The arborification-coarborification transform: analytic, combinatorial, and algebraic aspects.

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Abstract : This expository paper is devoted to the so-called *arborification-coarborification transform* which, by automatically carrying out suitable regroupings, often manages to restore convergence in multiple expansions that, in raw form, would seem hopelessly divergent. We first unravel the underlying combinatorics. Then we review 14 applications to complex analysis and holomorphic dynamics. Lastly, we present some new algebraic material : a bevy of some twenty richly structured "σ-functions", which are defined simultaneously on all symmetric groups $\mathbb{S}_r$. Since all these objects originate in arborification, their 'distinctiveness' rubs off on that particular transform, reinforcing its privileged status among all possible alternatives.

Résumé : Nous tâchons de faire le point sur l'arborification-coarborification. Il s'agit là d'une transformation générale qui effectue, au sein de séries multiples divergentes, des regroupements judicieux susceptibles d' instaurer la

convergence. Nous examinons la méthode tour à tour sous trois angles : combinatoire, analyse, algèbre. La partie algébrique présente une multitude de "σ-fonctions" (i.e. de fonctions définies simultanément sur tous les groupes de permutations) apparemment nouvelles et aux propriétés très riches. Tous ces objets, liés qu'ils sont à l'arborification, confirment indirectement le statut privilégié de cette dernière parmi toutes les transformations concurrentes.

1 Arborification-coarborification as a special case of fusion-fission.

1.1 Introduction. Why arborify ?

Analysis often presents us with so-called *mould-comould expansions*, i.e. infinite series of the form :

$$SS := \sum A^{\bullet}\, B_{\bullet} = \sum A^{\boldsymbol{\omega}}\, B_{\boldsymbol{\omega}} = \sum_{0\leq r\leq\infty} \sum_{\omega_i} A^{\omega_1,\dots,\omega_r}\, B_{\omega_1,\dots,\omega_r} \tag{1}$$

which, despite being divergent, somehow *ought to converge*, or at least to be *re-arrangeable into convergent shape.* But let us be a bit more specific. These expansions SS typically involve three ingredients :
– a highly multiple indexation, with " $\bullet$ " running through an infinite set of sequences [1] of arbitrary lengths $r = r(\bullet)$.
– a *mould part* $A^{\bullet}$, usually consisting of scalars, or scalar functions of some variables x_i or parameters t_i.
– a *comould part* $B_{\bullet}$, usually consisting of operators, which most of the time are ordinary differential operators in the variables x_i, but of high degree d.[2]

Unfortunately, as pointed out, these mould-comould expansions SS tend to be *normally divergent* [3] even when there are strong reasons to suspect that the corresponding power series $S_i := SS.x_i$ do, in fact, *have positive convergence radii.* No contradiction here : since a great many terms $A^{\bullet}B_{\bullet}$ in SS contribute to any given Taylor coefficient of S_i, there is ample scope for mutual cancellations or compensations *within* each Taylor coefficient. The challenge, therefore, is to regroup – preferably, in a conceptually appealing and universally valid manner – the terms in SS so as to make the suspected cancellations manifest. Clearly, these regroupings should be carried out adroitly,

[1] usually, " $\bullet$ " runs through a monoid freely generated by a *countable* index reservoir Ω such as $\mathbb{N}$ or $\mathbb{Z}$ or $\mathbb{N}^{\nu}$ or $\mathbb{Z}^{\nu}$.

[2] quite often, the $B_{\omega_1,\dots,\omega_r}$ are simple products $B_{\omega_r}\dots B_{\omega_1}$ of first-order differential operators, in which case *length* and *degree* coincide : $r = d$.

[3] i.e. $\sum |A^{\bullet}|.\|B_{\bullet}\| = +\infty$ for any reasonable norm or semi-norm $\|.\|$

and be exactly the right size: *neither too vast*, for then we would get unmanageably large expressions and the mechanisms responsible for compensation would remain as opaque as they are "inside" the Taylor cofficients of the S_i, *nor too constricted*, for in that case there would be no opportunity for compensation to take place.

One extremely general way of re-ordering our expansions SS to achieve promising re-groupings is to move from the "$\bullet$"-indexation by *totally ordered sequences* to some "#"-indexation by *partially ordered sequences*, for some specified type of partial order.

The idea translates into the general *fusion-fission transform*:

$$SS = \sum_{\bullet} A^{\bullet} B_{\bullet} \longmapsto SS = \sum_{\#} A^{\#} B_{\#} \tag{2}$$

with dual rules for the mould and comould parts:

$$\textit{Fusion rule:} \qquad A^{\#} \; := \; \sum_{\bullet} F_{\bullet}^{\#} A^{\bullet} \; := \sum_{\bullet \geq \#} A^{\bullet} \tag{3}$$

$$\textit{Fission constraint:} \qquad B_{\bullet} \; := \; \sum_{\#} F_{\bullet}^{\#} B_{\#} \; := \sum_{\bullet \geq \#} B_{\#} \tag{4}$$

which automatically ensure that SS remains globally unchanged. Here, the coefficients $F_{\bullet}^{\#}$ are either 1 or 0 and the notation $\bullet \geq \#$ says that, while both sequences $\bullet$ and $\#$ consist of exactly the same elements ω_i with exactly the same multiplicities, the second sequence has on it a partial[4] order *weaker than, but compatible with* the total order of the first.

As a special case, we have the *arborification-coarborification transform*:

$$SS = \sum_{\bullet} A^{\bullet} B_{\bullet} \longmapsto SS = \sum_{\prec} A^{\prec} B_{\prec} \tag{5}$$

with the dual rules:

$$\textit{Arborification rule:} \qquad A^{\prec} \; := \; \sum_{\bullet} F_{\bullet}^{\prec} A^{\bullet} := \sum_{\bullet \geq \prec} A^{\bullet} \tag{6}$$

$$\textit{Coarborification constraint:} \qquad B_{\bullet} \; := \; \sum_{\prec} F_{\bullet}^{\prec} B_{\prec} := \sum_{\bullet \geq \prec} B_{\prec} \tag{7}$$

which correspond to the choice of *arborescent orders*. In other words, we work here with partially ordered sequences $\prec$, each element ω_i of which possesses

[4]non-strictly, of course: that partial order may on occasion be total!

at most *one* antecedent, which we denote ω_{i_-}. Minimal elements, or *roots*, are not assumed to be unique.[5]

There are three distinct angles - *analytic, combinatorial, algebraic* - for approaching our "regrouping" transforms, and all three point to the same conclusion : *among all fusion-fission transforms, arborification-coarborification, for innumerable reasons, towers as the most important and the most useful.* The present paper is devoted to showing why this is so, by successively adopting the three viewpoints :

- **Analysis**, of course, remains the main *raison d'être* for these regrouping techniques. In §4, we shall review no less than fourteen genuinely distinct situations, ranging from holomorphic dynamics to KAM theory to resurgence calculus, where arborification *can* be harnessed to great effect – and often *must.*

- **Combinatorics**, on the other hand, lays bare the mechanisms at work, and explains why the technique succeeds. Here, the mould-comould duality is very helpful in sorting out the difficulties. As we shall see in §3, it is the comould part that leads us, rather naturally, to single out the *coarborification constraints* (7) among all *fusion constraints* (4). But it is in the *mould part* that the really subtle phenomena, those that hold the key to compensation, do occur, as will be shown in §2 on some rich mould material

- **Algebra** here is something of a side-show, but a fascinating one. As we shall see, to each fusion-fission transform one may attach a string of algebraic objects, mainly *arithmetical moulds* and *σ-functions* (i.e functions that are defined, simultaneously and *uniformly*, on all permutation groups $\mathbb{S}_r$) which encapsulate all that is most distinctive about each given transform. Now, the first surprise is that the particular moulds and σ-functions attached to arborification-coarborification (they constitute what we call the *haukian* family) are replete with structure, symmetries, and all manner of highly improbable properties, which are listed in §5 and illustrated in the tables of §7. And the second surprise is that all this structure comes crashing down as soon as we move on to the moulds or σ-functions associated with the other transforms : unlike the *haukian* prototypes, they seem to be utterly unremarkable.

[5]so that, technically, our arborescent sequences $\prec$ must be viewed as "weighted forests" rather than "weighted trees".

The arborification-coarborification technique has been around for quite some time; so here we merely present a unified treatment, catalogue some typical applications, and refer to the literature for details. The algebraic part, on the other hand, is quite new, or appears to be[6], but here the material is so abundant that the exposition had to be both sketchy (with only the barest hints at proofs) and lacunary (with many developments left out). Thus, damaging as the admission may sound, the present paper is partly a review, and partly a formulary. But this is all that the limited format allowed. And there will be, circumstances permitting, a sequel.

We wish to thank M. Kouider, C. Delorme, and D. Forge for helpful discussions, also for guidance in the literature on group representations.

1.2 Straight / contracting arborification-coarborification.

A brief reminder about mould calculus has been prefaced to the next section (§2). Here we simply recall how moulds and comoulds from the basic *symmetry types* interact in (5) and what sort of objects they produce:

$$\begin{array}{llll}
\mathbf{1}: & (A^\bullet, B_\bullet) = (\textit{symmetral, cosymmetral}) & \Rightarrow & SS = \textit{formal diffeomorphism}\\
\mathbf{2}: & (A^\bullet, B_\bullet) = (\textit{alternal, cosymmetral}) & \Rightarrow & SS = \textit{formal derivation}\\
\mathbf{3}: & (A^\bullet, B_\bullet) = (\textit{symmetrel, cosymmetrel}) & \Rightarrow & SS = \textit{formal diffeomorphism}\\
\mathbf{4}: & (A^\bullet, B_\bullet) = (\textit{alternel, cosymmetrel}) & \Rightarrow & SS = \textit{formal derivation}\\
\mathbf{5}: & (A^\prec, B_\prec) = (\textit{separative, coseparative}) & \Rightarrow & SS = \textit{formal diffeomorphism}\\
\mathbf{6}: & (A^\prec, B_\prec) = (\textit{atomic, coseparative}) & \Rightarrow & SS = \textit{formal derivation}
\end{array}$$

As it happens, depending on the symmetry types involved (whether they are of the straight sort, with the vowel a, or of the contracting sort, with the vowel e) one should resort to one or the other of two slightly different variants of arborification-coarborification:

Straight arborification-coarborification: for case 1 or 2

$$\textit{Arborification rule:} \qquad A^\prec := \sum_\bullet F_\bullet^\prec A^\bullet \tag{8}$$

$$\textit{Coarborification constraint:} \qquad B_\bullet := \sum_\prec F_\bullet^\prec B_\prec \tag{9}$$

$$\textit{Standard coarborification rule:} \qquad B_\prec := \sum_\bullet Stan_\prec^\bullet B_\bullet \tag{10}$$

[6] we cannot vouch for its newness, because the literature on groups and group functions is bottomless. But so far all our checks and inquiries have drawn a blank. Yet if some reader knows of previous connections, we would appreciate hearing from him.

Here, the arborification tensor $F_{\boldsymbol{\omega}}^{\boldsymbol{\omega}^{\prec}}$ is equal to 1 if there exists a *bijection* of $\boldsymbol{\omega}^{\prec}$ into $\boldsymbol{\omega}$ which :

(i) respects[7] the order on $\boldsymbol{\omega}^{\prec}$ and $\boldsymbol{\omega}$

(ii) leaves the indices ω_i unchanged

and $F_{\boldsymbol{\omega}}^{\boldsymbol{\omega}^{\prec}} := 0$ in all other cases. Thus (8) translates into such relations as :

$$A^{\left(\substack{\omega_1 \\ \omega_2} \to \omega_3\right)^{\prec}} := A^{\omega_1,\omega_2,\omega_3} + A^{\omega_2,\omega_1,\omega_3} + A^{\omega_2,\omega_3,\omega_1}$$
$$A^{\left(\omega_1 \to \omega_2 \lessgtr \substack{\omega_3 \\ \omega_4}\right)^{\prec}} := A^{\omega_1,\omega_2,\omega_3,\omega_4} + A^{\omega_1,\omega_2,\omega_4,\omega_3}$$

Whereas the arborification rule (8) completely defines $A^{\prec}$, the dual relation (9) merely constrains $B_{\prec}$. However, in the important case when the comoulds are differential operators, there is a natural[8] way to define $B_{\prec}$ which not only agrees with the constraints (9), but also meets the conditions C_3, C_4 below, which ensure the transparent (term by term) conservation of the nature (i.e. being a derivation or an automorphism) of the expansion SS.[9] When the comoulds belong to free associative algebras, there exists no such compelling choice, but several competing possibilities (see §1.5-9).

Let us sum up the pattern for case 1 and 2 :

$\mathbf{C_1}$:	*Straight arborification:*	$A^{\bullet}$ = *symmetral*	$\Rightarrow$	$A^{\prec}$ = *separative*
$\mathbf{C_2}$:	*Straight arborification:*	$A^{\bullet}$ = *alternal*	$\Rightarrow$	$A^{\prec}$ = *atomic*
$\mathbf{C_3}$:	*Standard coarborification:*	$B_{\bullet}$ = *cosymmetral*	$\Rightarrow$	$B_{\prec}$ = *coseparative*
$\mathbf{C_4}$:	*Standard coarborification:*	$B_{\bullet}$ = *coalternal*	$\Rightarrow$	$B_{\prec}$ = *coatomic*

Contracting arborification-coarborification : for case 3 or 4.

Contracting arborification rule : $$A^{\prec} := \sum_{\bullet} CF_{\bullet}^{\prec} A^{\bullet} \quad (11)$$

Contracting coarborification constraint : $$B_{\bullet} := \sum_{\prec} CF_{\bullet}^{\prec} B_{\prec} \quad (12)$$

Standard coarborification rule : $$B_{\prec} := \sum_{\bullet} Stan_{\prec}^{\bullet} B_{\bullet} \quad (13)$$

Here, the arborification tensor $CF_{\boldsymbol{\omega}}^{\boldsymbol{\omega}^{\prec}}$ is equal to 1 if there exists a *surjection* of $\boldsymbol{\omega}^{\prec}$ onto $\boldsymbol{\omega}$ which :

[7]non-comparable elements in $\boldsymbol{\omega}^{\prec}$ may become comparable in $\boldsymbol{\omega}$, but comparable elements have to remain so.

[8]even canonical, up to the choice of variables x_i.

[9]its *global* conservation is not an issue : it automatically follows from the dualness of the rules (8) and (9).

(i) respects the order on $\boldsymbol{\omega}^{\prec}$ and $\boldsymbol{\omega}$

(ii) contracts the indices, in the sense that each ω_i in $\boldsymbol{\omega}$ has to be the sum of all its pre-images ω_j in $\boldsymbol{\omega}^{\prec}$

and $CF_{\boldsymbol{\omega}}^{\boldsymbol{\omega}^{\prec}} := 0$ in all other cases. Thus (11) translates into such relations as:

$$A^{\left({\omega_1 \atop \omega_2} \to \omega_3\right)^{\prec}} := A^{\omega_1,\omega_2,\omega_3} + A^{\omega_2,\omega_1,\omega_3} + A^{\omega_2,\omega_3,\omega_1} + A^{\omega_1+\omega_2,\omega_3} + A^{\omega_2,\omega_1+\omega_3}$$
$$A^{\left(\omega_1 \to \omega_2 \lessgtr {\omega_3 \atop \omega_4}\right)^{\prec}} := A^{\omega_1,\omega_2,\omega_3,\omega_4} + A^{\omega_1,\omega_2,\omega_4,\omega_3} + A^{\omega_1,\omega_2,\omega_3+\omega_4}$$

Here again, we have to supply some coarborification rule compatible with the constraints (12) and, if possible, with conditions C_3', C_4'. Remarkably, it turns out that, in the case of differential operators at any rate, one and the same *standard arborification rule* (cf §1.4 and §3) applies equally in both contexts: *straight* or *contracting.*

Summing up, here is the general pattern for case 3 and 4:

$\mathbf{C_1'}$: *Contracting arborification:* $A^\bullet =$ *symmetrel* $\Rightarrow$ $A^\prec =$ *separative*
$\mathbf{C_2'}$: *Contracting arborification:* $A^\bullet =$ *alternel* $\Rightarrow$ $A^\prec =$ *atomic*
$\mathbf{C_3'}$: *Standard coarborification:* $B_\bullet =$ *cosymmetrel* $\Rightarrow$ $B_\prec =$ *coseparative*
$\mathbf{C_4'}$: *Standard coarborification:* $B_\bullet =$ *coalternel* $\Rightarrow$ $B_\prec =$ *coatomic*

1.3 The reason why arborification-coarborification works.

As far as analytic applications are concerned, the whole point of arborification-coarborification is to re-arrange expansions of the form $\sum A^\bullet B_\bullet$, which are usually hopelessly divergent, because they typically admit no better bounds than:

$$|A^\bullet| \le a_1\, a_2^r \quad ; \quad \|B_\bullet\|_{\mathcal{D}} \le r!\, a_3\, a_4^r \quad (\textit{with} \quad r := r(\bullet)\,) \tag{14}$$

into formally identical expansions $\sum A^\prec B_\prec$, which are often convergent, because they usually admit bounds of the form:

$$|A^\prec| \le c_1\, c_2^r \quad ; \quad \|B_\prec\|_{\mathcal{D}} \le \ c_3\, c_4^r \quad (\textit{with} \quad r := r(\prec)\,) \tag{15}$$

with fixed constants c_1, c_2, but with adjustable constants c_3, c_4 that depend on a neighbourhood $\mathcal{D}$ of the origin, and go to 0 as this neighbourhood shrinks.

The key here is not so much the disappearance of the factorial in the comould estimates as its non-appearance in the mould estimates. The *disappearance* is not really surprising, because the coarborification constraints enable us to spread the 'load' of any given $B_\bullet$ more or less evenly among a great many $B_\prec$. What calls for an explanation is the *non-appearance* of $r!$ in $A^\prec$, since under the arborification rule (8) or (11), and for very weakly ordered arborescent sequences, $A^\prec$ is equal to a sum of almost $r!$ distinct $A^\bullet$, which have no *a priori* reason of cancelling or compensating each other, and in fact don't cancel nor compensate for moulds $A^\bullet$ picked "at random". But for moulds of "natural origin", i.e. for the ones that spontaneously occur in the expansions $\sum A^\bullet B_\bullet$ that originate, not in our whims, but in analysis, such cancellations, on the contrary, tend to take place with providential regularity. Why so? Because of *case-specific identities*, which ensure that the norms of natural moulds don't increase significantly under arborification. A more precise mechanism, which accounts for this small miracle, is the frequent phenomenon of *form preservation*: after arborification, many moulds retain their outward analytical expression, except that in this expression all sums, differences, etc, of indices ω_i have to be re-interpreted in terms of the new arborescent order. But the ultimate reason lies is the fact that "useful" or "natural" moulds almost invariably conform to some "template" (usually, one or several relations involving some of the many operations that are defined on moulds) and that arborification ordinarily preserves the "template" in question, for the simple reason that nearly all mould operations "arborify", i.e. extend painlessly to arborescent moulds.

Summing up, we may say that the arborification technique works so well because arborification *usually* respects "norm", "form", and "template", with *usually* almost meaning *whenever needed.*

The section §2 *infra* enumerates a long list of natural moulds, which shall all be required for the applications to analysis of section §4, and which, barring two (explainable) exceptions, all possess the above properties. But take any of these moulds, and tinker ever so slightly with its definition, and everything immediately unravels: arborification no longer preserves norm, nor form, nor template. To grasp this stark dichotomy between the behaviour of natural-useful and artificial-random moulds, we may reach for an analogy: whereas a random Taylor series with convergence radius one will, with probability one, possess a natural boundary on the unit circle, most series encountered in real life tend, on the contrary, to possess only isolated singularities and endless continuability.

1.4 Standard coarborification.

Pending the precise desciption of coarborification in §3 (with the exact bounds), let us give a rough description with the heuristics behind it. Consider what is perhaps the most frequent situation. Take some comould $B_\bullet$ consisting of differential operators, with the following factorisation property :

$$B_{\boldsymbol{\omega}} = B_{\omega_1,\omega_2,\ldots,\omega_r} \;=\; B_{\omega_r}\ldots B_{\omega_2}B_{\omega_1} \;=\; x^{n_r}B^*_{\omega_r}\ldots x^{n_2}B^*_{\omega_2}\,x^{n_1}B^*_{\omega_1} \qquad (16)$$

with each factor B_{ω_i} separating into a homogeneous monomial x^{n_i} and a differential operator $B^*_{\omega_i}$ of homogeneity 0 :

$$B_{\omega_i} = x^{n_i}B^*_{\omega_i} \quad \textit{with} \quad B_{\omega_i}\ : x^m\,\mathbb{C} \to x^{m+n_i}\,\mathbb{C}\ ;\ B^*_{\omega_i}\ : x^m\,\mathbb{C} \to x^m\,\mathbb{C} \qquad (17)$$

In view of the Leibniz rules, a natural way to coarborify our comould is to define the action of the sought-after operator $B_{\boldsymbol{\omega}^{\prec}}$ on any test function $\varphi(x)$ as follows. We write $B_{\boldsymbol{\omega}^{\prec}}\,\varphi(x) = \big(x^{n_r}B^*_{\omega_r}\ldots x^{n_2}B^*_{\omega_2}\,x^{n_1}B^*_{\omega_1}\big)_{\prec}\,\varphi(x)$ and decree that :
(i) if ω_i is a root of $\boldsymbol{\omega}^{\prec}$, then $B^*_{\omega_i}$ should act on $\varphi(x)$ alone
(ii) if ω_i has an immediate antecedent $\omega_{i\_}$ in $\boldsymbol{\omega}^{\prec}$, then $B^*_{\omega_i}$ should act on the homogeneous monomial $x^{n_{i\_}}$ that accompanies the corresponding $B^*_{\omega_{i\_}}$.

If we start from a cosymmetral comould $B_\bullet$ with factor operators that are first-order derivations, then the Leibniz rules clearly ensure the desired decomposition $B_{\boldsymbol{\omega}} = \sum_{\boldsymbol{\omega}^{\prec}\le\boldsymbol{\omega}} B_{\boldsymbol{\omega}^{\prec}}$. But that decomposition also holds, less obviously so, when we start from a cosymmetrel comould.

1.5 Quadratic coarborification.

It applies above all to the case of comoulds with values in *free* associative algebras. Its true significance lies in the fact that it clears the way for the *algebraic* developments of section §5. But it also has *analytic* implications, namely for the notion of *free-analyticity* in §6.2.

Its quickest definition is by means of the tensor contractions[10] :

$$B_{\prec} := B_\bullet\,K^\bullet_\bullet\,F^\bullet_\prec \qquad \textit{with} \qquad K^\bullet_\bullet := \big(H^\bullet_\bullet\big)^{-1}\ ,\ \ H^\bullet_\bullet := F^\bullet_\prec\,F^\prec_\bullet \qquad (18)$$

where $F^\bullet_\prec\,F^\prec_\bullet$, short for $\sum_\prec F^\bullet_\prec\,F^\prec_\bullet$, denotes the symmetric tensor obtained by contracting both $\prec$ and leaving the two $\bullet$ alone. Viewed as a square matrix, the tensor $H^\bullet_\bullet$ so produced is invertible, with real-positive spectrum, and admits an inverse $K^\bullet_\bullet$.

[10] covariant indices contract with contravariant ones in proximate positions.

There is a more conceptual characterisation of quadratic coarborification: it is the one that minimises the quadratic 'coarborification norm'

$$\|B_\bullet\|^2_{\text{coarb}} := \sum_{\prec\,\leq\,\bullet} < B_\prec, B_\prec > \tag{19}$$

for the natural scalar product on the free algebra generated by the B_{ω_i}.[11]

1.6 Instances of over- and undershooting.

Overshooting: We may take *all* possible oders. But the regroupings are then too large to be helpful or to illuminate the compensation mechanisms.

