A leisurely walk through the resurgence landscape.

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A1. Not to be neglected: resurgence's algebraic apparatus.

• Resurgence is primarily about **alien differentiation**, and only secondarily about **resummation**.

• **Resurgent functions live in three models:**
(i) In the *formal model*, as formal power series or transseries $\widetilde{\varphi}(z)$ of z^{-1}.
(ii) In the *convolution model* or *Borel plane*, as analytic germs $\widehat{\varphi}(\zeta)$ at 0, endlessly continuable (laterally along any finitely broken line).
(iii) In the *geometric models*, as sectorial germs $\varphi_\theta(z)$ at ∞ in z.

$$
\begin{array}{lll}
(i) & \widetilde{\varphi}(z) = \sum a_n\, z^{-n} & \textit{multiplicative} \\
 & \downarrow \textit{Borel} & \\
(ii) & \widehat{\varphi}(\zeta) = \sum a_n \frac{\zeta^{n-1}}{(n-1)!} \quad (\textit{also} \ \ \check{\varphi}) & \textit{convolutive} \ \left\{ \begin{array}{l} (\widehat{\varphi}_1 * \widehat{\varphi}_2)(\zeta) := \\ \int_0^\zeta \widehat{\varphi}_1(\zeta_1)\, \widehat{\varphi}_2(\zeta - \zeta_1) d\zeta_1 \end{array} \right. \\
 & \downarrow \textit{Laplace} & \\
(iii) & \varphi_\theta(z) = \int_{\arg \zeta = \theta} e^{-z\zeta} \widehat{\varphi}(\zeta) d\zeta & \textit{multiplicative}
\end{array}
$$

The singularities of $\widehat{\varphi}(\zeta)$ carry the Stokes constants and are responsible for the divergence of $\widetilde{\varphi}(z)$. So they deserve close attention.
The tools for measuring them are the so-called *alien derivations* Δ_ω.

A2. The algebraic toolkit.

• **Alien derivations:** *measure singularities.*
$\Delta_\omega \hat{\varphi}(\zeta) =$ weighted average of determinations of $\hat{\varphi}$ over $\zeta + \omega$
Must verify Leibniz: $\Delta_\omega(\hat{\varphi} * \hat{\psi}) \equiv (\Delta_\omega \hat{\varphi}) * \hat{\psi} + \hat{\varphi} * (\Delta_\omega \hat{\psi})$

• **Convolution averages:** *clear the way for Laplace or acceleration.*
$(\mu\, \hat{\varphi})(\zeta) =$ weighted average of determinations of $\hat{\varphi}$ over ζ
Must verify: $\mu(\hat{\varphi} * \hat{\psi}) \equiv \mu(\hat{\varphi}) * \mu(\hat{\psi})$

• **Pseudo-variables:** *enter the definition of the* ***display*** *(see infra)*.
Indexed by ω_i-strings: $\mathbb{Z}^{\omega_1,\dots,\omega_r}$. Dual to the alien derivations, with which they interact according to $\Delta_{\omega_0} \mathbb{Z}^{\omega_1,\dots,\omega_r} = \delta_{\omega_0}^{\omega_1}\, \mathbb{Z}^{\omega_2,\dots,\omega_r}$.
Multiply according to the shuffle product.

• **Resurgence monomials and monics:** *are the "analytical arm" of resurgence.* Res. monomials are elementary yet all-generating res. functions $\widetilde{\mathcal{W}}^{\omega_1,\dots,\omega_r}$, and the monics are the (mostly transcendental) scalars they produce under ordinary or alien differentiation.

A3. **The algebraic toolkit** (continued).

• **Alien derivations and convolution averages:** two main systems. The **standard** system: Simplest of all. With weights that don't depend on the ω_i (singularities), only on the signs ϵ_i's (singul. circumvention) The **organic** system: on top of the mandatory algebraic constraints on the weights, they verify subtler, non-algebraic 'compensation constraints' that make them more pliant tools.

