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Toeplitz Operators, Analytic Torsion, and the Hypoelliptic Laplacian

À la mémoire de Louis Boutet de Monvel

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Abstract. The purpose of this paper is to explain how Toeplitz operators can be used in studying asymptotic torsion, and also in the theory of the hypoelliptic Laplacian. The role of the hypoelliptic Laplacian in the explicit computation of orbital integrals will be described. The geodesic flow will be viewed as implementing a dynamical version of Fourier transform.

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1. Introduction

The purpose of this paper is to review some applications of the theory of Toeplitz operators to questions connected with analytic torsion, and also from the point of view of the theory of the hypoelliptic Laplacian.

In the first two sections of the paper, we describe the results of [14] on the asymptotics of analytic torsion, when a class of corresponding flat vector bundles tends to infinity in the proper sense. The next two sections are devoted to the hypoelliptic Laplacian as a scalar operator, to the hypoelliptic Laplacian in Hodge theory, to hypoelliptic torsion, and to applications of the hypoelliptic Laplacian to Selberg's trace formula. In the last section, we connect the hypoelliptic Laplacian to a geometric version of Fourier transform, and to the wave equation.

The paper is written in an informal way. We only sketch the proofs, referring to the original papers when necessary.

We will now describe in more detail the content of this paper, and we will give the proper historical perspective to the results described in the paper.

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1.1. ANALYTIC TORSION AND COMBINATORIAL TORSION

Given a compact odd dimensional manifold X, and a flat unimodular vector bundle F, combinatorial torsion or Reidemeister torsion [37] is an invariant which can be computed from the combinatorial complex associated with a triangulation. Given metrics on TX and F, the Ray–Singer analytic torsion [38] is an invariant obtained via the spectrum of the corresponding Hodge Laplacian. This invariant does not depend on the choice of metrics.

A conjecture of Ray and Singer asserts that the two above invariants are equal. This conjecture was proved by Cheeger [20] and Müller [32,33]. A different proof, based on the Witten deformation, was given by Bismut and Zhang [15,16].

1.2. ASYMPTOTIC TORSION AND TOEPLITZ OPERATORS

As long as the flat vector bundle F is algebraic, Reidemeister torsion detects important information on the order of the torsion in the cohomology groups $H^{\cdot}(X,F)$. This is especially true when $H^{\cdot}(X,F)\otimes\mathbf{C}=0$, in which case Reidemeister torsion is the alternate product of the order of the torsion groups in $H^{\cdot}(X,F)$.

Bergeron and Venkatesh [2] initiated the study of the asymptotic torsion of locally symmetric spaces under finite coverings, in order to obtain asymptotic information of the order of torsion subgroups. They used the trace formula to obtain the proper control of analytic torsion, while a similar control on combinatorial torsion would have been difficult.

In [34], Müller developed the theory in another direction. In the case of hyperbolic 3-folds equipped with a natural flat vector bundle F, Müller [34] studied instead the asymptotics of the analytic torsion for the symmetric powers S^pF as $p\to+\infty$.

With Ma and Zhang, [14], we extended Müller's work in several directions. We considered the case of arbitrary compact manifolds, equipped with a flat fibration by compact complex manifolds equipped with a positive holomorphic line bundle L. In this case, F_p is taken to be the holomorphic direct image of L^p. We studied the behaviour of the analytic torsion associated with F_p. Toeplitz operators along the fibres play an important role in estimating certain 0-order terms that appear in the corresponding Weitzenböck formula. In Section 3, this point of view will be reviewed in some detail.

In [35], Müller and Pfaff have also studied the asymptotics of the analytic torsion of hyperbolic spaces using flat bundles associated with rays of representation of the underlying reductive group, a case also considered in [14].

1.3. THE HYPOELLIPTIC LAPLACIAN

On a Riemannian manifold, in its simplest form, the hypoelliptic Laplacian is a family of hypoelliptic operators $L^X_b|_{b>0}$, acting on the total space $\mathcal{X}$ of the tangent bundle TX, that interpolates in the proper sense between $-\Delta^X/2$ and $-Z$, where Z is the generator of the geodesic flow. The theory has been first developed in the context of Hodge theory [6]. With Lebeau [13], we showed that the elliptic and hypoelliptic torsions coincide.

In the present paper, we will emphasize the connections of this theory with Toeplitz operators. Also, we will review various results on the hypoelliptic Laplacian in Hodge theory.

1.4. HYPOELLIPTIC LAPLACIAN AND THE TRACE FORMULA

In [9], we have constructed another hypoelliptic Laplacian associated with symmetric spaces and their compact quotients. We gave an explicit local formula for the semisimple orbital integrals associated with the heat kernel, thus recovering Selberg's trace formula by interpolation. A remarkable property of this hypoelliptic Laplacian is that the spectrum of the original elliptic Laplacian remains rigidly embedded in the spectrum of its hypoelliptic deformation.

In the present paper, we will review some significant aspects of the construction of the hypoelliptic Laplacian on symmetric spaces. We will also explain the main geometric formula in [9] for the semisimple orbital integrals of elliptic heat kernels.

1.5. GEODESIC FLOW AND THE FOURIER TRANSFORM

In the last section of the present paper, we show that the geodesic flow implements a geometric form of Fourier transform. We will also emphasize the connections between the hypoelliptic Laplacian and the wave equation.

1.6. THE ORGANISATION OF THE PAPER

This paper is organized as follows. In Section 2, we give the main properties of analytic torsion and of combinatorial torsion.

In Section 3, we describe results that we obtained with Ma and Zhang on asymptotic torsion.

In Section 4, we introduce the hypoelliptic Laplacian, and we describe its main properties.

In Section 5, we consider the hypoelliptic Laplacian in relation with Selberg's trace formula.

Finally, in Section 6, in the context of the theory of the hypoelliptic Laplacian, we discuss the geodesic flow and its connections with Fourier transform, and also the relation of the hypoelliptic Laplacian to the wave equation.

We have excluded from the present paper connections of the hypoelliptic Laplacian with probability theory. These connections have been explained at length in [4,7,11].

2. Analytic Torsion and Combinatorial Torsion

Let X be a compact Riemannian manifold, let F be a complex flat vector bundle on X. The purpose of this section is to recall the definition of the Ray–Singer torsion in de Rham theory [38], of the combinatorial or Reidemeister torsion [31,37], and to state the Cheeger–Müller theorem [20,32,33].

This section is organized as follows. In Section 2.1, we recall the definition of the Ray Singer torsion.

In Section 2.2, the combinatorial torsion is defined.

In Section 2.3, given metrics g^{TX}, g^F, we give the Weitzenböck formula for the Hodge Laplacian acting on $\Omega^{\cdot}(X, F)$. Of special importance will be the section ω^F of $T^*X \otimes_{\mathbf{R}} \mathrm{End}(F)$ that measures the local variation of the metric g^F with respect to the flat connection.

Finally, in Section 2.4, we give the Weitzenböck formula for the Witten Laplacian associated with a smooth function $f : X \to \mathbf{R}$, and we briefly explain the arguments of [15,16] that are used in the proof of a general form of the Cheeger–Müller theorem.

2.1. THE RAY–SINGER ANALYTIC TORSION

Let X be a compact manifold of dimension m, let (F, ∇^F) be a complex flat vector bundle on X. Equivalently, F can be obtained via a complex representation of $\pi_1(X)$.

Let $(\Omega^{\cdot}(X, F), d^X)$ be the de Rham complex of smooth sections of $\Lambda^{\cdot}(T^*X) \otimes_{\mathbf{R}} F$.

Let $H^{\cdot}(X, F)$ denote the cohomology of the above complex. In the sequel, for simplicity, we will assume that $H^{\cdot}(X, F) = 0$, even though most statements that follow make sense in the general case.

EXAMPLE 2.1. We take $X = S^1 = \mathbf{R}/\mathbf{Z}$, and, given $\alpha \in \mathbf{C}$, F is the trivial line bundle $\mathbf{C}$ equipped with the connection $\nabla^F = d + \alpha dt$. If $\alpha \in \mathbf{C} \backslash 2i\pi\mathbf{Z}$, then $H^{\cdot}(X, F) = 0$.

Let g^{TX} be a Riemannian metric on X, let g^F be a Hermitian metric on F. We equip $\Omega^{\cdot}(X, F)$ with the obvious L_2 Hermitian product associated with g^{TX}, g^F. Let d^{X*} be the formal adjoint to d^X. Then, the Hodge Laplacian $\square^X$ is given by

$$\square^X = [d^X, d^{X*}]. \tag{2.1}$$

In the above formula, d^X, d^{X*} are viewed as odd endomorphisms of $\Omega^{\cdot}(X, F)$, and the right-hand side is their supercommutator, which in this case is an anticommutator. For $0 \leq q \leq m$, let $\square_q^X$ be the restriction of $\square^X$ to $\Omega^q(X, F)$.

Since $H^{\cdot}(X, F) = 0$, the Hodge theorem asserts that $\square^X$ is invertible.

DEFINITION 2.2. For $0 \leq q \leq m$, $s \in \mathbf{C}$, $\mathrm{Re} s > m/2$, set

$$\zeta_q(s) = \mathrm{Tr}\left[\square_q^{X,-s}\right]. \tag{2.2}$$

Set

$$\vartheta(s) = \sum_{q=0}^{m} (-1)^{q+1} q \zeta_q(s). \tag{2.3}$$

Then, the $\zeta_q(s)$ are holomorphic functions that extend to meromorphic functions on $\mathbf{C}$, which are holomorphic at 0. Now, we follow Ray and Singer [38].

DEFINITION 2.3. We define the Ray–Singer analytic torsion by the formula

$$T_{\mathrm{an}} = \frac{1}{2}\vartheta'(0). \tag{2.4}$$

If m is odd, Ray and Singer [38] have shown that T_{an} does not depend on g^{TX}, g^F. They used the fact that if A is a smooth section of $\mathrm{End}(\Lambda^{\cdot}(T^*X) \otimes_{\mathbf{R}} F)$, the asymptotic expansion of $\mathrm{Tr}\left[A \exp\left(-t\square^X\right)\right]$ as $t \to 0$ does not contain a constant term.

EXAMPLE 2.4. If we make the same assumptions as in Example 2.1, by an easy computation, we get

$$T_{\mathrm{an}} = -\log\left|2\sinh(\alpha/2)\right|. \tag{2.5}$$

2.2. THE COMBINATORIAL TORSION

From now on, m is assumed to be odd[1]. Let $o(TX)$ be the orientation bundle of TX. Put

$$\det F = \Lambda^{\max} F. \tag{2.6}$$

Then, $\det F$ is a complex line bundle. Let $g^{\det F}$ be the metric on $\det F$ that is induced by g^F.

Now, we follow [16, section 1 b)]. Let K be a triangulation of X. If σ is a simplex in K, let $x_\sigma \in \sigma$ denote its barycentre. Let $(C^{\cdot}(K,F), \partial)$ denote the combinatorial complex generated by locally flat sections of $F \otimes_{\mathbf{Z}_2} o(TX)$ on the cells of the dual triangulation K^*, whose cohomology can be canonically identified with $H^{\cdot}(K,F)$. In our case, our finite dimensional complex is exact, i.e. its cohomology is reduced to 0. The restriction of g^F to the barycentres x_σ induces a Hermitian

[1] This is a simplifying assumption. Analytic torsion and combinatorial torsion can also be made sense of when m is even.

structure on the complex $C^{\cdot}(K,F)$. If the metric g^F is flat, this Hermitian structure does not depend on the choice of the barycentres.

Let ∂^* be the adjoint of ∂. By proceeding as in Section 2.1, but in a finite dimensional context, we can define the combinatorial torsion τ_K by a formula similar to (2.4). If the metric g^F is flat, and more generally, as explained in [33], if the metric $g^{\det F}$ induced by g^F on $\det F$ is flat, τ_K does not depend on the choice of barycentres. In this case, it is a fundamental nontrivial fact that τ_K does not depend on K. The proof consists in proving the invariance of τ_K by subdivision of the triangulation K. This way, we obtain a combinatorial invariant $\tau_{\mathbf{R}}$, the Reidemeister torsion [37]. If $g^{\det F}$ is not flat, τ_K depends explicitly on K and on the choice of barycentres.

A related construction of the Reidemeister torsion was given by Milnor [31]. Indeed if $f : X \to \mathbf{R}$ is a Morse function, let Y be a Thom–Smale gradient vector field for f. A Thom–Smale gradient vector field is such that the ascending and descending cells for Y are transverse. This condition is generic. In this case, Thom [44] and Smale [41,42] have proved that the descending cells produce a CW complex. By proceeding formally as in the case of a triangulation, Milnor [31] has shown that one can defined an associated torsion, the Milnor torsion, which has exactly the same properties as the combinatorial torsion. In particular, when $g^{\det F}$ is flat, it coincides with the Reidemeister torsion $\tau_{\mathbf{R}}$.