Undershooting: We may take all *laminations*, i.e. all partial orders that allow to each element at most one successor and at most one predecessor. A lamination clearly splits a set into subsets ("branches") which (i) are mutually non-comparable (ii) carry each a total order. Here, the regroupings are too small to permit compensation to come into its own, at least if we insist that to each d-branched lamination there should correspond an operator of differential order d. But despite its uselessness as far as restoring convergence is concerned, lamination has interesting combinatorial-algebraic aspects. We shall briefly review two instances in §1.8 and §1.9. For now, let us note in passing that laminations lead to a decomposition of the space $\mathbb{B}_r$ spanned by all $r!$ products of r distinct, non-commuting operators B_i

(i) first into subspaces ${}^d\mathbb{B}_r$ consisting of derivations of order d

(i) then into subspaces $\mathbb{B}^{\binom{d_1\ ,\ d_2\ ,\ \dots\ ,\ d_k}{r_1\ ,\ r_2\ ,\ \dots\ ,\ r_k}}$ spanned by associative products of d_1 Lie elements of homogeneity r_1 , d_2 Lie elements of homogeneity r_2 , etc...

$$\mathbb{B}_r = \oplus_{1\leq d\leq r} \quad {}^d\mathbb{B}_r = \oplus^{r_1<r_2<\cdots<r_k}_{d_1\,r_1+\cdots+d_k\,r_k=r} \quad \mathbb{B}^{\binom{d_1\ ,\ d_2\ ,\ \dots\ ,\ d_k}{r_1\ ,\ r_2\ ,\ \dots\ ,\ r_k}} \tag{20}$$

Since these cells correspond one-to-one to the sets of all order-respecting laminations $\mathbf{r}^\#$ of $\mathbf{r} := (1,\dots,r)$ which have d_1 branches of length r_1, d_2 branches of length r_2, etc, the corresponding dimensions clearly are:

$$\dim\binom{d_1\,,d_2\,,\dots,d_k}{r_1\,,r_2\,,\dots,r_k} := \dim\left(\mathbb{B}^{\binom{d_1\ ,\ d_2\ ,\ \dots\ ,\ d_k}{r_1\ ,\ r_2\ ,\ \dots\ ,\ r_k}}\right) = \frac{r!}{\prod d_i!\,\prod r_i^{d_i}} \tag{21}$$

$$\dim_{r,d} := \dim\left({}^d\mathbb{B}_r\right) = \sum_{k\geq 1}\ \sum_{\sum d_i=d}\ \sum_{\sum d_i\,r_i=r} \dim\binom{d_1\,,d_2\,,\dots,d_k}{r_1\,,r_2\,,\dots,r_k} \tag{22}$$

$$= \#\,\{\mathbf{r}^\prec\ :\ \mathbf{r}^\prec \leq \mathbf{r}\ ,\ \mathbf{r}^\prec\ \textit{with}\ d\ \textit{roots}\} \tag{23}$$

[11] with $< B_{\boldsymbol\omega}, B_{\boldsymbol\omega'} >:= \delta_{\boldsymbol\omega,\boldsymbol\omega'}$.

The reason for the last identity is that ${}^d\mathbb{B}_r$ also possesses a basis whose elements correspond one-to-one to the various order-respecting arborescent orders $\mathbf{r}^{\prec}$ on the sequence $\mathbf{r} := (1, \ldots, r)$.

1.7 Lamination-colamination on a free algebra.

We consider the associative algebra $\mathbb{B}$ freely generated by the symbols $B_1, B_2, \ldots$ viewed as formal, order-one derivations, and we use the customary notations: $B_{\mathbf{n}} = B_{n_1,\ldots,n_r} := B_{n_r} \ldots B_{n_1}$.

Whereas $\mathbb{B}$ admits a unique filtration

$$\textit{associative algebra} = \mathbb{B} = {}^{\infty}\mathbb{B} \ldots {}^{3}\mathbb{B} \supset {}^{2}\mathbb{B} \supset {}^{1}\mathbb{B} = \textit{Lie algebra} \tag{24}$$

into the subspaces ${}^d\mathbb{B}$ consisting of formal derivations of order at most d, there exist several more or less natural ways of converting this into a gradation $\oplus\, {}^d\mathbb{B}_*$ with privileged projections $\mathbb{B} \to {}^d\mathbb{B}_*$:

$$\mathbb{B}_* \;=\; {}^1\mathbb{B}_* \oplus {}^2\mathbb{B}_* \oplus {}^3\mathbb{B}_* \ldots \qquad \textit{with}\ {}^1\mathbb{B}_* = {}^1\mathbb{B}\ ,\ {}^{d+1}\mathbb{B}_* \sim {}^{d+1}\mathbb{B}/\,{}^d\mathbb{B} \tag{25}$$

$$B_{\mathbf{n}} \;=\; {}^1B_{\mathbf{n}} + {}^2B_{\mathbf{n}} + {}^3B_{\mathbf{n}} \ldots \qquad \forall B_{\mathbf{n}} = B_{n_1,\ldots,n_r} := B_{n_r} \ldots B_{n_1} \tag{26}$$

$$B_{\mathbf{n}} \;\mapsto\; {}^dB_{\mathbf{n}} = \sum_{\mathbf{n}'} {}^dH_{\mathbf{n}}^{\mathbf{n}'}\ B_{\mathbf{n}'} \in {}^d\mathbb{B}_* \qquad \textit{with}\ \ {}^dH_{\mathbf{n}}^{\mathbf{n}'} \in \mathbb{Q} \tag{27}$$

depending on which set of conditions C_i we impose:

C_1: *economy:*
The projection tensors ${}^dH_{\mathbf{n}}^{\mathbf{n}'}$ should vanish unless the sequences $\mathbf{n}$ and $\mathbf{n}'$ have same length r, same elements n_i and n'_i (with the same multiplicities in case of repetitions), and differ only as to the order of these elements.

C_2: *isotropy (or universality):*
The projection tensors should depend only on the permutation σ that turns the ordered sequence $\mathbf{n}$ into $\mathbf{n}'$, ie:

$${}^dH_{n_1,\ldots,n_r}^{n'_1,\ldots,n'_r} \;\equiv\; {}^dh(\sigma) \qquad \textit{with} \quad n'_i \equiv n_{\sigma(i)} \quad \forall i \tag{28}$$

C_3: *symmetry:*
The projection tensors should be symmetric: ${}^dH_{\mathbf{n}}^{\mathbf{n}'} \equiv {}^dH_{\mathbf{n}'}^{\mathbf{n}}$.
In combination with condition C_2, this translates into: ${}^dh(\sigma) \equiv {}^dh(\sigma^{-1})$.

C_4: *orthogonality:*
The gradation subspaces ${}^d\mathbb{B}_*$ should be pairwise orthogonal, relative to the

natural scalar product :

$$< B_{\mathbf{n}}, B_{\mathbf{n}'} > := 1 \ \ (resp \ \ 0) \ \ if \ \ \ \mathbf{n} = \mathbf{n}' \ \ (resp \ \ \mathbf{n} \neq \mathbf{n}') \tag{29}$$

C_5: *order-compatibility:*
The first projection tensor ${}^1H_{\mathbf{n}}^{\mathbf{n}'}$ should depend only on the number of compatibilities/ incompatibilities in the orders of $\mathbf{n}$ and $\mathbf{n}'$. More concretely, and assuming condition C_2, this means that ${}^1h(\sigma)$ should depend only on the numbers p and q of $+$ and $-$ signs in the sequence $\sigma(i+1) - \sigma(i)$.

C_6: *lamination-compatibility:*
The higher projection tensors ${}^dH_{\mathbf{n}}^{\mathbf{n}'}$ should be simply deducible from the first one. Ideally, we should have :

$$ {}^dB_{\mathbf{n}} \ \ \equiv \ \ \frac{1}{d!} \sum_{\text{sha}(\mathbf{n}^1, \mathbf{n}^2, \ldots, \mathbf{n}^s) = \mathbf{n}} {}^1B_{\mathbf{n}^1} \, {}^1B_{\mathbf{n}^2} \ldots {}^1B_{\mathbf{n}^d} \tag{30}$$

leading to a natural co-lamination $B_{\mathbf{n}} \equiv \sum_{\mathbf{n}^\# \leq \mathbf{n}} B_{\mathbf{n}^\#}$

C_1, C_2 are minimum demands in this free algebra context but, as it turns out, there is some incompatibility between the further conditions.

1.8 Uniform lamination-colamination.

Imposing C_1, C_2 and C_5 (order compatibility) totally fixes the first projection tensor. If the sequences $\mathbf{n}, \mathbf{n}'$ are repetition-free, we get :

$$ {}^1H_{\mathbf{n}}^{\mathbf{n}'} = {}^1h(\sigma) = (-1)^q \, \frac{p! \, q!}{(p+q+1)!} = (-1)^q \, \frac{p! \, q!}{r!} \tag{31}$$

If $\mathbf{n}, \mathbf{n}'$ involve repetitions, with multiplicities $k_1, k_2 \ldots$, we must consider all $k_1! \, k_2! \ldots$ sequences $\underline{\mathbf{n}}, \underline{\mathbf{n}}'$ that coincide with $\mathbf{n}, \mathbf{n}'$, except that identical terms are now regarded as distinct, in all possibe ways, and then set ${}^1H_{\mathbf{n}}^{\mathbf{n}'} := \sum {}^1H_{\underline{\mathbf{n}}}^{\underline{\mathbf{n}}'}$ with $H_{\underline{\mathbf{n}}}^{\underline{\mathbf{n}}'}$ calculated according to the rule (31) Then condition C_6 is automatically fulfilled, in its strong form (30), leading to a natural colamination. But we have neither C_3 (symmetry) nor C_4 (orthogonality).

1.9 Quadratic lamination-colamination.

If we now add C_3 (orthogonality) to C_1, C_2, all projection tensors ${}^dH_{\mathbf{n}}^{\mathbf{n}'}$ are fixed at once. Although they lack simple, closed expressions, the associated σ-function ${}^dh(\sigma)$, especially the first one ($d = 1$) possess remarkable properties (see §5.18 and §7.9). Condition C_3 (symmetry) *is then automatically fulfilled* (the implication is non-trivial), as well as a weaker form of C_6: the right-hand side of (30) may involves partial sequences $\mathbf{n}^i$ which are not always order-compatible with $\mathbf{n}$.

2 Combinatorial aspects of arborification.

2.1 Basic mould operations.

Moulds are *functions of a variable number of variables*: they depend on sequences $\boldsymbol{\omega} := (\omega_1, \ldots, \omega_r)$ of arbitrary length $r = r(\boldsymbol{\omega})$. The sum $\|\boldsymbol{\omega}\|$ of a sequence is simply $\sum_1^r \omega_i$. Sequences are systematically written in boldface, with upper indexation when such is called for, and with the product denoting concatenation: e.g. $\boldsymbol{\omega} = \boldsymbol{\omega}^1.\boldsymbol{\omega}^2$. The elements ω_i which make up these sequences are written in normal print, with lower indexation. The sequences themselves are affixed to the moulds as upper indices $A^\bullet = \{A^{\boldsymbol{\omega}}\}$, since moulds are meant to be contracted

$$A^\bullet, B_\bullet \quad \mapsto \quad < A^\bullet, B_\bullet > := \sum A^{\boldsymbol{\omega}} B_{\boldsymbol{\omega}}$$

with dual objects (often differential operators or elements of an associative algebra), the so-called comoulds $B_\bullet = \{B_{\boldsymbol{\omega}}\}$, which carry their own indices in lower position. Moulds may be *added, multiplied, composed.*
Mould addition is what you expect: components of equal length get added.
Mould multiplication (*mu* or $\times$) is associative, but non-commutative:

$$C^\bullet = A^\bullet \times B^\bullet \Longleftrightarrow C^{\boldsymbol{\omega}} = \sum_{\boldsymbol{\omega} = \boldsymbol{\omega}^1.\boldsymbol{\omega}^2} A^{\boldsymbol{\omega}^1} B^{\boldsymbol{\omega}^2} \tag{32}$$

(This includes the trivial decompositions $\boldsymbol{\omega} = \boldsymbol{\omega}.\emptyset$ and $\boldsymbol{\omega} = \emptyset.\boldsymbol{\omega}$).

Mould composition ($\circ$) too is associative and non-commutative:

$$C^\bullet = (A^\bullet \circ B^\bullet) \iff C^{\boldsymbol{\omega}} = \sum_{\boldsymbol{\omega} = \boldsymbol{\omega}^1 \ldots \boldsymbol{\omega}^s} A^{\|\boldsymbol{\omega}^1\|, \ldots, \|\boldsymbol{\omega}^s\|} B^{\boldsymbol{\omega}^1} \ldots B^{\boldsymbol{\omega}^s} \tag{33}$$

with a sum extending to all possible decompositions of $\boldsymbol{\omega}$ into $s \leq r(\boldsymbol{\omega})$ non-empty factor sequences $\boldsymbol{\omega}^i$

The operations $(+, \times, \circ)$ on moulds interact in exactly the same way as their namesakes for power series. Thus $(A^\bullet \times B^\bullet) \circ C^\bullet \equiv (A^\bullet \circ C^\bullet) \times (B^\bullet \circ C^\bullet)$.

2.2 Basic mould symmetries.

Nearly all useful moulds fall into a few basic *symmetry types.*
A mould $A^\bullet$ is said to be *symmetral* (resp. *alternal*) iff :

$$\sum_{\boldsymbol{\omega}\in \mathrm{sha}(\boldsymbol{\omega^1},\boldsymbol{\omega^2})} A^{\boldsymbol{\omega}} = A^{\boldsymbol{\omega^1}} A^{\boldsymbol{\omega^2}} \quad (resp.\ 0) \qquad \forall\, \boldsymbol{\omega^1} \neq \emptyset\,, \forall\, \boldsymbol{\omega^2} \neq \emptyset \tag{34}$$

A mould $A^\bullet$ is said to be *symmetrel* (resp. *alternel*) iff :

$$\sum_{\boldsymbol{\omega}\in \mathrm{she}(\boldsymbol{\omega^1},\boldsymbol{\omega^2})} A^{\boldsymbol{\omega}} = A^{\boldsymbol{\omega^1}} A^{\boldsymbol{\omega^2}} \quad (resp.\ 0) \qquad \forall\, \boldsymbol{\omega^1} \neq \emptyset\,, \forall\, \boldsymbol{\omega^2} \neq \emptyset \tag{35}$$

Here $sha(\boldsymbol{\omega^1},\boldsymbol{\omega^2})$ (resp. $she(\boldsymbol{\omega^1},\boldsymbol{\omega^2})$) denotes the set of all sequences $\boldsymbol{\omega}$ obtained from $\boldsymbol{\omega^1}$ and $\boldsymbol{\omega^2}$ under ordinary (resp. contracting) shuffling. In a contracting shuffle, two adjacent indices ω_i and ω_j stemming from $\boldsymbol{\omega^1}$ and $\boldsymbol{\omega^2}$ respectively may coalesce to $\omega_{ij} := \omega_i + \omega_j$.

Thus, for a sequence $\boldsymbol{\omega^1} := (\omega_1)$ of length 1 and a sequence $\boldsymbol{\omega^2} := (\omega_2, \omega_3)$ of length 2, the symmetrality (resp alternality) condition reads:

$$\begin{aligned} A^{\omega_1,\omega_2,\omega_3} + A^{\omega_2,\omega_1,\omega_3} + A^{\omega_2,\omega_3,\omega_1} \ &\equiv \ A^{\omega_1}\ A^{\omega_2,\omega_3} \\ (resp \ &\equiv \ 0) \end{aligned}$$

and the symmetrelity (resp alternelity) condition reads:

$$\begin{aligned} A^{\omega_1,\omega_2,\omega_3} + A^{\omega_2,\omega_1,\omega_3} + A^{\omega_2,\omega_3,\omega_1} + A^{\omega_1+\omega_2,\omega_3} + A^{\omega_2,\omega_1+\omega_3} \ &\equiv \ A^{\omega_1}\ A^{\omega_2,\omega_3} \\ (resp \ &\equiv \ 0) \end{aligned}$$

For *arbomoulds*, i.e. moulds $A^\prec$ with an arborescent order on their indices, two new symmetries come into play: separativity and atomicity.

Separativity means that whenever $\boldsymbol{\omega}^\prec$ is many-rooted, with one-rooted subsequences $\boldsymbol{\omega^{i\prec}}$, the arbomould factors accordingly:

$$A^{\boldsymbol{\omega}^\prec} \equiv \prod A^{\boldsymbol{\omega^{i\prec}}} \quad \textit{if} \quad \boldsymbol{\omega}^\prec = \oplus\, \boldsymbol{\omega^{i\prec}} \quad \textit{with} \quad \boldsymbol{\omega^{i\prec}} \ \textit{one-rooted} \tag{36}$$

Atomicity means that whenever $\boldsymbol{\omega}^\prec$ has more than one root, the arbomould vanishes:

$$A^{\boldsymbol{\omega}^\prec} \equiv 0 \quad \textit{if} \quad \boldsymbol{\omega}^\prec \ \textit{is many-rooted} \tag{37}$$

Mould-comould contractions.

Let B_ω be the homogeneous components of some local-analytic, ν-dimensional vector field X (resp of the postcomposition operator F associated with some local-analytic ν-dimensional diffeomorphism f) and let

$$B_{\boldsymbol{\omega}} = B_{\omega_1,\dots,\omega_r} := B_{\omega_r} \dots B_{\omega_1} \qquad\qquad (reversion\,!) \tag{38}$$

The comould $B_\bullet$ so defined is said to be co-symmetral (resp co-symmetrel) if its action on a product $\varphi_1\varphi_2$ obeys the Leibniz rule:

$$B_{\boldsymbol{\omega}}(\varphi_1\,\varphi_2) = \sum (B_{\boldsymbol{\omega^1}}\,\varphi_1)\,(B_{\boldsymbol{\omega^2}}\,\varphi_2) \tag{39}$$

with a sum extending to all pairs $(\boldsymbol{\omega^1}, \boldsymbol{\omega^2})$ such that $\boldsymbol{\omega} \in sha(\boldsymbol{\omega^1}, \boldsymbol{\omega^2})$ (resp $\boldsymbol{\omega} \in she(\boldsymbol{\omega^1}, \boldsymbol{\omega^2})$).

The four main symmetry types admit a simple characterisation in terms of mould-comould contractions:

$$A^\bullet \;:\quad B_\bullet \mapsto C_\bullet \quad with \quad C_{\omega_0} := \sum_{\|\boldsymbol{\omega}\|=\omega_0} A^{\boldsymbol{\omega}}\,B_{\boldsymbol{\omega}} \tag{40}$$

Indeed:

$A^\bullet$				
$A^\bullet$	:	$B_\bullet$	$\to$	$C_\bullet$
alternal	:	*field*	$\to$	*field*
symmetral	:	*field*	$\to$	*diffeo*
alternel	:	*diffeo*	$\to$	*field*
symmetrel	:	*diffeo*	$\to$	*diffeo*

Most stability properties follow from this interpretation. Thus:

$$\begin{array}{ccccc} symmetral^\bullet & \times & symmetral^\bullet & = & symmetral^\bullet \\ symmetrel^\bullet & \times & symmetrel^\bullet & = & symmetrel^\bullet \\ alternal^\bullet & \circ & alternal^\bullet & = & alternal^\bullet \\ symmetrel^\bullet & \circ & symmetrel^\bullet & = & symmetrel^\bullet \end{array}$$

2.3 Constant-type moulds.

mould	value	symmetry type	associated series
$\mathbf{1}^\bullet$	$1 \;\; if\; r = 0 \; (0 \; otherwise)$	*symmetral*	1
$\mathrm{I}^\bullet$	$1 \;\; if\; r = 1 \; (0 \; otherwise)$	*alternal*	x
$\log^\bullet$	$\frac{(-1)^{r-1}}{r}$	*alternel*	$\log(1+x)$
$\exp_a^\bullet$	$\frac{a^r}{r!}$	*symmetral*	$e^{a\,x}$
$\mathrm{tu}_a^\bullet$	$\frac{(-1)^r}{r!}\frac{\Gamma(r-a)}{\Gamma(-a)}$	*symmetrel*	$(1+x)^a$

2.4 Difference-type flat moulds.

For any $\mathbf{t} = (t_1, .., t_r) \in \mathbb{R}^r, t_i \neq t_j$, we set $p := \sum_{t_i<t_{i+1}} 1,\; q := \sum_{t_i>t_{i+1}} 1$ and define the symmetral mould $sad_a^\bullet$ (special case: $sad^\bullet = sad_1^\bullet$) and the alternal mould $lad^\bullet$ as follows:

$$\begin{aligned}
\mathrm{sad}^{\emptyset} &:= 1 =: \mathrm{sad}_a^{\emptyset} \quad ; \quad \mathrm{lad}^{\emptyset} := 0\\
\mathrm{sad}^{t_1,\dots,t_r} &:= 1 \ \ (resp\, 0) \ \ if \ t_1 < t_2 \dots < t_r \ \ (resp \ otherwise)\\
\mathrm{sad}_a^{t_1,\dots,t_r} &:= \frac{a}{r!} \prod_{1\le i\le p} (a+i) \prod_{1\le j\le q} (a-j)\\
\mathrm{lad}^{t_1,\dots,t_r} &:= (-1)^q \, \frac{p!\,q!}{(p+q+1)!} = (-1)^q \, \frac{p!\,q!}{r!}
\end{aligned}$$

2.5 Difference-type polar moulds.

$$\begin{aligned}
\mathrm{tas}_{a,b}^{\emptyset} &:= 1 \quad ; \quad \mathrm{tas}_{a,b}^{t_1} := \frac{a-b}{(a-t_1)(t_1-b)}\\
\mathrm{tas}_{a,b}^{t_1,\dots,t_r} &:= \frac{a-b}{(a-t_1)(t_1-t_2)\dots(t_{r-1}-t_r)(t_r-b)}\\
\mathrm{tas}_{\star}^{\emptyset} &:= 0 \quad ; \quad \mathrm{tas}_{\star}^{t_1} := \frac{1}{(-t_1)(t_1)}\\
\mathrm{tas}_{\star}^{t_1,\dots,t_r} &:= \frac{1}{(-t_1)(t_1-t_2)\dots(t_{r-1}-t_r)(t_r)}\\
\mathrm{tas}_{\star\star}^{\emptyset} &:= 0 \quad ; \quad \mathrm{tas}_{\star\star}^{t_1} := 1\\
\mathrm{tas}_{\star\star}^{t_1,\dots,t_r} &:= \frac{1}{(t_1-t_2)\dots(t_{r-1}-t_r)}
\end{aligned}$$

$$\begin{aligned}
\mathrm{tas}_{a,b}^{\bullet} \times \mathrm{tas}_{b,c}^{\bullet} &= \mathrm{tas}_{a,c}^{\bullet}\\
\mathrm{tas}_{a,b}^{\bullet} \times \mathrm{tas}_{b,a}^{\bullet} &= \mathbf{1}^{\bullet}
\end{aligned}$$

2.6 Sum-type flat moulds.

We first settle some notations, then define our moulds:

$$\boldsymbol{x} := (x_1, \dots, x_r) \tag{41}$$

$$\check{x}_i := x_1 + \dots + x_i \tag{42}$$

$$\hat{x}_i := x_i + \dots + x_r \tag{43}$$

$$\|\boldsymbol{x}\| := x_1 + \dots + x_r = \hat{x}_1 = \check{x}_r \tag{44}$$

$$\sigma_+(x) := 1 \ \ if \ \ x > 0 \ \ (resp \ \ := 0 \ \ if \ \ x < 0) \tag{45}$$

$$\sigma_-(x) := 1 \ \ if \ \ x < 0 \ \ (resp \ \ := 0 \ \ if \ \ x > 0) \tag{46}$$

$$\delta(x) := 1 \ \ if \ \ x = 0 \ \ (resp \ \ := 0 \ \ if \ \ x \neq 0) \tag{47}$$

$$
\begin{aligned}
\mathrm{sofo}_{\pm}^{\boldsymbol{x}} &:= (-1)^r\ \sigma_{\pm}(\check{x}_1)\dots\sigma_{\pm}(\check{x}_r)\\
\mathrm{antisofo}_{a}^{\boldsymbol{x}} &:= (-1)^r\ \sigma_{\pm}(\hat{x}_1)\dots\sigma_{\pm}(\hat{x}_r)\\
\mathrm{sefo}_{\pm}^{\boldsymbol{x}} &:= (-1)^{r-1}\sigma_{\pm}(\check{x}_1)\dots\sigma_{\pm}(\check{x}_{r-1})\ \sigma_{\mp}(\check{x}_r)\\
\mathrm{antisefo}_{\pm}^{\boldsymbol{x}} &:= (-1)^{r-1}\sigma_{\mp}(\hat{x}_1)\ \sigma_{\pm}(\hat{x}_{r-1})\dots\sigma_{\pm}(\hat{x}_r)\\
\mathrm{lefo}_{\pm}^{\boldsymbol{x}} &:= (-1)^r\sigma_{\pm}(\check{x}_1)\dots\sigma_{\pm}(\check{x}_{r-1})\ \delta(\check{x}_r)\\
\mathrm{antilefo}_{\pm}^{\boldsymbol{x}} &:= (-1)^r\delta(\hat{x}_1)\ \sigma_{\pm}(\hat{x}_{r-1})\dots\sigma_{\pm}(\hat{x}_r)
\end{aligned}
$$

2.7 Sum-type polar moulds. The "organic" family.

$$
\begin{aligned}
\mathrm{sa}_a^{\boldsymbol{\omega}} &:= \textstyle\prod_{i=1}^{i=r} \frac{\omega_i}{\check{\omega}_i} & \mathrm{musa}_a^{\boldsymbol{\omega}} &:= (-1)^r \textstyle\prod_{i=1}^{i=r} \frac{\omega_i}{\hat{\omega}_i}\\
\mathrm{romo}_a^{\boldsymbol{\omega}} &:= \textstyle\prod_{i=1}^{i=r}(a\,\frac{\omega_i}{\check{\omega}_i}-1) & \mathrm{antiromo}_a^{\boldsymbol{\omega}} &:= \textstyle\prod_{i=1}^{i=r}(a\,\frac{\omega_i}{\hat{\omega}_i}-1)\\
\mathrm{remo}_a^{\boldsymbol{\omega}} &:= a\,\frac{\omega_r}{\check{\omega}_r}\textstyle\prod_{i=1}^{i=r-1}(a\,\frac{\omega_i}{\check{\omega}_i}-1) & \mathrm{antiremo}_a^{\boldsymbol{\omega}} &:= a\,\frac{\omega_1}{\hat{\omega}_1}\textstyle\prod_{i=2}^{i=r}(a\,\frac{\omega_i}{\hat{\omega}_i}-1)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{somo}_{a,b}^{\bullet} &:= \mathrm{remo}_a^{\bullet}\times\mathrm{antiromo}_{1-b}^{\bullet} && (48)\\
&:= \mathrm{romo}_a^{\bullet}\times\mathrm{antiremo}_{1-b}^{\bullet} && (49)\\
&= \mathrm{romo}_{a/b}^{\bullet}\times\mathrm{remo}_b^{\bullet} && (50)
\end{aligned}
$$

$$
\mathrm{somo}^{\bullet}_{\left[\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right]} := \mathrm{somo}^{\bullet}_{\frac{c-b}{d-b},\,\frac{a-b}{d-b}} \qquad (51)
$$

$$
\mathrm{somo}^{\bullet}_{\left[\begin{smallmatrix} b & 0\\ a & 1\end{smallmatrix}\right]} := \mathrm{somo}^{\bullet}_{a,b} \qquad (52)
$$

2.8 Main properties.

Symmetry types :[12]

All the above moulds fall into one or the other of the main symmetry types.