• **Resurgence monomials:** two main, roughly dual systems:
∂-friendly: behave simply under *ordinary*, less so under *alien* $\text{differ.}^{\text{ion}}$
Δ-friendly: behave simply under *alien*, less so under *ordinary* $\text{differ.}^{\text{ion}}$

• **Spherical resurgence monomials:** Crucially depend on a small parameter $c > 0$; live on the Riemann sphere; and behave roughly the same at both poles (∞ and 0).

Altogether, a rich and versatile, yet manageable toolkit, tidy, perfectly natural, easy to handle, and extremely useful.

A4. The display: what is it good for?

Definition: $\widetilde{\varphi} \mapsto \operatorname{displ}.\widetilde{\varphi} + \sum_r \sum_{\omega_i} \mathbb{Z}^{\omega_1,\ldots,\omega_r} \mathbb{\Delta}_{\omega_r} \ldots \mathbb{\Delta}_{\omega_1} \widetilde{\varphi}$

Relations between resurgent functions carry over to the *displays*:

$\{\mathcal{R}(\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_n) \equiv 0\} \Longrightarrow \{\mathcal{R}(\operatorname{displ}.\widetilde{\varphi}_1, \ldots, \operatorname{displ}.\widetilde{\varphi}_n) \equiv 0\}$

Useful for deriving independence relations (or a contrario for detecting unsuspected relations) between resurgent functions.

The display encodes *all* the information about the Borel Riemann surface in easily retrievable form. It also carries all the Stokes constants of $\widetilde{\varphi}$ arranged in the most meaningful way possible. With the display, the Stokes constants come alive: they cease to be a shapeless heap of inert numbers, to become an ordered system – capable of *interacting* with other *displays*.

A5. **Alien derivations:** what are they good for?

Apart from their primary function, they are indispensible for unravelling the **composition of singularities under convolution.**

Let $\mathcal{R} := \widetilde{\mathbb{Z} + i\mathbb{Z}}$ (universal covering) with a privileged origin $0_\bullet$ and convolution starting from $0_\bullet$. To each ramification point $Q \in \delta\mathcal{R}$ there is attached a difference operator

$$(\Delta_Q\, \hat{\varphi})(\zeta) = \hat{\varphi}(\zeta_Q^+) - \hat{\varphi}(\zeta_Q^-) \quad \textit{with} \quad \overrightarrow{0_\bullet, \zeta} = \overrightarrow{Q, \zeta_Q^\pm}$$

(first $\zeta \sim 0_\bullet$, then $\mathrm{cont}^{\mathrm{d}}$ in the large) with a co-commutative co-product

$$\Delta_Q(\hat{\varphi}_1 * \hat{\varphi}_2) \equiv \sum_{Q_1, Q_2 \prec Q} H_{Q_1, Q_2}^{n_1 | Q | n_2}\, (R^{n_1} \Delta_{Q_1} \hat{\varphi}_1) * (R^{n_2} \Delta_{Q_2} \hat{\varphi}_2) \quad (H_\bullet^\bullet \in \mathbb{Z})$$

leading to a natural order $\prec$ on $\delta\mathcal{R}$ and remarkable quadratic forms:

$$F_Q : \{x_{Q_i}\,;\ Q_i \prec Q\} \mapsto \sum H_{Q_1, Q_2}^{Q} x_{Q_i} x_{Q_j}$$

with non-trivial signatures etc (Think of the x_{Q_i}'s as residues at Q_i).

