# Index Theory and the Hypoelliptic Laplacian

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*Dedicated to Jeff Cheeger for his 65th birthday*

**Abstract.** We review various aspects of index theory, in connection with the hypoelliptic Laplacian and with orbital integrals.

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**Keywords.** Hypoelliptic equations, Index theory and related fixed point theorems.

Ah ! non ! c'est un peu court,
jeune homme!

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## Contents



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## Introduction

The purpose of this paper is to review various aspects of index theory in connection with the hypoelliptic Laplacian [B05, BL08, B08a], and also with our book [B11d] on orbital integrals. We have excluded from this survey the relations of local index theory to localization formulas and functional integrals, which were reviewed in [B05, B06, B08b], and also the construction on the hypoelliptic Laplacian in de Rham theory which was explained in some detail in [B08c]. Still the present paper can be read independently of the above references. One reason is that many of the supporting facts are elementary, and can be verified by simple computations. This is in particular the case of the examples involving the real line and the circle.

Let us review a few themes which are covered in this survey.

### 0.1. Gaussians and the index theorem

Consider the Gaussian integral

$$1 = \int_{\mathbf{R}} \exp\left(-y^2/2\right) \frac{dy}{\sqrt{2\pi}}. \tag{0.1}$$

For several interrelated reasons, we will consider equation (0.1) as the prototype of an index formula:

1. A first reason is that an index problem can be set up so that (0.1) is an index formula. Indeed 1 is the Euler characteristic of $\mathbf{R}$, and (0.1) is exactly the formula which is produced via local index theory, when considering the smooth de Rham Witten complex [W82] of $\mathbf{R}$.
2. The McKean-Singer index formula [McS67] for the index of a Dirac operator $D^X_+$,
$$\operatorname{Ind} D^X_+ = \operatorname{Tr_s}\left[\exp\left(-tD^{X,2}\right)\right], \tag{0.2}$$
also has an obvious Gaussian flavour. This equation was first considered as a tool to prove a local index theorem for Dirac operators [Gi74, ABoP73]. Here we will consider the full operator $\exp\left(-tD^{X,2}\right)$ as the genuine object to which index theory applies. One reason is that it verifies an obvious compatibility to products of manifolds, in the same way as the Gaussians distributions on Euclidean vector spaces are compatible to direct sums, a consequence of the Pythagorean theorem.
3. Quillen's theory of superconnections [Q85] gives a formalism putting the McKean-Singer formula and the Chern-Weil theory for the Chern character on the same footing, by exploiting the common Gaussian outlook of the McKean-Singer formula and of the Chern-Weil formula for the Chern character. Superconnections have been used in [B86] to prove a local version of the families index theorem of Atiyah-Singer [AS71].
4. The Atiyah-Singer index formula for the index of a Dirac operator says that
$$\operatorname{Ind} D^X_+ = \int_X \widehat{A}\left(TX, \nabla^{TX}\right) \operatorname{ch}\left(E, \nabla^E\right), \tag{0.3}$$
with the right-hand side written in Chern-Weil form. Formula (0.3) can be obtained by making $t \to 0$ in (0.2), while using the local 'fantastic cancellations' in the supertrace of the heat kernel associated with $\exp\left(-tD^{X,2}\right)$, which were anticipated by McKean-Singer [McS67]. We claim that (0.3) is also a Gaussian formula. This is clear for the Chern character form in the right-hand side of (0.3). The remarkable thing is that it is also true for $\widehat{A}\left(TX, \nabla^{TX}\right)$. Indeed superconnections can be used to reexpress the right-hand side as a Quillen like Chern character form, so that (0.2), (0.3) can be written as the doubly Gaussian formula

$$\operatorname{Ind} D^X_+ = \operatorname{Tr_s}\left[\exp\left(-tD^{X,2}\right)\right] = \int_X \varphi \operatorname{Tr_s}\left[\exp\left(-A^{S^{TX}\otimes E,2}\right)\right]. \tag{0.4}$$

In (0.4), $A^{S^{TX}\otimes E}$ is the Levi-Civita superconnection along the fibres $\widehat{TX}$ of the total space of $\mathcal{X}$ of $TX$, with fibre now called $\widehat{TX}$, which is precisely the superconnection one would need if we were to establish a local version of the families index theorem for the projection $\pi : \mathcal{X} \to X$ [B86]. A computational explanation for (0.4) is that $A^{S^{TX}\otimes E,2}$ is precisely the Getzler operator [Ge86] in local index theory. Also note that equation (0.3), which expresses a global quantity in local terms, looks like a kind of Fourier transform.

### 0.2. The hypoelliptic Laplacian

Equation (0.4) raises a number of natural questions.

1. First, one can ask whether the doubly Gaussian character of equation (0.4) can be viewed as an inspiring tautology.
2. The right-hand side of (0.4) has been obtained by making $t \to 0$ in (0.2). If (0.4) is indeed tautologous, is it possible to deform $A^{S^{TX}\otimes E}$ back to $D^X$?
3. Can this possibility unify the theory of operators and the theory of characteristic forms?

The construction of the hypoelliptic Laplacian gives a positive answer to the above questions. It is a family of hypoelliptic operators acting over $\mathcal{X}$, which, up to lower-order terms, is a weighted sum of the harmonic oscillator along the fibre $\widehat{TX}$ and of the generator of the geodesic flow.

### 0.3. Bargmann isomorphism and the harmonic oscillator

The harmonic oscillator appears naturally in the context of index theory. Indeed the Getzler operator [Ge86] is a harmonic oscillator along the fibre $\widehat{TX}$. When localizing analytically index theoretic data, like in the context of Morse inequalities [W82], or in questions related to analytic torsion [BZ92], the harmonic oscillator provides a universal model of localization.

Also, if $V$ is an Euclidean vector space, the Bargmann isomorphism identifies the Hilbert completion of the symmetric algebra $S^{\cdot}(V^*)$ with $L_2^V$. The Bargmann isomorphism also identifies the algebraic de Rham complex of $V$ with the de Rham Witten complex of $V$. In real or complex geometry, it expresses various aspects of the embedding of $\{0\} \to V$. The fundamental elementary fact is that via the Bargmann isomorphism, the operator which defines the $\mathbf{Z}$-grading of $S^{\cdot}(V^*)$ is just the harmonic oscillator of $V$.

If $X$ is a Riemannian manifold, one idea in the construction of the hypoelliptic Laplacian is to consider along each fibre $\widehat{TX}$ of $\pi : \mathcal{X} \to X$ the various forms of the Bargmann isomorphism, which provides different ways of expressing analytically the embedding $X \to \mathcal{X}$. This way, the fibres $\widehat{TX}$ carry universal infinite-dimensional algebraic and analytic objects. The question is then how to link geometrically the fibre $\widehat{TX}$ and the base manifold $X$.

### 0.4. Geodesic flow and the Fourier transform

The second key ingredient in the construction of the hypoelliptic Laplacian is the geodesic flow.

1. From an algebraic point of view, the geodesic flow can be interpreted as a kind of bosonic Hodge de Rham operator, in which the exterior algebra $\Lambda^{\cdot}(T^*X)$ is replaced by the symmetric algebra $S^{\cdot}(T^*X)$.
2. Geometrically, it provides the critical link between the fibre $\widehat{TX}$ and the physical tangent space $TX$. This way, the vector bundle $\widehat{TX}$ incorporates dynamic data.
3. Analytically, it is very much related to Fourier transform.

### 0.5. Operators and characteristic classes

Originally $\widehat{TX}$ is a vector bundle on $X$. The introduction of the geodesic flow treats $\widehat{TX}$ as a special vector bundle, since it is a copy of the tangent bundle $TX$, which can be moved around along geodesics by parallel transport. When considering the heat flow associated with the hypoelliptic Laplacian, the fibrewise harmonic oscillator makes this motion more erratic, until a final point is reached where in the fibre direction, only the kernel of the harmonic oscillator becomes relevant, the hypoelliptic Laplacian collapses to the Laplacian of the base, and the motion on $X$ becomes Brownian motion. The associated analytic data, which were the characteristic forms of $\widehat{TX}$, or superconnections associated with the projection $\pi : \mathcal{X} \to X$, ultimately become operators on $X$. The distinction between characteristic forms on $X$ and differential operators is now blurred.

### 0.6. The evaluation of orbital integrals

A significant application of the hypoelliptic Laplacian is the evaluation in [B11d] of semisimple orbital integrals associated with reductive groups, these integrals being a key ingredient in Selberg's trace formula. Here, we cover in detail the case of the real line and the case of the circle, for which the relevant Selberg formula is just the Poisson formula. In both cases, the computations can be done explicitly, and their relation to Fourier transform is obvious.

### 0.7. Heat and waves

A most remarkable aspect of the hypoelliptic Laplacian is its wave-like character. Indeed, observe that like the wave equation, the geodesic flow exhibits finite propagation speed. It turns out that, after projection on $X$, the heat equation for the hypoelliptic Laplacian exhibits a version of finite propagation speed, which plays an important role in its analysis. This wave-like aspect of the projected heat flow for the hypoelliptic Laplacian is a quantized version of the Hamiltonian-Lagrangian correspondence in the classical calculus of variations.

### 0.8. The organization of the paper

This paper is organized as follows. In Section 1, we describe the algebraic de Rham complex of a vector space, and the associated Bargmann isomorphism.

In Section 2, we review the Gaussian aspects of index theory.

In Section 3, we give an elementary construction of the degree 0 part of the hypoelliptic Laplacian in de Rham theory.

In Section 4, when $X$ is a complex manifold, the construction of the hypoelliptic Dirac operator is described in detail, in particular with regard to the deformation of the underlying characteristic forms of the tangent bundle. For other aspects of the construction, we refer to [B08c, Section 4].

Finally, in Section 5, we give an introduction to the role of the hypoelliptic Laplacian in the evaluation of orbital integrals, and to the wave-like character of this operator.

## 1. The algebraic de Rham complex and the Bargmann isomorphism

The purpose of this section is to review well-known facts on the algebraic de Rham complex and the Bargmann isomorphism. Connections with index theory are also emphasized, as well as the formal role of Gaussian formulas.

This section is organized as follows. In Subsection 1.1, we construct the algebraic de Rham complex of a real vector space.

In Subsection 1.2, the Hodge theory of the algebraic de Rham complex is explained.

In Subsection 1.3, the dual algebraic complex is constructed.

In Subsection 1.4, given an Euclidean vector space, we describe the Bargmann isomorphism of the Hilbert completion of the symmetric algebra of the dual of a vector space with the corresponding $L_2$ space.

In Subsection 1.5, the Bargmann isomorphism is shown to interchange Dirac masses and Gaussian distributions.

In Subsection 1.6, the Bargmann isomorphism is applied to the algebraic de Rham complex of a vector space.

In Subsection 1.7, various formulas are given for the Euler characteristic $\chi = 1$ of the algebraic de Rham complex. Their Gaussian character is emphasized.

Finally, in Subsection 1.8, we give various applications of the Bargmann isomorphism in differential geometry.

### 1.1. The algebraic de Rham complex

Let $\mathcal{C}$ be the category of real finite-dimensional vector spaces. If $V \in \mathcal{C}$, let $S^{\cdot}(V^*), \Lambda^{\cdot}(V^*)$ be the symmetric and exterior algebras of $V^*$. These are commutative and supercommutative $\mathbf{Z}$-graded algebras. Let $N^{S^{\cdot}(V^*)}, N^{\Lambda^{\cdot}(V^*)}$ be the number operators which define their $\mathbf{Z}$-grading. Let $Y \in V$ be the tautological section of $V$. Then $S^{\cdot}(V^*)$ can be identified with the polynomial algebra $\mathbf{R}[V]$, via the map $P \to P(Y)$.

Note that if $e \in V, f \in V^*$, the derivative $\nabla_e$ and the multiplication operator $P(Y) \to \langle f, Y\rangle P(Y)$ act on $S^{\cdot}(V^*)$, and respectively decrease and increase $N^{S^{\cdot}(V^*)}$ by 1. Similarly $i_e, f\wedge$ act on $\Lambda^{\cdot}(V^*)$, and decrease and increase $N^{\Lambda^{\cdot}(V^*)}$ by 1. Finally, we have the commutation relations,

$$[\nabla_e, \langle f, Y\rangle] = \langle f, e\rangle, \qquad [i_e, f\wedge] = \langle f, e\rangle. \tag{1.1}$$

In (1.1), the first bracket is an ordinary commutator, and the second bracket is a supercommutator. The operators $\nabla_e, i_e$ are called annihilation operators, and the operators $\langle f, Y\rangle, f\wedge$ are called creation operators. Also we have the identity of operators acting on $S^{\cdot}(V^*)$,

$$\nabla_Y = N^{S^{\cdot}(V^*)}. \tag{1.2}$$

Let $\mathcal{E}$ be automorphism of $S^{\cdot}(V^*) \otimes_{\mathbf{R}} \mathbf{C}$,

$$\mathcal{E} = \exp\left(-i\frac{\pi}{2}N^{S^{\cdot}(V^*)}\right) = -i^{N^{S^{\cdot}(V^*)}}. \tag{1.3}$$

Equivalently, if $P \in S^{\cdot}(V^*) \otimes_{\mathbf{R}} \mathbf{C}$,

$$\mathcal{E}P(Y) = P(-iY). \tag{1.4}$$

Set

$$\mathcal{A}^{\cdot}(V^*) = S^{\cdot}(V^*) \otimes \Lambda^{\cdot}(V^*). \tag{1.5}$$

We define $N^{\mathcal{A}(V^*)}$ by the formula

$$N^{\mathcal{A}(V^*)} = N^{S^{\cdot}(V^*)} \otimes 1 + 1 \otimes N^{\Lambda^{\cdot}(V^*)}. \tag{1.6}$$

In the sequel, we will write (1.5) in the form

$$N^{\mathcal{A}(V^*)} = N^{S^{\cdot}(V^*)} + N^{\Lambda^{\cdot}(V^*)}. \tag{1.7}$$

The algebra $\mathcal{A}^{\cdot}(V^*)$ is $\mathbf{Z}$-graded by $N^{\Lambda^{\cdot}(V^*)}$, and $N^{\mathcal{A}(V^*)}$ defines an increasing filtration on $\mathcal{A}^{\cdot}(V^*)$, so that $\mathcal{A}^{\cdot}(V^*)$ is a $\mathbf{Z}$-graded filtered supercommutative algebra. Also the operators in (1.1) act on $\mathcal{A}^{\cdot}(V^*)$ and verify the same commutation relations.

Observe that $V \to S^{\cdot}(V^*), V \to \Lambda^{\cdot}(V^*), V \to \mathcal{A}^{\cdot}(V^*)$ define multiplicative functors, so that if $V, V' \in \mathcal{C}$,

$$\begin{aligned} S^{\cdot}\left((V \oplus V')^*\right) &= S^{\cdot}(V^*) \otimes S^{\cdot}(V'^*), \\ \Lambda^{\cdot}\left((V \oplus V')^*\right) &= \Lambda^{\cdot}(V^*) \widehat{\otimes} \Lambda^{\cdot}(V'^*), \\ \mathcal{A}^{\cdot}\left((V \oplus V')^*\right) &= \mathcal{A}^{\cdot}(V^*) \widehat{\otimes} \mathcal{A}^{\cdot}(V'^*). \end{aligned} \tag{1.8}$$

In (1.8), $\otimes$ is the ordinary tensor product, and $\widehat{\otimes}$ is the $\mathbf{Z}$-graded tensor product of filtered algebras. In particular,

$$N^{\mathcal{A}^{\cdot}\left((V \oplus V')^*\right)} = N^{\mathcal{A}^{\cdot}(V^*)} \otimes 1 + 1 \otimes N^{\mathcal{A}^{\cdot}(V'^*)}. \tag{1.9}$$

As in (1.7), we rewrite (1.9) in the simpler form

$$N^{\mathcal{A}^{\cdot}((V\oplus V')^{*})} = N^{\mathcal{A}^{\cdot}(V^{*})} + N^{\mathcal{A}^{\cdot}(V'^{*})}. \tag{1.10}$$

Let $d$ be the de Rham operator acting on $\mathcal{A}^{\cdot}(V^{*})$. If $e_1, \dots, e_n$ is a basis of $V$, and if $e^1, \dots, e^n$ is the dual basis of $V^*$, then

$$d = \sum_{i=1}^{n} e^i \nabla_{e_i}. \tag{1.11}$$

Then $(\mathcal{A}^{\cdot}(V^{*}), d)$ is the algebraic de Rham complex of $V$. The de Rham operator $d$ increases the degree in $\mathcal{A}^{\cdot}(V^{*})$ by 1, and it commutes with $N^{\mathcal{A}(V^{*})}$, so that $(\mathcal{A}^{\cdot}(V^{*}), d)$ is also a filtered complex. We identify $Y$ with the tautological radial vector field. Let $i_Y$ denote the contraction by $Y$. Then $i_Y$ decreases the degree by 1, and it commutes with $N^{\mathcal{A}^{\cdot}(V^{*})}$. Let $L_Y$ be the corresponding Lie derivative operator. Cartan's formula asserts that

$$L_Y = [d, i_Y], \tag{1.12}$$

where [ ] still denotes a supercommutator. We have the trivial identity

$$L_Y = N^{\mathcal{A}^{\cdot}(V^{*})}. \tag{1.13}$$

The fact that $d, i_Y$ commute with $N^{\mathcal{A}^{\cdot}(V^{*})}$ can be viewed as a consequence of (1.13).

For $k \in \mathbf{N}$, let $\mathcal{A}^{\cdot}_k(V^{*})$ be the subcomplex of elements of $\mathcal{A}^{\cdot}(V^{*})$ which are of degree $k$ with respect to $N^{\mathcal{A}^{\cdot}(V^{*})}$. Then $(\mathcal{A}^{\cdot}(V^{*}), d)$ splits as a sum of finite-dimensional complexes

$$(\mathcal{A}^{\cdot}(V^{*}), d) = \bigoplus_{k\in\mathbf{N}} (\mathcal{A}^{\cdot}_k(V^{*}), d), \tag{1.14}$$

and $i_Y$ also acts on the splitting. By (1.12), (1.13), except for $k = 0$, the complexes $(\mathcal{A}^{\cdot}_k(V^{*}), d)$ are exact. This shows that the cohomology of $(\mathcal{A}^{\cdot}(V^{*}), d)$ is concentrated in degree 0 and is one-dimensional. In the above, we have used implicitly the homotopy on $(\mathcal{A}^{\cdot}_{>0}(V^{*}), d)$,

$$1 = \left[N^{\mathcal{A}^{\cdot}(V^{*})}\right]^{-1} [d, i_Y], \tag{1.15}$$

which is the algebraic version of Poincaré's lemma.

The argument is exactly the same when proving Poincaré's lemma on the smooth de Rham complex $(\Omega^{\cdot}(V), d)$. Indeed, let $\varphi_t(Y) = e^t Y$ be the group of diffeomorphisms associated with the radial vector field. If $\omega \in \Omega^{\cdot}(V)$, then

$$\frac{\partial}{\partial t} \varphi_t^* \omega = \varphi_t^* L_Y \omega, \tag{1.16}$$

so that

$$\omega - \omega^{(0)}(0) = \int_{-\infty}^{0} \varphi_t^* L_Y \omega dt. \tag{1.17}$$

One verifies easily that (1.17) is just the formal analogue of (1.15) in smooth de Rham theory. When $\omega$ is a polynomial form, these two equations are equivalent.

### 1.2. Algebraic Hodge theory

Let $h^V$ be a nondegenerate symmetric bilinear form on $V$. Then $h^V$ induces nondegenerate symmetric bilinear forms on $S^{\cdot}(V^*)$ and $\Lambda^{\cdot}(V^*)$. Let $h^{\mathcal{A}^{\cdot}(V^*)}$ be the corresponding symmetric bilinear form on $\mathcal{A}^{\cdot}(V^*)$.

Let $d^*$ denote the adjoint of $d$ with respect to $h^{\mathcal{A}^{\cdot}(V^*)}$.

