Arithmetical dimorphy for multizetas. Canonical irreducibles.

Warning: the various *symmetry types* are defined in the bottom section.

Coloured multizetas, actual and formal. Dimorphy.

In the *first basis*, the multizetas are given by polylogarithmic integrals:

$$\mathrm{Wa}_*^{\alpha_1,\ldots,\alpha_l} := (-1)^{l_0} \int_0^1 \frac{dt_l}{(\alpha_l - t_l)} \ldots \int_0^{t_3} \frac{dt_2}{(\alpha_2 - t_2)} \int_0^{t_2} \frac{dt_1}{(\alpha_1 - t_1)}$$

with α_j either 0 or a unit root, and l_0 the number of zeros in $\{\alpha_1, ..., \alpha_l\}$.
In the *second basis*, multizetas are expressed as harmonic sums :

$$\mathrm{Ze}_*^{\binom{\epsilon_1\ ,\ldots,\ \epsilon_r}{s_1\ ,\ldots,\ s_r}} := \sum_{n_1 > \cdots > n_r > 0} n_1^{-s_1} \ldots n_r^{-s_r}\ e_1^{-n_1} \ldots e_r^{-n_r}$$

with $s_j \in \mathbb{N}^*$ and unit roots $e_j := \exp(2\pi i \epsilon_j)$ with 'logarithms' $\epsilon_j \in \mathbb{Q}/\mathbb{Z}$. The star $*$ in $Wa_*^\bullet$ or $Ze_*^\bullet$ signals restriction to the convergent case. Its removal denotes extension to the divergent case. The conversion rule

$$\mathrm{Wa}_*{}^{e_1, 0^{[s_1-1]}, \ldots, e_r, 0^{[s_r-1]}} := \mathrm{Ze}_*^{\binom{\epsilon_r\ ,\ \epsilon_{r-1:r}\ ,\ldots,\ \epsilon_{1:2}}{s_r\ ,\ s_{r-1}\ ,\ldots,\ s_1}}$$

together with the bimould symmetries (see *infra*, in the bottom section)

$\mathrm{Wa}_*^\bullet$	*is symmetral with a unique symmetral extension*	$\mathrm{Wa}^\bullet$
$\mathrm{Ze}_*^\bullet$	*is symmetrel with a unique symmetrel extension*	$\mathrm{Ze}^\bullet$

and the multiplication rules they encode, are the essence of *multizeta dimorphy.* It is conjectured that these three rules exhaust all algebraic relations between multizetas. Pending a proof, the symbols $wa^\bullet$ and $ze^\bullet$ subject to the three rules, are known as *formal multizetas.* They are known to span a polynomial subring $\mathbb{Q}[\cup_j irr_j]$ of $\mathbb{C}$ generated by contably many irreducibles irr_j, and the challenge is to describe these irreducibles.

Generating series. Dimorphy rephrased.

The generating series

$$\mathrm{zag}^{\binom{u_1\ ,\ldots,\ u_r}{\epsilon_1\ ,\ldots,\ \epsilon_r}} := \sum_{1 \leqslant s_j} \mathrm{wa}^{e_1, 0^{[s_1-1]}, \ldots, e_r, 0^{[s_r-1]}}\ u_1^{s_1-1} u_{12}^{s_2-1} \ldots u_{12\ldots r}^{s_r-1}$$

$$\mathrm{zig}^{\binom{\epsilon_1\ ,\ldots,\ \epsilon_r}{v_1\ ,\ldots,\ v_r}} := \sum_{1 \leqslant s_j} \mathrm{ze}^{\binom{\epsilon_1\ ,\ldots,\ \epsilon_r}{s_1\ ,\ldots,\ s_r}}\ v_1^{s_1-1} \ldots v_r^{s_r-1}$$

define bimoulds $zag^\bullet$, $zig^\bullet$ of type *as/as* and *as/is* (see bottom section). Moreover, $zag^\bullet$ and $zig^\bullet$ are essentially exchanged by the involution *swap*:

$$\mathrm{swap}(\mathrm{zig}^\bullet) \begin{cases} = \mathrm{zag}^\bullet \times \mathrm{mana}^\bullet \\ = \mathrm{gari}(\mathrm{zag}^\bullet, \mathrm{mana}^\bullet) \\ = \mathrm{gari}(\mathrm{mana}^\bullet, \mathrm{zag}^\bullet) \end{cases} \quad \textit{with} \quad \begin{cases} \mathrm{zag}^\bullet \in \mathrm{GARI}^{\mathrm{as/as}} \\ \mathrm{zig}^\bullet \in \mathrm{GARI}^{\mathrm{as/is}} \\ \mathrm{mana}^\bullet \in \mathrm{center(GARI)} \end{cases}$$

