)$ is in general not locally integrable, the current $Td^{-1}(g^E)\tilde e(g^E)$ is locally integrable.

(h) *The current $\tilde e(g^E)$ as a function of g^E.* Let now g^E, g'^E be two Hermitian metrics on E. In [BGS1, Theorem 1.29], we have defined a unique class $\tilde e(g^E, g'^E) \in P^M/P^{M,0}$ which is such that

$$\frac{\bar\partial\partial}{2i\pi}\tilde e(g^E, g'^E) = e(g'^E) - e(g^E). \tag{3.82}$$

$\tilde e(g^E, g'^E)$ lifts naturally to M^E.

Theorem 3.19. *The following identity holds*

$$\tilde e(g'^E) - \tilde e(g^E) = -\tilde e(g^E, g'^E) \quad \text{in } P_M^{M^E}/P_M^{M^{E,0}} \tag{3.83}$$

Proof. The proof of Theorem 3.18 is closely related to the proof of Theorem 2.5. In fact we lift E to $M \times \mathbf{P}^1$, and we consider a metric g on

E which restricts to g^E, g'^E on $M \times \{0\}$, $M \times \{\infty\}$ respectively. We then use equation (3.64) for $\tilde{e}(g^E)$ on $M \times \mathbf{P}^1$ in the obvious analogue of (2.15). To obtain (3.83), the key fact is that $\int_{\mathbf{P}^1} \mathrm{Log}|z|^2 \delta_M = 0$. Details are left to the reader. □

(i) *A transitivity property of the currents* $\tilde{e}(g^E)$. Let now E and E' be two holomorphic vector bundles on M. E and E' are vector subbundles of $E \oplus E'$. Then the manifolds M^E and $M^{E'}$ are vector submanifolds of $M^{E\oplus E'}$ which intersect transversally, and $M^E \cap M^{E'} = M$.

The vector bundles E and E' lift naturally to $M^{E\oplus E'}$. If $z = (y, y') \in E \oplus E'$, set

$$\sigma(z) = y; \qquad \sigma'(z) = y'$$

Then σ and σ' are holomorphic sections of E and E', which exactly vanish on $M^{E'}$ and M^E respectively. On $M^{E\oplus E'}$, the sections σ and σ' clearly possess the transversality properties which were described in Remark 3.5.

Let g^E and $g^{E'}$ be Hermitian matrics on E and E'. We equip $E \oplus E'$ with the metric $g^{E\oplus E'}$, which is the orthogonal sum of the metrics g^E and $g^{E'}$. By Remark 3.16, the currents $\tilde{\sigma}^* \tilde{e}(g^E)$ and $\tilde{\sigma}'^* \tilde{e}(g^{E'})$ are well-defined currents on $M^{E\oplus E'}$. Also the forms $e(g^E)$ and $e(g^{E'})$ lift naturally to $M^{E\oplus E'}$.

Theorem 3.20. *The following identities of currents hold*

$$\begin{aligned} \tilde{e}(g^{E\oplus E'}) &= e(g^{E'})\sigma^* \tilde{e}(g^E) + \tilde{e}(g^{E'})\delta_{M^{E'}} \\ &\quad \text{in } P^{M^{E\oplus E'}}_{M^E \cup M^{E'}} / P^{M^{E\oplus E'},0}_{M^E \cup M^{E'}}. \\ \tilde{e}(g^{E\oplus E'}) &= e(g^E)\sigma'^* \tilde{e}(g^{E'}) + \tilde{e}(g^E)\delta_{M^E} \\ &\quad \text{in } P^{M^{E\oplus E'}}_{M^E \cup M^{E'}} / P^{M^{E\oplus E'},0}_{M^E \cup M^{E'}}. \end{aligned} \tag{3.84}$$

Proof. Our Theorem can be given a direct proof by using formula (3.61) and by proceeding as in the proof of Theorem 2.7. We will instead use our previous results. Namely by Theorem 3.17, we know that

$$\tilde{e}(g^{E\oplus E'}) = Td(g^{E\oplus E'})\tilde{T}(g^{E\oplus E'}) \text{ in } P^{M^{E\oplus E'}}_{M^E \cup M^{E'}} / P^{M^{E\oplus E'},0}_{M^E \cup M^{E'}}. \tag{3.85}$$

and that corresponding equalities hold for the currents $\tilde{e}(g^E)$ and $\tilde{e}(g^{E'})$.

Also we have the obvious

$$Td(g^{E\oplus E'}) = Td(g^E)Td(g^{E'}). \tag{3.86}$$

Finally, by Theorem 2.7, we know that

(3.87)
$$\tilde{T}(g^{E\oplus E'}) = \det\left(I - \exp\left(\frac{\Omega^{E'}}{2i\pi}\right)\right)\sigma^*\tilde{T}(g^E) + (Td^{-1})(g^E)\tilde{T}(g^{E'})\delta_{M^{E'}}$$
$$\text{in } P^{M^{E\oplus E'}}_{M^E\cup M^{E'}}/P^{M^{E\oplus E'},0}_{M^E\cup M^{E'}}.$$

Using (3.85)–(3.87), the first line of (3.84) follows. The second line is obvious by interchanging the roles of E and E'. □

Remark 3.21. Let s, s' be holomorphic sections of E, E' on M. Set

$$M' = \{x \in M; s(x) = 0\}$$
$$\tilde{M}' = \{x \in M; s'(x) = 0\}.$$

Assume that s, s' both verify the assumptions of Remark 3.5 and that M' and $\tilde{M}'$ intersect transversally. Then the section $s'' = (s, s')$ of $E \oplus E'$ also verifies the assumptions of Remark 3.5. Set $M'' = M' \cap \tilde{M}'$. From (3.84), we deduce

(3.88)
$$s''^*\tilde{e}(g^{E\oplus E'}) = e(g^{E'})s^*(\tilde{e}(g^E)) + s'^*(\tilde{e}(g^{E'}))\delta_{M'} \text{ in } P^M_{M'\cup\tilde{M}'}/P^{M,0}_{M'\cup\tilde{M}'}.$$

(j) *The arithmetic Euler class.* We first recall some definitions from [GS1] and [GS2]. Let (A, Σ, F_∞) be an *arithmetic ring* i.e. A is an excellent regular noetherian integral domain, Σ a finite nonempty set of monomorphisms $\sigma: A \to \mathbb{C}$, and F_∞ a conjugate linear involution of the $\mathbb{C}$-algebra $\mathbb{C}^\Sigma$ which fixes A (embedded diagonally into $\mathbb{C}^\Sigma$).

Let X be an *arithmetic variety* over A, i.e. a regular quasi-projective flat scheme over $\operatorname{Spec} A$. Denote by $X_\sigma = X \otimes_\sigma \mathbb{C}$ the complex variety obtained by extending scalars from A to $\mathbb{C}$ by $\sigma \in \Sigma$, and by X_∞ the analytic space $X_\infty = \coprod_{\sigma\in\Sigma} X_\sigma(\mathbb{C})$.

For any integer $p \geq 0$, let $A^{pp}(X_{\mathbb{R}})$ (resp. $\mathcal{D}^{pp}(X_{\mathbb{R}})$) be the real vector space of real forms (resp. currents) α of type (p,p) on X_∞ such that $F^*_\infty(\alpha) = (-1)^p\alpha$. We also introduce the quotient spaces

(3.89)
$$\tilde{A}^{pp}(X_{\mathbb{R}}) = A^{pp}(X_{\mathbb{R}})/(\operatorname{Im}\partial + \operatorname{Im}\bar{\partial})$$
$$\tilde{\mathcal{D}}^{pp}(X_{\mathbb{R}}) = \mathcal{D}^{pp}(X_{\mathbb{R}})/(\operatorname{Im}\partial + \operatorname{Im}\bar{\partial}).$$

Let now X^p be the set of points of codimension p of the scheme X and $Z^p(X)$ the group of codimension p cycles on X, i.e. the free abelian group generated by X^p. Given $Z \in Z^p(X)$, let Z_∞ be the analytic cycle on X_∞

attached to Z, and $\delta_Z \in \mathcal{D}^{pp}(X_{\mathbf{R}})$ the current given by integration on Z_∞: if $Z = \sum_\alpha n_\alpha Z_\alpha$, with $Z_\alpha \subset X$ and $n_\alpha \in \mathbf{Z}$, for any smooth form ω on X_∞,

$$\delta_Z(\omega) = \sum_\alpha n_\alpha \int_{Z_{\alpha,\infty}} \omega. \tag{3.90}$$

A *Green current* for $Z \in Z^p(X)$ is an element $g \in \tilde{\mathcal{D}}^{p-1,p-1}(X_{\mathbf{R}})$ such that

$$-dd^c g = \delta_Z - \omega,$$

with ω a smooth form (here, as usual, $d = \partial + \bar\partial$ and $d^c = (\partial - \bar\partial)/(4i\pi)$). The pair (Z,g) is called an *arithmetic cycle.* The *arithmetic Chow group* $\widehat{CH}^p(X)$ is the group generated by codimension p arithmetic cycles with the relation $(\mathrm{div}(f), -\mathrm{Log}\,|f|^2) = 0$, for every nonzero rational function f on a closed irreducible subscheme Y of codimension $p-1$ on X, where $\log|f|^2$ is the current defined by the formula

$$\mathrm{Log}\,|f|^2(\omega) = \int_{Y_\infty^{n-s}} \mathrm{Log}\,|f|^2 \omega, \tag{3.91}$$

Y_∞^{n-s} denoting the smooth locus of Y_∞. We refer to [GS1] for the study of these groups. Let

$$\widehat{CH}(X) = \bigoplus_{p\geq 0} \widehat{CH}^p(X).$$

If F is the fraction field of A, $X_F = X \bigotimes_A F$ the generic fiber of X and $CH_f^p(X)$ the Chow group of X with support in special fibers, we may view $\widehat{CH}^p(X)$ as a subquotient of

$$CH_f^p(X) \oplus Z^p(X_F) \oplus \tilde{\mathcal{D}}^{p-1,p-1}(X_{\mathbf{R}})$$

(see [GS1]).

Now let (E,h) be an *Hermitian vector bundle* on X, i.e. E is an algebraic vector bundle on X and h is an Hermitian metric on the holomorphic vector bundle E_∞ on X_∞ attached to E; furthermore h is invariant by F_∞. Let r be the rank of E and $\Phi(T_1,\dots,T_r)$ a symmetric polynomial in r variables with coefficients in a ring B. In [GS2] a characteristic class

$$\hat\Phi(E,h) \in \widehat{CH}(X)_B = \widehat{CH}(X) \bigotimes_{\mathbf{Z}} B$$

is defined. In particular we get Chern classes $\hat{c}_p(E,h) \in \widehat{CH}^p(X)$ where $c_p(T_1,\ldots,T_r)$ is the p-th elementary symmetric polynomial.

Even if E is generated by its global sections, we do not know in general an explicit arithmetic cycle representing $\hat{\Phi}(E,h)$ (see [GS2], §5). However, we shall give below a description of the *Euler class* (i.e. the top Chern class) $\hat{c}_r(E,h) \in \widehat{CH}^r(X)$.

For this, let us assume we are given an algebraic section s of E over X which satisfies over X_∞ the conditions of Remark 3.5. Call $Z_F \in Z^r(X_F)$ the zero set of s on the generic fiber. Using Remark 3.16 we see that $g = -s_\infty^*(\tilde{e}(h))$ is a Green current for Z_F. On the other hand, viewing s as a morphism from X to the total space X^E of E, we may consider $e_f = s^*([X]) \in CH_f^r(X)$, where $[X] \in CH_f^r(X^E)$ is the part of the zero section supported on special fibers. The triple (e_f, Z_F, g) defines a class $\hat{e}(E,h)$ in $\widehat{CH}^r(X)$.