A6. Practical calculation of the co-product.

$$
\begin{array}{lll}
\textit{atomic basis} & \Delta_Q & \\
 & \downarrow & \\
\textit{alien basis} & \sum R^{n(\boldsymbol{\omega})} \Delta_{\boldsymbol{\omega}} & \quad (\textit{with } \Delta_{\boldsymbol{\omega}} = \Delta_{\omega_r} \ldots \Delta_{\omega_1}) \\
\textit{co-product} & \downarrow & \begin{cases} R & \mapsto \; R \otimes R \\ \Delta_{\omega_i} & \mapsto \Delta_{\omega_i} \otimes id + id \otimes \Delta_{\omega_i} \\ \Delta_{\boldsymbol{\omega}} & \mapsto \sum \Delta_{\boldsymbol{\omega}'} \otimes \Delta_{\boldsymbol{\omega}''} \end{cases} \\
\textit{alien basis} & \sum H^{n(\boldsymbol{\omega}^1), n(\boldsymbol{\omega}^2)}_{\boldsymbol{\omega}^1, \boldsymbol{\omega}^2} & R^{n(\boldsymbol{\omega}^1)} \Delta_{\boldsymbol{\omega}^1} \otimes R^{n(\boldsymbol{\omega}^2)} \Delta_{\boldsymbol{\omega}^2} \\
 & \downarrow & \\
\textit{atomic basis} & \sum H^{n_1 | Q | n_2}_{Q_1, Q_2} & (R^{n_1} \Delta_{Q_1}) \otimes (R^{n_2} \Delta_{Q_2})
\end{array}
$$

Easy to program. Whereas the calculation based on SSS paths (self-symmetrically shrinkable paths) soon becomes radically impractical: for a sufficiently contorted path Γ of length 100, the corresponding SSS path Γ^* may exceed the Earth-to-Mars distance. So, roll your sleeves up!

The method also breaks down convolution on winding paths to a sum of straight path integrals, leading to optimal estimates.

A7. **Resurgence monomials:** what are they good for?

• **Monomial expansions: analysis.** ∂-friendly monomials $\widetilde{\mathcal{V}}^{\bullet}$ are handy for expanding the resurgent solutions of ODEs etc: $\sum B_{\omega_1,..,\omega_r}\widetilde{\mathcal{V}}^{\omega_1,\ldots,\omega_r}$ and 'absorbing' the brunt of the divergence.

• **Monomial expansions: synthesis.** Δ-friendly monomials $\widetilde{\mathcal{U}}^{\bullet}$, especially of the spherical sort $\widetilde{\mathcal{U}}_c^{\bullet}$, permit the construction of solutions $\sum A_{\omega_1,..,\omega_r}\widetilde{\mathcal{U}}_c^{\omega_1,\ldots,\omega_r}$ with a pre-assigned set of Stokes constants; hence the construction of analytic moduli for ODEs etc.

• **Alien Taylor expansions.** Using the convolution-respecting projectors $E^{\widetilde{\mathcal{U}}_c}(\varphi) := \widetilde{\varphi} + \sum(-1)^r\,\mathcal{U}_c^{\omega_1,\ldots,\omega_r}\,\Delta_{\omega_1}...\Delta_{\omega_r}\,\widetilde{\varphi}$ from the algegra of res. functions onto the algebra of res. constants (i.e. convergent functions), we arrive at an alien analogue of Taylor expansions:
$\widetilde{\varphi} = E^{\mathcal{U}_c}(\varphi) + \sum E^{\mathcal{U}_c}(\Delta_{\omega_r}\ldots.\Delta_{\omega_1})\,\mathcal{U}_c^{\omega_1,\ldots,\omega_r}$ (mark the indexation order)
Fascinating convergence issues there!

• **Monics** come with a natural indexation that reflects their symmetries and interrelations (esp. for hyperlogarithms).

B1. The Bridge equation, a marvel of conciseness.

- **Critical variables (or critical 'times').**

Start from an equation $E(\varphi) = 0$ (differential, difference, functional etc). Form its *full (parameter saturated) solution* $\widetilde{\varphi}(z, \boldsymbol{t})$ with $\boldsymbol{t} = (t_1, ..., t_s)$.
Rule of thumb: there are as many *critical times* $z_\alpha = z^\alpha$ as there are exponential blocks $e^{-\omega z^\alpha}$ co-present with positive powers of z in $\widetilde{\varphi}(z, \boldsymbol{t})$.

