

LOOP SPACES AND THE HYPOELLIPTIC LAPLACIAN

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ABSTRACT. The purpose of this paper is to give an introduction to some ideas which motivated the construction of the hypoelliptic Laplacian as a deformation of Hodge theory, which interpolates between Hodge theory and the geodesic flow. Results obtained with Lebeau on the analysis of the hypoelliptic Laplacian are also presented.

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INTRODUCTION

The purpose of this paper is to give an introduction to the ideas which motivated the construction of the hypoelliptic Laplacian [B05], and also to describe some results obtained jointly with Lebeau [BL06] on this operator.

One initial motivation was to provide a construction of the Hodge theory on the loop space LX of a Riemannian manifold X , and also of the corresponding Witten deformation [W82], which would interpolate between the Hodge theory of LX and the Morse theory of the energy functional E .

Constructing directly the Hodge theory on LX is notoriously difficult, in particular because of the issues related to the choice of a suitable L^2 scalar product on the de Rham complex. We will sidestep these delicate points.

We will give three interrelated approaches to the construction of the hypoelliptic Laplacian, as a substitute for the Hodge theory of LX :

- In a first approach, we replace X by X^m equipped with an action of U_m , the group of m -th roots of unity, we construct a Witten Laplacian on X^m and we make $m \rightarrow +\infty$.
- In a second approach, by extending Chern-Gauss-Bonnet to infinite dimensions, we propose the construction of a new Hodge theory based on a path integral where the gradient of the energy functional on LX should appear.
- In a third approach, we view the measure on LX associated to the hypoelliptic Laplacian as the local limit (in the sense of local index theory) of the local supertrace of a non existing heat kernel on LX .

The object which is finally obtained is a second order hypoelliptic operator of order 2 on the cotangent bundle T^*X of a Riemannian manifold X . This operator depends on a parameter $b > 0$. We give the details of its rigorous construction in [B05], and we explain in what sense it does interpolate between classical Hodge theory for $b \rightarrow 0$ and the geodesic flow for $b \rightarrow +\infty$.

Finally we describe a few results obtained with Lebeau [BL06] on the analysis of the hypoelliptic Laplacian, and also on the hypoelliptic torsion.

This paper is organized as follows. In section 1, we introduce the classical Witten Laplacian [W82], and we develop various non rigorous approaches to the construction of the Witten Laplacian on LX . The Thom forms of Mathai-Quillen [MQ86] play an important role in the whole argument.

In section 2, we give the rigorous construction of an exotic Hodge theory on T^*X , where the corresponding Laplacian is an hypoelliptic second order operator on T^*X . We also give the arguments in [B05] showing that this new Hodge theory has the suggested interpolation properties.

Finally in section 3, we state some analytic properties of the hypoelliptic Laplacian established with Lebeau in [BL06], and we present in particular our results on the hypoelliptic torsion.

1. THE WITTEN DEFORMATION

The purpose of this section is to describe the Witten deformation of classical Hodge theory on a compact Riemannian manifold X , and to explain its possible applications to LX , the loop space of X . This way, we will produce a second order hypoelliptic operator on T^*X , which will eventually turn out to be exactly the hypoelliptic Laplacian acting on 0-forms.

This section is organized as follows. In subsection 1.1, we recall some results on the Witten Laplacian on X , which interpolates between Hodge theory and Morse theory.

In subsection 1.2, we replace X by X^m equipped with the obvious cyclic action, we construct the Witten Laplacian on X^m associated to a natural smooth function on X^m , and we make $m \rightarrow +\infty$. We produce this way the dynamics of a random path in X , depending on a parameter $T \in \mathbf{R}$.

In subsection 1.3, we recall the construction of the Mathai-Quillen Thom forms.

In subsection 1.4, we interpret the localization of Witten eigenforms near the critical points of a Morse function as a formal consequence of Chern-Gauss-Bonnet on LX associated to the natural lift of a Morse function f on X to an S^1 -invariant function F on LX .

In subsection 1.5, we write a functional integral on LX associated with any Lagrangian $L(x, \dot{x})$. We observe that if this Lagrangian is just the energy, the corresponding functional integral should converge to a classical Brownian integral for $T \rightarrow 0$, and should localize on the closed geodesics when $T \rightarrow +\infty$. It also produces a dynamics for a random path x which is the same as the one which was produced in subsection 1.2. This dynamics is described by a second order differential operator on T^*X . In section 2, we will show that this operator is in fact a Laplacian associated with an exotic Hodge theory on T^*X .

Finally in subsection 1.6, we give still another approach to the construction of the hypoelliptic Laplacian via a non existing local index theory on LX , in connection with the theory of V -invariants which was developed in [BG04].

Many arguments used in this section are not rigorous. Still they provide a powerful motivation for the rigorous constructions of section 2.

The construction of the hypoelliptic Laplacian was announced in [B04a, B04b] and developed in [B05]. Our results with Lebeau were announced in [BL05] and are explained in detail in [BL06].

1.1. The Witten Laplacian. Let X be a compact connected manifold of dimension n , let g^{TX} be a Riemannian metric on TX , and let $d^X(\cdot, \cdot)$ be the corresponding Riemannian distance on X . Let ∇^{TX} be the Levi-Civita connection on (TX, g^{TX}) . Let (F, ∇^F, g^F) be a complex flat vector bundle on X equipped with a non necessarily flat metric. Let $(\Omega(X, F), d^X)$ be the de Rham complex of smooth forms on X with coefficients in F . Let $H^*(X, F)$ be the cohomology of this complex. It is a finite dimensional \mathbf{Z} -graded vector space.

Let $\langle \cdot, \cdot \rangle_{\Lambda(T^*X) \otimes F}$ be the Hermitian product on $\Lambda(T^*X) \otimes F$ which is associated to g^{TX}, g^F , let dv_X be the volume form on X associated to g^{TX} . We equip $\Omega(X, F)$ with the L^2 Hermitian product $\langle \cdot, \cdot \rangle_{\Omega(X, F)}$ associated to g^{TX}, g^F . If $s, s' \in \Omega(X, F)$, then

$$(1.1) \quad \langle s, s' \rangle_{\Omega(X, F)} = \int_X \langle s, s' \rangle_{\Lambda(T^*X) \otimes F} dv_X.$$

Let d^{X*} be the formal adjoint of d^X with respect to $\langle \cdot, \cdot \rangle_{\Omega(X, F)}$. Set

$$(1.2) \quad \square^X = [d^X, d^{X*}].$$

The operator \square^X is the Hodge Laplacian. It is a second order elliptic self-adjoint nonnegative operator.

Set

$$(1.3) \quad \mathcal{H} = \ker \square^X.$$

Equivalently,

$$(1.4) \quad \mathcal{H} = \ker d^X \cap \ker d^{X*}.$$

Hodge theory tells us that

$$(1.5) \quad \mathcal{H} \simeq H^*(X, F).$$

Equivalently any cohomology class in $H^*(X, F)$ is uniquely represented by a form in \mathcal{H} .

Let now $f : X \rightarrow \mathbf{R}$ be a smooth function. In [W82], Witten proposed a deformation of Hodge theory associated to f . Indeed for $T \in \mathbf{R}$, set

$$(1.6) \quad d_T^X = e^{-Tf} d^X e^{Tf}.$$

Let d_T^{X*} be the formal adjoint of d_T^X with respect to $\langle \cdot \rangle_{\Omega \cdot (X, F)}$, so that

$$(1.7) \quad d_T^{X*} = e^{Tf} d^{X*} e^{-Tf}.$$

The corresponding Laplacian \square_T^X is given by

$$(1.8) \quad \square_T^X = [d_T^X, d_T^{X*}].$$

An equivalent construction is to replace $\langle \cdot \rangle_{\Omega \cdot (X, F)}$ by $\langle \cdot \rangle_{\Omega \cdot (X, F), T}$ given by

$$(1.9) \quad \langle s, s' \rangle_{\Omega \cdot (X, F), T} = \int_X \langle s, s' \rangle_{\Lambda \cdot (T^*X) \otimes F} e^{-2Tf} dv_X.$$

Let \bar{d}_T^{X*} be the adjoint of d^X with respect to $\langle \cdot \rangle_{\Omega \cdot (X, F), T}$, so that

$$(1.10) \quad \bar{d}_T^{X*} = e^{2Tf} d^{X*} e^{-2Tf}.$$

Let $\bar{\square}_T^X$ be the associated Laplacian. Clearly

$$(1.11) \quad \square_T^X = e^{-Tf} \bar{\square}_T^X e^{Tf}.$$

These two constructions are essentially equivalent. The second one can be interpreted as one in which the trivial line bundle \mathbf{R} is equipped with the non trivial metric e^{-2Tf} .

In any case the Laplacians $\square_T^X, \bar{\square}_T^X$ are still second order elliptic self-adjoint nonnegative operators. For $T = 0$, they coincide with the standard Laplacian \square^X . For simplicity, for the moment we only consider the Laplacian \square_T^X .

Put

$$(1.12) \quad \mathcal{H}_T = \ker \square_T^X.$$

Then the obvious analogues of (1.4)-(1.5) still hold. In particular,

$$(1.13) \quad \mathcal{H}_T \simeq H^*(X, F).$$

Let e_1, \dots, e_n be an orthonormal basis of TX , let e^1, \dots, e^n be the corresponding dual basis of T^*X . Then we have the Weitzenböck formula for \square_T^X ,

$$(1.14) \quad \square_T^X = \square^X + T^2 |\nabla f|^2 + 2Te^i i_{e_j} \langle \nabla_{e_i}^{TX} \nabla f, e_j \rangle - T\Delta^X f.$$

In (1.14), e^i, i_{e_j} are the obvious creation and annihilation operators, and Δ^X is the Laplace-Beltrami operator.

The idea in [W82] is to make $T \rightarrow +\infty$. Indeed assume that f is a Morse function. The main assertion in [W82] is that as $T \rightarrow +\infty$, most of the eigenvalues of \square_T^X tend to $+\infty$, a finite number of them, counted with multiplicity, tend to 0. Among those, there are the ones which are exactly 0, which correspond to \mathcal{H}_T , and others which are exponentially small, i.e. they are dominated by e^{-cT} , with $c > 0$. Let (F_T, d_T^X) be the finite dimensional complex of eigenspaces associated with small eigenvalues. Then Witten shows that F_T localizes near the critical points of f . More precisely for $1 \leq i \leq n$, he shows that F_T^i localizes near the critical points of index i . If M_i is the number of critical points of index i , we find that for T large enough,

$$(1.15) \quad \dim F_T^i = M_i \operatorname{rk}(F).$$

From (1.15), one gets immediately a proof of the Morse inequalities.

In [W82], Witten goes one step further. He suggests that when $T \rightarrow +\infty$, the complex (F_T, d_T^X) can be identified with a combinatorial complex built out of the instanton integral trajectories of $-\nabla f$ connecting the critical points, which he interprets as causing tunnelling effects between critical points. If ∇f verifies the Thom-Smale transversality conditions [T49, Sm61], this combinatorial complex is in fact the complex described by Thom [T49] and Milnor [Mi65]. This conjecture by Witten was first proved by Helffer and Sjöstrand [HS85]. A simpler proof was given by Bismut-Zhang [BZ94, section 6], which is based on the de Rham map of Laudenbach [BZ92]. In fact in [BZ92, Appendix], Laudenbach shows that under adequate assumptions, the unstable or stable cells can be compactified in submanifolds with conical singularities, and that these compactified cells have essentially the same properties as the simplexes of a triangulation.

The Witten deformation has been used in [BZ92, BZ94] to give a proof of the Cheeger-Müller theorem [Ch79, Mül78], which asserts the equality of the Ray-Singer analytic torsion [RS71] with the corresponding Reidemeister torsion [Re35].

Let us also make a final observation on the dynamics associated to the semigroup $\exp(-t\bar{\square}_T^X/2)$ restricted to $\Omega^0(X, \mathbf{R})$. The stochastic differential equation describing the diffusion x associated to this semigroup is given by

$$(1.16) \quad \dot{x} = -T\nabla f(x) + \dot{w},$$

where w is a classical Brownian motion. When $f = 0$, we recover the equation for classical Brownian motion.

1.2. The action of U_m on X^m . Let H be a connected Lie group, and let \mathfrak{h} be its Lie algebra. Let V be a complex finite dimensional representation of H . For $m \in \mathbf{N}^*$, then H acts on $V^{\otimes m}$.

We will denote $\mathbf{Z}/m\mathbf{Z}$ multiplicatively. Equivalently we identify $\mathbf{Z}/m\mathbf{Z}$ to the group U_m of m -th roots of unity.

Then U_m acts on $V^{\otimes m}$, the action of $e^{2i\pi/m} \in U_m$ being given by

$$(1.17) \quad v_1 \otimes v_2 \dots \otimes v_m \rightarrow v_2 \otimes v_3 \dots \otimes v_1.$$

The actions of H and U_m commute, so that $U_m \times H$ acts on $V^{\otimes m}$.