When g^F is flat, a conjecture by Ray–Singer [38] asserts that

$$T_{\text{an}} = \tau_{\mathbf{R}}. \tag{2.7}$$

This was proved by Cheeger [20] and Müller [32], using different methods. This result was extended by Müller [33] in the case where $g^{\det F}$ is flat.

The Cheeger–Müller theorem was given another proof by Bismut and Zhang [15,16], and extended to the case where the metric $g^{\det F}$ is not necessarily flat. In this case, a formula obtained by [15,16] compares the analytic torsion to the Milnor torsion associated with a Thom–Smale gradient vector field, the difference being given by an explicit local formula. The proof in [15,16] is based on the Witten deformation of classical Hodge theory [45].

2.3. A WEITZENBÖCK FORMULA FOR THE HODGE LAPLACIAN

DEFINITION 2.5. Put

$$\omega^F = (g^F)^{-1} \nabla^F g^F. \tag{2.8}$$

Then, ω^F is a section of $T^*X \otimes_{\mathbf{R}} \operatorname{End}(F)$. More precisely, it is a 1-form with values in self-adjoint endomorphisms of F with respect to g^F. It exactly measures the extent to which the metric g^F is nonflat. Then, $\omega^{F,2}$ is a 2-form with values in skew-adjoint elements of $\operatorname{End}(F)$.

Let $\nabla^{F,u}$ be the unitary connection on F given by

$$\nabla^{F,u}=\nabla^{F}+\frac{1}{2}\omega^{F}. \tag{2.9}$$

Its curvature is given by

$$\nabla^{F,u,2}=-\frac{1}{4}\omega^{F,2}. \tag{2.10}$$

Also

$$\nabla^{F}\omega^{F}=-\omega^{F,2}, \quad \nabla^{F,u}\omega^{F}=0. \tag{2.11}$$

Let ∇^{TX} be the Levi–Civita connection on TX, and let $\nabla^{\Lambda^{\cdot}(T^{*}X)}$ be the induced connection on $\Lambda^{\cdot}(T^{*}X)$. Let $\nabla^{\Lambda^{\cdot}(T^{*}X)\otimes_{\mathbf{R}}F}, \nabla^{\Lambda^{\cdot}(T^{*}X)\otimes_{\mathbf{R}}F,u}$ be the connection on $\Lambda^{\cdot}(T^{*}X)\otimes_{\mathbf{R}}F$ induced by $\nabla^{\Lambda^{\cdot}(T^{*}X)}, \nabla^{F}$ and $\nabla^{\Lambda^{\cdot}(T^{*}X)}, \nabla^{F,u}$.

Let $e_1,\dots,e_m$ be an orthonormal basis of TX, and let $e^1,\dots,e^m$ be the corresponding dual basis of $T^{*}X$. The operators d^{X}, d^{X*} are given by

$$d^{X}=e^{i}\nabla_{e_i}^{\Lambda^{\cdot}(T^{*}X)\otimes_{\mathbf{R}}F}, \quad d^{X*}=-i_{e_i}\left(\nabla_{e_i}^{\Lambda^{\cdot}(T^{*}X)\otimes_{\mathbf{R}}F}+\omega^{F}(e_i)\right). \tag{2.12}$$

Let R^{TX} be the curvature of ∇^{TX}, and let K^{X} be the scalar curvature of X. Let $\Delta^{X,u}$ denote the Bochner Laplacian acting on $\Omega^{\cdot}(X,F)$,

$$\Delta^{X,u}=-\nabla_{e_i}^{\Lambda^{\cdot}(T^{*}X)\otimes_{\mathbf{R}}F,u,*}\nabla_{e_i}^{\Lambda^{\cdot}(T^{*}X)\otimes_{\mathbf{R}}F,u}. \tag{2.13}$$

In (2.13), $*$ is a notation for the adjoint. With our conventions, $-\Delta^{X,u}$ is nonnegative.

Put

$$|\omega^{F}|^{2}=\sum_{i=1}^{m}(\omega^{F}(e_i))^{2}. \tag{2.14}$$

Then, $|\omega^{F}|^{2}$ is a self-adjoint nonnegative section of $\mathrm{End}(F)$.

If $e\in TX$, if $e^{*}\in T^{*}X$ corresponds to e by the metric g^{TX}, put

$$c(e)=e^{*}\wedge-i_{e}, \quad \widehat{c}(e)=e^{*}\wedge+i_{e}. \tag{2.15}$$

The operators in (2.15) act on $\Lambda^{\cdot}(T^{*}X)$.

By [15, Theorem 4.13], we get

$$\begin{aligned}\square^{X}=&-\Delta^{X,u}+\frac{K^{X}}{4}-\frac{1}{8}\langle R^{TX}(e_i,e_j)e_k,e_\ell\rangle c(e_i)c(e_j)\widehat{c}(e_k)\widehat{c}(e_\ell)\\&+\frac{1}{4}|\omega^{F}|^{2}-\frac{1}{8}(c(e_i)c(e_j)-\widehat{c}(e_i)\widehat{c}(e_j))\omega^{F,2}(e_i,e_j)\\&-\frac{1}{2}c(e_i)\widehat{c}(e_j)\nabla_{e_i}^{TX\otimes F,u}\omega^{F}(e_j).\end{aligned} \tag{2.16}$$

The term $\frac{1}{4}\left|\omega^{F}\right|^{2}$ is of special interest. As we shall see later, it may well dominate the other terms of order 0 and guarantee the existence of a spectral gap in $\square^{X}$.

2.4. ANALYTIC TORSION AND THE WITTEN LAPLACIAN

The proof given in [15,16] of the Cheeger–Müller theorem is based on the Witten deformation [45].

Indeed let $f : X \to \mathbf{R}$ be a smooth function. Given $T \in \mathbf{R}$, in the above constructions, we replace g^F by $e^{-2Tf} g^F$. The corresponding form ω_T^F is given by

$$\omega_T^F = \omega^F - 2T \mathrm{d} f. \tag{2.17}$$

From (2.17), we get

$$\frac{1}{4} \left| \omega_T^F \right|^2 = \frac{1}{4} \left| \omega^F \right|^2 + T^2 \left| \nabla f \right|^2 - T \omega^F \left(\nabla f \right). \tag{2.18}$$

Let $\square_T^X$ be the Hodge Laplacian associated with the metrics g^{TX}, g_T^F. Set

$$\widetilde{\square}_T^X = e^{-Tf} \square_T^X e^{Tf}. \tag{2.19}$$

Then, $\widetilde{\square}_T^X$ is self-adjoint with respect to the Hermitian product associated with g^{TX}, g^F. By (2.16), we get

$$\begin{aligned}\widetilde{\square}_T^X = & -\Delta^{X,u} + \frac{K^X}{4} - \frac{1}{8} \langle R^{TX}(e_i, e_j) e_k, e_\ell \rangle c(e_i) c(e_j) \widehat{c}(e_k) \widehat{c}(e_\ell) \\ & + \frac{1}{4} |\omega_T^F|^2 - \frac{1}{8} (c(e_i) c(e_j) - \widehat{c}(e_i) \widehat{c}(e_j)) \omega^{F,2}(e_i, e_j) \\ & - \frac{1}{2} c(e_i) \widehat{c}(e_j) \nabla_{e_i}^{TX \otimes F, u} \omega_T^F(e_j).\end{aligned} \tag{2.20}$$

By (2.18), (2.20), the terms in the right-hand side that depend on T are given by

$$T^2 \left| \nabla f \right|^2 - T \omega^F \left(\nabla f \right) + T c\left(e_i\right) \widehat{c}\left(e_j\right) \nabla_{e_i}^{TX} \nabla_{e_j} f. \tag{2.21}$$

Assume now that f is a Morse function, i.e. it has a finite number of critical points, and these critical points are nondegenerate. By (2.20), (2.21), by proceeding as in Witten [45], it is elementary to verify that as $T \to +\infty$, the eigenforms associated with small eigenvalues localize near the critical points of f, the forms of degree p localizing near the critical points of index p. As shown in [45], by making $T \to +\infty$, one can obtain this way an analytic proof of the strong Morse inequalities.

Assume now that ∇f is a Morse–Smale vector field. It was shown by Helffer and Sjöstrand [24] that the asymptotics of the eigenvalues of $\square_T^X$ can be exactly determined in terms of the eigenvalues of the associated combinatorial Laplacian. For a simple geometric derivation of these results, we refer to Bismut and Zhang [16, section 6]. By elaborating on such considerations, Bismut and Zhang [15,16] gave a proof of the Cheeger–Müller theorem, which is also valid in the case of an arbitrary Hermitian metric g^F. In this context, the difference between the analytic torsion and the Milnor torsion is given by an explicit local formula that vanishes when $g^{\det F}$ is flat.

3. Asymptotic Torsion and Toeplitz Operators

We use the notation of Section 2. The purpose of this section is to explain the results obtained by Bismut et al. [14] in connection with Toeplitz operators. Let $q:\mathcal{N}\to X$ be a flat fibration by compact complex manifolds N, and let L be a fibrewise holomorphic positive line bundle. The idea is to consider the family of flat vector bundles $F_p = H^0(N, L^p)$ as $p\to+\infty$. We will express the tensors that appear in the Weitzenböck formula of Section 2.3 as Toeplitz operators, so that their asymptotics as $p\to+\infty$ can be suitably controlled.

This section is organized as follows. In Section 3.1, as a special case, we consider a flat vector bundle F, and its symmetric powers S^pF, in which case the fibres N are the projective bundles $\mathbf{P}(F^*)$.

In Section 3.2, we construct the flat vector bundles $F_p|_{p\in\mathbf{N}}$ in the case of a general fibration $\pi:\mathcal{N}\to X$.

In Section 3.3, we express the forms ω^{F_p} as Toeplitz operators in the sense of geometric quantization [1,17,27].

In Section 3.4, we describe the behaviour of certain heat equation supertraces as $p\to+\infty$.

In Section 3.5, we construct the real invariant W obtained in [14, subsection 9.6].

Finally, in Section 3.6, still following [14], we describe the asymptotics as $p\to+\infty$ of the analytic torsion of F_p in terms of the W-invariant.

As explained in the introduction, the subject of asymptotic torsion has been initiated by Bergeron and Venkatesh [2]. For work which is directly relevant to this section, we also refer to Müller [34] and Müller and Pfaff [35].

3.1. THE CASE OF PROJECTIVE BUNDLES

For the moment, F denotes a complex finite dimensional vector space. Let $\mathbf{P}(F^*)$ be the projectivization of F^*, and L be the canonical holomorphic complex line bundle on $\mathbf{P}(F^*)$. Then, F can be canonically identified with $H^0(\mathbf{P}(F^*), L)$, the holomorphic cohomology of L. More generally, for $p\in\mathbf{N}$, the symmetric power S^pF can be identified with $H^0(\mathbf{P}(F^*), L^p)$.

If g^F is a Hermitian metric on F, it induces the Fubini–Study metric on $\mathbf{P}(F^*)$, and also a Hermitian metric on L. The corresponding L_2 metric on $H^0(\mathbf{P}(F^*), L^p)$ is proportional to the canonical metric on S^pF.

If F is a complex flat vector bundle on X as in Section 2, let $\mathcal{P}(F^*)$ be the total space of the fibres $\mathbf{P}(F^*)$. Then, $\mathcal{P}(F^*)$ is a flat fibration over X with compact complex fibres $\mathbf{P}(F^*)$. Also L is a complex line bundle on $\mathcal{P}(F^*)$ equipped with a horizontal flat connection. If g^F is a Hermitian metric on F, it induces a Hermitian metric along the fibres $\mathbf{P}(F^*)$, and also a Hermitian metric on the line bundle L. For $p\in\mathbf{N}$, we have the identification of flat vector bundles $S^pF = H^0(\mathbf{P}(F^*), L^p)$. The L_2 metric on $H^0(\mathbf{P}(F^*), L^p)$ is proportional to the canonical metric of S^pF.