Theorem 3.22. *Under the above hypotheses,*

$$\hat{e}(E,h) = \hat{c}_r(E,h) \quad \text{in } \widehat{CH}^r(X). \tag{3.92}$$

Proof. Let h_0, h_1 be two Hermitian metrics on E. We will first prove that

$$\hat{e}(E,h_0) - \hat{e}(E,h_1) = \hat{c}_r(E,h_0) - \hat{c}_r(E,h_1). \tag{3.93}$$

Let $\tilde{e}(h_0,h_1) \in \tilde{A}^{r-1,r-1}(X_{\mathbb{R}})$ be the Bott–Chern class of forms constructed in [BGS1] associated with the polynomial $e(A) = \det(A)$ and the metrics h_0 and h_1. In particular, we know that

$$\frac{\bar{\partial}\partial}{2i\pi}\tilde{e}(h_0,h_1) = e(h_1) - e(h_0).$$

Let X^E be the total space of E. Let p be the projection $X^E \to X$. By Theorem 3.19, we know that

$$\tilde{e}(h_1) - \tilde{e}(h_0) = -p^*\tilde{e}(h_0,h_1). \tag{3.94}$$

By using Remark 3.5, we get from (3.94)

$$s^*\tilde{e}(h_1) - s^*\tilde{e}(h_0) = -\tilde{e}(h_0,h_1).$$

By using [GS2, Theorem 4.8], we obtain (3.93).

We now prove that $\hat{e}(E,h)$ does not depend on the choice of s. On X^E, the element $\hat{e}(p^*E,\, p^*h)$ of $\widehat{CH}(X^E)$ is the class of the arithmetic cycle $(s_0(X), -\tilde{e}(h))$. Using (3.93) we shall prove in 4e) below that

$$\hat{e}(E,h) = s_0^*\big(\hat{e}(p^*E,p^*h)\big) \tag{3.95}$$

where s_0^* on $\widehat{CH}$ is defined as in [GS1]. Therefore we conclude that

$$\hat{e}(E,h) = s^*\big(\hat{e}(p^*E,p^*h)\big) = s_0^*\big(\hat{e}(p^*E,p^*h)\big)$$

does not depend on the choice of s. The definition of the class $\hat{e}(E,h) = s_0^*\big(\hat{e}(p^*E,p^*h)\big)$ makes sense for any Hermitian vector bundle (E,h) on X (without assuming that E has a good global section). We shall see that this class $\hat{e}(E,h)$ has the same properties as $\hat{c}_r(E,h)$ ([GS2, par. 4]).

Property 1: the class $\hat{e}(E,h)$ is functorial. That is, given a morphism $f: Y \to X$ of arithmetic varieties,

$$f^*\hat{e}(E,h) = \hat{e}(f^*E, f^*h).$$

To see that, we first consider the commutative diagram

$$\begin{array}{ccc} Y^{f^*(E)} & \xrightarrow{\tilde{f}} & X^E \\ \downarrow & & \downarrow \\ Y & \xrightarrow{f} & X \end{array}$$

so that we may replace f by $\tilde{f}$. The functoriality follows from the formula for $\tilde{e}$ in Theorem 3.15.

Property 2: $\hat{e}(E,h)$ is multiplicative under direct sum. Let (E,h) and (E',h') be two hermitian vector bundles on X, and $(E \oplus E', h \oplus h')$ their orthogonal direct sum. Then $\hat{e}(E \oplus E',\, h \oplus h') = \hat{e}(E,h)\hat{e}(E',h')$ in $\widehat{CH}(X)$. By the definition of the product in $\widehat{CH}$ (see [GS1]) this follows from Remark 3.21.

Property 3: The image of $\hat{e}(E,h)$ into the (classical) Chow group $CH^r(X)$ is the Euler class of E. This is clear since it is given as the self-intersection of X in X^E.

Property 4: Let $\omega: \widehat{CH}^r \to A^{rr}(X_{\mathbf{R}})$ be the map sending the class of (Z,g) to the smooth form $\delta_Z + dd^c g$. Then $\omega(\hat{e}(E,h)) = e(E,h)$ is the Euler form of (E,h). Indeed, this follows from (3.64).

Property 5: When (L,h) is an Hermitian line bundle, $\hat{e}(L,h) = s_0^*(\operatorname{div}(y), -\operatorname{Log}|y|^2)$, where y is the tautological section of p^*L on X^L. This follows from (3.67) and the definition of $\hat{e}$ given above.

From [GS2] we know that $\hat{c}_r(E,h)$ enjoys also the properties 1, 2, 3, 4, 5. From this we shall conclude that $\hat{e}(E,h) = \hat{c}_r(E,h)$. Since X has ample line bundles, any bundle on X is the pull-back of a bundle on the product of a Grassmannian by a projective space over A, hence, using Property 1, we may assume that X is a complete variety. Let then

$$H^{r-1,r-1}(X_{\mathbf{R}}) = \operatorname{Ker}\big(dd^c\colon \tilde{A}^{r-1,r-1}(X_{\mathbf{R}}) \to A^{rr}(X_{\mathbf{R}})\big).$$

Since X_∞ is Kähler, this group is the subspace of the real cohomology of X_∞ of degree $(r-1,r-1)$ where F_∞^* is $(-1)^{r-1}$. We define a map

$$a\colon H^{r-1,r-1}(X_{\mathbf{R}}) \to \widehat{CH}^r(X)$$

by sending the element η to the class of $(0,\eta)$. From Properties 3 and 4 and the exact sequence of [GS1, Theorem 3.3.5] we conclude that

$$\delta(E) = \hat{e}(E,h) - \hat{c}_r(E,h)$$

lies in the image of a.

Let

$$0 \to S \to E \to Q \to 0$$

be an exact sequence of vector bundles on X with metrics h', h, h'' respectively. We shall prove that

$$\delta(E) = \delta(S) + \delta(Q). \tag{3.96}$$

From Property 2 we know that (3.96) holds when (E,h) is the orthogonal direct sum of (S,h') with (Q,h''). In general, on $X \times \mathbf{P}^1$ there is an Hermitian vector bundle $(\tilde{E},\tilde{h})$ whose restriction to $X \times \{0\}$ (resp. $X \times \{\infty\}$) is (E,h) (resp. $(S,h') \oplus (Q,h'')$) (see [BGS1, Theorem 1.29]). Since the restriction maps i_0^* and i_∞^* from $H^{r-1,r-1}((X \times \mathbf{P}^1)_\infty)$ to $H^{r-1,r-1}(X_\infty)$ coincide, we get

$$\delta(E) = \delta((S,h') \oplus (Q,h'')) = \delta(S) + \delta(Q).$$

In particular $\delta(E)$ does not depend on h.

(6) From Property 5 we know that $\delta(E) = 0$ when E has rank one. By induction on the rank of E we shall conclude that $\delta(E) = 0$. Indeed let $\mathbf{P}(E)$

be the projective space of E and $p: \mathbf{P}(E) \to X$ the projection. Since p^* is injective on $\widehat{CH}(X)$ ([GS2]3) we just need to check that $p^*(\delta(E)) = \delta(p^*E)$ vanishes. Using the standard exact sequence

$$0 \to S \to p^*E \to \mathcal{O}(1) \to 0$$

we get $\delta(p^*E) = \delta(\mathcal{O}(1)) + \delta(S) = 0$ by induction hypothesis. This concludes the proof of Theorem 3.22. □

4. Deformation to the normal cone

In this paragraph we study the deformation W to the normal cone of a closed immersion $i: Y \to X$ of complex manifolds. This construction is due to MacPherson [BaFM]. The variety W maps onto $\mathbf{P}^1$ and all its fibers W_t, $t \in \mathbf{P}^1$, except the fiber at infinity, are isomorphic to X. We prove in Theorem 4.8 that, given a bundle η on Y and a resolution ξ of $i_*\eta$ on X, there exists a complex $\tilde{\xi}$ of bundles on W such that $\tilde{\xi}$ restricted to $W_0 = X$ is ξ and $\tilde{\xi}$ restricted to $\mathbf{P}(N \oplus \mathbf{C}) \subset W_\infty$ (where N is the normal bundle to Y in X) maps onto the Koszul resolution of the direct image of η, with acyclic kernel. We then show in Theorem 4.11 how $T(h^\xi)$, for appropriate choices of metrics, relates to the singular Bott-Chern current $T(h^{\tilde{\xi}})$ and an Euler–Green current $\tilde{e}(g^{H^*})$. In e) we use deformation to the normal cone to end the proof of (3.95), which had been delayed in the previous paragraph. Finally, with Theorem 4.13, we get a version of an arithmetic Riemann–Roch–Grothendieck Theorem for immersions. The proof relies upon essentially all our previous results.

(a) *Projective bundles and Koszul complexes.* This section and the next one is a brief review of material from [SGA6], [Ha], and [BaFM]. If $Y \subset X$ is a closed imbedding of complex manifolds we write $\mathcal{I}_{Y/X}$ for the sheaf of ideals of Y in X; if Y is a divisor we also write $\mathcal{O}_X(-Y)$. Note that $\mathcal{O}_X(nY) = \mathcal{O}_X(-Y)^{\otimes -n}$, that $\mathcal{O}_X$ is a subsheaf of $\mathcal{O}_X(Y)$, and that $\mathcal{O}_X \subset \mathcal{O}_X(Y)$ canonically.

The conormal sheaf $\mathcal{N}^*_{Y/X} \cong \mathcal{I}_{Y/X}/\mathcal{I}^2_{Y/X}$ is the dual of the sheaf of sections of the normal bundle $N_{Y/X}$. (Here we follow [Ha] rather than [SGA6].) Note also $\mathcal{N}^*_{Y/X} \cong i^*\mathcal{I}_{Y/X}$ where $i: Y \to X$ is the inclusion. More generally, if Y is a submanifold of X, there is an isomorphism

$$Tor_i^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Y) = \Lambda^i \mathcal{N}_{Y/X}.$$

of sheaves on Y [SGA6].

If $p: E \to X$ is a vector bundle with sheaf of holomorphic sections $\mathcal{E}$, there is a canonical homomorphism $\iota: p^*\mathcal{E}^* \to \mathcal{O}_E$ dual to the tautological section of p^*E. This homomorphism vanishes along the zero section $s: X \to E$, and the associated Koszul complex $K(\iota) = (\Lambda^\bullet p^*\mathcal{E}^*)$ is a resolution of $s_*\mathcal{O}_X$. If $\varphi: \mathcal{V} \to \mathcal{O}_X$ is a homomorphism with $\mathcal{V}$ locally free, the differential in the associated Koszul complex $K(\varphi)$ shall be

$$d_i^{\varphi}: \Lambda^i\mathcal{V} \to \Lambda^{i-1}\mathcal{V}$$

$$e_1 \wedge \cdots \wedge e_i \to \sum_{k=1}^{i}(-1)^{k-1}\varphi(e_k)e_1 \wedge \cdots \wedge \hat{e}_k \wedge \cdots \wedge e_i$$

so that the chain map d_i^{φ} is interior multiplication by φ.

Let $P = \mathbf{P}(E \oplus \mathbf{C})$. This parametrizes lines in $E \oplus \mathbf{C}$ or, equivalently, rank one quotients of $\mathcal{E}^* \oplus \mathcal{O}_X$. On P we have the universal exact sequence:

$$0 \to \mathcal{H} \to \hat{p}^*\mathcal{E}^* \oplus \mathcal{O}_P \to \mathcal{O}_P(1) \to 0$$

($\hat{p}: \mathbf{P}(E \oplus \mathbf{C}) \to X$ is the projection). $\mathbf{P}(E)$ embeds as a divisor in $\mathbf{P}(E \oplus \mathbf{C})$, given by the vanishing of the map $\mathcal{O}_P \to \mathcal{O}_P(1)$ induced by the inclusion $\mathcal{O}_P \subset \hat{p}^*\mathcal{E}^* \oplus \mathcal{O}_P$.