If $\sigma \in U_m$, let $d \in \mathbf{N}$ be the order of σ , so that $\sigma^d = 1$, and $d|m$. If $h \in H$, one has the easy formula

$$(1.18) \quad \operatorname{Tr}^{V^{\otimes m}}[\sigma h] = (\operatorname{Tr}^V[h^d])^{m/d}.$$

In particular if $\sigma \in U_m$ is primitive, and if $A \in \mathfrak{h}, t \in \mathbf{R}$,

$$(1.19) \quad \mathrm{Tr}^{V^{\otimes m}} [\sigma e^{tA}] = \mathrm{Tr}^V [e^{mtA}].$$

We rewrite (1.19) in the form,

$$(1.20) \quad \mathrm{Tr}^{V^{\otimes m}} [\sigma e^{tA/m}] = \mathrm{Tr}^V [e^{tA}],$$

which does not depend on m .

Let G be a compact Lie group, and let \mathfrak{g} be its Lie algebra. Let X be a compact Riemannian connected manifold as in subsection 1.1, and let (F, ∇^F, g^F) be a flat Hermitian vector bundle on X . Assume that G acts isometrically on X , and that the action lifts to (F, ∇^F, g^F) . Then G acts on $H^*(X, F)$.

In the sequel, Tr_s is our notation for the supertrace. The Lefschetz formula asserts that if $g \in G$, if X_g is the fixed point manifold of g and if $e(TX_g)$ is the Euler class of TX_g , then

$$(1.21) \quad \mathrm{Tr}_s^{H^*(X, F)} [g] = \int_{X_g} e(TX_g) \mathrm{Tr}^F [g].$$

Take $m \in \mathbf{N}^*$. For $1 \leq i \leq m$, let $\pi_i : X^m \rightarrow X$ be the obvious projection. Put

$$(1.22) \quad F^{[m]} = \bigotimes_{i=1}^m \pi_i^* F.$$

Let $\nabla^{F^{[m]}}, g^{F^{[m]}}$ be the obvious flat connection and the obvious metric on $F^{[m]}$. Note that $U_m \times G$ acts isometrically on X^m and that this action lifts to $(F^{[m]}, \nabla^{F^{[m]}}, g^{F^{[m]}})$.

Clearly,

$$(1.23) \quad \Omega(X^m, F^{[m]}) = \Omega(X, F)^{\otimes m}, \quad H(X^m, F^{[m]}) = H(X, F)^{\otimes m}.$$

If $\sigma \in U_m$ is of order d , if $g \in G$, by (1.18), (1.23), we get

$$(1.24) \quad \mathrm{Tr}_s^{H^*(X^m, F^{[m]})} [\sigma g] = \left(\mathrm{Tr}_s^{H^*(X, F)} [g^d] \right)^{m/d}.$$

The Lefschetz fixed point formula (1.21) applied to X^m also leads easily (1.24).

Incidentally observe that if one uses the above formalism in the context of the Atiyah-Singer index formula or of Riemann-Roch-Hirzebruch, the identity in (1.21) reflects trivial identities on cyclotomic polynomials. Part of what we will say in the sequel will be valid also in this more general context, without further mention.

The Mc-Kean-Singer formula [McKS67] asserts that for any $t > 0$,

$$(1.25) \quad \mathrm{Tr}_s^{H^*(X, F)} [g] = \mathrm{Tr}_s [g \exp(-t\Box^X)].$$

Moreover,

$$(1.26) \quad \Box^{X^m} = \Box^X \otimes 1 \otimes \dots + 1 \otimes \Box^X \otimes 1 \dots + \dots$$

By (1.19), (1.26), if $\sigma \in U_m$ is of order d ,

$$(1.27) \quad \mathrm{Tr}_s [\sigma g \exp(-t\Box^{X^m})] = \left(\mathrm{Tr}_s [g^d \exp(-dt\Box^X)] \right)^{m/d}.$$

Of course (1.24), (1.25), (1.27) are compatible.

Note here that we could as well have taken a usual trace instead of a supertrace in (1.27). The fact that the supertrace in (1.27) has a topological interpretation does not play any role for the moment.

By (1.27), we find that if $\sigma \in U_m$ is primitive, if $A \in \mathfrak{g}$,

$$(1.28) \quad \mathrm{Tr}_s \left[\sigma e^{A/m} \exp \left(-t \square^{X^m} / m \right) \right] = \mathrm{Tr}_s^{H(X,F)} \left[e^A \exp \left(-t \square^X \right) \right] = \mathrm{Tr}_s^{H(X,F)} \left[e^A \right].$$

Again (1.28) does not depend on m .

Let us now give a geometric interpretation of an identity like (1.28). We will work here using the classical heat kernel $\exp(t\Delta^X/2)$ instead of $\exp(-t\square^X)$, but since all the arguments we give are algebraic, this is perfectly legitimate. Incidentally note that if $F = \mathbf{R}$, the restriction of \square^X to $\Omega(X, \mathbf{R})$ is just $-\Delta^X$.

The heat operator $e^{t\Delta^{X^m}/2}$ on X^m is associated to the motion of m independent Brownian motions in X . We take here $\sigma = e^{2i\pi/m}$. A simple computation shows that if $p_t(x, y)$ is the smooth heat kernel associated to $e^{t\Delta^X/2}$, then

$$(1.29) \quad \mathrm{Tr} \left[\sigma g e^{t\Delta^{X^m}/2m} \right] = \int_{X^m} p_{t/m}(x_1, gx_2) p_{t/m}(x_2, gx_3) \dots p_{t/m}(x_m, gx_1) dx_1 \dots dx_m.$$

Using the fact that G commutes with Δ^X and the semigroup property of the heat kernel, we get from (1.29),

$$(1.30) \quad \mathrm{Tr} \left[\sigma g e^{t\Delta^{X^m}/2m} \right] = \int_X p_t(x, g^m x) dx,$$

which is precisely the abstract content of (1.28).

The interpretation of (1.30) is that the dynamics of m independent Brownian motions in X on the time interval $[0, t/m]$ is equivalent to a single Brownian motion on X on the time interval $[0, t]$.

Let us point out here that the time scaling in (1.30) is natural. Indeed there are m independent points in X^m . The total randomness on the time interval $[0, t/m]$ is then $m \times t/m = t$.

Let now $f : X^m \rightarrow \mathbf{R}$ be a smooth function which is G and U_m invariant. For $T \in \mathbf{R}$, let $\square_T^{X^m}$ be the corresponding Witten Laplacian. Of course by (1.25), we get

$$(1.31) \quad \mathrm{Tr}_s^{H(X^m, F^{[m]})} [\sigma g] = \mathrm{Tr}_s \left[\sigma g \exp \left(-t \square_T^{X^m} \right) \right].$$

However no identity like (1.26) holds any more for $\square_T^{X^m}$, except in the case where f is of the form

$$(1.32) \quad f(x_1, \dots, x_m) = \sum_1^m h(x_i).$$

In the sequel, we take f of the form

$$(1.33) \quad f_m(x_1, \dots, x_m) = \sum_{i=1}^m \log p_{1/m}(x_i, x_{i+1}),$$

with the convention that $x_{m+1} = x_1$. Obviously the function f has the required invariance properties.

One can then construct the Witten Laplacian $\square_T^{X^m}$ associated to the function f_m . The dynamics of the m particles x_i associated formally to the Witten Laplacian $\square_T^{X^m}$ are now correlated. If f_m happens to be a Morse function, when $T \rightarrow +\infty$, the eigenforms associated to small eigenvalues of $\square_T^{X^m}$ will concentrate near the critical points of f_m .

Now we will make $m \rightarrow +\infty$ in the above construction. Let LX be the loop space of X , i.e. the set smooth maps $s \in S^1 \rightarrow x_s \in X$. Let us accept the fact that LX is the ‘limit’ of X^m as $m \rightarrow +\infty$. This means that given $(x_1, \dots, x_m) \in X^m$, we think of these m points as being such that there is $x \in LX$ for which if $1 \leq k \leq m$, $x_k = x_{(k-1)/m}$. Equivalently, if $x \in LX$, when m is large enough, we may replace x by its piecewise geodesic approximation which interpolates between $x_{(k-1)/m}$ and $x_{k/m}$ for $1 \leq k \leq n$. Now for $x, y \in X$, with $d^X(x, y)$ small, as $t \rightarrow 0$, then

$$(1.34) \quad p_t(x, y) \simeq \frac{e^{-d^X(x, y)^2/2t}}{(2\pi t)^{n/2}}.$$

By (1.33), (1.34), we find that if $x \in LX$,

$$(1.35) \quad f_m(x_0, \dots, x_{(m-1)/m}) + \frac{n}{2} \log(2\pi/m) \rightarrow -\frac{1}{2} \int_{S^1} |\dot{x}|^2 ds.$$

Observe that adding a constant to f does not change the Witten Laplacian. By (1.35), we find that when $m \rightarrow +\infty$, when replacing X^m by LX , then f_m should be replaced by $-E$, where E is the energy functional on LX given by

$$(1.36) \quad E(x) = \frac{1}{2} \int_{S^1} |\dot{x}|^2 ds.$$

The above approach is more than disingenuous. Indeed by (1.33),

$$(1.37) \quad e^{f_m}(x_1, \dots, x_m) = \prod_1^m p_{1/m}(x_i, x_{i+1}),$$

and the right-hand side of (1.37) is the obvious discrete time approximation of the Brownian measure μ on LX . This Brownian measure can be represented formally as being given by

$$(1.38) \quad \mu = \exp(-E) dx,$$

which ultimately explains why f_m should be replaced by $-E$.

Admittedly if $F = \mathbf{R}$, the limit of the Witten Laplacian $\square_T^{X^m}$ associated to f_m should then be the Witten Laplacian \square_T^{LX} associated to $-E$. Moreover,

$$(1.39) \quad \nabla E(x) = -\ddot{x}.$$

The critical points of $-E$ are the closed geodesics in X . One can then say that as $m \rightarrow +\infty$, the critical set of f_m on X^m converges in some sense to the closed geodesics on X .

We already indicated that in (1.30), the scaling of t by the factor $1/m$ is natural so as to keep total randomness constant. Indeed if one expect that the limit as $m \rightarrow +\infty$ of the motion of m independent Brownian motions describes the Brownian motion of a string on a time interval $[0, t]$, it is necessary to scale the time interval of evolution of each of the m independent Brownian motions in X^m by a factor $1/m$ so as to keep the total randomness (which in this case is equal to the variance of the underlying Brownian sheet) equal to t .

Also we want to understand what dynamics of the loop x associated to the semi-group $\exp(-t\square_T^{LX}/2)$. Recall that $s \in S^1$ describes the parametrization of a given loop. Extending equation (1.16) to infinite dimensions and using (1.39) means that

the process $t \in \mathbf{R}_+ \rightarrow x_t \in LX$ should be a solution of the stochastic differential equation,

$$(1.40) \quad \frac{\partial x}{\partial t} = -T \frac{\partial^2 x}{\partial s^2} + \dot{w}_{s,t}.$$

In (1.40), \dot{w} is a Gaussian process whose covariance on $L^2(S^1 \times \mathbf{R}_+)$ is just the identity.

However we will here go back to equation (1.29). The left-hand side describes the motion of a collection m independent Brownian motions (x_1, \dots, x_m) such that

$$(1.41) \quad (x_1, \dots, x_m) |_{t/m} = (x_2, \dots, x_1) |_0.$$

To obtain the right-hand side of (1.29), we constructed a single Brownian motion on the time interval $[0, t]$ which coincides with x_1 on $[0, t/m]$, with x_2 on $[t/m, 2t/m]$...

If $y = (x_1, \dots, x_m) \in X^m$, the dynamics of y associated to the semigroup $e^{-t\Box^{X^m}/2}$ is now

$$(1.42) \quad \dot{y} = -T\nabla f_m(y) + \dot{w}^m,$$

where w^m is a collection of m independent Brownian motions. The idea is now to consider a process x_t which coincides with x_1 on $[0, t/m]$, with x_2 on $[t/m, 2t/m]$... and then to take the limit as $m \rightarrow +\infty$. Keeping in mind the limit remains formal, we find that x should verify the equation

$$(1.43) \quad \dot{x} = -T\ddot{x} + \dot{w}.$$

Observe that equation (1.43) looks like a degenerate version of (1.40), where we have made $s = t$. Also note that while we pieced together various Brownian motions with a drift (the drift is the local deviation from mean 0), which are nowhere differentiable, the resulting equation for $T > 0$ indicates the process x should become C^1 in the time variable. Note that there is the implicit constraint $x_t = x_0$.

For $T = 0$, as it should be, equation (1.43) becomes,

$$(1.44) \quad \dot{x} = \dot{w}$$

which is the classical equation for Brownian motion, and for $T = +\infty$, equation (1.43) becomes

$$(1.45) \quad \ddot{x} = 0,$$

which is the equation for closed geodesics.

The above reasoning indicates that studying equation (1.40) for a Brownian motion with drift on LX is equivalent to studying equation (1.45), which is the equation for a single diffusion in X . Also equation (1.27) indicates that the evaluation of the expectation for certain observables associated to the Brownian motion with drift on LX can be reduced to the evaluation of other observables associated to the standard diffusion (1.43).

The program carried through in [B05, BL06] consists in precisely disregarding the infinite dimensional picture by concentrating on the finite dimensional equation (1.43).