3.2. FLAT VECTOR BUNDLES AND DOLBEAULT COHOMOLOGY

We make the same assumptions as in Section 2, and we assume m to be odd. We will explain the formalism developed in [14, section 9]. Let $q : \mathcal{N} \to X$ be a flat fibration over X by compact complex manifolds N of complex dimension n. Let L be a line bundle on $\mathcal{N}$ which is holomorphic along the fibres N. Also we assume that the flat connection on the fibration $\mathcal{N}$ lifts to the line bundle L. In particular, the fibration by the fibres (N, L) is locally trivial, i.e. X can be covered by open sets U such that over U, if $x_0 \in U$, $q^{-1}U$ is just $U \times N_{x_0}$, and L is the pull back of its restriction to N_{x_0}.

For $p \in \mathbf{N}$, set

$$F_p = H^0(N, L^p). \tag{3.1}$$

Then, F_p is a complex flat vector bundle on X. The flat connection ∇^{F_p} is just induced by the flat structure on the fibration $\mathcal{N}$.

We assume that g^L is a Hermitian metric on L which is fibrewise positive, i.e. if r^L is the curvature of its fibrewise Chern connection, if $U \in T^{(1,0)}N$, then $r^L\left(U, \overline{U}\right) > 0$. For the existence of such a metric g^L, it is enough to assume that such a metric exists for each fibre N_{x_0}, or even for just one fibre when X is connected. Let g^{TN} be a Hermitian metric on the fibrewise holomorphic tangent bundle TN. The metrics g^{TN}, g^L induce a corresponding L_2 metric g^{F_p} on F_p.

In the sequel, we use the notation

$$\xi^V = -ir^L. \tag{3.2}$$

Then, ξ^V is a fibrewise symplectic form along the fibres N. If $\mathcal{H} \in C^\infty(N, \mathbf{R})$, let $X_{\mathcal{H}}$ denote the fibrewise Hamiltonian vector field along N, so that

$$d^N\mathcal{H} + i_{X_{\mathcal{H}}}\xi^V = 0. \tag{3.3}$$

If $\mathcal{H}, \mathcal{H}' \in C^\infty(N, \mathbf{R})$, let $\left\{\mathcal{H}, \mathcal{H}'\right\}$ denote their fibrewise Poisson bracket, i.e.

$$\left\{\mathcal{H}, \mathcal{H}'\right\} = \xi^V\left(X_{\mathcal{H}}, X_{\mathcal{H}'}\right). \tag{3.4}$$

3.3. THE FORMS ω^{F_p} AS TOEPLITZ OPERATORS

Let ω^L denote the horizontal variation of the metric g^L. More precisely, if $U \in TX$, if $U^H \in T\mathcal{N}$ denotes the horizontal lift of U, set

$$\omega^L(U) = (g^L)^{-1}\nabla^L_{U^H} g^L. \tag{3.5}$$

Then, ω^L is a smooth section of q^*T^*X. By construction, it is horizontally closed, i.e. it vanishes under the horizontal lift of the de Rham operator d^X. Equivalently, its restriction to horizontal leaves is closed.

If $U \in TX$, let $\operatorname{div}_N(U)$ denote the divergence of U with respect to the fibrewise volume form dv_N along the fibres N. More precisely, if $U \in TX$, we have the identity

$$L_{U^H} dv_N = \operatorname{div}_N(U)\, dv_N. \tag{3.6}$$

Let $\mathcal{F}$ be the infinite dimensional vector bundle on X,

$$\mathcal{F} = C^\infty(N, L|_N). \tag{3.7}$$

Then, $\mathcal{F}$ is a flat vector bundle on X, equipped with the L_2 metric $g^{\mathcal{F}}$ associated with g^{TN}, g^L.

We use the notation $F = F_1$. Let P denote the fibrewise orthogonal projection operator from $\mathcal{F}$ on F with respect to the metric $g^{\mathcal{F}}$.

Now, we have the elementary result in [14, Theorem 9.25].

PROPOSITION 3.1. *The following identity holds:*

$$\omega^{\mathcal{F}} = \omega^L + \operatorname{div}_N. \tag{3.8}$$

Also

$$\omega^F = P\omega^{\mathcal{F}} P. \tag{3.9}$$

When replacing L by L^p, we add the extra subscript p. In particular, for $p \in \mathbf{N}$, P_p denotes the orthogonal projection from $\mathcal{F}_p$ on F_p. By (3.8), (3.9), we get

$$\omega^{\mathcal{F}_p} = p\omega^L + \operatorname{div}_N, \quad \omega^{F_p} = P_p \omega^{\mathcal{F}_p} P_p. \tag{3.10}$$

By the above, ω^{F_p} is a Toeplitz operator in the sense of Berezin [1], Boutet de Monvel and Sjöstrand [19], Boutet de Monvel and Guillemin [18], Bordemann et al. [17], Ma and Marinescu [27].

In the sequel, we use the notation

$$\theta = -\omega^L/2, \quad \eta^N = -\operatorname{div}_N/2. \tag{3.11}$$

We will rewrite the second identity in (3.10) in the form

$$\frac{1}{2p}\omega^{F_p} = -T_{p,\theta+\eta^N/p}, \tag{3.12}$$

T_p being a shortcut for Toeplitz.

We use the notation

$$\left|T_{p,\theta+\eta^N/p}\right|^2 = \sum_{i=1}^m \left(T_{p,(\theta+\eta^N/p)(e_i)}\right)^2. \tag{3.13}$$

By (3.12), we deduce that

$$\frac{1}{4p^2}\left|\omega^{F_p}\right|^2=\left|T_{p,\theta+\eta^N/p}\right|^2,\quad \frac{1}{4p}\omega^{F_p,2}=pT^2_{p,\theta+\eta^N/p}. \tag{3.14}$$

Similarly, by [14, Theorem 9.27], if $U,V\in TX$, we get

$$-\frac{1}{2p}\nabla_U^{F_p,u}\omega^{F_p}(V)=\nabla_U^{F_p,u}T_{p,(\theta+\eta^N/p)(V)}. \tag{3.15}$$

The main advantage of Equations (3.13)–(3.15) is that they reexpress the considered tensors in terms of Toeplitz operators associated with functions or 1-forms.

Let θ^{*2} denote the section of $q^*\Lambda^2(T^*X)$ which is such that

$$\theta^{*2}(U,V)=\{\theta(U),\theta(V)\}. \tag{3.16}$$

As explained before, the right-hand side of (3.16) is a fibrewise Poisson bracket. Put

$$|\theta|^2=\sum_{i=1}^m\theta(e_i)^2. \tag{3.17}$$

Then, $|\theta|^2$ is a smooth function on $\mathcal{N}$.

Set

$$\mathcal{A}=C^\infty(N,\mathbf{R}). \tag{3.18}$$

Let $\nabla^{\mathcal{A}}$ be the obvious flat connection on $\mathcal{A}$. In [14, Definition 9.6], another connection $\nabla^{\mathcal{A},u}$ on $\mathcal{A}$ is defined that depends only on the metric g^L, and is such that the Poisson bracket in (3.4) is parallel.

We are now ready to use the full strength of the properties of the Toeplitz algebra in the context of Berezin quantization. By the results of [1,17–19,27], as $p\to+\infty$,

$$\begin{aligned}\left|T_{p,\theta+\eta^N/p}\right|^2&=T_{p,|\theta|^2}+\mathcal{O}(1/p),\\ pT^2_{p,\theta+\eta^N/p}&=iT_{p,\theta^{*2}}+\mathcal{O}(1/p),\\ \nabla_U^{F_p,u}T_{p,\theta+\eta^N/p}(V)&=T_{p,\nabla_U^{\mathcal{A},u}\theta(V)}+\mathcal{O}(1/p).\end{aligned} \tag{3.19}$$

In (3.19), $\mathcal{O}(1/p)$ is taken in the sense of the norm of operators with respect to the L_2 norm on $\mathcal{F}_p$.

EXAMPLE 3.2. In the case where N is of dimension 0, and L is the trivial flat bundle $\mathbf{C}$ on X, let f be the smooth function on X such that

$$\|1\|^2_{g^L}=e^{-2f}. \tag{3.20}$$

Then

$$\theta = \mathrm{d}f. \tag{3.21}$$

From (3.21), we deduce that

$$|\theta|^2 = |\nabla f|^2. \tag{3.22}$$

When replacing L by L^p, f is replaced by pf. In this case, the integral parameter p plays the role of T in the Witten deformation considered in Section 2.4.

We go back to the general case. We follow [14, Definition 9.13].

DEFINITION 3.3. We will say that the metric g^L is nondegenerate if $|\theta|^2$ does not vanish.

Remark 3.4. In [14, Proposition 8.12], when X is locally symmetric, and when the fibration $q : \mathcal{N} \to X$ is defined in group theoretic terms, natural conditions are given under which the metric g^L is nondegenerate. In Example 2.1, $X = S^1$. The canonical metric on $\mathbf{C}$ is nondegenerate if and only if $\mathrm{Re}\,\alpha \neq 0$.

When replacing F by F_p, we denote by $\square_p^X$ the Hodge Laplacian acting on $\Omega^{\cdot}\left(X, F_p\right)$, and by $\Delta_p^{X,u}$ the corresponding Bochner Laplacian. By Equation (2.16), we get

$$\begin{aligned}\square_p^X = & -\Delta_p^{X,u} + \frac{K^X}{4} - \frac{1}{8}\langle R^{TX}(e_i, e_j)e_k, e_\ell\rangle c(e_i)c(e_j)\widehat{c}(e_k)\widehat{c}(e_\ell) \\ & + \frac{1}{4}|\omega^{F_p}|^2 - \frac{1}{8}(c(e_i)c(e_j) - \widehat{c}(e_i)\widehat{c}(e_j))\omega^{F_p,2}(e_i, e_j) \\ & - \frac{1}{2}c(e_i)\widehat{c}(e_j)\nabla_{e_i}^{TX\otimes F_p,u}\omega^{F_p}(e_j).\end{aligned} \tag{3.23}$$

Now, we give a result established in [14, Theorem 4.4 and section 9.10].

THEOREM 3.5. *If the metric g^L is nondegenerate, there exist $C > 0, C' > 0$ such that for $p \in \mathbf{N}$,*

$$\square_p^X \geq Cp^2 - C'. \tag{3.24}$$

In particular, for $p \in \mathbf{N}$ large enough, $H^{\cdot}(X, F_p) = 0$.

Proof. The proof is obtained by combining (3.12), (3.14), (3.19) and (3.23). □

3.4. THE BEHAVIOUR OF $\mathrm{Tr_s}\left[N^{\Lambda^{\cdot}(T^*X)}\exp\left(-t\square_p^X/p^2\right)\right]$ AS $p\to+\infty$

For the moment, we do not assume the metric g^L to be nondegenerate. Let $N^{\Lambda^{\cdot}(T^*X)}$ be the number operator of $\Lambda^{\cdot}(T^*X)$.

If $\mathcal{H}\in\mathcal{A}$, if $dv_{N,0}$ is the symplectic volume form along the fibre N with respect to ξ^V, set

$$\mathrm{Tr}[\mathcal{H}]=(2\pi)^{-n}\int_N \mathcal{H}dv_{N,0}. \tag{3.25}$$

Then, Tr is a trace on the Poisson algebra $\mathcal{A}$.

Recall that by Ma and Marinescu [26, eq. (4.1.84), Lemma7.2.4], as $p\to+\infty$,

$$\mathrm{Tr}[T_{p,\mathcal{H}}]=p^n\mathrm{Tr}[\mathcal{H}]+\mathcal{O}(p^{n-1}). \tag{3.26}$$

In the sequel if $A\in\mathrm{End}(\Omega^{\cdot}(X,F_p))$ is trace class, we denote by $\mathrm{Tr_s}[A]$ its supertrace, i.e.

$$\mathrm{Tr_s}[A]=\mathrm{Tr}[(-1)^{N^{\Lambda^{\cdot}(T^*X)}}A]. \tag{3.27}$$

Take $t>0$. By (3.19), (3.23), and (3.26), as $p\to+\infty$, we can study the asymptotics of the supertrace $\mathrm{Tr_s}\left[N^{\Lambda^{\cdot}(T^*X)}\exp\left(-t\square_p^X/p^2\right)\right]$ using semiclassical methods on the base X and the asymptotics of Toeplitz operators described in Section 3.3. This is done in detail in [14, proof of Theorem 9.30]. The singularity p^m associated with the singularity of the standard heat kernel as the time parameter tends to 0 can be compensated by suitable Getzler rescalings [23] on the Clifford algebras $c(TX),\widehat{c}(TX)$.

3.5. THE INVARIANT W

We assume again the metric g^L to be nondegenerate.