Equivalently, $\mathbf{P}(E)$ is the locus on which $\mathcal{O}_P \subset \mathcal{H}$. The complement of $\mathbf{P}(E)$ is canonically isomorphic to E, since a line $L \subset E \oplus \mathbf{C}$ which maps surjectively to $\mathbf{C}$ is equivalent to a homomorphism $\mathbf{C} \to E$. Specifically a homomorphism $\ell: \mathbf{C} \to E$ determines an injective homomorphism $\hat{e}: \mathbf{C} \to \mathsf{E} \oplus \mathbf{C}$, by $\hat{\ell}(x) = (\ell(x), x)$ i.e. we embed $E \subset \mathbf{P}(E \oplus \mathbf{C})$ by $\underline{x} \to (\underline{x}, 1)$.

The map $\theta: \mathcal{H} \to \hat{p}^*\mathcal{E}^*$ induced by the projection from $\hat{p}^*\mathcal{E}^* \oplus \mathcal{O}_P$ is an isomorphism on E, and if, on E, we compose θ^{-1} with the homomorphism induced by *minus* the second projection, we obtain ι. Let $\varphi: \mathcal{H} \to \mathcal{O}_P$ be the homomorphism induced by minus the projection $\hat{p}^*\mathcal{E}^* \oplus \mathcal{O}_P \to \mathcal{O}_P$. Since it is surjective on $\mathbf{P}(E)$ and equal to ι on E, the associated Koszul complex $K(\varphi)$ is a resolution of $s_*\mathcal{O}_X$, where $s: X \to E \subset \mathbf{P}(E \oplus \mathbf{C})$ is the zero section.

(b) *Deformation to the normal bundle.* Let $i: Y \to X$ be a closed embedding of complex manifolds. We suppose that Y is of pure codimension n in X.

Let $W = W_{Y/X}$ be the blow up of $X \times \mathbf{P}^1$ along $Y \times \{\infty\}$. Since Y and $X \times \mathbf{P}^1$ are manifolds, W is itself a manifold. The map $\pi: W \to X \times \mathbf{P}^1$ is an isomorphism away from $Y \times \{\infty\}$; we shall write P for the exceptional divisor in W of the blow up. Recall that

$$P \cong \mathbf{P}(N_{Y \times \{\infty\}/X \times \mathbf{P}^1}).$$

Since $N_{Y\times\{\infty\}/X\times\mathbf{P}^1} \cong p_Y^* N_{Y/X} \oplus p_\infty^* N_{\infty/\mathbf{P}^1}$, where $p_Y: Y \times \{\infty\} \to Y$ and $p_\infty: Y \times \{\infty\} \to \{\infty\}$ are the projections, we may also write

$$P \cong \mathbf{P}\left(N_{Y/X} \otimes N_{\infty/\mathbf{P}^1}^{-1} \oplus \mathbf{C}\right).$$

Note that the bundle $N_{\infty/\mathbf{P}^1}$, while trivial, is not canonically trivial. Hence P is the projective completion of $N_{Y/X} \otimes N_{\infty/\mathbf{P}^1}^{-1}$ with divisor at ∞, $\mathbf{P}\left(N_{Y/X} \otimes N_{\infty/\mathbf{P}^1}^{-1}\right) \cong \mathbf{P}(N_{Y/X})$.

The map $q_W : W \to \mathbf{P}^1$, obtained by composing π with the projection $q: X \times \mathbf{P}^1 \to \mathbf{P}^1$, is flat, and for $z \in \mathbf{P}^1$:

$$q^{-1}(z) = \begin{cases} X & z \neq \infty \\ P \cup \tilde{X} & z = \infty . \end{cases}$$

Here $\tilde{X}$ is the blow up of X along Y, and $P \cap \tilde{X}$ is the divisor at ∞ on P, identified with the exceptional divisor on $\tilde{X}$.

We shall use the following notations:

$$\begin{array}{ccc} P & \xrightarrow{f} & W \\ \pi_P \downarrow & & \downarrow \pi \\ Y \times \{\infty\} & \xrightarrow{i_\infty} & X \times \mathbf{P}^1 \end{array}$$

$$\begin{aligned} i: Y &\to X \\ W = \pi^{-1}(\infty) &= P \cup \tilde{X}. \\ q: X \times \mathbf{P}^1 &\to \mathbf{P}^1 && \text{the projection,} \\ q_W &= q \cdot \pi, \\ p: X \times \mathbf{P}^1 &\to X && \text{the projection,} \\ p_W &= p \cdot \pi, \\ q_Y: Y \times \mathbf{P}^1 &\to \mathbf{P}^1 && \text{the natural projections} \\ p_Y: Y \times \mathbf{P}^1 &\to Y && \text{the natural projections} \end{aligned}$$

Note that the proper transform of $Y \times \mathbf{P}^1 \subset X \times \mathbf{P}^1$ is isomorphic to $Y \times \mathbf{P}^1$. We write

$$j: Y \times \mathbf{P}^1 \to W$$

for the induced map.

(c) *Deformation of resolutions.* Now suppose that η is a locally free sheaf on Y and that $\xi \to i_* \eta$ is a resolution of η by a bounded complex of locally free sheaves on X. We shall deform ξ through a complex $\tilde{\xi}$ on W to a Koszul type resolution of $s_*\eta$ on P, where $s: Y \to N_{Y/X} \otimes N^{-1}_{\infty/\mathbf{P}^1} \subset P$ is the zero section.

First observe that there is an exact sequence of sheaves on $\mathbf{P}^1$:

$$0 \to \mathcal{O}_{\mathbf{P}^1}(-\infty) \to \mathcal{O}_{\mathbf{P}^1} \to \mathcal{O}_\infty \to 0. \tag{4.3}$$

Hence if we write K^∞ for the complex $\mathcal{O}_{\mathbf{P}^1}(-\infty) \to \mathcal{O}_{\mathbf{P}^1}$, with $\mathcal{O}_{\mathbf{P}^1}$ in degree zero, $p^*\xi \otimes q^*K^\infty$ is a resolution of $i_{\infty *}\eta$. Let us write $\mathcal{G} = \pi^*(p^*\xi \otimes q^*K^\infty)$ which is a bounded complex of locally free sheaves on W.

Lemma 4.1. *Let $\mathcal{H}$ be the locally free sheaf on P which is the kernel of the map*

$$\pi_P^*\left(\mathcal{N}^*_{Y/X} \oplus \mathcal{N}^*_{\infty/\mathbf{P}^1}\right) \to \mathcal{O}_P(1).$$

Then

$$\mathcal{H}_p(\mathcal{G}) \cong f_*\left(\Lambda^p\mathcal{H} \otimes_{\mathcal{O}_P} \pi_P^*\eta\right).$$

Proof. This follows immediately from [SGA6, exp. VII, lemma 3.2]. □

The sheaves $f_*\left(\Lambda^p\mathcal{H} \otimes_{\mathcal{O}_P} \pi_p^*\eta\right)$ are locally direct sums of copies of $f_*\mathcal{O}_P$ hence, since P is a divisor in W, they are locally of projective dimension one. Therefore if

$$a: \mathcal{E} \to f_*\left(\Lambda^p\mathcal{H} \otimes_{\mathcal{O}_P} \pi_P^*\eta\right)$$

is an epimorphism, with $\mathcal{E}$ a locally free $\mathcal{O}_W$- module the kernel of α is again locally free. It follows, by induction on $i \geq 0$, that $\mathrm{Ker}(d_i^{\mathcal{G}})$ is a locally free sheaf on W.

Definition 4.2. For $i \geq 0$, $\tilde{\xi}_i$ is the locally free $\mathcal{O}_W$ module $\ker(d_i^{\mathcal{G}}) \otimes_{\mathcal{O}_W} \mathcal{O}_W(i\infty)$. Here $\mathcal{O}_W(\infty) = q_W^*\mathcal{O}_{\mathbf{P}^1}(\infty)$ and $\mathcal{O}_W(i\infty) = \mathcal{O}_W(\infty)^{\otimes i}$ contains $\mathcal{O}_W$ naturally as a subsheaf.

Observe that $\mathcal{G}_i \cong p_W^*\xi_i \oplus p_W^*\xi_{i-1}(-\infty)$, and that $d_i^{\mathcal{G}}(x,y) = (d_i^{\xi}(x) + (-1)^i y,\ d_{i-1}^{\xi}(y))$, where we identify $y \in p_W^*\xi_{i-1}(-\infty)$ with its image in $p_W^*\xi_i$ under the natural inclusion. The projection $\mathcal{G}_i \to \xi_i$ identifies $\mathrm{Ker}(d_i^{\mathcal{G}})$ with the subsheaf of $p_W^*\xi_i$ consisting of those sections x such that dx lies in $p_W^*\xi_{i-1}(-\infty)$. Hence $\tilde{\xi}_i$ is isomorphic to the fibre product of the diagram:

$$\begin{array}{ccc}
\tilde{\xi}_i & \longrightarrow & p_W^*\xi_{i-1}((i-1)\infty) \\
\Big| & & \Big| \\
p_W^*\xi_i(i\infty) & \xrightarrow{d_i^{\xi}} & p_W^*\xi_{i-1}(i\infty)
\end{array}$$

Remark. (i) $\tilde{\xi}_i$ contains $p_W^*\xi_i$ as a subsheaf, the inclusion being induced by the "graph" map

$$\begin{aligned} p_W^*\xi_i &\to p_W^*\xi_i(i\infty) \oplus p_W^*\xi_{i-1}((i-1)\infty) \\ x &\to \left(x, (-1)^i d_i^\xi(x)\right). \end{aligned}$$

Clearly $\tilde{\xi}_i|_{W-W_\infty} = p_W^*\xi_i|_{W-W_\infty}$.

(ii) There is an epimorphism of locally free $\mathcal{O}_P$ modules $f^*\tilde{\xi}_i \to \Lambda^i(\mathcal{H}\otimes \pi_P^*\eta \otimes f^*\mathcal{O}_W(\infty))$.

(iii) There is an exact sequence

$$0 \to \mathrm{Ker}(p_W^* d_i^\xi)(i\infty) \to \tilde{\xi}_i \to \mathrm{Im}(p_W^* d_i^\xi)((i-1)\infty) \to 0. \tag{4.4}$$

Lemma 4.3. *The differential $p_W^* d_i : p_W^*\xi_i \to p_W^*\xi_{i-1}$ extends to a homomorphism $d_i^{\tilde{\xi}} : \tilde{\xi}_i \to \tilde{\xi}_{i-1}$ which makes $(\tilde{\xi}_\bullet, d_\bullet^{\tilde{\xi}})$ a complex of $\mathcal{O}_W$ modules.*

Proof. $p_W^* d_i$ extends trivially to the homomorphism

$$p_W^* d_i \otimes \mathrm{Id}_{\mathcal{O}_W(i\infty)} : p_W^*\xi_i(i\infty) \to p_W^*\xi_{i-1}(i\infty).$$

By definition this homomorphism, restricted to $\tilde{\xi}_i$, has image lying in $p_W^*\xi_{i-1}((i-1)\infty)$. Since $d_i^{\xi 2} = 0$, the image is contained in $\tilde{\xi}_{i-1} \subset p_W^*((i-1)\infty)$. Since $d_i^{\tilde{\xi}}|_{W-W_\infty} = p_W^*(d_i^\xi)$, $d_i^{\tilde{\xi}2}$ vanishes on $W - W_\infty$ and hence on W. □

Remark 4.4. The construction of the complex $\tilde{\xi}_\bullet = (\tilde{\xi}_\bullet, d_\bullet^{\tilde{\xi}})$ is local on X and is compatible with finite sums since both twisting by $\mathcal{O}_W(\infty)$ and fibre products are local operations compatible with finite sums.