Alternal : $\mathrm{lad}^{\bullet}$, $\mathrm{tas}_{\star}^{\bullet}$, $\mathrm{tas}_{\star\star}^{\bullet}$

Symmetral : $\mathrm{exp}_a^{\bullet}$, $\mathrm{sad}^{\bullet}$, $\mathrm{sad}_a^{\bullet}$, $\mathrm{tas}_{a,b}^{\bullet}$, $\mathrm{sa}^{\bullet}$, $\mathrm{musa}^{\bullet}$

Alternel : $\mathrm{log}^{\bullet}$, $\mathrm{lefo}_{\pm}^{\bullet}$, $\mathrm{redo}_{\pm}^{\bullet}$, $\mathrm{redom}^{\bullet}$

Symmetrel : $\mathrm{tu}_a^{\bullet}$, $\mathrm{sofo}_{\pm}^{\bullet}$, $\mathrm{sefo}_{\pm}^{\bullet}$, $\mathrm{romo}_a^{\bullet}$, $\mathrm{remo}_a^{\bullet}$, $\mathrm{somo}_{a,b}^{\bullet}$

[12] *flat* moulds should be regarded as distribution-valued : for them the *symmetries* hold *almost everywhere*, not necessarily *everywhere*.

All pairs ($mould^\bullet, antimould^\bullet$) have the same symmetry type.

Useful identities and closure properties :

$$\mathrm{sofo}^\bullet_+ \times \mathrm{sefo}^\bullet_- = \mathbf{1}^\bullet \tag{53}$$

$$\mathrm{antisofo}^\bullet_+ \times \mathrm{antisefo}^\bullet_- = \mathbf{1}^\bullet \tag{54}$$

$$\mathrm{remo}^\bullet_a \times \mathrm{antiromo}^\bullet_{1-a} = \mathbf{1}^\bullet \tag{55}$$

$$\mathrm{romo}^\bullet_a \times \mathrm{antiremo}^\bullet_{1-a} = \mathbf{1}^\bullet \tag{56}$$

$$\begin{array}{llcll}
\textit{multplicative inverse}: & \mathrm{somo}^\bullet_{\left[\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right]} & \leftrightarrow & \mathrm{somo}^\bullet_{\left[\begin{smallmatrix} c & b \\ a & d\end{smallmatrix}\right]} & (\ a\ ,\ c\ \textit{exchanged}\ ) \\
\textit{composition inverse}: & \mathrm{somo}^\bullet_{\left[\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right]} & \leftrightarrow & \mathrm{somo}^\bullet_{\left[\begin{smallmatrix} b & a \\ d & c\end{smallmatrix}\right]} & (\ \textit{columns exchanged}\ ) \\
\textit{sequence reversal}: & \mathrm{somo}^\bullet_{\left[\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right]} & \overset{\mathbf{anti}}{\leftrightarrow} & \mathrm{somo}^\bullet_{\left[\begin{smallmatrix} c & d \\ a & b\end{smallmatrix}\right]} & (\ \textit{rows exchanged}\ )
\end{array}$$

$$\begin{array}{lll}
\textit{multplication}: & \mathrm{somo}^\bullet_{a_1,a_2} \times \mathrm{somo}^\bullet_{a_2,a_3} & = \mathrm{somo}^\bullet_{a_1,a_3} \\
\textit{composition}: & \mathrm{somo}^\bullet_{a_1,b_1} \circ \mathrm{somo}^\bullet_{a_2,b_2} & = \mathrm{somo}^\bullet_{(a_2-b_2)a_1+b_2\,,\,(a_2-b_2)b_1+b_2}
\end{array}$$

$$\begin{array}{lll}
\textit{multplication}: & \mathrm{somo}^\bullet_{\left[\begin{smallmatrix} a_0 & b_1 \\ a_2 & b_2\end{smallmatrix}\right]} \times \mathrm{somo}^\bullet_{\left[\begin{smallmatrix} a_1 & b_1 \\ a_0 & b_2\end{smallmatrix}\right]} & = \mathrm{somo}^\bullet_{\left[\begin{smallmatrix} a_1 & b_1 \\ a_2 & b_2\end{smallmatrix}\right]} \\
\textit{composition}: & \mathrm{somo}^\bullet_{\left[\begin{smallmatrix} a_1 & b_1 \\ a_2 & b_2\end{smallmatrix}\right]} \circ \mathrm{somo}^\bullet_{\left[\begin{smallmatrix} b_1 & c_1 \\ b_2 & c_2\end{smallmatrix}\right]} & = \mathrm{somo}^\bullet_{\left[\begin{smallmatrix} a_1 & c_1 \\ a_2 & c_2\end{smallmatrix}\right]}
\end{array}$$

2.9 Smooth and form-preserving arborification.

Smooth or size-preserving arborification.

All the above moulds possess the property of smooth arborification (meaning that their arborified variants admit essentially the same type of bounds) the only exception being the moulds $log^\bullet$ and $tu^\bullet_a$ for $a \notin \mathbb{Z}$ and in particular for $a = 1/2$. This is in relation with the fact that the *standard* alien derivations (which admit $log^\bullet$ as their left-lateral mould) and the *standard* or *median* convolution average (which admits $tu^\bullet_{1/2}$ as its right- *and* left-lateral mould) are not *well-behaved.*[13]

Of course, for alternal or symmetral (resp alternel or symmetrel) moulds, one should take the ordinary (resp contracting) form of arborification.

Form-preserving arborification.

All the sum-type moulds listed above, i.e. all those moulds whose definition

[13]See §4.10, §4.11 and [E11].

involves forward sums $\hat{x}_i$ or $\hat{\omega}_i$ (resp backward sums $\check{x}_i$ or $\check{\omega}_i$) have the stronger and very useful property of form-preserving arborification. This means that they retain their *outward analytical expression*, except that the sums $\hat{x}_i$ or $\hat{\omega}_i$ (resp $\check{x}_i$ or $\check{\omega}_i$) are now relative to the arborescent (resp antiarborescent) order. The same holds for the difference-type moulds $tas^{\bullet}_{a,\infty}$ and $tas^{\bullet}_{\infty,b}$.

Thus, it is an easy matter to check that for any arborescent sequence $\boldsymbol{\omega}^{\prec}$ (resp antiarborescent sequence $\boldsymbol{\omega}^{\succ}$) we still have :

$$\mathrm{sa}_a^{\boldsymbol{\omega}^{\succ}} := \prod_{i=1}^{i=r} \frac{\omega_i}{\check{\omega}_i} \qquad \mathrm{musa}_a^{\boldsymbol{\omega}^{\prec}} := (-1)^r \prod_{i=1}^{i=r} \frac{\omega_i}{\hat{\omega}_i}$$

exactly as in §2.1.10, except that $\hat{\omega}_i$ (resp $\check{\omega}_i$) now denotes the sum of all indices ω_j that follow (resp precede) ω_i *inside* $\boldsymbol{\omega}^{\prec}$ (resp $\boldsymbol{\omega}^{\succ}$). Of course, as in the case of totally ordered sequences, ω_i itself should be included in that sum.

2.10 Mould mixing and arborification.

For any pair $A^{\bullet}, B^{\bullet}$ of moulds carrying real indices ω_i, the mould *mixture* $C^{\bullet} := A^{\bullet}\,\mathrm{mix}\,B^{\bullet}$ is defined by :

$$C^{\omega_1,\ldots,\omega_r} := \sum_{\pi\in\mathbb{S}_r}\sum_{1\le m\le r} MIX^{\omega_1,\ldots,\omega_r}_{\pi,m}\ \widetilde{B}^{\omega_{\pi(1)},\ldots,\omega_{\pi(m)}} A^{\omega_{\pi(m+1)},\ldots,\omega_{\pi(r)}} \tag{57}$$

with a sum extending to all permutations π of the sequence $(1,\ldots,r)$. This sum involves the mould $A^{\bullet}$ itself and the conjugate $\widetilde{B}^{\bullet}$ of the mould $B^{\bullet}$:

$$\widetilde{B}^{\,\omega_1,\ldots,\omega_r} \ := \ (-1)^r\,\widetilde{B}^{\,\omega_r,\ldots,\omega_1} \tag{58}$$

as well as a *'disorder coefficient'* which is defined as follows :

$$MIX^{\,\omega_1,\ldots,\omega_r}_{\pi,m} \ := \ \epsilon_1\,\epsilon_2\ldots\epsilon_r\ \sigma_{\epsilon_1}(\hat{\omega}_1)\,\sigma_{\epsilon_2}(\hat{\omega}_2)\ldots\sigma_r(\hat{\omega}_r) \tag{59}$$

and assumes the values $0, \pm 1$. Here, the sign function $\sigma_{\pm}$ and the forward sums $\hat{\omega}_i := \omega_i + \ldots \omega_r$ are as in §2.6, and the signs ϵ_i are given by :

$$\begin{aligned}
\epsilon_1 \ &:= \ + \quad \textit{if} \qquad m < \pi^{-1}(1)\\
&:= \ - \quad \textit{if} \qquad m \ge \pi^{-1}(1)\\
\epsilon_i \ &:= \ + \quad \textit{if} \quad \pi^{-1}(i-1) < \pi^{-1}(i) \qquad (\textit{for}\ \ i>1)\\
&:= \ - \quad \textit{if} \quad \pi^{-1}(i-1) > \pi^{-1}(i) \qquad (\textit{for}\ \ i>1)
\end{aligned}$$

The usefulness of *mix* derives from the automatic *sign separation* which it brings about in the index sequences. Indeed, the sum on the right-hand side of (57) involves only terms of the form $A^{\alpha_1,\ldots,\alpha_{r_1}}$ and $B^{\beta_1,\ldots,\beta_{r_2}}$ such that :

$$\hat{\alpha}_i := \alpha_1 + \cdots + \alpha_{r_1} \ge 0 \quad ; \quad \hat{\beta}_i := \beta_1 + \cdots + \beta_{r_2} \le 0 \tag{60}$$

Mould mixing also respects symmetrality (in particular, *self-mixing* leaves symmetral moulds unchanged) and commutes with pre-multiplication by a third mould:

$$\{A^\bullet \;\; and \;\; B^\bullet \;\; symmetral\} \implies \{A^\bullet \;\; \mathrm{mix} \;\; B^\bullet \;\; symmetral\} \tag{61}$$

$$\{A^\bullet \;\; symmetral\} \implies \{A^\bullet \;\; \mathrm{mix} \;\; A^\bullet = A^\bullet\} \tag{62}$$

$$(C^\bullet \times A^\bullet)\,\mathrm{mix}\,(C^\bullet \times B^\bullet) \quad = \quad C^\bullet \times (A^\bullet\,\mathrm{mix}\,B^\bullet) \tag{63}$$

Moreover – and this is essential for the sequel – the *mixing* operation retains its form under arborification. Indeed, if we construct $C^\bullet := A^\bullet\,\mathrm{mix}\,B^\bullet$ as in (57), then the standard (non-contracting) arborification $C^{\prec}$ is given by a straightforward variant of (57):

$$C^{(\omega_1,\ldots,\omega_r)^{\prec}} \quad := \sum_{\pi\in S_r}\sum_{1\leq m\leq r} MIX_{\pi,m}^{(\omega_1,\ldots,\omega_r)^{\prec}} \; \widetilde{B}^{\omega_{\pi(1)},\ldots,\omega_{\pi(m)}} A^{\omega_{\pi(m+1)},\ldots,\omega_{\pi(r)}} \tag{64}$$

with disorder coefficients $MIX_{\pi,m}^{\boldsymbol{\omega}^{\prec}}$ still given by (59), except that the forward sums $\hat{\omega}_i$ are now relative to the arborescent order on $\boldsymbol{\omega}^{\prec}$, and with a suitable redefinition of the signs ϵ_i:

$$\begin{array}{lllll} \epsilon_1 & := & + & if & m < \pi^{-1}(1) \qquad\qquad (for \;\; i \;\; root \;\; of \;\; \boldsymbol{\omega}^{\prec}) \\ & := & - & if & m \geq \pi^{-1}(1) \\ \epsilon_i & := & + & if & \pi^{-1}(i_-) < \pi^{-1}(i) \qquad (for \;\; i \;\; not \;\; a \;\; root \;\; \boldsymbol{\omega}^{\prec} \\ & := & - & if & \pi^{-1}(i_-) > \pi^{-1}(i) \qquad\; and \;\; i_- \;\; antecedent \;\; of \;\; i) \end{array}$$

2.11 Mould flattening and arborification.

Let us also mention two more mould transforms which turn alternel (resp symmetrel) moulds $A^\bullet$ into alternal (resp symmetral) moulds $B^\bullet$. The first transform is quite elementary and applies to all cases. The second transform is more subtle, but also more relevant to the present investigation. It applies only to moulds $A^\bullet$ with indices n_i in $\mathbb{N}$ and turns them into 'flat' or 'piecewise-constant' moulds $B^\bullet$ with indices t_i in $\mathbb{R}$. Both transforms respect multiplication in the sense that $transf(A_1^\bullet \times A_2^\bullet) \equiv transf(A_1^\bullet) \times transf(A_2^\bullet)$. Here is how they are defined:

First mould transform:

$$direct: \qquad A^\bullet \;\mapsto\; B^\bullet := A^\bullet \circ \exp_1^\bullet \tag{65}$$

$$inverse: \qquad B^\bullet \;\mapsto\; A^\bullet := B^\bullet \circ \log^\bullet \tag{66}$$

Second mould transform:

$$
\begin{array}{rcl}
A^\bullet \leftrightarrow B^\bullet & \textit{with} & B^{\,t_1,\dots,t_r} := SA^{\,\epsilon_1,\dots,\epsilon_{r-1},+} \quad \textit{and} \\
\epsilon_1 := \mathrm{sign}(t_1-t_2)\;, & \dots & ,\ \epsilon_{r-1} := \mathrm{sign}(t_{r-1}-t_r) \\
SA^{+} & := & -A^{\,1} \\
SA^{+,+} & := & +A^{\,1,1} \\
SA^{-,+} & := & +A^{\,1,1}+A^{\,2} \\
SA^{+,+,+} & := & -A^{\,1,1,1} \\
SA^{+,-,+} & := & -A^{\,1,1,1}-A^{\,1,2} \\
SA^{-,+,+} & := & -A^{\,1,1,1}-A^{\,2,1} \\
SA^{-,-,+} & := & -A^{\,1,1,1}-A^{\,1,2}-A^{\,2,1}-A^{\,3} \\
etc\dots & & \textit{Generally:} \\
\textit{direct}\ :\qquad SA^{\boldsymbol{\epsilon^1},\dots,\boldsymbol{\epsilon^s}} & := & (-1)^r \sum^{\star} A^{\mathbf{n^1},\dots,\mathbf{n^s}} \\
\textit{inverse}\ :\qquad A^{\,r_1,\dots,\mathbf{r_s}} & := & (-1)^s \sum^{\star\star} \epsilon_1\dots\epsilon_r\ SA^{\,\epsilon_1,\dots,\epsilon_r}
\end{array}
$$

In the last but one identity, all sign subsequences $\boldsymbol{\epsilon^i}$ consist of (r_i-1) initial $-$ signs and one final $+$ sign (r_i may be $=1$) and $\sum^\star$ extends to all integer sequences $\mathbf{n^i}$ of sum r_i , whereas in the last (reverse) identity the sum $\sum^{\star\star}$ extends to all $\epsilon_j \in \{+,-\}$ except when $j\in\{r_1, r_1{+}r_2,\dots,r_1{+}\dots{+}r_s\}$, in which case ϵ_j has to be $+$.

3 Combinatorial aspects of coarborification.

3.1 The standard coarborification rule.

Let $\{B_\omega,\ \omega\in\Omega\}$ be any system of ordinary differential operators in the variables $x_1,\dots,x_\nu$ and define the comould $B_\bullet$ as usual by setting:

$$B_{\,\omega_1,\dots,\omega_r} := B_{\,\omega_r}\dots B_{\,\omega_1} \tag{67}$$

Then *there exists a privileged arborescent comould $B_{\bullet^\prec}$, the so-called* standard *or* homogeneous *coarborification of $B_\bullet$, which is entirely characterised by the following three properties:*
P1 $B_{\bullet^\prec}$ is coseparative[14] i.e.:

$$B_{\boldsymbol{\omega}^\prec}\,(\varphi_1\,\varphi_2) \equiv \sum_{\boldsymbol{\omega^{1}}^\prec\,\oplus\,\boldsymbol{\omega^{2}}^\prec\,=\,\boldsymbol{\omega}^\prec} B_{\boldsymbol{\omega^1}^\prec}\,(\varphi_1)\ B_{\boldsymbol{\omega^2}^\prec}\,(\varphi_2) \tag{68}$$

[14] $\boldsymbol{\omega^1}^\prec\oplus\boldsymbol{\omega^2}^\prec$ denotes the tree obtained by juxtaposition of $\boldsymbol{\omega^1}^\prec$ and $\boldsymbol{\omega^2}^\prec$, with no other order relations than those inherited from the sub-trees $\boldsymbol{\omega^i}^\prec$. The sum (68) extends also to the trivial juxtapositions, with one summand $\boldsymbol{\omega^i}^\prec$ equal to $\boldsymbol{\omega}^\prec$ and the other one empty.

P2 If $deg(\boldsymbol{\omega}^{\prec}) = d$ i.e. if the tree $\boldsymbol{\omega}^{\prec}$ has exactly d roots, then the operator is homogeneous in the $\partial_i := \partial_{x_i}$ with total degree d
P3 If $\boldsymbol{\omega} = \omega_1\, \boldsymbol{\omega}^*$ (in other words, if $\boldsymbol{\omega}$ is of degree one, with a root element ω_1 followed by some arborescent sequence $\boldsymbol{\omega}^{*\prec}$) the corresponding operator factors as:

$$B_{\boldsymbol{\omega}^{\prec}}\ \ x_j \ \equiv\ B_{\boldsymbol{\omega}^{*\prec}}\ \ B_{\omega_1}\ \ \log x_j \qquad (j = 1, 2, \ldots, \nu) \tag{69}$$

Moreover, if $B_{\bullet}$ is cosymmetral [15] *(resp cosymmetrel* [16]*), then $B_{\bullet}$ and $B_{\bullet}^{\prec}$ are indeed correlated according to $B_{\boldsymbol{\omega}} := \sum_{\boldsymbol{\omega}^{\prec} < \boldsymbol{\omega}} B_{\boldsymbol{\omega}^{\prec}}$ (resp $B_{\boldsymbol{\omega}} := \sum_{\boldsymbol{\omega}^{\lessdot} \ll \boldsymbol{\omega}} B_{\boldsymbol{\omega}^{\lessdot}}$). In other words, whereas symmetral and symmetrel moulds obey different arborification rules (simple/contracting), the standard co-arborification rules are exactly the same for a cosymmetral comould and a cosymmetrel one.*

Let us check, by induction on the length r of $\boldsymbol{\omega}^{\prec}$, the fact that **P1, P2, P3** together do determine $\mathbb{B}_{\boldsymbol{\omega}^{\prec}}$.

Either $d(\boldsymbol{\omega}^{\prec}) = 1$, which means that $\boldsymbol{\omega}^{\prec}$ is of the form (70), in which case $B_{\boldsymbol{\omega}^{\prec}}$ is as in (71) below:

$$\boldsymbol{\omega}^{\prec} \ = \ (\omega_1, \boldsymbol{\omega}^{*\prec}) \tag{70}$$

$$B_{\boldsymbol{\omega}^{\prec}} \ = \ \sum_{1 \le i \le \nu} (B_{\omega^{*\prec}}.B_{\omega_1{}^{\prec}}.\log x_j)(x_j\, \partial_j) \tag{71}$$

Or $deg(\boldsymbol{\omega}^{\prec}) = d \geq 2$, which means that $\boldsymbol{\omega}^{\prec}$ is of the form (72), with s clusters of $d_1, \ldots, d_s$ identical, irreducible summands $\boldsymbol{\omega}^{\boldsymbol{i_1}\prec}, \ldots, \boldsymbol{\omega}^{\boldsymbol{i_s}\prec}$, in which case $\mathbb{B}_{\boldsymbol{\omega}^{\prec}}$ is as below:

$$\begin{aligned}
\boldsymbol{\omega}^{\prec} \ &= \ \boldsymbol{\omega}^{\boldsymbol{1}\prec} \oplus \cdots \oplus \boldsymbol{\omega}^{\boldsymbol{d}\prec} && (\boldsymbol{\omega}^{\boldsymbol{i}\prec} \neq \emptyset\ ,\ deg(\boldsymbol{\omega}^{\boldsymbol{i}\prec}) = 1)\\
&= \ (\boldsymbol{\omega}^{\boldsymbol{i_1}\prec})^{\oplus d_1} \oplus \cdots \oplus (\boldsymbol{\omega}^{\boldsymbol{i_s}\prec})^{\oplus d_s} && (d_1 + \cdots + d_s = d) \qquad (72)\\
B_{\boldsymbol{\omega}^{\prec}} \ &= \ \frac{1}{d_1! \ldots d_s!} \sum_{\substack{1 \le s \le d \\ 1 \le j_s \le \nu}} (B_{\boldsymbol{\omega}^{1\prec}}.\log x_{j_1}) \ldots (B_{\boldsymbol{\omega}^{d\prec}}.\log x_{j_d})\ (x_{j_1}\, \partial_{j_1}) \ldots (x_{j_d}\, \partial_{j_d})
\end{aligned}$$

3.2 Interpretation for cosymmetral/el comoulds.

To see how one and the same operation works equally well in the seemingly so different contexts of *cosymmetrality* and *cosymmetrelity*, the reader may

[15] see §3.2.
[16] see §3.2.

examine the simplest non-trivial examples of cosymmetral and cosymmetrel comoulds, with one variable x only and the factorisation property :

$$
\begin{array}{lll}
B^{(a)}_{\bullet} : & B^{(a)}_{n_1,\dots,n_r} := B^{(a)}_{n_r} \dots B^{(a)}_{n_1} & \\
B^{(e)}_{\bullet} : & B^{(e)}_{n_1,\dots,n_r} := B^{(e)}_{n} \dots B^{(e)}_{n_1} & \\
B^{(a)}_n := & b_n\, x^{n+1}\, \partial_n & B^{(a)}_n \ : \ x^m\, \mathbb{C} \to x^{m+n}\, \mathbb{C} \ \ (\forall m, n) \\
\sum_{n\geq 0} B^{(e)}_n := & \exp\big(\sum_{n\geq 1} b_n\, x^{n+1}\, \partial_n\big) & B^{(e)}_n \ : \ x^m\, \mathbb{C} \to x^{m+n}\, \mathbb{C} \ \ (\forall m, n)
\end{array}
$$

and then use the corresponding cosymmetries[17] :

$$
B^{(a)}_n \overset{\text{coproduct}}{\longrightarrow} 1 \otimes B^{(a)}_n + B^{(a)}_n \otimes 1 \qquad \textit{(cosymmetrality)} \tag{73}
$$

$$
B^{(e)}_n \overset{\text{coproduct}}{\longrightarrow} \sum_{n_1+n_1=n} B^{(e)}_{n_1} \otimes B^{(e)}_{n_2} \qquad \textit{(cosymmetrelity)} \tag{74}
$$

to check that in both cases the same standard procedure of §3.1 leads to comoulds $B^{(a)}_{\prec}$ and $B^{(e)}_{\prec}$ which are both *coseparative*, but verify the distinct coarborification constraints (9) and (12).