There is a canonical Hermitian connection $\nabla^{L,u}$ on the line bundle (L,g^L) that coincides with the Chern connection along the fibres N, and which is such that if $U\in TX$,

$$\nabla^{L,u}_{U^H}=\nabla^L_{U^H}+\frac{1}{2}\omega^L(U). \tag{3.28}$$

Let $c_1\left(L,g^L\right)$ be the first Chern form associated with the connection ∇^L.

The form θ can be viewed as a 1-form on $\mathcal{N}$, which is in general not closed. Let $\widehat{TX}$ be another copy of TX, and let $\nabla^{\widehat{TX}}$ be the analogue of ∇^{TX}. Let ψ denote the $m-1$ current on the total space $\widehat{\mathcal{X}}$ of $\widehat{TX}$ with values in the orientation bundle $o(\widehat{TX})$ of $\widehat{TX}$ that was constructed by Mathai and Quillen [29] which is such that

$$d\psi=-\delta_X. \tag{3.29}$$

In (3.29), δ_X denotes the current of integration on X. Here, we have used the fact that X is odd dimensional. If X was even dimensional, there would be an extra

Euler form $e\left(TX,\nabla^{TX}\right)$ in the right-hand side of (3.29). The restriction of $-\psi$ to the fibres $\widehat{TX}$ is just the solid angle form of total volume 1.

In particular, since $\widehat{\theta}$ does not vanish, the $n-1$ form $\widehat{\theta}^*\psi$ on $\mathcal{N}$ with values in $o(TX)$ is well defined. By (3.29), this form is closed.

DEFINITION 3.6. Let $W\in\mathbf{R}$ be given by

$$W=\int_{\mathcal{N}}\left[\theta\left(\widehat{\theta}^*\psi\right)\exp\left(c_1\left(L,g^L\right)\right)\right]. \tag{3.30}$$

In [14, subsection 9.6], another construction of W is given in the Berezin integral formalism of Mathai–Quillen [29].

The following result is established in [14, Theorem 9.23].

THEOREM 3.7. *The invariant W does not depend on g^{TX} or on infinitesimal variations of the metric g^L.*

3.6. THE ASYMPTOTICS OF $T_{\mathrm{an},p}$

We assume the metric g^L to be nondegenerate. By Theorem 3.5, for $p\in\mathbf{N}$ large enough, $H^{\cdot}\left(X,F_p\right)=0$.

For $p\in\mathbf{N}$ large enough, let $T_{\mathrm{an},p}$ denote the analytic torsion associated with the vector bundle F_p. By reexpressing the zeta functions ζ_q as Mellin transforms of the trace of the corresponding heat kernels, we will rewrite $T_{\mathrm{an},p}$ informally as:

$$T_{\mathrm{an},p}=-\frac{1}{2}\int_0^{+\infty}\mathrm{Tr_s}\left[N^{\Lambda^{\cdot}(T^*X)}\exp\left(-t\square_p^X/p^2\right)\right]\frac{dt}{t}. \tag{3.31}$$

The integral in (3.31) converges as $t\to+\infty$, but has to be suitably regularized as $t\to 0$.

We will now describe the results on the asymptotics of $T_{\mathrm{an},p}$ as $p\to+\infty$ that were obtained in [14, Theorem 9.32].

THEOREM 3.8. *As $p\to+\infty$,*

$$p^{-n-1}T_{\mathrm{an},p}=W+\mathcal{O}\left(1/p\right). \tag{3.32}$$

Proof. The proof consists in expressing $T_{\mathrm{an},p}$ as in (3.31) and in taking the proper limit in the integrand, along the lines indicated in Section 3.4. □

Remark 3.9. Since $T_{\mathrm{an},p}$ does not depend on the metric g^L, equation (3.32) shows that W does not depend on g^L. The proof in [14, section 9] is formally very close to the proof given in [15,16] of the Cheeger–Müller theorem. From this point of view, one can work as if the Morse function f in Section 2.4 did not have critical points. This is compensated by the extra complication due to the presence of Toeplitz operators.

As we saw in Remark 3.4, in the context of Example 2.1, the metric g^L on $L=\mathbf{C}$ is nondegenerate if and only if $\mathrm{Re}\,\alpha \neq 0$. In this case, by (2.5),

$$T_{\mathrm{an},p} = -\log\left|2\sinh(\alpha p/2)\right|. \tag{3.33}$$

By (3.33), as $p \to +\infty$,

$$T_{\mathrm{an},p} = -\frac{p}{2}|\mathrm{Re}\,\alpha|(1+\mathcal{O}(e^{-\mathrm{Re}\,\alpha p})). \tag{3.34}$$

Note that here, $n=0, \theta=(\mathrm{Re}\,\alpha)dt$, $m=1$, and ψ is minus half of the form of degree 0 that defines the orientation of S^1. By (3.30), we get

$$W = -|\mathrm{Re}\,\alpha|/2. \tag{3.35}$$

By (3.35), Equations (3.32) and (3.34) are compatible.

4. The Hypoelliptic Laplacian

Let X be a compact Riemannian manifold. The purpose of this section is to give a short introduction to the theory of the hypoelliptic Laplacian [6,13]. In its simplest form, the hypoelliptic Laplacian is a family of scalar hypoelliptic operators $L_b^X|_{b>0}$ acting on the total space of the tangent bundle $\mathcal{X}$ that interpolates between $-\Delta^X/2$ and the vector field $-Z$, where Z is the generator of the geodesic flow. The ordinary Laplacian $\Delta^X/2$ will be viewed as a Toeplitz operator, the effect of the Toeplitz compression being to eliminate the fibres TX in $\mathcal{X}$.

This section is organized as follows. In Section 4.1, we introduce the family of operators $L_b^X|_{b>0}$.

In Section 4.2, we show that $-\Delta^X/2$ can be viewed as a Toeplitz operator.

In Section 4.3, we describe the asymptotics of the resolvent of L_b^X as $b\to 0$ in terms of the resolvent of $-\Delta^X/2$.

In Section 4.4, we recall the results of [13] that describe the behaviour as $b\to 0$ of the hypoelliptic heat kernels for $\exp\left(-tL_b^X\right)$.

In Section 4.5, the operators L_b^X are shown to be self-adjoint with respect to a bilinear symmetric nondegenerate form of signature (∞,∞).

In Section 4.6, we show that as $b\to+\infty$, L_b^X converges in a proper sense to $-Z$.

In Section 4.7, we extend the construction of the scalar hypoelliptic Laplacian to hypoelliptic Hodge Laplacians that deform the classical elliptic Hodge Laplacian.

In Section 4.8, we briefly connect the hypoelliptic Laplacian to the more classical theory of the Witten deformation.

In Section 4.9, we state the main result in Bismut and Lebeau [13] that asserts that for odd dimensional manifolds, the elliptic and hypoelliptic analytic torsions coincide.

Finally, in Section 4.10, we describe the results of Shen [39,40], who gave a direct proof of the equality of hypoelliptic and Reidemeister torsions.

4.1. THE SCALAR HYPOELLIPTIC LAPLACIAN

Let X be a compact Riemannian manifold of dimension m. Let Δ^X be the corresponding Laplace–Beltrami operator.

Let $\pi : \mathcal{X} \to X$ be the total space of its tangent bundle TX. Let Δ^V denote the Laplacian along the fibres TX.

Let H be the harmonic oscillator along the fibre TX. If Y is the tautological section of $\pi^* TX$ on $\mathcal{X}$, then

$$H = \frac{1}{2}\left(-\Delta^V + |Y|^2 - m\right). \tag{4.1}$$

Let Z be the vector field on $\mathcal{X}$ which is the generator of the geodesic flow. If $x \in X$, in local geodesic coordinates centred at x, then

$$Z(x, Y) = Y^i \frac{\partial}{\partial x^i}. \tag{4.2}$$

DEFINITION 4.1. For $b > 0$, the scalar hypoelliptic Laplacian L_b^X with parameter b is given by

$$L_b^X = \frac{H}{b^2} - \frac{Z}{b}. \tag{4.3}$$

By [25], L_b^X is hypoelliptic. Also we have the stronger result that if $t \in \mathbf{R}_+$ is an extra coordinate, the operator $\frac{\partial}{\partial t} + L_b^X$ is hypoelliptic. This shows that as soon as the heat operators $\exp\left(-tL_b^X\right)$ are properly defined, they are given by smooth kernels.

4.2. THE OPERATOR $\Delta^X/2$ AS A TOEPLITZ OPERATOR

Recall that fibrewise, H is a self-adjoint elliptic nonnegative operator with spectrum $\mathbf{N}$, that $\ker H$ is 1 dimensional and spanned by the Gaussian function $\exp(-|Y|^2/2)$, and that more generally, the eigenfunctions of H are the products of Hermite polynomials by the Gaussian weight $\exp(-|Y|^2/2)$. Also Z is an antisymmetric operator.

The Gaussian function $\exp(-|Y|^2/2)$ is a section of $\ker H$. Equivalently

$$\ker H = C^\infty(X, \mathbf{R}) \otimes \left\{\exp(-|Y|^2/2)\right\} \simeq C^\infty(X, \mathbf{R}). \tag{4.4}$$

Let P denote the (fibrewise) L_2 orthogonal projection on $\ker H$.

PROPOSITION 4.2. *The following identities hold:*

$$PZP=0, \quad PZ^2P=\Delta^X/2. \tag{4.5}$$

Proof. Since Z is odd in the variable Y, the first identity is obvious. The second identity follows from (4.2) and from the fact that

$$\int_{\mathbf{R}} e^{-y^2} y^2 \frac{dy}{\sqrt{\pi}} = \frac{1}{2}. \tag{4.6}$$

□

We write L_b^X as a $(2,2)$ matrix of unbounded operators with respect to the orthogonal splitting $L_2 = \ker H \oplus \operatorname{Im} H$. Because of the first identity in (4.5), as $b \to 0$, we get

$$L_b^X \simeq \begin{bmatrix} 0 & -Z/b \\ -Z/b & H/b^2 \end{bmatrix}. \tag{4.7}$$

As the notation indicates, in the right-hand side of (4.7), we only wrote the leading terms.

4.3. THE RESOLVENT OF L_b^X AS $b \to 0$

Let us now pretend that the operator L_b^X is a linear operator acting on a finite dimensional Euclidean vector space E, that H is a symmetric nonnegative matrix and that $Z \in \operatorname{End}(E)$ is an antisymmetric morphism that maps $\ker H$ into $\operatorname{Im} H$. For $b > 0$, put

$$L_b = \frac{H}{b^2} - \frac{Z}{b}. \tag{4.8}$$

Then, we can write L_b as a $(2,2)$ matrix as in (4.7).

Let P still denote the orthogonal projection on $\ker H$. Since Z maps $\ker H$ in $\operatorname{Im} H$, the operator $PZH^{-1}ZP$ is well defined. An elementary computation shows that for $\lambda \in \mathbf{C} \backslash \mathbf{R}_+$, as $b \to 0$,

$$(\lambda - L_b)^{-1} \to P(\lambda + PZH^{-1}ZP)^{-1}P. \tag{4.9}$$

Because of (4.9), we will say that in the proper sense, as $b \to 0$, L_b is a deformation of $-PZH^{-1}ZP$. From (4.9), we deduce that as $b \to 0$, except for the eigenvalues $\lambda \in \mathbf{C}$ of L_b that are such that $|\lambda| \to +\infty$, as $b \to 0$,

$$\operatorname{Sp} L_b \to \operatorname{Sp}(-PZH^{-1}ZP). \tag{4.10}$$

Let us now pretend that the previous considerations apply to the operator L_b^X in (4.3). Recall that by (4.4), $\ker H \simeq C^\infty(X, \mathbf{R})$.

PROPOSITION 4.3. *The following identity holds:*

$$PZH^{-1}ZP = \Delta^X/2. \tag{4.11}$$

Proof. Note that the $Y^i \exp(-|Y|^2/2)$ span the eigenspace of H associated with the eigenvalue 1. Using Proposition 4.2 completes the proof of our proposition. □

Remark 4.4. Equations (4.5) and (4.11) express the elliptic Laplacian $\Delta^X/2$ as a Toeplitz operator. The context is different than in classical geometric Toeplitz operator theory, since there is no line bundle L and no parameter $p \in \mathbf{N}$, but instead a parameter $b > 0$. The context will be more the one of index theory. In the sequel, like in the theory of Toeplitz operators, the problem will be to exploit the above identities, in order to understand more about the elliptic Laplacian $-\Delta^X/2$.

