Bimoulds, ARI/GARI, and the flexion structure.

Bimoulds $M^{\binom{u_1\ ,\dots,\ u_r}{v_1\ ,\dots,\ v_r}}$, together with their two-tier indexation $\boldsymbol{w} = \binom{\boldsymbol{u}}{\boldsymbol{v}}$, crystallized out of the intricate combinatorics that underpins coequational resurgence, and inherited therefrom a plethora of structure. The fact is that they can be subjected to an incredibly rich system of operations, unary and binary, that all rely on so-called flexions, i.e. changes of type $(u_i, v_i) \mapsto (u_i', v_i')$ that typically add the u_i's cluster-wise, subtract the v_i's pair-wise, and keep the products $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \sum u_i v_i$ invariant. This results in what is known as *flexion polyalgebra*. Its most salient feature is perhaps the existence:

- of a central involution *swap*:

$$(\mathrm{swap.M})^{\binom{u_1\ ,\dots,\ u_r}{v_1\ ,\dots,\ v_r}} := \mathrm{M}^{\binom{v_r\ ,\dots,\ v_2-v_3\ ,\ v_1-v_2}{u_1+\dots+u_r\ ,\dots,\ u_1+u_2\ ,\ u_1}}$$

- of bimoulds possessed of a double symmetry, such as *bialternality*, meaning that both $Ma^\bullet$ and its 'swappee' $Mi^\bullet := swap(Ma^\bullet)$ are simultaneously alternal:

$$\sum_{\boldsymbol{w} \in \mathrm{sha}(\boldsymbol{w^1}, \boldsymbol{w^2})} \mathrm{Ma}^{\boldsymbol{w}} \equiv \sum_{\boldsymbol{w} \in \mathrm{sha}(\boldsymbol{w^1}, \boldsymbol{w^2})} \mathrm{Mi}^{\boldsymbol{w}} \equiv 0 \qquad \begin{cases} \forall\ \boldsymbol{w^1}, \boldsymbol{w^2} \\ \text{sha } \textit{for} \text{ shuffle} \end{cases}$$

  or *bisymmetrality*, with $Ma^\bullet$ and $Mi^\bullet$ simultaneously symmetral:

$$\sum_{\boldsymbol{w} \in \mathrm{sha}(\boldsymbol{w^1}, \boldsymbol{w^2})} \mathrm{Ma}^{\boldsymbol{w}} \equiv \mathrm{Ma}^{\boldsymbol{w^1}} \mathrm{Ma}^{\boldsymbol{w^2}} \quad ; \quad \sum_{\boldsymbol{w} \in \mathrm{sha}(\boldsymbol{w^1}, \boldsymbol{w^2})} \mathrm{Mi}^{\boldsymbol{w}} \equiv \mathrm{Mi}^{\boldsymbol{w^1}} \mathrm{Mi}^{\boldsymbol{w^2}}$$

- of binary operations that preserve the double symmetries. These operations are chiefly the bracket *ari* (behind the Lie algebra *ARI*) and the associative product *gari* (behind the Lie group *GARI*).

Deserving of special attention are the monogenous polyalgebras $\mathrm{Flex}(\mathfrak{E}^\bullet)$ generated by a *flexion unit* $\mathfrak{E}^\bullet$, e.g. a depth-1 bimould that verifies the identities:

$$\mathfrak{E}^{\binom{-u_1}{-v_1}} \equiv -\mathfrak{E}^{\binom{u_1}{v_1}} \qquad ; \qquad \mathfrak{E}^{\binom{u_1}{v_1}} \mathfrak{E}^{\binom{u_2}{v_2}} \equiv \mathfrak{E}^{\binom{u_1+u_2}{v_1}} \mathfrak{E}^{\binom{u_2}{v_2-v_1}} + \mathfrak{E}^{\binom{u_1+u_2}{v_2}} \mathfrak{E}^{\binom{u_1}{v_1-v_2}}$$

Although there exist wildly different realisations of $\mathfrak{E}^\bullet$, all structures $Flex(\mathfrak{E}^\bullet)$ are isomorphic. The 'polar' specialisations stand out: they are $Flex(Pa^\bullet)$ and $Flex(Pi^\bullet)$, corresponding to the flexion units $Pa^{\binom{u_1}{v_1}}{:=}1/u_1$, $Pi^{\binom{u_1}{v_1}}{:=}1/v_1$, and containing the bisymmetral bimould $pal^\bullet/pil^\bullet$, of central importance to the theory.

As a supremely versatile structure, flexion polyalgebra fully deserves to be studied for its own sake. Its involution *swap* and its dextrous handling of double symmetries also make it an ideal framework for unpicking *arithmetical dimorphy*, a phenomenon in no way confined to the ring of *multizetas*, but preeminently manifest there, due the two basic encodings of multizetas and the two multiplication tables that go with them.