1.3. Chern-Gauss Bonnet and the Mathai-Quillen Thom form. To make our notation simpler, the coefficient systems of the cohomology groups which are considered later will always be tensored by \mathbf{R} .

Let M be a manifold, let $\pi : (E, g^E, \nabla^E) \rightarrow M$ be a real vector bundle of dimension n equipped with a metric and a metric preserving connection. Let R^E be the curvature of ∇^E . Let $o(E)$ be the orientation bundle of E . Let $e(E) \in H^n(M, o(E))$ be the real Euler class of E . If n is odd then $e(E) = 0$, if n is even, it is represented by Chern's form [C44],

$$(1.46) \quad e(E, \nabla^E) = \text{Pf} \left[\frac{R^E}{2\pi} \right].$$

By definition, we make the right-hand side vanish when n is odd, so that (1.46) will be valid in all cases.

Let \mathcal{E} be the total space of E . Let $H^{c,\cdot}(\mathcal{E}, \mathbf{R})$ be the compactly cohomology of \mathcal{E} . A similar notation is used when replacing \mathbf{R} by $o(E)$. The Thom class $[\Phi^E] \in H^{c,n}(\mathcal{E}, \pi^*o(E))$ is characterized by the fact that

$$(1.47) \quad \pi_* [\Phi^E] = 1.$$

Let $i : M \rightarrow \mathcal{E}$ be the embedding of M as the zero section of E . Then

$$(1.48) \quad i^* [\Phi^E] = e(E).$$

Moreover we have the Thom isomorphism $H^*(M, o(E)) \simeq H^{c,\cdot+n}(\mathcal{E}, \mathbf{R})$ given by

$$(1.49) \quad \alpha \in H^*(M, o(E)) \rightarrow \pi^* \alpha \wedge [\Phi^E].$$

In [MQ86], Mathai and Quillen gave an explicit construction of a Thom form $\Phi^E \in \Omega^n(\mathcal{E}, \pi^*o(E))$ which depends on (g^E, ∇^E) , which represents canonically the Thom class $[\Phi^E]$. The form Φ^E is Gaussian shaped. Actually its restriction to the fibre is a Gaussian. The identity

$$(1.50) \quad \pi_* \Phi^E = 1$$

just reflects the known identity for the Gaussian distribution. Also corresponding to (1.48), we now have

$$(1.51) \quad i^* \Phi^E = e(E, \nabla^E).$$

Let y the generic element in E . We write Φ^E in the form,

$$(1.52) \quad \Phi^E = \exp \left(-\frac{|y|^2}{2} + \dots \right),$$

the expression \dots containing the geometric information involving the connection ∇^E and its curvature R^E .

Let s be a smooth section of E . Then $(Ts)^* \Phi^E$ is a smooth closed n -form on M whose cohomology class in $H^n(M, o(E))$ is equal to $e(E)$. By (1.52),

$$(1.53) \quad (Ts)^* \Phi^E = \exp \left(-T^2 \frac{|s|^2}{2} + \dots \right).$$

Equation (1.53) makes clear that as $T \rightarrow +\infty$, $(Ts)^* \Phi^E$ concentrates on the zero locus Y of s . If s is generic, Y is a submanifold of M , and the limit as $T \rightarrow +\infty$ can be explicitly evaluated [BGS90]. This leads in particular to a proof of Chern-Gauss-Bonnet which is in fact exactly the proof by Chern [C44].

1.4. Witten's localization and Chern-Gauss-Bonnet. We now make the same assumptions as in subsection 1.1. We assume here that $F = \mathbf{R}$ equipped with its trivial metric. Let Φ^{TX} be the associated Thom form on the total space of TX . We assume that $f : X \rightarrow \mathbf{R}$ is Morse. Then ∇f is a generic section of TX , whose zero set Y consists of the critical points of f .

By (1.53),

$$(1.54) \quad (T\nabla f)^* \Phi^{TX} = \exp\left(-T^2 \frac{|\nabla f|^2}{2} + \dots\right).$$

When $T \rightarrow +\infty$, the currents in (1.54) converge to a sum of \pm Dirac masses at the critical points, which gives a special case of Hopf's formula for the Euler characteristic.

Now we will briefly show that aspects of Witten localisation of eigenforms can be understood via an infinite dimensional version of the localisation in (1.54).

Note that the metric g^{TX} induces a natural L^2 -metric on TLX . Namely if $x \in LX$, if U, V are two smooth sections of TX along x ., set

$$(1.55) \quad \langle U, V \rangle_{g^{TLX}} = \int_{S^1} \langle U_s, V_s \rangle_{g^{TX}} ds.$$

Also S^1 acts on LX . Namely if $t \in S^1, x \in LX$, set $k_t x = x_{t+}$. The generating vector field K for this action is given by

$$(1.56) \quad K(x) = \dot{x}.$$

The action of S^1 on LX is isometric, so that K is a Killing vector field. Its zero set is just $X \in LX$.

Note that the function f lifts naturally to the S^1 -invariant function $\mathcal{F} : LX \rightarrow \mathbf{R}$ given by

$$(1.57) \quad \mathcal{F}(x) = \int_{S^1} f(x_s) ds.$$

Then $\nabla \mathcal{F}$ vanishes exactly at $Y \subset LX$.

To explain the Witten localisation in its simplest form, we start from the McKean-Singer formula [McKS67], which asserts that for any $t > 0, T > 0$, the Euler characteristic $\chi(X)$ of X is given by

$$(1.58) \quad \chi(X) = \text{Tr}_s \left[\exp\left(-t \square_{T/\sqrt{t}}^X / 2\right) \right].$$

Recall that F is assumed to be the trivial \mathbf{R} . The Weitzenböck formula for \square^X says that

$$(1.59) \quad \square^X = -\Delta^H + \langle R^{TX}(e_i, e_j) e_k, e_\ell \rangle e^i e_j e^k e_\ell.$$

In (1.59), Δ^H is the horizontal or Bochner Laplacian. From (1.14) and (1.59), we get a formula form $t \square_{T/\sqrt{t}}^X$. The leading term in $t \square_{T/\sqrt{t}}^X$ is the negative of the Bochner Laplacian $-t \Delta^H$, the remainder consists of zero order terms. In particular the principal symbol of $t \square_{T/\sqrt{t}}^X$ is $t |\xi|^2$.

The dynamics associated to $t \Delta^H / 2$ is just parallel transport with respect to the Levi-Civita connection over a Brownian trajectory x .. The paths of Brownian motion are nowhere differentiable, but the stochastic calculus shows that parallel transport along such paths is still well defined.

The above formula form $\square_{T/\sqrt{t}}^X$ indicates that the heat kernel for $\exp\left(-\square_{T/\sqrt{t}}^X/2\right)$ can be calculated using a Feynman-Kac formula evaluated over the Brownian path x . Ultimately simple probabilistic arguments shows that there is a signed measure $\mu_{t,T}$ on L^0X , the set of continuous loops in X , such that

$$(1.60) \quad \text{Tr}_s \left[\exp \left(-t \square_{T/\sqrt{t}}^X / 2 \right) \right] = \int_{L^0X} d\mu_{t,T}.$$

Integrating on a loop space reflects the fact that we integrate a heat kernel on the diagonal, so that ultimately we have Brownian paths starting and ending at the same point. Tautologically, the measure $\mu_{t,T}$ is S^1 -invariant.

The right-hand side of (1.60) has no obvious cohomological content. In particular the fact that the integral does not depend on t, T is mysterious from that point of view. However Atiyah and Witten [A85] taught us how to transform the well-defined integral of a measure on L^0X into an ill defined integral of a differential form on LX , with an obvious cohomological interpretation. We will not give the detail of our calculation in this context, but we state simply the final product. The formula can be written as follows,

$$(1.61) \quad \text{Tr}_s \left[\exp \left(-t \square_{T/\sqrt{t}}^X / 2 \right) \right] = \int_{LX} \alpha_t \wedge (T\nabla\mathcal{F})^* \Phi^{TLX}.$$

In the right-hand side of (1.61) appear two explicitly defined series of forms, which are both S^1 -invariant, but also d_K closed, with $d_K = d + i_K$. Note that the property of being d_K closed cannot be read degree by degree.

Let K' be the 1-form dual to K . Then

$$(1.62) \quad \alpha_t = \exp \left(- (d_K K' / 2t) \right).$$

We can rewrite (1.62) as

$$(1.63) \quad \alpha_t = \exp \left(-E/t - dK'/2t \right).$$

The form Φ^{TLX} is the equivariant Thom form of Mathai-Quillen [MQ86] associated to $(TLX, g^{TLX}, \nabla^{TLX})$ and to the action of S^1 on LX . The construction of Φ^{TLX} is a trivial modification of the construction of Mathai-Quillen [MQ86]. By (1.53), we get

$$(1.64) \quad (T\nabla\mathcal{F})^* \Phi^{TLX} = \exp \left(-\frac{T^2}{2} \int_{S^1} |\nabla f(x_s)|^2 ds + \dots \right).$$

The difficulty in making sense of (1.61) is that we do not know precisely what is the integral of a series of forms, since we should select the term of infinite degree corresponding to the dimension of LX , which is not well-defined. However, from another point of view, this is irrelevant since the integral (1.60) is well-defined anyway.

Equation (1.61) explains why the right-hand side is independent of t, T . For $T = 0$, we recover a classical integral with respect to Brownian motion, which, as we know, corresponds to classical Hodge theory. However as $T \rightarrow +\infty$, (1.64) makes clear that the integral should localize near the critical loops of \mathcal{F} , i.e. near the critical points of f . This is precisely what happens for the heat kernel $\exp\left(-t \square_{T/\sqrt{t}}^X / 2\right)(x, x)$ here evaluated on the diagonal. Actually, a direct proof of this fact can be easily given using (1.14). Indeed $T^2 |\nabla f|^2$ appears there as a potential, and when $T \rightarrow +\infty$, the heat kernel on the diagonal localizes near the

potential well, where this potential vanishes. Of course localization of the heat kernel on the diagonal implies the localization of the corresponding eigenforms.

When making $t \rightarrow 0$ first in the integral in (1.58)-(1.61), we recover an integral over X of the form $(T\nabla f)^* \Phi^{TX}$. One can then write an obvious commutative diagram in which the limits $t \rightarrow 0, T \rightarrow +\infty$ are interchanged. Local index theory shows that as $t \rightarrow 0$,

$$(1.65) \quad \text{Tr}_s \left[\exp \left(-t \square_{T/\sqrt{t}}^X / 2 \right) (x, x) \right] \rightarrow [(T\nabla f)^* \Phi^{TX}]^{\max},$$

which is compatible with (1.58) and with Chern-Gauss-Bonnet.

1.5. Functional integration and the energy functional. Let $L(x, \dot{x})$ be a Lagrangian, i.e. a smooth function $TX \rightarrow \mathbf{R}$. If $x \in LX$, set

$$(1.66) \quad I(x) = \int_{S^1} L(x, \dot{x}) ds.$$

Then I is a S^1 -invariant function on LX . Of course \mathcal{F} in (1.57) is a special case of I , with $L(x, \dot{x}) = f(x)$. Among the Lagrangians are those coming from classical mechanics, of the type

$$(1.67) \quad L(x, \dot{x}) = \frac{1}{2} |\dot{x}|^2 - V(x).$$

The function \mathcal{F} looks like a functional I attached to L in (1.67), in which the most important part, the energy $E(x) = \frac{1}{2} \int_{S^1} |\dot{x}|^2 ds$ has been omitted.

For the functional I associated to the Lagrangian L as in (1.66), we know that

$$(1.68) \quad \nabla I = \frac{\partial L}{\partial x}(x, \dot{x}) - \frac{D}{Dt} \frac{\partial L}{\partial \dot{x}}(x, \dot{x}).$$

The key idea is to consider a functional integral like the one in (1.61), in which \mathcal{F} is replaced by I . For the moment we make $t = 1$, and we set

$$(1.69) \quad \alpha = \alpha_1.$$

By (1.52), (1.68), the path integral to be considered is of the form,

$$(1.70) \quad \int_{LX} \alpha \wedge (T\nabla I)^* \Phi^{TLX} = \int_{LX} \exp \left(-\frac{1}{2} \int_{S^1} |\dot{x}|^2 ds - \frac{T^2}{2} \int_{S^1} \left| \frac{D}{Dt} \frac{\partial L}{\partial \dot{x}}(x, \dot{x}) - \frac{\partial L}{\partial x}(x, \dot{x}) \right|^2 ds + \dots \right).$$

Since I is S^1 -invariant $\langle \nabla I, \dot{x} \rangle = 0$. We can rewrite (1.70) in the form

$$(1.71) \quad \int_{LX} \alpha \wedge (T\nabla I)^* \Phi^{TLX} = \int_{LX} \exp \left(-\frac{1}{2} \int_{S^1} \left| \dot{x} + T \left(\frac{D}{Dt} \frac{\partial L}{\partial \dot{x}}(x, \dot{x}) - \frac{\partial L}{\partial x}(x, \dot{x}) \right) \right|^2 ds + \dots \right).$$

Let us make a few simple considerations on (1.70). Indeed if $\frac{D}{Dt} \frac{\partial L}{\partial \dot{x}} = 0$, we recover an integral of the type (1.61). But if $\frac{D}{Dt} \frac{\partial L}{\partial \dot{x}} \neq 0$, the differential $\frac{D}{Dt} \frac{\partial L}{\partial \dot{x}}(x, \dot{x})$ contains \ddot{x} , in which case the functional integral changes fundamentally of nature. For this last condition to be true, we need that $\frac{\partial^2 L}{\partial \dot{x}^2}(x, \dot{x}) \neq 0$, which implies that the map $\dot{x} \rightarrow p = \frac{\partial L}{\partial \dot{x}}(x, \dot{x})$ is a local diffeomorphism.