4.4. CONVERGENCE OF THE HEAT KERNELS AS $b \to 0$

To give a concrete consequence of the convergence results of Section 4.3, we will state a result established by Bismut and Lebeau [13, section 3.4]. Given $t > 0$, let $p_t^X(x, x')$ be the smooth kernel associated with $\exp(t\Delta^X/2)$ with respect to the volume dx'. For $b > 0, t > 0$, let $q_{b,t}^X((x, Y), (x', Y'))$ be the smooth kernel associated with $\exp(-tL_b^X)$ with respect to the volume $\mathrm{d}x'\mathrm{d}Y'$.

THEOREM 4.5. *Given $t > 0$, as $b \to 0$, we have the uniform convergence of smooth kernels over compact subsets of $\mathcal{X}$ together with their derivatives of arbitrary order,*

$$\begin{aligned} &q_{b,t}^X((x, Y), (x', Y')) \to q_{0,t}^X((x, Y), (x', Y')) \\ &\quad = \pi^{-m/2} p_t^X(x, x') \exp\left(-\frac{1}{2}(|Y|^2 + |Y'|^2)\right). \end{aligned} \tag{4.12}$$

4.5. SELF-ADJOINTNESS

The operator L_b^X is not classically self-adjoint because of the presence of the antisymmetric operator Z. Let $dv_{\mathcal{X}}$ be the natural volume form on $\mathcal{X}$. Let γ be the symmetric nondegenerate bilinear form on $C^{\infty,c}(\mathcal{X}, \mathbf{R})$ that is given by

$$\gamma(f, g) = \int_{\mathcal{X}} f(x, Y)\, g(x, -Y)\, dxdY. \tag{4.13}$$

Then, γ is a bilinear form of signature (∞, ∞), which is positive on the even functions of Y, and negative on the odd functions of Y. Then, L_b^X is formally self-adjoint with respect to γ. All the hypoelliptic Laplacians which we have constructed share this property of being self-adjoint with respect to such a bilinear form.

4.6. THE HYPOELLIPTIC LAPLACIAN AS $b \to +\infty$

After conjugation by the map $(x, Y) \to (x, bY)$, as $b \to +\infty$,

$$L_b^X \simeq \frac{1}{2}|Y|^2 - Z. \tag{4.14}$$

The two operators in the right-hand side of (4.14) commute. Equation (4.14) indicates that as $b \to +\infty$, the heat flow $\exp\left(-tL_b^X\right)$ propagates more and more along the trajectories of the vector field Z[2]. When considering the trace of the hypoelliptic heat kernel, as $b \to +\infty$, this forces the localization of the trace near closed geodesics.

4.7. THE HYPOELLIPTIC LAPLACIAN AS A HODGE THEORETIC OBJECT

The operator $-\Delta^X/2$ is the restriction of the Hodge Laplacian $\square^X/2$ to smooth functions. A natural question is to ask whether the Hodge Laplacian $\square^X/2$ can be deformed to a hypoelliptic Hodge Laplacian $\mathcal{L}_b^X$ acting on $\Omega^{\cdot}(\mathcal{X}, \mathbf{R})$, whose restriction to forms of degree 0 would be precisely the scalar hypoelliptic Laplacian L_b^X. This is precisely the question which is completely solved in [6]. For detailed surveys of the construction of the hypoelliptic Hodge Laplacian, we refer to [5,8].

Let us briefly make a few remarks on this construction. First, as explained in [8, section 2.2], if (M, ω) is a symplectic manifold, the symplectic form ω defines a bilinear pairing between compactly supported forms on M, which is neither symmetric nor antisymmetric. Let $\overline{d}^M$ denote the formal adjoint of d^M with respect to this pairing. It is elementary to verify that

$$\left[d^M, \overline{d}^M\right] = 0, \tag{4.15}$$

i.e. the symplectic Laplacian vanishes. Let $\mathcal{H}: M \to \mathbf{R}$ be a smooth function. Let $Y^{\mathcal{H}}$ be the Hamiltonian vector field associated with $\mathcal{H}$ so that

$$d\mathcal{H} + i_{Y^{\mathcal{H}}}\omega = 0. \tag{4.16}$$

Let $L_{Y^{\mathcal{H}}}$ denote the Lie derivative operator associated with $Y^{\mathcal{H}}$. When introducing the extra weight $e^{-2\mathcal{H}}$ in the formula for the pairing of two forms, we obtain this way the symplectic Witten Laplacian $\left[d^M, \overline{d}^M_{2\mathcal{H}}\right]$. By [8, Proposition 2.2], the symplectic Witten Laplacian is given by

$$\left[d^M, \overline{d}^M_{2\mathcal{H}}\right] = -2L_{Y^{\mathcal{H}}}. \tag{4.17}$$

---

[2] From equation (4.14), one should conclude that the family of operators $L_b^X|_{b>0}$ interpolates between $-\Delta^X/2$ and $\frac{1}{2}|Y|^2 - Z$, which seems to contradict the assertion made in the paper that as $b \to +\infty$, we should get instead $-Z$. The contradiction disappears when one realizes that two conjugate versions of L_b^X may have different limits as $b \to +\infty$. This observation applies to the remainder of the paper.

Let us now return to the case of the manifold $\mathcal{X}$. By identifying TX and T^*X by the metric g^{TX}, $\mathcal{X}$ is also a symplectic manifold. If $\mathcal{H}=|Y|^2/2$, the corresponding Hamiltonian vector field $Y^{\mathcal{H}}$ is precisely the vector field Z.

Therefore, $-L_Z$, the natural extension of $-Z$ to smooth forms, is half of a symplectic Witten Laplacian. The interpolation between $-\Delta^X/2$ and $-Z$ lifts to the question of properly interpolating between $\square^X/2$ and $-L_Z$, which is an interpolation between a Riemannian Laplacian and a symplectic Witten Laplacian. As explained in [6, section 2.12], [9, section 3.2], the construction of the hypoelliptic Hodge Laplacian is obtained by interpolating between the scalar product $\langle\rangle$ on TX and the symplectic form ω on $T\mathcal{X}$. Namely for $b>0$, $U, V\in T\mathcal{X}$, set

$$\eta_b(U,V)=\langle\pi_*U,\pi_*V\rangle_{g^{TX}}+b\omega(U,V). \tag{4.18}$$

Then, η_b is a nondegenerate bilinear form. By proceeding exactly as before, it defines a nondegenerate bilinear pairing on $\Omega^c(\mathcal{X},\mathbf{R})$. Still correcting the volume form by the extra weight $\exp(-2\mathcal{H})$, it is shown in [6,9] that we obtain this way a Hodge hypoelliptic Laplacian $\mathcal{L}_b^X$ that restricts on functions to the scalar operator L_b^X. As $b\to 0$, $\mathcal{L}_b^X$ deforms $\square^X/2$, and as $b\to+\infty$, $\mathcal{L}_b^X$ converges in the proper sense to $-L_Z$. Also as shown in [6, section 2.12], [9, section 3.4], $\mathcal{L}_b^X$ is also self-adjoint with respect to a symmetric bilinear form of signature (∞,∞). Similar considerations are valid when considering instead the Hodge Laplacian $\square^X$ acting on $\Omega^\cdot(X,F)$. In this case, $\mathcal{L}_b^X$ acts on $\Omega^\cdot(\mathcal{X},\pi^*F)$.

4.8. HYPOELLIPTIC LAPLACIAN AND THE WITTEN DEFORMATION

As explained in [6, section 0] and in [7], the hypoelliptic Laplacian should be thought as a semi-classical limit of a nonexisting Witten Laplacian on the loop space LX. This implies that the hypoelliptic Laplacian should share many properties of the Witten deformation.

4.9. THE HYPOELLIPTIC TORSION

We make the same assumptions as in Section 2.1, and we assume m to be odd. In particular, $\square^X$ is invertible. By [13, Theorem 3.5.1], for $b>0$ small enough, the Hodge hypoelliptic Laplacian is invertible. As shown in [13, chapter 6], for $b>0$ small enough, it is still possible to define the hypoelliptic torsion $T_{\mathrm{an},b}$ in spite of the fact that in general the spectrum of $\mathcal{L}_b^X$ is not real. The key fact is that the eigenvalues of $\mathcal{L}_b^X$ still come by conjugate pairs. For arbitrary values of $b>0$, by [13, Theorem 3.6.2], except for a discrete set in $\mathbf{R}_+^*$ not accumulating at 0, $\mathcal{L}_b^X$ is invertible. For $b>0$ not belonging to this set, one can still define the hypoelliptic analytic torsion $T_{\mathrm{an},b}$ as before. As explained in [13, chapter 6], the theory of Quillen metrics [12,36] is enough to suitably extend the definition of $T_{\mathrm{an},b}$ to arbitrary $b>0$. There are questions of signs which we will not address here since, in this case, only $\exp(2T_{\mathrm{an},b})$ is unambiguously defined. In the sequel, we will disregard this subtlety.

We now state one key result of Bismut and Lebeau [13, Theorem 9.0.1].

THEOREM 4.6. *For any $b>0$, the following identity holds:*

$$\mathcal{T}_{\mathrm{an}}=\mathcal{T}_{\mathrm{an},b}. \tag{4.19}$$

The proof of Theorem 4.6 uses the full strength of the analysis which is needed to show that as $b\to 0$, the hypoelliptic Laplacian $\mathcal{L}_b^X$ deforms $\square^X/2$.

Remark 4.7. We will now give the proper perspective to Theorem 4.6. The Fried conjecture [21,22] predicts that for manifolds with strictly negative curvature, the analytic torsion can be expressed in terms of special values of a combination of values at 0 of Ruelle's zeta functions. In the introduction to [6], it was shown that at least formally, the Fried conjecture is an equivariant version of the Cheeger–Müller theorem on the loop space LX, if one takes the view in [15,16] that the Cheeger–Müller can be proved using the Witten deformation. Here, the relevant Morse–Bott function on LX would be the energy E. Theorem 4.6 is nothing else than a result that expresses the invariance of analytic torsion under the Witten deformation. The proof of the Fried conjecture would be completed if one could make $b\to+\infty$ in (4.19). The behaviour of the spectrum of $\mathcal{L}_b^X$ as $b\to+\infty$ makes this difficult.

By combining (2.7) and (4.19), we get

$$\mathcal{T}_{\mathrm{an}}=\mathcal{T}_{\mathrm{an},b}=\tau_{\mathrm{R}}. \tag{4.20}$$

4.10. THE RESULTS OF SHEN ON THE HYPOELLIPTIC TORSION

In [39,40], Shen gave a direct proof of the fact that

$$\mathcal{T}_{\mathrm{an},b}=\tau_{\mathrm{R}}. \tag{4.21}$$

Combining (2.7) and (4.21) gives a different proof of Theorem 4.6. The proof of Shen consists in deforming the hypoelliptic torsion to the combinatorial torsion, without going through the hard analysis involved in passing from the hypoelliptic Laplacian $\mathcal{L}_b^X$ to the elliptic Laplacian $\square^X/2$. To do this, Shen uses the full strength of the hypoelliptic theory, replacing the Hamiltonian $\mathcal{H}=|Y|^2/2$ by a linear combination of $|Y|^2/2$ and of a Morse function f on X. Shen's proof gives a new confirmation of the fundamental connection between the classical Witten deformation and the hypoelliptic Laplacian.

5. Hypoelliptic Laplacian, Orbital Integrals and the Trace Formula

The purpose of this section is to give a short introduction to the results of [9] on a different version of the hypoelliptic Laplacian on locally symmetric spaces, which is such that the spectrum of the original elliptic Laplacian remains rigidly embedded in the spectrum of the hypoelliptic deformation. This new version has been used in [9] to give another approach to Selberg's trace formula.

This section is organized as follows. In Section 5.1, we consider in detail the case where $X = S^1$, and we describe the spectrum of the scalar hypoelliptic Laplacian $L_b^{S^1}$.

In Section 5.2, we show how to derive the usual Poisson formula from a supersymmetric modification $\mathcal{L}_b^{S^1}$ of $L_b^{S^1}$.

In Section 5.3, we introduce the locally symmetric space $X = G/K$. The hypoelliptic Laplacian $\mathcal{L}_b^X$ now acts on the total space $\widehat{\mathcal{X}}$ of $TX \oplus N$, where N is an extra natural vector bundle on X. We only give the formula for $\mathcal{L}_b^X$, referring to [9] for details on its construction.

In Section 5.4, we explain the results of [9] according to which the trace of heat kernel for the elliptic Casimir operator is preserved under the hypoelliptic deformation. This property is extended to the corresponding semisimple orbital integrals.

In Section 5.5, we explain the uniform estimates of [9] on the hypoelliptic heat kernels on $\widehat{\mathcal{X}}$ for fixed $t > 0$ and bounded $b > 0$. These estimates play a key role in proving the invariance of semisimple orbital integrals under the hypoelliptic deformation.