3.3 Standard coarborification and norm reduction.

Coarborification automatically diminishes comould norms. This of course is its main property, its main justification, and the reason for its usefulness in analysis. The phenomenon takes place for any reasonable norm on local differential operators, for instance :

$$
\|B\| = \|B\|_{\mathcal{D}_1,\mathcal{D}_2} := \sup_{\varphi\neq 0} \frac{\|B\,\varphi\|_{\mathcal{D}_1}}{\|\varphi\|_{\mathcal{D}_2}} \qquad \textit{with} \quad 0 \in \mathcal{D}_1\,,\ \bar{\mathcal{D}}_1 \subset \mathcal{D}_2 \subset \mathbb{C}^\nu \tag{75}
$$

with $\mathcal{D}_1, \mathcal{D}_2$ two small open neighbourhoods of 0 and $\|\varphi\|_{\mathcal{D}_i}$ the uniform norm on $\mathcal{D}_i$. To illustrate norm reduction, i.e. the improvement from (76) to (77) :

$$
\|B_{\boldsymbol{\omega}}\| \ \leq \ r(\boldsymbol{\omega}^{\prec})!\ C^{N(\boldsymbol{\omega}^{\prec})}\ \|B_{\omega_1}\| \dots \|B_{\omega_r}\| \tag{76}
$$

$$
\|B_{\boldsymbol{\omega}^{\prec}}\| \ \leq \ C^{N(\boldsymbol{\omega}^{\prec})}\ \|B_{\omega_1}\| \dots \|B_{\omega_r}\| \tag{77}
$$

let us fix a non-resonant spectrum $\lambda \in \mathbb{C}^\nu$ and consider first-order differential operators of the form :

$$
B_{\omega_i} := x^{n_i}\, B^*_{\omega_i} \quad \textit{with} \quad B^*_{\omega_i} := \sum_{1\leq j\leq \nu} b^j_{\omega_i}\, x_j\, \partial_{x_j}\ ,\ \omega_i := < n_i, \lambda >\ ,\ b^j_{\omega_j} \in \mathbb{C}
$$

[17] Since $B^{(e)}_0 = 1$ the sum (74) includes as extreme terms $B^{(e)}_n \otimes 1$ and $1 \otimes B^{(e)}_n$.

Next, let us carry out homogeneous coarborification for three extreme types of arborescent sequences:

$$\begin{array}{llllll} \boldsymbol{\omega} & := & (\omega_1,\ldots,\omega_r) & ; \text{ total order} & ; \text{ all } \omega_i \text{ distinct} \\ \boldsymbol{\omega'}^{\prec} & := & (\omega_1,\ldots,\omega_r) & ; \text{ total order} & ; \text{ all } \omega_i \text{ distinct} \\ \boldsymbol{\omega''}^{\prec} & := & (\omega_1\oplus\cdots\oplus\omega_r) & ; \text{ no order} & ; \text{ all } \omega_i \text{ distinct} \\ \boldsymbol{\omega'''}^{\prec} & := & (\omega_1\oplus\cdots\oplus\omega_r) & ; \text{ no order} & ; \text{ all } \omega_i \text{ identical} \end{array}$$

We find:

$$B_{\boldsymbol{\omega}} \quad := \quad B_{\omega_r}\ldots B_{\omega_1} \tag{78}$$

$$B_{\boldsymbol{\omega'}^{\prec}} \quad := \quad x^{n_r}(B^*_{\omega_r}\,x^{n_{r-1}})\,(B^*_{\omega_{r-1}}\,x^{n_{r-2}})\ldots(B^*_{\omega_3}\,x^{n_2})\,(B^*_{\omega_2}\,x^{n_1})B^*_{\omega_1} \tag{79}$$

$$B_{\boldsymbol{\omega''}^{\prec}} \quad := \quad x^{n_1+\cdots+n_r}\,B^*_{\omega_1}\ldots B^*_{\omega_r} \tag{80}$$

$$B_{\boldsymbol{\omega'''}^{\prec}} \quad := \quad \frac{1}{r!}\,x^{n_1+\cdots+n_r}\,B^*_{\omega_1}\ldots B^*_{\omega_r} \tag{81}$$

and in all three cases we observe the disappearance of the factor $r!$, though for rather distinct reasons:

– in (79) we have a first-order differential operator $B^*_{\omega_1}$ preceded by innocuous scalar factors $B^*_{\omega_i}\,x^{n_{i-1}}$

– in (80) we have a differential operator $B^*_{\omega_1}\ldots B^*_{\omega_r}$ (all terms commute) of order r and of factorially large norm, but with a more than factorially small front factor $x^{\|\mathbf{n}\|}$ since x is small and $\|\mathbf{n}\| \geq const\,.\,r^{1+\frac{1}{\nu}}$

– in (81) we have again a differential operator $B^*_{\omega_1}\ldots B^*_{\omega_r}$ (all terms are equal) of order r and of factorially large norm, but with a multiplicity factor $\frac{1}{r!}$ in front.

4 The arborification-coarborification transform. Fourteen applications to analysis.

4.1 Application 1: Linearisation of vector fields with diophantine spectra.

A local analytic vector field X with diophantine, non-resonant spectrum $\lambda := (\lambda_1,\ldots,\lambda_\nu)$:

$$\begin{aligned} X &:= X^{\text{lin}} + \sum B_n & & \textit{with} \\ X^{\text{lin}} &:= \sum_{1\leq i\leq\nu} \lambda_i\,x_i\,\partial_{x_i} & & \textit{and} \\ B_n &:= \textit{homog. part of deg. } n = (n_1,..,n_\nu)\,(n_i \geq -1, \textit{at most one neg. } n_i) \end{aligned} \tag{82}$$

admits a formal linearisation Θ_{ent} which in operatorial form reads:

$$
\begin{aligned}
X &= \Theta_{\text{ent}}\, X^{\text{lin}}\, \Theta_{\text{ent}}^{-1} \qquad \textit{with} && (83)\\
\Theta_{\text{ent}} &:= \sum Sa^{\bullet}\, B_{\bullet} \equiv \sum Sa^{\prec}\, B_{\prec} && (84)\\
\Theta_{\text{ent}}^{-1} &:= \sum {}^{\text{inv}}Sa^{\bullet}\, B_{\bullet} \equiv \sum {}^{\text{inv}}Sa^{\prec}\, B_{\prec} && (85)\\
Sa^{\boldsymbol{\omega}} &:= (-1)^r \prod_{1\le i\le r} \frac{1}{\omega_1+\cdots+\omega_i} && (86)\\
{}^{\text{inv}}Sa^{\boldsymbol{\omega}} &:= \prod_{1\le i\le r} \frac{1}{\omega_i+\cdots+\omega_r} && (87)
\end{aligned}
$$

Prior to arborification, the normalising series (84),(85) are usually divergent. After arborification, they are always convergent, because both moulds $Sa^{\bullet}$ and ${}^{\text{inv}}Sa^{\bullet}$ suffer *no significant norm increase.* And the reason why they don't is that *one of them*, namely ${}^{\text{inv}}Sa^{\bullet}$ actually retains its *form*, i.e. its outward analytical expression, under arborification.[18]
N.B. Here and in the sequel, we take advantage of the non-resonance of the λ_i's to substitute an indexation by $\omega_i =< \lambda, n_i >\in \mathbb{C}$ for the original indexation by $n_i \in \mathbb{Z}^{\nu}$.

4.2 Application 2: Linearisation of diffeos with diophantine spectra.

A local analytic diffeo X with diophantine and (multiplicitively) non-resonant spectrum $l := (l_1, \ldots, l_{\nu})$:

$$
\begin{aligned}
F &:= \big(1 + \sum B_n\big).\, F^{\text{lin}} \qquad \textit{with} && (88)\\
F^{\text{lin}} &:= \varphi(x_1,\ldots,x_{\nu}) \mapsto \varphi(l_1\, x_1,\ldots,l_{\nu}\, x_{\nu})\\
B_n &:= \textit{homog. part of deg. } n = (n_1,..,n_{\nu})\ (n_i \ge -1, \textit{at most one neg. } n_i)
\end{aligned}
$$

[18] For details, see [E3],[E9], also [Sie1],[Sie2],[Br] for the historical background.

admits a formal entire linearisation Θ_{ent} which in operatorial form reads:

$$F \quad = \quad \Theta_{\text{ent}}\, F^{\text{lin}}\, \Theta_{\text{ent}}^{-1} \qquad\qquad \textit{with} \tag{89}$$

$$\Theta_{\text{ent}} \quad := \quad \sum \quad Se^{\bullet}\, B_{\bullet} \;\equiv\; \sum \quad Se^{\prec}\, B_{\prec} \tag{90}$$

$$\Theta_{\text{ent}}^{-1} \quad := \quad \sum {}^{\text{inv}}Se^{\bullet}\, B_{\bullet} \;\equiv\; \sum {}^{\text{inv}}Se^{\prec}\, B_{\prec} \tag{91}$$

$$Se^{\boldsymbol{\omega}} \quad := \quad (-1)^r \prod_{1\leq i\leq r} \frac{e^{-\omega_i}}{1-e^{-\omega_1\cdots-\omega_i}} \tag{92}$$

$${}^{\text{inv}}Se^{\boldsymbol{\omega}} \quad := \quad \prod_{1\leq i\leq r} \frac{1}{e^{\omega_i+\cdots+\omega_r}-1} \tag{93}$$

As before, and for the same reasons, arborification restores convergence in the normalising series $\Theta_{\text{ent}}^{\pm 1}$. [19]

4.3 Application 3: Normalisation of vector fields with resonant spectra.

Here *normalisation* rather than *linearisation* is the order of the day, with normalising transformations Θ_{res} that are generally divergent but resurgent. To simplify, assume the resonance to be of degree 1 (only one relation between the λ_i's), in which case one single 'normal' variable z bears the whole burden of divergence and resurgence.

$$X \quad := \quad X^{\text{nor}} + \sum B_n \qquad\qquad \textit{with} \tag{94}$$

$$X^{\text{nor}} \quad := \quad \sum_{1\leq i\leq \nu} \lambda_i\, x_i\, \partial_{x_i} + \; x^m \sum_{1\leq i\leq \nu} \tau_i\, x_i\, \partial_{x_i} \quad \textit{with} \quad < m,\tau >= -1 \quad \textit{and}$$

$$B_n \quad := \quad \textit{homog. part of deg. } n=(n_1,..,n_\nu)\; (n_i\geq -1, \textit{at most one neg. } n_i)$$

In operatorial form, the resurgent normalising transformations $\Theta_{\text{res}}^{\pm 1}$ read:

$$X \quad = \quad \Theta_{\text{res}}\, X^{\text{nor}}\, \Theta_{\text{res}}^{-1} \qquad\qquad \textit{with} \tag{95}$$

$$\Theta_{\text{res}} \quad := \quad \sum \quad \mathcal{V}e(z)^{\bullet}\, B_{\bullet} \;\equiv\; \sum \quad \mathcal{V}e(z)^{\prec}\, B_{\prec} \tag{96}$$

$$\Theta_{\text{res}}^{-1} \quad := \quad \sum {}^{\text{inv}}\mathcal{V}e(z)^{\bullet}\, B_{\bullet} \;\equiv\; \sum {}^{\text{invinv}}\mathcal{V}e(z)^{\prec}\, B_{\prec} \tag{97}$$

with mould elements $\mathcal{V}e(z)^{\boldsymbol{\omega}}, \mathcal{V}e(z)^{\boldsymbol{\omega}}$ that are elementary resurgent monomials. The normalising transformations being usually divergent, the only

[19] For details, see [E3],[E9], also [Sie1],[Sie2],[Br],[Rü] for the background.

question that arises is of course whether the $\Theta_{\text{res}}^{\pm 1}$ are *convergent* as series of *resurgent functions*. Sometimes they already are, prior to arborification; sometimes arborification is called for.[20]

4.4 Application 4: Normalisation of diffeos with resonant spectra.

The picture is much the same as in the previous example.

$$\begin{aligned} F &:= \big(1+\sum B_n\big).\, F^{\text{nor}} \qquad \textit{with} && (98)\\ F^{\text{nor}} &:= \varphi(x_1,\dots,x_\nu) \;\mapsto\; \varphi(l_1\, x_1...,\dots,l_\nu\, x_\nu...) \\ B_n &:= \textit{homog. part of deg. } n=(n_1,..,n_\nu)\,(n_i\geq -1, \textit{at most one neg. } n_i) \end{aligned}$$

with resurgent normalising transformations $\Theta_{\text{res}}^{\pm 1}$ of the form :

$$F \;=\; \Theta_{\text{nor}}\, F^{\text{lin}}\, \Theta_{\text{nor}}^{-1} \qquad \textit{with} \tag{99}$$

$$\Theta_{\text{nor}} \;:=\; \sum \;\mathcal{W}e(z)^{\bullet}\, B_{\bullet} \;\equiv\; \sum \;\mathcal{W}e(z)^{\prec}\, B_{\prec} \tag{100}$$

$$\Theta_{\text{ent}}^{-1} \;:=\; \sum {}^{\text{inv}}\mathcal{W}e(z)^{\bullet}\, B_{\bullet} \;\equiv\; \sum {}^{\text{inv}}\mathcal{W}e(z)^{\prec}\, B_{\prec} \tag{101}$$

and with suitable resurgent monomials $\mathcal{W}e(z)^{\bullet}$ and ${}^{\text{inv}}\mathcal{W}e(z)^{\bullet}$.[21]

4.5 Application 5: Ramified linearisation of vector fields with quasi-resonant spectra.

Here, we assume *pure* quasi-resonance. In other words, we have no (exact) resonance, but a violation of Bryuno's classical diophantine condition.

$$\begin{aligned} X &:= X^{\text{lin}} + \sum B_n \qquad \textit{with} && (102)\\ X^{\text{lin}} &:= \sum_{1\leq i\leq \nu} \lambda_i\, x_i\, \partial_{x_i} \qquad \textit{and} \\ B_n &:= \textit{homog. part of deg. } n=(n_1,..,n_\nu)\,(n_i\geq -1, \textit{at most one neg. } n_i) \end{aligned}$$

Quasi-resonance doesn't prevent *formal entire* linearisation, but it usually renders $\Theta_{\text{ent}}^{\pm 1}$ divergent. To get hold of something convergent, we must harness the phenomenon of *compensation* and work with *ramified* transformations $\Theta_{\text{ram}}^{\pm 1}$. These are 'ramified' in the sense that they involve positive,

[20] For details, see [E2],[E3],[E5].
[21] For details, see [E2],[E3],[E5].

irrational powers of at least one, but sometimes two or three variables x_i. Moreover, instead of being defined on ordinary (uniform) neighbourhoods of the origin $0 \in \mathbb{C}^\nu$, they are defined in spiral-like, ramified neighbourhoods. The operatorial expansions for $\Theta^{\pm 1}_{\mathrm{ram}}$ are always of the form :

$$X \;=\; \Theta_{\mathrm{ram}}\, X^{\mathrm{lin}}\, \Theta^{-1}_{\mathrm{ram}} \qquad \textit{with} \tag{103}$$

$$\Theta_{\mathrm{ram}} \;:=\; \Theta_{\mathrm{ent}}\,\Theta^{-1}_{\mathrm{colin}} = \sum \; Sa^{\bullet}_{\mathrm{ram}}(z)\; B_{\bullet} \;\equiv\; \sum \; Sa^{\prec}_{\mathrm{ram}}(z)\; B_{\prec} \tag{104}$$

$$\Theta^{-1}_{\mathrm{ram}} \;:=\; \Theta_{\mathrm{colin}}\,\Theta^{-1}_{\mathrm{ent}} = \sum {}^{\mathrm{inv}}Sa^{\bullet}_{\mathrm{ram}}(z)\; B_{\bullet} \;\equiv\; \sum {}^{\mathrm{inv}}Sa^{\prec}_{\mathrm{ram}}(z)\; B_{\prec} \tag{105}$$

but the analysis very much depends on the 'badness' of the quasiresonance.

Case 1: Real, semi-mixed spectrum :
This is the case when $\lambda_1 < 0$ but $0 < \lambda_2, \lambda_3, \ldots, \lambda_\nu$.
Then *one* ramification suffices :

$$z \;:=\; x_1^{-1/\lambda_1} \tag{106}$$

$$Sa^{\bullet}_{\mathrm{ram}}(z) \;:=\; Sa^{\bullet}_{\mathrm{co}}(z) := ({}^{\mathrm{inv}}Sa^{\bullet}\, z^{\|\bullet\|}) \times Sa^{\bullet} \tag{107}$$

$${}^{\mathrm{inv}}Sa^{\bullet}_{\mathrm{ram}}(z) \;:=\; {}^{\mathrm{inv}}Sa^{\bullet}_{\mathrm{co}}(z) := {}^{\mathrm{inv}}Sa^{\bullet} \times (Sa^{\bullet}\, z^{\|\bullet\|}) \tag{108}$$

with $z^{\|\bullet\|}$ used as short-hand for $z^{\|\boldsymbol{\omega}\|} := z^{\sum \omega_i}$. Here the expansions for $\Theta^{\pm 1}_{\mathrm{ram}}$ are already convergent *before arborification.*[22]

Case 2 : Real, mixed spectrum:
This is the case when we have at least two negative and two positive λ_i. Here, *two* ramifications become necessary, attached to two eigenvalues of our own choosing, but of opposite signs, say $\lambda_1 < 0 < \lambda_2$, and we must resort to the sophisticated operation of *mould mixing*, which is described in §2.10. The mould ingredients for $\Theta^{\pm 1}_{ram}$ now read :

$$z_1 \;:=\; x_1^{-1/\lambda_1} \;\;,\;\; z_2 \;:=\; x_2^{-1/\lambda_2} \tag{109}$$

$${}^{\mathrm{inv}}Sa^{\bullet}_{\mathrm{ram}}(z) \;:=\; {}^{\mathrm{inv}}Sa^{\bullet}_{\mathrm{co}}(z_1) \;\; \mathrm{mix} \;\; {}^{\mathrm{inv}}Sa^{\bullet}_{\mathrm{co}}(z_2) \tag{110}$$

$$\equiv\; {}^{\mathrm{inv}}Sa^{\bullet} \times \big((Sa^{\bullet}\; z_1^{\|\bullet\|}) \;\; \mathrm{mix} \;\; (Sa^{\bullet}\; z_2^{\|\bullet\|})\big) \tag{111}$$

but the novelty is that now $\Theta^{\pm 1}_{ram}$ requires arborification to become convergent (in a suitable space of ramified functions, of course).

[22]of course, they remain so after arborification : arborification is sometimes *unnecessary*, but never *harmful.*

Case 3 : Full-blown quasiresonance with complex spectrum .
The same approach as above applies, but with *three* ramifications and more intricate forms of *mixing*. Here again, one cannot avoid arborification.

Link with the so-called 'compensators'.
The mould ingredients $Sa^{\bullet}_{\mathrm{ram}}(x)$ and ${}^{\mathrm{inv}}Sa^{\bullet}_{\mathrm{ram}}(x)$ which enter the construction of $\Theta^{\pm}_{ram}$ are actually sums of *compensators* of the form :

$$z^{\sigma_0,\sigma_1,\ldots,\sigma_r} \quad := \quad \sum_{0\leq i\leq r} z^{\sigma_i} \prod_{j\neq i} \frac{1}{\sigma_i - \sigma_j} \qquad (z \in \mathbb{C},\ \sigma_i \in \mathbb{R}^+) \tag{112}$$

which remain bounded even when the σ_i's get dangerously close to one another. This simple remark underpins the whole theory of compensation. [23]

4.6 Application 6: "Correction" of vector fields with resonant spectra.

This section and the two that follow deal with a remarkable, often misunderstood phenomenon : the *non-appearance of supermultiple small denominators*[24] when *resonance* interacts with *diophantine small denominators*. The present section tackles the phenomenon in its purest form and at the simplest level. Take a resonant vector field X with diophantine spectrum. Since resonance generally precludes linearisation (even formal), that leaves two options. In the first one, we add a resonant series to the linear part X^{lin} to get a normal or prenormal form, leading to an entire, but divergent and resurgent conjugation of X to that normal form, as in §4.3. In the second option, we subtract a resonant series (the 'correction') from the field X to force formal conjugation with X^{lin}. But this time, despite the deceptive symmetry of the two approaches, the formal conjugation turns out to be analytic as well.

$$X \sim X^{\mathrm{lin}} \qquad \textit{(non-resonant case)} \tag{113}$$

$$X \sim X^{\mathrm{lin}} + X^{\mathrm{pre}} \quad \textit{with} \quad [X^{\mathrm{pre}}, X^{\mathrm{lin}}] = 0 \quad \textit{(resonant case)} \tag{114}$$

$$X - X^{\mathrm{cor}} \sim X^{\mathrm{lin}} \quad \textit{with} \quad [X^{\mathrm{cor}}, X^{\mathrm{lin}}] = 0 \quad \textit{(resonant case)} \tag{115}$$

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[23] For details, see [E8], also [E6].

[24] very roughly : small denominators with such abnormally high multiplicities that their presence would automatically thwart convergence.