The corrective term is an elementary, u_i-independent bimould $mana^\bullet$ whose only non-zero components are expressible in terms of monozetas:
$1 + \sum_{r \geqslant 2} \mathrm{mana}^{\binom{u_1\ ,\ldots,\ u_r}{0\ ,\ldots,\ 0}}\ t^r := \exp\left(\sum_{s \geqslant 2} (-1)^{s-1} \zeta(s)\ \frac{t^s}{s}\right)$
The above relations amount to an exact rephrasing of multizeta dimorphy in the more flexible *ARI/GARI* framework.

continued $\Longrightarrow$

Multizeta parsing and canonical irreducibles.

The generating series $zag^\bullet$ neatly factors as:

$$\mathrm{zag}^\bullet = \mathrm{gari}(\mathrm{zag}^\bullet_{\rm I}, \mathrm{zag}^\bullet_{\rm II}, \mathrm{zag}^\bullet_{\rm III}) \qquad \textit{with} \qquad \begin{cases} \mathrm{zag}^\bullet_{\rm I} \in \mathrm{GARI}^{\mathrm{as/is}}_{\mathrm{e.w.}} \\ \mathrm{zag}^\bullet_{\rm II} \in \mathrm{GARI}^{\underline{\mathrm{as/is}}}_{\mathrm{e.w.}} \\ \mathrm{zag}^\bullet_{\rm III} \in \mathrm{GARI}^{\underline{\mathrm{as/is}}}_{\mathrm{o.w.}} \end{cases}$$

with factors $zag^\bullet_{I}$, $zag^\bullet_{II}$ (even weights), $\mathrm{zag}^\bullet_{\rm III}$ (odd weights) that break down to:

$$\begin{aligned} \mathrm{zag}^\bullet_{\rm I} &= \mathrm{gari}^\bullet\,(\mathrm{tal}^\bullet, \mathrm{invgari}(\mathrm{pal}^\bullet), \mathrm{expari}(\mathrm{roma}^\bullet)) \\ act(\mathrm{zag}^\bullet_{\rm II}) &= 1 + \sum \ \mathrm{irr}_{\rm II}{}^{s_1,\ldots,s_r}\ act(\tau^{s_1}\mathrm{loma}^\bullet)\ldots act(\tau^{s_r}\mathrm{loma}^\bullet) \\ act(\mathrm{zag}^\bullet_{\rm III}) &= 1 + \sum \ \mathrm{irr}_{\rm III}{}^{s_1,\ldots,s_r}\ act(\tau^{s_1}\mathrm{loma}^\bullet)\ldots act(\tau^{s_r}\mathrm{loma}^\bullet) \end{aligned}$$

If for simplicity we limit ourselves to *uncoloured* multizetas (i.e. $\epsilon_i \equiv 0$), then:

- $loma^\bullet$, $roma^\bullet$ are elements of $ARI^{\underline{al/il}}$ with rational coefficients
- τ^s is the projector $M^{\left(\begin{smallmatrix} u_1 & ,\ldots, & u_r \\ v_1 & ,\ldots, & v_r \end{smallmatrix}\right)} \mapsto M^{\left(\begin{smallmatrix} u_1 & ,\ldots, & u_r \\ v_1 & ,\ldots, & v_r \end{smallmatrix}\right)}\|_{\boldsymbol{u}\textit{-part of degree } s-r}$
- In both sums $\sum$, the indices s_i run through all odd integers $\geqslant 3$
- *"act"* is *any* transitive action of $ARI/GARI$ in $BIMU$ – no matter which.
- *Together with* $\mathrm{irr}^2_{I} = \sim$ "π^2"*, the symmetral moulds* $\mathrm{irr}^\bullet_{\rm II}$ *and* $\mathrm{irr}^\bullet_{\rm III}$

  $$\begin{cases} \mathrm{irr}^\bullet_{\rm II} = \left\{\ \mathrm{irr}^{s_1,s_2,\ldots,s_r} \in \mathbb{C}\ ; \quad \textit{with} \ \ r \in \{2,4,6...\} \ \textit{and} \ s_i \in \{3,5,7,9...\}\right\} \\ \mathrm{irr}^\bullet_{\rm III} = \left\{\ \mathrm{irr}^{s_1,s_2,\ldots,s_r} \in \mathbb{C}\ ; \quad \textit{with} \ \ r \in \{1,2,3...\} \ \textit{and} \ s_i \in \{3,5,7,9...\}\right\} \end{cases}$$

  jointly constitute a system, complete and free, of multizeta irreducibles.