Finally, in Section 5.6, we describe the main result of [9], in which arbitrary semisimple orbital integrals are evaluated. As a special case, when $G = \mathrm{SL}_2(\mathbf{R})$, we rederive the original Selberg trace formula for compact Riemann surfaces of constant negative curvature.

5.1. THE CASE OF S^1

We use the formalism of Section 4. Assume that $X = S^1$, so that $\widehat{\mathcal{X}} = S^1 \times \mathbf{R}$. This case has been extensively covered in [6, section 3.10], [8, section 1.3], [10, section 5.1]. The importance of the case $X = S^1$ is that S^1 is the simplest compact manifold, and also because S^1 is the model of a closed geodesic.

If (x, y) is the generic element in $S^1 \times \mathbf{R}$, then

$$L_b^{S^1} = \frac{1}{2b^2}\left(-\frac{\partial^2}{\partial y^2} + y^2 - 1\right) - \frac{y}{b}\frac{\partial}{\partial x}. \tag{5.1}$$

Also, we have the identity

$$\exp\left(b\frac{\partial^2}{\partial x \partial y}\right) L_b^{S^1} \exp\left(-b\frac{\partial^2}{\partial x \partial y}\right) = \frac{1}{2b^2}\left(-\frac{\partial^2}{\partial y^2} + y^2 - 1\right) - \frac{1}{2}\frac{\partial^2}{\partial x^2}. \tag{5.2}$$

Equation (5.2) should be viewed as an equation in the Lie algebra of differential operators in the variables x, y whose total weight obtained by giving weight 1 to the action of $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, y is lower or equal to 2. The question is how to interpret (5.2) as an identity of unbounded partial differential operators. The obvious difficulty is that $b\frac{\partial^2}{\partial x \partial y}$ is a hyperbolic operator, which has no heat kernel. As explained in [9, sections 10.3 and 10.5] and in [10, section 5.1], equation (5.2) can be thought of as a version of Egorov's theorem, in which the symplectic transformation to be quantized is imaginary.

Still by exploiting the fact that the eigenfunctions of the harmonic oscillator are analytic, one can show that an obvious formal consequence of (5.2) is indeed correct, i.e.

$$\mathrm{Sp}\left(L_b^{S^1}\right) = \left\{2k^2\pi^2\right\}_{k\in\mathbf{Z}} + \frac{\mathbf{N}}{b^2}. \tag{5.3}$$

In (5.3), the + sign indicates that elements in each set are added to each other.

From (5.3), we deduce that the spectrum of $-\frac{1}{2}\partial^2/\partial x^2$ on S^1 remains rigidly embedded in $\mathrm{Sp}\, L_b^{S^1}$. Moreover, by (5.3), as $b \to 0$, we have the convergence

$$\mathrm{Sp}\left(L_b^{S^1}\right) \to \mathrm{Sp}\left(-\frac{1}{2}\frac{\partial^2}{\partial x^2}\right). \tag{5.4}$$

Equation (5.4) can be viewed as a consequence of the convergence of resolvents described in Section 4.3.

5.2. THE HYPOELLIPTIC LAPLACIAN AND THE POISSON FORMULA

We will now show how to extract the spectrum of $-\frac{1}{2}\frac{\partial^2}{\partial x^2}$ from the spectrum of L_b^X. We introduce the exterior algebra $\Lambda^\cdot(\mathbf{R})$ and the corresponding number operator $N^{\Lambda^\cdot(\mathbf{R})}$. Set

$$\mathcal{L}_b^{S^1} = L_b^{S^1} + \frac{N^{\Lambda^\cdot(\mathbf{R})}}{b^2}. \tag{5.5}$$

By (5.3), (5.5), except for questions of multiplicity, we have

$$\mathrm{Sp}\,\mathcal{L}_b^{S^1} = \left\{2k^2\pi^2\right\}_{k\in\mathbf{Z}} + \frac{\mathbf{N}}{b^2}. \tag{5.6}$$

Now, we state a result that was established in [6, section 3.10], [8, section 1.2], [10, section 5.1]. Recall that the supertrace was defined in equation (3.27).

THEOREM 5.1. *For $b > 0, t > 0$, we have the identity:*

$$\mathrm{Tr}\left[\exp\left(\frac{t}{2}\frac{\partial^2}{\partial x^2}\right)\right] = \mathrm{Tr_s}\left[\exp\left(-t\mathcal{L}_b^{S^1}\right)\right]. \tag{5.7}$$

Proof. This is an easy consequence of (5.3), (5.6). □

Using the results of Section 4.6, as $b \to +\infty$, the supertrace in the right-hand side of (5.7) localizes along the trajectories of closed geodesics on S^1. As was explained in [6, section 3.10], we obtain this way a proof of the Poisson formula for the trace of the heat kernel on S^1.

Much more is true. It is shown in [10, Theorem 5.3] that an analogue of equation (5.7) still makes sense when $X = \mathbf{R}$. More precisely, equation (5.7) lifts to the equality of the individual terms in the Poisson sum. This observation is the starting point of our work on the hypoelliptic Laplacian and orbital integrals [9].

5.3. A LOCALLY SYMMETRIC SPACE

Let G be a reductive group of noncompact type, let $\theta \in \mathrm{Aut}(G)$ be the Cartan involution, and let $K \subset G$ be the corresponding maximal compact subgroup. Let $\mathfrak{g}, \mathfrak{k}$ be the Lie algebras of G, K, and let $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ be the Cartan splitting of $\mathfrak{g}$. Let B be a θ-invariant nondegenerate symmetric bilinear form on $\mathfrak{g}$ which is positive on $\mathfrak{p}$ and negative on $\mathfrak{k}$, the Cartan splitting being orthogonal. Let $C^{\mathfrak{g}}$ be the Casimir operator on G. This is a biinvariant differential operator of order 2 on G, whose principal symbol is given by $B(\xi, \xi)$. It can be thought of as the sum of a Laplacian along $\mathfrak{k}$ and minus the Laplacian in the directions $\mathfrak{p}$.

Let $X = G/K$ be the associated symmetric space. The principal K-bundle $\pi : G \to X$ is naturally equipped with a connection. Also X is a Riemannian manifold with nonpositive covariantly constant curvature, whose tangent bundle is given by $TX = G \times_K \mathfrak{p}$. Moreover, G acts isometrically and transitively on the left on X. The Casimir operator $C^{\mathfrak{g}}$ descends to the operator $-\Delta^X$ on X.

Let $\Gamma \subset G$ be a discrete cocompact torsion-free subgroup. Then, $Y = \Gamma \backslash X$ is a compact smooth manifold, which is locally symmetric.

EXAMPLE 5.2. The simplest nontrivial example is when $G = \mathrm{SL}_2(\mathbf{R})$, $\theta g = \widetilde{g}^{-1}$, so that $K = \mathrm{SO}(2)$. Then, $\mathfrak{g}$ is the Lie algebra of trace-free $(2,2)$ matrices, $\mathfrak{p}, \mathfrak{k}$ being the symmetric and antisymmetric matrices in $\mathfrak{g}$. We may take as B the bilinear form $2\mathrm{Tr}[AB]$ in the natural representation of $\mathrm{SL}_2(\mathbf{R})$. Then, X is the upper half-plane, and the Y are all the Riemann surfaces of genus $g \geq 2$ with constant scalar curvature -2.

Let us now summarize the constructions in [9]. As we saw before, $\mathfrak{p}$ descends to the tangent bundle TX. Similarly $\mathfrak{k}$ descends to the vector bundle $N = G \times_K \mathfrak{k}$, so that $TX \oplus N = G \times_K \mathfrak{g}$. Then, $TX \oplus N$ is a bundle of Lie algebras. Let $\rho : K \to \mathrm{Aut}(E)$ be a finite dimensional irreducible unitary representation of K, and let $F = G \times_K E$ be the corresponding Hermitian vector bundle on X. Let $C^{\mathfrak{g},X}$ be the action of $C^{\mathfrak{g}}$ on $C^\infty(X, F)$. Up to a constant, $C^{\mathfrak{g},X}$ coincides with minus the Bochner Laplacian $\Delta^{X,F}$.

Let $\widehat{\pi} : \widehat{\mathcal{X}} \to X$ be the total space of $TX \oplus N$. Now, we follow [9, section 2.13].

DEFINITION 5.3. For $b>0$, let $\mathcal{L}_b^X$ be the operator acting on

$$C^\infty\left(\widehat{\mathcal{X}}, \widehat{\pi}^*\left(\Lambda^\cdot\left(T^*X \oplus N^*\right) \otimes F\right)\right)$$

that is given by

$$\begin{aligned}\mathcal{L}_b^X = & \frac{1}{2}|[Y^N, Y^{TX}]|^2 + \frac{1}{2b^2}(-\Delta^{TX\oplus N} + |Y|^2 - m - n) + \frac{N^{\Lambda^\cdot(T^*X\oplus N^*)}}{b^2} \\ & + \frac{1}{b}\Big(\nabla^{C^\infty(TX\oplus N, \widehat{\pi}^*(\Lambda^\cdot(T^*X\oplus N^*)\otimes F))}_{Y^{TX}} + \widehat{c}(\mathrm{ad}(Y^{TX})) \\ & - c(\mathrm{ad}(Y^{TX}) + i\theta \mathrm{ad}(Y^N)) - i\rho^E(Y^N)\Big). \end{aligned} \tag{5.8}$$

We will not explain in detail the terms in equation (5.8). Let us just mention that $Y = Y^{TX} + Y^N$ is the generic section of $\widehat{\pi}^*(TX \oplus N)$. The reader should have recognized the harmonic oscillator along the fibre $TX \oplus N$, and also the generator of the geodesic flow $\nabla_{Y^{TX}}$ lifted to $\widehat{\mathcal{X}}$. The terms that depend linearly on Y^{TX}, Y^N are matrices acting on $\Lambda^\cdot(T^*X \oplus N^*)$ or F. The quartic term $\frac{1}{2}\left|\left[Y^{TX}, Y^N\right]\right|^2$ is the square of the norm of a fibrewise Lie bracket. Observe the fundamental fact that $\mathcal{L}_b^X$ acts on $\widehat{\mathcal{X}}$ and not on $\mathcal{X}$.

One key result proved in [9] is that as $b \to 0$, $\mathcal{L}_b^X$ deforms the elliptic operator $\left(C^{\mathfrak{g},X} - c\right)/2$, where c is an explicit constant.

5.4. THE PRESERVATION OF THE TRACE OF THE ELLIPTIC HEAT KERNEL

Let $Y = \Gamma\backslash X$ be a compact locally symmetric space as in Section 5.3. The operators $C^{\mathfrak{g},X}, \mathcal{L}_b^X$ descend to operators $C^{\mathfrak{g},Y}, \mathcal{L}_b^Y$.

The following analogue of Theorem 5.1 was established in [9, Theorem 4.8.1].

THEOREM 5.4. *For any $b>0, t>0$, then*

$$\mathrm{Tr}\left[\exp\left(-t\left(C^{\mathfrak{g},Y} - c\right)/2\right)\right] = \mathrm{Tr_s}\left[\exp\left(-t\mathcal{L}_b^Y\right)\right]. \tag{5.9}$$

The proof of Theorem 5.4 is relatively easy. However, making $b \to +\infty$ would force localization on an infinity of manifolds of closed geodesics in Y. In [9], one of the main points is to split the identity in (5.9) as an identity of orbital integrals. These orbital integrals are analogues of Fourier coefficients. The advantage is that given a semisimple element $\gamma \in G$, one can refine (5.9) to an identity of orbital integrals

$$\mathrm{Tr}^{[\gamma]}\left[\exp\left(-t\left(C^{\mathfrak{g},Y} - c\right)/2\right)\right] = \mathrm{Tr_s}^{[\gamma]}\left[\exp\left(-t\mathcal{L}_b^Y\right)\right]. \tag{5.10}$$

In the context of Section 5.2, this corresponds to replacing an identity of traces over S^1 to an identity of smooth kernels over $\mathbf{R}$.

The price to pay is that instead of working with ordinary traces over Y, we need to consider the corresponding smooth heat kernels over X or $\widehat{\mathcal{X}}$ and, more specifically, we need to control their long distance decay.