**Proposition 1.1.** *The following identity holds:*

$$d^* = i_Y. \tag{1.18}$$

*In particular $d^*$ does not depend on $h^V$. Moreover,*

$$[d, d^*] = N^{\mathcal{A}^{\cdot}(V^*)}. \tag{1.19}$$

*Proof.* Equation (1.18) follows from an obvious computation. By (1.13), we get (1.19). □

That $d^*$ does not depend on $h^V$ is strange. This is because $h^V$ is used in both factors $S^{\cdot}(V^*), \Lambda^{\cdot}(V^*)$. If $h^V$ is a scalar product, then $[d, d^*]$ is a Hodge Laplacian. It follows from the above that $L_Y$, a differential operator of order 1, is a Hodge Laplacian. The fact that $N^{\mathcal{A}^{\cdot}(V^*)}$ is nonnegative appears as a consequence of Hodge theory. Equation (1.15) is then a version of the standard assertion in Hodge theory that on the direct sum of eigenspaces associated with positive eigenvalues of the Hodge Laplacian, the corresponding complex is exact.

The above tends to indicate that the classical proof of Poincaré's lemma is nothing else than a standard application of Hodge theory.

### 1.3. Duality

The algebraic dual of $S^{\cdot}(V^*)$ is given by $S^{\cdot}_{\infty}(V) = \prod_{p=0}^{+\infty} S^p(V)$. Then $S^{\cdot}(V)$ embeds in the algebraic dual to $S^{\cdot}_{\infty}(V)$, $\Lambda^{\cdot}(V)$ is the algebraic dual to $\Lambda^{\cdot}(V^*)$. Also $S^{\cdot}(V)$ can be identified with the algebra of polynomials $\mathbf{R}[V^*]$ on $V^*$. Finally, $\mathcal{A}^{\cdot}(V)$ embeds in the algebraic dual $\mathcal{A}^{\cdot}_{\infty}(V)$ to $\mathcal{A}(V^*)$.

Let $Z$ be the tautological section of $V^*$. The complex $(\mathcal{A}^{\cdot}(V), i_Z)$ is dual to the algebraic de Rham complex $(\mathcal{A}(V^*), d)$, and the algebraic de Rham complex $(\mathcal{A}^{\cdot}(V), d)$ is dual to $(\mathcal{A}^{\cdot}(V^*), i_Y)$. The complex $(\mathcal{A}^{\cdot}(V), i_Z)$ is $\mathbf{Z}$-graded by $n - N^{\Lambda^{\cdot}(V)}$, and is filtered by

$$N^{\mathcal{A}^{\cdot}(V)} = N^{S^{\cdot}(V)} + N^{\Lambda^{\cdot}(V)}. \tag{1.20}$$

Clearly,

$$\Lambda^{\cdot}(V) = \Lambda^{n-\cdot}(V^*) \otimes \Lambda^n(V). \tag{1.21}$$

Set

$$\mathcal{B}^{\cdot}(V) = S^{\cdot}(V) \otimes \Lambda^{\cdot}(V^*) \otimes \Lambda^n(V). \tag{1.22}$$

Then $(\mathcal{B}^{\cdot}(V), Z\wedge)$ is a complex which is $\mathbf{Z}$-graded by $N^{\Lambda^{\cdot}(V^*)}$, and filtered by

$$N^{\mathcal{B}^{\cdot}(V)} = N^{S^{\cdot}(V)} + n - N^{\Lambda^{\cdot}(V^*)}. \tag{1.23}$$

Set

$$d_* = -\sum_{i=1}^{n} i_{e_i} \nabla_{e^i}. \tag{1.24}$$

From the above, we have the identification of **Z**-graded filtered complexes

$$\begin{aligned} \left(\mathcal{B}^{\cdot}(V), Z\wedge\right) &= \left(\mathcal{A}^{n-\cdot}(V), i_Z\right), \\ \left(\mathcal{B}(V), -d_*\right) &= \left(\mathcal{A}^{\cdot}(V), d\right). \end{aligned} \tag{1.25}$$

The Lie derivative operator $L_Z$ acts on $\mathcal{A}^{\cdot}(V)$, and moreover,

$$L_Z = [d, i_Z] = N^{\mathcal{A}^{\cdot}(V)}. \tag{1.26}$$

The action of $L_Z$ on $\mathcal{B}(V)$ is given by

$$L_Z = [Z\wedge, -d_*] = \nabla_Z + n - N^{\Lambda^{\cdot}(V^*)} = N^{\mathcal{B}^{\cdot}(V)}. \tag{1.27}$$

By proceeding as before, we find that the cohomology of $(\mathcal{B}(V), Z\wedge)$ is concentrated in degree $n$ and is one-dimensional, and $1 \in \Lambda^n(V^*) \otimes \Lambda^n(V)$ is a canonical section of the cohomology.

### 1.4. The Bargmann isomorphism

Let $\mathcal{D}$ be the category of real Euclidean vector spaces.

Let $\left(V, g^V\right) \in \mathcal{D}$. Set $n = \dim V$. Let $dY$ be the corresponding Lebesgue measure. Let $m^V$ be the Gaussian measure on $V$,

$$dm^V(Y) = \exp\left(-|Y|^2/2\right) \frac{dY}{(2\pi)^{n/2}}. \tag{1.28}$$

Then

$$\int_V dm^V(Y) = 1. \tag{1.29}$$

In the sequel we will often omit $g^V$, so that $V$ comes with its Euclidean product $g^V$. Then $V \to m^V$ is multiplicative, in the sense that if $V, V' \in \mathcal{D}$,

$$m^{V\oplus V'} = m^V \otimes m^{V'}. \tag{1.30}$$

Conversely, if $V \to n^V$ is a map into the set of $\mathrm{O}(V)$-invariant nonnegative measures on $V$ such that (1.29), (1.30) hold, then $n^V$ can be deduced from $m^V$ by the action of a universal dilation $\delta_s : s \geq 0$. In particular the Dirac mass $\delta_0^V$ is also such a measure.

Let $L_2^V\left(m^V\right)$ be the real $L_2$-space associated with the measure $m^V$. Again,

$$L_2^{V\oplus V'}\left(m^{V\oplus V'}\right) = L_2^V\left(m^V\right) \otimes L_2^{V'}\left(m^{V'}\right). \tag{1.31}$$

Let $\Delta^V$ be the Laplacian on $V$ which is associated with $g^V$. Here $\Delta^V$ is a sum of second derivatives. Note that even though the operator $\exp\left(-\Delta^V/2\right)$ is not well defined, it acts on $S^{\cdot}(V^*)$ by expanding $\exp\left(-\Delta^V/2\right)$ as a power series. Using the operator $\mathcal{E}$ in (1.3), (1.4), we get

$$\exp\left(-\Delta^V/2\right) = \mathcal{E} \exp\left(\Delta^V/2\right) \mathcal{E}^{-1}. \tag{1.32}$$

Equivalently, if $P \in S^{\cdot}(V^*)$,

$$\exp\left(-\Delta^V/2\right) P(Y_0) = (2\pi)^{-n/2} \int_V P(Y_0 - iY) \exp\left(-|Y|^2/2\right) dY$$
$$= (2\pi)^{-n/2} \int_V \exp\left(\frac{1}{2}|Y_0|^2 - \frac{1}{2}|Y|^2 + i\langle Y_0, Y\rangle\right) P(-iY)\, dY. \quad (1.33)$$

Let $F^{\cdot}S^{\cdot}(V^*)$ be the increasing filtration on $S^{\cdot}(V^*)$ which is associated with $N^{S^{\cdot}(V^*)}$, and let $\mathrm{Gr}^{\cdot}(V^*) = F^{\cdot}S^{\cdot}(V^*)/F^{\cdot-1}S^{\cdot}(V^*)$ be the corresponding $\mathrm{Gr}^{\cdot}$. We have the canonical isomorphism of algebras,

$$\mathrm{Gr}^{\cdot}S^{\cdot}(V^*) \simeq S^{\cdot}(V^*). \quad (1.34)$$

Then $\exp\left(-\Delta^V/2\right) : S^{\cdot}(V^*) \to S^{\cdot}(V^*)$ preserves the filtration of $S^{\cdot}(V^*)$, and induces the identity map on $\mathrm{Gr}^{\cdot}S^{\cdot}(V^*)$. Let $g^{S^{\cdot}(V^*)}$ be the scalar product on $S^{\cdot}(V^*)$ which is induced by $g^V$. Then one verifies easily that if $P \in S^{\cdot}(V^*)$,

$$\left\|\exp\left(-\Delta^V/2\right) P\right\|_{L_2^V(m^V)} = \|P\|_{g^{S^{\cdot}(V^*)}}. \quad (1.35)$$

The map $\exp\left(-\Delta^V/2\right)$ does not preserve the product on $S^{\cdot}(V^*)$. The pull-back of the product on $S^{\cdot}(V^*)$ by the map $\exp\left(-\Delta^V/2\right)$ can be computed using the Wick rules.

Let $\overline{S^{\cdot}(V^*)}$ be the Hilbert completion of $S^{\cdot}(V^*)$ with respect to $g^{S^{\cdot}(V^*)}$. A basic result is that the map $\exp\left(-\Delta^V/2\right)$ extends to an isometry

$$\overline{S^{\cdot}(V^*)} \simeq L_2^V\left(m^V\right). \quad (1.36)$$

If $V = \mathbf{R}$ is the canonical Euclidean vector space of dimension 1, for $m \in \mathbf{N}$,

$$P_m(x) = \exp\left(-\frac{1}{2}\frac{\partial^2}{\partial x^2}\right) x^m \quad (1.37)$$

is called the Hermite polynomial of order $m$. Also

$$\|P_m\|^2_{L_2^V(m^V)} = m!, \quad (1.38)$$

and the $P_m, m \in \mathbf{N}$ form an orthogonal basis of $L_2^V\left(m^V\right)$. Of course, the above extends to the case of arbitrary dimension.

If $f \in V$, let $f^* \in V^*$ correspond to $f$ by $g^V$. Then

$$\begin{aligned}
&\exp\left(-\Delta^V/2\right) \nabla_e \exp\left(\Delta^V/2\right) = \nabla_e,\\
&\exp\left(-\Delta^V/2\right) \langle f^*, Y\rangle \exp\left(\Delta^V/2\right) = -\nabla_f + \langle f^*, Y\rangle,\\
&\exp\left(-\Delta^V/2\right) \nabla_Y \exp\left(\Delta^V/2\right) = -\Delta^V + \nabla_Y.
\end{aligned} \quad (1.39)$$

Clearly $\nabla_e$ maps $F^{\cdot}S^{\cdot}(V^*)$ into $F^{\cdot-1}S^{\cdot}(V^*)$, $-\nabla_f + \langle f^*, Y\rangle$ maps $F^{\cdot}S^{\cdot}(V^*)$ into $F^{\cdot+1}S^{\cdot}(V^*)$, and $-\Delta^V + \nabla_Y$ preserves the filtration $F^{\cdot}S(V^*)$.

Let $L_2^V$ be the ordinary $L_2$ space of $V$ with respect to $dY$. Let $T$ denote the isometry from $L_2^V\left(m^V\right) \to L_2^V$,

$$Tf(Y) = \frac{1}{\pi^{n/4}} \exp\left(-|Y|^2/2\right) f\left(\sqrt{2}Y\right). \quad (1.40)$$

**Definition 1.2.** Let $B : \overline{S^{\cdot}(V^*)} \to L_2^V$ be given by

$$B = T\exp\left(-\Delta^V/2\right). \tag{1.41}$$

The isometry $B$ is called the Bargmann isomorphism. An easy computation shows that if $P \in S^{\cdot}(V^*)$, $f \in L_2^V$,

$$BP(Y_0) = \int_V \left(2^{-2}\pi^{-3}\right)^{n/4} \exp\left(\frac{1}{2}|Y_0|^2 - \frac{1}{2}|Y|^2 + i\sqrt{2}\langle Y_0, Y\rangle\right) P(-iY)\, dY, \tag{1.42}$$

$$B^{-1}f(Y_0) = \int_V \pi^{-n/4} \exp\left(-\frac{1}{2}|Y_0|^2 - \frac{1}{2}|Y|^2 + \sqrt{2}\langle Y_0, Y\rangle\right) f(Y)\, dY.$$

As in (1.39), we get

$$\begin{aligned} B\nabla_e B^{-1} &= \frac{1}{\sqrt{2}}\left(\nabla_e + \langle e^*, Y\rangle\right), \\ B\langle f^*, Y\rangle B^{-1} &= \frac{1}{\sqrt{2}}\left(-\nabla_f + \langle f^*, Y\rangle\right), \\ B\nabla_Y B^{-1} &= \frac{1}{2}\left(-\Delta^V + |Y|^2 - n\right). \end{aligned} \tag{1.43}$$

In the sequel, we will use the notation

$$\mathcal{H}^V = \frac{1}{2}\left(-\Delta^V + |Y|^2 - n\right), \tag{1.44}$$

so that the last identity in (1.43) takes the form

$$B\nabla_Y B^{-1} = \mathcal{H}^V. \tag{1.45}$$

The operator $\mathcal{H}^V$ in the right-hand side of (1.43) is called the harmonic oscillator. By the above, it is self-adjoint, and its spectrum is $\mathbf{N}$.

Using $\mathcal{H}^V$, one can define corresponding abstract Sobolev spaces $H^s$. It is well known that $H^\infty$ coincides with the Schwartz space $\mathcal{S}(V)$. Of course, $B^{-1}H^\infty$ is a vector space of formal power series with rapidly decreasing coefficients. When $V = \mathbf{R}$, an orthonormal basis of $S^{\cdot}(V^*)$ is given by the $x^m/\sqrt{m!}$, and the coefficients of the formal power series are evaluated with respect to this basis.

By the above, tempered distributions on $V$ can be expressed as formal power series on $V$. Equivalently, tempered distributions can be written as linear combinations of Hermite polynomials, and from the coefficients of the expansion, we obtain a corresponding power series in a suitable completion of $S^{\cdot}(V^*)$.

Let $\mathcal{F}$ be the Fourier transform acting on $L_2^V$,

$$\mathcal{F}f(Y) = (2\pi)^{-n/2}\int_V \exp\left(-i\langle Y, Y'\rangle\right) f(Y')\, dY'. \tag{1.46}$$

By (1.42), one gets easily

$$\mathcal{F} = B\mathcal{E}B^{-1}. \tag{1.47}$$

By (1.2), (1.3), (1.45), and (1.47), we obtain

$$\mathcal{F} = \exp\left(-i\frac{\pi}{2}\mathcal{H}^V\right) = (-i)^{\mathcal{H}^V}. \tag{1.48}$$

Equation (1.48) explains the invariance of $\mathcal{H}^V$ under Fourier transform. It can also be obtained via Mehler's formula.

### 1.5. Gaussians and Dirac masses

Take $P \in S^{\cdot}(V^*)$. If $Y_0 \in V$, one verifies easily that

$$\begin{aligned} P(Y_0) &= \langle \exp(\langle Y_0, Y\rangle), P\rangle_{S^{\cdot}(V^*)} \\ &= \left\langle \exp\left(\langle Y_0, Y\rangle - \frac{1}{2}|Y_0|^2\right), \exp\left(-\Delta^V/2\right)P\right\rangle_{L_2^V(m^V)} \\ &= \left\langle \pi^{-n/4}\exp\left(-\frac{1}{2}|Y_0|^2 - \frac{1}{2}|Y|^2 + \sqrt{2}\langle Y_0, Y\rangle\right), BP\right\rangle_{L_2^V}. \end{aligned} \tag{1.49}$$

Similarly, if $P \in S^{\cdot}(V^*)$, then $P \in L_2^V\left(m^V\right)$, and moreover,

$$\begin{aligned} P(Y_0) &= (2\pi)^{n/2}\exp\left(|Y_0|^2/2\right)\langle \delta_{Y_0}, P\rangle_{L_2^V(m^V)} \\ &= \left\langle \exp\left(\langle Y, Y_0\rangle - \frac{1}{2}|Y|^2\right), \exp\left(\Delta^V/2\right)P\right\rangle_{\overline{S^{\cdot}(V^*)}} \\ &= \left\langle \pi^{-n/4}\exp\left(-\frac{1}{2}|Y_0|^2 - \frac{1}{2}|Y|^2 + \sqrt{2}\langle Y_0, Y\rangle\right), B^{-1}P\right\rangle_{\overline{S^{\cdot}(V^*)}} \end{aligned} \tag{1.50}$$

The last identity in (1.42) is equivalent to the last identity in (1.49). By the above, the Bargmann isomorphism exchanges Dirac masses and Gaussians. It is similar to a Fourier transform, in the sense that localization in one variable is necessarily delocalized in the other variable.

Finally, observe that

$$B1 = \pi^{-n/4}\exp\left(-|Y|^2/2\right), \qquad B^{-1}1 = 2^{n/2}\pi^{n/4}\exp\left(\frac{1}{2}|Y|^2\right). \tag{1.51}$$

### 1.6. Algebraic de Rham complex and Bargmann isomorphism

We use the same notation as in Subsection 1.4. The map $\exp\left(-\Delta^V/2\right) : S^{\cdot}(V^*) \to S^{\cdot}(V^*)$ extends to an automorphism of the $\mathbf{Z}$-graded filtered complex $(\mathcal{A}^{\cdot}(V^*), d)$. This automorphism induces the identity on $\mathrm{Gr}^{\cdot} S^{\cdot}(V^*)$.

Let $e_1, \ldots, e_n$ be an orthonormal basis of $V$. Set

$$d^* = -\sum_{i=1}^n i_{e_i}\nabla_{e_i}. \tag{1.52}$$

By (1.39),

$$\begin{aligned}\exp\left(-\Delta^V/2\right) d \exp\left(\Delta^V/2\right) &= d,\\ \exp\left(-\Delta^V/2\right) i_Y \exp\left(\Delta^V/2\right) &= d^* + i_Y,\\ \exp\left(-\Delta^V/2\right) L_Y \exp\left(\Delta^V/2\right) &= -\Delta^V + L_Y.\end{aligned} \tag{1.53}$$

Also by (1.43),

$$\begin{aligned}BdB^{-1} &= \frac{1}{\sqrt{2}}\left(d + Y^*\wedge\right),\\ Bi_Y B^{-1} &= \frac{1}{\sqrt{2}}\left(d^* + i_Y\right),\\ BL_Y B^{-1} &= \mathcal{H}^V + N^{\Lambda^{\cdot}(V^*)}.\end{aligned} \tag{1.54}$$

The fact that up to a constant, in the right-hand side of (1.54), we recover the Witten twist [W82] of $d, d^*$ associated with the function $|Y|^2/2$ is not a surprise in view of (1.40), (1.41).

The strangest aspect of (1.53) is that with the proper proviso, the algebraic de Rham complex $(\mathcal{A}^{\cdot}(V^*), d)$ is isomorphic to a version of the smooth de Rham complex $(\Omega^{\cdot}(V), d)$, but this isomorphism does not preserve the product.

The kernel of $L_Y$ in $\overline{S^{\cdot}(V^*)}$ is spanned by 1, and the kernel of $\mathcal{H}^V + N^{\Lambda^{\cdot}(V^*)}$ in $L_2^V \otimes \Lambda^{\cdot}(V^*)$ is spanned by $\pi^{-n/4}\exp\left(-|Y|^2/2\right)$. This is compatible with the first identity in (1.51).