Translating the second option into mould expansions, we find:

$$X - X^{\text{cor}} \;=\; \Theta_{\text{cor}}\, X\, \Theta_{\text{cor}}^{-1} \tag{116}$$

$$X^{\text{cor}} \;=\; \sum Carr^{\bullet}\, B_{\bullet} \;=\; \sum Carr^{\omega_1,\dots,\omega_r} B_{n_r} \dots B_{n_1} \tag{117}$$

$$\Theta_{\text{cor}} \;=\; \sum Scarr^{\bullet}\, B_{\bullet} \;=\; \sum Scarr^{\omega_1,\dots,\omega_r} B_{n_r} \dots B_{n_1} \tag{118}$$

The key ingredient here is a mould $Carr^{\bullet}$ inductively defined by:

$$Carr^{\emptyset} \;=\; 0 \;;\;\; Carr^{0} \;=\; 1 \;;\;\;\; Carr^{\omega_1} \;=\; 0 \;\; \textit{if} \;\; \omega_1 = 0 \tag{119}$$

$$\text{var}_i\, Carr^{\boldsymbol{\omega}} \;=\; \sum_{\boldsymbol{\omega}^1 \omega_i \boldsymbol{\omega}^2 \boldsymbol{\omega}^3 = \boldsymbol{\omega}} Carr^{\boldsymbol{\omega}^1 \omega_i \boldsymbol{\omega}^3} Carr^{\boldsymbol{\omega}^2} - \sum_{\boldsymbol{\omega}^1 \boldsymbol{\omega}^2 \omega_i \boldsymbol{\omega}^3 = \boldsymbol{\omega}} Carr^{\boldsymbol{\omega}^1 \omega_i \boldsymbol{\omega}^3} Carr^{\boldsymbol{\omega}^2} \tag{120}$$

with a variation operator var_i that acts as follows:

$$\text{var}_i\, M^{\,\omega_1,\dots,\omega_r} \;:=\; \omega_i\, M^{\,\omega_1,\dots,\omega_r} + M^{\,\omega_1,\dots,\omega_i+\omega_{i+1},\dots,\omega_r} - M^{\,\omega_1,\dots,\omega_{i-1}+\omega_i,\dots,\omega_r} \tag{121}$$

We have analogous formulas for $Scarr^{\bullet}$. Two points must be emphasised here. The first is that the above induction leaves us sufficient latitude (through the choice of the index i) to prevent the occurence of supermultiple small denominators. The second point is that it takes *arborification* to make the expansions (117) and(118) convergent.[25]

4.7 Application 7: Floquet theory.

Floquet theory concerns itself with differential equations with quasi-periodic coefficients. A test case is the system:

$$\partial_t\, X(t) \;=\; U(t)\, X(t) \qquad \textit{with} \tag{122}$$

$$U(t) \;:=\; l\, A + \sum_{\omega \in \Omega} e^{i\omega t}\, U_\omega \qquad (A, U_\omega\ \textit{const}\,;\; l, t \in \mathbb{R}\;, l \gg 1) \tag{123}$$

$$\omega \in \Omega \;:=\; \lambda_1\, \mathbb{Z} + \dots + \lambda_\nu\, \mathbb{Z} \qquad (\lambda_1, \lambda_2 \dots \textit{non-resonant}) \tag{124}$$

In order to reduce (122) to an elementary, 'self-solving' equation:

$$\partial_t\, Y(t) \;=\; V\, Y(t) \qquad \textit{with}\;\; V = \textit{const} \tag{125}$$

by means of a change of unknown $X(t) = \Theta(t)\, Y(t)$, we must solve:

$$V + \Theta^{-1}(t)\;\; \partial_t \Theta(t) = \Theta^{-1}(t)\;\; U(t)\;\; \Theta(t) \tag{126}$$

[25] There exists a parallel theory for diffeos. For details, see [EV1],[EV2].

with a constant matrix V whose spectrum $(iv_1, \ldots, iv_\nu)$ can be read off the asymptotic behaviour of the solution of (122). The next steps are broadly parallel to those in the preceding section, except that now *multiplication or division by the frequencies* ω_i *must be replaced respectively by the action of the operators :*

$$\overline{V} := (\partial_t - ad(V)) \qquad ; \qquad \underline{V} := (\partial_t - ad(V))^{-1} \tag{127}$$

The elementary identities :

$$\frac{1}{\omega_1\,(\omega_1+\omega_2)} + \frac{1}{\omega_2\,(\omega_1+\omega_2)} \equiv \frac{1}{\omega_1\,\omega_2} \tag{128}$$

upon whose repeated use the induction (120) rests, give way to the identities :

$$\underline{V}\,((\underline{V}\,B^*_{\omega_1})\;B^*_{\omega_2}) + \underline{V}(B^*_{\omega_1}(\underline{V}B^*_{\omega_2})) \;\equiv\; (\underline{V}B^*_{\omega_1})\,(\underline{V}B^*_{\omega_2}) \tag{129}$$

$$\textit{with} \qquad B^*_{\omega_i} := e^{i\omega t}B_{\omega_i} \qquad \textit{and} \qquad B_{\omega_i} = \textit{const} \tag{130}$$

The last step – arborification – is not required in all cases : whether it is or not depends on the group we work in.[26]

4.8 Application 8: KAM theory and the survival of invariant tori.

Working under the classical (analytic) KAM assumptions, we perturb an integrable hamiltonian h:

$$h(y) =< \lambda, y > + < y, Q, y >= \sum \lambda_i\, y_i + \sum Q_{i,j}\, y_i\, y_j \tag{131}$$

(with $\mathbb{Q}$-independent basic frequencies λ_i) into a non-integrable H :

$$H(x,y) \;=\; h(y) + \epsilon\; b(x,y) \qquad (x \in \mathbb{T}^\nu, y \in \mathbb{R}^\nu_0\,) \tag{132}$$

$$\;=\; < \lambda, y > + \sum_{m,n} H_{m,n}(x,y) \tag{133}$$

The whole point is to start from Bryuno's (not Siegel's) diophantine assumptions on the λ_i's and to prove the convergence, for $y = 0$ and a small enough perturbation parameter ϵ, of the *uncorrected* Lindstedt series :

$$\sum H_{m,n}(x,y) \;=\; \sum c_{m,n}(\epsilon)\; e^{2\pi i <x,m>}\; y^n = \sum c_{m,n}(\epsilon)\; e^{2\pi i <\lambda,m>\,t}\; y^n \tag{134}$$

$$\omega \;:=\; < m, \lambda >= \textit{`frequency'} \qquad (m \in \mathbb{Z}^\nu)$$

$$\eta \;:=\; -1 + \|n\| = -1 + \sum n_i = \textit{`grade'} \qquad (n \in \mathbb{N}^\nu, \eta \geq -1)$$

[26] For some details, see [E10].

Going over from the potential H to the vector field X^H, we must partially *correct* and partially *normalise* our field X^H :

$$X^H - X^{\text{cor}} \overset{conj}{\sim} X^{\text{lin}} + X^{\text{nor}} \tag{135}$$

$$\text{frequency}(X^{\text{cor}}) = 0 \quad , \quad \text{frequency}(X^{\text{nor}}) = 0 \tag{136}$$

$$\text{grade}(X^{\text{cor}}) = 0 \quad , \quad \text{grade}(X^{\text{nor}}) \neq 0 \tag{137}$$

by allowing only terms of zero (resp non-zero) grade on the left- (resp right-hand) side of (135). Like in §4.3 the correction still possesses a mould expansion of type:

$$X^{\text{cor}} = \sum_{r\geq 1} \sum Bicarr^{\binom{\omega_1\ ,\dots,\ \omega_r}{\eta_1\ ,\dots,\ \eta_r}}\, X^{H_{m_r,n_r}} \dots X^{H_{m_2,n_2}} X^{H_{m_1,n_1}} \tag{138}$$

with frequencies $\omega_i :=<m_i, \lambda>$ and grades $\eta_i := -1 + \|n_i\|$. The normal part X^{nor} also has a similar mould expansion, but we need not worry about it, since it vanishes for $y = 0$ and so does not contribute to the Lindstedt series.

The alternal mould $Bicarr^\bullet$ is more complex than, but essentially similar to, the mould $Carr^\bullet$ of §4.3. In fact, $Bicarr^\bullet$ reduces to $Carr^\bullet$ when all the grades η_i are 0 or, more generally, when to each vanishing partial sum $\omega_i + \dots + \omega_j = 0$ there corresponds a vanishing partial sum $\eta_i + \dots + \eta_j = 0$.

We can duplicate in this case all the steps of §4.6 and prove, once again, the non-occurence of *supermultiple small denominators*, except that now the *formal multiplicity* of a divisor is exactly *twice* what it was in §4.6. That apart, precious little changes. We still must *arborify* to get the convergence of X^{cor}. This establishes, for a small enough perturbation parameter ϵ, the convergence of the Lindstedt series for the *corrected hamiltonian*. Then a standard argument going back to Poincaré (known as "killing the constants" and using the possibility of changing the integration constants) readily yields the convergence of the Lindstedt series for the *given* hamiltonian itself.[27]

4.9 Application 9: Well-behaved alien derivations.

Roughly speaking, a system $\mathbf{\Delta} = \{\mathbf{\Delta}_\omega\, , \, \omega \in \mathrm{R}^+\}$ of alien derivations is said to be *well-behaved* if, getting them to act on natural resurgent functions φ, we get exponential bounds of type $\|\mathbf{\Delta}_\omega \varphi\| \leq c_o\, e^{c_1\, \omega}$. This condition, which is useful in certain (not all) applications, is *not* fulfilled by the simplest and oldest system – that of *standard* alien derivations. Now, a system $\mathbf{\Delta}$ is completely characterised by a system of *weights* $\mathbf{d}^{\binom{\epsilon_1\ ,\dots,\ \epsilon_1}{\omega_1\ ,\dots,\ \omega_1}}$ with $\epsilon_i \in \{+, -\}$

[27]For some details, see [E10].

and $\omega_i \in \mathbb{R}^+$. Further, due to so-called *self-consistency constraints*, knowing these weights reduces to knowing any one of the three following moulds [28]:

$$\begin{array}{rcll}
\mathbf{red}^{\omega_1,\dots,\omega_r} & := & (-1)^r\, \mathbf{d}^{\binom{+\ ,\dots,\ +}{\omega_1\ ,\dots,\ \omega_r}} & (\text{“right-lateral mould”}) \quad (139)\\
\mathbf{led}^{\omega_1,\dots,\omega_r} & := & (-1)^r\, \mathbf{d}^{\binom{-\ ,\dots,\ -}{\omega_1\ ,\dots,\ \omega_r}} & (\text{“left-lateral mould”}) \quad (140)\\
\mathbf{nad}^{t_1,\dots,t_r}_{\omega_*,t_*} & := & \epsilon_1 \dots \epsilon_r\, \mathbf{d}^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{\omega_*\ ,\dots,\ \omega_*}} & (\text{“neutral mould”}) \quad (141)\\
\mathit{with}\quad \epsilon_i & := & \mathit{sign}(t_i - t_{i-1}) \quad (\forall i < r) & \mathit{and}\quad \epsilon_r := \mathit{sign}(t_r - t_*)
\end{array}$$

and we have this very useful criterion: *the system* $\mathbb{\Delta}$ *is well-behaved iff, after arborification, one of these moulds (and therefore all three) admit exponential bounds.*[29]

4.10 Application 10: Well-behaved uniformising averages.

For uniformising convolution averages[30] $\mathbf{m}$ the requirement of being *well-behaved* is even more essential than for alien derivations. These averages were first devised to overcome the vexing phenomenon of faster-than-exponential growth in the Borel plane along singularity-carrying axes. Like alien derivations, averages admit a description in terms of weights $\mathbf{m}^{\binom{\epsilon_1\ ,\dots,\ \epsilon_1}{\omega_1\ ,\dots,\ \omega_1}}$ that are subject to severe self-consistency constraints, and all the information can be compressed into either of three moulds [31]:

$$\begin{array}{rcll}
\mathbf{rem}^{\omega_1,\dots,\omega_r} & := & (-1)^r\, \mathbf{m}^{\binom{+\ ,\dots,\ +}{\omega_1\ ,\dots,\ \omega_r}} & (\text{“right-lateral mould”}) \quad (142)\\
\mathbf{lem}^{\omega_1,\dots,\omega_r} & := & (-1)^r\, \mathbf{m}^{\binom{-\ ,\dots,\ -}{\omega_1\ ,\dots,\ \omega_r}} & (\text{“left-lateral mould”}) \quad (143)\\
\mathbf{nam}^{t_1,\dots,t_r}_{\omega_*,t_*} & := & \epsilon_1 \dots \epsilon_r\, \mathbf{m}^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{\omega_*\ ,\dots,\ \omega_*}} & (\text{“neutral mould”}) \quad (144)\\
\mathit{with}\quad \epsilon_i & := & \mathit{sign}(t_i - t_{i-1}) \quad (\forall i < r) & \mathit{and}\quad \epsilon_r := \mathit{sign}(t_r - t_*)
\end{array}$$

Here again, well-behavedness has a simple characterisation: *the uniformising average* $\mathbf{m}$ *is well-behaved iff, after arborification, one of these moulds (and therefore all three) admit exponential bounds.* [32]

[28] the first two are alternel; the last one is alternal.

[29] For details, see [E11].

[30] they turn multivalued functions $\hat{\varphi}$ over $\mathbb{R}^+$ into uniform ones and respect convolution: $\mathbf{m}(\hat{\varphi}_1 * \hat{\varphi}_2) \equiv \mathbf{m}(\hat{\varphi}_1) * \mathbf{m}(\hat{\varphi}_2)$

[31] the first two are symmetrel; the last one is symmetral.

[32] For details, see [Me1],[EM],[E11].

4.11 Application 11: 'Display' of a resurgent function.

The *display* of a resurgent function f is defined by:

$$f \mapsto \text{display}(f) := f + \sum_{r \geq 1} \sum_{\omega_i} \mathbb{Z}^{\omega_1,\ldots,\omega_r} \mathbb{\Delta}_{\omega_r} \ldots \mathbb{\Delta}_{\omega_1} f \tag{145}$$

It encapsulates in user-friendly form all the information about f. It involves all (successive) alien derivatives of f, along with dual objects, the so-called *pseudovariables*, which multiply according to the shuffle product, behave predictably under alien derivation, and remain inert under natural derivation :

$$\mathbb{Z}^{\boldsymbol{\omega}^1}\, \mathbb{Z}^{\boldsymbol{\omega}^2} = \sum_{\boldsymbol{\omega} \in \text{sha}(\boldsymbol{\omega}^1, \boldsymbol{\omega}^2)} \mathbb{Z}^{\boldsymbol{\omega}} \tag{146}$$

$$\mathbb{\Delta}_{\omega_0}\, \mathbb{Z}^{\omega_1,\ldots,\omega_r} = \mathbb{Z}^{\omega_2,\ldots,\omega_r} \;\; (\textit{resp}\; 0) \;\; \textit{if}\;\; \omega_0 = \omega_1 \;\; (\textit{resp}\; \omega_0 \neq \omega_1) \tag{147}$$

$$\partial_z\, \mathbb{Z}^{\boldsymbol{\omega}} = 0 \tag{148}$$

These rules ensure that the *display* commutes with all operations (addition, multiplication, ordinary and alien derivation) and makes it an extremely useful tool for
(a) writing down in compact form all the obstructions to convergence[33]
(b) proving transcendence results[34].

There are precautions to take, however: although the display may be written down in any dual bases of *ALIEN* and *PSEUDO*, if we want the expansion (145) to be convergent[35] we must
(a) work with a *well-behaved basis* of *ALIEN* and *PSEUDO*
(b) *arborify* the expansion (145).[36]

4.12 Application 12: Canonical-spherical Object Synthesis.

Object Analysis is concerned with finding the analytic invariants $\{\mathbb{A}_\omega\}$ of *local analytic objects* **Ob**.[37] *Object Synthesis*, conversely, starts from some

[33] i.e. all the Stokes constants, whose non-vanishing prevents f from being convergent.
[34] since any relation $R(f_1, f_2, \ldots) = 0$ immediately translates into a corresponding relation between the *displays*, whose impossibility is often conspicuous, in view of the huge mass of constraints which it implies.
[35] relative to the natural topology of *RESUR* $\otimes$ *PSEUDO*
[36] For some details, see [E10].
[37] these are mostly, but not only, vector fields or diffeomorphisms.

(admissible!) system of invariants $\{\mathbb{A}_\omega\}$ and endeavours to produce an object **Ob** with precisely those prescribed invariants. The beauty is that there exists:
(a) a canonical solution $\mathbf{Ob}^{\mathrm{can}}$
(b) an entirely explicit, easy-to-handle expression of $\mathbf{Ob}^{\mathrm{can}}$ in terms of mould-comould expansions which involve (on the comould side) the invariants $\{\mathbb{A}_\omega\}$ and (on the mould side) a special system of resurgence monomials, the so-called 'spherical' or 'twisted' monomials.
Here again, the mould-comould expansions always *can*, and often *must* be arborified to achieve convergence.[38]

4.13 Application 13: Non-linear q-equations (F.Menous).

The technique of arborification has recently been used to great effect by F. Menous [39] to prove that the q-difference equation:

$$x\,\sigma_q\,y = y + b(x,y) \qquad\qquad (\,b(0,0) = \partial_y b(0,0) = 0\,) \tag{149}$$

with analytic right-hand side and $(\sigma_q\,f)(x) := f(q\,x)$, is *analytically* conjugate to one of the following normal forms:

$$x\,\sigma_q\,y = y \qquad , \qquad x\,\sigma_q\,y = y + x \tag{150}$$

4.14 Application 14: The "sandwich equation".

The "sandwich equation" of unknown f :

$$f^{n_1} \circ g_1 \circ f^{n_2} \circ g_2 \circ \ldots f^{n_r} \circ g_r = id \qquad\qquad \textit{with} \quad n_i \in \mathbb{Z} \tag{151}$$

is clearly the most general equation that may be considered on an unspecified group G. If we now take G to be the group of local diffeos of $\mathbb{C}$ and assume the data g_i to be quasirotations, i.e. of the form $x \mapsto c_i\,x + o(x)$ with $|c_i| = 1$, then, barring global resonance and quasiresonance and assuming $\sum n_i \neq 0$, the unique formal solution of (151) is also *analytic*. To establish this fact, massive arborification of the 'template-preserving' sort [40] is required. [41]

[38] For details, see [E10].
[39] For details, see [Me2].
[40] see §1.3
[41] For some details, see [EV3].

5 Algebraic aspects of arborification-coarborification. Haukian moulds and haukian σ-functions.

5.1 Quadratic coarborification and quadratic fission: induced matrices, induced σ-functions, induced moulds.

We recall that the general fusion-fission transform:

$$SS = \sum_{\bullet} B_{\bullet}\, A^{\bullet} \longmapsto SS = \sum_{\#} B_{\#}\, A^{\#} \tag{152}$$

involves a *fusion rule* and a dual *fission constraint.* The latter leaves considerable latitude. In a differential operator context, there is a natural way of satisfying it.[42] In a free-associative context, there exists another natural answer, which is the *quadratic fission rule.*[43] In matrix notations:

$$\begin{array}{lrcl} \textit{Fusion rule:} & A^{\#} & := & F^{\#}_{\bullet}\, A^{\bullet} \\ \textit{Fission constraint:} & B_{\bullet} & := & B_{\#}\, F^{\#}_{\bullet} \\ \textit{Quadratic fission rule:} & B_{\#} & := & B_{\bullet}\, (F^{\bullet}_{\#}\, F^{\#}_{\bullet})^{-1}\, F^{\bullet}_{\#} \end{array} \begin{array}{r} (153) \\ (154) \\ (155) \end{array}$$

$$A^{\bullet} := \textit{column matrix of type } (r!, 1) \tag{156}$$
$$A^{\#} := \textit{column matrix of type } (r!!, 1) \tag{157}$$
$$B_{\bullet} := \textit{row matrix of type } (r!, 1) \tag{158}$$
$$B_{\#} := \textit{row matrix of type } (r!!, 1) \tag{159}$$
$$F^{\#}_{\bullet} := \textit{rectangular matrix of type } (r!!, r!) \tag{160}$$
$$F^{\bullet}_{\#} := \textit{rectangular matrix of type } (r!, r!!) := \ {}^{\mathrm{tr}}(F^{\#}_{\bullet}) \tag{161}$$
$$H^{\bullet}_{\bullet} := \textit{square matrix of type } (r!, r!) := F^{\bullet}_{\#}\, F^{\#}_{\bullet} \tag{162}$$
$$K^{\bullet}_{\bullet} := \textit{square matrix of type } (r!, r!) := (F^{\bullet}_{\#}\, F^{\#}_{\bullet})^{-1} \tag{163}$$

For definiteness, we concentrate on the case when all r indices inside $\bullet$ are distinct. Then $r!!$ denotes some integer larger than $r!$ that only depends on the chosen type of order. For the arborescent order, there exist exactly $r!$ arborescent $\#$ compatible with a given $\bullet$ and so $r! < r!! < r!^2$.

Among all the fission rules compatible with the fission constraints, quadratic fission stands out as the only one that admits a simple matrix expression.

[42] the so-called standard coarborification rule, studied at length in §3.
[43] it minimizes the quadratic fission norm $\|B_{\bullet}\|^2_{\mathrm{fission}} := \sum_{\# \leq \bullet} < B_{\#}, B_{\#} >$.

Likewise, among all the fusion-fission transforms of type (152), the special case of arborification-coarborification stands out in at least three respects :

(i) it gives rise, in the group algebra $\mathbb{A}(\mathbb{S}_r)$ *of the symmetric group* $\mathbb{S}_r$*, to a pair of elements* $(\underline{\mathbf{has}}, \underline{\mathbf{kas}})$ *which, despite being mutually inverse, are both expressible by simple, totally explicit formulae.*

(ii) after normalisation to $(\mathbf{has}, \mathbf{kas})$ *under the condition* $\sum_{\sigma\in\mathbb{S}_r} \mathbf{has}(\sigma) = \sum_{\sigma\in\mathbb{S}_r} \mathbf{kas}(\sigma) = 1$*, these elements in turn give rise to a pair of moulds* $(has^\bullet, kas^\bullet)$ *which are unexpectedly simple, extend to the whole of* $\mathbb{N}$ *and even* $\mathbb{Z}$*, and are of symmetral type.*

(iii) both as moulds and σ*-functions, the above objects extend naturally to a two-parameter family, the* haukian *objects, which possess a wealth of rather improbable properties, all the more remarkable for completely disappearing when we substitute for the arborescent order any other type of order.*

5.2 The symmetric group algebras and σ-functions.

Throughout, $\mathbb{S}_r$ shall denote the group of all permutations σ of $\{1, \ldots, r\}$ and $\mathbb{A}(\mathbb{S}_r)$ shall be the corresponding group algebra, relative to the standard convolution product $*$. As for the σ-functions, they are functions $\sigma \mapsto \mathbf{h}(\sigma)$ that are defined simultaneously and uniformly on all groups $\mathbb{S}_r$. Most of the σ-functions $\mathbf{h}, \mathbf{k}$ we shall encounter will stand in natural relation to integer-indexed moulds $h^\bullet, k^\bullet$. They will also possess simple invariance properties under a finite group of order 8 that acts on all $\mathbb{S}_r$. This "octo-group" consists of the following operations $\{o_0, o_1, \ldots, o_7\}$:

$$o_0 : \quad \sigma \mapsto o_0\,\sigma := \sigma \tag{164}$$
$$o_1 : \quad \sigma \mapsto o_1\,\sigma := \sigma^{-1} \tag{165}$$
$$o_2 : \quad \sigma \mapsto o_2\,\sigma := rev\ \ \sigma\ \ rev \tag{166}$$
$$o_3 : \quad \sigma \mapsto o_3\,\sigma := rev\ \ \sigma^{-1}\ rev \tag{167}$$
$$o_4 : \quad \sigma \mapsto o_4\,\sigma := rev\ \ \sigma \tag{168}$$
$$o_5 : \quad \sigma \mapsto o_5\,\sigma := \sigma\ \ rev \tag{169}$$
$$o_6 : \quad \sigma \mapsto o_6\,\sigma := rev\ \ \sigma^{-1} \tag{170}$$
$$o_7 : \quad \sigma \mapsto o_7\,\sigma := \sigma^{-1}\ rev \tag{171}$$

with $rev = rev_r \in \mathbb{S}_r$ denoting the particular permutation ("reversion") such that $rev(i) + i \equiv r + 1$. We shall refer to $o_1\,\sigma$ and $o_2\,\sigma$ as the *inverse* and *reverse* of σ. Apart from the unit element o_0, the *octo-group* comprises five involutions $o_1, \ldots, o_5$ and two elements o_6, o_7 of order 4.

5.3 Quadratic coarborification and the fully explicit σ-functions has, kas .

σ-functions $(\mathbf{has},\mathbf{kas})$ induced by the matrices $(H^{\bullet}_{\bullet},K^{\bullet}_{\bullet})$:
For any fusion-fission transform, the matrices $H^{\bullet}_{\bullet}$ and $K^{\bullet}_{\bullet}$ are clearly invertible[44], symmetric, and of the form :

$$H^{n'_1,\dots,n'_r}_{n_1,\dots,n_r}=\underline{\mathbf{h}}(\sigma)\quad,\quad K^{n'_1,\dots,n'_r}_{n_1,\dots,n_r}=\underline{\mathbf{k}}(\sigma)\quad \textit{with}\quad n'_i\equiv n_{\sigma(i)}\ ,\ \sigma\in\mathbb{S}_r \tag{172}$$

So, knowing $(H^{\bullet}_{\bullet},K^{\bullet}_{\bullet})$ reduces to knowing the induced σ-functions $(\underline{\mathbf{h}},\underline{\mathbf{k}})$. Moreover, since $H^{\bullet}_{\bullet}$ and $K^{\bullet}_{\bullet}$ are symmetric and mutually inverse, we have :

$$\underline{\mathbf{h}}(\sigma^{-1})\equiv\underline{\mathbf{h}}(\sigma)\quad,\quad \underline{\mathbf{k}}(\sigma^{-1})\equiv\underline{\mathbf{k}}(\sigma)\quad,\quad \underline{\mathbf{h}}*\underline{\mathbf{k}}=\mathbf{1}_{\mathbb{A}(\mathbb{S}_r)} \tag{173}$$

For a general fusion-fission transform, this is about all there is to say. But for the arborification-coarborification transform, $(\underline{\mathbf{h}},\underline{\mathbf{k}})$ specialises to a highly remarkable pair $(\underline{\mathbf{has}},\underline{\mathbf{kas}})$, which becomes easier to handle when normalised to $(\mathbf{has},\mathbf{kas})$ under the condition :

$$\sum_{\sigma\in\mathbb{S}_r}\mathbf{has}(\sigma)=\sum_{\sigma\in\mathbb{S}_r}\mathbf{kas}(\sigma)=1 \tag{174}$$

We shall now succinctly describe these two objects and their teeming progeny.