5.5. UNIFORM ESTIMATES ON THE HYPOELLIPTIC HEAT KERNELS

Let d denote the Riemannian distance on X. For $t>0$, let $p_t^X(x,x')$ be the smooth kernel associated with $\exp(-t(C^{\mathfrak{g},X}-c)/2)$. It is well known that given $\epsilon>0, M>0, \epsilon\leq M$, there exist $C>0, C'>0$ such that for $\epsilon\leq t\leq M, x, x'\in X$,

$$|p_t^X(x,x')|\leq C\exp(-C'd^2(x,x')). \tag{5.11}$$

For $b>0, t>0$, let $q_{b,t}^X((x,Y),(x',Y'))$ be the smooth kernel associated with $\exp(-t\mathcal{L}_b^X)$. Set

$$m=\dim\mathfrak{p},\quad n=\dim\mathfrak{k}. \tag{5.12}$$

Let $\mathbf{P}$ denote the projection from $\Lambda^{\cdot}(T^*X\oplus N^*)$ on $\Lambda^0(T^*X\oplus N^*)=\mathbf{R}$.

To establish (5.10), the proof consists that the right-hand side does not depend on $b>0$, and also in proving that as $b\to 0$, the right-hand side converges to the left-hand side. In that respect, we state a key result established in [9, Theorem 4.5.2].

THEOREM 5.5. *Given $\epsilon>0, M>0, \epsilon\leq M$, there exist $C>0, C'>0$ such that for $0<b\leq M, \epsilon\leq t\leq M$, $(x,Y),(x',Y')\in\widehat{\mathcal{X}}$,*

$$|q_{b,t}^X((x,Y),(x',Y'))|\leq C\exp(-C'(\mathrm{d}^2(x,x')+|Y|^2+|Y'|^2)). \tag{5.13}$$

Moreover, as $b\to 0$,

$$\begin{aligned} q_{b,t}^X((x,Y),(x',Y'))&\to q_{0,t}^X((x,Y),(x',Y'))\\ &=\mathbf{P}p_t^X(x,x')\pi^{-(m+n)/2}\exp\left(-\frac{1}{2}(|Y|^2+|Y'|^2)\right)\mathbf{P}. \end{aligned}\tag{5.14}$$

The proof of Theorem 5.5 given in [9] is based on probabilistic considerations. A uniform version as $b\to 0$ of the Malliavin calculus [3,28,43] plays a key role in the proof. Needless to say, the estimates in Theorem 5.5 are also valid for the scalar kernels that were considered in Section 4.4, when X is taken to be our symmetric space. More generally, such estimates still hold on noncompact manifolds with uniform geometry.

5.6. THE LIMIT AS $b\to+\infty$

In [9], we take the limit of (5.10) as $b\to+\infty$. As was explained before, on the compact manifold Y, $\mathrm{Tr_s}\left[\exp\left(-t\mathcal{L}_b^Y\right)\right]$ localizes on closed geodesics. Similarly if

$\gamma \in G$ is semisimple, as $b \to +\infty$, $\mathrm{Tr_s}^{[\gamma]}\left[\exp\left(-t\mathcal{L}_b^X\right)\right]$ localizes near $X(\gamma) \subset X$, the minimizing set for the displacement function $d_\gamma(x) = d(x, \gamma x)$. This minimizing set is just the symmetric space associated with the centralizer $Z(\gamma) \subset G$ of γ. Ultimately, the limit can be explicitly computed in terms of the local geometry of $X(\gamma)$ and the action of γ on various vector bundles over $X(\gamma)$.

Let us briefly describe the formula obtained in [9]. After conjugation, we can write γ in the form

$$\gamma = e^a k^{-1}, \tag{5.15}$$

with

$$a \in \mathfrak{p}, \quad k \in K, \quad \mathrm{Ad}(k)\, a = a. \tag{5.16}$$

Here, $|a|$ will eventually represent the length of a closed geodesic in Y.

Let $\mathfrak{z}(\gamma) = \mathfrak{p}(\gamma) \oplus \mathfrak{k}(\gamma)$ be the Lie algebra of $Z(\gamma)$. Let $\mathfrak{z}_0 \subset \mathfrak{g}$ be the kernel of $\mathrm{ad}(a)$, and let $\mathfrak{z}_0^\perp \subset \mathfrak{g}$ be its orthogonal with respect to B. Note that $\mathfrak{z}(\gamma) \subset \mathfrak{z}_0$. Let $\mathfrak{z}_0^\perp(\gamma)$ be the orthogonal to $\mathfrak{z}(\gamma)$ in $\mathfrak{z}_0$.

Now, we follow [9, Theorem 5.5.1].

DEFINITION 5.6. Let $J_\gamma\left(Y_0^{\mathfrak{k}}\right)$ be the function on $\mathfrak{k}(\gamma)$ given by

$$J_\gamma\left(Y_0^{\mathfrak{k}}\right) = \frac{1}{\left|\det\left(1 - \mathrm{Ad}(\gamma)\right)|_{\mathfrak{z}_0^\perp}\right|^{1/2}} \frac{\widehat{A}\left(i\,\mathrm{ad}\left(Y_0^{\mathfrak{k}}\right)|_{\mathfrak{p}(\gamma)}\right)}{\widehat{A}\left(i\,\mathrm{ad}\left(Y_0^{\mathfrak{k}}\right)|_{\mathfrak{k}(\gamma)}\right)} \times \left[\frac{1}{\det\left(1 - \mathrm{Ad}\left(k^{-1}\right)\right)|_{\mathfrak{z}_0^\perp(\gamma)}} \frac{\det\left(1 - \exp\left(-i\,\mathrm{ad}\left(Y_0^{\mathfrak{k}}\right)\right)\mathrm{Ad}\left(k^{-1}\right)\right)|_{\mathfrak{k}_0^\perp(\gamma)}}{\det\left(1 - \exp\left(-i\,\mathrm{ad}\left(Y_0^{\mathfrak{k}}\right)\right)\mathrm{Ad}\left(k^{-1}\right)\right)|_{\mathfrak{p}_0^\perp(\gamma)}}\right]^{1/2}. \tag{5.17}$$

The fact that the square roots in (5.17) are unambiguously defined is established in [9, chapter 5]. The function J_γ is the quotient of two functions associated with $\mathfrak{p}$ and $\mathfrak{k}$. Geometrically, this represents the quotient of two equivariant genera associated with TX, N respectively.

Now, we state the result in [9, Theorem 6.1.1], which was obtained from (5.10) by making $b \to +\infty$. Set $p = \dim \mathfrak{p}(\gamma), q = \dim \mathfrak{k}(\gamma)$.

THEOREM 5.7. *For any $t > 0$, the following identity holds:*

$$\mathrm{Tr}^{[\gamma]}\left[\exp\left(-t\left(C^{\mathfrak{g},X} - c\right)/2\right)\right] = \frac{\exp\left(-|a|^2/2t\right)}{(2\pi t)^{p/2}} \int_{\mathfrak{k}(\gamma)} J_\gamma\left(Y_0^{\mathfrak{k}}\right) \mathrm{Tr}^E\left[\rho^E\left(k^{-1}\right)\exp\left(-i\rho^E\left(Y_0^{\mathfrak{k}}\right)\right)\right] \exp\left(-\left|Y_0^{\mathfrak{k}}\right|^2/2t\right) \frac{dY_0^{\mathfrak{k}}}{(2\pi t)^{q/2}}. \tag{5.18}$$

Remark 5.8. In [9, Theorems 6.2.2 and 6.3.2], it is shown how to derive a corresponding formula for more general kernels that include the wave kernel. One remarkable aspect of equation (5.18) is that the orbital integral is expressed as an integral over a part of the Lie algebra $\mathfrak{k}$, which is partly hidden from the analyst who would work on the total space of the tangent bundle $\mathcal{X}$. This explains in retrospect why one is forced to introduce the larger space $\widehat{\mathcal{X}}$.

Let us now make the same assumptions as in Example 5.2. Here, Y is a Riemann surface of constant scalar curvature -2. Then, Selberg's original trace formula [30, p. 233] can be written in the form:

$$\operatorname{Tr}\left[\exp\left(t\Delta^{Y}/2\right)\right]=\frac{\exp(-t/8)}{2\pi t}\operatorname{Vol}(Y)\int_{\mathbf{R}}\exp\left(-y^2/2t\right)\frac{y/2}{\sinh(y/2)}\frac{dy}{\sqrt{2\pi t}}$$
$$+\sum_{\gamma\neq 0}\frac{\operatorname{Vol}_{\gamma}}{\sqrt{2\pi t}}\frac{\exp\left(-\ell_{\gamma}^2/2t-t/8\right)}{2\sinh\left(\ell_{\gamma}/2\right)}. \tag{5.19}$$

In (5.19), the γ described the nontrivial closed geodesics in Y of length l_{γ}, and $\operatorname{Vol}_{\gamma}$ is the length of the corresponding primitive closed geodesic. Except for the volume factors, the objects appearing in the right-hand side are orbital integrals that can be reobtained using Theorem 5.7. When γ is the identity, then $\mathfrak{k}(\gamma)=\mathbf{R}$, which explains the integral over $\mathbf{R}$. When $\gamma\in\Gamma$ is nontrivial, then $\mathfrak{k}(\gamma)=\{0\}$, which explains why there is no integral. Ultimately, Selberg's trace formula in (5.19) has been obtained via an interpolation process between the left-hand side and the right-hand side via the hypoelliptic Laplacian.

6. Geodesic Flow, Fourier Transform, and the Wave Equation

The purpose of this section is to explain the appearance of the geodesic flow in the hypoelliptic Laplacian as part of a dynamical version of Fourier transform. Also, we exhibit relations of the hypoelliptic Laplacian to the wave equation.

This section is organized as follows. In Section 6.1, we connect the geodesic flow to Fourier transform.

In Section 6.2, we develop the considerations of [9] showing that as $b\to 0$, the projection on X of the hypoelliptic heat kernel can be viewed as the solution of an 'intelligent' wave equation.

We make the same assumptions as in Section 4, and we use the corresponding notation. Also we use the fact that the symbol of a vector field can be identified with the vector field itself.

6.1. THE GEODESIC FLOW AND ITS FOURIER SYMBOL

Let i be the embedding of the fibres TX in $\mathcal{X}$. We have the exact sequence

$$0\to\pi^{*}TX\xrightarrow{i_{*}}T\mathcal{X}\xrightarrow{\pi_{*}}\pi^{*}TX\to 0. \tag{6.1}$$

In (6.1), the first $\pi^* TX$ is identified with the tangent bundle to the fibre TX. By (6.1), we have the dual exact sequence

$$0 \to \pi^* T^* X \xrightarrow{\pi^*} T^* \mathcal{X} \xrightarrow{i^*} \pi^* T^* X \to 0. \tag{6.2}$$

Let η denote the canonical section of $T^*\mathcal{X}$ on the total space of $T^*\mathcal{X}$. The symbol of the radial vector field Y along the fibres TX is given by $\sqrt{-1}\langle Y, i^*\eta\rangle$.

Let g^{TX} be a Riemannian metric on TX, and let ∇^{TX} denote the corresponding Levi-Civita connection on TX. The connection ∇^{TX} induces a splitting

$$T\mathcal{X} = \pi^*(TX \oplus TX). \tag{6.3}$$

The first copy $\pi^* TX$ corresponds to horizontal vectors in $T\mathcal{X}$, and the second copy to vertical vectors. We also have the dual splitting

$$T^*\mathcal{X} = \pi^*\left(T^*X \oplus T^*X\right). \tag{6.4}$$

The first copy π^*T^*X consists of the pullbacks of 1-forms on X, and the second to the 1-forms that are vertical, i.e. they vanish on horizontal vectors.

The effect of (6.3), (6.4) is to split the exact sequences (6.1), (6.2). In particular, if $\eta \in T^*\mathcal{X}$, then η splits as:

$$\eta = (\xi, \upsilon). \tag{6.5}$$

The symbol of the radial vector field Y is the function $\sqrt{-1}\langle Y, \upsilon\rangle$ that does not depend on g^{TX}.

PROPOSITION 6.1. *The symbol of the vector field Z is given by*

$$\sigma(Z) = \sqrt{-1}\langle Y, \xi\rangle. \tag{6.6}$$

Proof. Since Z is a horizontal vector field, this just follows from (4.2). □

Remark 6.2. As explained before, the function $\sqrt{-1}\langle Y, \upsilon\rangle$ corresponds to the radial vector field Y and is well defined without any choice of g^{TX}. However, the function $\sqrt{-1}\langle Y, \xi\rangle$ makes sense as a symbol on $\mathcal{X}$ only once the choice of a metric g^{TX} has been made. The function $\sqrt{-1}\langle Y, \xi\rangle$ will be called a Fourier symbol.

