Topology of rationally and polynomially convex domains

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Stanford University

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Conference in honor of Jean-Michel Bismut

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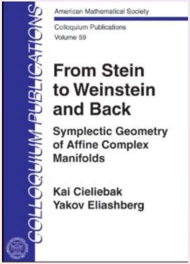

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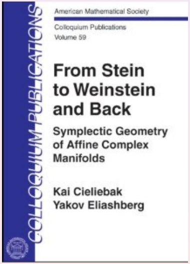

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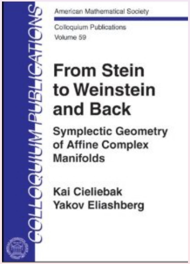

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Polynomial, Rational and Holomorphic hulls

For a compact set $K \subset \mathbb{C}^n$, one defines its *polynomial hull* as

$$\widehat{K}_{\mathcal{P}} := \{z \in \mathbb{C}^n \Big| \, |P(z)| \leq \max_{u \in K} |P(u)|, \ P : \mathbb{C}^n \to \mathbb{C} \ \text{ is a polynomial}\}$$

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Given an open set $U \supset K$, its *holomorphic hull in U* is

$$\widehat{K}_{\mathcal{H}}^{U} := \{z \in U \; \Big| \; |f(z)| \leq \max_{u \in K} |f(u)|; \; f : U \to \mathbb{C} \text{ is holomorphic}\}.$$

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A compact set $K \subset \mathbb{C}^n$ is called *rationally* (resp. *polynomially*) *convex* if

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The rational convexity of K is equivalent to the following condition:

(P) for every point $a \in \mathbb{C}^n \setminus K$ there exists a polynomial P_a such that $P_a(a) = 0$ and $P_a|_K \neq 0$.

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polynomially convex $\Longrightarrow$ rationally convex $\Longrightarrow$ holomorphically convex.

For a real valued function $\phi : U \to \mathbb{R}$ on an open subset $U \subset \mathbb{C}^n$, we denote $d^{\mathbb{C}}\phi := d\phi \circ i$ and set

$$\omega_\phi := -dd^{\mathbb{C}}\phi = 2i\partial\bar{\partial}\phi = 2i \sum_{i,j} \frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j} dz_i \wedge d\bar{z}_j.$$

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A cooriented hypersurface $\Sigma \subset \mathbb{C}^n$ (of real codimension 1) is called *i-convex* if there exists an i-convex function ϕ defined on some neighborhood of Σ such that $\Sigma = \{\phi = c\}$, and Σ is cooriented by a vector field v satisfying $d\phi(v) > 0$.

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More generally in a complex manifold (V, J) we use the term J-convexity.

Traditionally J-convexity for functions and hypersurfaces is called *strict plurisubharmonicity*, and *strict pseudoconvexity*, respectively.

Terminology

A (compact) cobordism W between $\partial_- W$ and $\partial_+ W$ we call a *domain* $\partial_- W = \varnothing$, so that $\partial W = \partial_+ W$.

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A function $\phi : W \to \mathbb{R}$ on a cobordism W is called *defining* if $\partial_\pm W$ are regular level sets and $\phi|_{\partial_- W} = \min\limits_W \phi$, $\phi|_{\partial_+ W} = \max\limits_W \phi$.

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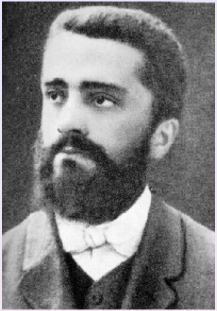
A domain $W \subset \mathbb{C}^n$ is called *i-convex* if its boundary is i-convex. Any weakly i-convex domain in $\mathbb{C}^n$ can be C^∞-approximated by a slightly smaller i-convex one.

Holomorphic vs *i*-convexity

According to a theorem of E. Levi any holomorphically convex domain $W \subset \mathbb{C}^n$ is *weakly* *i*-convex.