Now we briefly consider the Bargmann transform of the algebraic dual complex $(\mathcal{B}^{\cdot}(V), Z\wedge)$. First of all, we identify $V$ and $V^*$ by the scalar product, so that

$$\mathcal{B}^{\cdot}(V) \simeq S^{\cdot}(V^*) \otimes \Lambda^{\cdot}(V^*) \otimes \Lambda^n(V). \tag{1.55}$$

By (1.55), we get

$$\mathcal{B}^{\cdot}(V) \simeq \mathcal{A}^{\cdot}(V^*) \otimes \Lambda^n(V). \tag{1.56}$$

Also the scalar product trivializes $\Lambda^n(V)$ up to a $\mathbf{Z}_2$ ambiguity. The canonical sections $Y$ and $Z$ are also identified. By (1.56), we have the identification of complexes,

$$(\mathcal{B}^{\cdot}(V), Z\wedge) \simeq (\mathcal{A}^{\cdot}(V^*) \otimes \Lambda^n(V), Y\wedge). \tag{1.57}$$

In what follows, we use the notation $L_Z$ instead of $L_Y$ to point out that here $L_Z$ is given by (1.27). As in (1.53), we get

$$\begin{aligned}\exp\left(-\Delta^V/2\right) Y^*\wedge \exp\left(\Delta^V/2\right) &= -d + Y^*\wedge,\\ \exp\left(-\Delta^V/2\right)(-d_*) \exp\left(\Delta^V/2\right) &= -d^*,\\ \exp\left(-\Delta^V/2\right) L_Z \exp\left(\Delta^V/2\right) &= -\Delta^V + \nabla_Y + n - N^{\Lambda^{\cdot}(V^*)}.\end{aligned} \tag{1.58}$$

Moreover,

$$BY^* \wedge B^{-1} = \frac{1}{\sqrt{2}} \left( -d + Y^* \wedge \right), \qquad B \left( -d_* \right) B^{-1} = -\frac{1}{\sqrt{2}} \left( d^* - i_Y \right),$$
$$BL_Z B^{-1} = \mathcal{H}^V + n - N^{\Lambda^{\cdot}(V^*)}. \tag{1.59}$$

The operators in the right-hand side of (1.59) come just from the Witten twist associated with the function $-|Y|^2/2$. Incidentally, because the $L_2$ norm involves the choice of the volume form associated with the metric of $V$, it is better to view the Bargmann transform of $\overline{\mathcal{B}^{\cdot}(V)}$ as being the vector space of $L_2$ sections of $\Lambda^{\cdot}(V^*) \otimes o(V)$, where $o(V)$ is the orientation line of $V$.

As we saw in Subsection 1.3, the kernel of $L_Z$ in $\mathcal{B}^{\cdot}(V)$ is concentrated in degree $n$ and spanned by $1 \in \Lambda^n(V^*) \otimes \Lambda^n(V)$.

Similarly the kernel of $\mathcal{H}^V + n - N^{\Lambda^{\cdot}(V^*)}$ is concentrated in degree $n$ and spanned by the density $\pi^{-n/4} \exp\left(-|Y|^2/2\right) dY$. This fits with the first identity in (1.51).

### 1.7. Euler characteristic and the Pythagorean theorem

We saw in Subsection 1.1 that the cohomology of the complex $(\mathcal{A}^{\cdot}(V^*), d)$ is concentrated in degree 0 and is one-dimensional. In particular its Euler characteristic $\chi$ is equal to 1.

For $t > 0$, the operator $\exp\left(-tN^{S^{\cdot}(V^*)}\right)$ is self-adjoint and trace class, and moreover,

$$\mathrm{Tr}\left[\exp\left(-tN^{S^{\cdot}(V^*)}\right)\right] = \left(1 - e^{-t}\right)^{-n}. \tag{1.60}$$

The supertrace of $\exp\left(-tN^{\Lambda^{\cdot}(V^*)}\right)$ is given by

$$\mathrm{Tr_s}\left[\exp\left(-tN^{\Lambda^{\cdot}(V^*)}\right)\right] = \left(1 - e^{-t}\right)^{n}. \tag{1.61}$$

By (1.60), (1.61), we conclude that the operator $\exp\left(-tN^{\mathcal{A}^{\cdot}(V^*)}\right)$ is trace class, and that

$$1 = \mathrm{Tr_s}\left[\exp\left(-tN^{\mathcal{A}^{\cdot}(V^*)}\right)\right]. \tag{1.62}$$

Note that because of (1.10), equation (1.62) is natural with respect to direct sums. More precisely, the operator $\exp\left(-tN^{\mathcal{A}^{\cdot}(V^*)}\right)$ behaves multiplicatively with respect to direct sums, which is also a reflection of the fact that the exponential maps sums into products.

We have the splitting of $(\mathcal{A}^{\cdot}(V^*), d)$ in (1.14). By (1.15), for $k > 0$, the complex $(\mathcal{A}^{\cdot}_k(V^*), d)$ is exact. We get

$$\begin{aligned} \mathrm{Tr_s}^{\mathcal{A}^{\cdot}_k(V^*)}\left[\exp\left(-tN^{\mathcal{A}^{\cdot}(V^*)}\right)\right] &= 1 \text{ for } k = 0, \\ &= 0 \text{ for } k > 0. \end{aligned} \tag{1.63}$$

Of course (1.63) explains (1.62). The above suggests that equation (1.62) can be rewritten in the form

$$1 = \left[1 + \left(e^{-t} - e^{-t}\right) + \left(e^{-2t} - e^{-2t}\right) + \cdots\right]^n. \tag{1.64}$$

Set

$$D^{V} = d + i_{Y}. \tag{1.65}$$

By (1.12), (1.13), we get

$$N^{\mathcal{A}^{\cdot}(V^{*})} = L_{Y} = D^{V,2}. \tag{1.66}$$

Then (1.62) can be rewritten in the form

$$\chi = \operatorname{Tr_s}\left[\exp\left(-tD^{V,2}\right)\right]. \tag{1.67}$$

Using the Bargmann isomorphism and (1.54), we can rewrite (1.67) in the form

$$\chi = \operatorname{Tr_s}\left[\exp\left(-t\left(\mathcal{H}^{V} + N^{\Lambda^{\cdot}(V^{*})}\right)\right)\right]. \tag{1.68}$$

Let us now make $t \to 0$ in equation (1.67). By using classical local index theory on the right-hand side of (1.67), that is by working on smooth differential forms on $V$, we find that in this case the index formula takes the form

$$\chi = \operatorname{Tr_s}\left[\exp\left(-tD^{V,2}\right)\right] = \int_{V} \exp\left(-\left|Y\right|^{2}/2\right)\frac{dY}{(2\pi)^{n/2}}. \tag{1.69}$$

It is remarkable that the last two terms in (1.69) have a similar Gaussian character.

### 1.8. Symmetric algebras, the geodesic flow and Brownian motion

Let $X$ be a smooth manifold of dimension $n$. The exterior algebra $\Lambda^{\cdot}(T^{*}X)$ and the de Rham complex $\left(\Omega^{\cdot}(X), d^{X}\right)$ have been universally used in geometry. One cannot say the same for the symmetric algebra $S^{\cdot}(T^{*}X)$. Symmetric algebras are important in $K$-theory and representation theory, but their specific geometric role is difficult to point out.

If $TX$ is equipped with a Riemannian metric, classical Hodge theory on the de Rham complex $\left(\Omega^{\cdot}(X), d^{X}\right)$ is inescapable. However, the geometric role of the induced scalar product on $S^{\cdot}(T^{*}X)$, of its completion $\overline{S^{\cdot}(T^{*}X)}$, and of the Bargmann isomorphism with the Hilbert bundle $L_{2}(TX)$ are not so much used.

Still, one can say that in geometric analysis, the harmonic oscillator appears recurrently in a number of related questions:

- In problems involving localization. For instance, in Witten's proof of Morse inequalities [W82], the local model of the deformed Witten complex near a critical point is precisely the one described in the right-hand side of (1.54). In complex geometry, the Koszul complex and its Hodge theory [B90, BL91] produce another version of a harmonic oscillator.
- The computation by Getzler [Ge86] of the local index theorem also involves a harmonic oscillator which is responsible for the appearance of the class $\widehat{A}(TX)$ in the Atiyah-Singer index theorem for Dirac operators.
- The evaluation of a number of exotic genera appears in explicit computations involving the harmonic oscillator [B90, B94, BG00, BG01, BG04], still in questions connected with localization.

The fact that the harmonic oscillator appears repeatedly in various unrelated questions has prompted us to try putting it at centre stage in our construction of the hypoelliptic Laplacian, which will be reviewed in Sections 3, 4, and 5.

Let us briefly mention two easy constructions involving $S^{\cdot}(T^*X)$ when $TX$ is equipped with a Riemannian metric. First we proceed by analogy with the case of the de Rham complex. Let $d^{X*}$ be the formal adjoint of $d^X$ with respect to the obvious $L_2$ scalar product on $\Omega^{\cdot}(X)$. Let $\nabla^{\Lambda^{\cdot}(T^*X)}$ be the connection on $\Lambda^{\cdot}(T^*X)$ which is induced by the Levi-Civita connection $\nabla^{TX}$ on $TX$. If $e_1, \dots, e_n$ is an orthonormal basis of $TX$, and if $e^1, \dots, e^n$ is the corresponding dual basis, then

$$d^X - d^{X*} = \sum_{i=1}^{n} \left(e^i + i_{e_i}\right) \nabla_{e_i}^{\Lambda^{\cdot}(T^*X)}. \tag{1.70}$$

We will now obtain the $S^{\cdot}(T^*X)$ analogue of (1.70). Let $\nabla^{S^{\cdot}(T^*X)}$ be the connection on $S^{\cdot}(T^*X)$ which is induced by $\nabla^{TX}$. If $e \in TX$, let $\nabla_e^V$ denotes differentiation along the fibre $TX$. Using the results of Subsection 1.1, we find that the analogue of $d^X - d^{X*}$ is the operator $E$ acting on $C^\infty(X, S^{\cdot}(T^*X))$ which is given by

$$E = \sum_{i=1}^{n} \left(\left\langle e^i, Y\right\rangle + \nabla_{e_i}^V\right) \nabla_{e_i}^{S^{\cdot}(T^*X)}. \tag{1.71}$$

Let $\pi : \mathcal{X} \to X$ be the total space of $TX$, with fibre $\widehat{TX}$, another copy of the tangent bundle $TX$. The notation $\widehat{TX}$ is used instead of $TX$, because we want to emphasize that $\widehat{TX}$ is distinct from the physical tangent bundle $TX$. Then $C^\infty\left(X, S^{\cdot}\left(\widehat{T^*X}\right)\right)$ is the vector space of smooth functions on $\mathcal{X}$ which are polynomial along the fibre. Via the Bargmann isomorphism, a proper completion of this vector space can be identified with the Schwartz space $\mathcal{S}(\mathcal{X})$ of smooth rapidly decreasing functions on $\mathcal{X}$ together with their derivatives of arbitrary order.

Let $\widehat{Y}$ be the tautological section of $p^*\widehat{TX}$ on $\mathcal{X}$, and let $Y$ be the corresponding section of $p^*TX$. Let $B$ be the fibrewise Bargmann transform. Set

$$F = BEB^{-1}. \tag{1.72}$$

Then $F$ acts on the Schwartz space $\mathcal{S}(\mathcal{X})$. In the sequel, if $e \in TX$, we identify $e$ with the corresponding horizontal vector field on $\mathcal{X}$. In particular $\nabla_e^X$ denotes horizontal differentiation in the direction $e$. From (1.43), (1.71), and (1.72), we get

$$F = \sqrt{2} \sum_{i=1}^{n} \left\langle e^i, Y\right\rangle \nabla_{e_i}^X. \tag{1.73}$$

By following the previous conventions, $\nabla_Y^X$, which differentiates horizontally, is the geodesic vector field on $\mathcal{X}$. We can rewrite (1.73) in the form

$$F = \sqrt{2} \nabla_Y^X, \tag{1.74}$$

i.e., $F/\sqrt{2}$ is the geodesic vector field $\nabla_Y^X$ on $\mathcal{X}$. The geodesic vector field appears then to be a kind of bosonic Dirac operator, similar to the fermionic operator

$d^X - d^{X*}$. This should not be too much of a surprise. After all, the number of canonical objects one can produce by simple algebra is limited.

Observe that the principal symbol $\sigma\left(\nabla^X_Y\right)$ of $\nabla^X_Y$ is given by

$$\sigma\left(\nabla^X_Y\right) = i\left\langle \xi, Y\right\rangle, \tag{1.75}$$

with $\xi$ the canonical section of $T^*X$. Equation (1.75) has an obvious Fourier transform quality. Moreover, when identifying $TX$ and $T^*X$ by the metric, both $Y$ and $\xi$ represent two distinct forms of differentiation on $X$. One can then view $Y$ and $i\xi$ as lying in the real and imaginary parts of the complexified tangent bundle $T_{\mathbf{C}}X$.

Another construction is related to the embedding of $X$ as the zero section of $\mathcal{X}$. Note that $\left(\mathcal{A}^{\cdot}\left(\widehat{T^*X}\right), d\right)$ is the algebraic de Rham complex along the fibre, and $\left(\mathcal{B}\left(\widehat{TX}\right), Z\wedge\right)$ is the dual complex.

The cohomology of $\left(\mathcal{A}^{\cdot}\left(\widehat{T^*X}\right), d\right)$ is concentrated in degree 0 and represented by $1 \in S^0\left(T^*X\right)$. Via the Bargmann isomorphism, the function 1 is replaced by the fibrewise Gaussian function $\pi^{-n/4}\exp\left(-\left|Y\right|^2/2\right)$, and the compactly supported cohomology of the fibre is represented by the Gaussian measure $dm^{\widehat{TX}}$ along the fibre $\widehat{TX}$.

The family of Gaussian measures $dm^{\widehat{TX}}$ along the fibres $\widehat{TX}$ has been related to the embedding of $X$ as the zero section of $\mathcal{X}$. Now we give a dynamical interpretation of this family of Gaussian measures.

A vector field $U$ on $X$ is a smooth section of $TX$. When identifying $TX$ and $\widehat{TX}$, we can identify $U$ with the family of Dirac masses $\delta_{U(x)}$ in the fibres $\widehat{TX}$. From the above, we see that the family $dm^V$ represents another kind of family of measures on $\widehat{TX}$. What is its dynamical or vector field counterpart? It has to be Brownian motion on the manifold $X$, since the speed of Brownian motion, even if it is infinite, can be thought of as being given by independent Gaussians. On the other hand, the natural differential operator associated with Brownian motion is the Laplace-Beltrami operator $\Delta^X$.

It follows from the above that the Laplace-Beltrami operator has been made to be related to the embedding of $X$ into $\mathcal{X}$. As we shall see in Section 3, this idea reappears in the construction of the hypoelliptic Laplacian.

## 2. Gaussian index theory

The purpose of this section is explain the relation of index theory to the Gaussian formalism. In particular, we extend to general Dirac operators the results of Section 1 on the algebraic de Rham complex.

This section is organized as follows. In Subsection 2.1, we recall elementary facts on the index of Dirac operators, among which the McKean-Singer formula [McS67] for the index.

In Subsection 2.2, using the superconnection formalism of [Q85], we give a Gaussian expression for the Atiyah-Singer index class, which ultimately 'explains' the multiplicativity of the index formula.

In Subsection 2.3, we consider the case of the Lefschetz formulas.

Finally, in Subsection 2.4, we briefly describe other natural multiplicative constructions, which include the Brownian measure.

### 2.1. The index of the Dirac operator

Let $X$ be a compact even-dimensional Riemannian manifold, which we assume to be oriented and spin. Let $\nabla^{TX}$ be the Levi-Civita connection on $TX$, and let $R^{TX}$ be its curvature. Let $\left(E, g^E, \nabla^E\right)$ be a complex Hermitian vector bundle on $X$ equipped with a unitary connection, and let $R^E$ be the curvature of the connection.

Let $S^{TX} = S_+^{TX} \oplus S_-^{TX}$ be the Hermitian $\mathbf{Z}_2$-graded vector bundle of $\left(TX, g^{TX}\right)$ spinors on $X$, and let $\nabla^{S^{TX}} = \nabla^{S_+^{TX}} \oplus \nabla^{S_-^{TX}}$ be the Levi-Civita connection on this vector bundle.

Let $D^X$ be the Dirac operator acting on $C^\infty\left(X, S^{TX} \otimes E\right)$. Then $D^X$ is an odd elliptic first-order operator, which can be written in matrix form as

$$D^X = \begin{bmatrix} 0 & D_-^X \\ D_+^X & 0 \end{bmatrix}. \tag{2.1}$$

Moreover, $D_+^X$ is a Fredholm operator, whose index is denoted $\operatorname{Ind} D_+^X$.

The McKean-Singer formula [McS67] asserts that if $\operatorname{Tr_s}$ still denotes the supertrace, for any $t > 0$,

$$\operatorname{Ind} D_+^X = \operatorname{Tr_s}\left[\exp\left(-tD^{X,2}\right)\right]. \tag{2.2}$$

The proof of (2.2) is well known. The eigenspaces of $D^{X,2}$ define an increasing filtration on $C^\infty\left(X, S^{TX} \otimes E\right)$, and $D^X$ preserves the filtration. The eigenspace splitting of $C^\infty\left(X, S^{TX} \otimes E\right)$ is strictly similar to (1.14), and the proof of (2.2) can be obtained by proceeding as in (1.63). The analogy is not perfect in the sense that the set of eigenvalues of $D^{X,2}$ does not have an additive structure.

Let $X', E'$ be another pair similar to $X, E$. Let $p, p'$ be the projections $X \times X' \to X, X \times X' \to X'$. Then $(X \times X', p^*E \otimes p'^*E')$ is also such a pair. Moreover,

$$\begin{aligned} &C^\infty\left(X \times X', S^{T\left(X\times X'\right)} \otimes p^*E \otimes p'^*E'\right) \\ &\quad = C^\infty\left(X, S^{TX} \otimes E\right) \widehat{\otimes} C^\infty\left(X', S^{TX'} \otimes E'\right). \end{aligned} \tag{2.3}$$

Also,

$$D^{X\times X'} = D^X \widehat{\otimes} 1 + 1 \widehat{\otimes} D^{X'}. \tag{2.4}$$

In the sequel, we will write (2.4) in the simpler form

$$D^{X\times X'} = D^X + D^{X'}. \tag{2.5}$$

By (2.4), (2.5), we get

$$\operatorname{Ind} D_+^{X\times X'} = \operatorname{Ind} D_+^X \operatorname{Ind} D_+^{X'}. \tag{2.6}$$

Also the anticommutator of $D^X$ and $D^{X\prime}$ vanishes. If $\left[D^X, D^{X\prime}\right]$ is the supercommutator of $D^X$ and $D^{X\prime}$, we have the equivalent identity

$$\left[D^X, D^{X\prime}\right] = 0. \tag{2.7}$$

By (2.7), we get the Pythagorean theorem for Dirac operators,

$$D^{X\times X',2} = D^{X,2} + D^{X',2}. \tag{2.8}$$

By (2.8), we obtain

$$\exp\left(-tD^{X\times X',2}\right) = \exp\left(-tD^{X,2}\right)\widehat{\otimes}\exp\left(-tD^{X',2}\right). \tag{2.9}$$

By (2.9), we get

$$\operatorname{Tr_s}\left[\exp\left(-tD^{X\times X',2}\right)\right] = \operatorname{Tr_s}\left[\exp\left(-tD^{X,2}\right)\right]\operatorname{Tr_s}\left[\exp\left(-tD^{X',2}\right)\right]. \tag{2.10}$$

Of course, (2.2), (2.6), and (2.10) are compatible.

In equation (2.2), one can replace $\exp(-t\cdot)$ by any function $F \in \mathcal{S}(\mathbf{R})$ such that $F(0) = 1$, so that instead of (2.2), we have the identity

$$\operatorname{Ind}\left(D_+^X\right) = \operatorname{Tr_s}\left[F\left(D^{X,2}\right)\right]. \tag{2.11}$$

However, in general, the operator $F\left(D^{X,2}\right)$ does not behave multiplicatively.

The real heroes of equation (2.2) are the integer $\operatorname{Ind} D_+^X$ and the operator $\exp\left(-tD^{X,2}\right)$, which, by (2.6), (2.9), exhibit multiplicativity in their category.

Let $\mathcal{D}(X)$ be the algebra of differential operators acting on $C^\infty(X,E)$, and let $\mathcal{P}(X)$ be the algebra of pseudodifferential operators acting on the same vector space. Note that

$$\mathcal{D}(X\times X') = \mathcal{D}(X)\widehat{\otimes}\mathcal{D}(X'). \tag{2.12}$$

However the algebra $\mathcal{P}(X)$ does not behave properly with respect to products.