Direct expression of $\mathbf{has}(\sigma)$:

$$\underline{\mathbf{has}}(\sigma)\ :=\ \prod_{1\le j\le r}\beta_j(\sigma)\qquad\in\mathbb{N}\qquad(\forall\sigma\in\mathbb{S}_r) \tag{175}$$

$$\mathbf{has}(\sigma)\ :=\ \prod_{1\le j\le r}\frac{2\,\beta_j(\sigma)}{j\,(j+1)}=\frac{1}{h_r}\,\underline{\mathbf{has}}(\sigma)\quad\in\mathbb{Q}^+\qquad(\forall\sigma\in\mathbb{S}_r) \tag{176}$$

$$\textit{with}\quad \beta_j(\sigma)\ :=\ \#\,\{\,i\ :\ 1\le i\le j\ ,\ \sigma(i)\le\sigma(j)\} \tag{177}$$

$$\textit{and}\qquad h_r\ :=\ 2^{-r}\,r!\,(r+1)!\qquad\in\mathbb{N} \tag{178}$$

These formulas easily follow from the interpretation of $\underline{\mathbf{has}}(\sigma)$ as the number of arborescent sequences that are order-compatible with both $\{1,\dots,r\}$ and $\{\sigma(1),\dots,\sigma(r)\}$. More unexpected is the existence of a closed expression for the convolution inverse $\underline{\mathbf{kas}}(\sigma)$.

[44] because of their interpretation in terms of norm minimisation. See §5.1

Direct expression of kas(σ) **.** We have :

$$\underline{\mathbf{kas}}(\sigma) \;:=\; \frac{1}{h_r} \sum_{\mathcal{P}(0,\sigma)\in \mathrm{Coher}(0,\sigma)} \mathrm{sign}(\mathcal{P}(0,\sigma))\,.\,\mathcal{P}(0,\sigma)! \qquad \in \mathbb{Q} \quad (179)$$

$$= \frac{1}{h_r}\, \boldsymbol{\xi}(\sigma) \sum_{\mathcal{P}(0,\sigma)\in \mathrm{Coher}(0,\sigma)} \mathcal{P}(0,\sigma)! \qquad (180)$$

$$\mathbf{kas}(\sigma) \;:=\; \boldsymbol{\xi}(\sigma) \sum_{\mathcal{P}(0,\sigma)\in \mathrm{Coher}(0,\sigma)} \mathcal{P}(0,\sigma)! \qquad \in \mathbb{Z} \quad (181)$$

with elementary summands defined by :

$$\mathcal{P}(0,\sigma)! \;:=\; \frac{r!\,(r-1)!}{\prod_i\,[p_i(p_i-1)]^*.\,[q_i(q_i-1)]^*} \qquad (182)$$

$$\textit{with} \qquad [x]^* := x \;(\textit{resp}\ 1)\ \textit{if}\ \ x>0\ \ (\textit{resp}\ \textit{if}\ \ x=0) \qquad (183)$$

or equivalently :

$$\mathcal{P}(0,\sigma)! \;:=\; \prod_i \;\mathrm{ca}^*_{p_i,q_i} \qquad (184)$$

$$\textit{with}\quad \mathrm{ca}^*_{p,q} \;:=\; \frac{(p+q-1)!\,(p+q-2)!}{p!(p-1)!\,q!\,(q-1)!} \qquad (185)$$

The sums extend to all *maximal coherent binary bracketings*[45] of the sequence $0, \sigma(1), \ldots, \sigma(r)$. *Maximal binary bracketings* are systems of nested pairs of brackets. They correspond one-to-one to *binary trees.* The *coherence* condition means that the integers within each bracket should be some permutation of *consecutive* integers $(s, s+1, \ldots)$. Thus, 'holes' are prohibited. As for the products (182),(184), they extend to all pairs i of nested brackets or, equivalently, to all *nodes* i in the associated binary tree. Each of these pairs (or nodes) involves a sequence $\mathbf{p^i}$ of length p_i in the left bracket and a sequence $\mathbf{q^i}$ of length q_i in the right bracket, and gives rise to two factors :
(i) the integer factor $ca^*_{p_i,q_i}$ defined above
(ii) a sign factor which is 1 (resp -1) if $\mathbf{p^i} < \mathbf{q^i}$ (resp. $\mathbf{p^i} > \mathbf{q^i}$), meaning of course that each element of $\mathbf{p^i}$ is less (resp. greater) than each element of $\mathbf{q^i}$. Multiplied together, the factors $ca^*_{p_i,q_i}$ yield the "factorial" $\mathcal{P}(0,\sigma)!$ and the sign factors yield the global $sign(\mathcal{P}(0,\sigma))$. This global sign is actually independent of the bracketing $\mathcal{P}$. It depends solely on the permutation σ. So it

[45] when no such bracketings exist (which becomes possible for $r \geq 4$, and tends to occur with a probability approaching 1 as r increases), then of course the right-hand side of (179) should be taken as 0.

may be denoted as $\boldsymbol{\xi}(\sigma)$ and factored out of the sum on the right-hand side of (180). Beware that $\boldsymbol{\xi}(\sigma)$ is *not* the permutation's *signature* $\boldsymbol{\epsilon}(\sigma)$.

Let us show on two examples how the above rules work.
First, let $r = 4$ and $(0, \sigma) = (0, 2, 1, 4, 3)$. We find only two coherent bracketings. Here they are, along with the attached factors :

$$\Big(0\Big)\Big(\big((2)(1)\big)\big((4)(3)\big)\Big) \quad \Longrightarrow \quad \mathrm{ca}^*_{1,4}.\mathrm{ca}^*_{2,2}.\mathrm{ca}^*_{1,1}.\mathrm{ca}^*_{1,1} = 3 \tag{186}$$

$$\Big((0)\big((2)(1)\big)\Big)\Big((4)(3)\Big) \quad \Longrightarrow \quad \mathrm{ca}^*_{3,2}.\mathrm{ca}^*_{1,2}.\mathrm{ca}^*_{1,1}.\mathrm{ca}^*_{1,1} = 6 \tag{187}$$

The global sign factor being $(-1) \times (-1) = 1$, we find $\mathbf{kas}(\sigma) = 9$.
Now, consider the case $r = 4$ and $(0, \sigma) = (0, 3, 1, 4, 2)$. It is easy to check that there exits no coherent bracketing here. Therefore $\mathbf{kas}(\sigma) = 0$.

Normalisation :
The reason for normalising $(\underline{\mathbf{has}}, \underline{\mathbf{has}})$ to $(\mathbf{has}, \mathbf{has})$ is that the latter form alone leads to an interesting mould extension. In this context, let us record the two parallel formulas :

$$\sum_{\sigma\in\mathbb{S}_r} \underline{\mathbf{has}}(\sigma) \quad = \quad \Big(\sum_{\sigma\in\mathbb{S}_r} \underline{\mathbf{kas}}(\sigma)\Big)^{-1} = 2^{-r}\, r!\,(r+1)! \tag{188}$$

$$\sum_{\sigma\in\mathbb{S}_r} \boldsymbol{\epsilon}(\sigma)\, \underline{\mathbf{has}}(\sigma) \quad = \quad \Big(\sum_{\sigma\in\mathbb{S}_r} \boldsymbol{\epsilon}(\sigma)\, \underline{\mathbf{kas}}(\sigma)\Big)^{-1} = \mathrm{ent}(\frac{r}{2})\,!\ \mathrm{ent}(\frac{r+1}{2})\,! \tag{189}$$

with $\boldsymbol{\epsilon}(\sigma) :=$ *signature of* σ and *ent*$(x) :=$ *integer part of* x.

5.4 The associated moulds $has^\bullet, kas^\bullet$.

Definition of $has^{\mathbf{n}}$ and $kas^{\mathbf{n}}$ for arbitrary positive sequences $\mathbf{n}$:
The relations

$$has^{\sigma(1),\ldots,\sigma(r)} := \mathbf{has}(\sigma) \quad , \quad kas^{\sigma(1),\ldots,\sigma(r)} := \mathbf{kas}(\sigma) \quad , \tag{190}$$

define $has^{\mathbf{n}}, kas^{\mathbf{n}}$ for any *standard* sequence $\mathbf{n}$ of length r, i.e. for any permutation of $\{1, \ldots, r\}$. Now, any sequence of positive integers $\mathbf{n}$, of length r, coherent or not, but *without repetitions*, may, for r^* large enough, be embedded in a *standard* sequence $\mathbf{n}^*$ of length r^*. Surprisingly, the following two sums :

$$has^{\mathbf{n}} := \sum_{\mathbf{n}^*\in Standard(r^*),\mathbf{n}^*\ni\mathbf{n}} has^{\mathbf{n}^*} \qquad (independent\ of\, r^*) \tag{191}$$

$$kas^{\mathbf{n}} := \sum_{\mathbf{n}^*\in Standard(r^*),\mathbf{n}^*\ni\mathbf{n}} has^{\mathbf{n}^*} \qquad (independent\ of\, r^*) \tag{192}$$

which range through all $r^*!/(r^*-r)!$ standard sequences $\mathbf{n}^*$ containing $\mathbf{n}$, *do not depend* on the choice of r^*. Thus $has^\bullet$ and $kas^\bullet$ possess a natural extension to all positive, repetition-free sequences $\mathbf{n}$.

Symmetrality of $has^\bullet$ (conditional) and $kas^\bullet$ (unconditional) :
The two moulds so defined are *symmetral*:

$$\sum_{\mathbf{n}\in \mathrm{sha}(\mathbf{n^1},\mathbf{n^2})} has^{\mathbf{n}} \equiv has^{\mathbf{n^1}}\, has^{\mathbf{n^2}} \qquad \mathit{if}\ \mathbf{n^1},\mathbf{n^2},\mathbf{n^1n^2} \in \mathit{coherent} \tag{193}$$

$$\sum_{\mathbf{n}\in \mathrm{sha}(\mathbf{n^1},\mathbf{n^2})} kas^{\mathbf{n}} \equiv kas^{\mathbf{n^1}}\, kas^{\mathbf{n^2}} \qquad \forall\, \mathbf{n^1}\neq\emptyset, \forall\, \mathbf{n^2}\neq\emptyset \tag{194}$$

but whereas the first identity is conditional on all three sequences $\mathbf{n^1},\mathbf{n^2},\mathbf{n^1n^2}$ being *coherent*[46], the second identity holds in all cases, at least whenever it makes sense, i.e. for any repetition-free sequences $\mathbf{n^1},\mathbf{n^2},\mathbf{n^1n^2}$.

Form preservation under arborification :
The direct expressions for $\mathbf{has}(\sigma),\mathbf{kas}(\sigma)$ carry over trivially to $has^{\mathbf{n}},kas^{\mathbf{n}}$, at least for standard $\mathbf{n}$, but they also carry over, almost unchanged, to the arborified variants $has^{\mathbf{n}^\prec},kas^{\mathbf{n}^\prec}$. For instance, (181) remains in force, with maximal binary bracketings as in (181), with the very same Catalan factors and sign rule, and a "coherence" condition which demands that each parenthesis should contain
(i) some *coherent* subsequence
(ii) some *connected* portion of the original tree $\mathbf{n}^\prec$

Factorisation properties of $kas^{\mathbf{n}}$:
Any sequence $\mathbf{n}$ of positive integers factors uniquely into a product of maximal coherent sequences $\mathbf{n^1n^2}\dots\mathbf{n^k}$ and so too does the mould $kas^\bullet$:

$$kas^{\mathbf{n}} \equiv kas^{\mathbf{n^1}}kas^{\mathbf{n^2}}\dots kas^{\mathbf{n^k}} \ \ \mathit{if}\ \ \mathbf{n^1}<\mathbf{n^2}\dots<\mathbf{n^k} \tag{195}$$

$$\equiv 0 \quad \mathit{otherwise} \tag{196}$$

No such rule holds for $has^{\mathbf{n}}$, but this is immaterial, as the direct definition is so simple.

Shift parameter of $has^{\mathbf{n}}$ and $kas^{\mathbf{n}}$:
For any sequence $\mathbf{n}=(n_1,\dots,n_r)$ and any shift parameter $s\in\mathbb{N}$ let us set ${}^{\mathbf{s}}\mathbf{n} := (s+n_1,\dots,s+n_r)$. The shift-dependence of $has^{{}^s\mathbf{n}}$ and $kas^{{}^s\mathbf{n}}$ turns out to be remarkably simple. It is:

[46] i.e. permutations of *unbroken* integer sequences.

(i) rational[47] of degree at most $2.r$ for the former,
(ii) polynomial of degree at most r for the latter.

Extension of $has^{\mathbf{n}}$ and $kas^{\mathbf{n}}$ to arbitrary integer sequences $\mathbf{n}$:
Simply write any (repetion-free) sequence $\mathbf{n}$ as ${}^{s}\mathbf{m}$ for some positive $\mathbf{m}$ and negative s, and using rational (resp. polynomial) shift-continuation, set:

$$has^{\mathbf{n}} := has^{{}^{s}\mathbf{m}} \qquad , \qquad kas^{\mathbf{n}} := kas^{{}^{s}\mathbf{m}} \tag{197}$$

The result won't depend on the pair $(s, \mathbf{m})$, but on $\mathbf{n}$ alone. *Symmetrality* also is guaranteed by construction, and so too is the persistence of the factorisation (195). The only hurdle, namely the occurence of s-poles which may render $has^{\mathbf{n}}$ (but not $kas^{\mathbf{n}}$) infinite for certain sequences $\mathbf{n}$ of *mixed signs*, will be removed by the introduction of a 'twist' parameter t. See below.

5.5 The twist parameter t and the shift parameter s.

Introduction of a 'twist' parameter t and survival of all essential properties of $has^{\mathbf{n}}, kas^{\mathbf{n}}$.
Fixing a real or complex parameter t, we first define $\underline{\mathbf{has}}_t$ and its normalised variant $\mathbf{has}_t$ by formulae closely patterned on (175) and (176):

$$\underline{\mathbf{has}}_t(\sigma) := \prod_{1\le j\le r} \Big(\frac{t}{2} + \beta_j(\sigma)\Big) \qquad (\sum_\sigma \underline{\mathbf{has}}_t(\sigma) \neq 1) \tag{198}$$

$$\mathbf{has}_t(\sigma) := \frac{1}{r!} \prod_{1\le j\le r} \frac{t + 2\,\beta_j(\sigma)}{t+j+1} \qquad (\sum_\sigma \mathbf{has}_t(\sigma) = 1) \tag{199}$$

$$\textit{with} \quad \beta_j(\sigma) := \#\,\{\, i \;:\; 1 \le i \le j\ ,\ \sigma(i) \le \sigma(j)\}$$

Next, we derive $\underline{\mathbf{kas}}_t$ and $\mathbf{kas}_t$ by straightforward inversion in the group algebra $\mathbb{A}(\mathbb{S}_r)$. We then construct the moulds $has_t^{\bullet}$ and $kas_t^{\bullet}$ exactly as before, successively for sequences $\mathbf{n}$ of *standard*, then *positive*, then *arbitrary* type. For this last step, we use the same trick as before, introducing a shift-parameter s and setting:

$$has_t^{\mathbf{n}} := has_t^{{}^{s}\mathbf{m}} =: has_{t,s}^{\mathbf{m}} \qquad , \qquad kas_t^{\mathbf{n}} := kas_t^{{}^{s}\mathbf{m}} =: kas_{t,s}^{\mathbf{m}} \tag{200}$$

As before, we get the bonus:
(i) of *conditional symmetrality* for $has_t^{\bullet}$ and $has_{t,s}^{\bullet}$
(ii) of *unconditional symmetrality* for $kas_t^{\bullet}$ and $kas_{t,s}^{\bullet}$.

[47] with simple poles at the points $s = -2, -3, \dots, -r-1$.

Broadly speaking, all known properties of $has^\bullet$ and $kas^\bullet$ seem to survive the introduction of the 'twist' parameter t. The t-dependence itself closely resembles the s-dependence: *rational* for $has_t^\bullet$ and *polynomial* for $kas_t^\bullet$. Actually, the *shift* and *twist*[48] parameters coexist and commingle amicably, and the t-dependence even turns out to be the simpler of the two.

Twist- and shift-dependence of $has^\bullet$ **.**
$has^{\mathbf{n}}_{t,s}$ is a rational function of t, s, of total degree no larger than $2\,r'$, and with at most r' simple poles of the form $t+s+1+k$, $\inf(\mathbf{n}) \leq k \leq \sup(\mathbf{n})$. Note that here r' is not the *length* r of $\mathbf{n}$, but its *span* $:= 1+\sup(\mathbf{n})-\inf(\mathbf{n})$.

Twist- and shift-dependence of $kas^\bullet$ **.**
$kas^{\mathbf{n}}_{t,s}$ is a polynomial in (t,s), of t-degree at most $r-1$, of s-degree at most $2\,r-2$, and of total (t,s)-degree also no larger than $2\,r-2$. The main thing, however, is the existence of a closed expresssion for $kas^{\mathbf{n}}_{t,s}$. First, we set[49]:

$$\mathrm{ca}^*_{p,q} := \frac{(p+q-2)!\,(p+q-1)!}{(p-1)!\,(q-1)!\,(p)!\,q!} \tag{201}$$

$$\mathrm{ca}^*_{p,q}(t) := \frac{(p+q-2)!\,(p+q-1+t)!}{(p-1)!\,(q-1)!\,(p+t)!\,q!} \in \mathbb{Z}[t] \tag{202}$$

$$\mathrm{ca}^*_{p,q}(t,s) := \frac{(p+q-2+s)!\,(p+q-1+t+s)!}{(p-1+s)!\,(q-1)!\,(p+t+s)!\,q!} \in \mathbb{Z}[t,s] \tag{203}$$

Next, we define mappings $P_{t,s}$ by the following induction:

$$P_{t,s} \;:\; \mathbf{n} \to \mathrm{P}_{t,s}(\mathbf{n}) \in \mathbb{Z}[t,s] \tag{204}$$

$$\mathrm{P}_{t,s}(\mathbf{n}) := 1 \quad \textit{if} \quad \mathbf{n} \;\; \textit{has length one. Otherwise}: \tag{205}$$

$$\mathrm{P}_{t,s}(\mathbf{n}) := \sum_{\mathbf{n^1}.\mathbf{n^2}=\mathbf{n}} \mathrm{ca}^*_{r_1,r_2}(s,t)\; \xi(\mathbf{n^1},\mathbf{n^2})\; \mathrm{P}_{t,s}(\mathbf{n^1})\; \mathrm{P}_{0,0}(\mathbf{n^2}) \tag{206}$$

with a sum extending to all factorisations of $\mathbf{n}$ into non-empty sequences $\mathbf{n^1}, \mathbf{n^2}$ of length r_1, r_2; and with sign coefficients defined in this way:

$$\xi(\mathbf{n^1},\mathbf{n^2}) := +1 \quad \textit{if} \quad \max(\mathbf{n^1}) < \min(\mathbf{n^2}) \tag{207}$$

$$:= -1 \quad \textit{if} \quad \max(\mathbf{n^2}) < \min(\mathbf{n^1}) \tag{208}$$

$$:= 0 \quad \textit{otherwise} \tag{209}$$

We should pay attention to the highly dissymmetric role assigned to $\mathbf{n^1}$ and $\mathbf{n^2}$ on the right-hand side of (206). Now, with all the ingredients in place,

[48] this is a mere label, of course: the *twist* attached to $has^\bullet$ and $kas^\bullet$ bears no relation to the one attached to the resurgence monomials.

[49] of course, for $x \notin \mathbb{N}$, $x!$ means $\Gamma(x+1)$.

we may write down the required formulas for any $\sigma \in \mathbb{S}_r$. They read :

$$\begin{aligned} \mathbf{kas}(\sigma) &= kas^{\mathbf{n}} &:=& \ \mathrm{P}_{0,0}({}^{-}\mathbf{n}) && (210)\\ \mathbf{kas}_{t}(\sigma) &= kas^{\mathbf{n}}_{t} &:=& \ \mathrm{P}_{t,0}({}^{-}\mathbf{n}) && (211)\\ \mathbf{kas}_{t,s}(\sigma) &= kas^{\mathbf{n}}_{t,s} &:=& \ \mathrm{P}_{t,s}({}^{-}\mathbf{n}) && (212) \end{aligned}$$

with $\mathbf{n} := (\sigma(1), \ldots, \sigma(r))$, ${}^{-}\mathbf{n} := (0, \sigma(1), \ldots, \sigma(r))$.

For future use let us also define a related, parameter-free σ-function **ka** :

$$\mathbf{ka}(\sigma) \ = \ ka^{\mathbf{n}} \ := \ \mathrm{P}_{0,0}(\mathbf{n}) \qquad (\mathbf{n} \ \textit{here, not} \ {}^{-}\mathbf{n}\,!) \qquad (213)$$

The corresponding mould $ka^{\bullet}$ turns out to be *alternal.*

Let us point out, lastly, that $\mathbf{has}_{t,\,s}, \mathbf{kas}_{t,\,s}$ are mutually inverse in $\mathbb{A}(\mathbb{S}_r)$ only for $s = 0$. For other values of s, the inverse of $\mathbf{has}_{t,\,s}$ is unremarkable, and that of $\mathbf{kas}_{t,\,s}$ is remarkable (i.e. factorisable and explicitable) only for $s \in \{0, -1, \ldots, -r\}$.