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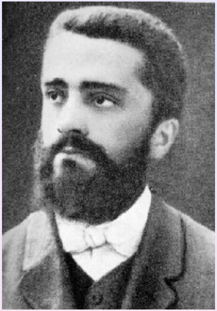

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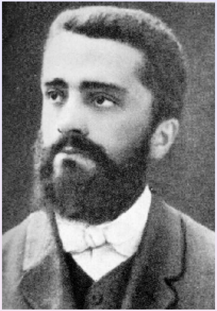

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Stein domains and cobordisms

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More generally, a Stein cobordism (W, J) is a smooth cobordism between $\partial_- W$ and $\partial_+ W$ with a complex structure J which admits a defining J-convex function $\phi : W \to \mathbb{R}$.

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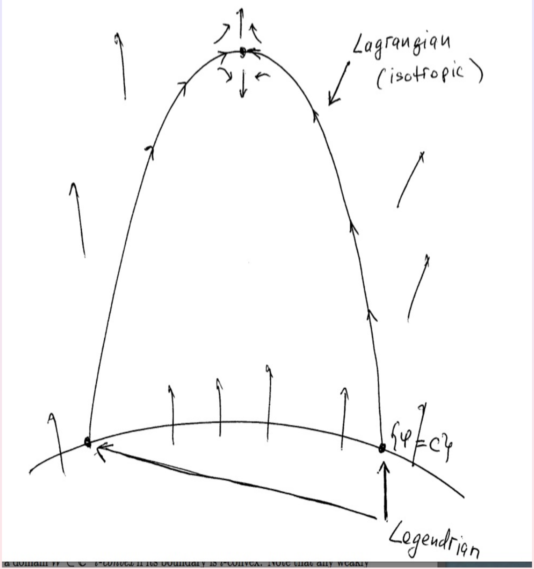
The union $K_\phi \subset W$ is called the skeleton. The flow X_ϕ^{-t} retracts W onto an arbotrarily small neighborhood of K_ϕ.

Symplectic aspects of J-convexity



Lagrangian stable manifold of a critical point of a J-convex function ϕ intersects a contact level set $\{\phi = c\}$ along a Legendrian submanifold of the level sets.

Theorem (E.,1990)

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Topological characterization of Stein domains of dim > 2

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The key analytic ingredient in the proof is the following

Proposition

Suppose that the i-convex domain $W \subset \mathbb{C}^n$ is rationally convex and $\Delta \subset \mathbb{C}^n \setminus \mathrm{Int}\, W$ an n-disc which intersects ∂W transversely along a Legendrian sphere $\partial\Delta$. if Δ is totally real then $W \cup \Delta$ has an arbitrary small i-convex neighborhood

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Oka:

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Duval-Sibony, Nemirovski:

An i-convex domain $W \subset \mathbb{C}^n$ is rationally convex if and only if the following condition holds:

(R) There exists an i-convex function $\phi : W \to \mathbb{R}$ such that $W = \{\phi \leq 0\}$, and the form $-dd^{\mathbb{C}}\phi$ on W extends to a Kähler form ω on the whole $\mathbb{C}^n$.

Rational convexity of the isotropic skeleton

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Corollary

Let W be a Stein domain and $\phi : W \to \mathbb{R}$ a defining J-convex Morse. Suppose that there exists a symplectic embedding $h : (W, \omega_\phi) \to (\mathbb{C}^n, \omega_{\mathrm{st}})$. Then the image $h(K_\phi)$ of the skeleton K_ϕ admits an arbitrary small rationally convex neighborhood.

Topological characterization of polynomially and rationally convex domains

Main Theorem

Consider a domain $W \subset \mathbb{C}^n$, $n > 2$.

1. Then W is isotopic to a rationally convex domain if and only if it admits a defining function without critical points of index $> n$.

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(a) If W is simply connected, then the condition of Theorem 2 is equivalent to the existence of a defining Morse function without critical points of index $\geq n$.

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(b) For any $n \geq 3$ there exists a (non-simply connected) domain W satisfying the condition of Theorem 2 but which does not admit a defining function without critical points of index $\geq n$.