The index theorem of Atiyah-Singer [AS68] is formulated for elliptic pseudodifferential operators. However if $P \in \mathcal{P}(X), P' \in \mathcal{P}(X')$, in general $P\widehat{\otimes}1 + 1\widehat{\otimes}P'$ is not an element of $\mathcal{P}(X\times X')$. This means that in the abstract framework of index theory, the considerations we made before for Dirac operators are not valid. It is because Dirac operators are differential operators which are compatible to products of manifolds, and are such that their square verifies the Pythagorean formula in (2.8) that the McKean-Singer index formula, restricted to such operators, has this universal form. Needless to say, the operators we met in Section 1 are also differential operators, and this fact plays a critical role in their naturality with respect to direct sums.

**2.2. The index formula as a Gaussian formula**

The index formula of Atiyah-Singer [AS68] asserts that

$$\operatorname{Ind} D_{+}^{X} = \int_X \widehat{A}(TX) \operatorname{ch}(E). \tag{2.13}$$

The heat equation proof of (2.13) consists in making $t \to 0$ in the McKean-Singer formula (2.2), and to use the 'fantastic cancellation' mechanism anticipated by McKean-Singer [McS67] to show that as $t \to 0$, the local supertrace of the heat kernel associated with $\exp\left(-tD^{X,2}\right)$, instead of being singular as it should be, has a limit as $t \to 0$, which can ultimately be identified with a local Chern-Weil polynomial.

Observe that the right-hand side of (2.13) is compatible to products. In particular, the two following multiplicativity properties play a crucial role:

$$\widehat{A}(F \oplus F') = \widehat{A}(F)\,\widehat{A}(F'), \qquad \operatorname{ch}(E \otimes E') = \operatorname{ch}(E)\operatorname{ch}(E'). \tag{2.14}$$

Let us think more about (2.14). First note that the Chern character form $\operatorname{ch}\left(E, \nabla^E\right)$ associated with the connection $\nabla^E$ is given by

$$\operatorname{ch}\left(E, \nabla^E\right) = \varphi \operatorname{Tr}\left[\exp\left(-R^E\right)\right]. \tag{2.15}$$

In (2.15), $\varphi$ denotes the standard $2i\pi$ normalization of characteristic forms in Chern-Weil theory. The curvature $R^E$ can be written in the form

$$R^E = \nabla^{E,2}, \tag{2.16}$$

so that (2.15) has the equivalent Gaussian form

$$\operatorname{ch}\left(E, \nabla^E\right) = \varphi \operatorname{Tr}\left[\exp\left(-\nabla^{E,2}\right)\right]. \tag{2.17}$$

Let $\left(E', g^{E'}, \nabla^{E'}\right)$ be another Hermitian vector bundle with connection. Let $\nabla^{E\otimes E'}$ be the connection on $E \otimes E'$ which is induced by $\nabla^E, \nabla^{E'}$, i.e.,

$$\nabla^{E\otimes E'} = \nabla^E \otimes 1 + 1 \otimes \nabla^{E'}. \tag{2.18}$$

Then we have the Pythagorean formula

$$\nabla^{E\otimes E',2} = \nabla^{E,2} \otimes 1 + 1 \otimes \nabla^{E',2}, \tag{2.19}$$

from which we get

$$\operatorname{ch}\left(E \otimes E', \nabla^{E\otimes E'}\right) = \operatorname{ch}\left(E, \nabla^E\right)\operatorname{ch}\left(E', \nabla^{E'}\right), \tag{2.20}$$

which in turn implies the second equation in (2.14).

Let us now concentrate on the first equation in (2.14). This first equation can be refined to Chern-Weil forms, i.e., if $\left(F, g^F, \nabla^F\right), \left(F', g^{F'}, \nabla^{F'}\right)$ are real Euclidean vector bundles with metric connections, then

$$\widehat{A}\left(F \oplus F', \nabla^{F\oplus F'}\right) = \widehat{A}\left(F, \nabla^F\right)\widehat{A}\left(F', \nabla^{F'}\right). \tag{2.21}$$

We will now give a Gaussian explanation for (2.21). Let $\left(F, g^F, \nabla^F\right)$ be a real Euclidean oriented spin vector bundle of even dimension $n$, which is equipped with a metric connection. The curvature of $\nabla^F$ is denoted $R^F$. Let $\mathcal{F}$ be the total space of $F$. Then $\pi : \mathcal{F} \to X$ defines a fibration with fibre $F$. Also the fibres are equipped with the metric $g^F$, and moreover, the connection $\nabla^F$ defines a horizontal subbundle $T^H\mathcal{F} \subset T\mathcal{F}$. Let $Y$ be the tautological section of $\pi^*F$ on $\mathcal{F}$.

Let $S^F = S^F_+ \oplus S^F_-$ be the vector bundle of $\left(F, g^F\right)$ spinors on $X$. The above data define unambiguously a Levi-Civita superconnection $A^{S^F}$ [B86] on the vector bundle $C^\infty\left(F, \pi^*S^F\right)$ over $X$. Let $D^F$ be the obvious family of Dirac operators along the fibres $F$. As explained in [B90, Section 3(b)], with the proper normalization, $A^{S^F}$ takes the form

$$A^{S^F} = \nabla^{C^\infty\left(F,\pi^*S^F\right)} + \frac{D^F}{\sqrt{2}} - \frac{c\left(R^F Y\right)}{2\sqrt{2}}. \tag{2.22}$$

In (2.22), $\nabla^{C^\infty\left(F,\pi^*S^F\right)}$ is the connection on $C^\infty\left(F, \pi^*S^F\right)$ which is induced by $\nabla^F$, $D^F$ is the family of Dirac operators along the fibres, and $c$ denotes Clifford multiplication. The discrepancy by a factor $1/\sqrt{2}$ with respect to [B90, eq. (3.12] comes from the fact that in this reference, the family of Dirac operators along the fibres is just $D^F/\sqrt{2}$.

A remarkable fact established in [B90, Theorem 3.6] is that if $e_1, \dots, e_n$ is an orthonormal basis of $F$,

$$A^{S^F,2} = -\frac{1}{2}\sum_{i=1}^n \left(\nabla_{e_i} + \frac{1}{2}\left\langle R^F Y, e_i\right\rangle\right)^2. \tag{2.23}$$

Let us comment on equation (2.23). First if $F = TX$, the reader will have recognized the Getzler operator [Ge86], which appears as a renormalized limit of $tD^{X,2}/2$ as $t \to 0$, after rescaling of the local coordinates and of the Clifford variables. The fact that the Getzler operator is a square is a miracle. The fact that it is the curvature of a Levi-Civita superconnection [B86], whose purpose was to prove a local version of the families index theorem of Atiyah-Singer [AS71], is even more surprising. It indicates that when proving the index theorem for a single manifold, the formalism of the families index theorem is already there, even if it is, for the moment, difficult to interpret this coincidence. This is another manifestation of the fact that algebra severely constrains the structure of universal objects. In our opinion, this coincidence is the most important fact in the index theory of Dirac operators[1]. For an interpretation in terms of localization formulas in equivariant cohomology, we refer to [B11c, Remark 1.4 and Section 4.2].

Let us now describe the consequences of (2.23). First, the right-hand side of (2.23) does not contain Clifford multiplication operators. Except for the exterior algebra $\Lambda^\cdot\left(T^*X\right)$ which is present in $R^F$, $A^{S^F,2}$ is a scalar operator. Besides its component of degree 0 in $\Lambda^\cdot\left(T^*X\right)$ is just $-\frac{1}{2}\Delta^F$, which indicates that the heat operator $\exp\left(-A^{S^F,2}\right)$ is not trace class. This is not surprising. Indeed the fibres $F$ are noncompact, there is no natural families index associated with the projection

[1] This is, admittedly, a minority view.

$\pi : \mathcal{F} \to X$. Let $P^{S^F}(Y,Y')$ be the smooth kernel along the fibre associated with the operator $\exp\left(-A^{S^F,2}\right)$ with respect to the volume form $dY'/(2\pi)^{n/2}$. As explained before, $P^{S^F}(Y,Y')$ is a differential form on $X$, and does not contain Clifford variables. Therefore its classical supertrace over $S^F$ vanishes. Moreover, one verifies easily that $P^{S^F}(Y,Y)$ does not depend on $Y$.

From the above it is natural to view $P^{S^F}(Y,Y) = P^{S^F}(0,0)$ as a generalized von Neumann supertrace of $\exp\left(-A^{S^F,2}\right)$. We will write this in the form

$$\mathrm{Tr_s}\left[\exp\left(-A^{S^F,2}\right)\right] = P^{S^F}(0,0). \tag{2.24}$$

Now the basic computation of the local index theorem by Getzler [Ge86] says that

$$\widehat{A}\left(F,\nabla^F\right) = \varphi \mathrm{Tr_s}\left[\exp\left(-A^{S^F,2}\right)\right]. \tag{2.25}$$

Equation (2.25) indicates that $\widehat{A}\left(F,\nabla^F\right)$ has a Gaussian expression which in turn implies (2.21).

If in the above construction, we twist $S^F$ by $\left(E, g^E, \nabla^E\right)$, we obtain a superconnection $A^{S^F\otimes E}$ on $C^\infty\left(F, \pi^*\left(S^F\otimes E\right)\right)$ such that

$$A^{S^F\otimes E,2} = A^{F,2}\otimes 1 + 1\otimes \nabla^{E,2}. \tag{2.26}$$

If $P^{S^F\otimes E}(Y,Y')$ is the kernel associated with $\exp\left(-A^{S^F\otimes E,2}\right)$, then

$$P^{S^F\otimes E}(Y,Y') = P^{S^F}(Y,Y')\otimes \exp\left(-R^E\right). \tag{2.27}$$

Now we define $\mathrm{Tr_s}\left[\exp\left(-A^{S^F\otimes E,2}\right)\right]$ by the formula

$$\mathrm{Tr_s}\left[\exp\left(-A^{S^F\otimes E,2}\right)\right] = \mathrm{Tr}^E\left[P^{S^F\otimes E}(0,0)\right]. \tag{2.28}$$

Then

$$\widehat{A}\left(F,\nabla^F\right)\mathrm{ch}\left(E,\nabla^E\right) = \varphi\mathrm{Tr_s}\left[\exp\left(-A^{S^F\otimes E,2}\right)\right]. \tag{2.29}$$

For $b>0$, set

$$K_b s(Y) = s(bY). \tag{2.30}$$

Observe that for $t>0$,

$$K_{\sqrt{t}} t A^{S^F\otimes E,2} K_{1/\sqrt{t}} = -\frac{1}{2}\sum_{i=1}^n \left(\nabla_{e_i} + \frac{1}{2}\left\langle tR^{TX}Y, e_i\right\rangle\right)^2 + tR^E. \tag{2.31}$$

For $s\in\mathbf{R}$, let $\psi_s$ the endomorphism of $\Lambda^\cdot(T^*X)$ such that if $\alpha\in\Lambda^p(T^*X)$, then

$$\psi_s\alpha = s^p\alpha. \tag{2.32}$$

We can rewrite (2.31) in the form

$$K_{\sqrt{t}} t A^{S^F\otimes E,2} K_{1/\sqrt{t}} = \psi_{\sqrt{t}} A^{S^F\otimes E,2}\psi_{1/\sqrt{t}}. \tag{2.33}$$

We will denote by $P_t^{S^F\otimes E}(Y,Y')$ the smooth kernel associated with

$$\exp\left(-tA^{S^F\otimes E,2}\right)$$

with respect to the volume $dY'/(2\pi)^{n/2}$. By (2.33), we deduce that

$$P_t^{S^F\otimes E}(0,0)=\psi_{\sqrt{t}}\frac{1}{t^{n/2}}P^{S^F\otimes E}(0,0)\,\psi_{1/\sqrt{t}}. \tag{2.34}$$

We define $\mathrm{Tr_s}\left[\exp\left(-tA^{S^F\otimes E,2}\right)\right]$ by the formula

$$\mathrm{Tr_s}\left[\exp\left(-tA^{S^F\otimes E,2}\right)\right]=\mathrm{Tr}^E\left[P_t^{S^F\otimes E}(0,0)\right]. \tag{2.35}$$

By (2.34), (2.35), we get

$$\mathrm{Tr_s}\left[\exp\left(-tA^{S^F\otimes E,2}\right)\right]=\frac{1}{t^{n/2}}\psi_{\sqrt{t}}\mathrm{Tr_s}\left[\exp\left(-A^{S^F\otimes E,2}\right)\right]. \tag{2.36}$$

If $\left(F,g^F,\nabla^F\right)$ is taken to be $\left(TX,g^{TX},\nabla^{TX}\right)$, then the index formula (2.13) takes the form

$$\mathrm{Ind}\,D_+^X=\int_X\varphi\mathrm{Tr_s}\left[\exp\left(-A^{S^{TX}\otimes E,2}\right)\right]. \tag{2.37}$$

More generally, by (2.36), (2.37), for any $t>0$, we get

$$\mathrm{Ind}\,D_+^X=\int_X\varphi\mathrm{Tr_s}\left[\exp\left(-tA^{S^{TX}\otimes E,2}\right)\right]. \tag{2.38}$$

The fact that the right-hand side of (2.38) does not depend on $t>0$ should not come as a surprise, since it just reflects the fact that in the McKean-Singer formula, the function $\exp(-s)$ can be replaced by $\exp(-ts)$ for any $t>0$.

Let us now put together the McKean-Singer formula in (2.2) and (2.38). For any $t>0$, we get

$$\mathrm{Ind}\,D_+^X=\mathrm{Tr_s}\left[\exp\left(-tD^{X,2}/2\right)\right]=\int_X\varphi\mathrm{Tr_s}\left[\exp\left(-tA^{S^{TX}\otimes E,2}\right)\right]. \tag{2.39}$$

Comparing with (1.69), we find that equation (2.39) for $\mathrm{Ind}\,D_+^X$ has exactly the same doubly Gaussian character.

Also, we pointed out before that the hero of the McKean-Singer formula in (2.2) is the operator $\exp\left(-tD^{X,2}/2\right)$, and not the equation itself. Equation (2.39) suggests that $\exp\left(-tA^{S^F\otimes E,2}\right)$ is another such hero. Actually, the equality of the two last expressions in (2.39) suggests that the operators $D^{X,2}/2$ and $A^{S^F\otimes E,2}$ are just the two faces of a Janus like object, which we do not immediately identify as identical because of our imperfect understanding of their similarity.

The Getzler operator $A^{S^{TX}\otimes E,2}$ is obtained via a local rescaled deformation of $tD^{X,2}/2$ as $t\to 0$. The theory of the hypoelliptic Dirac operator is an attempt to reverse the above limit, i.e., to obtain the Dirac operator $D^X/\sqrt{2}$ as a limit of a deformation of a version of $A^{S^{TX}\otimes E}$.

Let $f(t) : \mathbf{R}_+ \to \mathbf{R}$ be a smooth function with compact support not including 0, and such that

$$\int_0^{+\infty} f(t)\,dt = 1. \tag{2.40}$$

Set

$$F(x) = \int_0^{+\infty} \exp(-tx) f(t)\,dt. \tag{2.41}$$

Then $F(x)$ is an analytic function of $x$ such that $F(0) = 1$.

By integrating (2.39), we get

$$\operatorname{Ind} D_+^X = \operatorname{Tr_s}\left[F\left(\frac{D^{X,2}}{2}\right)\right] = \int_X \varphi \operatorname{Tr_s}\left[F\left(A^{S^F \otimes E,2}\right)\right]. \tag{2.42}$$

We have now artificially destroyed the multiplicative nature of the index formula, by replacing the exponential by some other analytic function. Still equation (2.42) indicates how similar $D^{X,2}/2$ and $A^{S^F \otimes E,2}$ are.

Finally, let us point out that there is a Fourier transform quality to the index formula of Atiyah-Singer in (2.13), if only because it is a formula expressing the global quantity $\operatorname{Ind} D_+^X$ in local terms. This facts fits with the arguments developed in subsection 1.5 on the local versus global aspects of the Bargmann isomorphism, as well as with the considerations of Subsection 1.7. For other connections of Fourier transform with index theory, we refer to [B11c].

### 2.3. The Lefschetz fixed point formulas

We make the same assumptions as in Subsection 2.1. Assume that $G$ is a compact Lie group acting on $X$ and preserving all the data, including the metrics. Then $G$ acts isometrically on $C^\infty\left(X, S^{TX} \otimes E\right)$, and this action commutes with $D^X$. In particular $G$ acts on $\ker D^X$. If $g \in G$, let $\operatorname{Tr_s}^{\ker D^X}[g]$ denote the supertrace of the action of $g$. This character verifies exactly the same functorial properties as $\operatorname{Ind} D_+^X$. In this case, the McKean-Singer formula asserts that for any $t > 0$,

$$\operatorname{Tr_s}^{\ker D^X}[g] = \operatorname{Tr_s}\left[g \exp\left(-tD^{X,2}\right)\right]. \tag{2.43}$$

Needless to say, for $g = 1$, equations (2.2) and (2.43) coincide.

The Lefschetz formulas of Atiyah-Bott [ABo67, ABo68] for $\operatorname{Tr_s}^{\ker D^X}[g]$ are local formulas for $\operatorname{Tr_s}^{\ker D^X}[g]$, which are integrals of characteristic classes on the fixed point $X_g$ of $g$ in $X$. They can be obtained by making $t \to 0$ in (2.43).

Let $\mathfrak{g}$ be the Lie algebra of $G$. If $A \in \mathfrak{g}$, let $L_A^X$ denote the corresponding Lie derivative operator acting on $C^\infty\left(X, S^{TX} \otimes E\right)$. Then $L_A^X$ commutes with $D^{X,2}$. Moreover, if $g = e^A$, then

$$\operatorname{Tr_s}^{\ker D^X}[g] = \operatorname{Tr_s}\left[\exp\left(-L_A^X - tD^{X,2}\right)\right]. \tag{2.44}$$

The considerations of Subsection 2.2 remain valid for the Atiyah-Bott formulas.

The above formalism applies to the case where $D^X = d^X + d^{X*}$ or $D^X = \sqrt{2}\left(\overline{\partial}^X + \overline{\partial}^{X*}\right)$. Then $\ker D^X$ can be identified with the cohomology of the corresponding complex, and $\mathrm{Tr_s}^{\ker D^X}[g]$ is just the supertrace of the action of $g$ on the cohomology, which we will also call a Lefschetz trace.

### 2.4. Other multiplicative constructions

If $X$ is a Riemannian manifold, the Laplace-Beltrami operator $\Delta^X$ is natural with respect to products, i.e.,

$$\Delta^{X\times X'} = \Delta^X \otimes 1 + 1 \otimes \Delta^{X'}. \tag{2.45}$$

If $\sigma\left(-\Delta^X\right)$ is the principal symbol of $-\Delta^X$, then

$$\sigma\left(-\Delta^X\right) = |\xi|^2 . \tag{2.46}$$

In (2.46), $|\xi|^2$ is just the square of the norm of $\xi \in T^*X$. Taking the principal symbol of (2.45) gives the Pythagorean theorem.

By (2.45), we find that for $t > 0$,

$$\exp\left(t\Delta^{X\times X'}/2\right) = \exp\left(t\Delta^X/2\right) \otimes \exp\left(t\Delta^{X'}/2\right). \tag{2.47}$$

Let $LX$ be the smooth loop space of $X$, that is $LX$ is the set of smooth maps $x : S^1 \to X$. Let $E^X(x)$ be the energy of $x \in LX$, which is given by

$$E^X(x) = \frac{1}{2}\int_{S^1} |\dot{x}|^2 \, ds. \tag{2.48}$$

Note that

$$L(X \times X') = LX \times LX'. \tag{2.49}$$

Then

$$E^{X\times X'} = p^* E^X + p'^* E^{X'}. \tag{2.50}$$

The associated Brownian motion on $X$ defines a canonical family of measures $P_t^X$ on the continuous loop space $L^0X$, which is multiplicative with respect to products, i.e.,

$$P_t^{X\times X'} = P_t^X \otimes P_t^{X'}. \tag{2.51}$$

At least formally, one can write $P_t^X$ in the form

$$P_t^X = \exp\left(-\frac{E^X}{t}\right)\mathcal{D}x. \tag{2.52}$$

In (2.52), $\mathcal{D}x$ is a non-existing Lebesgue measure on $LX$, and $E^X$ takes the value $+\infty$ on the support of $P_t^X$. Still, the measures $P_t^X$ have the same universal Gaussian flavour as the other formulas we presented before. As was already explained in Subsection 1.8, Brownian motion can be thought of as the integral curve of the family of Gaussian measures $m^{TX}$. This point will be clarified in equation (5.34).