It is then natural to assume in the sequel that $L(x, \dot{x})$ has a smooth Legendre transform $\mathcal{H}(x, p)$, which is a smooth function on T^*X , and that it verifies the non degeneracy condition

$$(1.72) \quad \frac{\partial^2 \mathcal{H}}{\partial p^2}(x, p) \neq 0.$$

If $x \in LX$, put

$$(1.73) \quad p = \frac{\partial L}{\partial \dot{x}}(x, \dot{x}).$$

The path integral in (1.71) can be written in the form

$$(1.74) \quad \int_{LX} \exp \left(-\frac{1}{2} \int_{S^1} \left| \dot{x} + T \left(\dot{p} + \frac{\partial \mathcal{H}}{\partial x}(x, p) \right) \right|^2 ds + \dots \right).$$

Let us ignore the ‘fermionic’ part \dots in (1.74) and concentrate on the bosonic part containing the scalar action. The probabilistic content of (1.74) is that the process $(x, p) \in T^*X$ verifies a stochastic differential equation of the type

$$(1.75) \quad \dot{x} = \frac{\partial \mathcal{H}}{\partial p}(x, p), \quad \dot{p} = -T \left(\dot{p} + \frac{\partial \mathcal{H}}{\partial x}(x, p) \right) + \dot{w}.$$

In (1.75), w is a classical Brownian motion in a fibre T^*X , which is parallel transported with respect to the Levi-Civita connection along the path x . Of course since $x \in LX$, we should also have

$$(1.76) \quad x_1 = x_0, \quad p_1 = p_0.$$

Let $Y^{\mathcal{H}}$ be the Hamiltonian vector field associated to the Hamiltonian \mathcal{H} on T^*X , so that

$$(1.77) \quad Y^{\mathcal{H}} = \left(\frac{\partial \mathcal{H}}{\partial p}(x, p), -\frac{\partial \mathcal{H}}{\partial x}(x, p) \right).$$

Let $\widehat{\nabla^V \mathcal{H}}$ be the vector field along the fibre T^*X associated to the differential of \mathcal{H} along the fibre.

Using stochastic calculus, we find that the second order differential operator \mathcal{L}_T which is associated to the dynamics in (1.77) is given by

$$(1.78) \quad \mathcal{L}_T = -\frac{\Delta^V}{2T^2} + \frac{1}{T} \nabla_{\widehat{\nabla^V \mathcal{H}}} - \nabla_{Y^{\mathcal{H}}}.$$

Let us now make several remarks here. If $\mathcal{H} = |p|^2/2$, we can rewrite (1.70), (1.74) in the form

$$(1.79) \quad \int_{LX} \alpha \wedge (-T \nabla E)^* \Phi^{TLX} = \int_{LX} \exp \left(-\frac{1}{2} \int_{S^1} |\dot{x}|^2 ds - \frac{T^2}{2} \int_{S^1} |\dot{x}^2|^2 ds + \dots \right) = \int_{LX} \exp \left(-\frac{1}{2} \int_{S^1} |\dot{x} + T \ddot{x}|^2 ds + \dots \right).$$

Also we can rewrite (1.75) in the form

$$(1.80) \quad \dot{x} = -T \ddot{x} + \dot{w}.$$

This is precisely the equation we obtained in (1.43) by an entirely different argument.

Also observe that by a theorem by Hörmander [Hö67], if $u \in \mathbf{R}$ is an extra coordinate, a sufficient condition for hypoellipticity of $\frac{\partial}{\partial u} - \mathcal{L}_T$ is that $\frac{\partial^2 \mathcal{H}}{\partial p^2}(x, p) \neq 0$, which is almost equivalent to saying that \mathcal{H} has a nice Legendre transform, which is L in this case.

Things start falling into place. We will try to solve the following *well-defined problem*: is there an exotic deformation Hodge theory whose Laplacian looks like the operator \mathcal{L}_T in (1.78)? Note that this question does not involve path integrals any more. The answer to this section will be the exotic Hodge theory corresponding to the hypoelliptic Laplacian.

But before its effective construction, let us relate the above considerations to local index theory over LX .

1.6. Some aspects of local index theory on LX . Recall that $p_t(x, y)$ is the heat kernel associated to $\exp(t\Delta^X/2)$. As $t \rightarrow 0$, the heat kernel $p_t(x, x)$ on the diagonal is equivalent to $t^{-n/2}$. The fact that this singularity depends on the dimension n is one of the reasons why there are no heat kernels in infinite dimensions, since all measures tend to be mutually singular.

As we explain in (1.65), local index theory tells us that there are ‘fantastic cancellations’ in the local supertrace $\text{Tr}_s \left[\exp \left(-t\Box_{T/\sqrt{t}}^X/2 \right) \right]$ as $t \rightarrow 0$, which make that, in spite of the fact that the heat kernel is singular as $t \rightarrow 0$, with a singularity like $t^{-n/2}$, the supertrace itself is non singular. Tautologically, this cancellation mechanism is by definition dimension independent. This is why we can hope to produce directly local ‘densities’ on LX which should be related to the asymptotics as $t \rightarrow 0$ of some heat kernel supertrace on the diagonal. Of course none of this should be taken literally, since densities do not really exist in infinite dimensions, they are replaced by corresponding measures.

Now consider the path integral in (1.70). If one could apply (1.65) to LX , we would get

$$(1.81) \quad \text{Tr}_s \left[\exp \left(-t\Box_{T/\sqrt{t}}^{LX} \right) (x, x) \right] \rightarrow (\nabla I)^* \Phi^{TLX},$$

where \Box_T^{LX} would be the Witten Laplacian associated to the functional I in (1.66). However this disregards the fact we deal indeed with d_K closed forms on LX , and not with ordinary closed forms. Worst still, at least when $I = -E$, the term α is really needed to make the integral over LX converge in (1.70), even at a formal level.

Let X be a finite dimensional Riemannian manifold equipped with an action of S^1 associated to a Killing vector field K , and also equipped with a K -invariant smooth function $f : X \rightarrow \mathbf{R}$. Joint work with Goette [BG04] shows that it is possible to produce a natural elliptic operator on X such that the limit as a new time t' tends to 0 of the local supertrace of its heat kernel is given by

$$(1.82) \quad [\alpha_t \wedge (T\nabla f)^* \Phi^{TX}]^{\max},$$

where the forms in (1.82) are d_K -closed. So it seems that the local ‘density’ on LX of the path integral (1.71) can be viewed as a ‘local index density’ for an index problem on LX where the action of S^1 on LX should be incorporated, and this for any functional like \mathcal{F}, I or $-E$.

Incidentally observe that the action of the cyclic group U_m was already present in subsection 1.2, as a substitute for the action of S^1 .

Reference [BG04] played a basic conceptual role in the whole construction. Indeed the purpose of this reference is to give a formula for the difference of two natural versions of the equivariant Ray-Singer analytic torsion [RS71]. This difference is expressed as a new invariant, the V -invariant of a manifold equipped with a Killing vector field K .

Our point of view is to consider the analytic torsion of X as being the V -invariant of LX equipped with its action of S^1 , so that the main result of [BG04] is an illustration of a functoriality principle for V -invariants.

A remarkable property of V -invariants is that as shown in [BG04], they localize on critical points of invariant Morse functions. From our point of view, this explains the compatibility of the main result of [BG04] to the Cheeger-Müller theorem, where the smooth function to be considered on LX is precisely \mathcal{F} in (1.57). To our surprise, when replacing \mathcal{F} by $-E$, at least formally, we obtained a result closely related to Fried's conjectures [F86, F88] on relations between analytic torsion and Ruelle dynamical zeta functions. We will say more about this in section 3. However the above represented still another incentive to understand what Hodge theory would correspond to the path integral (1.79).

Our hope is to have convinced the reader that there is an array of facts which makes unavoidable the existence of the hypoelliptic Laplacian in its relation with LX .

A final point which we should emphasize is that the above does not provide any hint on how to put our hand on the general Dirac operator on LX , nor on the construction of the elliptic genus.

2. A CONSTRUCTION OF THE HYPOELLIPTIC LAPLACIAN

The purpose of this section is to explain the rigorous construction of the hypoelliptic Laplacian [B05].

This section is organized as follows. In subsection 2.1, we define the adjoint of the de Rham operator with respect to a non degenerate bilinear form on the tangent bundle.

In subsection 2.2, if X is a compact Riemannian manifold, we define a nontrivial bilinear form on TT^*X , and we construct the adjoint of the de Rham operator d^{T^*X} on T^*X with respect to that form.

In subsection 2.3, given a Hamiltonian function $\mathcal{H} : T^*X \rightarrow \mathbf{R}$, we introduce an extra Witten twist associated to \mathcal{H} . By taking the half-sum of d^{T^*X} and its 'adjoint', we obtain an operator $A_{\phi, \mathcal{H}}$.

In subsection 2.4, we show that $A_{\phi, \mathcal{H}}$ is self-adjoint with respect to a Hermitian form of signature (∞, ∞) .

In subsection 2.5, we give the Weitzenböck formula for the Laplacian $A_{\phi, \mathcal{H}}^2$. In degree 0, this formula coincides with the one we dreamt about in section 1.

In subsection 2.6, we obtain our hypoelliptic Laplacian, which depends on a parameter $b > 0$.

In subsection 2.7, we show that as $b \rightarrow 0$, this Laplacian should converge in the proper sense to $\square^X/4$.

In subsection 2.8, we show that as $b \rightarrow +\infty$, our Laplacian converges towards the generator of the Hamiltonian flow associated to \mathcal{H} .

Finally in subsection 2.9, we consider the case where X is a circle.

2.1. The adjoint of the de Rham operator with respect to a bilinear form.

In subsection 1.5, we saw the Hamiltonian vector field $Y^{\mathcal{H}}$ appear. This indicates that the symplectic structure of T^*X should play some role in the construction.

Let M be a manifold. Let η be a bilinear nondegenerate form on TM . Let $\phi : TM \rightarrow T^*M$ be the isomorphism canonically attached to η so that if $U, V \in TM$,

$$(2.1) \quad \eta(U, V) = \langle U, \phi V \rangle.$$

Let η^* be the bilinear form on T^*M which corresponds to η by the isomorphism ϕ , so that if $f, f' \in T^*M$,

$$(2.2) \quad \eta^*(f, f') = \langle \phi^{-1}f, f' \rangle.$$

The bilinear form η^* on T^*M extends to a bilinear form on $\Lambda(T^*M)$, which is obtained using the obvious extension of (2.2). Also η induces a volume form dv_M on M .

Let $(\Omega^*(M), d^M)$ be the de Rham complex of smooth forms on M with compact support. If $s, s' \in \Omega^*(M)$, put

$$(2.3) \quad \langle s, s' \rangle_\phi = \int_M \eta^*(s, s') dv_M.$$

Let \bar{d}^M be the formal adjoint of d^M with respect to $\langle \rangle_\phi$, so that

$$(2.4) \quad \langle s, d^M s' \rangle_\phi = \langle \bar{d}^M s, s' \rangle_\phi.$$

Note that since $\langle \rangle_\phi$ is in general not symmetric, the formal adjoint of \bar{d}^M is not equal to d^M .

Of course $\bar{d}^{M,2} = 0$. Then $[d^M, \bar{d}^M]$ is a generalized Laplacian.

Assume now that M is even dimensional, and that ω is a symplectic form on M . The symplectic form defines a nondegenerate bilinear form on TM . Therefore the above formalism can be applied. We denote by \bar{d}^M the formal adjoint of d^M which is associated to the symplectic form ω .

Proposition 2.1. *The following identity holds,*

$$(2.5) \quad [d^M, \bar{d}^M] = 0.$$

Proof. By Darboux's theorem, we can suppose that ω has constant coefficients. Then (2.5) follows from the vanishing of ω on the diagonal. \square

Remark 2.2. Identity (2.5) is responsible for some of the commutation relations in Kähler geometry.

Let (F, ∇^F, g^F) be a flat Hermitian vector bundle on M . Note that g^F is not supposed to be flat. The above construction is still possible when the de Rham complex $(\Omega^*(M), d^M)$ is replaced by $(\Omega^*(M, F), d^M)$. The bilinear form $\langle \rangle_\phi$ is now a skew-linear form, and incorporates the metric g^F in the obvious way. Still Proposition 2.1 only holds at the level of principal symbols.