- **Bridge equation: one per critical time** $z_\alpha \to \zeta_\alpha$

$$\boldsymbol{\Delta}_\omega \varphi(z, \boldsymbol{t}) = \mathbb{A}_\omega \varphi(z, \boldsymbol{t}) \quad \textit{with} \quad \begin{cases} \omega \in \Omega \quad (\textit{res. support}) \\ \mathbb{A}_\omega \quad \textit{ordinary diff. operators in}\, (z, \boldsymbol{t}). \end{cases}$$

The operators A_ω carry the Stokes contants as coefficients. They are subject to no other constraints than 'making sense', i.e. meaningfully pairing off the exponentials on both sides of the Bridge equation.
The B.E. keeps Analysis down to a minimum, and covers huge ground.
The **"Bridge principle"** actually exceeds the scope of the B.E.:

$$ALIEN^{act.} := ALIEN/ALIEN^{nih.} \overset{\textit{nearly always}}{\sim} \begin{cases} \textit{algebra of ordinary} \\ \textit{differential operators} \end{cases}$$

C1. Taking germ formalization to its utmost limit.

• The multicritical case: accelerations/pseudo-decelerations.

Acceleration from z_1 to z_2 (with $z_2/z_1 \to +\infty$)

$$\begin{cases} \hat{\varphi}_2(\zeta_2) = \int_0^{+\infty} C_F(\zeta_2,\zeta_1)\hat{\varphi}_1(\zeta_1)\,d\zeta_1 \quad \textit{with} \\ C_F(\zeta_2,\zeta_1) := \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{z_2\zeta_2 - z_1\zeta_1}\,dz_2 \;\; \textit{and} \;\; z_1 = F(z_2) \end{cases}$$

Pseudo-deceleration from z_1 to z_{1*} $\left(\text{with } \begin{cases} z_1 \sim z_{1*} \\ z_1 - z_{1*} \to +\infty \end{cases}\right)$

$$\begin{cases} \hat{\varphi}_{1*}(\zeta_{1*}) = \int_0^{\zeta_1} C_{id+F}(\zeta_{1*},\zeta_1)\,\hat{\varphi}_1(\zeta_1)d\zeta_1 \quad \textit{with} \\ C_{id+F}(\zeta_{1*},\zeta_1) := C_F(\zeta_{1*}-\zeta_1,\zeta_1) \;\; \textit{and} \;\; z_1 = z_{1*} + F(z_{1*}) \end{cases}$$

Accelerations totally overturn the lanscape, by destroying all given singularities and calling new ones into existence. Pseudo-decelerations keep all singularities in place and merely smoothen them.

C2. Taking germ formalization to its utmost limit. (Cont-d)

The accelero-summation scheme. First, identify the critical times and order them from slower to faster: $z_1 \prec z_2 \prec \dots z_{n-1} \prec z_n$. Next:

$$\begin{array}{ccccccccc}
\widetilde{\varphi}_1(z_1) & \equiv & \widetilde{\varphi}(z) & & & \varphi(z) & \equiv & \varphi(z_n) \\
\downarrow \textit{Borel} & & & & & & & \uparrow \textit{Laplace} \\
\hat{\varphi}_1(\zeta_1) & \overset{accel.}{\rightarrow} & \hat{\varphi}_2(\zeta_2) & \dots & \dots & \hat{\varphi}_{n-1}(\zeta_{n-1}) & \overset{accel.}{\rightarrow} & \hat{\varphi}_n(\zeta_n)
\end{array}$$

At each stage, we get on the ζ_i-axis a *cohesive* (i.e. analytic or regular quasi-ana.) germ $\hat{\varphi}_i$ at 0^+ that must be continued up to $+\infty$ to prepare for the next integration (accelelation or final Laplace). That means by-passing whatever singularities stand in the way. In the analytic case, no problem: the operations $\exp(\pm\,\epsilon\,\zeta_i\,\partial_{\zeta_i})$, then $\exp(\mp\,\epsilon\,\zeta_i\,\partial_{\zeta_i})$ will do. In the cohesive case, we must replace $\exp(\pm\,\epsilon\,\zeta_i\,\partial_{\zeta_i})$ by finely calibrated *subexponential* operators that restore cohesiveness at the intervening singularities (left- or rightwards).