All these natural constructions are related. Let just give an example. For $t > 0$, let $p_t^X(x, x')$ be the smooth kernel associated with the operator $\exp\left(t\Delta^X/2\right)$. Then

$$\operatorname{Tr}\left[\exp\left(t\Delta^X/2\right)\right] = \int_X p_t^X(x, x)\, dx. \tag{2.53}$$

Now using the Itô calculus, we have the rigorous equality

$$\operatorname{Tr}\left[\exp\left(t\Delta^X/2\right)\right] = \int_X p_t^X(x, x)\, dx = \int_{L^0 X} dP_t^X, \tag{2.54}$$

which, using (2.52), can be complemented by

$$\operatorname{Tr}\left[\exp\left(t\Delta^X/2\right)\right] = \int_X p_t^X(x, x)\, dx = \int_{L^0 X} dP_t^X = \int_{LX} \exp\left(-\frac{E^X}{t}\right) \mathcal{D}x. \tag{2.55}$$

The objects which appear in (2.55), whether they exist or not, are multiplicative in their category.

As explained in [B11c, Section 3.5], for $t > 0$, $\operatorname{Tr_s}\left[\exp\left(-tD^{X,2}/2\right)\right]$ can be expressed as the integral of a signed measure on $L^0 X$ which has a density with respect to $P_t^X$. Atiyah [A85] has shown how to relate these formulas to the localization formulas of Duistermaat-Heckman [DH82, DH83]. For a systematic exposition of equivariant cohomology on loops spaces, we refer to [B11c], where other natural objects over $LX$ are also constructed. Whether they do exist rigorously or not, they are still naturally multiplicative in their own category, the multiplicativity having the same Gaussian character as before.

In particular, using (2.23), (2.25), the form $\widehat{A}\left(F, \nabla^F\right)$ can be expressed as a Gaussian integral involving the flat Brownian motion, whose explicit evaluation is known as Lévy's stochastic area formula [Le51]. The unavoidability of this identity in connection with the index formula in (2.13), (2.37) and equivariant cohomology is explained in [B11c].

## 3. The hypoelliptic Laplacian in de Rham theory

The purpose of this section is to briefly describe the construction of the hypoelliptic Laplacian in de Rham theory, which was introduced in [B05].

This section is organized as follows. In Subsection 3.1, we describe the action of the hypoelliptic Laplacian on smooth functions.

In Subsection 3.2, we briefly explain how to extend this action to smooth forms of arbitrary degree.

Finally, in Subsection 3.3, we describe the functorial and functional integral aspects of the hypoelliptic Laplacian in de Rham theory.

### 3.1. The hypoelliptic Laplacian in degree 0

Let $\left(X, g^{TX}\right)$ be a compact Riemannian manifold of dimension $n$, and let $\Delta^X$ be the Laplace-Beltrami operator. Let $\square^X = \left[d^X, d^{X*}\right]$ be the Hodge Laplacian. Then $-\Delta^X$ is just the restriction of $\square^X$ to differential forms of degree 0, i.e.,

$$-\Delta^X = d^{X*} d^X|_{\Omega^0(X)}. \tag{3.1}$$

Let $\pi : \mathcal{X} \to X$ be the total space of $TX$. As in subsection 1.8, we denote by $\widehat{TX}$ the fibre of $\pi$, and we use the corresponding notation.

In de Rham theory [B05], the hypoelliptic Laplacian $L_b^X$ is a second-order operator acting on smooth forms over $\mathcal{X}$, which is associated with an exotic Hodge theory on $\mathcal{X}$. As $b \to 0$, $L_b^X$ is supposed to converge in the proper sense to $\square^X/2$, and as $b \to +\infty$, one expects $L_b^X$ to converge to minus the Lie derivative operator associated with the geodesic flow.

Let $\Delta^V$ be the Laplacian along the fibre $\widehat{TX}$. Up to lower degree terms, $L_b^X$ can be written in the form

$$L_b^X = \frac{1}{2b^2}\left(-\Delta^V + |Y|^2 - n\right) - \frac{1}{b}\nabla_Y^X + \cdots \tag{3.2}$$

In the right-hand side of (3.2), the first operator is a scaled version of the harmonic oscillator along the fibre, and the second operator is the generator of the geodesic flow. The analytic properties of the hypoelliptic Laplacian have been studied in detail in Bismut-Lebeau [BL08]. We refer to the surveys [B08b, B08c] for more details on the hypoelliptic Laplacian in de Rham theory.

For the moment we just consider the scalar version $M_b^X$ of the operator $L_b^X$ in (3.2), i.e.,

$$M_b^X = \frac{1}{2b^2}\left(-\Delta^V + |Y|^2 - n\right) - \frac{1}{b}\nabla_Y^X. \tag{3.3}$$

Using the notation in (1.44), we can write (3.3) in the form

$$M_b^X = \frac{\mathcal{H}^{TX}}{b^2} - \frac{1}{b}\nabla_Y^X. \tag{3.4}$$

Observe that in (3.2)–(3.4), the operator $\nabla_Y^X$ appears. As explained in Subsection 1.8, this is a kind of bosonic Hodge de Rham operator, which implements a version of Fourier transform. We will elaborate more on this point in Subsection 5.1.

Here we will limit ourselves to write $M_b^X$ in a form similar to (3.1). Let $\nabla^{TX}$ be the Levi-Civita connection on $TX$. Then $T\mathcal{X}$ splits as

$$T\mathcal{X} = \pi^*\left(TX \oplus \widehat{TX}\right). \tag{3.5}$$

In (3.5), $\pi^*TX$ is the horizontal subbundle of $T\mathcal{X}$, and $\pi^*\widehat{TX}$ is the tangent bundle to the fibre of $\pi$.

Consider the Witten twist $d_b^{\mathcal{X}}$ of the de Rham operator $d^{\mathcal{X}}$ by the function $\exp\left(-|Y|^2/2b^2\right)$, i.e.,

$$d_b^{\mathcal{X}} = d^{\mathcal{X}} + \frac{\widehat{Y}^*}{b^2}\wedge . \tag{3.6}$$

In (3.6), $\widehat{Y}^*$ is just the vertical form along the fibre associated with the tautological section $\widehat{Y}$.

The de Rham operator acting on $\Omega^0(\mathcal{X})$ can be written in the form

$$d^{\mathcal{X}}|_{\Omega^0(\mathcal{X})} = d^H + d^V. \tag{3.7}$$

In (3.7), $d^H, d^V$ correspond to horizontal and vertical differentiation. By (3.7), we get

$$d_b^{\mathcal{X}}|_{\Omega^0(\mathcal{X})} = d^H + d^V + \frac{\widehat{Y}^*}{b^2}\wedge . \tag{3.8}$$

We equip $T\mathcal{X} = \pi^*\left(TX \oplus \widehat{TX}\right)$ with the bilinear symmetric form associated with the matrix $\mathfrak{f}$ given by

$$\mathfrak{f} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}. \tag{3.9}$$

Then $\mathfrak{f}$ is a bilinear form of signature $(n,n)$. It induces on $\mathcal{X}$ its obvious volume form $dv_{\mathcal{X}}$. Moreover, it also defines a nondegenerate symmetric bilinear pairing ( ) on the $\Lambda^p(T^*\mathcal{X}), 1 \le p \le 2n$. We equip $\Omega^{\cdot,c}(\mathcal{X})$ with the nondegenerate symmetric bilinear form $\eta$ given by

$$\eta(s,s') = \int_{\mathcal{X}} (s(x,Y), s'(x,-Y))\, dv_{\mathcal{X}}. \tag{3.10}$$

Let $d_b^{\mathcal{X}*}$ be the formal adjoint of $d_b^{\mathcal{X}}$ with respect to $\eta$. Let $e_1,\dots,e_n$ be an orthonormal basis of $TX$, which we identify with the corresponding basis $\pi^*TX$ in the right-hand side of (3.5). Let $\widehat{e}_1,\dots,\widehat{e}_n$ be the corresponding basis of $\pi^*\widehat{TX}$. The associated dual bases are denoted with upper indices. Let $\nabla^{\Lambda^{\cdot}(T^*X)\widehat{\otimes}\Lambda^{\cdot}\left(\widehat{T^*X}\right)}$ be the connection on $\Lambda^{\cdot}(T^*X)\widehat{\otimes}\Lambda^{\cdot}\left(\widehat{T^*X}\right)$ which is induced by $\nabla^{TX}$. Finally, if $\widehat{e} \in \widehat{TX}$, let $\nabla^V_{\widehat{e}}$ denote the corresponding derivative operator along the fibre $\widehat{TX}$. By (3.8), one verifies easily that

$$d_b^{\mathcal{X}*}|_{\Omega^1(X)} = -i_{\widehat{e}_i}\nabla_{e_i}^{\Lambda^{\cdot}(T^*X)\widehat{\otimes}\Lambda^{\cdot}\left(\widehat{T^*X}\right)} + i_{e_i-\widehat{e}_i}\nabla^V_{\widehat{e}_i} - i_{Y-\widehat{Y}}/b^2. \tag{3.11}$$

By (3.7), (3.11), we get

$$\frac{1}{2}d_b^{\mathcal{X}*}d_b^{\mathcal{X}}|_{\Omega^0(\mathcal{X})} = \frac{1}{2}\left(-\Delta^V + \frac{1}{b^4}|Y|^2 - \frac{n}{b^2}\right) - \frac{1}{b^2}\nabla^X_Y. \tag{3.12}$$

Let $K_b$ be the map $s(x,Y) \to s(x,bY)$. By (3.12), we get

$$M_b^X = K_b\frac{1}{2}d_b^{\mathcal{X}*}d_b^{\mathcal{X}}K_b^{-1}. \tag{3.13}$$

### 3.2. The hypoelliptic de Rham Laplacian in arbitrary degree

Equation (3.13) makes clear that $-2M_b^X$ is an exotic Laplace-Beltrami operator on $\mathcal{X}$. But it also suggests how to construct $L_b^X$. Indeed, (3.13) should extend to

$$L_b^X = K_b \frac{1}{2} \left[ d_b^{\mathcal{X}}, d_b^{\mathcal{X}*} \right] K_b^{-1}. \tag{3.14}$$

This is precisely one of the definitions of $L_b^X$ which is taken in [B05, Section 2.7], [BL08, Section 2.1], and [B08c, Section 3.4]. A formula for $L_b^X$ can be written explicitly, which coincides with (3.3) in degree 0. In higher degree, the right-hand side contains a third term where the curvature $R^{TX}$ of $\nabla^{TX}$ appears explicitly. This complicates the formula for $d_b^{\mathcal{X}*}$ and for $L_b^X$. For a detailed discussion, we refer to [B05].

### 3.3. Functorial aspects of the hypoelliptic Laplacian

First note that $M_b^X$ verifies the same naturality properties as the Laplacian $\Delta^X$ in (2.45), i.e.,

$$M_b^{X \times X'} = M_b^X \otimes 1 + 1 \otimes M_b^{X'}. \tag{3.15}$$

Of course the two components of $M_b^X$ in (3.3) verify similar functoriality properties. This is in particular the case for the vector field $\nabla_Y^X$ which defines the geodesic flow.

For the functional integral aspects of the hypoelliptic Laplacian, and its connection with Chern-Gauss-Bonnet, we refer to [B06, B08b]. Let us just give the analogue of equation (2.55). By the results of [BL08, Chapter 3], for $t > 0$, the operator $\exp\left(-tM_b^X\right)$ is trace class, and there is an associated smooth kernel $p_{b,t}^X(z, z')$. Then

$$\operatorname{Tr}\left[\exp\left(-tM_b^X\right)\right] = \int_{\mathcal{X}} p_{b,t}^X(z, z)\, dv_{\mathcal{X}}. \tag{3.16}$$

For $b \geq 0, t > 0, x \in LX$, set

$$H_{b,t}^X(x) = \frac{1}{2t}\int_{S^1} |\dot{x}|^2 ds + \frac{b^4}{2t^3}\int_{S^1} |\ddot{x}|^2 ds = \frac{1}{2t}\int_{S^1} \left|\dot{x} + b^2\frac{\ddot{x}}{t}\right|^2 ds. \tag{3.17}$$

Note that

$$H_{0,t}^X(x) = \frac{E^X(x)}{t}. \tag{3.18}$$

For $b > 0, t > 0$, let $P_{b,t}^X$ be the measure on $LX$ which is given formally by

$$P_{b,t}^X = \exp\left(-H_{b,t}^X(x)\right)\mathcal{D}x. \tag{3.19}$$

The measure $P_{b,t}^X$ exists, and is carried by $L^1X$, the set of $C^1$ maps $s \in S^1 \to X$. The analogue of (2.55) just says that

$$\begin{aligned} \operatorname{Tr}\left[\exp\left(-tM_b^X\right)\right] &= \int_{\mathcal{X}} p_{b,t}^X\left((x,Y),(x,Y)\right) dv_{\mathcal{X}} \\ &= \int_{LX} dP_{b,t}^X = \int_{LX} \exp\left(-H_{b,t}^X\right)\mathcal{D}x. \end{aligned} \tag{3.20}$$

In (3.20), only the last expression is formal. In Subsection 5.2, we will come back to the interpretation of (3.20).

By (3.18), as $b \to 0$, the quantities in (3.20) should converge to the ones in (2.55). Equivalently, as $b \to 0$, one should have

$$\operatorname{Tr}\left[\exp\left(-tM_b^X\right)\right] \to \operatorname{Tr}\left[\exp\left(t\Delta^X/2\right)\right]. \tag{3.21}$$

The convergence result in (3.21) is proved in [BL08, Section 3.4]. This is an example of the mysterious predictive power of the functional integral, even if the mathematics behind the rigorous proof is much deeper than what the formal arguments suggest. For a probabilistic proof of this result, we also refer to [B11d, Section 12.8].

## 4. The hypoelliptic Dirac operator for complex manifolds

If $X$ is a complex Kähler manifold, the purpose of this section is to explain the construction of the hypoelliptic deformation of the Dirac operator $D^X = \sqrt{2}\left(\overline{\partial}^X + \overline{\partial}^{X*}\right)$ which is given in [B08a, Section 3]. Roughly speaking, the starting point of the construction is a modified version of the Levi-Civita superconnection $A^{S^{TX}\otimes E}$ (whose square is the Getzler operator in (2.23)). This superconnection is modified for reasons which are explained in some detail. By combining this modified superconnection with the Koszul complex along the fibre, we obtain a family of hypoelliptic operators $\mathcal{A}_b$, which deforms the Dirac operator $D^X/\sqrt{2}$. Also $\mathcal{A}_b^2$ is a hypoelliptic Laplacian, in the sense that it has the same structure as the operators $L_b^X, M_b^X$ in (3.2), (3.3).

This section is organized as follows. In Subsection 4.1, we describe a natural superconnection associated with a holomorphic Hermitian vector bundle.

In Subsection 4.2, we construct the elliptic Dirac operator $D^X$.

In Subsection 4.3, for $b > 0$, we obtain the hypoelliptic Dirac operator $\mathcal{A}_b$.

In Subsection 4.4, we give formal arguments showing that $\mathcal{A}_b$ is a deformation of $D^X/\sqrt{2}$.

In Subsection 4.5, we relate the operator $\mathcal{A}_b$ to the Levi-Civita superconnection of $TX$.

Finally, in Subsection 4.6, we construct a suitable deformation of the Levi-Civita superconnection of a vector bundle. This construction produces a microlocal analogue of the hypoelliptic Dirac operator.

For a survey of other properties of the hypoelliptic Dirac operator, we refer to [B08c, Section 4].

### 4.1. A superconnection associated with a vector bundle

Let $X$ be a compact complex manifold. Let $TX$ be the holomorphic tangent bundle to $X$, let $T_{\mathbf{C}}X = TX \oplus \overline{TX}$ be the complexification of the real tangent bundle $T_{\mathbf{R}}X$.

Let $\left(F, g^F\right)$ be a holomorphic Hermitian vector bundle of dimension $n$ on $X$, let $\nabla^F$ be the holomorphic Hermitian connection on $F$, and let $R^F$ be its curvature.

Let $\pi : \mathcal{F} \to X$ be the total space of $F$. Then $\mathcal{F}$ is a complex manifold. Let $i : X \to \mathcal{F}$ be the embedding of $X$ as the zero section of $F$. Using the connection $\nabla^F$, we have the identification of smooth vector bundles,

$$T\mathcal{F} \simeq \pi^* \left(TX \oplus F\right). \tag{4.1}$$

From (4.1), we get the smooth identification

$$\Lambda^{\cdot} \left(\overline{T^*\mathcal{F}}\right) = \pi^* \left(\Lambda^{\cdot} \left(\overline{T^*X}\right) \widehat{\otimes} \Lambda^{\cdot} \left(\overline{F}^*\right)\right). \tag{4.2}$$

Let $\left(\Omega^{(0,\cdot)}\left(\mathcal{F}\right), \overline{\partial}^{\mathcal{F}}\right)$ be the Dolbeault complex of smooth antiholomorphic forms on $\mathcal{F}$. Let $\mathbf{I}$ be the vector bundle on $X$ of the smooth sections of $\pi^*\Lambda^{\cdot}\left(\overline{F}^*\right)$ along the fibre $F$. It is well known that the operator $\overline{\partial}^{\mathcal{F}}$ splits as

$$\overline{\partial}^{\mathcal{F}} = \nabla^{\mathbf{I}\prime\prime} + \overline{\partial}^V. \tag{4.3}$$

In (4.3), $\overline{\partial}^V$ is the Dolbeault operator along the fibre $F$, and $\nabla^{\mathbf{I}\prime\prime}$ is the horizontal part of $\overline{\partial}^{\mathcal{F}}$. By (4.3), $\overline{\partial}^{\mathcal{F}}$ can also be viewed as a holomorphic superconnection $A''$ on $\mathbf{I}$.

The adjoint superconnection $A'$ of $A''$ is given by

$$A' = \nabla^{\mathbf{I}\prime} + \overline{\partial}^{V*}. \tag{4.4}$$

In (4.4), $\nabla^{\mathbf{I}\prime}$ is the holomorphic part of the horizontal connection $\nabla^{\mathbf{I}}$ on $\mathbf{I}$, and $\overline{\partial}^{V*}$ is the fibrewise $L_2$-adjoint of $\overline{\partial}^V$ with respect to the metric $g^F$.

Set

$$A = A'' + A'. \tag{4.5}$$

By (4.3)–(4.5), we get

$$A = \nabla^{\mathbf{I}} + \overline{\partial}^V + \overline{\partial}^{V*}. \tag{4.6}$$

Then $A$ is a standard superconnection on $\mathbf{I}$.

Still $A$ is not the Levi-Civita superconnection on $\mathbf{I}$ in the sense of (2.22). We will briefly describe this last superconnection in more detail.