Lemma 4.5. *If $\xi_\bullet$ is acyclic (i.e. $\eta \cong 0$ or Y is empty), then $\tilde{\xi}_\bullet$ is the pullback, via π, of a complex on $X \times \mathbf{P}^1$ which becomes split acyclic when restricted to $X \times \{\infty\}$.*

Proof. If $\xi_\bullet$ is acyclic, then

$$0 \to \mathrm{Im}(d_{i+1}) \to \xi_i \to \mathrm{Im}(d_i) \to 0$$

is an exact sequence of locally free sheaves. Hence the fibre product of the diagram

$$\begin{array}{ccc} & & q^*\mathrm{Im}(d_i)(-\infty) \\ & & \downarrow \\ q^*(\xi_i) & \xrightarrow{d_i} & q^*\mathrm{Im}(d_i) \end{array}$$

is already locally free on $X \times \mathbf{P}^1$, and is preserved under π_W^*. As discussed in [BGS1, Theorem 1.29] this pullback has the property that over $X \times \{\infty\}$ it splits as the direct sum $q^*\mathrm{Im}(d_{i+1}) \oplus q^*\mathrm{Im}(d_i)(-\infty)$. □

Corollary 4.6. *If $U = X - Y$, the restriction of $\tilde{\xi}$ to $U \times \mathbf{P}^1 \subset W$ restricts to a split acyclic complex on $U \times \{\infty\} \subset \tilde{X} \subset W_\infty$.*

Since $j_*\mathcal{O}_{Y \times \mathbf{P}^1}$ and $\mathcal{O}_{W_\infty}$ are Tor independent, the formation of $\tilde{\xi}$ commutes with restriction to $Y \times \mathbf{P}^1 \subset W$. The restriction of $\xi_\bullet$ to Y has locally free homology:

$$\begin{aligned}\mathcal{H}_p(i^*\xi_\bullet) &= \mathcal{T}or_p^{\mathcal{O}_X}(\mathcal{O}_Y, \eta) \\ &\cong \mathcal{T}or_p^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Y) \otimes_{\mathcal{O}_Y} \eta \\ &\cong \Lambda^p \mathcal{N}_{Y/X}^* \otimes_{\mathcal{O}_Y} \eta.\end{aligned}$$

Hence $i^*\xi_\bullet$ breaks up into short exact sequences of locally free sheaves:

$$\begin{gathered}0 \to \mathcal{Z}_i \to i^*\xi_i \to \mathcal{B}_i \to 0 \\ 0 \to \mathcal{B}_{i+1} \to \mathcal{Z}_i \to \mathcal{H}_i(i^*\xi_\bullet) \to 0.\end{gathered}$$

Therefore $\tilde{\xi}_i|_{Y \times \mathbf{P}^1}$ is obtained by pulling back the extension (or rather its inverse image under p_Y^*) by the inclusion $\mathcal{B}_i(-\infty) \to \mathcal{B}_i$ and twisting by $\mathcal{O}(i\infty)$. On restricting to $Y \times \{\infty\} \subset Y \times \mathbf{P}^1$, we find that, since $\mathcal{O}_{\mathbf{P}^1}(\infty)|_\infty \cong \mathcal{N}_{\infty/\mathbf{P}^1}$,

$$\tilde{\xi}_i|_{Y \times \mathbf{P}^1} \cong \left(\mathcal{Z}_i \otimes \mathcal{N}_{\infty/\mathbf{P}^1}^i\right) \oplus \left(\mathcal{B}_i \otimes \mathcal{N}_{\infty/\mathbf{P}^1}^{i-1}\right). \tag{4.5}$$

The differential $d_i^{\tilde{\xi}}$ restricts to the map $d_i: (x, y) \to (\gamma_i(y), 0)$, where

$$\gamma_i: \mathcal{B}_i \otimes \mathcal{N}_{\infty/\mathbf{P}^1}^{i-1} \to \mathcal{Z}_{i-1}^{i-1} \otimes \mathcal{N}_{\infty/\mathbf{P}^1}^{i-1}$$

is the natural inclusion. If we set $\mathcal{L}$ equal to the split acyclic complex with:

$$\mathcal{L}_i = \left(\mathcal{B}_{i+1} \otimes \mathcal{N}_{\infty/\mathbf{P}^1}^i\right) \oplus \left(\mathcal{B}_i \otimes \mathcal{N}_{\infty/\mathbf{P}^1}^{i-1}\right) \tag{4.6}$$

with differential $d_i(x, y) = (y, 0)$, then there is an obvious inclusion $\mathcal{L}_\bullet \to \tilde{\xi}|_{Y \times \{\infty\}}$ with cokernel the direct sum $\bigoplus_{i \geq 0} H_i(i^*\xi_\bullet) \otimes \mathcal{N}_{\infty/\mathbf{P}^1}^i$ viewed as a complex with zero differentials. Hence we have an exact sequence:

$$0 \to \mathcal{L}_\bullet \to \tilde{\xi}|_{Y \times \{\infty\}} \to \bigoplus_{i \geq 0} \Lambda^i\left(\mathcal{N}_{Y/X}^* \otimes \mathcal{N}_{\infty/\mathbf{P}^1}\right) \otimes_{\mathcal{O}_Y} \eta \to 0. \tag{4.7}$$

Suppose that $i: Y \to X$ is defined by equations $x_1 = \ldots = x_n = 0$ with the x_i part of a system of coordinates on X. Let $\xi_\bullet = K_\bullet(\underline{x})$ be the Koszul complex associated to the map $\mathcal{O}_X^n \to \mathcal{O}_X$ sending $\underline{a}$ to $\underline{a} \cdot \underline{x}$, where $\underline{x} = (x_1, \ldots, x_n)$. Then $K_\bullet(\underline{x})$ is a resolution of $i_*\mathcal{O}_Y$. On W we have an epimorphism

$$\mathcal{O}_W^n \oplus \mathcal{O}_W(-\infty) \to \mathcal{O}_W(-P)$$
$$(\underline{a}, b) \to \underline{a} \cdot \underline{x} + b.$$

Since $\mathcal{O}_W(-P)$ is invertible, the kernel $\mathcal{I}$ of this map is a locally free sheaf.

Lemma 4.7. *There is a canonical isomorphism of complexes* $\tilde{\xi}_\bullet \to K_\bullet(\varphi)$ *where* $\varphi: p_w^*\mathcal{I}(\infty) \to \mathcal{O}_W$ *is the map induced by minus the projection map* $\mathcal{O}_W^n \oplus \mathcal{O}_W(-\infty) \to \mathcal{O}_W(-\infty)$. *Furthermore*

(i) $\tilde{\xi}_\bullet|_{\tilde{X}}$ *is split acyclic.*

(ii) $\tilde{\xi}_\bullet|_{N_{Y/X} \otimes N_{\infty/\mathbf{P}CP}^{-1}}$ *is the tautological Koszul complex.*

(iii) $\tilde{\xi}_\bullet$ *is a resolution of* $j_*\mathcal{O}_{Y \times \mathbf{P}^1}$.

Proof. The complex $p_W^*\xi_\bullet \otimes K_\bullet^\infty$ is the Koszul complex associated to the map of sheaves $\mathcal{O}_W^n \oplus \mathcal{O}_W(-\infty) \to \mathcal{O}_W$ induced by the inclusion $\mathcal{O}_W(-P) \subset \mathcal{O}_W$. Hence the Koszul differential

$$\Lambda^i(\mathcal{O}_W^n \oplus \mathcal{O}_W(-\infty)) \to \Lambda^{i-1}(\mathcal{O}_W^n \oplus \mathcal{O}_W(-\infty))$$

is the composition of the canonical map

$$\Lambda^i(\mathcal{O}_W^n \oplus \mathcal{O}_W(-\infty)) \to \Lambda^{i-1}(\mathcal{O}_W^n \oplus \mathcal{O}_W(-P))$$

which has kernel $\Lambda^i\mathcal{I}$, the injective multiplication map

$$\Lambda^{i-1}(\mathcal{I}) \otimes \mathcal{O}_W(-P) \to \Lambda^{i-1}(\mathcal{I})$$

and the natural inclusion

$$\Lambda^{i-1}(\mathcal{I}) \to \Lambda^{i-1}(\mathcal{O}_W^n \oplus \mathcal{O}_W(-\infty)).$$

Hence the inclusion $\Lambda^i(\mathcal{I}) \to \Lambda^i(\mathcal{O}_W^n \oplus \mathcal{O}_W(-\infty))$ identifies $\Lambda^i(\mathcal{I})$ with $\mathrm{Ker}(d_i^{\mathcal{G}})$ and therefore, equivalently, identifies $\Lambda^i(\mathcal{I}(\infty))$ with $\tilde{\xi}_i$.

Minus the projection $\mathcal{O}_W^n \oplus \mathcal{O}_W(-\infty) \to \mathcal{O}_W(-\infty)$ induces a surjective map $\mathcal{O}_W(-\infty) \oplus \mathcal{O}_W \to \mathcal{O}_W$ and hence a map $\mathcal{I}(\infty) \to \mathcal{O}_W$. There is an induced map between Koszul complexes $\Lambda^\bullet(\mathcal{I}(\infty)) \to \Lambda^\bullet(\mathcal{O}_W^n(\infty) \oplus \mathcal{O}_W)$ which gives a commutative diagram

$$\begin{array}{ccccccc}
\Lambda^i(\mathcal{O}_W^n) = \xi_i & \longrightarrow & \Lambda^i(\mathcal{I}(\infty)) & \longrightarrow & \Lambda^i(\mathcal{O}_W^n(\infty) \oplus \mathcal{O}_W) & = & \Lambda^i(\mathcal{O}_W^n(\infty)) \oplus \Lambda^{i-1}(\mathcal{O}_W^n(\infty)) \\
\downarrow & & \downarrow & & & & \downarrow \\
\Lambda^{i-1}(\mathcal{O}_W^n) = \xi_{i-1} & \longrightarrow & \Lambda^{i-1}(\mathcal{I}(\infty)) & \longrightarrow & \Lambda^{i-1}(\mathcal{O}_W^n(\infty) \oplus \mathcal{O}_W) & = & \Lambda^{i-1}(\mathcal{O}_W^n(\infty)) \oplus \Lambda^{i-2}(\mathcal{O}_W^n(\infty))
\end{array}$$

$$\begin{array}{ccc} a & \longrightarrow & (a,(-1)^i d_i(a)) \\ \downarrow & & \downarrow \\ da & \longrightarrow & (d_i(a),0) \end{array}$$

(n.b. The Koszul differential induced by *minus* the projection $\mathcal{O}_W^n(\infty)\oplus \mathcal{O}_W \to \mathcal{O}_W$ is $e_1\wedge\cdots\wedge e_i + e_1\wedge\cdots\wedge e_{i-1}\wedge f \to (-1)^{i-2} f\, e_1\wedge\cdots\wedge e_{i-1}$ for $e_1,\ldots,e_n \in \mathcal{O}_W^n(\infty)$ and $f\in\mathcal{O}_W$.

We turn now to items (i), (ii), and (iii).