C3. Taking germ formalization to its utmost limit. (Cont-d)

Resummation for real transseries, with closure under $+, \times, \partial^{\pm 1}$.

Here are the main takeaways:

- Multiple critical times (z_i, ζ_i), calling for accelero-summation.
- Gradual de-formalization of the transseries' z_i-subexponential parts.
- Cohesiveness becomes unavoidable, at *two places.*
- The seemingly impossible (the constructive circumvention of real singularities without leaving $\mathbb{R}^+$) turns out to be possible.
- The "formal-to-geometric" correspondence (from unique $\widetilde{\varphi}$ to polarized, hence multiple φ) changes in nature. Here, the formal object itself becomes polarized, hence multiple, depending on the convolution averages $\boldsymbol{\mu} = \{\mu_1, ..., \mu_n\}$ used during accelero-summation. The proper thing to consider, on the formal side, is now the multicritical display *displ*.$\widetilde{\varphi}\|_{\boldsymbol{\mu}_0}$ relative to the harmonic averaging. One goes from *displ*.$\widetilde{\varphi}\|_{\boldsymbol{\mu}_0}$ to any *displ*.$\widetilde{\varphi}\|_{\boldsymbol{\mu}}$ via universal constants that depend only on $(\boldsymbol{\mu}_0, \boldsymbol{\mu})$.

D1. Resurgence's bearing on two major watersheds in Analysis.

Resurgence has something to say about two fundamental dichotomies:

• Real functions: the divide *cohesive/non-cohesive* *(dislocated)*
Cohesive: *having the property of unique continuation*.

• Entire functions: the divide *autarchic/non-autarchic* *(anarchic, chaotic)*
Autarchic: roughly, *with resurgent asymptotic expansions in all sectors, and a closed system of resurgence equations.*
Like $1/\Gamma(s)$. Unlike $\Xi(s)$, the classical companion of $\zeta(s)$.

It is about nothing less than answering these two ultra basic questions:

• Which real functions are *of one piece*?

• Which entire functions do admit an *exhaustive description*?

• **The divide** *cohesive/non-cohesive (= dislocated)*.

$$(*) \quad f^{(n)}(t) < c_0 c_1^n \big(n \log n \log_2 n \ldots \log_\alpha n\big)^n = c_0 c_1^n \left(\log'_{\alpha+1}(n)\right)^{-n}$$
$$(**) \; f^{(n)}(t) < c_0 c_1^n \big(n \log n \log_2 n \ldots (\log_\alpha n)^{1+\epsilon}\big)^n$$
$$(*) \Longrightarrow \{f \textit{ cohesive}\} \quad ; \quad (**) \not\Longrightarrow \{f \textit{ cohesive}\}$$

• With the classical Carleman-Mandelbrojt definitions, the quasi-analytic classes don't constitute an ever expanding sequence ($\Rightarrow$ **instability** etc).

• For $\alpha = 0$ we get the analytic class.

• For $\alpha \in [1, \omega[$ we get the Denjoy quasi-analytic classes.

• For $\alpha \in [\omega, \omega^\omega[$ the growth types are well-defined up to equivalence:

$$c_1 < \log_\alpha(t)/\underline{\log}_\alpha(t) < c_2 \quad , \quad c_3 < \log'_\alpha(t)/\underline{\log}'_\alpha(t) < c_4 \quad \textit{as } t \to +\infty$$

leading to the equally well-defined class of **cohesive functions.**

• For $\alpha \in [\omega^\omega, \omega^{\omega^{\omega^{\cdots}}}[$ there is no compelling definition of the growth types $\log_\alpha, \log'_\alpha$, even up to equivalence. Fortunately, the corresponding quasi-analytic classes **never** spontaneously occur in Analysis.