Flexible Stein domains

There is class of Stein domains (cobordisms) in $\mathbb{C}^n$, $n > 2$, called flexible, with the following properties:

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4. *Let $W \subset \mathbb{C}^n$ be a flexible Stein domain and $\phi : W \to \mathbb{R}$ a defining i-convex function. Then the inclusion $W \hookrightarrow \mathbb{C}^n$ is isotopic to a symplectic embedding $h : (W, \omega_\phi) \to (\mathbb{C}^n, \omega_{\mathrm{st}})$.*

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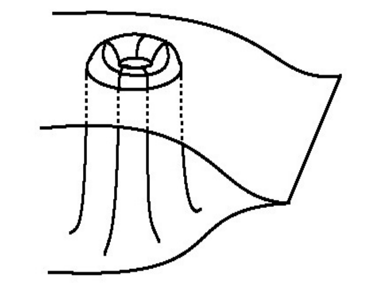

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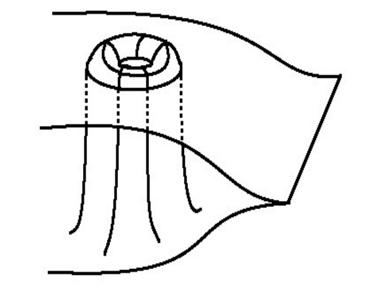

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Theorem (Murphy's h-principle)

For loose Legendrian knots formal Legendrian isotopy implies genuine Legendrian isotopy.

Lagrangian caps

Question. Let B be the round ball in the standard symplectic $\mathbb{R}^{2n}$. *Is there an embedded Lagrangian disc* $\Delta \subset \mathbb{R}^{2n} \setminus \operatorname{Int} B$ *with* $\partial\Delta \subset \partial B$ *such that* $\partial\Delta$ *is a Legendrian submanifold and* Δ *transversely intersects* ∂B *along its boundary?*

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Given an i-convex domain $W \subset \mathbb{C}^n$ and an n-dimensional compact manifold L with boundary, for Lagrangian embeddings $f : (L, \partial L) \to \mathbb{C}^n \setminus \mathrm{Int}\, W)$ with loose Legendrian boundary $f(\partial L) \subset \partial W$ one has an h-principle.

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If $n = 2$ then such a Lagrangian disc does not exist, because its existence would contradict the slice Thurston-Bennequin inequality (Lee Rudolph).

Theorem

Given an i-convex domain $W \subset \mathbb{C}^n$ and an n-dimensional compact manifold L with boundary, for Lagrangian embeddings $f : (L, \partial L) \to \mathbb{C}^n \setminus \mathrm{Int}\, W)$ with loose Legendrian boundary $f(\partial L) \subset \partial W$ one has an h-principle. In particular, the triviality of the complexified tangent bundle $TL \otimes \mathbb{C}$ is a necessary and sufficient condition for existence of a Lagrangian embedding $f : (L, \partial L) \to (\mathbb{R}^{2n} \setminus \mathrm{Int}\, W, \partial W)$ with Legendrian boundary $f(\partial L) \subset \partial W$.

Flexible Stein cobordisms

A Stein cobordism (W, J, ϕ) together with a defining J-convex function ϕ is called elementary if there are no gradient trajectories of $X_{J,\phi}$ connecting critical points of ϕ. Any cobordism can be sliced into elementary ones.

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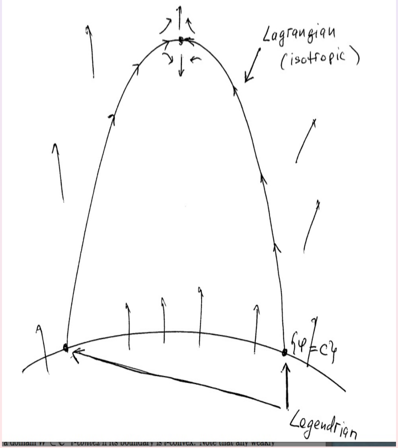

An elementary Stein cobordism (W, J, ϕ) is called flexible if the attaching spheres of stable discs of all index n critical points form a loose Legendrian link. There are no constraints on index $< n$ critical points.