Observe that $\mathcal{O}_W(-P)\otimes\mathcal{O}_W(-\infty) = \mathcal{O}_W(\tilde{X})$, so that we have an exact sequence

$$(4.8)\qquad 0\to\mathcal{I}(\infty)\to\mathcal{O}_W^n(\infty)\oplus\mathcal{O}_W\to\mathcal{O}_W(\tilde{X})\to 0.$$

Hence if we restrict to $\tilde{X}$, the map $\mathcal{O}_{W|\tilde{X}} \subset \left(\mathcal{O}_W^n(\infty)\oplus\mathcal{O}_W\right)\big|_{\tilde{X}}$ to $\mathcal{O}_W(\tilde{X})\big|_{\tilde{X}}$ vanishes. Therefore, on restricting to $\tilde{X}$, $\mathcal{I}(\infty)$ splits as a direct sum

$$\mathcal{I}(\infty)\big|_{\tilde{X}} = \mathcal{I}_1\oplus\mathcal{O}_{\tilde{X}} \subset \mathcal{O}_W(\infty)^n\big|_{\tilde{X}}\oplus\mathcal{O}_{\tilde{X}}.$$

The Koszul complex associated to $\mathcal{I}(\infty)\to\mathcal{O}_W$ therefore restricts to the Koszul complex for minus the projection $\mathcal{I}_1\oplus\mathcal{O}_{\tilde{X}}\to\mathcal{O}_{\tilde{X}}$, which is split acyclic.

Turning to (ii) observe that on $P = \mathbf{P}(N_{Y/X}\otimes N_{\infty/\mathbf{P}^1}^{-1}\oplus\mathbf{C}), \mathcal{I}(\infty)$ restricts to the kernel of the map

$$\pi_P^*(\mathcal{N}_{Y/X}^*\otimes\mathcal{N}_{\infty/\mathbf{P}^1})\oplus\mathcal{O}_P\to\mathcal{O}_P(\tilde{X}\cap P)$$

i.e. to the analogue of the locally free sheaf $\mathcal{H}$ of Lemma 4.1; (ii) now follows from the discussion of loc.cit.

Finally, to prove (iii), we need only show that the map $\tilde{\xi}_\bullet\to j_*\mathcal{O}_{Y\times\mathbf{P}^1}$, induced by the map $\tilde{\xi}_0 = \mathcal{O}_W\to j_*\mathcal{O}_{Y\times\mathbf{P}^1}$, is a quasi-isomorphism of complexes. We know that $\tilde{\xi}_\bullet\big|_{W-W_\infty} = p_W^*\xi_\bullet$ is a resolution of $j_*\mathcal{O}_{Y\times\mathbf{C}}$, and we know that $\xi_\bullet\big|_{\tilde{X}}$ is acyclic; it suffices therefore to verify (iii) in a neighbourhood U of $Y\times\{\infty\}\subset W$. If we choose a local equation $t=0$ for ∞ in $\mathbf{P}^1$, then $x_1=\ldots=x_n=t=0$ is a local system of equations for $Y\times\{\infty\}\subset X\times\mathbf{P}^1$. Hence $\mathcal{I}$ is isomorphic to the kernel of map $\mathcal{O}_W^{n+1}\to\mathcal{O}_W$ sending $(a_1,\ldots,a_n,b)$ to $\sum_{i=1}^n a_ix_i+bt$. We can choose U so that $\frac{x_1}{t},\ldots,\frac{x_n}{t}$, t is part of a system of coordinates on U, with $Y\times\mathbf{P}^1\subset U$ given by the equations $\frac{x_1}{t}=\ldots=\frac{x_n}{t}=0$. (This is a part of the standard description of coordinates on a blow up.) The locally free sheaf $\mathcal{I}(\infty)$ is the kernel of the map

$$\mathcal{O}_W^{n+1}\to\mathcal{O}_W$$

$$(a_1,\ldots,a_n,b)\to\sum_{i=1}^n a_ix_i/t+b.$$

Therefore the map $\mathcal{O}_W^n \to \mathcal{O}_W^{n+1}$ sending $(a_1, \ldots, a_n)$ to $(a_1, \ldots, a_n, -\sum a_i x_i/t))$ is an isomorphism onto $\mathcal{I}(\infty)$. Composing with minus the projection $\mathcal{O}_W^{n+1} = \mathcal{O}_W^n \oplus \mathcal{O}_W \to \mathcal{O}_W$, we identify the Koszul complex $\Lambda^\bullet \mathcal{I}(\infty)$ with the Koszul complex $K_\bullet(\frac{x_1}{t}, \ldots, \frac{x_n}{t})$, which is a resolution of $j_* \mathcal{O}_{Y \times \mathbf{P}^1}$. □

We return now to the general case of a resolution $\xi_\bullet \to i_* \eta$. Note that there is a natural map $\tilde{\xi}_0 \to j_* \eta$.

Theorem 4.8.
(1) $\tilde{\xi}_\bullet$ *is a resolution of* $j_* p_Y^* \eta$.
(2) $\tilde{\xi}_\bullet|_{\tilde{X}}$ *is split acyclic.*
(3) *There is a natural exact sequence of complexes of locally free sheaves on* P:

$$0 \to \pi_P^* \mathcal{L}_\bullet \to f^* \tilde{\xi} \to K_\bullet(\varphi) \otimes \pi_P^* \eta \to 0 \tag{4.9}$$

where $K_\bullet(\varphi)$ *is the Koszul complex on* $P = \mathbf{P}(N_{Y/X} \otimes N_{\infty/\mathbf{P}^1}^{-1} \oplus \mathbf{C})$ *described in* a), *and* $\mathcal{L}_\bullet$ *is the split acyclic complex described in* (4.6).

Proof. (1) is a local question on X, so we may suppose that $\xi_\bullet = \zeta_\bullet \oplus (K_\bullet(\underline{x}) \otimes \mathcal{V})$ where $\zeta_\bullet$ is acyclic, $K_\bullet(\underline{x})$ is the Koszul complex associated to equations $x_1 = \ldots = x_n = 0$ of $Y \subset X$ and $\mathcal{V}$ is locally free on X such that $j^* \mathcal{V} \cong \eta$. Then $\tilde{\xi}_\bullet \cong \tilde{\zeta}_\bullet \oplus \widetilde{K_\bullet(\underline{x})} \otimes p_W^* \mathcal{V}$. By Lemma 4.5, $\tilde{\zeta}_\bullet$ is acyclic. By Lemma 4.7 $\widetilde{K_\bullet(\underline{x})}$ is a resolution of $j_* \mathcal{O}_{Y \times \mathbf{P}^1}$, and hence $\tilde{\xi}_\bullet$ is a resolution of $\mathcal{O}_{Y \times \mathbf{P}^1} \otimes p_W^* \mathcal{V} \cong j_*(p_Y^* \eta)$, thus proving part (1). Next, observe that by $\tilde{\zeta}_\bullet|_{W_\infty}$ is split acyclic, and hence $\tilde{\zeta}_\bullet|_{\tilde{X}}$ is split acyclic too, while $\widetilde{K_\bullet(x_1, \ldots, x_n)}|_{\tilde{X}}$ is split acyclic by part (i) of Lemma 4.7. Hence $\tilde{\zeta}_\bullet|_{\tilde{X}}$ is split acyclic, at least locally on X. However $\tilde{\xi}|_{\tilde{X} - \tilde{X} \cap P}$ is split acyclic by Lemma 4.5, and the local splittings described above are compatible with this splitting, and hence uniquely determined, and patch together into a global splitting.

Finally, to prove (iii), we start by observing that by Remark 4.2 (ii), there is an epimorphism of locally free sheaves on P:

$$\varepsilon_i : f^* \tilde{\xi}_i \to H_i(p_W^* \xi_\bullet \otimes (\mathcal{O}_W(-\infty) \to \mathcal{O}_W))(i\infty) \cong \Lambda^i(\mathcal{H}) \otimes \pi_P^* \eta. \tag{4.10}$$

Here $\mathcal{H}$ is the kernel of the map $\mathcal{N}_{Y/X}^* \otimes \mathcal{N}_{\infty/\mathbf{P}^1} \oplus \mathcal{O}_P \to \mathcal{O}_P(\tilde{X})$. In order to show that this map is compatible with differentials, we can work locally on X, and write $\xi_\bullet = \zeta_\bullet \oplus (K_\bullet(\underline{x}) \otimes \mathcal{V})$ as above. Then $f^*(\tilde{\xi}) \cong f^*(\tilde{\zeta}) \oplus f^* \widetilde{K_\bullet(\underline{x})} \otimes \pi_P^* \eta$, and since $\tilde{\zeta}$ is acyclic, the map ε factors through the projection onto the second factor in the direct sum. But by Lemma 4.7, the map $f^*\widetilde{K_\bullet(\underline{x})}\otimes\pi_P^*\eta \to \Lambda^*(\mathcal{H})\otimes\pi_P^*(\eta)$ is an isomorphism of complexes. Hence ε is a map of complexes, and its kernel is, locally on X, $\tilde{\zeta}_\bullet|_P$. There is a map of complexes $f^*\pi_P^*\eta \to f^*(\tilde{\xi}_\bullet \otimes (\mathcal{O}_W \to \mathcal{O}_W(\infty)))$ given by the inclusions

$$\pi_P^*(\mathcal{B}_{i+1}\otimes\mathcal{N}^i_{\infty/\mathbf{P}^1}) \subset \pi_P^*(i^*\xi_i\otimes\mathcal{O}_{\mathbf{P}^1}(\infty)) = f^*\pi^*(\xi_i(i\infty)).$$

It suffices to show that the image of this map is contained in $\tilde{\xi}_\bullet$, and equals the kernel of ε. Again this is a local question, and writing, as usual, $\xi_\bullet \cong \zeta_\bullet \oplus (K_\infty(\underline{x})\otimes\mathcal{V})$, we see that $\mathcal{L}_\bullet \cong \tilde{\zeta}|_{Y\times\{\infty\}} = \operatorname{Ker}\varepsilon$ as desired. The complex $\tilde{\xi}_\bullet$ on W constructed above coincides (if $\eta \neq 0$) with the complex obtained by MacPherson's Grassmannian graph construction, which is described in [BaFM]. In [BaFM] it is shown that the parameter space for the Grassmannian graph construction applied to $\xi_\bullet$ is *locally* isomorphic to W. It then follows from the fact that the map from the Grassman bundle to $X \times \mathbf{P}^1$ is separated, that the two constructions are globally isomorphic. □

Lemma 4.9. *The normal bundle of $Y \times \mathbf{P}^1$, in W,*

$$\mathcal{N}_{Y\times\mathbf{P}^1/W} \cong p_Y^*\mathcal{N}_{Y/X}\otimes q_Y^*\mathcal{O}_{\mathbf{P}^1}(-\infty).$$

Proof. (This also follows directly from description $\mathcal{N}^*$ as $\mathcal{I}/\mathcal{I}^2$; see 1.1).

$$\mathcal{N}^*_{Y\times\mathbf{P}^1/W} \cong \mathcal{T}or_1^{\mathcal{O}_W}(\mathcal{O}_Y,\mathcal{O}_Y).$$

Let $\tilde{\xi}_\bullet \to i_*\mathcal{O}_Y$ be a resolution; then

$$\mathcal{T}or_1^{\mathcal{O}_W}(\mathcal{O}_Y,\mathcal{O}_Y) \cong \mathcal{H}_1(j^*\tilde{\xi}_\infty) \cong \mathcal{H}_1(p_Y^*\xi_\bullet)\otimes q_Y^*\mathcal{O}_{\mathbf{P}^1}(\infty)$$

and by the discussion after Corollary 4.6

$$\cong\ p_Y^*\mathcal{N}^*_{Y/X}\otimes q_Y'^*\mathcal{O}_{\mathbf{P}^1}(\infty). \qquad \square$$

Remark 4.10. All the results of a), b), c) remain valid for arithmetic varieties over any base instead of the complex numbers.