We denote by $y$ the tautological section of $\pi^*F$ on $\mathcal{F}$, by $Y$ the tautological section of $\pi^*F_{\mathbf{R}}$, so that $Y = y + \overline{y}$, and $|Y|^2_{g^F} = 2\,|y|^2_{g^F}$. Set

$$\omega^{\mathcal{F}} = i\overline{\partial}^{\mathcal{F}} \partial^{\mathcal{F}} |y|^2_{g^F}. \tag{4.7}$$

Let $\omega^{\mathcal{F},V}$ be the Kähler form along the fibres, i.e., $\omega^{\mathcal{F},V}$ is the restriction of $\omega^{\mathcal{F}}$ to the fibres $F$. Put

$$\omega^{\mathcal{F},H} = i\left\langle R^F y, \overline{y}\right\rangle_{g^F}. \tag{4.8}$$

Then a simple computation shows that

$$\omega^{\mathcal{F}} = \omega^{\mathcal{F},V} + \omega^{\mathcal{F},H}. \tag{4.9}$$

Equation (4.9) gives the splitting of $\omega^{\mathcal{F}}$ into its vertical and horizontal parts. The same equation indicates that $\pi : \mathcal{F} \to X$ is a Kähler fibration in the sense of [BGS88], i.e., the horizontal vector bundle $T^H\mathcal{F}$ is just the orthogonal bundle to the vertical vector bundle $F$ with respect to $\omega^{\mathcal{F}}$.

**Definition 4.1.** Set

$$\begin{aligned} B'' &= A'', \qquad B' = e^{i\omega^{\mathcal{F},H}} A' \exp\left(-i\omega^{\mathcal{F},H}\right), \\ C'' &= \exp\left(-i\omega^{\mathcal{F},H}/2\right) A'' \exp\left(i\omega^{\mathcal{F},H}/2\right), \\ C' &= \exp\left(i\omega^{\mathcal{F},H}/2\right) A' \exp\left(-i\omega^{\mathcal{F},H}/2\right), \\ B &= B'' + B', \qquad C = C'' + C'. \end{aligned} \tag{4.10}$$

Then $B, C$ are superconnections on $\mathbf{I}$, and moreover,

$$C = \exp\left(-i\omega^{\mathcal{F},H}/2\right) B \exp\left(i\omega^{\mathcal{F},H}/2\right). \tag{4.11}$$

By (4.8), (4.10), we get

$$\begin{aligned} B &= \nabla^{\mathbf{I}} + \overline{\partial}^V + \overline{\partial}^{V*} + i_{\overline{R^F y}}, \\ C &= \nabla^{\mathbf{I}} + \overline{\partial}^V + \overline{\partial}^{V*} - \frac{1}{2}\left(R^F y^* \wedge - i_{\overline{R^F y}}\right). \end{aligned} \tag{4.12}$$

Let $c\left(F_{\mathbf{R}}\right)$ be the Clifford algebra of $\left(F_{\mathbf{R}}, g^F\right)$. Then $c\left(F_{\mathbf{R}}\right)$ acts naturally on $\Lambda^{\cdot}\left(\overline{F}^*\right)$. Set

$$D^F = \sqrt{2}\left(\overline{\partial}^V + \overline{\partial}^{V*}\right). \tag{4.13}$$

Then $D^F$ is a standard Dirac operator along the fibre $F$.

We can rewrite the second identity in (4.12) in the form

$$C = \nabla^{\mathbf{I}} + \frac{D^F}{\sqrt{2}} - \frac{c\left(R^F Y\right)}{2\sqrt{2}}. \tag{4.14}$$

Comparing with (2.22), we find that $C$ is indeed the Levi-Civita superconnection associated with the fibration $\pi : \mathcal{F} \to X$.

Still, as we shall see, $C$ is not the right object to consider in the construction of the hypoelliptic Dirac operator.

## 4.2. The elliptic Dirac operator

Let $E$ be a complex holomorphic vector bundle on $X$. Let $\left(\Omega^{(0,\cdot)}(X, E), \overline{\partial}^X\right)$ be the Dolbeault complex of smooth antiholomorphic forms with coefficients in $E$.

Let $g^{TX}, g^E$ be Hermitian metrics on $TX, E$. Let $\overline{\partial}^{X*}$ denote the formal adjoint of $\overline{\partial}^X$ with respect to the obvious Hermitian product on $\Omega^{(0,\cdot)}(X, E)$. Set

$$D^X = \sqrt{2}\left(\overline{\partial}^X + \overline{\partial}^{X*}\right). \tag{4.15}$$

If the metric $g^{TX}$ is Kähler, it is well known that $D^X$ is a standard Dirac operator.

### 4.3. The hypoelliptic Dirac operator

Here we follow [B08a, Section 3]. Let $\widehat{TX}$ be another copy of $TX$, and let $\mathcal{X}$ be the total space of $\widehat{TX}$. Let $g^{\widehat{TX}}$ be a Hermitian metric on $\widehat{TX}$, let $\nabla^{\widehat{TX}}$ be the corresponding holomorphic Hermitian connection, and let $R^{\widehat{TX}}$ be its curvature. We now use the notation of Subsection 4.1, with $\left(\widehat{TX}, g^{\widehat{TX}}\right) = \left(F, g^F\right)$.

Note that $\Lambda^{\cdot}\left(T^*X\right)$ is the holomorphic part of the exterior algebra of $X$. Recall that $y$ is the tautological section of $\pi^*\widehat{TX}$. Since $\widehat{TX}$ and $TX$ are isomorphic, the operator $i_y$ acts on $\pi^*\Lambda^{\cdot}\left(T^*X\right)$. The Koszul complex $\left(\mathcal{O}_{\mathcal{X}}\pi^*\Lambda^{\cdot}\left(T^*X\right), i_y\right)$ provides a resolution of the sheaf $i_*\mathcal{O}_X$.

Set

$$\mathcal{E} = \pi^*\left(\Lambda^{\cdot}\left(T^*X\right)\otimes E\right), \qquad \mathscr{E} = \Lambda^{\cdot}\left(T^*_{\mathbf{C}}X\right)\widehat{\otimes}\Lambda^{\cdot}\left(\overline{\widehat{T^*X}}\right)\otimes E. \tag{4.16}$$

In the sequel, our operators will act on $\Omega^{(0,\cdot)}\left(\mathcal{X}, \pi^*\mathcal{E}\right) = C^{\infty}\left(\mathcal{X}, \pi^*\mathscr{E}\right)$.

Recall that the superconnection $A'' = \overline{\partial}^{\mathcal{X}}$ was defined in Subsection 4.1. For $b > 0$, set

$$A''_b = A'' + i_y/b^2. \tag{4.17}$$

By (4.3), (4.17), we get

$$A''_b = \nabla^{\mathbf{I}''} + \overline{\partial}^V + i_y/b^2. \tag{4.18}$$

Since $y$ is a holomorphic section of $\pi^*TX$, we get

$$A''^2_b = 0. \tag{4.19}$$

We will take the 'adjoint' of $A''_b$, partly in the sense of superconnections. The operator $i_{\overline{y}}$ acts on $\Lambda^{\cdot}\left(\overline{T^*X}\right)$, and so it acts on $\mathscr{E}$. Set

$$A'_b = A' + i_{\overline{y}}/b^2. \tag{4.20}$$

Then $A'_b$ also acts on $\Omega^{(0,\cdot)}\left(\mathcal{X}, \pi^*\mathcal{E}\right)$. Moreover,

$$A'^2_b = 0. \tag{4.21}$$

Set

$$A_b = A''_b + A'_b. \tag{4.22}$$

When identifying $Y \in \widehat{T_{\mathbf{R}}X}$ to the corresponding section of $T_{\mathbf{R}}X$, we get

$$A_b = A + i_Y/b^2. \tag{4.23}$$

Note that $A_b$ is no longer a superconnection on $\mathbf{I}$, but a genuine operator acting on $\Omega^{(0,\cdot)}\left(\mathcal{X}, \pi^*\mathcal{E}\right)$.

One verifies easily that $A^2_b$ is hypoelliptic, because the anticommutator of $\nabla^{\mathbf{I}}$ with $i_Y/b^2$ produces the critical term $\nabla^{\mathbf{I}}_Y/b^2$. This term is obviously related to Cartan's formula in (1.12). If the metric induced by $g^{\widehat{TX}}$ on $TX$ is Kähler, this is the generator of the geodesic flow on $\mathcal{X}$.

Still $A_b$ is not the right hypoelliptic Dirac operator, because $A^2_b$ does not contain a potential which is quadratic in $Y$. Let $g^{TX}$ be a Hermitian metric on

$TX$, unrelated to $g^{\widehat{TX}}$, and let $\omega^X$ be the Kähler form on $X$ associated with the metric $g^{TX}$. If $J$ is the complex structure of $T_{\mathbf{R}}X$, if $U, V \in T_{\mathbf{R}}X$, then

$$\omega^X(U,V) = \langle U, JV \rangle_{g^{TX}} . \tag{4.24}$$

We will view $\omega^X$ as a section of $\Lambda^{\cdot}(T^*_{\mathbf{C}}X)$.

Put

$$\mathcal{A}''_b = A''_b, \qquad \mathcal{A}'_b = e^{i\omega^X} A'_b e^{-i\omega^X}, \qquad \mathcal{A}_b = \mathcal{A}''_b + \mathcal{A}'_b. \tag{4.25}$$

Assume that the metric $g^{TX}$ is Kähler, i.e., $\omega^X$ is closed. Let $\overline{y}^* \in T^*X$ be dual to $\overline{y} \in \overline{TX}$ with respect to the metric $g^{TX}$. Then

$$\mathcal{A}'_b = A'_b + \overline{y}^* \wedge /b^2. \tag{4.26}$$

By (4.23), (4.25), and (4.26), we get

$$\mathcal{A}_b = A_b + \overline{y}^* \wedge /b^2. \tag{4.27}$$

Let $\Delta^V_{g^{\widehat{TX}}}$ be the Laplacian along the fibre $\widehat{TX}$ with respect to $g^{\widehat{TX}}$. Let $\nabla^V$ denote differentiation along the fibre $\widehat{TX}$. One then verifies easily that

$$\mathcal{A}_b^2 = \frac{1}{2}\left(-\Delta^V_{g^{\widehat{TX}}} + \frac{1}{b^4}|Y|^2_{g^{TX}}\right) + \frac{1}{b^2}\nabla^{\mathbf{I}}_Y - \nabla^V_{R^{\widehat{TX}}Y} + \cdots , \tag{4.28}$$

where $\ldots$ denotes a differential operator of order 0. The effect of the addition of $\overline{y}^* \wedge /b^2$ in (4.26) is precisely to produce the desired $|Y|^2_{g^{TX}}/2b^4$ in $\mathcal{A}_b^2$.

When $g^{TX}$ is Kähler and $g^{\widehat{TX}} = g^{TX}$, the operator $\mathcal{A}_b^2$ is precisely the hypoelliptic Dirac operator constructed in [B08a, Section 3]. It has the preferred structure of a hypoelliptic Laplacian, i.e., its principal part is a scaled sum of a harmonic oscillator and of the generator of the geodesic flow. When $g^{\widehat{TX}}$ is unrelated to $g^{TX}$ and $g^{TX}$ is Kähler, this operator was constructed in [B08a, Section 10]. When no assumption is made on $g^{TX}$ and $g^{\widehat{TX}}$, this construction is developed in [B11b, Section 7].

### 4.4. The operator $\mathcal{A}_b$ as a deformation of $D^X/\sqrt{2}$

We still follow [B08a, Section 3]. We assume that $g^{TX}$ is Kähler, and that $g^{\widehat{TX}} = g^{TX}$. Let $w_1, \ldots, w_n$ be an orthonormal basis of $TX$. Let $\widehat{w}_1, \ldots, \widehat{w}_n$ be the corresponding basis of $\widehat{TX}$. Put

$$\Lambda = \sqrt{-1}\, i_{\overline{w}_i} i_{w_i}. \tag{4.29}$$

Set

$$\mathcal{B}_b = \exp(i\Lambda)\,\mathcal{A}_b \exp(-i\Lambda). \tag{4.30}$$

For $b > 0$, we define $K_b$ as in (2.30). Put

$$\mathcal{C}_b = K_b \mathcal{B}_b K_b^{-1}. \tag{4.31}$$

Let $\nabla^{\mathscr{E}}$ be the unitary connection on $\mathscr{E}$ which is induced by the connections $\nabla^{TX}, \nabla^{\widehat{TX}}, \nabla^E$. Let $K$ be the operator acting on $C^\infty(\mathcal{X}, \mathscr{E})$,

$$K = \left(\overline{w}^i \wedge + i_{w_i}\right)\nabla^{\mathscr{E}}_{\overline{w}_i} + \left(w^i \wedge - i_{\overline{w}_i}\right)\nabla^{\mathscr{E}}_{w_i}. \tag{4.32}$$

In [B08a, Eq. (3.68)], it is shown that

$$\mathcal{C}_b = K + \frac{1}{b}\left(\overline{\partial}^V + i_y + \overline{\partial}^{V*} + \overline{y}^* \wedge\right). \tag{4.33}$$

Set

$$\widehat{\omega}^{\mathcal{X},V} = -\sqrt{-1} w^i \wedge \widehat{\overline{w}}^i. \tag{4.34}$$

By results obtained in [B90, Proposition 1.5 and Theorem 1.6], the kernel of $\overline{\partial}^V + i_y + \overline{\partial}^{V*} + \overline{y}^* \wedge$ acting on the Schwartz space $\mathcal{S}^{(0,\cdot)}\left(\widehat{TX}, \pi^* \Lambda^{\cdot}(T^*X)\right)$ is one-dimensional and spanned by $\beta = \exp\left(i\widehat{\omega}^{\mathcal{X},V} - \frac{|Y|^2_{g^{\widehat{TX}}}}{2}\right)$. Let $P$ be the orthogonal projection operator on this kernel. We embed $\Omega^{(0,\cdot)}(X,E)$ into $C^\infty(\mathcal{X}, \pi^*\mathscr{E})$ by the embedding $\alpha \to \pi^*\alpha \wedge \beta$. This embedding is isometric for a suitable normalization of the $L_2$ Hermitian product.

The following result was established in [B08a, Theorem 3.12].

**Theorem 4.2.** *The following identity holds:*

$$PKP = \overline{\partial}^X + \overline{\partial}^{X*}. \tag{4.35}$$

*Proof.* Using (4.32), we get easily

$$PKP = \overline{w}^i \nabla^{\Lambda^{\cdot}(\overline{T^*X}) \otimes E}_{\overline{w}_i} - i_{\overline{w}_i} \nabla^{\Lambda^{\cdot}(\overline{T^*X}) \otimes E}_{w_i}, \tag{4.36}$$

which is just (4.35). □

The decomposition (4.33) and the identity (4.35) are the key algebraic facts which ultimately explain that as $b \to 0$, $\mathcal{A}_b$ converges in the proper sense to $D^X/\sqrt{2}$. For more details, we refer to [B08a], and also to [BL08, Chapter 3], where a similar problem is dealt with involving de Rham cohomology.

In [B11a, B11b], the above results have been extended to the case where $\omega^X$ is not closed. A motivation for doing this is that certain Riemann-Roch-Grothendieck theorems in Bott-Chern cohomology [D09] cannot be proved in the elliptic world, because of the absence of the relevant local cancellations.

### 4.5. The operator $\mathcal{A}_b$ and the Levi-Civita superconnection of $\widehat{TX}$

If we make $b = +\infty$ in (4.23), (4.27), we get

$$\mathcal{A}_\infty = A. \tag{4.37}$$

Comparing (4.6) and (4.12) shows that $\mathcal{A}_\infty$ does not coincide with the Levi-Civita superconnection $C$, or with its conjugate $B$. Let us briefly explain why it would be wrong to try forcing the use of the Levi-Civita superconnection in this context.

Indeed instead of (4.25), set

$$\begin{aligned} \mathscr{A}''_b &= A''_b, \\ \mathscr{A}'_b &= \exp\left(i\omega^X + i\omega^{\mathcal{X},H}\right) A'_b \exp\left(-i\left(\omega^X + \omega^{\mathcal{X},H}\right)\right), \\ \mathscr{A}_b &= \mathscr{A}''_b + \mathscr{A}'_b. \end{aligned} \tag{4.38}$$

By (4.25), (4.38), we get

$$\mathscr{A}_b' = \exp\left(i\omega^{\mathcal{X},H}\right)\mathcal{A}_b' \exp\left(-i\omega^{\mathcal{X},H}\right). \tag{4.39}$$

By (4.8), (4.20), (4.26), and (4.39), we obtain

$$\mathscr{A}_b' = \mathcal{A}_b' + i_{\widehat{R^{TX}y}} + \left\langle R^{TX}\left(\overline{y},\cdot\right)y,\overline{y}\right\rangle_{g^{TX}}/b^2. \tag{4.40}$$

By (4.18), (4.38), and (4.40), in $\mathscr{A}_b^2$, there is a quartic term

$$\frac{1}{b^4}\left\langle R^{TX}\left(\overline{y},y\right)y,\overline{y}\right\rangle_{g^{TX}}. \tag{4.41}$$

This term destroys the fibrewise harmonic oscillator character one expects from a hypoelliptic Laplacian. If (4.41) is negative, the operator $\mathscr{A}_b^2$ is hopeless anyway. This is why the hypoelliptic Dirac operator is defined to be $\mathcal{A}_b$ and not $\mathscr{A}_b$.

Incidentally, in the fibre direction, quartic nonnegative potentials in the variable $Y$ have also their role in the theory of the hypoelliptic Laplacian. In [B11d], the hypoelliptic Laplacian which is used in the evaluation of orbital integrals contains such a quartic term. This is also the case for the version of the hypoelliptic Laplacian which appears in [B11b], a quadratic potential being there more a nuisance than an asset. Such quartic terms are needed in [B11b] in spite of the fact they violate the Pythagorean principle of Section 2.

### 4.6. The hypoelliptic deformation of the local index theoretic data

As shown in [B08a, Sections 2 and 3], it is also possible to deform the superconnections attached to a family of complex manifolds whose curvature is elliptic to superconnections whose curvature is fibrewise hypoelliptic. This we will only explain in the context of vector bundles.

Let us consider the Levi-Civita superconnection $C$ in (4.12) which is attached to the Hermitian vector bundle $\left(F,g^F\right)$. Its curvature $C^2$ is an elliptic operator along the fibre $F$. Let $e_1,\ldots,e_{2n}$ be an orthonormal basis of $F_{\mathbf{R}}$ with respect to $g^F$. By (2.23),

$$C^2 = -\frac{1}{2}\sum_{i=1}^{2n}\left(\nabla_{e_i} + \frac{1}{2}\left\langle R^F Y, e_i\right\rangle\right)^2 + \frac{1}{2}\mathrm{Tr}\left[R^F\right]. \tag{4.42}$$

The presence of the term $\frac{1}{2}\mathrm{Tr}\left[R^F\right]$ just reflects the fact that $S^{F_{\mathbf{R}}}$ is replaced by $\Lambda^{\cdot}\left(\overline{F}^*\right)$.

Let $\widehat{F}$ denote the tangent bundle to the fibre $F$, and let $\widehat{\mathcal{F}}$ be the total space of this tangent bundle. Equivalently $\widehat{\mathcal{F}}$ is the total space $F\oplus\widehat{F}$, where $\widehat{F}$ is another copy of $F$. Let $\sigma$ be the projection $\widehat{\mathcal{F}}\to X$, and let $p:\widehat{\mathcal{F}}\to\mathcal{F}$ be the obvious projection with fibre $\widehat{F}$. Let $y,z$ be the tautological sections of $\sigma^*F,\sigma^*\widehat{F}$ on $\widehat{\mathcal{F}}$.

Let $g^{\widehat{F}}$ be a Hermitian metric on $\widehat{F}$. Let $\nabla^{\widehat{F}}$ be the holomorphic Hermitian connection on $\widehat{F}$, and let $R^{\widehat{F}}$ be its curvature. We will view $\sigma^*\widehat{F}$ as a holomorphic vector bundle on $\mathcal{F}$, to which we will apply the constructions of Subsection 4.1.