• **The divide** *cohesive/non-cohesive (= dislocated)*. (*Continued*)

The transition from *cohesive* to *dislocated*, more than a discontinuity, even a sharp one, is a **radical reversal.** Thus:

$Cohes([0,1])$ "$\supset$" $Cohes([0,2])$ *but* $\mathcal{C}^{\infty}([0,1])$ "$\subset$" $\mathcal{C}^{\infty}([0,2])$
"Measures" are localised on *Non-cohes*, diffuse on *Cohes*. Etc etc.

How gratifying to find this reversal reflected in the twin statements:

• **Accelerations** can produce any **cohesive function**, no matter how close to the divide (the weaker, the closer).
• **Pseudo-decelerations** can produce any **non-cohesive function**, no matter how close to the divide (the stronger, the closer).

Moreover, the theory yields a scheme for **cohesive continuation:**

$$\hat{\varphi} \in Cohes([0,\epsilon]) \stackrel{weak\ decel.}{\longrightarrow} \hat{\varphi}_{\_} \in Cohes([0,\infty]) \stackrel{weak\ accel.}{\longrightarrow} \hat{\varphi} \in Cohes([0,\omega])$$

($\omega =$ cohesive singularity closest to 0) that compares favourably with the classical procedures (Carleman et al.) for quasi-analytic continuation.

• **The divide** *autarchic/non-autarchic* *(anarchic, chaotic)* **for entire functions.**

Autarchic: very roughly, *functions with resurgent asymptotic expansions in all sectors, and a closed system of resurgence equations.*

Like $1/\Gamma(s)$. Unlike $\Xi(s)$, the classical companion of $\zeta(s)$.

Most Stokes constants, viewed as entire functions of any given Taylor coefficient of the "object" at hand (diff. eq., vector field, hol. mapping etc) appear to fall into the autarchic class. These quasi-algebraic qualities of 'neatness', 'finiteness', 'self-closure' set them apart from functions like Ξ that may well eternally defy exhaustive description (given that Ξ can "approximate anything" on the critical strip).

That makes the *autarch/non-autarch* divide arguably as fundamental as the *algebraic/transcendental* divide. It practically dovetails with the *knowable/unknowable* divide.

E0. Three main sources of divergence and/or resurgence.

The chief sources of divergence (for local dynamical systems) are three:

• **Resonance:** Additive (resp. multiplicative) resonance for the multipliers (eigenvalues of the linear part) of a local vector field (resp. self-mapping) of $\mathbb{C}^{\nu}_{,0}$. Covers the operations $(\partial_z - \omega)^{-1}$ for ODE's and $(e^{\partial_z} - e^{\omega})^{-1}$ for difference equations. Plus other functional equations, e.g. the "sandwich equation":
$g_1 \circ f \circ g_2 \circ f \dots g_n \circ f = id \;\; (f \text{ unknown} \,,\; g_i \text{ given})$.

• **Quasi-resonance:** Liouvillian small denominators.

• **Symplecticity** and isochoricity: the so-called small denominators of celestial mechanics, KAM theory etc.

E1. First main source of divergence: resonance.

• Generates **resurgence** and is amenable to **resummation** (with a minor caveat: see question Q2 in section F)

• Of the aforementioned "three sources" of divergence, resonance is incidentally the only one for which the analytic moduli are characterizable by a system of **"holomorphic invariants"** ("holomorphic" in the sense of non-formal, but holomorphic in the Taylor coefficients.)

E2. Second main source of divergence: quasi-resonance.