5.6 Basic symmetries for has, kas.

These σ-functions present a large number of symmetries, which involve the 'octo-group' (see §5.2) and become easier to write down after suitable parameter changes $(t, s) \to (t', s')$ *or* (t'', s'') that mix up *twist*, *shift*, and *length.*

First, we have the parity relations in σ (or o_1-invariance):

$$\begin{aligned} \mathbf{has}_t(\sigma) &\equiv \mathbf{has}_t(\sigma^{-1}) && \forall \sigma \in \mathbb{S}_r && (214)\\ \mathbf{kas}_t(\sigma) &\equiv \mathbf{kas}_t(\sigma^{-1}) && \forall \sigma \in \mathbb{S}_r && (215)\\ \mathbf{has}_{t,s}(\sigma) &\neq \mathbf{has}_{t,s}(\sigma^{-1}) && \textit{generally} && (216)\\ \mathbf{kas}_{t,s}(\sigma) &\equiv \mathbf{kas}_{t,s}(\sigma^{-1}) && \forall \sigma \in \mathbb{S}_r && (217) \end{aligned}$$

Now to the symmetries proper. It is convenient to set :

$$\begin{aligned} \mathbf{has}_{\{t',s'\}}(\sigma) &:= \mathbf{has}_{t'-1,s'-\frac{r}{2}-\frac{1}{2}}(\sigma) && \forall \sigma \in \mathbb{S}_r && (218)\\ \mathbf{kas}_{\{t'',s''\}}(\sigma) &:= \mathbf{kas}_{2\,t''-1,s''-t''-\frac{r}{2}}(\sigma) && \forall \sigma \in \mathbb{S}_r && (219) \end{aligned}$$

The σ-function **has** is invariant under one involution only:

$$\{t', s', \sigma\} \ \longmapsto \ \{-t', -s', o_4\sigma\} \qquad (\textit{recall that } o_4\sigma := \textit{rev} \,.\, \sigma) \ (220)$$

but the σ-function **kas** is invariant under 11 involutions (4 independent):

$$\{t'',s'',\sigma\} \longmapsto \{t'',s'',o_1\sigma\} \qquad (\textit{recall that } o_1\sigma := \sigma^{-1}) \tag{221}$$
$$\{t'',s'',\sigma\} \longmapsto \{-t'',s'',o_0\,\sigma\} \qquad (\textit{recall that } o_0\sigma := \sigma) \tag{222}$$
$$\{t'',s'',\sigma\} \longmapsto \{-t'',s'',o_1\sigma\} \tag{223}$$
$$\{t'',s'',\sigma\} \longmapsto \{t'',-s'',o_2\sigma\} \qquad (\textit{recall that } o_2\sigma := \textit{rev}\,\sigma^{-1}\,\textit{rev}) \tag{224}$$
$$\{t'',s'',\sigma\} \longmapsto \{t'',-s'',o_3\sigma\} \tag{225}$$
$$\{t'',s'',\sigma\} \longmapsto \{-t'',-s'',o_2\sigma\} \tag{226}$$
$$\{t'',s'',\sigma\} \longmapsto \{-t'',-s'',o_3\sigma\} \tag{227}$$
$$\{t'',s'',\sigma\} \longmapsto \{s'',t'',o_0\sigma\} \qquad \textit{if } \boldsymbol{\xi}(\sigma)=- \textit{ and with factor } (-1)^{r-1} \tag{228}$$
$$\{t'',s'',\sigma\} \longmapsto \{s'',t'',o_1\sigma\} \qquad \textit{if } \boldsymbol{\xi}(\sigma)=- \textit{ and with factor } (-1)^{r-1} \tag{229}$$
$$\{t'',s'',\sigma\} \longmapsto \{s'',t'',o_2\sigma\} \qquad \textit{if } \boldsymbol{\xi}(\sigma)=- \textit{ and with factor } (-1)^{r-1} \tag{230}$$
$$\{t'',s'',\sigma\} \longmapsto \{s'',t'',o_3\sigma\} \qquad \textit{if } \boldsymbol{\xi}(\sigma)=- \textit{ and with factor } (-1)^{r-1} \tag{231}$$

The next symmetries involve a σ-function **lokas** derived from **kas** by taking the (mould) logarithm of the corresponding moulds, but *after reversion to the original* (s,t) *parameters*, like this:

σ-functions		*moulds*		*moulds*		*σ-functions*
$\mathbf{kas}_{t,s}$	$\longrightarrow$	$kas^{\bullet}_{t,s}$	$\overset{\textit{mould logarithm}}{\longrightarrow}$	$lokas^{\bullet}_{t,s}$	$\longrightarrow$	$\mathbf{lokas}_{t,s}$
$\downarrow$						$\downarrow$
$\mathbf{kas}_{\{t'',s''\}}$		(*symmetral*)		(*alternal*)		$\mathbf{lokas}_{\{t'',s''\}}$

This σ-function **lokas** is invariant under 7 involutions (3 independent) :

$$\{t'',s'',\sigma\} \longmapsto \{t'',s'',o_5\sigma\} \qquad \textit{with factor } (-1)^{r-1} \tag{232}$$
$$\{t'',s'',\sigma\} \longmapsto \{-t'',s'',o_0\,\sigma\} \tag{233}$$
$$\{t'',s'',\sigma\} \longmapsto \{-t'',s'',o_5\sigma\} \qquad \textit{with factor } (-1)^{r-1} \tag{234}$$
$$\{t'',s'',\sigma\} \longmapsto \{t'',-s'',o_2\sigma\} \tag{235}$$
$$\{t'',s'',\sigma\} \longmapsto \{t'',-s'',o_4\sigma\} \qquad \textit{with factor } (-1)^{r-1} \tag{236}$$
$$\{t'',s'',\sigma\} \longmapsto \{-t'',-s'',o_0\sigma\} \tag{237}$$
$$\{t'',s'',\sigma\} \longmapsto \{-t'',-s'',o_5\sigma\} \qquad \textit{with factor } (-1)^{r-1} \tag{238}$$

These new symmetries are as unexpected as the previous ones. In particular, they are *no direct consequences* of the symmetries for **kas** [50]

[50] indeed, due to the non-linearity of the taking of mould logarithms, the $(t,s) \leftrightarrow (t'',s'')$ shuttle has the effect of mixing up quite distinct sequence lengths.

5.7 Factorisation properties for has, kas.

The factorisaton property for **kas** already encountered in §5.4 survives the introduction of the twist and shift parameters. For any repetition-free integer sequence $\mathbf{n}$ with its decomposition $\mathbf{n^1} \ldots \mathbf{n^k}$ into a product of coherent factor sequences, we still have:

$$\begin{aligned} kas^{\mathbf{n}}_{t,\,s} \;&\equiv\; kas^{\mathbf{n}^1}_{t,\,s}\, kas^{\mathbf{n}^2}_{t,\,s} \ldots kas^{\mathbf{n}^k}_{t,\,s} \qquad && \textit{if} \quad \mathbf{n}^1 < \mathbf{n}^2 \ldots < \mathbf{n}^k && (239)\\ &\equiv\; 0 && \textit{otherwise} && (240)\end{aligned}$$

In combination with the formula (212), which already settles the case of coherent sequences $\mathbf{n}$, the rule (239) covers all possible cases.

Moreover, if a sequence $\mathbf{n}$ contains indices n_i of both signs, we have a further factorisation result:

$$\begin{aligned} kas^{\mathbf{n}}_{t,\,s} \;&\equiv\; kas^{\mathbf{n}^1}_{t,\,s}\, kas^{\mathbf{n}^2}_{t,\,s} \qquad && \textit{if} \quad \mathbf{n} = \mathbf{n}^1 . \mathbf{n}^2 \;\; \textit{with} \;\; \mathbf{n}^1 \leq 0 < \mathbf{n}^2 && (241)\\ &=\; 0 && \textit{otherwise} && (242)\end{aligned}$$

5.8 Proofs: main steps.

Catalan numbers and polynomials.

$$\begin{aligned} \mathrm{ca}_n \;&:=\; \frac{(2\,n)!}{n!(n+1)!} \qquad & \mathrm{ca}_{p,q} \;&:=\; \frac{(p+q)!}{p!\,q!}\,\frac{(p+q+1)!}{(p+1)!\,(q+1)!}\\ \mathrm{ca}_n(t) \;&:=\; \frac{(2\,n+t)!}{n!(n+1+t)!} & \mathrm{ca}_{p,q}(t) \;&:=\; \frac{(p+q)!}{p!\,q!}\,\frac{(p+q+1+t)!}{(p+1+t)!\,(q+1)!}\\ \mathrm{ca}_n(t,s) \;&:=\; \frac{(2\,n+t+s)!}{n!(n+1+t+s)!} & \mathrm{ca}_{p,q}(t,s) \;&:=\; \frac{(p+q+s)!}{(p+s)!\,q!}\,\frac{(p+q+1+t+s)!}{(p+1+t+s)!\,(q+1)!}\end{aligned}$$

They relate under $\mathrm{ca}_n = \mathrm{ca}^*_{n+1}$ and $\mathrm{ca}_{p,q} = \mathrm{ca}^*_{p+1,q+1}$ to the earlier coefficients and polynomials, but are sometimes more convenient. Useful identities:

$$\mathrm{ca}_n \equiv \sum_{\substack{p+q\,=\,n-1,\\ p\geq 0\,,\,q\geq 0}} \mathrm{ca}_{p,q} \qquad ; \qquad \mathrm{ca}_n(t) \equiv \sum_{\substack{p+q\,=\,n-1,\\ p\geq 0\,,\,q\geq 0}} \mathrm{ca}_{p,q}(t) \qquad (243)$$

Induction for $has^\bullet$ and $has^\bullet_t$.
It is implicit in the factorisation rule

Induction for $kas^\bullet$ and $kas^\bullet_t$.
Thanks to the factorisation property (239) we may limit ourselves to *coherent* sequences $\mathbf{n}$, and by playing on the shift parameter s, we may even assume $\mathbf{n}$ to be some permutation of the basic sequence $(1, \ldots, r)$. That leaves the distinction between *normal* and *antinormal* sequences, depending on whether the smallest element 1 precedes or follows the largest element r. The simpler

induction rules apply for antinormal sequences. As usual, we have the choice between two (non-trivially equivalent) variants, the one privileging the smallest element, the other the largest. They go like this:

For antinormal sequences $\mathbf{n}=(...r...1...)=(\mathbf{a},1,\mathbf{b})=(\mathbf{c},r,\mathbf{d})$ of length r :

$$kas_{t,s}^{\mathbf{a},1,\mathbf{b}} := \mathrm{ca}_{s,r-1}(t)\,(-1)^{r(\mathbf{a})}\,kas_{t,-1}^{\tilde{\mathbf{a}}}\,kas_{t,-1}^{\mathbf{b}} \tag{244}$$

$$kas_{t,s}^{\mathbf{c},r,\mathbf{d}} := \mathrm{ca}_{s,r-1}(t)\,(-1)^{r(\mathbf{d})}\,kas_{t,r}^{-\tilde{\mathbf{c}}}\,kas_{t,r}^{-\mathbf{d}} \tag{245}$$

with $\;^{\sim}(n_1,\ldots,n_r) := (n_r,\ldots,n_1)$ and $\;^{-}(n_1,\ldots,n_r) := (-n_1,\ldots,-n_r)$.
For normal sequences $\mathbf{n} = (...1...r...)$, $\tilde{\mathbf{n}}$ is antinormal[51] and the rule is:

$$kas_{t,s}^{\mathbf{n}} := (-1)^{r-1}kas_{t,s}^{\tilde{\mathbf{n}}} + \sum_{2\le k\le r(\mathbf{n})} (-1)^{r-k} \sum_{\mathbf{n^1 n^2 \ldots n^k}=\mathbf{n}} kas_{t,s}^{\tilde{\mathbf{n}}^{\mathbf{1}}}\,kas_{t,s}^{\tilde{\mathbf{n}}^{\mathbf{2}}} \ldots kas_{t,s}^{\tilde{\mathbf{n}}^{\mathbf{k}}} \tag{246}$$

Main steps: One checks that the elementary induction for **has** and $\mathbf{has}_t$ translates into the above induction for **kas** and $\mathbf{kas}_t$. Then one shows that the latter agrees (is equivalent) with the direct expressions (181) for **kas** and (211) for $\mathbf{kas}_t$.

5.9 Factorisation properties for the connecting functions hak , häk.

Fix t_1, t_2. Since $\mathbf{has}_{t_1}$, $\mathbf{kas}_{t_2}$ are *even* σ-functions[52], it is readily seen that all $2 \times 8 \times 8$ convolution products of the form $(o_i\,\mathbf{has}_{t_1}) * (o_j\,\mathbf{kas}_{t_2})$ and $(o_j\,\mathbf{kas}_{t_2}) * (o_i\,\mathbf{has}_{t_1})$, with $0 \le i,j \le 7$, actually reduce, modulo the o_i-action of the octo-group, to just two of them, e.g. $\mathbf{hak}_{t_1,t_2}$ and $\mathbf{h\ddot{a}k}_{t_1,t_2}$:

$$\mathbf{hak}_{t_1,t_2}(\sigma) := \mathbf{has}_{t_1} * \mathbf{kas}_{t_2}(\sigma) = \sum_{\sigma_1\sigma_2=\sigma} \mathbf{has}_{t_1}(\sigma_1)\,\mathbf{kas}_{t_2}(\sigma_2) \tag{247}$$

$$\mathbf{h\ddot{a}k}_{t_1,t_2}(\sigma) := \mathbf{has}_{t_1} * (o_4\mathbf{kas}_{t_2})(\sigma) = (o_5\mathbf{has}_{t_1}) * \mathbf{kas}_{t_2}(\sigma) \tag{248}$$

$$= \sum_{\sigma_1 .\, rev .\, \sigma_2=\sigma} \mathbf{has}_{t_1}(\sigma_1)\,\mathbf{kas}_{t_2}(\sigma_2) \tag{249}$$

But the real surprise is that both these "connecting" σ-functions should enjoy the property of *maximal factorisation*, which $\mathbf{has}_{t_1}$ already possesses, but not

[51]so the first term in (246) may be calculated according to the rule (244) or (245).
[52]i.e. invariant under the change $\sigma \mapsto \sigma^{-1}$

$\mathbf{kas}_{t_2}$.[53] Indeed, we have :

$$\mathbf{hak}_{\,t_1,t_2}(\sigma) \;:=\; \frac{1}{r!}\prod_{1\leq j\leq r}\frac{t_1+\gamma_j(\sigma)\,t_2+\delta_j(\sigma)}{t_1+j+1} \qquad \forall\sigma\in\mathbb{S}_r \tag{250}$$

$$\mathbf{häk}_{\,t_1,t_2}(\sigma) \;:=\; \frac{1}{r!}\prod_{1\leq j\leq r}\frac{t_1+\gamma_j^*(\sigma)\,t_2+\delta_j^*(\sigma)}{t_1+j+1} \qquad \forall\sigma\in\mathbb{S}_r \tag{251}$$

with coefficients γ_j , δ_j given by :

$$\begin{array}{llll}
\textit{if} \;\; \sigma(j-1)<\sigma(j)<\sigma(j+1) : & \gamma_j(\sigma) := j-1 & \delta_j(\sigma) := 2\,\beta_j(\sigma)+j^2-j \\
\textit{if} \;\; \sigma(j-1)<\sigma(j)>\sigma(j+1) : & \gamma_j(\sigma) := -1 & \delta_j(\sigma) := 2\,\beta_j(\sigma) \;\; -2\,j \\
\textit{if} \;\; \sigma(j-1)>\sigma(j)<\sigma(j+1) : & \gamma_j(\sigma) := \;\;0 & \delta_j(\sigma) := 2\,\beta_j(\sigma) \\
\textit{if} \;\; \sigma(j-1)>\sigma(j)>\sigma(j+1) : & \gamma_j(\sigma) := -j & \delta_j(\sigma) := 2\,\beta_j(\sigma)-j^2-j
\end{array}$$

and with the same $\beta_j(\sigma)$ as in the definition (175), (177) of **has**.

For $j=1$ or r, the above inequalities involve numbers $\sigma(0)$ or $\sigma(r+1)$ which are not defined, since $\sigma\in\mathbb{S}_r$, but even then one does get the correct answer by setting $\sigma(0):=0$ or $\sigma(r{+}1):=r+1$. We may also note that there is always a factor[54] t_1+2 on the numerator of (250) , which cancels the t_1+2 on the denominator. Similarly, unless $\sigma=id$, there always has to be at least one factor[55] t_1-t_2 on the numerator of (250) since $\mathbf{hak}_{t,t}(\sigma)\equiv 0$ when $\sigma\neq id$.

Analogous formulas hold for the coefficients γ_j^* , δ_j^* . In fact :

$$\gamma_j^*(\sigma) \;\equiv\; -\gamma_j(\sigma) \qquad \forall\sigma\in\mathbb{S}_r\ ,\ \forall j\in\{1,\ldots,r\} \tag{252}$$

$$\delta_j^*(\sigma) \;\equiv\; -\delta_j(\sigma)+2+2\,\sigma(j) \qquad \forall\sigma\in\mathbb{S}_r\ ,\ \forall j\in\{1,\ldots,r\} \tag{253}$$

Here again, there is always a factor[56] t_1+2 on the numerator of (251) , which cancels the one on the denominator. But since generally $\mathbf{häk}_{t,t}(\sigma)\neq 0$ there is no 'permanent' factor t_1-t_2 on the numerator of (251).

Proofs : These factorisation properties haven't been proved yet in all cases, but they have been systematically checked on a computer up to $r=9$.

[53] at least not in that sense. Its own factorisation properties (239) are of a markedly different nature.

[54] it corresponds to the *largest* value of j such that $\sigma(1)>\sigma(2)\cdots>\sigma(j)$.

[55] it corresponds to the value of j such that $\sigma(1)<\sigma(2)<\cdots<\sigma(j)>\sigma(j{+}1)$ $(\sigma\neq id)$.

[56] it corresponds to the *largest* value of j such that $\sigma(1)>\sigma(2)\cdots>\sigma(j)$.

Moreover, for a large proportion of permutations σ, they result from the three, clearly equivalent identities that follow:

$$\mathbf{kas}_t * \mathbf{ha}(\sigma) = \lambda_r(t)\ \mathbf{ka}(\sigma) \qquad \forall \sigma \in \mathbb{S}_r \tag{254}$$

$$\mathbf{has}_t * \mathbf{ka}(\sigma) = \lambda_r^{-1}(t)\ \mathbf{ha}(\sigma) \qquad \forall \sigma \in \mathbb{S}_r \tag{255}$$

$$\mathbf{hak}_{t_1,t_2} * \mathbf{ha}(\sigma) = \frac{\lambda_r(t_2)}{\lambda_r(t_1)}\ \mathbf{ha}(\sigma) \qquad \forall \sigma \in \mathbb{S}_r \tag{256}$$

$$\textit{with} \qquad \lambda_r(t) := \frac{\Gamma(t+r+2)}{\Gamma(r)\Gamma(t+3)} = \frac{(t+3)\dots(t+r+1)}{(r-1)!} \tag{257}$$

These identities involve new σ-functions $\mathbf{ha}, \mathbf{ka}$. The first is elementary, and can be read off the defining identity:

$$\sum_{\sigma \in \mathbb{S}_r} \mathbf{ha}(\sigma)\, e_{\sigma(1)} \dots e_{\sigma(r)} := [..[e_1, e_2] \dots e_r] \tag{258}$$

The other one, $\mathbf{ka}$, has already received a direct definition in (213). It is closely related to the leading t-terms in $\mathbf{kas}_t$ and $\mathbf{kas}_{t,s}$. Indeed:

$$\mathbf{ka}(\sigma) \equiv r\,(\frac{d}{dt})^{r-1}\mathbf{kas}_t(\sigma) \qquad \forall t \tag{259}$$

$$\equiv \frac{r}{\Gamma(r)}\frac{\Gamma(s+r)}{\Gamma(s+1)}\,(\frac{d}{dt})^{r-1}\mathbf{kas}_{t,s}(\sigma) \qquad \forall t, \forall s \tag{260}$$

It displays maximal symmetry under the action of the octo-group:

$$\mathbf{ka}(\sigma) = \mathbf{ka}(o_i\sigma) \qquad \forall \sigma \in \mathbb{S}_r\ ,\ \forall i \in \{0,1,2,3\} \tag{261}$$

$$\mathbf{ka}(\sigma) = (-1)^{r-1}\mathbf{ka}(o_i\sigma) \qquad \forall \sigma \in \mathbb{S}_r\ ,\ \forall i \in \{4,5,6,7\} \tag{262}$$

The corresponding moulds $ha^\bullet, ka^\bullet$ are clearly alternal.[57]

Convolution group. Link with the 'organic' family.
From the construction of the connecting functions there follow the identities:

$$\mathbf{hak}_{t_1,t_2} * \mathbf{hak}_{t_2,t_3} = \mathbf{häk}_{t_1,t_2} * \mathbf{häk}_{t_2,t_3} = \mathbf{hak}_{t_1,t_3} \qquad \forall t_1, t_2, t_3 \tag{263}$$

$$\mathbf{hak}_{t_1,t_2} * \mathbf{häk}_{t_2,t_3} = \mathbf{häk}_{t_1,t_2} * \mathbf{hak}_{t_2,t_3} = \mathbf{häk}_{t_1,t_3} \qquad \forall t_1, t_2, t_3 \tag{264}$$

$$\mathbf{hak}_{t,t} = \mathbf{häk}_{t,t} * \mathbf{häk}_{t,t} = \mathbf{1}_{\mathbb{A}(S_r)} \qquad \forall t \tag{265}$$

To derive from these a true convolution group we must take the limits:

$$\mathbf{hok}_t := \lim_{t_1 \to \infty} \mathbf{hak}_{t_1,\, t\, t_1} \tag{266}$$

$$\mathbf{hök}_t := \lim_{t_1 \to \infty} \mathbf{häk}_{t_1,\, t\, t_1} \tag{267}$$

[57] $\mathbf{ha}(\sigma) =: ha^{\sigma(1),\dots,\sigma(r)}$, $\mathbf{ka}(\sigma) =: ka^{\sigma(1),\dots,\sigma(r)}$.

We end up with much simpler σ-functions :

$$\mathbf{hok}_t(\sigma) \;\equiv\; \mathbf{h\ddot{o}k}_{-t}(\sigma) \equiv \frac{1}{r!} \prod_{1 \le j \le r} (1 + \gamma_j(\sigma)\, t) \tag{268}$$

with automatic stability under convolution ;

$$\mathbf{hok}_{t_1} * \mathbf{hok}_{t_2} \;\equiv\; \mathbf{hok}_{t_1\, t_2} \qquad\qquad \forall t_1, t_2 \tag{269}$$

and an unexpected connection with the organic mould family : see §5.19.

5.10 Yet more factorisation properties.

Two 'dual objects', namely the scalar products $\sum_{\sigma\in\mathbb{S}_r} \mathbf{has}_{t_1}(\sigma)\,\mathbf{has}_{t_2}(\sigma)$ and the convolution products $\sum_{\sigma_1\sigma_2=\sigma} \mathbf{has}_{t_1}(\sigma_1)\,\mathbf{has}_{t_2}(\sigma_2)$ evaluated at $\sigma = id$ also display, as functions of the twist parameters t_1, t_2, quite unexpected factorisation properties. Actually, this holds for all k-linear sums :

$$\sum_{\sigma\in\mathbb{S}_r} \mathbf{has}_{t_1}(\sigma)\,\mathbf{has}_{t_2}(\sigma) \ldots \mathbf{has}_{t_k}(\sigma) \;=\; \frac{N_{r,k}}{D_{r,k}} \tag{270}$$

$$\sum_{\sigma\in\mathbb{S}_r} \boldsymbol{\epsilon}(\sigma)\; \mathbf{has}_{t_1}(\sigma)\,\mathbf{has}_{t_2}(\sigma) \ldots \mathbf{has}_{t_k}(\sigma) \;=\; \frac{N'_{r,k}}{D'_{r,k}} \tag{271}$$

and also for convolutions evaluated at more general permutations $\sigma \in \mathbb{S}_r$, like for instance those acting like *rev* on $\{1, .., j_0\}$ and like *id* on $\{1{+}j_0, .., r\}$:

$$(\mathbf{has}_{t_1} * \mathbf{has}_{t_2})(\sigma) \;=\; \frac{N^*_{r,j_0}}{D^*_{r,j_0}} \tag{272}$$

Indeed, the numerators $N_{r,k}$ and $N'_{r,k}$ factor into products of r polynomials, each of total degree k, and the numerator N^*_{r,j_0} factors into a product of r quadratic polynomials [58]

There is no point in either writing down or proving the above factorisations, since they will turn out to be special cases of a more general result. Indeed, the factorisations (270),(271) will reduce to (275),(276) *infra* with $t_{ij} := \frac{1}{2}\, t_i + j$, and the factorisation (272) will reduce to (277) *infra* with $a_j := \frac{1}{2}\, t_1 + j$ and $b_j := \frac{1}{2}\, t_2 + j$.

[58] The denominators $D_{r,k}$ and D^*_{r,j_0} also break down into simple factors, but this was entirely predictable, since all terms in the sums being considered already share the same elementary factors.

Though not nearly as deep as the factorisations of the previous sections, in particular those for the mould $kas^\bullet$ or the σ-functions $\mathbf{hak}, \mathbf{häk}$, the unexpected splitting phenomenon occurring in (270),(271),(272) has one merit: when looking for the underlying mechanism, one is led quite naturally to a generalisation of the σ-functions $\mathbf{has}_t, \mathbf{kas}_t$ under with the twist parameter t is replaced by a parameter sequence $T = \{t_j\}$. The next section shall be devoted to this extension, and the subsequent sections to a search for those particular sequences T that yield the most interesting σ-functions.