(d) *Bott-Chern currents and deformation to the normal cone.* We keep the notations of the previous paragraph, except that we denote by $N = N_{Y/X}$ the normal bundle of Y in X, and by $N' = N_{Y\times\mathbf{P}^1/W}$ the normal bundle of $Y \times \mathbf{P}^1$ in W. We also call φ the projection $p_W: W \to X$, and $\psi = \pi_P: P \to Y$ its restriction to P.

We now put metrics on the bundles considered so far: We fix metrics g^N g^η on N, η, and h^ξ on ξ. We assume that h^ξ satisfies assumption (A) with respect to g^N, g^η.

On $\mathcal{O}_{\mathbf{P}^1}(-\infty)$ we put the standard metric, and on N' the metric $g^{N'}$ coming from the isomorphism of Lemma 4.9.

On $\tilde{\xi}$ we choose a metric $h^{\tilde{\xi}}$ which satisfies assumption (A) with respect to g^η and $g^{N'}$, whose restriction to $W_0 = X \times \{0\}$ coincides with h^ξ, and whose restriction to the blow up $\tilde{X} \subset W_\infty$ of X along Y is split acyclic (these assumptions are compatible since (ξ, h) satisfies assumption (A) and $\tilde{X}$ does not meet $Y \times \{\infty\}$).

On the bundle H on P whose sheaf of sections is $\mathcal{H}$, we put the metric g^H induced from its inclusion into $\psi^*(\mathcal{N}^*_{Y/X} \oplus \mathcal{N}^*_{\infty/\mathbf{P}^1})$ (see Lemma 4.1). Note that H restricted to Y is isometric to N^*. Let σ be the canonical section of H^* and $\sigma^*(\tilde{e}(g^{H^*}))$ the corresponding Euler-Green current on P (see Remark 3.16).

For each $j \geq 0$ consider the exact sequence of sheaves

$$A_j: 0 \to \pi_P^* \mathcal{L}_j \to f^* \tilde{\xi}_j \xrightarrow{\varepsilon} K_j(\varphi) \otimes \pi_P^* \eta \to 0$$

defined in Theorem 4.8 (3). On the bundle L_j corresponding to

$$\mathcal{L}_j = (\mathcal{B}_{j+1} \otimes \mathcal{N}^j_{\infty/\mathbf{P}^1}) \oplus (\mathcal{B}_j \otimes \mathcal{N}^{j-1}_{\infty/\mathbf{P}^1})$$

we put the orthogonal direct sum of the induced metrics, so that the complex $L_\bullet$ attached to $\mathcal{L}_\bullet$ becomes split acyclic as a complex of Hermitian holomorphic vector bundles. On $K_j(\varphi) = \Lambda^j H$ we put the metric induced from g^H. Let g^{A_j} be the corresponding metric on the complex A_j. Then $T(g^{A_j})$ is a smooth current on P.

Finally, let z be the standard coordinate on $\mathbf{P}^1$ and $\operatorname{Log}|z|^2$ the function on W obtained by pulling back $\operatorname{Log}|z|^2$ on $\mathbf{P}^1$ via $p_W: W \to \mathbf{P}^1$. Notice that $\operatorname{Log}|z|^2$ is integrable on W, since near $\mathbf{P}(N)$ we have the equations

$$z^{-1} = \frac{x_0}{x_i} y_i$$

(where $y_i = 0$ are local equations of Y in X and $x_0, \ldots, x_e$ are coordinates in $N \oplus 1$). Therefore

$$\operatorname{Log}|z|^2 = -\operatorname{Log}|y_i|^2 - \operatorname{Log}\left|\frac{x_i}{x_0}\right|^2 \tag{4.11}$$

is locally integrable.

Theorem 4.11. *Let C be the current defined by*

(4.12)

$$C = T(h^{\xi}) + \varphi_*\left[\mathrm{Log}|z|^2 ch(h^{\tilde{\xi}})\right] - \psi_*\left[Td^{-1}(g^{H^*})\,\sigma^*(\tilde{e}(g^{H^*}))\right] ch(g^{\eta})\delta_Y + \psi_*\left[\sum_{j=0}^{m}(-1)^j T(g^{A_j})\right]\delta_Y.$$

Then $C \in P_Y^0$. Furthermore the integral $\psi_*\left[Td^{-1}(g^{H^*})\sigma^*(\tilde{e}(g^{H^*}))\right]$ along the fibers of ψ is a smooth closed differential form on Y, whose cohomology class does not depend on the choice of the metric g^N.

Proof. By Theorem 1.9 the wave front set $WF(T(h^{\tilde{\xi}}))$ in $T^*_{\mathbf{R}}W$ of the current $T(h^{\tilde{\xi}})$ is included in $N'^*_{\mathbf{R}}$. Also the projection $q_W: W \to \mathbf{P}^1$ is a submersion near $Y \times \mathbf{P}^1 \subset W$. In particular we deduce from [H, Theorem 8.2.4] that

$$WF(T(h^{\tilde{\xi}})) \cap WF(-\mathrm{Log}|z|^2) = \emptyset.$$

By [H, Theorem 8.2.10], the product of currents $\mathrm{Log}|z|^2 T(h^{\tilde{\xi}})$ is well-defined, and we can use on this product the usual rules of differential calculus. In particular

$$(4.13)\qquad \begin{aligned} &\frac{1}{2i\pi}\bar{\partial}\partial(\mathrm{Log}|z|^2)T(h^{\tilde{\xi}}) - \mathrm{Log}|z|^2\frac{\bar{\partial}\partial}{2i\pi}T(h^{\tilde{\xi}}) \\ &\quad = \frac{\bar{\partial}}{2i\pi}((\partial\,\mathrm{Log}|z|^2)T(h^{\tilde{\xi}})) + \frac{\partial}{2i\pi}(\mathrm{Log}|z|^2(\bar{\partial}T(h^{\tilde{\xi}}))) \end{aligned}$$

and so

$$\frac{1}{2i\pi}\bar{\partial}\partial(\mathrm{Log}|z|^2)T(h^{\tilde{\xi}}) - \mathrm{Log}|z|^2\frac{\bar{\partial}\partial}{2i\pi}T(h^{\tilde{\xi}})$$

lies in $P^{W,0}$. Using (2.16) and (4.11) (which is of special interest where q_W is not a submersion), we find that

$$(4.14)\qquad \frac{\bar{\partial}\partial}{2i\pi}\mathrm{Log}|z|^2 = \delta_{W_0} - \delta_{W_\infty}.$$

It follows from our previous considerations on wave front sets that the restrictions of the current $T(H^{\tilde{\xi}})$ to W_0 and W_∞ are well-defined. Also, by construction $T(h^{\tilde{\xi}})|_{W_0} = T(h^{\xi})$. We deduce from Theorem 1.9 and equations (4.13) and (4.14) that

(4.15)

$$T(h^{\xi})\delta_{W_0} - T(h^{\tilde{\xi}})\delta_{W_\infty} - \operatorname{Log}|z|^2 (Td^{-1}(g^{N'})ch(g^{\eta}))\delta_{Y\times \mathbf{P}^1} - ch(\tilde{\xi})) = \frac{\bar{\partial}}{2i\pi}((\partial \operatorname{Log}|z|^2)T(h^{\tilde{\xi}})) + \frac{\partial}{2i\pi}((\operatorname{Log}|z|^2)\bar{\partial}T(h^{\tilde{\xi}}))$$

lies in P_0^W. Using Lemma 3.10, we find easily that,

$$Td^{-1}(g^{N'}) = Td^{-1}(g^N) + (Td^{-1})'(g^N)\cdot\left(\frac{-R}{2i\pi}\right), \tag{4.16}$$

where R denotes the curvature of the holomorphic Hermitian connection on $\mathcal{O}(-1)$. By integrating (4.15) along the fibers of $\varphi: W \to X$ and using (4.16), we find that

$$\begin{aligned} &T(h^{\xi}) - \varphi_*\left[T(h^{\tilde{\xi}})\delta_{W_\infty}\right] + \varphi_*\left[\operatorname{Log}|z|^2 ch(h^{\tilde{\xi}})\right] \\ &\quad - \left[\int_{\mathbf{P}^1} \operatorname{Log}(|z|^2)\cdot\left(\frac{-R}{2i\pi}\right)\right](Td^{-1})'(g^N)ch(g^{\eta})\delta_Y \\ &\quad = \frac{\bar{\partial}}{2i\pi}\varphi_*(\partial(\operatorname{Log}|z|^2)T(h^{\tilde{\xi}})) + \frac{\partial}{2i\pi}\varphi_*((\operatorname{Log}|z|^2)\bar{\partial}T(h^{\tilde{\xi}})). \end{aligned} \tag{4.17}$$

We now calculate $\varphi_*\left(T(h^{\tilde{\xi}})\delta_{W_\infty}\right)$.

Let $\tilde{A}_u$ be the superconnection (1.10) associated with the chain complex of Hermitian vector bundles $(\tilde{\xi}, \tilde{v})$. Let k be the imbedding $W_\infty \to W$. Note that $\mathbf{P}(N)$ is of measure zero in W_∞. Near $\mathbf{P}(N)$, W_∞ is the union of two smooth manifolds intersecting transversally along $\mathbf{P}(N)$. Therefore, if α is a smooth form on W, the form $k^*\alpha$ is unambiguously defined on $W \setminus \mathbf{P}(N)$, and defines an integrable current on W_∞. Also, as a current on M, $k^*(\alpha)\delta_{W_\infty}$ is exactly the product of the currents α and δ_{W_∞}.

Note that Y is a submanifold of W_∞. Also, by Section 4b) and our choice of metrics, the normal bundle to Y in W_∞ coincides—as a holomorphic Hermitian vector bundle—with N.

For $0 < \operatorname{Re}(s) < 1$ let $\zeta_{\tilde{\xi}}^{\infty}(s)$ be the current on W_∞

$$\begin{aligned} \zeta_{\tilde{\xi}}^{\infty}(s) = \frac{1}{\Gamma(s)}\int_0^{+\infty} u^{s-1}\Big[&k^* Tr_s\left[N_H \exp(-\tilde{A}_u^2)\right] \\ &+ \varphi^{-1}\left[(Td^{-1})'(g^N)ch(\eta)\right]\delta_Y\Big]du. \end{aligned} \tag{4.18}$$

Note that since the forms $Tr_s\left[N_H \exp(-\tilde{A}_u^2)\right]$ decay exponentially fast as $u \to +\infty$ on compact subsets of $W_\infty \setminus Y$, and in particular near $\mathbf{P}(N)$, the arguments of [BGS4] can be used verbatim to prove the existence of $\zeta_{\tilde{\xi}}^{\infty}(s)$. Furthermore this function of s extends holomorphically at $s = 0$. Let

$$T^{\infty}(h^{\tilde{\xi}}) = \varphi[\zeta_{\tilde{\xi}}^{\infty\prime}(0)]. \tag{4.19}$$

We claim that

$$T(h^{\tilde{\xi}})\delta_{W_\infty} = T^{\infty}(h^{\tilde{\xi}})\delta_{W_\infty}. \tag{4.20}$$

In fact, we can replace in both sides of (4.20) integration in u from 0 to $+\infty$ by integration from 0 to T $(0 < T < +\infty)$, and then equality is an obvious consequence of the previous considerations. We now make $T \to +\infty$. Since by Theorem 1.9, the truncated integrals approximate $T(h^{\tilde{\xi}})$ in $\mathcal{D}'_{N'_{\bullet}}(W)$, and since multiplication by δ_{W_∞} maps continuously $\mathcal{D}'_{N'_{\bullet}}(W)$ in $\mathcal{D}'(W)$, we obtain (4.20).