2.2. An exotic Hodge theory on T^*X . Let X be a compact connected Riemannian manifold as in section 1, let (F, ∇^F, g^F) be a flat Hermitian vector bundle as in that section. Put

$$(2.6) \quad \omega(\nabla^F, g^F) = (g^F)^{-1} \nabla^F g^F.$$

Then $\omega(\nabla^F, g^F)$ is a 1-form with values in self-adjoint sections of $\text{End}(F)$. Set

$$(2.7) \quad \nabla^{F,u} = \nabla^F + \frac{1}{2} \omega(\nabla^F, g^F).$$

Then $\nabla^{F,u}$ is a unitary connection on F .

Let $\pi : T^*X \rightarrow X$ be the cotangent bundle of X . Let p be the generic element of the fibre T^*X . Let $\theta = \pi^*p$ be the canonical 1-form on T^*X , let $\omega = d^{T^*X}\theta$ be the canonical symplectic 2-form on T^*X . On $\Omega(T^*X, \pi^*F)$, we consider the symplectic adjoint \bar{d}^{T^*X} of d^{T^*X} . If F is trivial, by (2.5), we get

$$(2.8) \quad [d^{T^*X}, \bar{d}^{T^*X}] = 0.$$

To construct the hypoelliptic Laplacian, we have to go one step further. Let ∇^{TX} be the Levi-Civita connection on TX , and let R^{TX} be its curvature. Let ∇^{T^*X} be the connection induced by ∇^{TX} on T^*X , and let R^{T^*X} be its curvature. When identifying TX and T^*X by the metric g^{TX} , R^{TX} and R^{T^*X} correspond.

The connection ∇^{T^*X} induces a splitting of TT^*X , so that

$$(2.9) \quad TT^*X = \pi^*(TX \oplus T^*X).$$

Elements of the second factor in (2.9) or its dual will usually wear hats. From (2.9), we get the isomorphism,

$$(2.10) \quad \Lambda(T^*T^*X) = \pi^*(\Lambda(T^*X) \widehat{\otimes} \widehat{\Lambda}(TX)).$$

Let $\nabla^{\Lambda(T^*T^*X)}$ be the connection on $\Lambda(T^*T^*X)$ induced by ∇^{TX} .

Let e_1, \dots, e_n be a basis of TX , let e^1, \dots, e^n be the corresponding dual basis. Then

$$(2.11) \quad \omega = e^i \wedge \widehat{e}_i.$$

Let I be the vector bundle on X of smooth sections of $\Lambda(TX)$ along the fibre T^*X . By (2.10), and disregarding supports, we get

$$(2.12) \quad \Omega(T^*X, \pi^*F) = \Omega(X, I \otimes F).$$

Classically, using (2.10), we can write d^{T^*X} in the form,

$$(2.13) \quad d^{T^*X} = \widehat{d}^{T^*X} + \nabla^I + i_{\widehat{R^{TX}p}}.$$

In (2.13), \widehat{d}^{T^*X} is the de Rham operator along the fibre T^*X , ∇^I is the obvious connection on I , and $i_{\widehat{R^{TX}p}}$ denotes interior multiplication by the vertical vector $\widehat{R^{TX}p}$. We can rewrite (2.13) in the form,

$$(2.14) \quad d^{T^*X} = e^i \wedge \nabla_{e_i}^{\Lambda(T^*T^*X) \widehat{\otimes} F} + \widehat{e}_i \wedge \nabla_{\widehat{e}_i} + i_{\widehat{R^{TX}p}}.$$

On TT^*X , we consider the following bilinear form, so that if $U, V \in TT^*X$,

$$(2.15) \quad \eta(U, V) = \langle \pi_*U, \pi_*V \rangle_{g^{TX}} + \omega(U, V).$$

The isomorphism $\phi : TT^*X \rightarrow T^*T^*X$ associated to η is given by

$$(2.16) \quad \phi = \begin{pmatrix} g^{TX} & -1|_{T^*X} \\ 1|_{TX} & 0 \end{pmatrix}.$$

The volume form on T^*X associated to η is just the symplectic volume form dv_{T^*X} . Let $\bar{d}_\phi^{T^*X}$ be the formal adjoint of d^{T^*X} with respect to the bilinear form η as in subsection 2.1.

Put

$$(2.17) \quad \lambda_0 = \langle g^{TX} e_i, e_j \rangle e^i \wedge i_{\widehat{e}^j}, \quad \delta^{T^*X, V} = -\langle g^{TX} e_i, e_j \rangle i_{\widehat{e}^i} \nabla_{\widehat{e}^j}.$$

In the sequel, we will assume that the basis e_1, \dots, e_n is orthonormal, so that

$$(2.18) \quad \lambda_0 = e^i \wedge i_{\widehat{e}^i}, \quad \delta^{T^*X, V} = -i_{\widehat{e}^i} \nabla_{\widehat{e}^i}.$$

Set

$$(2.19) \quad R^{TX} p \wedge = \frac{1}{2} i_{\widehat{e}^i} i_{\widehat{e}^j} R^{TX} (e_i, e_j) p \wedge.$$

We now have the result established in [B05, Proposition 2.10].

Proposition 2.3. *The following identity holds,*

$$(2.20) \quad \bar{d}_\phi^{T^*X} = \bar{d}^{T^*X} - [\bar{d}^{T^*X}, \lambda_0].$$

Also,

$$(2.21) \quad \begin{aligned} [\bar{d}^{T^*X}, \lambda_0] &= -\delta^{T^*X, V}, \\ \bar{d}_\phi^{T^*X} &= -i_{\widehat{e}^i} \left(\nabla_{e_i}^\Lambda (T^*T^*X) \widehat{\otimes}^F + \omega(\nabla^F, g^F)(e_i) \right) \\ &\quad + i_{e_i} \nabla_{\widehat{e}^i} + R^{TX} p \wedge - i_{\widehat{e}^i} \nabla_{\widehat{e}^i}. \end{aligned}$$

2.3. A Hamiltonian function. Let $\mathcal{H} : T^*X \rightarrow \mathbf{R}$ be a smooth function. Let $Y^\mathcal{H}$ be the associated Hamiltonian vector field, so that $d\mathcal{H} + i_{Y^\mathcal{H}}\omega = 0$. Using (2.11), we get

$$(2.22) \quad Y^\mathcal{H} = (\nabla_{\widehat{e}^i} \mathcal{H}) e_i - (\nabla_{e_i} \mathcal{H}) \widehat{e}^i.$$

Recall that $\widehat{\nabla^V} \mathcal{H}$ is the fibrewise gradient field of \mathcal{H} .

Definition 2.4. Set

$$(2.23) \quad d_{\mathcal{H}}^{T^*X} = e^{-\mathcal{H}} d^{T^*X} e^{\mathcal{H}}, \quad \bar{d}_{\phi, \mathcal{H}}^{T^*X} = e^{\mathcal{H}} \bar{d}_\phi^{T^*X} e^{-\mathcal{H}}.$$

Observe that $\bar{d}_{\phi, \mathcal{H}}^{T^*X}$ is the adjoint of $d_{\mathcal{H}}^{T^*X}$ with respect to the Hermitian form $\langle \cdot \rangle_\phi$ in (2.3), with ϕ given by (2.16). Also, if $s, s' \in \Omega(T^*X, \pi^*F)$, put

$$(2.24) \quad \langle s, s' \rangle_{\phi, \mathcal{H}} = \int_{T^*X} \eta^*(s, s')_{g^F} e^{-2\mathcal{H}} dv_{T^*X}.$$

Then $\bar{d}_{\phi, 2\mathcal{H}}^{T^*X}$ is the adjoint of d^{T^*X} with respect to $\langle \cdot \rangle_{\phi, \mathcal{H}}$.

Definition 2.5. Set

$$(2.25) \quad \begin{aligned} A_{\phi, \mathcal{H}} &= \frac{1}{2} \left(\bar{d}_{\phi, 2\mathcal{H}}^{T^*X} + d^{T^*X} \right), & B_{\phi, \mathcal{H}} &= \frac{1}{2} \left(\bar{d}_{\phi, 2\mathcal{H}}^{T^*X} - d^{T^*X} \right), \\ \mathfrak{A}_{\phi, \mathcal{H}} &= \frac{1}{2} \left(\bar{d}_{\phi, \mathcal{H}}^{T^*X} + d_{\mathcal{H}}^{T^*X} \right), & \mathfrak{B}_{\phi, \mathcal{H}} &= \frac{1}{2} \left(\bar{d}_{\phi, \mathcal{H}}^{T^*X} - d_{\mathcal{H}}^{T^*X} \right). \end{aligned}$$

Then

$$(2.26) \quad \mathfrak{A}_{\phi, \mathcal{H}} = e^{-\mathcal{H}} A_{\phi, \mathcal{H}} e^{\mathcal{H}}, \quad \mathfrak{B}_{\phi, \mathcal{H}} = e^{-\mathcal{H}} B_{\phi, \mathcal{H}} e^{\mathcal{H}}.$$

We have the identities,

$$(2.27) \quad d^{T^*X, 2} = 0, \quad \bar{d}_{\phi, \mathcal{H}}^{T^*X, 2} = 0.$$

From (2.27), we deduce that

$$(2.28) \quad A_{\phi, \mathcal{H}}^2 = -B_{\phi, \mathcal{H}}^2 = \frac{1}{4} \left[d^{T^*X}, \bar{d}_{\phi, 2\mathcal{H}}^{T^*X} \right], \quad [A_{\phi, \mathcal{H}}, B_{\phi, \mathcal{H}}] = 0, \\ \left[d^{T^*X}, A_{\phi, \mathcal{H}}^2 \right] = 0, \quad \left[\bar{d}_{\phi, 2\mathcal{H}}^{T^*X}, A_{\phi, \mathcal{H}}^2 \right] = 0.$$

Let $\nabla^{\Lambda(T^*T^*X) \otimes F, u}$ be the connection on $\Lambda(T^*T^*X) \otimes F$ which is associated to $\nabla^{T^*T^*X}$ and $\nabla^{F, u}$. We have the result established in [B05, Proposition 2.18].

Proposition 2.6. *The following identities hold,*

$$(2.29) \quad A_{\phi, \mathcal{H}} = \frac{1}{2} (e^i - i_{\widehat{e}^i}) \nabla_{e_i}^{\Lambda(T^*T^*X) \otimes F, u} - \frac{1}{4} (e^i + i_{\widehat{e}^i}) \omega(\nabla^F, g^F)(e_i) \\ + \frac{1}{2} (\widehat{e}_i + i_{e_i - \widehat{e}^i}) \nabla_{\widehat{e}^i} + \frac{1}{2} \left(R^{TX} p \wedge + i_{\widehat{R^{TX} p}} \right) \\ + i_{\widehat{e}^i} \nabla_{e_i} \mathcal{H} + i_{\widehat{e}^i - e_i} \nabla_{\widehat{e}^i} \mathcal{H}, \\ \mathfrak{A}_{\phi, \mathcal{H}} = \frac{1}{2} (e^i - i_{\widehat{e}^i}) \nabla_{e_i}^{\Lambda(T^*T^*X) \otimes F, u} - \frac{1}{4} (e^i + i_{\widehat{e}^i}) \omega(\nabla^F, g^F)(e_i) \\ + \frac{1}{2} (\widehat{e}_i + i_{e_i - \widehat{e}^i}) \nabla_{\widehat{e}^i} + \frac{1}{2} \left(R^{TX} p \wedge + i_{\widehat{R^{TX} p}} \right) \\ + \frac{1}{2} (e^i + i_{\widehat{e}^i}) \nabla_{e_i} \mathcal{H} + \frac{1}{2} (\widehat{e}_i + i_{\widehat{e}^i - e_i}) \nabla_{\widehat{e}^i} \mathcal{H}.$$

Set

$$(2.30) \quad \mu_0 = \widehat{e}_i \wedge i_{e_i}.$$

Put

$$(2.31) \quad \mathfrak{A}'_{\phi, \mathcal{H}} = e^{-\mu_0} \mathfrak{A}_{\phi, \mathcal{H}} e^{\mu_0}.$$

The operator $\mathfrak{A}'_{\phi, \mathcal{H}}$ will also be considered in the sequel.

Remark 2.7. Let M be a symplectic manifold as in Proposition 2.1, and let $\mathcal{H} : M \rightarrow \mathbf{R}$ be a smooth function. Put

$$(2.32) \quad \bar{d}_{\mathcal{H}}^M = e^{\mathcal{H}} \bar{d}^M e^{-\mathcal{H}}.$$

Let $Y^{\mathcal{H}}$ still denote the Hamiltonian vector field associated to \mathcal{H} . One verifies easily that

$$(2.33) \quad \bar{d}_{\mathcal{H}}^M = \bar{d}^M - i_{Y^{\mathcal{H}}}.$$

Combining (2.5) with (2.33), we obtain

$$(2.34) \quad \left[d^M, \bar{d}_{\mathcal{H}}^M \right] = -L_{Y^{\mathcal{H}}}.$$

Equation (2.34) shows that the operator $-L_{Y^{\mathcal{H}}}$ can be considered as a generalized Witten Laplacian. This fact plays an important role in our construction of the hypoelliptic Laplacian.