Let $\mathbf{J}^{\cdot}$ be the vector bundle on $\mathcal{F}$ of the smooth sections of $\Lambda^{\cdot}\left(\widehat{\overline{F}}^{*}\right)$ along the fibre $\widehat{F}$. Let $\mathfrak{A}'', \mathfrak{A}', \mathfrak{A}$ be the analogues of $A'', A', A$. Let $\overline{\partial}^{\widehat{V}}$ be the $\overline{\partial}$ operator along $\widehat{F}$, and let $\overline{\partial}^{\widehat{V}*}$ be its fibrewise adjoint with respect to $g^{\widehat{F}}$. Then

$$\begin{aligned} \mathfrak{A}'' &= \nabla^{\mathbf{J}^{\cdot}\prime\prime} + \overline{\partial}^{V} + i_{\overline{R^F y}} + \overline{\partial}^{\widehat{V}}, \\ \mathfrak{A}' &= \nabla^{\mathbf{J}^{\cdot}\prime} + \overline{\partial}^{V} + i_{R^{TX} y} + \overline{\partial}^{\widehat{V}*}, \\ \mathfrak{A} &= \mathfrak{A}'' + \mathfrak{A}'. \end{aligned} \tag{4.43}$$

The first three terms in the right-hand side of the first two equations in (4.43) are the contributions of $\overline{\partial}^{\mathcal{F}}, \partial^{\mathcal{F}}$. Also $\mathfrak{A}^2$ is fibrewise elliptic along $\widehat{F}$.

Let $\mathbf{K}^{\cdot}$ be the vector bundle on $X$ of smooth sections along the fibre $F \oplus \widehat{F}$ of $\sigma^{*}\left(\Lambda^{\cdot}(F^{*}_{\mathbf{C}})\widehat{\otimes}\Lambda^{\cdot}\left(\widehat{\overline{F}}^{*}\right)\right)$. We will now imitate (4.17), (4.20), and (4.25). Let $\overline{z}^{*} \in F^{*}$ be dual to $\overline{z} \in \overline{F}$ with respect to $g^{F}$. Because of the identification $\widehat{F} \simeq F$, the operators $i_z, i_{\overline{z}}, \overline{z}^{*}\wedge$ act on $\Lambda^{\cdot}(F^{*}_{\mathbf{C}})$. Recall that the form $\omega^{\mathcal{F}}$ on $\mathcal{F}$ was defined in (4.7). Set

$$\begin{aligned} \mathfrak{A}''_b &= \mathfrak{A}'' + i_z/b^2, \\ \mathfrak{A}'_b &= e^{i\omega^{\mathcal{F}}}\left(\mathfrak{A}' + i_{\overline{z}}/b^2\right)e^{-i\omega^{\mathcal{F}}}, \\ \mathfrak{A}_b &= \mathfrak{A}''_b + \mathfrak{A}'_b. \end{aligned} \tag{4.44}$$

By proceeding as in Subsection 4.4, one can show that the operator $\mathfrak{A}_b$ is a deformation of the Levi-Civita superconnection $C$ for $\left(F, g^{F}\right)$.

When $F = TX$, this deformation is a microlocal version of the deformation of $D^{X}/\sqrt{2}$ to $\mathcal{A}_b$. It deforms local index theoretic data for $D^{X}/\sqrt{2}$ to what turns out to be local index theoretic data for $\mathcal{A}_b$. More precisely as shown in [B08a, Theorem 6.12], when $\omega^{X}$ is closed and $g^{\widehat{TX}} = g^{TX}$, the superconnection $\mathfrak{A}_b$ associated with $\left(TX, g^{TX}, g^{\widehat{TX}}\right)$ appears when computing the limit as $t \to 0$ of $\mathrm{Tr_s}\left[\exp\left(-t\mathcal{A}^2_{b\sqrt{t}}\right)\right]$.

## 5. Orbital integrals and the hypoelliptic heat kernel

The purpose of this section is to survey some aspects of the application of the hypoelliptic Laplacian to the evaluation of semisimple orbital integrals for reductive groups [B11d]. The orbital integrals are the main ingredient which appears in the geometric side of Selberg's trace formula.

One of the main points made in [B11d] is that the evaluation of the trace of a heat kernel can be made formally similar to the evaluation of a Lefschetz formula. We will prove this in the trivial case where the symmetric space is $\mathbf{R}$. Also we describe in some detail the uniform estimates on the hypoelliptic heat kernel which were established in [B11d]. Finally, we explain the link made in [B11d] between the heat equation for the hypoelliptic Laplacian and the wave equation, as a version of quantization of the Hamiltonian-Lagrangian correspondence in the classical calculus of variations.

This section is organized as follows. In Subsection 5.1, we explain the role of the hypoelliptic Laplacian in the evaluation of the orbital integrals on $\mathbf{R}$.

In Subsection 5.2, we describe some results on the hypoelliptic heat kernel in the case where the base manifold is a symmetric space of noncompact type, and the role of corresponding action functionals.

Finally, in Subsection 5.3, we explain the wave like character of the hypoelliptic Laplacian.

### 5.1. Orbital integrals: the case of the real line

Here, we adopt temporarily the formalism of Sections 2 and 3 with $X = \mathbf{R}$. The fact that $X$ is noncompact will be irrelevant. Note that $\mathcal{X} = \mathbf{R} \times \mathbf{R}$. We will denote by $(x, y)$ the generic element in $\mathcal{X}$. By (3.3), we get

$$\Delta^{\mathbf{R}} = \frac{\partial^2}{\partial x^2}, \qquad M_b^{\mathbf{R}} = \frac{1}{2b^2}\left(-\frac{\partial^2}{\partial y^2} + y^2 - 1\right) - \frac{1}{b} y \frac{\partial}{\partial x}. \tag{5.1}$$

The heat kernel $p_t^{\mathbf{R}}(x, x')$ associated with $\exp\left(t\Delta^{\mathbf{R}}/2\right)$ is given by

$$p_t^{\mathbf{R}}(x, x') = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(x'-x)^2}{2t}\right). \tag{5.2}$$

The heat kernel $p_{b,t}^{\mathbf{R}}((x, y), (x', y'))$ associated with $\exp\left(-tM_b^{\mathbf{R}}\right)$ depends only on $x' - x, y, y'$. It has been computed explicitly in [B11d, Proposition 10.5.1].

Take $a \in \mathbf{R}$. In the sequel we use the notation

$$\begin{aligned} \operatorname{Tr}^a\left[\exp\left(t\Delta^{\mathbf{R}}/2\right)\right] &= p_t^{\mathbf{R}}(0, a), \\ \operatorname{Tr}^a\left[\exp\left(-tM_b^{\mathbf{R}}\right)\right] &= \int_{\mathbf{R}} p_{b,t}^{\mathbf{R}}((0, y), (a, y))\, dy. \end{aligned} \tag{5.3}$$

The following result was established in [B11d, Eq. (10.6.13)].

**Proposition 5.1.** *The following identity holds:*

$$\operatorname{Tr}^a\left[\exp\left(-tM_b^{\mathbf{R}}\right)\right] = \left(1 - e^{-t/b^2}\right)^{-1} \operatorname{Tr}^a\left[\exp\left(t\Delta^{\mathbf{R}}/2\right)\right]. \tag{5.4}$$

*Proof.* We give a formal proof which can be ultimately easily justified. Note that

$$M_b^{\mathbf{R}} = \frac{1}{2b^2}\left(-\frac{\partial^2}{\partial y^2} + \left(y - b\frac{\partial}{\partial x}\right)^2 - 1\right) - \frac{1}{2}\frac{\partial^2}{\partial x^2}. \tag{5.5}$$

By (5.5), we get

$$\exp\left(b\frac{\partial^2}{\partial x \partial y}\right) M_b^{\mathbf{R}} \exp\left(-b\frac{\partial^2}{\partial x \partial y}\right) = \frac{1}{2b^2}\left(-\frac{\partial^2}{\partial y^2} + y^2 - 1\right) - \frac{1}{2}\frac{\partial^2}{\partial x^2}. \tag{5.6}$$

Using the fact that $p^{\mathbf{R}}_{b,t}\left((x,Y),(x',Y')\right)$ only depends on $x'-x,y,y'$, we deduce from (5.6) that

$$\operatorname{Tr}^a\left[\exp\left(-tM^{\mathbf{R}}_b\right)\right]=\operatorname{Tr}\left[\exp\left(-\frac{t}{b^2}\mathcal{H}^{\mathbf{R}}\right)\right]\operatorname{Tr}^a\left[\exp\left(t\Delta^{\mathbf{R}}/2\right)\right]. \tag{5.7}$$

Since the spectrum of the harmonic oscillator $\mathcal{H}^{\mathbf{R}}$ in (1.44) is just $\mathbf{N}$, by (5.7), we get (5.4). □

*Remark* 5.2. As explained in [B11d, Subsection 10.5], equation (5.6) should be thought of as a version of Egorov's theorem, the main difference being that the conjugating operator $\exp\left(b\frac{\partial^2}{\partial x\partial y}\right)$ is not well defined, because the operator $\frac{\partial^2}{\partial x\partial y}$ is hyperbolic. As was shown in [B11d, Chapter 10], while Egorov's theorem is related to the quantization of symplectic transformations, here the conjugating operator $\exp\left(b\frac{\partial^2}{\partial x\partial y}\right)$ quantizes formally an imaginary symplectic transformation.

The above has nothing to do with the idea of a Wick rotation. In the Wick rotation, a well-defined Schrödinger operator is changed into a well-defined heat operator, by complexifying the time variable. Here one could consider that if $b$ was changed into $ib$, the conjugating operator would be well defined, and would quantize a real symplectic transformation. However, the operator $M^{\mathbf{R}}_{ib}$ is no longer hypoelliptic, and its conjugate by the conjugation in (5.6) is now hyperbolic! The power of equation (5.6) is that because it does not make literal sense, and still its consequences are true, it can lead to exotic results, which cannot be anticipated by traditional physical considerations.

The role of the operator $y\frac{\partial}{\partial x}$ in $M^{\mathbf{R}}_b$ is to implement a version of Fourier transform. Indeed, we proceed by analogy, by simply giving the main steps in the computation of the Fourier transform of the Gaussian distribution, while introducing also the parameter $b>0$. We get

$$\begin{aligned}\int_{\mathbf{R}}&\exp\left(iy\xi/b-y^2/2b^2\right)\frac{dy}{b\sqrt{2\pi}}\\&=\int_{\mathbf{R}}\exp\left(-\left(y-ib\xi\right)^2/2b^2-\xi^2/2\right)\frac{dy}{b\sqrt{2\pi}}\\&=\int_{\mathbf{R}}\exp\left(-y^2/2b^2-\xi^2/2\right)\frac{dy}{b\sqrt{2\pi}}=\exp\left(-\xi^2/2\right).\end{aligned} \tag{5.8}$$

The analyticity of the function $\exp\left(-y^2/2\right)$ plays a key role in (5.8). The various identities in (5.8) should be compared with (5.5), (5.6). One should also remember that $\exp\left(-y^2/2\right)$ is the ground state of the harmonic oscillator $\mathcal{H}^{\mathbf{R}}$.

Now we introduce the exterior algebra $\Lambda^{\cdot}\left(\mathbf{R}^*\right)$ and its corresponding number operator $N^{\Lambda^{\cdot}(\mathbf{R}^*)}$. Set

$$N^{\mathbf{R}}_b=M^{\mathbf{R}}_b+\frac{N^{\Lambda^{\cdot}(\mathbf{R}^*)}}{b^2}. \tag{5.9}$$

Let $q_{b,t}^{\mathbf{R}}((x,y),(x',y'))$ be the smooth kernel associated with $\exp\left(-tN_b^{\mathbf{R}}\right)$. Then

$$q_{b,t}^{\mathbf{R}}((x,y),(x',y')) = p_{b,t}^{\mathbf{R}}((x,y),(x',y'))\exp\left(-tN^{\Lambda^{\cdot}(\mathbf{R}^*)}/b^2\right). \tag{5.10}$$

We define $\mathrm{Tr_s}^a\left[\exp\left(-tN_b^{\mathbf{R}}\right)\right]$ by a formula similar to (5.3).

**Theorem 5.3.** *For any $a \in \mathbf{R}, b > 0, t > 0$,*

$$\mathrm{Tr}^a\left[\exp\left(t\Delta^{\mathbf{R}}/2\right)\right] = \mathrm{Tr_s}^a\left[\exp\left(-tN_b^{\mathbf{R}}\right)\right]. \tag{5.11}$$

*Proof.* This is an obvious consequence of Proposition 5.1. □

Since (5.11) is valid for any $a \in \mathbf{R}$, we will rewrite (5.11) in the form

$$\exp\left(t\Delta^{\mathbf{R}}/2\right) = \mathrm{Tr_s}\left[\exp\left(-tN_b^{\mathbf{R}}\right)\right]. \tag{5.12}$$

Equation (5.12) is an equality of operators acting on smooth functions on $\mathbf{R}$. For a detailed analysis of (5.11), (5.12), we refer to [B11d, Chapter 10].

The first striking fact about equation (5.12) is that the right-hand side does not depend on $b > 0$. The second fact is that when $b \to 0$, the right-hand side converges obviously to the left-hand side. Indeed note that

$$N_b^{\mathbf{R}} = \frac{1}{b^2}\left(\mathcal{H}^{\mathbf{R}} + N^{\Lambda^{\cdot}(\mathbf{R}^*)}\right) - \frac{1}{b}y\frac{\partial}{\partial x}. \tag{5.13}$$

As we saw in Subsection 1.6, the kernel of $\mathcal{H}^{\mathbf{R}} + N^{\Lambda^{\cdot}(\mathbf{R}^*)}$ is one-dimensional and spanned by $\pi^{-1/4}\exp\left(-y^2/2\right)$. Also multiplication by $y$ maps this kernel in its $L_2$ orthogonal. It follows that when splitting the $L_2$ space of $\mathbf{R}^2$ into the kernel of $\mathcal{H}^{\mathbf{R}} + N^{\Lambda^{\cdot}(\mathbf{R}^*)}$ and its orthogonal, the operator $N_b^{\mathbf{R}}$ has the preferred matrix structure

$$N_b^{\mathbf{R}} = \begin{bmatrix} 0 & -\frac{y}{b}\frac{\partial}{\partial x} \\ -\frac{y}{b}\frac{\partial}{\partial x} & \frac{\mathcal{H}}{b^2} - \frac{1}{b}y\frac{\partial}{\partial x} \end{bmatrix}. \tag{5.14}$$

Now we deal with this matrix as if it were a finite-dimensional matrix. As explained in [B05, Theorem 3.14], [BL08, Section 17.1], and [B08c, Section 3.7], if we consider (5.14) as a finite-dimensional matrix, one finds that if $P$ is orthogonal projection operator on $\ker\left(\mathcal{H}^{\mathbf{R}} + N^{\Lambda^{\cdot}(\mathbf{R}^*)}\right)$, then

$$\exp\left(-tN_b^{\mathbf{R}}\right) \to P\exp\left(t\Delta^{\mathbf{R}}/2\right)P. \tag{5.15}$$

In this case, equation (5.15) can be easily justified by explicit computations.

We define $K_b$ as in equation (2.30). Note that

$$K_bN_b^{\mathbf{R}}K_b^{-1} = \frac{y^2}{2} - y\frac{\partial}{\partial x} + \frac{1}{b^2}\left(-\frac{1}{2b^2}\frac{\partial^2}{\partial y^2} - 1 + N^{\Lambda^{\cdot}(\mathbf{R}^*)}\right). \tag{5.16}$$

By (5.16), as $b \to +\infty$, the heat kernel for the operator in (5.16) propagates more and more along the geodesic flow generated by the vector field $y\frac{\partial}{\partial x}$. As explained in [B05, Section 3.10], from (5.12), one can obtain this way the explicit formula for $p_t^{\mathbf{R}}(x,x')$ in (5.2). In other words, the right-hand side of (5.12) interpolates between the operator $\exp\left(t\Delta^{\mathbf{R}}/2\right)$ for $b \to 0$, and its local expression as a heat kernel for $b \to +\infty$.

Let us now replace $\mathbf{R}$ by $S^1$. The same argument as in (5.6) shows that

$$\exp\left(b\frac{\partial^2}{\partial x\partial y}\right)N_b^{S^1}\exp\left(-b\frac{\partial^2}{\partial x\partial y}\right)=\frac{1}{b^2}\left(\mathcal{H}^{\mathbf{R}}+N^{\Lambda^{\cdot}(\mathbf{R}^*)}\right)-\frac{1}{2}\Delta^{S^1}. \tag{5.17}$$

As explained in [B08c, Section 1.2], from (5.17), one can deduce that the operator $N_b^{S^1}$ is isospectral to the operator in the right-hand side of (5.17). This can be done first by using Fourier transform in the variable $x$, and also by noting that the eigenfunctions of $\mathcal{H}^{\mathbf{R}}$ are analytic in the variable $y$. Ultimately the operator $N_b^{S^1}$ can be explicitly diagonalized, and its eigenfunctions form a complete set in the corresponding $L_2$ space.

The consequence is that

$$\operatorname{Sp}N_b^{S^1}=\left\{2\pi^2k^2\right\}_{k\in\mathbf{Z}}+\frac{\mathbf{N}}{b^2}. \tag{5.18}$$

The remarkable feature of (5.18) is that $\left\{2\pi^2k^2\right\}_{k\in\mathbf{Z}}$ remains fixed in $\operatorname{Sp}N_b^{S^1}$ while $\frac{\mathbf{N}}{b^2}$ moves around, essentially disappearing as $b\to 0$, except for 0, and accumulating near 0 as $b\to+\infty$.

From the above, we find that instead of (5.11), we now have

$$\operatorname{Tr}\left[\exp\left(t\Delta^{S^1}/2\right)\right]=\operatorname{Tr_s}\left[\exp\left(-tN_b^{S^1}\right)\right]. \tag{5.19}$$

Equation (5.19) is an equality of genuine traces or supertraces. Of course (5.19) is weaker than (5.11). If we make $b\to 0$ in (5.19), the right-hand side becomes the left-hand side. Making $b\to+\infty$ localizes the heat kernel along the closed geodesics in $S^1$, and ultimately expresses the trace of the heat kernel in terms of a Poisson summation formula.

We will now compare the considerations of Subsection 2.3 with the ones we just made. Indeed we can think of $g=\exp\left(t\Delta^{S^1}/2\right)$ as an element of a group. Let $L_2^{S_1}$ be the $L_2$ space of $S^1$. Tautologically,

$$\operatorname{Tr}\left[\exp\left(t\Delta^{S^1}/2\right)\right]=\operatorname{Tr}^{L_2^{S^1}}[g]. \tag{5.20}$$

Now think of $L_2^{S^1}$ as the cohomology of some complex, so that (5.20) is just the computation of a Lefschetz number. Formula (5.19) begs to be compared with the McKean-Singer formula in (2.44) for a Lefschetz number. The fact that the right-hand side does not depend on $b>0$ is just the analogue of the right-hand side of (2.44) not depending on $t>0$. When comparing with (2.44), it is natural to take $t=1/b^2$, so that making $b\to 0$ is like making $t\to+\infty$ in (2.44), and making $b\to+\infty$ is like making $t\to 0$ in (2.44). Making $b\to 0$ localizes $\operatorname{Tr_s}\left[\exp\left(-tN_b^{S^1}\right)\right]$ on $\ker\mathcal{H}^{\mathbf{R}}\simeq L_2^{S^1}$. We saw that making $b\to+\infty$ localizes the right-hand side of (5.19) near the closed geodesics, which can be thought of as the fixed points of the return map for the geodesic flow in $S^1\times\mathbf{R}$.

From this point of view, $L_2^{S_1}$ should be identified with $\ker D_+^X$, and $t\Delta^{S^1}/2$ with $-L_A^X$. The spectral formula in (5.18) should be viewed as the analogue of the computation of the spectrum of $\operatorname{Sp}\left(L_A^X + tD^{X,2}\right)$ which also depends on $t > 0$, but is such that the supertrace of the associated heat kernel does not depend on $t > 0$. A minor discrepancy is that in (2.44), $L_A^X$ contributes to the spectrum by imaginary eigenvalues, while the contribution of $-t\Delta^{S^1}/2$ is real. This is because $g = \exp\left(t\Delta^{S^1}/2\right)$ is not unitary but self-adjoint.

Ultimately, we hope to have shown that equation (5.19) has all the features of a McKean-Singer formula for a Lefschetz number, except that we have not produced the associated Dirac operator. Let us just do that.