A general Stein cobordism (W, J) is called *flexible* if there exists a defining function $\phi : W \to \mathbb{R}$ such that (W, J, ϕ) can be sliced into elementary flexible Stein cobordisms.

Flexible Stein cobordisms

Theorem

1. *Any domain in $\mathbb{C}^n$, $n > 2$, which admits a defining function without critical points of index $> n$ is isotopic to a flexible Stein domain.*
2. *Moreover, any two smoothly isotopic flexible Stein domains in $\mathbb{C}^n$ are isotopic through i-convex domains.*
3. *Given any flexible Stein cobordism $W \subset \mathbb{C}^n$, any defining Morse function $\phi : W \to \mathbb{R}$ without critical points of index $> n$ is equivalent to an i-convex function, i.e. there exists isotopic to the identity diffeomorphisms $h : W \to W$ and $g :\to \mathbb{R}$ such that $g \circ \phi \circ h$ is i-convex. In particular, for flexible Stein cobordisms one has the J-convex h-cobordism theorem.*
4. *Let $W \subset \mathbb{C}^n$ be a flexible Stein domain and $\phi : W \to \mathbb{R}$ a defining i-convex function. Then the inclusion $W \hookrightarrow \mathbb{C}^n$ is isotopic to a symplectic embedding $h : (W, \omega_\phi) \to (\mathbb{C}^n, \omega_{\rm st})$.*

Proof of Math Theorem: polynomial convexity

Theorem 3,

- Given any flexible Stein cobordism $W \subset \mathbb{C}^n$, any defining Morse function $\phi : W \to \mathbb{R}$ without critical points of index $> n$ is equivalent to an i-convex function, i.e. there exists isotopic to the identity diffeomorphisms $h : W \to W$ and $g :\to \mathbb{R}$ such that $g \circ \phi \circ h$ is i-convex.

implies Main Theorem in the polynomially convex domains using the following

Topological lemma

Let $W \subset \mathbb{C}^n$, $n > 2$, be a domain which admits a defining Morse function $\phi : W \to \mathbb{R}$ without critical points of index $> n$ and which satisfies condition (T), i.e. $H_n(W) = 0$ and $H_{n-1}(W)$ has no torsion. Then ϕ extends to a Morse function $\widehat{\phi} : \mathbb{C}^n \to \mathbb{R}$ without critical points of index $> n$ and which is equal to $|z|^2$ at infinity.

Proof of Math Theorem: rational convexity

Theorem 4,

- Let $W \subset \mathbb{C}^n$ be a flexible Stein domain and $\phi : W \to \mathbb{R}$ a defining i-convex function. Then the inclusion $W \hookrightarrow \mathbb{C}^n$ is isotopic to a symplectic embedding $h : (W, \omega_\phi) \to (\mathbb{C}^n, \omega_{\mathrm{st}})$.

together with the surrounding of isotropic skeletons theorems implies

> Any domain $W \subset \mathbb{C}^n$, $n > 2$, which admits a defining function without critical points of index $> n$ is isotopic to a flexible rationally convex domain. Moreover, if W is itself a flexible Stein domain, then the isotopy can be chosen through Stein domains.

Generalizations, open problems and conjectures

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- **Conjecture:** Any polynomially convex domain in $\mathbb{C}^n, n > 2$ is flexible.
- **Question:** Is the same holds for simply connected rationally convex domains?
- **Conjecture:** Let $D^*(S)$ be the unit cotangent bundle of a 2-dimensional surface D. An embedding $h : D^*(S) \to (\mathbb{C}^2, \omega_{\mathrm{st}})$ is isotopic to an embedding onto a rationally convex domain if and only if $h|_S : S \to \mathbb{C}^2$ is isotopic to a Lagrangian embedding.