2.4. A self-adjointness property. The sesquilinear form $\langle \cdot \rangle_{\phi, \mathcal{H}}$ is in general not a Hermitian form, that is exchanging the two arguments does not produce the conjugate expression. Following [B05, section 2.7], we will produce a Hermitian form with respect to which $A_{\phi, \mathcal{H}}$ will be self-adjoint.

Set

$$(2.35) \quad f = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad F = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}.$$

Then f defines a scalar product on \mathbf{R}^2 , and F is an involution of \mathbf{R}^2 , which is an isometry with respect to f . Its $+1$ eigenspace is spanned by $(1, 0)$, and the -1 eigenspace is spanned by $(1, -1)$. Finally, the volume form on \mathbf{R}^2 which is attached to f is just the original volume form of \mathbf{R}^2 .

Using the identifications in (2.9), we observe that f defines a metric \mathfrak{g}^{TT^*X} on TT^*X given by

$$(2.36) \quad \mathfrak{g}^{TT^*X} = \begin{pmatrix} g^{TX} & 1|_{T^*X} \\ 1|_{TX} & 2g^{T^*X} \end{pmatrix}.$$

Let $\mathfrak{p} : TT^*X \rightarrow T^*X$ be the obvious projection with respect to the splitting (2.9) of TT^*X . Then if $U \in TT^*X$,

$$(2.37) \quad \langle U, U \rangle_{\mathfrak{g}^{TT^*X}} = \langle \pi_* U, \pi_* U \rangle_{g^{TX}} + 2 \langle \pi_* U, \mathfrak{p}U \rangle + 2 \langle \mathfrak{p}U, \mathfrak{p}U \rangle_{g^{T^*X}}.$$

Then the volume form on T^*X which is attached to \mathfrak{g}^{TT^*X} is the symplectic volume form.

Similarly, we will identify F to the \mathfrak{g}^{TT^*X} isometric involution of TT^*X ,

$$(2.38) \quad F = \begin{pmatrix} 1|_{TX} & 2(g^{TX})^{-1} \\ 0 & -1|_{T^*X} \end{pmatrix}.$$

Then F acts as $\tilde{F}^{-1} = \tilde{F}$ on $\Lambda^{\cdot}(TT^*X)$.

Let $r : T^*X \rightarrow T^*X$ be the involution $(x, p) \rightarrow (x, -p)$.

Definition 2.8. Let $\langle \cdot \rangle_{\mathfrak{g}^{\Omega(T^*X, \pi^*F)}}$ be the Hermitian product on $\Omega^{\cdot}(T^*X, \pi^*F)$ which is naturally associated to the metrics \mathfrak{g}^{TT^*X} and g^F . Let u be the isometric involution of $\Omega^{\cdot}(T^*X, \pi^*F)$ with respect to $\langle \cdot \rangle_{\mathfrak{g}^{\Omega(T^*X, \pi^*F)}}$ such that if $s \in \Omega^{\cdot}(T^*X, \pi^*F)$,

$$(2.39) \quad us(x, p) = Fs(x, -p).$$

Let $\langle \cdot \rangle_{\mathfrak{h}^{\Omega(T^*X, \pi^*F)}}$ be the Hermitian form on $\Omega^{\cdot}(T^*X, \pi^*F)$,

$$(2.40) \quad \langle s, s' \rangle_{\mathfrak{h}^{\Omega(T^*X, \pi^*F)}} = \langle us, s' \rangle_{\mathfrak{g}^{\Omega(T^*X, \pi^*F)}}.$$

It should be pointed out that in (2.39), the change of variable $p \rightarrow -p$ is not made on the form part of s . So this action does not incorporate the full action of r^* . Set

$$(2.41) \quad \langle s, s' \rangle_{\mathfrak{h}^{\Omega(T^*X, \pi^*F)}} = \langle ue^{-2\mathcal{H}}s, s' \rangle_{\mathfrak{g}^{\Omega(T^*X, \pi^*F)}}.$$

If \mathcal{H} is r -invariant, then (2.41) is a Hermitian form.

Let g^{TT^*X} be the obvious natural metric on TT^*X which is associated to the splitting (2.9), and let $g^{\Omega(T^*X, \pi^*F)}$ be the Hermitian product on $\Omega^{\cdot}(T^*X, \pi^*F)$ associated to g^{TT^*X}, g^F . Let $h^{\Omega(T^*X, \pi^*F)}$ be the Hermitian form on $\Omega^{\cdot}(T^*X, \pi^*F)$,

$$(2.42) \quad \langle s, s' \rangle_{h^{\Omega(T^*X, \pi^*F)}} = \langle r^*s, s' \rangle_{g^{\Omega(T^*X, \pi^*F)}}.$$

Note that the Hermitian forms in (2.40) and (2.42) have signature (∞, ∞) . The same is true for (2.41) if \mathcal{H} is r -invariant.

We state a result established in [B05, Theorems 2.21 and 2.30].

Theorem 2.9. *If \mathcal{H} is r -invariant, then $A_{\phi, \mathcal{H}}$ is $\mathfrak{h}_{\mathcal{H}}^{\Omega(T^*X, \pi^*F)}$ self-adjoint, $\mathfrak{A}_{\phi, \mathcal{H}}$ is $\mathfrak{h}^{\Omega(T^*X, \pi^*F)}$ self-adjoint, and $\mathfrak{A}'_{\phi, \mathcal{H}}$ is $\mathfrak{h}^{\Omega(T^*X, \pi^*F)}$ self-adjoint.*

2.5. The Weitzenböck formula. We give the Weitzenböck formula established in [B05, Theorem 3.3].

Theorem 2.10. *The following identities hold,*

$$\begin{aligned}
(2.43) \quad A_{\phi, \mathcal{H}}^2 &= \frac{1}{4} \left(-\Delta^V - \frac{1}{2} \langle R^{TX}(e_i, e_j) e_k, e_l \rangle e^i e^j i_{\widehat{e}^k} i_{\widehat{e}^l} + 2L_{\widehat{\nabla^V \mathcal{H}}} \right) \\
&- \frac{1}{2} \left(L_{Y^{\mathcal{H}}} + \frac{1}{2} e^i i_{\widehat{e}^j} \nabla_{e_i}^F \omega(\nabla^F, g^F)(e_j) + \frac{1}{2} \omega(\nabla^F, g^F)(e_i) \nabla_{\widehat{e}^i} \right), \\
\mathfrak{A}_{\phi, \mathcal{H}}^2 &= \frac{1}{4} \left(-\Delta^V - \frac{1}{2} \langle R^{TX}(e_i, e_j) e_k, e_l \rangle e^i e^j i_{\widehat{e}^k} i_{\widehat{e}^l} + |\nabla^V \mathcal{H}|^2 \right. \\
&\quad \left. - \Delta^V \mathcal{H} + 2(\nabla_{\widehat{e}^i} \nabla_{\widehat{e}^j} \mathcal{H}) \widehat{e}_i i_{\widehat{e}^j} + 2(\nabla_{\widehat{e}^i} \nabla_{e_j} \mathcal{H}) e^j i_{\widehat{e}^i} \right) \\
&- \frac{1}{2} \left(L_{Y^{\mathcal{H}}} + \frac{1}{2} \omega(\nabla^F, g^F)(Y^{\mathcal{H}}) + \frac{1}{2} e^i i_{\widehat{e}^j} \nabla_{e_i}^F \omega(\nabla^F, g^F)(e_j) \right. \\
&\quad \left. + \frac{1}{2} \omega(\nabla^F, g^F)(e_i) \nabla_{\widehat{e}^i} \right).
\end{aligned}$$

Remark 2.11. Observe that if $F = \mathbf{R}$, the restriction of $2A_{\phi, \mathcal{H}}^2$ to $\Omega^0(T^*X)$ is given by,

$$(2.44) \quad 2A_{\phi, \mathcal{H}}^2|_{\Omega^0(T^*X)} = -\frac{1}{2} \Delta^V + \nabla_{\widehat{\nabla^V \mathcal{H}}} - \nabla_{Y^{\mathcal{H}}},$$

which is just the operator \mathcal{L}_1 in (1.78).

More generally set

$$(2.45) \quad \mathcal{H}_T(x, p) = T\mathcal{H}(x, p/T).$$

If \mathcal{H} is the Legendre transform of $L(x, \dot{x})$, then \mathcal{H}_T is the Legendre transform of $TL(x, \dot{x})$.

Let K_T be the map $s(x, p) \rightarrow s(x, Tp)$. Then

$$(2.46) \quad K_T 2A_{\phi, \mathcal{H}_T}^2|_{\Omega^0(T^*X)} K_T^{-1} = -\frac{\Delta^V}{2T^2} + \frac{1}{T} \nabla_{\widehat{\nabla^V \mathcal{H}}} - \nabla_{Y^{\mathcal{H}}},$$

which is just the operator \mathcal{L}_T in (1.78). The program we had outlined at the end of section 1 is now partially fulfilled. The operator $2A_{\phi, \mathcal{H}}^2$ is indeed the Laplacian of an exotic Hodge theory whose restriction to forms of degree 0 is precisely the operator \mathcal{L}_T .

2.6. The hypoelliptic Laplacian. For $c \in \mathbf{R}$, set

$$(2.47) \quad \mathcal{H}^c = c \frac{|p|^2}{2}.$$

Let $u \in \mathbf{R}$ be an extra variable. The following result was established in [B05, Theorems 3.4 and 3.6].

Theorem 2.12. *The following identities hold,*

$$\begin{aligned}
 (2.48) \quad A_{\phi, \mathcal{H}^c}^2 &= \frac{1}{4} \left(-\Delta^V + 2cL_{\widehat{p}} - \frac{1}{2} \langle R^{TX}(e_i, e_j) e_k, e_l \rangle e^i e^j i_{\widehat{e}^k} i_{\widehat{e}^l} \right) \\
 &\quad - \frac{1}{2} \left(L_{Y^{\mathcal{H}^c}} + \frac{1}{2} e^i i_{\widehat{e}^j} \nabla_{e_i}^F \omega(\nabla^F, g^F)(e_j) + \frac{1}{2} \omega(\nabla^F, g^F)(e_i) \nabla_{\widehat{e}^i} \right), \\
 \mathfrak{A}_{\phi, \mathcal{H}^c}^2 &= \frac{1}{4} \left(-\Delta^V + c^2 |p|^2 + c(2\widehat{e}_i i_{\widehat{e}^i} - n) - \frac{1}{2} \langle R^{TX}(e_i, e_j) e_k, e_l \rangle e^i e^j i_{\widehat{e}^k} i_{\widehat{e}^l} \right) \\
 &\quad - \frac{1}{2} \left(L_{Y^{\mathcal{H}^c}} + \frac{1}{2} \omega(\nabla^F, g^F)(Y^{\mathcal{H}^c}) + \frac{1}{2} e^i i_{\widehat{e}^j} \nabla_{e_i}^F \omega(\nabla^F, g^F)(e_j) \right. \\
 &\quad \left. + \frac{1}{2} \omega(\nabla^F, g^F)(e_i) \nabla_{\widehat{e}^i} \right).
 \end{aligned}$$

For $c \neq 0$, the operators $\frac{\partial}{\partial u} - A_{\phi, \mathcal{H}^c}^2, \frac{\partial}{\partial u} - \mathfrak{A}_{\phi, \mathcal{H}^c}^2$ are hypoelliptic.

Remark 2.13. Of course (2.48) follows from Theorem 2.10. Hypoellipticity follows from Hörmander [Hö67]. Also observe that the hypoellipticity result still holds if $\frac{\partial^2 \mathcal{H}}{\partial p^2}$ is non degenerate.

Any of the operators in Theorem 2.12 is called a hypoelliptic Laplacian.

2.7. An interpolation property: the limit $b \rightarrow 0$ and classical Hodge theory. In the sequel, we take $b > 0$, $T = b^2$, and we still define \mathcal{H}_T as in (2.45). For $T > 0$, set

$$(2.49) \quad \widehat{\mathcal{H}}_T(x, p) = T\mathcal{H}(x, p/\sqrt{T}).$$

For $a \in \mathbf{R}$, let $r_a : T^*X \rightarrow T^*X$ be given by $(x, p) \rightarrow (x, ap)$. Note that $r = r_{-1}$. By (2.43), we get

$$\begin{aligned}
 (2.50) \quad r_b^* A_{\phi, \mathcal{H}_{b^2}}^2 r_b^{*-1} &= \frac{1}{4b^2} \left(-\Delta^V - \frac{1}{2} \langle R^{TX}(e_i, e_j) e_k, e_l \rangle e^i e^j i_{\widehat{e}^k} i_{\widehat{e}^l} + 2L_{b\nabla^V \widehat{\mathcal{H}}(p/b)} \right) \\
 &\quad - \frac{1}{2b} \left(L_{Y^{\widehat{\mathcal{H}}_{b^2}}} + \frac{1}{2} e^i i_{\widehat{e}^j} \nabla_{e_i} \omega(\nabla^F, g^F)(e_j) + \frac{1}{2} \omega(\nabla^F, g^F)(e_i) \nabla_{\widehat{e}^i} \right).
 \end{aligned}$$

Now we study the behaviour of the operator in (2.50) as $b \rightarrow 0$. To make the argument simpler, we set

$$(2.51) \quad \mathcal{H} = \frac{|p|^2}{2}.$$

In this case,

$$(2.52) \quad \mathcal{H}_{b^2} = \mathcal{H}/b^2 = \mathcal{H}^{1/b^2}, \quad \widehat{\mathcal{H}}_{b^2} = \mathcal{H}.$$

The Hermitian form $\mathfrak{h}_{\pm \mathcal{H}_{b^2}}^{\Omega^0(T^*X, \pi^*F)}$ is given by

$$(2.53) \quad \langle s, s' \rangle_{\mathfrak{h}_{\pm \mathcal{H}_{b^2}}^{\Omega^0(T^*X, \pi^*F)}} = \int_{T^*X} \langle s(x, p), s'(x, -p) \rangle_{g^F} e^{\mp 2\mathcal{H}/b^2} dv_{T^*X}.$$

In degree 0, the self-adjointness of $A_{\phi, \pm \mathcal{H}_{b^2}}^2$ with respect to (2.53) is the exact reflection of the self-adjointness in degree 0 of the formal Witten Laplacian $\square_T^{X^m}$ of section 1.2 or of the non existing Laplacian $\square_b^{L^X}$ associated with $\mp E$.