Rule of thumb:

• Whenever there is **no clear geometric counterpart**, there can be no meaningful resummation. For instance, the formal linearization (linearizing mapping) of a quasi-resonant vector field cannot be resummed – nor does it deserve to be.[1]

• Whenever there is **a clear geometric counterpart**, resummation (via Borel-Laplace) is possible and **unique.** For instance, the local correspondence $x_1 \leftrightarrow x_2$ attached to a quasi-resonant vector field $X = -\lambda\, x_1\,(1 + \ldots)\,\partial_{x_1} + x_2\,(1 + \ldots)\,\partial_{x_2}$ (with λ Liouvillian) is Borel-Laplace summable. However, instead of *critical times* ($z \sim z'$) what we have here are wider *critical time windows* ($\log z \sim \log z'$), without isolated singularites in the Borel plane and without Stokes constants, which explains why in this case resummation produces a unique (non-polarized) result.

[1] There exist, though, resummable *ramified* linearizations: see references.

E3. Third main source of divergence: symplecticity.

• On its own, symplecticity **generates no resurgence.**

• It may, however, **coexist with resurgence**, like when a symplectic vector field, on top of its intrinsical resonance $(\lambda_i + \lambda_{i+\nu} = 0\,,\, \forall i \leq \nu)$, presents extrinsical resonance (say, $\lambda_1 + 3\,\lambda_2 = 0$). In that case, the **differential operators** $\mathbb{A}_\omega$ in the Bridge equation $\quad \Delta_\omega \widetilde{\varphi} = \mathbb{A}_\omega \widetilde{\varphi}$ **derive from an *alien potential*** $\mathcal{A}_\omega$**.**

• Returning to "pure symplecticity", it is a little known but remarkable fact that the convergence of the Lindstädt series on the invariant tori can be established by elementary combinatorics (completely by-passing KAM theory) via an inductive procedure that avoids the introduction of *fictitious small denominators* in their (i.e. the Lindstädt series') coefficients.

F. Four open questions, from vexing to daunting to haunting.

• **Q1: Impossibility of mixed composition identities** (vexing)

Many constructions relative to transseries, or *display*-based independence proofs, etc, would drastically simplify if we could prove the following:

Conjecture: Barring semi-trivial cases, no mixed identities of type

$$f_1 \circ g_1 \circ f_2 \circ g_2 \ldots f_r \circ g_r \equiv id \quad \textit{with} \quad f_i \in \mathbb{G} \ , \ g_i \in \mathbb{G}_* \qquad (1)$$

are possible with group pairs such as
$\mathbb{G} \ := \{f \ ; \ f(z) = z\,(1 + \sum a_d\, z^{-d}\ )\}$
$\mathbb{G}_* := \{g \ ; \ g(z) = z\,(1 + \sum b_d\, e^{-d\,z})\}$ or
$\mathbb{G}_* := \{g \ ; \ g(z) = z\,(1 + \sum c_d\, z^{-\alpha\,z})\}$ with α irrational.

The stalemate is vexing because the obstructions to (1) are so overwhelming: when the degree d increases, the number of constraints grows like $\mathcal{O}(d^2)$, that of free parameters like $\mathcal{O}(d)$. There is a ray of hope, though: D.Panazzolo[2] has proven the "essential independence" of *exp* and *real shifts* (in the framework of groupoids).

[2] *$PSL(2,\mathbb{C})$, the exponential and some new free group*, Quart. J. of Math, 2017 (DOI).

• **Q2: Dense distribution of singularities** (vexing)

• Singularities that project densely onto the Borel plane, yet form a **discrete set of ramification points** on the Borel Riemann surface — **such singularities are no problem at all.**

• **Not so when the singularities are dense on the "Riemann surface"** (an oxymoron in that case!) **itself**, as occurs for example when we formally normalize resonant vector fields of type

$$X = \sum_{i=1}^{r} \lambda_i \, x_i \, (1 + \tau_i \, x^{\boldsymbol{m}} + \dots) \, \partial_{x_i} \quad \textit{with} \quad \begin{cases} \sum m_i \lambda_i = 0 \ (m_i \in \mathbb{N}) \\ x^{\boldsymbol{m}} = \prod x_i^{m_i} \\ \sum \lambda_i \, \mathbb{N} \ \ \textit{dense in} \ \ \mathbb{C} \end{cases}$$

Depending on the value of $\Re(\sum m_i \lambda_i \tau_i)$, we get (resp. don't get) singularities of manageable "violence", and a *ramified version* of Emile Borel's theory of *monogenous functions* (dense distributions of poles) lets us get by: we can form the Bridge equation; calculate Stokes constants, etc. **But what about the case when we cannot?**

• **Q3 Alien Taylor expansions and Riemann surfaces** (daunting).