Perinomal algebra:

A function $\rho : \mathbb{N}^* \to \mathbb{Z}$ is *perinomal* of degrees $d_{i,j}$ if each $f(x_1,..,x_i{+}k\,x_j,..,x_r)$ is *polynomial* in k of degree $d_{i,j}$. The irreducibles $irr^\bullet$, $irr^\bullet_{\rm II}$, $irr^\bullet_{\rm III}$ are *perinomal numbers* $\rho^\#$ attached to remarkable perinomal functions ρ via the series:

$$\rho^\#(s_1,\ldots,s_r) \ \stackrel{ess}{:=} \ \sum_{n_i \in \mathbb{N}^*} \rho(n_1,\ldots,n_r)\ n_1^{-s_1}\ldots n_r^{-s_r}$$

N.B. Bimould symmetries (simple or double):

A bimould $A^\bullet$ is *symmetral,-el,-il* (resp. *alternal,-el,-il*) if for all $\boldsymbol{w^1}, \boldsymbol{w^2}$:

$$\sum_{\boldsymbol{w}} A^{\boldsymbol{w}} \equiv \begin{cases} A^{\boldsymbol{w^1}}\,A^{\boldsymbol{w^2}} \\ \textit{resp} \ \ 0 \end{cases} \qquad \begin{cases} \textit{symmetral}, \textit{alternal}: & \boldsymbol{w} \in \mathrm{sha}(\boldsymbol{w^1},\boldsymbol{w^2}) \\ \textit{symmetrel}, \textit{alternel}: & \boldsymbol{w} \in \mathrm{she}(\boldsymbol{w^1},\boldsymbol{w^2}) \\ \textit{symmetril}, \textit{alternil}: & \boldsymbol{w} \in \mathrm{shi}(\boldsymbol{w^1},\boldsymbol{w^2}) \end{cases}$$

Here $sha(\boldsymbol{w^1},\boldsymbol{w^2})$ (resp. $she(\boldsymbol{w^1},\boldsymbol{w^2})$) denotes the set of all ordinary (resp. contracting) shufflings of the sequences $\boldsymbol{w^1},\boldsymbol{w^2}$. Under *ordinary/contracting* shufflings, adjacent indices w_i, w_j stemming from different sequences are *forbidden/allowed* to merge into $w_i + w_j$. In the case *symmetril/alternil*, the straightforward addition $(w_i, w_j) \mapsto w_i + w_j$ makes way for the subtler contractions:

$$\left(A^{\binom{\dots,u_i,\dots}{\dots,v_i,\dots}}, A^{\binom{\dots,u_j,\dots}{\dots,v_j,\dots}}\right) \quad \mapsto \quad \frac{1}{v_i - v_j}\left(A^{\binom{\dots,u_i+u_j,\dots}{\dots,v_i,\dots}} - A^{\binom{\dots,u_i+u_j,\dots}{\dots,v_j,\dots}}\right)$$

A bimould is said to be of type *as/as* or *as/is* (resp. *al/al* or *al/il*) if it is symmetral with a symmetral or symmetril *swappee* (resp. alternal with an alternal or alternil *swappee*).

The sets $GARI^{\underline{as/as}}$, $GARI^{\underline{as/is}}$ of all *even* bimoulds of type *as/as* or *as/is* are subgroups of $GARI$.

The sets $ARI^{\underline{al/al}}$, $ARI^{\underline{al/il}}$ of all *even* bimoulds of type *al/al* or *al/il* are subalgebras of ARI.

(An *even* bimould is of course one that verifies $M^{-\boldsymbol{w}} \equiv M^{\boldsymbol{w}}$ for all $\boldsymbol{w}$. Bialternality or bisymmetrality automatically imply *parity* for components of depth $r \geqslant 2$ but not for $r = 1$).