Set

$$(2.54) \quad \mathbf{a}_{\pm} = \frac{1}{2} \left(-\Delta^V \pm 2L_{\widehat{p}} - \frac{1}{2} \langle R^{TX} (e_i, e_j) e_k, e_l \rangle e^i e^j i_{\widehat{e}^k} i_{\widehat{e}^l} \right),$$

$$\mathbf{b}_{\pm} = - \left(\pm L_{Y^{\mathcal{H}}} + \frac{1}{2} e^i i_{\widehat{e}^j} \nabla_{e_i}^F \omega (\nabla^F, g^F) (e_j) + \frac{1}{2} \omega (\nabla^F, g^F) (e_i) \nabla_{\widehat{e}^i} \right)$$

Note that \mathbf{a}_{\pm} commutes with r^* , and \mathbf{b}_{\pm} anticommutes with r^* . Also one checks easily that

$$(2.55) \quad \mathbf{a}_{\pm} = \frac{1}{2} \left[d^{T^*X}, \delta^{T^*X, V} \pm 2i_{\widehat{p}} \right], \quad \mathbf{b}_{\pm} = \frac{1}{2} \left[d^{T^*X}, \bar{d}^{T^*X} \mp 2i_{Y^{\mathcal{H}}} \right].$$

In particular $\mathbf{a}_{\pm}, \mathbf{b}_{\pm}$ commute with d^{T^*X} .

In (2.50), we replace \mathcal{H} by $\pm \mathcal{H}$. We get

$$(2.56) \quad r_b^* 2A_{\phi, \pm \mathcal{H}_{b^2}}^2 r_b^{*-1} = \frac{\mathbf{a}_{\pm}}{b^2} + \frac{\mathbf{b}_{\pm}}{b}.$$

Observe that the operator \mathbf{a}_{\pm} makes sense on any real Euclidean vector bundle with connection (E, g^E, ∇^E) , and not only on T^*X . To keep the discussion short, we will limit ourselves to the case where $E = T^*X$, but the fact that \mathbf{a}_{\pm} makes sense in full generality is important.

Let Φ^{T^*X} be the Thom form associated to $(T^*X, 2g^{T^*X}, \nabla^{T^*X})$ as in (1.52). The choice of $2g^{T^*X}$ instead of g^{T^*X} reflects a difference in scaling, that is, instead of (1.52), we have now

$$(2.57) \quad \Phi^{T^*X} = \exp \left(-|p|^2 + \dots \right).$$

The following result is established in [B05, Theorem 3.11].

Theorem 2.14. *The following identities hold,*

$$(2.58) \quad d^{T^*X} \Phi^{T^*X} = 0, \quad \left(\delta^{T^*X} - 2i_{\widehat{p}} \right) \Phi^{T^*X} = 0.$$

*The operator \mathbf{a}_{\pm} is semisimple. The kernel of \mathbf{a}_{+} is spanned over $\Lambda(T^*X)$ by the zero form 1. The corresponding spectral projection operator $Q_{+}^{T^*X}$ is given by*

$$(2.59) \quad \alpha \rightarrow Q_{+}^{T^*X} \alpha = \pi^* \pi_* \left(\alpha \wedge \Phi^{T^*X} \right).$$

*The kernel of \mathbf{a}_{-} is spanned over $\Lambda(T^*X)$ by Φ^{T^*X} , and the corresponding spectral projection operator $Q_{-}^{T^*X}$ is given by*

$$(2.60) \quad \alpha \rightarrow Q_{-}^{T^*X} \alpha = (\pi^* \pi_* \alpha) \wedge \Phi^{T^*X}.$$

Remark 2.15. Theorem 2.14 is remarkable. It asserts in particular that Φ^{T^*X} is a harmonic form with respect to a fibrewise exotic Hodge theory, as shown by equations (2.55) and (2.58). Together with (1.50), this characterizes the Mathai-Quillen form Φ^{T^*X} uniquely.

Since the operators \mathbf{a}_{\pm} are semisimple, we can write,

$$(2.61) \quad \Omega(T^*X, \pi^*F) = \ker \mathbf{a}_{\pm} \oplus \text{Im } \mathbf{a}_{\pm}.$$

Needless to say, one should be careful about the function spaces on which the operator \mathfrak{a}_\pm acts. However \mathfrak{a}_\pm is conjugate to a standard harmonic oscillator, so this question can be easily settled.

Since \mathfrak{b}_\pm anticommutes with r^* , it exchanges the invariant and antiinvariant parts of $\Omega^*(T^*X, \pi^*F)$, while \mathfrak{a}_\pm preserves these invariant and antiinvariant parts. By Theorem 2.14, $\ker \mathfrak{a}_\pm$ is either invariant or antiinvariant. It follows that \mathfrak{b}_\pm maps $\ker \mathfrak{a}_\pm$ into $\text{Im } \mathfrak{a}_\pm$.

Let us pretend for the moment that $\mathfrak{a}_\pm, \mathfrak{b}_\pm$ are endomorphisms of a finite dimensional vector space E , that \mathfrak{a}_\pm is semisimple, so that

$$(2.62) \quad E = \ker \mathfrak{a}_\pm \oplus \text{Im } \mathfrak{a}_\pm.$$

Let Q_\pm be the projector from E on $\ker \mathfrak{a}_\pm$ with respect to the splitting (2.62). We also assume that \mathfrak{b}_\pm maps $\ker \mathfrak{a}_\pm$ into $\text{Im } \mathfrak{a}_\pm$.

Let $u \in \text{End}(E)$. We write u in matrix form with respect to the splitting (2.62).

$$(2.63) \quad u = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

Assume that u is invertible. We will give a matrix expression for the inverse u^{-1} of u under the assumption that D is invertible. We will implicitly assume that other matrix expressions are invertible as well. These implicit assumptions will be obvious in the formula anyway.

In fact we have the following easy formula,

$$(2.64) \quad u^{-1} = \begin{bmatrix} (A - BD^{-1}C)^{-1} & -(A - BD^{-1}C)^{-1}BD^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & D^{-1} + D^{-1}C(A - BD^{-1}C)^{-1}BD^{-1} \end{bmatrix}.$$

Let \mathfrak{a}_\pm^{-1} be the inverse of \mathfrak{a}_\pm acting on $\text{Im } \mathfrak{a}_\pm$. Using (2.64), if $\lambda \in \mathbf{C}$, at least formally,

$$(2.65) \quad \left(\lambda - \frac{\mathfrak{a}_\pm}{b^2} - \frac{\mathfrak{b}_\pm}{b} \right)^{-1} = \begin{bmatrix} (\lambda + Q_\pm \mathfrak{b}_\pm \mathfrak{a}_\pm^{-1} \mathfrak{b}_\pm Q_\pm)^{-1} + \mathcal{O}(b) & \mathcal{O}(b) \\ \mathcal{O}(b) & \mathcal{O}(b^2) \end{bmatrix}.$$

By (2.65) we find that as $b \rightarrow 0$,

$$(2.66) \quad \left(\lambda - \frac{\mathfrak{a}_\pm}{b^2} - \frac{\mathfrak{b}_\pm}{b} \right)^{-1} = Q_\pm (\lambda + Q_\pm \mathfrak{b}_\pm \mathfrak{a}_\pm^{-1} \mathfrak{b}_\pm Q_\pm)^{-1} Q_\pm + \mathcal{O}(b).$$

In particular the relevant operator in the limit $b \rightarrow 0$ appears to be $-Q_\pm \mathfrak{b}_\pm \mathfrak{a}_\pm^{-1} \mathfrak{b}_\pm Q_\pm$ acting on $\ker \mathfrak{a}_\pm$.

Passing from the above finite dimensional argument to an infinite dimensional considered in (2.56) is a wild jump. However this is the sort of situation one encounters typically in adiabatic limit problems in the theory of Quillen metrics [BL91, BB94]. The major difference is that the operators considered in these references are elliptic and self-adjoint, which is not the case here.

We have given enough motivation for studying the operator $-Q_\pm^{T^*X} \mathfrak{b}_\pm \mathfrak{a}_\pm^{-1} \mathfrak{b}_\pm Q_\pm^{T^*X}$ in the context of (2.56).

We identify $\Omega^+(X, F)$ to $\ker \mathfrak{a}_+$ by the map $\alpha \rightarrow \pi^* \alpha$, and $\Omega^-(X, F \otimes o(TX))$ to $\ker \mathfrak{a}_-$ by the map $\alpha \rightarrow \pi^* \alpha \wedge \Phi^{T^*X}$. Let \square^X be the standard Hodge Laplacian acting on $\Omega^+(X, F)$ in the + case, on $\Omega^-(X, F \otimes o(TX))$ in the - case.

Now we state the crucial result established in [B05, Theorem 3.13].

Theorem 2.16. *The following identity holds,*

$$(2.67) \quad -Q_{\pm}^{T^*X} \mathbf{b}_{\pm} \mathbf{a}_{\pm}^{-1} \mathbf{b}_{\pm} Q_{\pm}^{T^*X} = \frac{1}{2} \square^X.$$

In the same way, it is shown in [B05, Theorem 3.8] that

$$(2.68) \quad K_b 2\mathfrak{A}_{\phi, \pm \mathcal{H}_{b^2}}^2 K_b^{-1} = \frac{\alpha_{\pm}}{b^2} + \frac{\beta_{\pm}}{b} + \gamma_{\pm}.$$

We just give the corresponding formulas for $\alpha_{\pm}, \beta_{\pm}$,

$$(2.69) \quad \alpha_{\pm} = \frac{1}{2} \left(-\Delta^V + |p|^2 \pm (2\tilde{e}^i i_{\tilde{e}^i} - n) \right),$$

$$\beta_{\pm} = -\nabla_{\pm Y^{\mathcal{H}}}^{\wedge (T^*X) \otimes F, u} + \frac{1}{2} \omega(\nabla^F, g^F)(e_i) \nabla_{\tilde{e}^i}.$$

The main point of (2.68), (2.69) is that contrary to \mathbf{a}_{\pm} , α_{\pm} is a standard self-adjoint harmonic oscillator. Then $\ker \alpha_{\pm}$ is spanned by the function $\exp(-|p|^2/2)$, and $\ker \alpha_{-}$ by $\exp(-|p|^2/2) \eta$, where η is a fibrewise n -form of norm 1.

We identify $\Omega(X, F)$ to $\ker \alpha_{+}$ by the map $\alpha \rightarrow \pi^* \alpha \exp(-|p|^2/2) / \pi^{n/4}$, and $\Omega(X, F \otimes o(TX))$ to $\ker \alpha_{-}$ by the map $\alpha \rightarrow \pi^* \alpha \exp(-|p|^2/2) \eta / \pi^{n/4}$. Let P_{\pm} be the standard L^2 -projector from $\Omega(T^*X, \pi^*F)$ on $\ker \alpha_{\pm}$. Note that β_{\pm} maps $\ker \alpha_{\pm}$ into its L^2 orthogonal.

In [B05, Theorem 3.14], the following analogue of Theorem 2.16 is established.

Theorem 2.17. *The following identity holds,*

$$(2.70) \quad P_{\pm} (\gamma_{\pm} - \beta_{\pm} \alpha_{\pm}^{-1} \beta_{\pm}) P_{\pm} = \frac{1}{2} \square^X.$$

Remark 2.18. Theorems 2.16 and 2.17 give another powerful argument in favour of the fact that up to conjugation, $A_{\phi, \mathcal{H}^c}^2$ is a deformation of $\square^X/4$. It is of an entirely different nature than the one discussed in (1.43)-(1.45).

Indeed the content of these equations can be made rigorous. What these equations say is that for a given $T = \pm b^2$, the process x is a motion whose speed $\dot{x} = p$ is what is known as an Ornstein-Uhlenbeck process (or autoregressive process in the statistics literature), with covariance is $\exp(-|t-s|/T)/2T$. When T to 0, the covariance tends to the Dirac $\delta_{t=s}$, when $T \rightarrow +\infty$, it tends to 0. This means that when $T \rightarrow 0$, the dynamics of x becomes Brownian, and when $T \rightarrow +\infty$, the speed of x becomes constant, i.e. it becomes a geodesic. Proving this convergence at the dynamics level was already done by Stroock and Varadhan [StV72], where instead they approximated Brownian motion by piecewise geodesic approximations. The key to the argument in [StV72] is seeing the Itô calculus as the proper limit of classical differential calculus on \mathbf{R}_{+} .