We now return to the **convolution-respecting projectors**

$$E^{\widetilde{\mathcal{U}}_c}(\widetilde{\varphi}) := \widetilde{\varphi} + \sum (-1)^r \, \widetilde{\mathcal{U}}_c^{\omega_1,\dots,\omega_r} \, \Delta_{\omega_1} \dots \Delta_{\omega_r} \, \widetilde{\varphi} \qquad (*)$$

from the algebra of res. functions onto the algebra of res. constants (i.e. convergent functions) and to the **'alien version' of Taylor expansions:**

$$\widetilde{\varphi} = E^{\widetilde{\mathcal{U}}_c}(\varphi) + \sum E^{\widetilde{\mathcal{U}}_c}(\Delta_{\omega_r} \dots . \Delta_{\omega_1}) \, \widetilde{\mathcal{U}}_c^{\omega_1,\dots,\omega_r} \qquad (**)$$

Establishing the convergence of $(*)$ and $(**)$ for a large class of interesting $\widetilde{\varphi}$ and c small ($c < c(\widetilde{\varphi})$) is hard enough, but doable.
But what about the continuability in the large of $(*)$, $(**)$ in $\{\widetilde{\mathcal{U}}_c\}$ which in our analogy assumes the role of the *variable* in ordinary Taylor series? Are there natural barriers? Or "alien counterparts" of *ramification points* and *Riemann surfaces*?

Though purely academic (– expansions $(*)$, $(**)$ are computationally hugely costly –), the question opens fascinating theoretical vistas.

• **Q4 The super-exponential range** (haunting).

Neither ODEs nor ordinary real Analysis will take us beyond the range of real transseries and finite exponential towers. But the moment we allow general composition equations, there is no avoiding transfinite iterates $exp^{\circ\,\alpha}$ with $\alpha < \omega^{\omega}$. **Are there privileged choices for these iterates?** (essentially: for $\alpha = \omega^{n}$). No purely asymptotic criterion can tell one choice of $exp^{\circ\,\omega^{n}}$ from another. The existence of analytic extensions to the complex domain might give us a handle, but on the other hand *generic cohesiveness* is unavoidable.[3]

The question is *haunting* because it is so basic. **It is ultimately about the chartability of the range** $[\omega, \omega^{\omega}[$. Are there nature-given landmarks beyond ω, as there are before it? Or does everything become inherently fuzzy? Coarse-grained beyond remedy?

[3]Thus, any *analytic* solution of $f(x+1) = e^{f(x)}$ $(1 \ll x)$ induces a *cohesive* solution of $P(x)+f(x+1) = Q(x).e^{f(x)}$ (P, Q any polynomials), and *vice versa*.

SOME REFERENCES:

• A brief historical survey:

Guided tour of resurgence theory, on Ecalle's Webpage

• Fairly compact general expositions:

J. E., *Six lectures on Transseries and Analysable Functions, etc*, in Bifurcations and Periodic Orbits etc, (1993), Kluwer Acad.

J.E., *Recent Advances in the Analysis of Divergence and Singularities*, Proceedings of the July 2002 Montreal Seminar on Bifurcations, Normal forms etc, (2004), Kluwer Acad.

• The foundational papers (bulky, poor typography: not recommended):

J.E., *Les fonctions résurgentes*, Vol. I,II,III, Publ. Math. Orsay, (1981) and (1985)

• On the organic family of averages, alien derivations, monomials:

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Thanks for the attention!