We claim that on $W_\infty \setminus P$ the smooth current $T^{\infty}(h^{\tilde{\xi}})$ vanishes identically. In fact, on $W_\infty \setminus P$ the complex $(\tilde{\xi}, \tilde{v})$ splits as a holomorphic Hermitian complex. We can now use the argument of [BGS1, Corollary 1.30]. On $W_\infty \setminus P$ we have

$$\tilde{A}_u^2 = (\nabla^{\tilde{\xi}})^2 + uI_{\tilde{\xi}}. \tag{4.21}$$

Using (4.18) and (4.21), we find that on $W_\infty \setminus P$, $\zeta_{\tilde{\xi}}^{\infty}(s)$ is the smooth form $Tr_s[N_H \exp(-(\nabla^{\tilde{\xi}})^2)]$, which does not depend on s, so that $T^{\infty}(h^{\tilde{\xi}}) = 0$ on $W_\infty \setminus P$.

Therefore the support of the current $T^{\infty}(h^{\tilde{\xi}})$ is included in the manifold P. More precisely the restriction of the current $T^{\infty}(h^{\tilde{\xi}})$ to the manifold P is exactly the singular current associated with the holomorphic complex $\tilde{\xi} \mid P$ of Hermitian vector bundles, which provides a resolution of the direct image $s_*\eta$ of η by the immersion $s: Y \to P$. We now consider $T^{\infty}(h^{\tilde{\xi}})$ as a current on P.

By (4.9) we have on P an exact sequence of complexes of sheaves

$$0 \to \pi_P^* \mathcal{L}_\bullet \to f^* \tilde{\xi} \xrightarrow{\varepsilon} K_\bullet(\varphi) \otimes \pi_P^* \eta \to 0$$

where $\mathcal{L}$ is split acyclic, and ε is a map of resolutions of $s_*\eta$. The Hermitian metrics on $K_\bullet(\varphi) \otimes \pi_P^*\eta$ and $\tilde{\xi}$ both verify assumption (A) with respect to the same couple of metrics on N and η. Finally, the holomorphic Hermitian complex $L_\bullet$ defined from $\mathcal{L}_\bullet$ splits. It now follows from Theorem 2.9 that

$$T^{\infty}(h^{\tilde{\xi}}) = T(K_\bullet(\varphi) \otimes \pi_P^*\eta) - \sum_{j=0}^{m} (-1)^j T(g^{A_j}) \quad \text{in } P_Y^P / P_Y^{P,0}. \tag{4.22}$$

By Theorem 3.17 we know that

$$T(K_\bullet(\varphi)\otimes\pi_P^*\eta) - Td^{-1}(g^{H^*})\sigma^*(\tilde{e}(g^{H^*}))\,ch(g^\eta) \tag{4.23}$$

lies in $\tilde{P}_Y^{P,0}$. From (4.20), (4.22), (4.23) we deduce that

$$\begin{aligned}\varphi_*\left[T(h^{\tilde{\xi}})\delta_{W_\infty}\right] &= \psi_*\left[Td^{-1}(g^{H^*})\sigma^*(\tilde{e}(g^{H^*}))\right]ch(g^\eta)\delta_Y \\ &\quad - \psi_*\left[\sum_{j=0}^m(-1)^jT(g^{A_j})\right]\delta_Y \ \text{ in } P_Y^X/P_Y^{X,0}.\end{aligned} \tag{4.24}$$

Also

$$\int_{\mathbf{P}^1}\operatorname{Log}|z|^2R = 0$$

(map z to $1/z$).

The conormal bundle to W_∞ in W is well defined out of $\mathbf{P}(N)$. Near $\mathbf{P}(N)$, W is made of two smooth submanifolds intersecting transversally along $\mathbf{P}(N)$. On $\mathbf{P}(N)$, we define the conormal bundle to W_∞ to be the union of the conormal bundles to these two submanifolds.

Using [H, Theorem 8.2.13] and the fact that the map of φ can be expressed as the composition of an immersion and a submersion, we find that if ω is a current on W

$$WF(\varphi_*\omega)\subset\left\{p\in T_R^*Y\setminus\{0\};\ \varphi^*p\in\{0\}\cup WF(\omega)\right\}.$$

Now by [H, Theorem 8.2.9], the wave front sets of the currents $\partial(\operatorname{Log}|z|^2T(h^{\tilde{\xi}}))$ and $\operatorname{Log}|z|^2\bar{\partial}T(h^{\tilde{\xi}})$ are included in the sum of the conormal bundles to W_∞ and to $Y\times\mathbf{P}^1$ in W. Using the local form of the map φ given in the proof of Lemma 4.7 (iii), it is clear that

$$\begin{aligned}&WF\big(\varphi_*\partial(\operatorname{Log}|z|^2)T(h^\xi)\big)\subset N_R^*\\ &WF\big(\varphi_*((\operatorname{Log}|z|^2)\bar{\partial}T(h^\xi))\big)\subset N_R^*.\end{aligned}$$

From (4.17), (4.24) and from the previous considerations, we find that C lies in $P_Y^{X,0}$.

We now prove that the form $\psi_*\left[Td^{-1}(g^{H^*})\sigma^*(\tilde{e}(g^{H^*}))\right]$ is closed. Let Q be the bundle of unitary frames in the vector bundle N. Then Q is a $U(e)$-principal bundle ($e=\operatorname{codim}(Y)$) which we equip with the connection ∇^N. We equip $\mathbf{C}^e$ with its canonical Hermitian metric. Then $U(e)$ acts on $\mathbf{C}^e$ and it also acts on $\mathbf{P}(\mathbf{C}^e\oplus 1)$ as a group of holomorphic transformations. One then immediately verifies that

$$P=\mathbf{P}(N\oplus 1)=Q\times_{U(e)}\mathbf{P}(\mathbf{C}^e\oplus 1).$$

Also we can form on the single "fiber" $\mathsf{P}(\mathbf{C}^e \oplus 1)$ the holomorphic Hermitian vector bundle H_0, with a morphism $\varphi_0 : \mathcal{H}_0 \to \mathcal{O}_{\mathsf{P}(\mathbf{C}^e \oplus 1)}$ and the holomorphic Hermitian Koszul chain complex $\Lambda H_0 = K(\varphi_0)$. The group $U(e)$ acts naturally on H_0 as a group of holomorphic unitary transformations, which preserve the map φ_0. One has

$$H = Q \times_{U(e)} H_0$$

and

$$K(\varphi) = Q \times_{U(e)} K(\varphi_0).$$

The connection ∇^N induces a connection on the fibration $P \to Y$. In particular the curvature T of the fibration $P \to Y$ is obtained by lifting the action of $(\nabla^N)^2$ on the fibers N to P; it is a two form on Y with values in vector fields along the fibers.

The previous considerations show that T lifts to a two form $\tilde{T}$ on Y with values in infinitesimal unitary transformations of H along the fibers. Let $\tilde{T}_0$ be the horizontal part of $\tilde{T}$ with respect to the connection ∇^H. Then $\tilde{T}_0$ is a two-form on Y with values in skew-adjoint endomorphisms of H, so that

$$\tilde{T} = -\nabla^H_T + \tilde{T}_0.$$

The connection ∇^N induces a splitting

$$T_{\mathbf{R}} P = \psi^* T_{\mathbf{R}} Y \oplus T^V_{\mathbf{R}} P.$$

Let R be the restriction of the curvature $(\nabla^H)^2$ to vectors of $T^V_{\mathbf{R}} P$. Then we find that

$$(\nabla^H)^2 = R + \tilde{T}_0$$

i.e.:

- on vectors of $T^V_{\mathbf{R}} P$, $(\nabla^H)^2$ coincides with R.
- on horizontal vectors, $(\nabla^H)^2$ coincides with $\tilde{T}_0$.
- if $U \in \psi^* \, T_{\mathbf{R}} Y$, and $V \in T^V_{\mathbf{R}} P$, then $(\nabla^H)^2(U, V) = 0$.

In the Chern–Weil formula for $Td(g^H)$, we now replace $(\nabla^H)^2$ by $R + \tilde{T}_0$. Similarly in the formula for $\tilde{e}(g^H)$, we also replace $(\nabla^H)^2$ by $R + \tilde{T}_0$. Remember that $\tilde{T}_0$ can be canonically expressed in terms of $(\nabla^N)^2$.

Let u_0 be a unitary frame in N. We consider u_0 as a linear isometry from $\mathbf{C}^e$ into N. Let $\mathcal{U}(e)$ be the Lie algebra of $U(e)$. The previous considerations show that for $A \in \mathcal{U}(e)$ there exists a smooth form $\omega(A)$ on $\mathsf{P}(\mathbf{C}^e \oplus 1)$ with the following properties:

- the map $A \to \int_{\mathsf{P}(\mathbf{C}^e \oplus 1)} \omega(A)$ is ad-invariant.

• $\psi_*\left[Td^{-1}(g^{H^*})\sigma^*(\tilde{e}(g^{H^*}))\right] = \int_{\mathbf{P}(\mathbf{C}^e\oplus 1)} \omega(u_0^{-1}(\nabla^N)^2 u_0)$.

Using the standard Chern–Weil theory, it is now clear that

$$\psi_*\left[Td^{-1}(g^{H^*})\,\sigma^*(\tilde{e}(g^{H^*}))\right]$$

is a closed form.

The fact that the cohomology class of $\psi_*\left[Td^{-1}(g^{H^*})\sigma^*(\tilde{e}(g^{H^*}))\right]$ does not depend on the metric g^H is now obtained by the usual argument in Chern–Weil theory.

(e) **Proof of (3.95).** We use the notations from (3.95). In particular X is an arithmetic variety, (E,h) an hermitian vector bundle over X, $p\colon X^E \to X$ the canonical projection from its total space to X, and $\hat{e}(p^*E,p^*h) \in \widehat{CH}^r(X^E)$ the Euler class defined before Theorem 3.22. Let s be an algebraic section of E such that the corresponding holomorphic section s_∞ satisfies the hypotheses of Remark 3.5. Using [GS1] and the fact that s is transverse to $s_0(X_F)$, we get

$$\hat{e}(E,h) = s^*\left(\hat{e}(p^*E,p^*h)\right).$$

We want to show that this class is also equal to

$$s_0^*\left(\hat{e}(p^*E,p^*h)\right).$$

Let $E\oplus 1$ be the direct sum on X of E with the trivial line bundle, and $\mathbf{P}(E\oplus 1)$ the associated projective space. Consider the map

$$\psi\colon X\times \mathbf{A}^1 \to X^E$$

sending (x,t) to $(x,ts(x))$. On $X\times\{0\}$ ψ coincides with s_0, and with s on $X\times\{1\}$. The closure of the image of ψ in $\mathbf{P}(E\oplus 1)$ is the variety W studied in paragraph (b) above. Let H be the kernel of the map $E^*\oplus 1 \to \mathcal{O}(1)$ on $\mathbf{P}(E\oplus 1)$, endowed with the induced metric, and let $\alpha\in\widehat{CH}^r(W)$ be the restriction of $\hat{e}(H^*)$. By the argument of [GS1, Theorem 4.4.5] (see also Lemma 4.12 below), and using the fact that the function $\mathrm{Log}|z/(z-1)|^2$ vanishes at infinity, we get the following equation in $\widehat{CH}^r(X)$:

$$i_0^*\alpha - i_1^*\alpha = -a\left(p_{W*}\left((\mathrm{Log}|z/(z-1)|^2 e(H^*)\right)\right) \tag{4.25}$$

where p_W is the projection from W to X, $e(H^*)$ is the Euler form of the dual of H, and $i_t\colon X\to\mathbf{P}(E\oplus 1)$ maps x to $(x,ts(x))\in X^E\subset\mathbf{P}(E\oplus 1)$.