Let $d_x, d_y$ be the de Rham operators acting on $C^\infty\left(\mathbf{R}, \Lambda^{\cdot}\left(\mathbf{R}^*\right)\right)$, the first operator acting on the variable $x \in \mathbf{R}$, the second on the variable $y \in \mathbf{R}$, the exterior algebra being the same algebra in both cases. Let $d_x^*, d_y^*$ be the formal adjoints of $d_x, d_y$. The operators $y\wedge, i_y$ also act on $\Lambda^{\cdot}\left(\mathbf{R}^*\right)$.

Set

$$\widehat{D}^{\mathbf{R}} = d_x - d_x^*. \tag{5.21}$$

Then

$$\widehat{D}^{\mathbf{R},2} = \Delta^{\mathbf{R}}. \tag{5.22}$$

For $b > 0$, put

$$\mathfrak{D}_b^{\mathbf{R}} = \frac{1}{\sqrt{2}}\left(-\widehat{D}^{\mathbf{R}} + \frac{1}{b}\left(d_y + y \wedge + d_y^* + i_y\right)\right). \tag{5.23}$$

An easy computation shows that

$$N_b^{\mathbf{R}} = \mathfrak{D}_b^{\mathbf{R},2} - \frac{1}{2}\Delta^{\mathbf{R}}. \tag{5.24}$$

By (5.23),

$$\left[\mathfrak{D}_b^{\mathbf{R}}, \Delta^{\mathbf{R}}\right] = 0, \qquad \left[\mathfrak{D}_b^{\mathbf{R}}, N_b^{\mathbf{R}}\right] = 0. \tag{5.25}$$

Then equation (5.12) can be written in the form

$$\exp\left(t\Delta^{\mathbf{R}}/2\right) = \operatorname{Tr_s}\left[\exp\left(\frac{t}{2}\Delta^{\mathbf{R}} - t\mathfrak{D}_b^{\mathbf{R},2}\right)\right]. \tag{5.26}$$

Similarly, if we replace $\mathbf{R}$ by $S_1$, we can still define the operator $\mathfrak{D}_b^{S_1}$ by a formula like (5.23). Equation (5.19) is just the identity

$$\operatorname{Tr}\left[\exp\left(t\Delta^{S_1}/2\right)\right] = \operatorname{Tr_s}\left[\exp\left(\frac{t}{2}\Delta^{S_1} - t\mathfrak{D}_b^{S_1,2}\right)\right]. \tag{5.27}$$

A direct proof of the fact that the right-hand sides of (5.12), (5.13) do not depend on $b > 0$ can be given, which is based on (5.26), (5.27), and on classical arguments of index theory. In the case of $\mathbf{R}$, we get the chain of equalities,

$$\begin{aligned}\frac{\partial}{\partial b}\mathrm{Tr_s}&\left[\exp\left(\frac{t}{2}\Delta^{\mathbf{R}}-t\mathfrak{D}_b^{\mathbf{R},2}\right)\right]\\&=-t\mathrm{Tr_s}\left[\left[\mathfrak{D}_b^{\mathbf{R}},\frac{\partial}{\partial b}\mathfrak{D}_b^{\mathbf{R}}\right]\exp\left(\frac{t}{2}\Delta^{\mathbf{R}}-t\mathfrak{D}_b^{\mathbf{R},2}\right)\right]\\&=-t\mathrm{Tr_s}\left[\left[\mathfrak{D}_b^{\mathbf{R}},\frac{\partial}{\partial b}\mathfrak{D}_b^{\mathbf{R}}\exp\left(\frac{t}{2}\Delta^{\mathbf{R}}-t\mathfrak{D}_b^{\mathbf{R},2}\right)\right]\right]=0.\end{aligned} \tag{5.28}$$

In (5.28), we have used the first identity in (5.25), and also the fact that supertraces vanish on supercommutators [Q85].

In [B11d], we have shown that if $G$ is a reductive group, if $K$ is a maximal compact subgroup, if $X=G/K$ is the corresponding symmetric space, there is an analogue of Theorem 5.3, which becomes an equality of semisimple orbital integrals. It is obtained by extending the construction of the operator $\mathfrak{D}_b^{\mathbf{R}}$, with $\widehat{D}^{\mathbf{R}}$ replaced by the Dirac operator of Kostant [K97] which is associated with $G$. Making $b\to+\infty$ then leads to an explicit local evaluation of these orbital integrals. The analogue of (5.19) produces a version of the Selberg trace formula. We refer to [B11d] for more details.

### 5.2. The hypoelliptic heat kernel and the action functional

We use here the notation of Subsections 2.4 and 3.3. Also $n$ denotes the dimension of $X$.

For $t>0, b\geq 0$, if $s\in[0,t]\to x_s\in X$ be a smooth path, set

$$\begin{aligned}E_t^X(x)&=\frac{1}{2}\int_0^t|\dot{x}|^2\,ds,\\H_{b,t}^X(x)&=\frac{1}{2}\int_0^t\left(|\dot{x}|^2+b^4|\ddot{x}|^2\right)ds,\\K_{b,t}^X(x)&=\frac{1}{2}\int_0^t\left|b^2\ddot{x}+\dot{x}\right|^2 ds.\end{aligned} \tag{5.29}$$

Then

$$H_{0,t}^X=K_{0,t}^X(x)=E_t^X,\qquad K_{b,t}(x)=H_{b,t}(x)+\frac{b^2}{2}\left(|\dot{x}_t|^2-|\dot{x}_0|^2\right). \tag{5.30}$$

Note that the map $x_\cdot\to x_{t\cdot}$ maps paths parametrized by $[0,t]$ to paths parametrized by $[0,1]$.

For $t>0$, let $L_tX$ be the set of smooth maps $s\in\mathbf{R}/t\mathbf{Z}\to X$. For $t=1$, $L_tX$ is just the space $LX$ defined in Subsection 2.4. Moreover, there is a canonical identification $L_tX\simeq LX$ via the map $x_\cdot\in L_tX\to x_{t\cdot}\in LX$. Via this map, $E_t^X$ corresponds to $E^X/t$, and $H_{b,t}^X$ is precisely the functional in (3.17). If $x,x'\in X$, we denote by $L_{t,x,x'}X$ the set of $x_\cdot\in L_tX$ such that $x_0=x, x_t=x'$.

Assume that $X$ is compact. Let $d$ be the distance function on $X$. Recall that for $t>0$, $p_t^X(x,x')$ is the smooth kernel associated with the operator $\exp\left(t\Delta^X/2\right)$.

For $M > 0$, for $t \in ]0, M]$, we have the uniform estimate,

$$p_t^X(x, x') \leq \frac{C_M}{t^{n/2}} \exp\left(-d^2(x, x')/2t\right), \tag{5.31}$$

The theory of Brownian motion shows that there is a canonical positive measure $P^X_{t,x,x'}$ on $L^0_{t,x,x'}X$, which is the obvious set of continuous paths, such that

$$p_t^X(x, x') = \int_{L^0_{t,x,x'}X} dP^X_{t,x,x'}, \tag{5.32}$$

Also $P^X_{t,x,x'}$ has the formal representation

$$P^X_{t,x,x'} = \exp\left(-E^X_t\right)\mathcal{D}x. \tag{5.33}$$

The content of (5.33) is that the Brownian motion $x.$ verifies the stochastic differential equation

$$\dot{x} = \dot{w}, \tag{5.34}$$

where $w.$ is a standard Brownian motion in $TX$ which is transported along $x.$ using the Levi-Civita connection. Equation (2.55) can be viewed as a consequence of (5.32), (5.33). Also equation (5.31) can be proved to be a consequence of (5.32).

Given $b > 0$, $z = (x, Y)$, $z' = (x', Y') \in \mathcal{X}$, let $L_{t,z,z'}X$ be the set of smooth maps $s \in [0, t] \to x_s \in X$ such that $(x_0, \dot{x}_0) = z, (x_t, \dot{x}_t) = z'$. Recall that $p^X_{b,t}(z, z')$ is the smooth kernel on $\mathcal{X}$ which is associated with $\exp\left(-tM^X_b\right)$. Let $L^1_{t,z,z'}$ be the obvious $C^1$ loop space in $X$. There is a canonical positive measure $P^X_{b,t,z,z'}$ on $L^1_{t,z,z'}X$ such that

$$b^n p^X_{b,t}((x, bY), (x', bY')) = \int_{L^1_{t,z,z'}X} dP^X_{b,t,z,z'}. \tag{5.35}$$

Also we have the formal representation

$$P^X_{b,t,z,z'} = \exp\left(-H^X_{b,t}\right)\mathcal{D}x. \tag{5.36}$$

Set

$$M^{X\prime}_b = \exp\left(|Y|^2/2\right) M^X_b \exp\left(-|Y|^2/2\right). \tag{5.37}$$

Then

$$M^{X\prime}_b = \frac{1}{2b^2}\left(-\Delta^V + 2\nabla^V_{\widehat{Y}}\right) - \frac{1}{b}\nabla^X_Y. \tag{5.38}$$

Let $p^{X\prime}_{b,t}(z, z')$ be the smooth kernel associated with $\exp\left(-tM^{X\prime}_{b,t}\right)$. Then

$$p^{X\prime}_{b,t}(z, z') = \exp\left(|Y|^2/2\right) p^X_{b,t}(z, z') \exp\left(-|Y'|^2/2\right). \tag{5.39}$$

There is a canonical positive measure $P^{X\prime}_{b,t,z,z'}$ on $L^0_{t,z,z'}X$ such that

$$b^n p^{X\prime}_{b,t}((x, bY), (x', bY')) = \int_{L^0_{t,z,z'}X} dP^{X\prime}_{b,t,z,z'}. \tag{5.40}$$

Also the measure $P^{X\prime}_{b,t,z,z'}$ can be formally represented in the form

$$P^{X\prime}_{b,t,z,z'} = \exp\left(-K^X_{b,t}\right)\mathcal{D}x. \tag{5.41}$$

By (5.30), (5.36), and (5.41), we obtain

$$P^{X\prime}_{b,t,z,z'} = \exp\left(\frac{b^2}{2}\left(|Y|^2 - |Y'|^2\right)\right)P^X_{b,t,z,z'}. \tag{5.42}$$

Equation (5.39) is compatible with (5.35), (5.40), and (5.42).

If $z_\cdot = (x_\cdot, Y_\cdot)$ is the stochastic process associated with the heat semigroup $\exp\left(-tM^{X\prime}_b\right)$, an elementary application of the Itô calculus shows that it verifies the stochastic differential equation

$$\dot{x} = \frac{Y}{b}, \qquad \dot{Y} = -\frac{Y}{b^2} + \frac{\dot{w}}{b}. \tag{5.43}$$

By (5.43), we get

$$b^2\ddot{x} = -\dot{x} + \dot{w}. \tag{5.44}$$

The formal content of (5.40), (5.41) is just (5.44). Also (3.20) is a consequence of (5.35), (5.36).

If we make $b = 0$ in (5.44), we recover (5.34). We already warned the reader against believing that making $b \to 0$ in (5.44) is innocuous.

Given $z, z' \in \mathcal{X}$, let $H^X_{b,t}(z, z')$ be the minimum of $H^X_{b,t}(x)$ over the paths $x_\cdot \in L_{t,z,z'}X$. Using large deviations, $H^X_{b,t}((x, Y/b), (x', Y'/b))$ should play for $p^X_{b,t}(z, z')$ the same role as $d^2/2$ for $p^X_t(x, x')$. This question has been studied by Lebeau [L05] on a compact manifold $X$ for fixed $b > 0$.

An elementary observation is that the function $H^X_{b,t}$ is not symmetric. More precisely,

$$H^X_{b,t}((x', Y'), (x, Y)) = H^X_{b,t}((x, -Y), (x', -Y')). \tag{5.45}$$

Moreover, the function $H^X_{b,t}$ does not vanish on the diagonal. This makes that contrary to the function $d^2/2$, the function $H^X_{b,t}$ is very far from defining a distance over $\mathcal{X}$. It is because it is as far as possible from a distance on $\mathcal{X}$ that it will be of any use, geometrically and analytically. This makes the operator $M^X_b$ fundamentally different from more classical hypoelliptic operators like the hypoelliptic Laplacian on the Heisenberg group. On such a space, the action functional still defines a Carnot-Carathéodory distance.

If $X$ is a locally symmetric space of non compact type, for a given $t > 0$, we have the global estimate,

$$p^X_t(x, x') \le c\exp\left(-Cd^2(x, x')\right). \tag{5.46}$$

One way of proving (5.31) is to use finite propagation speed for the wave equation on $X$. Also equations (5.32)–(5.34) are still valid.

When $X$ is a symmetric space, estimates similar to (5.46) have been established in [B11d, Chapter 14] for the hypoelliptic heat kernel $p^X_{b,t}$. Let us briefly explain these estimates.

First of all, assume temporarily that $X = \mathbf{R}^n$. An elementary result established in [B11d, Eq. (10.3.48)] says that as $b \to 0$,

$$\begin{aligned} H_{b,t}^{\mathbf{R}^n}\left(\left(x, Y/b\right), \left(x', Y'/b\right)\right) &\to \underline{H}_{0,t}^{X}\left(\left(x, Y\right), \left(x', Y'\right)\right) \\ &= \frac{1}{2t}\left|x'-x\right|^2 + \frac{1}{2}\left(\left|Y\right|^2 + \left|Y'\right|^2\right). \end{aligned} \tag{5.47}$$

The proof of (5.47) is completely elementary. In spite of the fact that as $b \to 0$, $H_{b,t}^{\mathbf{R}^n} \to E_t^{\mathbf{R}^n}$, still the corresponding minimum values do not converge, the defect being $\frac{1}{2}\left(\left|Y\right|^2 + \left|Y'\right|^2\right)$. A simple result established in [B11d, Proposition 10.3.3] is that given $0 < \epsilon < M < +\infty$, there exists $C > 0$ such that for $0 \le b \le M, \epsilon \le t \le M$,

$$H_{b,t}^{\mathbf{R}^n}\left(\left(x, Y/b\right), \left(x', Y'/b\right)\right) \ge C\left(\left|x'-x\right|^2 + \left|Y\right|^2 + \left|Y'\right|^2\right). \tag{5.48}$$

Equations (5.47), (5.48) show how far the function $H_{b,t}^{\mathbf{R}^n}\left(\left(x, Y/b\right), \left(x', Y'/b\right)\right)$ is from a distance on $\mathcal{X}$. With the exception of the defect $\frac{1}{2}\left(\left|Y\right|^2 + \left|Y'\right|^2\right)$, as $b \to 0$, it collapses to the normalized square of the distance on $X$.

No attempt has been made in [B11d] to establish similar results on the functional $H_{b,t}^X$ when $X$ is instead a Riemannian manifold. However, it has been shown in [B11d, Theorem 13.2.4] that when $X$ is a symmetric space of noncompact type, given $M > 0$, there exist $C > 0, C' > 0$ such that for $b \in ]0, M], \epsilon \le t \le M$,

$$p_{b,t}^X\left(\left(x, Y\right), \left(x', Y'\right)\right) \le C \exp\left(-C'\left(d^2\left(x, x'\right) + \left|Y\right|^2 + \left|Y'\right|^2\right)\right), \tag{5.49}$$

Proving (5.49) for a given $b > 0$ is rather easy. The real challenge is to prove uniformity in $b \in ]0, M]$.

In the case where $X$ is a compact manifold, it was shown in Bismut-Lebeau [BL08, Section 3.4] that as $b \to 0$,

$$\begin{aligned} p_{b,t}^X\left(\left(x, Y\right), \left(x', Y'\right)\right) &\to p_{0,t}^X\left(\left(x, Y\right), \left(x', Y'\right)\right) \\ &= p_t^X\left(x, x'\right)\pi^{-n/2}\exp\left(-\frac{1}{2}\left(\left|Y\right|^2 + \left|Y'\right|^2\right)\right). \end{aligned} \tag{5.50}$$

When $X$ is a symmetric space, this result was established in [B11d, Theorem 12.8.1]. Observe that in the last exponential, what we called the defect in the right-hand side of (5.47) appears. This is no accident.

The proof of the above results is difficult. In the case where $X$ is compact, the proof of (5.50) in [BL08] uses pseudodifferential operators. When $X$ is a symmetric space, the arguments in [B11d] are mostly probabilistic. Finite propagation speed arguments cannot be used on $M_{b,t}^X$, because this operator does not have a wave equation. The proof in [B11d] uses instead (5.47), (5.48), the Malliavin calculus [M78], and also the wave-like character of the hypoelliptic Laplacian, which we will review next.

### 5.3. The hypoelliptic Laplacian and the wave equation

Let us briefly consider the parabolic heat operator

$$P=\frac{\partial}{\partial t}-\Delta^X, \tag{5.51}$$

and also, for $b>0$, its hyperbolic deformation

$$P_b=b^2\frac{\partial^2}{\partial t^2}+\frac{\partial}{\partial t}-\Delta^X. \tag{5.52}$$

In the naive sense, $P_b$ deforms $P$. Also the operator $P_b$ has finite propagation speed $1/b$. As $b\to 0$, the propagation speed tends to $+\infty$, which is compatible with the fact that the heat equation has infinite propagation speed.

Consider the stochastic differential equations in (5.34), (5.44). Let us rewrite (5.44) by a simple change of notation

$$\left(b^2\frac{D^2}{Dt^2}+\frac{D}{Dt}\right)x-\dot{w}=0. \tag{5.53}$$

In (5.53), $\frac{D}{Dt}$ denotes covariant differentiation. The similarity between equation (5.52) for the operator $P_b$ and the dynamical equation (5.53) is impossible to miss. What the formal similarity betrays is a hidden wave-like character in the operator $M_b^{X\prime}$.

Indeed observe that contrary to Brownian motion in (5.34), the solution of the second-order differential equation (5.44), (5.53) has finite speed. More precisely $\dot{x}_\cdot$ is a continuous process. This means that on a given time interval, $x_\cdot$ cannot escape too much at infinity. However, this argument is misleading, since while Brownian motion in (5.53) has infinite speed, it does not escape at infinity either. The wave-like character of the hypoelliptic Laplacian is subtler than that.

Let us give an operator theoretic version of (5.53). Let $f:X\to\mathbf{R}$ be a smooth function. Then $f$ lifts to a smooth function $\mathcal{X}\to\mathbf{R}$. One verifies easily that

$$\left(b^2M_b^{X\prime,2}-M_b^{X\prime}\right)f=\nabla_Y^{TX}\nabla_Y f. \tag{5.54}$$

For $t>0$, put

$$S_t=\exp\left(-tM_b^{X\prime}\right). \tag{5.55}$$

If $f(x)$ is a smooth bounded function on $X$, by (5.54), we get

$$\left(b^2\frac{\partial^2}{\partial t^2}+\frac{\partial}{\partial t}\right)S_tf=S_t\nabla_Y^{TX}\nabla_Y f. \tag{5.56}$$

Equation (5.56) is just the analytic counterpart to (5.53). Set

$$U_t\left((x,Y),x'\right)=\int_{T_{x'}X}S_t\left((x,Y),(x',Y')\right)dY'. \tag{5.57}$$

By (5.56), as a function of $x'$, $U_t$ verifies a nonautonomous wave equation on $X$.

These considerations are developed in [B11d, Chapter 12]. They play an important role in establishing the estimate (5.49).

Also observe the first-order Hamiltonian differential equation for the geodesic flow on $\mathcal{X}$,

$$\dot{x} = Y, \qquad \dot{Y} = 0, \tag{5.58}$$

projects on $X$ to the second-order differential equation

$$\ddot{x} = 0. \tag{5.59}$$

That the parabolic equation associated with the heat flow for $M_b^{X\prime}$ projects approximately to a wave-like equation on $X$ should be viewed as a sort of quantization of the Hamiltonian-Lagrangian correspondence. The fact that for $b \to 0$, $M_b^{X\prime}$ collapses in the proper way to the classical $-\frac{1}{2}\Delta^X$, which is the genuine quantization of the geodesic flow, makes this intermediate quantization especially relevant.

## Index

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