What equations (2.67) and (2.70) reflect is of a different nature. They should be viewed as a functional analytic version of Itô calculus, where as $T \rightarrow 0$, besides its more and more erratic dynamics, we request the process x to also remember about Hodge theory...

The same arguments are still valid when instead of being quadratic, the Hamiltonian \mathcal{H} is such that $\frac{\partial^2 \mathcal{H}}{\partial p^2}$ is nondegenerate.

When $\mathcal{H} = |p|^2/2$, for $T \neq 0$, the functional integral in (1.79) can be viewed as a regularized version of the corresponding functional integral with $T = 0$ because

of the regularizing effect of the term $\frac{T^2}{2} \int_{S^1} |\dot{x}|^2 ds$. A byproduct of this regularization is that while, for $T = 0$, the trajectories of Brownian motion are nowhere differentiable, for $T \neq 0$, the trajectories of the solution of (1.80) will be C^1 .

Usually regularization is viewed as bad, since we replace the real physical theory by an approximation. We would like to take here the opposite view. In our context, regularization of the theory is excellent, since it will lead to a deformation of Hodge theory to the geodesic flow.

In joint work with Lebeau [BL06], the hard analysis involved in the analysis of the convergence of $A_{\phi, \mathcal{H}^c}^2$ is carried through in detail. The results of [BL06] will be briefly reviewed in section 3.

2.8. An interpolation property: the limit $b \rightarrow +\infty$ and the geodesic flow.

We assume for the moment \mathcal{H} to be arbitrary. Using (2.43), we get

$$(2.71) \quad r_{b^2}^* 2\mathfrak{A}_{\phi, b^2 \mathcal{H}_{1/b^2}}^2 r_{1/b^2}^* = \frac{1}{2} |\nabla^V \mathcal{H}|^2 - \left(L_{Y^{\mathcal{H}}} + \frac{1}{2} \omega(\nabla^F, g^F)(Y^{\mathcal{H}}) \right) + \mathcal{O}(1/b).$$

The dynamics associated to the operator in the right-hand side of (2.71) is associated to the Hamiltonian vector field $Y^{\mathcal{H}}$. In the case where $\mathcal{H} = \pm |p|^2/2$, this is just the geodesic flow.

From (2.71), we deduce that when $b \rightarrow +\infty$, the trace of an operator like $\exp(-A_{\phi, \mathcal{H}^c}^2)$ should localize around closed geodesics.

2.9. The case of the circle. Assume that $X = S^1$, with $S^1 = \mathbf{R}/\mathbf{Z}$ equipped with its standard metric. Then $T^*X = S^1 \times \mathbf{R}$. We take here $F = \mathbf{R}$ and $c > 0$. We will now find remarkable properties of the hypoelliptic Laplacian in this simple situation. Here we follow [B05, section 3.10].

By (2.31) and by (2.48), we get

$$(2.72) \quad \mathfrak{A}_{\phi, \mathcal{H}^c}^{\prime 2} = \frac{1}{4} \left(-\Delta^V + c^2 |p|^2 + c \left(2\widehat{N} - 1 \right) \right) - \frac{c}{2} \nabla_p^F.$$

In (2.72), \widehat{N} is just the number operator in $\Lambda \cdot (\mathbf{R})$.

An easy formal computation shows that for $c \neq 0$,

$$(2.73) \quad \exp\left(\frac{1}{c} \frac{\partial^2}{\partial p \partial x}\right) \mathfrak{A}_{\phi, \mathcal{H}^c}^{\prime 2} \exp\left(-\frac{1}{c} \frac{\partial^2}{\partial p \partial x}\right) \\ = \frac{1}{4} \left(-\Delta^V + c^2 |p|^2 + c \left(2\widehat{N} - 1 \right) \right) - \frac{1}{4} \Delta^X.$$

The conjugation in (2.73) is done with an unbounded operator, but still the conclusions one can derive from (2.73) are correct. In particular the spectrum of $\mathfrak{A}_{\phi, \mathcal{H}^c}^{\prime 2}$ is given by $c \frac{\mathbf{N}}{2} + 2k^2 \pi^2$, where $\frac{\mathbf{N}}{2}$ denotes the set of $a \in \mathbf{Q}$ with $2a \in \mathbf{N}$. The fundamental point about (2.73) is that the operator in the right-hand side is now elliptic.

In the next formula, Tr_s denotes the supertrace with respect to the vertical exterior algebra. By (2.73), we find that

$$(2.74) \quad \text{Tr}_s \left[\exp(-2t \mathfrak{A}_{\phi, \mathcal{H}^c}^{\prime 2}) \right] = \text{Tr} \left[\exp(t \Delta^X / 2) \right].$$

Indeed we use the conjugation formally to replace $\mathfrak{A}_{\phi, \mathcal{H}^c}^{\prime 2}$ by the right-hand side of (2.73). Using the explicit spectral decomposition of Δ^X , one can show easily that

this is indeed legitimate. The McKean-Singer formula [McKS67] applied in a very simple case shows that

$$(2.75) \quad \text{Tr}_s \left[\exp \left(-\frac{t}{2} \left(-\Delta^V + c^2 |p|^2 + c (2\widehat{N} - 1) \right) \right) \right] = 1,$$

which completes the proof of (2.74).

If we make $c \rightarrow \infty$ in (2.74) and follow the ideas in subsection 2.7 and in [BL06], we get a tautology, i.e. the operator $2\mathfrak{A}_{\phi, \mathcal{H}^c}^{\prime 2}$ tends in the proper sense to the operator $-\Delta^X/2$. Using (2.71), we find easily that as $c \rightarrow 0$, the local expression for the left-hand side of (2.75) converges to the classical Poisson sum for the heat kernel on S^1 . The interpolation property for the hypoelliptic Laplacian has then been used as a substitute to Poisson's summation formula.

Note that the operator (2.72) is closely related to the hypoelliptic operator whose heat kernel was evaluated by Kolmogorov [K34].

3. THE ANALYSIS OF THE HYPOELLIPTIC LAPLACIAN

The purpose of this section is to report on the results obtained jointly with Lebeau [BL06] on the analysis of the hypoelliptic Laplacian.

In subsection 3.1, we summarize some of the main analytic and spectral properties of the hypoelliptic Laplacian, of its resolvent and of its heat kernel.

In subsection 3.2, we give one of the important results in [BL06] which relates the hypoelliptic Ray-Singer metric on $\det H^*(X, F)$ to the corresponding classical elliptic Ray-Singer metric.

3.1. The resolvent of $A_{\phi, \mathcal{H}^c}^2$ and the spectral theory of $\mathfrak{A}_{\phi_b, \pm \mathcal{H}}^{\prime 2}$. In joint work with Lebeau [BL06], we have studied in detail the analytic properties of the hypoelliptic Laplacian for $c = \pm 1/b^2$, and shown precisely that in the proper sense, as $b \rightarrow 0$, $A_{\phi, \mathcal{H}^c}^2$ converges to $\frac{1}{4}\square^X$.

Let us now describe these results in more detail. We fix $b > 0$, and we take $c = \pm 1/b^2$. Let $\Omega(T^*X, \pi^*F)^0$ be the vector space of L^2 sections of $\Lambda(T^*T^*X) \otimes F$ on T^*X . Then $\mathfrak{A}_{\phi, \mathcal{H}^c}^2$ has discrete conjugation invariant spectrum and compact resolvent in $\text{End}(\Omega(T^*X, \pi^*F)^0)$. Moreover the resolvent maps the Schwartz space $\mathcal{S}(T^*X, \pi^*F)$ into itself.

Given constants $\lambda_0 > 0, c_0 > 0$, set

$$(3.1) \quad \mathcal{W} = \left\{ \lambda \in \mathbf{C}, \text{Re } \lambda + \lambda_0 \leq c_0 |\text{Im } \lambda|^{1/6} \right\}.$$

It is shown in [BL06] that, with an adequate decay choice of λ_0, c_0 , \mathcal{W} is included in the resolvent set.

Moreover for $t > 0$, the heat kernel $\exp(-t\mathfrak{A}_{\phi, \mathcal{H}^c}^2)$ is trace class, and has a smooth kernel on T^*X with adequate decay at infinity.

Put

$$(3.2) \quad \mathfrak{A}_{\phi_b, \pm \mathcal{H}}^{\prime 2} = K_b \mathfrak{A}_{\phi, \pm \mathcal{H}_{b^2}}^{\prime 2} K_b^{-1}.$$

It is proved in [BL06] that if $\lambda \in \mathbf{C}, \lambda \notin \text{Sp } \square^X/4$, as $b \rightarrow 0$, $\lambda \notin \text{Sp } \mathfrak{A}_{\phi_b, \pm \mathcal{H}}^{\prime 2}$, and moreover $(\lambda - \mathfrak{A}_{\phi_b, \pm \mathcal{H}}^{\prime 2})^{-1}$ converges in a very strong sense to $P_{\pm} (\lambda - \square^X/4)^{-1} P_{\pm}$, which justifies the anticipations of subsection 2.7.

Besides it is shown in [BL06] that for $b > 0$ small enough, the classical conclusions of Hodge theory hold, and moreover that the set of the $b > 0$ where these conclusions do not hold is discrete. The relevant cohomology $\mathfrak{H}^*(X, F)$ is the standard cohomology of T^*X for $c > 0$, and the compactly supported cohomology for $c < 0$.

But more is true. Indeed for $b > 0$ small enough, the spectrum has nonnegative real part, and moreover for any $M > 0$, for $b > 0$ small enough, the $\lambda \in \text{Sp} \mathfrak{A}_{\phi_b, \pm \mathcal{H}}^2$ such that $|\lambda| \leq M$ remain real. Note that the fact that the spectrum is conjugation invariant, and also these last results follow in particular from Theorem 2.9.

3.2. The hypoelliptic Ray-Singer metric. We now explain briefly a result established in [BL06]. Set

$$(3.3) \quad \lambda = \det \mathfrak{H}^*(X, F).$$

The determinant in the right-hand side of (3.3) should be understood as a tensor product of determinants of the $\mathfrak{H}^i(X, F)$ or their duals, the choice depending on the parity of i .

For $c > 0$, $\lambda = \det H^*(X, F)$, and for $c < 0$, $\lambda = (\det H^*(X, F \otimes o(TX)))^{(-1)^n}$. In any case λ can be equipped with the Hermitian metric defined via the Ray-Singer analytic torsion [RS71], which one obtains via the derivative at 0 of the zeta function of \square^X in various degrees. This metric, denoted $\|\cdot\|_{\lambda, 0}^2$, is called the Ray-Singer metric [BZ92].

It is shown in [BL06] that for $b > 0$, it is possible to define a generalized metric $\|\cdot\|_{\lambda, b}^2$ associated to the hypoelliptic Laplacian $A_{\phi, \mathcal{H}^c}^2$. The fact it is a generalized metric means that a priori, this metric has a sign, which is positive if it is a classical metric, negative if not. This metric is defined via the Hermitian form $\mathfrak{h}_{\mathcal{H}^c}^{\Omega(T^*X, \pi^*F)}$, and also using the analytic torsion of $A_{\phi, \mathcal{H}^c}^2$.

We now state a result established in [BL06].

Theorem 3.1. *For any $b > 0$, the following identity holds,*

$$(3.4) \quad \|\cdot\|_{\lambda, b}^2 = \|\cdot\|_{\lambda, 0}^2.$$

The proof of (3.4) is difficult. It requires all the results stating that $A_{\phi, \mathcal{H}^c}^2$ is a deformation of $\square^X/4$, the development of a local index theory for the hypoelliptic Laplacian, and also a careful study of the transition from the small time asymptotics for the heat kernel of $A_{\phi, \mathcal{H}^c}^2$ to the small time asymptotics of the heat kernel for $\square^X/4$. The equality should not be taken granted. In fact in the equivariant context, there is a topological defect in the corresponding formula.

Remark 3.2. One motivation for [B05] has been the conjectures made by Fried [F86, F88] on the relation between the Ray-Singer torsion to the value at 0 of the dynamical zeta function associated to certain dynamical systems. Equality was proved by Moscovici and Stanton [MoSt91] in the case of symmetric spaces, by using the Selberg trace formula, when the dynamical system is precisely the geodesic flow.

However we verified that at least formally, Fried's conjecture can be understood as a consequence of an infinite dimensional version of the proof by Zhang and ourselves [BZ92, BZ94] of the Cheeger-Müller theorem, where we used the Witten deformation to obtain this result. Theorem 3.1 should be understood as a first

step in giving a proof of Fried's conjecture, with a proof formally similar to the proof of the Cheeger-Müller theorem. Indeed a natural first step when using the Witten deformation is to show that the corresponding metric does not depend on the deformation parameter. This is precisely the content of Theorem 3.1.

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