When restricted to X^E, the bundle H^* becomes isomorphic to p^*E, and its canonical section becomes the tautological section of p^*E, but its metric is not p^*h. Let $\tilde{c}_r$ be the corresponding Bott-Chern class. From (3.93) we get

$$(4.26) \qquad i_0^*\alpha - i_1^*\alpha = (s_0^* - s^*)(\hat{e}(p^*E, p^*h) + \tilde{c}_r).$$

On the other hand, the restriction β of $\hat{c}_r(H^*)$ to W satisfies

$$(4.27) \qquad i_0^*\beta - i_1^*\beta = -a(p_{W*}(\mathrm{Log}|z/(z-1)|^2 e(H^*))$$

by the same argument as above, since $e(H^*)$ is the r-th Chern form of H^*. We also have

$$(4.28) \qquad i_0^*\beta - i_1^*\beta = (s_0^* - s^*)(\hat{c}_r(p^*E, p^*h) + \tilde{c}_r).$$

But the functoriality of $\hat{c}_r$ implies

$$(4.29) \qquad (s_0^* - s^*)(\hat{c}_r(p^*E, p^*h)) = \hat{c}_r(E, h) - \hat{c}_r(E, h) = 0.$$

Using (4.25), (4.26), (4.27), (4.28), (4.29) we conclude that (3.95) holds. □

(f) *Immersions and the arithmetic Chern character.*

1. Let (A, Σ, F_∞) be an arithmetic ring as in 3 j) and

$$\begin{array}{ccc} Y & \xrightarrow{i} & X \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle f} & \\ & B & \end{array}$$

a diagram of morphisms between arithmetic varieties over A. We assume that the morphisms f and g are smooth and projective, and that i is a closed immersion. Let η be an Hermitian vector bundle on Y and

$$\xi_\bullet \to i_*\eta$$

a resolution of its direct image on X. We choose a metric on the normal bundle N to Y in X and a metric on ξ such that hypothesis (A) is satisfied.

Let $\varphi: W \to X$ be the deformation of i to the normal cone, $j: Y \times \mathbf{P}^1 \to W$ the immersion extending i, $p_Y: Y \times \mathbf{P}^1 \to Y$ the first projection, $\tilde{\eta} = p_Y^*(\eta)$ and $\tilde{\xi}_\bullet \to j_*\tilde{\eta}$ a metrized resolution such that, on $W_0 = q_W^{-1}(0)$,

$\tilde{\xi}_\bullet$ coincides with $\xi_\bullet$ (we denote by $q_W: W \to \mathbf{P}^1$ the projection). We also assume that the metric on $\tilde{\xi}$ is compatible with the metric on the normal bundle $N(-1)$ of $Y \times \mathbf{P}^1$ in W as in 4d). As we saw in 4b) above, $W_\infty = q_W^{-1}(\infty)$ has two components, the blow up $\tilde{X}$ of X along Y and the projective space $P = \mathbf{P}(N \oplus 1)$ over Y. Let $\psi: \mathbf{P}(N \oplus 1) \to Y$ be the projection and ξ_∞ the restriction of $\tilde{\xi}_\bullet$ on P. We also assume that $\tilde{\xi}_\bullet$ restricted to $\tilde{X}$ is split acyclic as complex of hermitian vector bundles [BGS1]. We shall consider the Chern character

$$\widehat{ch}(\tilde{\xi}_\bullet) = \sum_{p \geq 0} (-1)^p \widehat{ch}(\tilde{\xi}_p)$$

in $\widehat{CH}(W)_{\mathbf{Q}}$ [GS2].

Let $a: \tilde{A}(X_{\mathbf{R}}) \to \widehat{CH}(X)$ and $\omega = \widehat{CH}(X) \to A(X_{\mathbf{R}})$ be defined as in the proof of Theorem 3.22. Finally let $ch(h^{\tilde{\xi}})$ be the Chern character of $\tilde{\xi}_\bullet$ and

$$\beta = \int_{W/X} ch(h^{\tilde{\xi}}) \mathrm{Log}|z|^2$$

its integral against the integrable function $\mathrm{Log}|z|^2$ (defined via $q_W: W \to \mathbf{P}^1$). We view β as a current in $\tilde{\mathcal{D}}(X_{\mathbf{R}})$.

We refer to [GS1] for the definition of f_* and i^* on $\widehat{CH}$.

Lemma 4.12. *For any $\alpha \in \widehat{CH}(X)$, the following identity holds:*

$$f_*\big(\widehat{ch}(\xi_\bullet)\alpha\big) = g_*\big(\psi_*(\widehat{ch}(\xi_\infty))i^*(\alpha)\big) + af_*(\beta\omega(\alpha)).$$

Proof. The proof below will imply that $dd^c f_*(\beta\omega(\alpha))$ is smooth, hence the current $f_*(\beta\omega(\alpha))$ lies in $\tilde{A}(B_{\mathbf{R}})$ and its image by a makes sense.

Let us represent $\widehat{ch}(\tilde{\xi}_\bullet)$ by an arithmetic cycle (Z, g). Since $\tilde{\xi}_\bullet$ is acyclic outside $Y \times \mathbf{P}^1$ we may assume (up to linear equivalence) that Z is supported on $Y \times \mathbf{P}^1$, and moreover that $Z = p_Y^*(T) + S \times \{1\}$, with S and T two cycles on Y.

By definition of $\widehat{CH}^*(W)$ we have $\widehat{\mathrm{div}}(z) = \big(\mathrm{div}(z), -\mathrm{Log}|z|^2\big) = 0$ in this group. Therefore

$$f_*\varphi_*\big(\widehat{ch}(\tilde{\xi}_\bullet)\widehat{\mathrm{div}}(z)\alpha\big) = 0. \tag{4.30}$$

Now $\widehat{ch}(\tilde{\xi}_\bullet)\widehat{\mathrm{div}}(z)$ is the class of

$$\begin{aligned}(Z, g)\big(\mathrm{div}(z), -\mathrm{Log}|z|^2\big) = \big(&T \times \{0\} - T \times \{\infty\}, \\ &g|_{W_0} - g|_{W_\infty} - ch(h^{\tilde{\xi}})\mathrm{Log}|z|^2\big)\end{aligned} \tag{4.31}$$

since $S \times \{1\}$ does not meet $\operatorname{div}(z) = W_0 - W_\infty$. On the blow-up $\tilde{X}$ of X along Y the restriction of $\tilde{\xi}_\bullet$ is split acyclic, therefore we may assume that $g\big|_{\tilde{X}} = 0$. Using (4.30) and (4.31) we get

$$\begin{aligned} 0 &= f_*\varphi_*\big(\widehat{ch}(\tilde{\xi}_\bullet)\widehat{\operatorname{div}}(z)\alpha\big) \\ &= f_*\big(\widehat{ch}(\tilde{\xi}_\bullet)\alpha\big) - g_*\big(\psi_*(\widehat{ch}(\xi_\infty))\big)i^*(\alpha) - f_*\varphi_*\big(0, ch(h^{\tilde{\xi}})\mathrm{Log}|z|^2\omega(\alpha)\big) \end{aligned}$$

and we get the lemma since, by definition,

$$\beta = \varphi_*\big(ch(h^{\tilde{\xi}})\mathrm{Log}|z|^2\big). \qquad \square$$

Let now $\widehat{Td}(N) \in \widehat{CH}(Y)_{\mathbf{Q}}$ be the Todd class of the normal bundle to Y in X (defined in [GS2]; $\widehat{Td}$ is not to be confused with the arithmetic Todd genus Td^A of [GS3]).

Theorem 4.13. *For any α in $\widehat{CH}(X)$, the following identity holds in $\widehat{CH}(B)_{\mathbf{Q}}$*

$$f_*\big(\widehat{ch}(\xi_\bullet)\alpha\big) = g_*\big(\widehat{Td}(N)^{-1}\widehat{ch}(\eta)i^*(\alpha)\big) - af_*\big(T(h^\xi)\omega(\alpha)\big).$$

Proof. From Lemma 4.12, we have

$$f_*\big(\widehat{ch}(\xi_\bullet)\alpha\big) = g_*\big(\psi_*(\widehat{ch}(\xi_\infty)i^*(\alpha))\big) + af_*(\beta\omega(\alpha)). \tag{4.32}$$

From Theorem 4.11

$$\begin{aligned} -f_*(\beta\omega(\alpha)) = f_*\big(T(h^\xi)\omega(\alpha)\big) \\ &- g_*\big(\psi_*(Td^{-1}(g^{H^*})\sigma^*(\tilde{e}(g^{H^*}))ch(g^\eta)i^*\omega(\alpha))\big) \\ &+ \sum_{j\geq 0}(-1)^j g_*\big(\psi_*(T(g^{A_j}))i^*\omega(\alpha)\big). \end{aligned} \tag{4.33}$$

From [GS2, Theorem 4.8(ii)] and the fact that $L_\bullet$ is split acyclic, we know that

$$\widehat{ch}(\xi_\infty) = \widehat{ch}\big(K(\varphi)\otimes\pi_P^*\eta\big) + \sum_{j=0}(-1)^j a(T(g^{A_j})). \tag{4.34}$$

Combining (4.32), (4.33), and (4.34) we obtain

$$\begin{aligned} f_*\big(\widehat{ch}(\xi_\bullet)\alpha\big) &= g_*\big(\psi_*\big(\widehat{ch}(K(\varphi)\otimes\pi_P^*\eta)\big)i^*\alpha\big) \\ &+ ag_*\big(\psi_*(Td^{-1}(g^{H^*})\sigma^*(\tilde{e}(g^{H^*}))ch(g^\eta)i^*\omega(\alpha))\big) \\ &- af_*\big(T(h^\xi)\omega(\alpha)\big). \end{aligned} \tag{4.35}$$

Since $\tilde{\pi}_P^* \eta = \psi^*(\eta)$ we get

$$\psi_* \big(\widehat{ch}(K(\varphi) \otimes \pi_P^* \eta)\big) = \psi_* \big(\widehat{ch}(K(\varphi))\big)\widehat{ch}(\eta) \tag{4.36}$$

and by a standard formula

$$\widehat{ch}(K(\varphi)) = \hat{c}_r(H^*)\widehat{Td}(H^*)^{-1}, \tag{4.37}$$

where r is the rank of H. From Theorem 3.22 we know that $\hat{c}_r(H^*) = \hat{e}(H^*)$ is the class of $(Y \times \{\infty\}, -\sigma^*(\tilde{e}(g^{H^*}))$ in $\widehat{CH}(P)$. Since the restriction of H^* to $Y \times \{\infty\}$ coincides with N, we conclude from (4.36) and (4.37) that (4.38)

$$g_*\psi_* \big(\widehat{ch}(K(\varphi) \otimes \pi_P^* \eta)\big) = g_* \big(\widehat{Td}(N)^{-1}\widehat{ch}(\eta)\big) - a g_* \psi_* \big(Td^{-1}(g^{H^*})\sigma^*(\tilde{e}(g^{H^*}))ch(g^\eta)\big).$$

From (4.35) and (4.38) we conclude that

$$f_* \big(\widehat{ch}(\xi_\bullet)\alpha\big) = g_* \big(\widehat{Td}(N)^{-1}\widehat{ch}(\eta) i^*(\alpha)\big) - a f_* \big(T(h^\xi)\omega(\alpha)\big).$$

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