Consider equation (4.3) for L_b^X. The heat kernel for L_b^X gives a dynamical interpretation of Fourier transform. From this point of view, the vector field Z is the only operator in L_b^X that connects the fibre TX to the base via the geodesic flow $\dot{x} = Y$. This is to be contrasted with the use of the Fourier transform in the theory of pseudodifferential operators, in which the Fourier transform is made locally via a choice of local coordinates.

The reason for the introduction of the geodesic flow in the context of the trace formula is precisely because it contributes to a global Fourier transform, which can be made to be exact in the case of locally symmetric spaces. The classical heat flow $\exp(t\Delta^X/2)$ can again be viewed as the exponential of a Toeplitz operator, obtained via a local Fourier transform of the function $-|\xi|^2/2$, while the hypoelliptic heat flow $\exp\left(-tL_b^X\right)$ is the exponential of an operator in which the Fourier transform is done dynamically.

6.2. THE HYPOELLIPTIC LAPLACIAN AND THE WAVE EQUATION

As we saw in Section 4, L_b^X interpolates between the $-\Delta^X/2$ and $-Z$. The elliptic heat flow $\exp\left(t\Delta^X/2\right)$ has infinite propagation speed, while the geodesic flow $\exp(tZ)$ propagates with finite constant speed on X. It is natural to ask how the hypoelliptic heat flow $\exp\left(-tL_b^X\right)$ propagates on X. Giving the proper answer to this question plays a crucial role in [9], and partly explains the uniform estimates in Theorem 5.5.

Set

$$M_b^X = \exp(|Y|^2/2) L_b^X \exp(-|Y|^2/2). \tag{6.7}$$

Let ∇_Y^V denote fibrewise differentiation with respect to the radial vector field Y. Then

$$M_b^X = \frac{1}{2b^2}\left(-\Delta^V + 2\nabla_Y^V\right) - \frac{Z}{b}. \tag{6.8}$$

Let $f \in C^\infty(X, \mathbf{R})$. Then, f lifts to a smooth real function on $\mathcal{X}$. One verifies easily that

$$\left(b^2 M_b^{X,2} - M_b^X\right) f = Z^2 f. \tag{6.9}$$

By (6.9), we deduce that

$$\left(b^2 \frac{\mathrm{d}^2}{\mathrm{d}t^2} + \frac{\mathrm{d}}{\mathrm{d}t}\right) \exp\left(-tM_b^X\right) f = \exp\left(-tM_b^X\right) Z^2 f. \tag{6.10}$$

We follow [9, section 12.3]. Given $b > 0, t > 0$, let $s_{b,t}((x,Y),(x',Y'))$ be the smooth kernel associated with $\exp\left(-tM_b^X\right)$. As explained in [9], $s_{b,t}$ is positive on $\mathcal{X} \times \mathcal{X}$. Put

$$\sigma_{b,t}((x,Y),x') = \int_{T_{x'}X} s_{b,t}((x,Y),(x',Y'))\mathrm{d}Y',$$
$$M_{b,t}((x,Y),x') = \frac{1}{\sigma_{b,t}((x,Y),x')} \int_{T_{x'}X} s_{b,t}^X((x,Y),(x',Y'))(Y' \otimes Y')\mathrm{d}Y'. \tag{6.11}$$

As a function of x', $\sigma_{b,t}$ is the projection on X of $s_{b,t}((x,Y),(x',Y'))$. Also $M_{b,t}((x,Y),x')$ is a symmetric positive quadratic form on $T_{x'}X$. To this quadratic form, we can associate the second-order elliptic operator acting on $C^\infty(X,\mathbf{R})$,

$$\mathbf{M}_{b,t}((x,Y),x')g(x') = \langle M_{b,t}((x,Y),x'), \nabla_\cdot^{TX}\nabla_\cdot\rangle g(x'). \tag{6.12}$$

By (6.10)–(6.12), we obtain

$$\left(b^2\frac{\partial^2}{\partial t^2}+\frac{\partial}{\partial t}\right)\sigma_{b,t}f-\sigma_{b,t}\mathbf{M}_{b,t}f=0. \tag{6.13}$$

Equation (6.13) is not an autonomous equation for the kernel $\sigma_{b,t}\left((x,Y),\cdot\right)$, and it is not time homogeneous. Still it is similar to a wave equation.

Moreover, by Theorem 4.5 and using the uniform bounds in [13, section 3.4], for $t>0$, we get

$$\sigma_{b,t}(((x,Y),x'))\to p_t^X(x,x'),\quad \mathbf{M}_{b,t}((x,Y),x')\to\Delta^X/2. \tag{6.14}$$

From the above, it follows that as $b\to 0$, the projected heat kernel $\sigma_{b,t}((x,Y),x')$ is the solution of a wave-like equation with propagation speed $1/b$.

The heat operators $\exp\left(-tL_b^X\right),\exp\left(-tM_b^X\right)$ produce the solution of an intelligent wave equation, which is programmed to look for geodesics as $b\to+\infty$. When taking their traces, this produces a localization on closed geodesics.

References

1. Berezin, F.A.: Quantization. Izv. Akad. Nauk SSSR Ser. Mat. **38**, 1116–1175 (1974)
2. Bergeron, N., Venkatesh, A.: The asymptotic growth of torsion homology for arithmetic groups. J. Inst. Math. Jussieu **12**(2), 391–447 (2013)
3. Bismut, J.-M.: Martingales, the Malliavin calculus and hypoellipticity under general Hörmander's conditions. Z. Wahrsch. Verw. Gebiete. **56**(4), 469–505 (1981)
4. Bismut, J.-M.: Equivariant immersions and Quillen metrics. J. Differ. Geom. **41**(1), 53–157 (1995)
5. Bismut. J.-M.: Le laplacien hypoelliptique. In: Séminaire: Équations aux Dérivées Partielles. 2003–2004. Sémin. Équ. Dériv. Partielles. pages Exp. No. XXII. 15. École Polytech. Palaiseau (2004)
6. Bismut, J.-M.: The hypoelliptic Laplacian on the cotangent bundle. J. Am. Math. Soc. **18**(2), 379–476 (2005). **(Electronic)**
7. Bismut, J.-M.: Loop spaces and the hypoelliptic Laplacian. Commun. Pure Appl. Math. **61**(4), 559–593 (2008)
8. Bismut, J.-M.: A survey of the hypoelliptic Laplacian. *Astérisque*. Géométrie différentielle, physique mathématique, mathématiques et société. II **322**, 39–69 (2008)
9. Bismut, J.-M.: Hypoelliptic Laplacian and orbital integrals. In: Annals of Mathematics Studies, vol. 177. Princeton University Press, Princeton (2011)
10. Bismut, J.-M.: Index theory and the hypoelliptic Laplacian. In: Metric and differential geometry. In: Progress in Mathematics, vol 297, pp. 181–232. Birkhäuser/Springer, Basel (2012)
11. Bismut, J.-M.: Hypoelliptic Laplacian and probability. J. Math. Soc. Jpn. **67**(4), 1317–1357 (2015)
12. Bismut, J.-M., Gillet, H., Soulé, C.: Analytic torsion and holomorphic determinant bundles. III. Quillen metrics on holomorphic determinants. Commun. Math. Phys. **115**(2), 301–351 (1988)
13. Bismut, J.-M., Lebeau, G.: The hypoelliptic Laplacian and Ray–Singer metrics. In: Annals of Mathematics Studies, vol. 167. Princeton University Press, Princeton (2008)

14. Bismut, J.-M., Ma, X., Zhang, W.: Asymptotic torsion and Toeplitz operators. J. Inst. Math. Jussieu 1–127 (2015)
15. Bismut, J.-M., Zhang, W.: An extension of a theorem by Cheeger and Müller. *Astérisque* **205**, 235 (1992). **(With an appendix by François Laudenbach)**
16. Bismut, J.-M., Zhang, W.: Milnor and Ray–Singer metrics on the equivariant determinant of a flat vector bundle. Geom. Funct. Anal. **4**(2), 136–212 (1994)
17. Bordemann, M., Meinrenken, E., Schlichenmaier, M.: Toeplitz quantization of Kähler manifolds and gl(N), $N \to \infty$ limits. Commun. Math. Phys. **165**(2), 281–296 (1994)
18. Boutet de Monvel, L., Guillemin, V.: The spectral theory of Toeplitz operators. In: Annals of Mathematics Studies, vol. 99. Princeton University Press, Princeton (1981)
19. Boutet de Monvel, L., Sjöstrand, J.: Sur la singularité des noyaux de Bergman et de Szegő. In: *Journées: Équations aux Dérivées Partielles de Rennes (1975)*, pp. 123–164. Astérisque, No. 34–35. Soc. Math. France, Paris (1976)
20. Cheeger, J.: Analytic torsion and the heat equation. Ann. Math. **2**(109), 259–322 (1979)
21. Fried, D.: Analytic torsion and closed geodesics on hyperbolic manifolds. Invent. Math. **84**(3), 523–540 (1986)
22. Fried, D.: The zeta functions of Ruelle and Selberg. I. Ann. Sci. École Norm. Sup. **4**(19), 491–517 (1986)
23. Getzler, E.: A short proof of the local Atiyah–Singer index theorem. Topology **25**(1), 111–117 (1986)
24. Helffer, B., Sjöstrand, J.: Puits multiples en mécanique semi-classique. IV. Étude du complexe de Witten. Commun. Part. Differ. Equ. **10**(3), 245–340 (1985)
25. Hörmander, L.: Hypoelliptic second order differential equations. Acta Math. **119**, 147–171 (1967)
26. Ma, X., Marinescu, G.: Holomorphic Morse inequalities and Bergman kernels. In: Progress in Mathematics, vol. 254. Birkhäuser Verlag, Basel (2007)
27. Ma, X., Marinescu, G.: Berezin–Toeplitz quantization on Kähler manifolds. J. Reine Angew. Math. **662**, 1–56 (2012)
28. Malliavin, P.: Stochastic calculus of variation and hypoelliptic operators. In: Proceedings of the International Symposium on Stochastic Differential Equations (Res. Inst. Math. Sci., Kyoto Univ., Kyoto, 1976), pp. 195–263. Wiley, New York (1978)
29. Mathai, V., Quillen, D.: Superconnections, Thom classes, and equivariant differential forms. Topology **25**(1), 85–110 (1986)
30. McKean, H.P.: Selberg's trace formula as applied to a compact Riemann surface. Commun. Pure Appl. Math. **25**, 225–246 (1972)
31. Milnor, J.: Whitehead torsion. Bull. Am. Math. Soc. **72**, 358–426 (1966)
32. Müller, W.: Analytic torsion and R-torsion of Riemannian manifolds. Adv. Math. **28**(3), 233–305 (1978)
33. Müller, W.: Analytic torsion and R-torsion for unimodular representations. J. Am. Math. Soc. **6**(3), 721–753 (1993)
34. Müller, W.: The asymptotics of the Ray–Singer analytic torsion of hyperbolic 3-manifolds. In: Metric and Differential Geometry. Progr. Math., vol. 297, pp. 317–352. Birkhäuser/Springer, Basel (2012)
35. Müller, W., Pfaff, J.: On the asymptotics of the Ray–Singer analytic torsion for compact hyperbolic manifolds. Int. Math. Res. Not. IMRN **13**, 2945–2983 (2013)
36. Quillen, D.: Determinants of Cauchy–Riemann operators on Riemann surfaces. Funct. Anal. Appl. **19**(1), 31–34 (1985)
37. Reidemeister, K.: Homotopieringe und Linsenraüm. In: Hamburger Abhandl., pp. 102–109 (1935)

38. Ray, D.B., Singer, I.M.: R-torsion and the Laplacian on Riemannian manifolds. Adv. Math. **7**, 145–210 (1971)
39. Shen, S.: The hypoelliptic Laplacian, analytic torsion and Cheeger–Müller theorem. C. R. Math. Acad. Sci. Paris **352**(2), 153–156 (2014)
40. Shen, S.: Laplacien hypoelliptique, torsion analytique, et théorème de Cheeger–Müller. J. Funct. Anal **270**(8), 2817–2999 (2016)
41. Smale, S.: On gradient dynamical systems. Ann. Math. **2**(74), 199–206 (1961)
42. Smale, S.: Differentiable dynamical systems. Bull. Am. Math. Soc. **73**, 747–817 (1967)
43. Stroock, D.W.: The Malliavin calculus, a functional analytic approach. J. Funct. Anal. **44**(2), 212–257 (1981)
44. Thom, R.: Sur une partition en cellules associée à une fonction sur une variété. C. R. Acad. Sci. Paris **228**, 973–975 (1949)
45. Witten, E.: Supersymmetry and Morse theory. J. Differ. Geom. **17**(4), 661–692 (1983)