5.11 Extending has, kas to haus, kaus.

Starting from any sequence $T = \{t_1, t_2, t_3 \dots\}$ we set:

$$\mathbf{haus}_T(\sigma) \;:=\; \prod_{1\le j\le r} \frac{t_{\beta_j(\sigma)}}{t_1 + t_2 \cdots + t_j} \qquad (\forall \sigma \in \mathbb{S}_r) \tag{273}$$

with $\beta_j(\sigma)$ as in (177). Normalisation is non-trivial but automatic:

$$\sum_{\sigma\in\mathbb{S}_r} \mathbf{haus}_T(\sigma) \;\equiv\; 1 \qquad \forall r \tag{274}$$

and the 'superficial' factorisations of the last section have exact analogues:

$$\sum_{\sigma\in\mathbb{S}_r} \mathbf{haus}_{T_1}(\sigma)\dots\mathbf{haus}_{T_k}(\sigma) \;=\; \prod_{1\le p\le r} \frac{\sum_{1\le j\le p}\prod_{1\le i\le k} t_{ij}}{\prod_{1\le i\le k}\sum_{1\le j\le p} t_{ij}} \tag{275}$$

$$\sum_{\sigma\in\mathbb{S}_r} \mathbf{haus}_{T_1}(\sigma)\dots\mathbf{haus}_{T_k}(\sigma)\,\epsilon(\sigma) \;=\; (-1)^{r'} \prod_{1\le p\le r} \frac{\sum_{1\le j\le p}(-1)^{1+j}\prod_{1\le i\le k} t_{ij}}{\prod_{1\le i\le k}\sum_{1\le j\le p} t_{ij}}$$

$$\textit{with}\;\; T_i = \{t_{ij}\} = \{t_{i1}, t_{i2}, t_{i3}\dots\} \quad \textit{and} \quad r = 2\,r' \quad \textit{or} \quad 2\,r'+1 \tag{276}$$

The factorisation (272) for convolution products also has an analogue:

$$\sum_{\sigma\in S_r} (\mathbf{haus}_A * \mathbf{haus}_B)(\sigma) \;=\; \prod_{1\le p\le r} \frac{\sum_{1\le j\le p} a_j\, b_{\sigma(j)}}{(\sum_{1\le j\le p} a_j).(\sum_{1\le j\le p} b_j)} \tag{277}$$

which holds for all sequences $A=\{a_1, a_2\dots\}, B=\{b_1, b_2\dots\}$ and all permutations σ of the form σ_{j_0}:

$$\sigma_{j_0}(j) = 1 + j_0 - j \quad (\textit{resp}\;\; j) \qquad \textit{if} \quad j \le j_0 \quad (\textit{resp}\quad j > j_0) \tag{278}$$

But we would also like the deeper properties of $\mathbf{has}, \mathbf{kas}$ to survive. In other words, we would like to come up with pairs $\mathbf{haus}_T, \mathbf{kaus}_T$ of mutually

inverse σ-functions such that :
(i) $\mathbf{kaus}_T(\sigma)$ has low degree denominators and is expressible in closed, transparent form
(ii) $\mathbf{kaus}_T(\sigma)$ vanishes for most [59] permutations σ
(iii) $\mathbf{haus}_T, \mathbf{kaus}_T$ admit natural mould extensions $haus_T^\bullet, kaus_T^\bullet$ with nice properties such as *symmetrality.*
(iv) there exist simple *connecting* σ-functions[60] with maximum factorisation.
(v) $\mathbf{haus}_T, \mathbf{kaus}_T$ possess simple images under most linear representations of the symmetric groups $\mathbb{S}_r$.

As it turns out, there are three types of sequences T, and only three, which answer this long wish list. They are :

$$\mathrm{Ta}_t := \left[\frac{t}{2} + n \right]_{n=1}^{n=+\infty} \quad \textit{`arithmetic sequence'} \tag{279}$$

$$\mathrm{Tu}_x := \left[\quad x^n \quad \right]_{n=1}^{n=+\infty} \quad \textit{`geometric sequence'} \tag{280}$$

$$\mathrm{Tu}_{x,t} := \left[\frac{x^n}{t} - \frac{t}{x^n} \right]_{n=1}^{n=+\infty} \quad \textit{`bigeometric sequence'} \tag{281}$$

Moreover, since $(\mathbf{haus}_T, \mathbf{kaus}_T)$ depend, not on the sequence T as such, but on its class $\widetilde{T}$ up to homotheties $\{t_1, t_2, \dots\} \mapsto \{c\,t_1, c\,t_2, \dots\}$, these three classes $\widetilde{Ta}_t$ $\widetilde{Tu}_x$, $\widetilde{Tu}_{x,t}$ constitute a two-dimensional *connected manifold.* Indeed :

$$\widetilde{\mathrm{Ta}}_t = \lim_{\epsilon \to 0} \widetilde{\mathrm{Tu}}_{1+2\,\epsilon\,,\,1-t\,\epsilon} \tag{282}$$

$$\widetilde{\mathrm{Tu}}_x = \widetilde{\mathrm{Tu}}_{x,\,0} := \lim_{t \to 0} \widetilde{\mathrm{Tu}}_{x,\,t} \tag{283}$$

Arithmetic sequences yield the familiar pair $(\mathbf{has}_t, \mathbf{kas}_t) = (\mathbf{haus}_{\mathrm{Ta}_t}, \mathbf{kaus}_{\mathrm{Ta}_t})$. So let us turn successively to the geometric and bigeometric sequences.

5.12 Restricting haus, kaus to hus, kus.

σ-functions $\mathbf{hus}_x$ and $\mathbf{kus}_x$. Setting :

$$(\mathbf{hus}_x, \mathbf{kus}_x) := (\mathbf{haus}_T, \mathbf{kaus}_T) \quad \textit{with} \quad T = \mathrm{Tu}_x := \{x, x^2, x^3, \dots\} \tag{284}$$

[59] more precisely, for all permutations that admit no *maximal coherent binary bracketing* : see §5.3

[60] i.e. σ-functions $\mathbf{hauk}_{T_1,T_2}$ such that $\mathbf{haus}_{T_1} \equiv \mathbf{hauk}_{T_1,T_2} * \mathbf{haus}_{T_2}$.

we get :

$$\mathbf{hus}_x(\sigma) \;=:\; \frac{\mathbf{hus}^*_x(\sigma)}{DH_r(x)} = \frac{x^{\beta^*(\sigma)}}{DH_r(x)} \tag{285}$$

$$\mathbf{kus}_x(\sigma) \;=:\; \frac{\mathbf{kus}^*_x(\sigma)}{DK_r(x)} \quad \textit{with} \quad \mathbf{kus}^*_x(\sigma) \in \mathbb{Z}[x^2] \;\; \textit{or} \;\; x\,\mathbb{Z}[x^2] \tag{286}$$

with simple, cyclotomic denominators :

$$DH_r(x) \;:=\; \prod_{1\leq k\leq r} \frac{1-x^k}{1-x} \tag{287}$$

$$DK_r(x) \;:=\; \prod_{2\leq k\leq r} \frac{(1-x)(1-x^{k\,(k-1)})}{(1-x^k)} \quad \textit{if} \;\; r\geq 2 \qquad (DK_1(x) := 1) \tag{288}$$

and with simple numerators. Those of $\mathbf{hus}_{\,x}$ are monomials of exponent :

$$\beta^*(\sigma) \;:=\; -r + \sum_{1\leq j\leq r} j\,\beta_j(\sigma) \qquad\qquad (\;\beta_j(\sigma)\;\; \textit{as in}\;\; (177)\;)$$

$$\equiv\;\; \#\,\{(i,j) : 1\leq i<j\leq r\;,\; \sigma(i)<\sigma(j)\} \tag{289}$$

and those of $\mathbf{kus}_{\,x}$ are *even*[61] polynomials of low degree.

Numerators of $\mathbf{hus}_x$ and $\mathbf{kus}_x$.
Unexpected as the simplicity of the denominators $DK_r(x)$ may be, the truly interesting part is the *numerators* $\mathbf{kus}^*_x$. Like with $\mathbf{kas}_{\,t}$, they depends on the *maximal coherent binary bracketings* of the sequence $\{\sigma(1),..,\sigma(r)\}$:
– when no such bracketings exist, the numerator vanishes
– when there is only one bracketing, we have maximal factorisation into cyclotomic factors
– when there are several bracketings, we get very peculiar superpositions of such products, with many residual aspects of 'cyclotomicity'.
All cases are covered by a completely explicit generalisation of formula (211) which involves the so-called *Gaussian polynomials* which are the q-analogues of the *binomial coefficients* so abundantly present in the definition of the operators $P_{t,\,s}$ of (211).

Symmetries of $\mathbf{hus}_x$ and $\mathbf{kus}_x$. With $\boldsymbol{\xi}(\sigma)$ as in (180), we have :

$$\mathbf{kus}_x(o_i\,\sigma) \;\equiv\; \mathbf{kus}_x(\sigma) \qquad \forall\sigma\in\mathbb{S}_r\,, \quad \forall i\in\{0,1,2,3\} \tag{290}$$

$$\mathbf{kus}_x(o_i\,\sigma) \;\equiv\; \mathbf{kus}_{\frac{1}{x}}(\sigma) \qquad \forall\sigma\in\mathbb{S}_r\,, \quad \forall i\in\{4,5,6,7\} \tag{291}$$

$$\mathbf{kus}_{\frac{1}{x}}(\sigma) \;\equiv\; (-1)^{r-1}\,x^{\,-\boldsymbol{\xi}(\sigma)\,\frac{r\,(r-1)}{2}}\;\mathbf{kus}_x(\sigma) \tag{292}$$

$$\mathbf{kus}^*_{-x}(\sigma) \;\equiv\; \epsilon_r\;\boldsymbol{\epsilon}(\sigma)\;\mathbf{kus}^*_x(\sigma) \tag{293}$$

[61]up to an occasional factor x, present whenever $\boldsymbol{\xi}(\sigma)=-1$.

with $\epsilon_r := 1$ if $r = 0$ or $1 \mod 4$ and $\epsilon_r := -1$ if $r = 2$ or $3 \mod 4$.

Connections between $\mathbf{kus}_x$ and $\mathbf{kas}_t$:

$$\mathbf{kas}_0(\sigma) \;\equiv\; \mathbf{kus}_1^*(\sigma^+) \qquad \forall\sigma \in \mathbb{S}_r \;,\; \sigma^+ \in \mathbb{S}_{r+1} \tag{294}$$

where σ^+ stands for the natural extension of σ to $\mathbb{S}_{r+1}$. [62]

5.13 Endowing hus, kus with a twist parameter t.

Turning now to the *bigeometric sequences*, we set :

$$(\mathbf{hus}_{x,t}, \mathbf{kus}_{x,t}) := (\mathbf{haus}_T, \mathbf{kaus}_T) \quad \textit{with} \quad T = \mathrm{Tu}_{x,t} := \left[\frac{x^n}{t} - \frac{t}{x^n}\right]_1^{+\infty} \tag{295}$$

As usual, the 'direct' σ-function $\mathbf{hus}_{x,t}$ holds no mysteries. Its numerator is elementary, and its denominator breaks up into simple factors that are immediately obtainable from the general formula (273) for $\mathbf{haus}_T$ after the substitution $t_n \rightsquigarrow \frac{x^n}{t} - \frac{t}{x^n}$.

More remarkable are the simplifications that occur with the σ-function $\mathbf{kus}_{x,t}$. Its denominator $DK_r(x,t)$ also breaks up into simple factors : we have on the one hand the cyclotomic factors of x alone, already present in the denominators $DK_r(x)$ of $\mathbf{kus}_x$, and on the other hand, in equal number, elementary factors that depend on both x and t. Explicitely :

$$\mathbf{hus}_{x,t}(\sigma) = \frac{\mathbf{hus}^*_{x,t}(\sigma)}{DH_r(x,\,t)} = \frac{\mathbf{hus}^*_{x,t}(\sigma)}{DH_r(x)\,DH^*_r(x,\,t)} \qquad \forall\sigma \in \mathbb{S}_r$$

$$\mathbf{kus}_{x,t}(\sigma) = \frac{\mathbf{kus}^*_{x,t}(\sigma)}{DK_r(x,\,t)} = \frac{\mathbf{kus}^*_{x,t}(\sigma)}{DK_r(x)\,DK^*_r(x,\,t)} \qquad \forall\sigma \in \mathbb{S}_r$$

$$DH^*_r(x,t) := \prod_{1\le k\le r} (t^2 - x^{k+1}) \tag{296}$$

$$DK^*_r(x,t) := \prod_{1\le k\le r} \frac{(t^{2k} - x^{k(k+1)})}{(t^2 - x^{k+1})} \tag{297}$$

The really non-trivial part of $\mathbf{kus}_{x,t}$, however, is its numerator. Like with $\mathbf{kas}_t$ and $\mathbf{kus}_{x,t}$, the new numerator $\mathbf{kus}^*_{x,t}(\sigma)$ depends on the *maximal coherent binary bracketings* of the sequence $\{\sigma(1),..,\sigma(r)\}$:
– when no such bracketings exist, the numerator vanishes
– when there is only one bracketing, the numerator breaks up completely into simple factors
– when there exist several bracketings, we get a superposition of such terms.
All cases are covered by a suitable generalisation of formula (212).

[62] ie $\sigma^+(j) := \sigma(j)$ for $j = 1, \ldots, r$ and $\sigma^+(r+1) := r+1$

5.14 Factorisation properties for the connecting functions huk, hük.

Their construction runs parallel to that of **hak**, **häk**. We set:

$$\mathbf{huk}_{x,t_1,t_2}(\sigma) := \mathbf{hus}_{x,t_1} * \mathbf{kus}_{x,t_2}(\sigma) = \sum_{\sigma_1\sigma_2=\sigma} \mathbf{hus}_{x,t_1}(\sigma_1)\,\mathbf{kus}_{x,t_2}(\sigma_2) \quad (298)$$

$$\mathbf{hük}_{x,t_1,t_2}(\sigma) := \mathbf{hus}_{x,t_1} * (o_4\mathbf{kus}_{x,t_2})(\sigma) = (o_5\mathbf{hus}_{x,t_1}) * \mathbf{kus}_{x,t_2}(\sigma) \quad (299)$$

$$= \sum_{\sigma_1.\,rev.\,\sigma_2=\sigma} \mathbf{hus}_{x,t_1}(\sigma_1)\,\mathbf{kus}_{x,t_2}(\sigma_2) \quad (300)$$

and we encounter once again the miracle of maximal factorisation, for both numerators and denominators:

$$\mathbf{huk}_{x,t_1,t_2}(\sigma) = \frac{\mathbf{huk}^*_{x,t_1,t_2}(\sigma)}{DHK_r(x,\,t_1,t_2)} \quad ; \quad \mathbf{hük}_{x,t}(\sigma) = \frac{\mathbf{hük}^*_{x,t_1,t_2}(\sigma)}{DHK_r(x,\,t_1,t_2)} \qquad \forall\sigma\in\mathbb{S}_r$$

$$DHK_r(x;\,t_1,t_2) := DH^*_r(x,\,t_1)\,DK^*_r(x,\,t_2) = \prod_{1\le k\le r}(t_1^2 - x^{k+1}) \prod_{1\le k\le r} \frac{(t_2^{2\,k} - x^{k(k+1)})}{(t_2^2 - x^{k+1})}$$

$$\mathbf{huk}^*_{x,t_1,t_2} = t_2^{\gamma(\sigma)}\ x^{\delta(\sigma)} \prod_{1\le j\le r}\Big(t_1\ t_2^{\gamma_j(\sigma)} + x^{\delta_j(\sigma)/2}\Big) \prod_{1\le j\le r}\Big(t_1\ t_2^{\gamma_j(\sigma)} - x^{\delta_j(\sigma)/2}\Big)$$

$$\mathbf{hük}^*_{x,t_1,t_2} = t_2^{\gamma^*(\sigma)}\ x^{\delta^*(\sigma)} \prod_{1\le j\le r}\Big(t_1\ t_2^{\gamma^*_j(\sigma)} + x^{\delta^*_j(\sigma)/2}\Big) \prod_{1\le j\le r}\Big(t_1\ t_2^{\gamma^*_j(\sigma)} - x^{\delta^*_j(\sigma)/2}\Big)$$

with the very same $\gamma_j, \delta_j, \gamma^*_j, \delta^*_j$ as in §5.9 [63] and with elementary corrective factors $t_2^{\gamma(\sigma)}x^{\delta(\sigma)}$ or $t_2^{\gamma^*(\sigma)}x^{\delta^*(\sigma)}$ which account for the global invariance under the change $(x,t_1,t_2)\to(x^{-1},t_1^{-1},t_2^{-1})$. To highlight this invariance, we may also write down our connecting functions as follows:

$$\mathbf{huk}_{x,t_1,t_2} \equiv \prod_{1\le j\le r} \frac{\mathrm{bigeo}(x^{\frac{j+1}{2}},t_2)}{\mathrm{bigeo}(x^{\frac{j+1}{2}},t_1)} \frac{\mathrm{bigeo}(x^{\frac{\delta_j(\sigma)}{2}},t_1\,t_2^{\gamma_j(\sigma)})}{\mathrm{bigeo}(x^{\frac{j(j+1)}{2}},t_2^j)} \quad (301)$$

$$\mathbf{hük}_{x,t_1,t_2} \equiv \prod_{1\le j\le r} \frac{\mathrm{bigeo}(x^{\frac{j+1}{2}},t_2)}{\mathrm{bigeo}(x^{\frac{j+1}{2}},t_1)} \frac{\mathrm{bigeo}(x^{\frac{\delta^*_j(\sigma)}{2}},t_1\,t_2^{\gamma^*_j(\sigma)})}{\mathrm{bigeo}(x^{\frac{j(j+1)}{2}},t_2^j)} \quad (302)$$

$$\textit{with}\ \ \mathrm{bigeo}(x,t) := \frac{x}{t} - \frac{t}{x} \quad (303)$$

[63] Note in passing that $\delta_j(\sigma)$ and $\delta^*_j(\sigma)$ always being *even* integers, the above products amount to *entire* factorisations.

When the parameters x, t_1, t_2 go to 1 simultaneously and with all three numbers $x-1, t_1-1, t_2-1$ in fixed ratios, we clearly retrieve as a special case the factorisations (250),(251) for the 'arithmetic' case:

$$\rightsquigarrow \prod_{1\leq j\leq r} \frac{(t_1+\gamma_j(\sigma)\,t_2\;+\;\delta_j(\sigma))}{j\;(j+1)}$$

$$\rightsquigarrow \prod_{1\leq j\leq r} \frac{(t_1+\gamma_j^*(\sigma)\,t_2\;+\;\delta_j^*(\sigma))}{j\;(j+1)}$$

5.15 The pair $\mathbf{hus}, \mathbf{kus}$ as a q-analogue of $\mathbf{has}, \mathbf{kas}$. The 'haukian' family of σ-functions.

The pairs $(\mathbf{hus}_x, \mathbf{kus}_x)$ and $(\mathbf{hus}_{x,t}, \mathbf{kus}_{x,t})$ may be looked upon as q-analogues of $(\mathbf{has}, \mathbf{kas})$ and $(\mathbf{has}_t, \mathbf{kas}_t)$ respectively, with x functioning as q-parameter. The associated moulds $(hus^\bullet_x, kus^\bullet_x)$ and $(hus^\bullet_{x,t}, kus^\bullet_{x,t})$ even display a symmetry *sui generis*, which resembles symmetrality and might be called q-symmetrality. But we cannot afford to go into these matters here. Be it enough to say that the three pairs:

$$\begin{array}{rcl} (\mathbf{has}_t, \mathbf{kas}_t) & : & \textit{'arithmetic'} \\ (\mathbf{hus}_x, \mathbf{kus}_x) & : & \textit{'geometric'} \\ (\mathbf{hus}_{x,t}, \mathbf{kus}_{x,t}) & : & \textit{'bigeometric.'} \end{array}$$

which due to (282), (283) constitute a connected manifold, seem to enjoy a unique status, not only among all pairs $(\mathbf{haus}_T, \mathbf{kaus}_T)$, but even among all pairs $(\mathbf{h}, \mathbf{k})$ of mutually inverse σ-functions. They fully deserve a name of their own: let us call them *haukian functions.*[64]

5.16 Representation theory of finite groups and 'haukian' σ-functions.

The existence of simple images $\sum_\sigma \mathbf{h}(\sigma)\,\boldsymbol{\rho}(\sigma)$, $\sum_\sigma \mathbf{k}(\sigma)\,\boldsymbol{\rho}(\sigma)$ under the elementary, one-dimensional representations $\boldsymbol{\rho}(\sigma) := 1$ (trivial) or $\boldsymbol{\rho}(\sigma) := \boldsymbol{\epsilon}(\sigma)$ (signature) is garanteed for all pairs $(\mathbf{haus}_T, \mathbf{kaus}_T)$ by the formulas (275), (276). But if we move on to general, higher-dimensional representations $\boldsymbol{\rho}$ of the symmetric groups $\mathbb{S}_r$, the *haukian* family once again stands out for the simplicity of its behaviour, in particular for the distribution pattern of its *standard factors* inside the determinants of the representations. Results

[64]the h stands for the direct function; the k for its convolution inverse; and the diphthong *au* refers to the a and u of the arithmetic and (bi)geometric cases.

here are still incomplete, so we mention just two formulae, relative to the r-dimensional representations :

$$\boldsymbol{\rho}_r(\sigma)\,.\,\mathbf{e}_i := \mathbf{e}_{\sigma(i)} \qquad\qquad \forall\sigma\in\mathbb{S}_r\ ,\ i\in\{1,\dots,r\} \tag{304}$$

Typically, we get the familiar factors but with altered multiplicities :

$$\det\Big(\sum_{\sigma\in S_r}\mathbf{has}_t(\sigma)\,\boldsymbol{\rho}_r(\sigma)\Big) = \prod_{1\le k\le r-1}(1+k)^{-1+2r-3k}\prod_{1\le k\le r-1}(t+2+k)^{3k} \tag{305}$$

$$\det\Big(\sum_{\sigma\in S_r}\mathbf{hus}_x(\sigma)\,\boldsymbol{\rho}_r(\sigma)\Big) = \prod_{1\le k\le r-1}(x^k-1)^{+1+2r-3k} \tag{306}$$

5.17 σ-functions originating in uniform lamination.

We now take leave of the *haukian* family and consider a few other σ-functions that arise in the context of our fusion-fission transforms. The first is connected with the *uniform lamination-colamination* described in §1.8. It involves the alternal mould $lad^\bullet$ (of flat, difference-type : see §2.4) which also occurs in the construction of the standard alien derivations. The closely related mould $sad_a^\bullet$ will resurface in §5.19.

5.18 σ-functions originating in quadratic lamination.

The *quadratic lamination-colamination* described in §1.9 also gives rise to interesting σ-functions **hes**, **kes**, **ke**. The first two are mutually inverse and all three are simple. Let $\mathbb{B}$ be the associative algebra freely generated by $\mathbf{e}_1, \mathbf{e}_2, \dots$ and let ${}^1\mathbb{B}$ be the corresponding Lie algebra, with its natural embedding in $\mathbb{B}$. The projection $proj_1 : \mathbb{B} \to {}^1\mathbb{B}$ characterised in §1.9 involves a σ-function **ke** which, though not invertible, is closely related to an invertible one, **kes**, whose inverse **hes** is unexpectedly simple : it assumes only zeros or powers of 2 as its values.

Projection $proj_1 : \mathbb{B} \to {}^1\mathbb{B}$ **:** We have five equivalent expressions :

$$
\begin{aligned}
proj_1(\mathbf{e}_1...\mathbf{e}_r) &= \sum_{\sigma\in S_r} \mathbf{ke}\,(\sigma)\,\mathbf{e}_{\sigma(1)}..\,\mathbf{e}_{\sigma(r)} && (307)\\
&= \sum_{\sigma\in S_r} \frac{1}{r}\,\mathbf{ke}\,(\sigma)\,[..[\mathbf{e}_{\sigma(1)},\mathbf{e}_{\sigma(2)}]\ldots\,\mathbf{e}_{\sigma(r)}] && (308)\\
&= \sum_{\sigma\in S_r} \frac{1}{r}\,\mathbf{ke}\,(\sigma)\,[\mathbf{e}_{\sigma(1)}\ldots[\mathbf{e}_{\sigma(r-1)},\mathbf{e}_{\sigma(r)}]..] && (309)\\
&= \sum_{\tau\in S_{r-1}} \mathbf{kes}\,(\tau)\,[..[\mathbf{e}_1,\mathbf{e}_{\tau(2)}]\ldots\,\mathbf{e}_{\tau(r)}] && (310)\\
&= \sum_{\tau\in S_{r-1}} \mathbf{kes}\,(\tau)\,[\mathbf{e}_{\tau(1)}\ldots[\mathbf{e}_{\tau(r-1)},A_r]..] && (311)
\end{aligned}
$$

which make manifest the one-to-one correspondance that exists between $\mathbf{kes}(\tau)$ as defined on $\mathbb{S}_{r-1}=\mathbb{S}_l$ and $\mathbf{ke}(\sigma)$ as defined on $\mathbb{S}_r$:

$$
\begin{aligned}
\mathbf{kes}(\tau) &\overset{\rightrightarrows}{=} \mathbf{ke}(\sigma) \quad \textit{with}\ \ \sigma(1):=\tau(1),\ldots,\sigma(l):=\tau(l),\sigma(r):=r && (312)\\
ke^{\mathbf{n^1},\,r,\,\mathbf{n^2}} &\overset{\rightrightarrows}{=} (-1)^{r_2} \sum_{\mathbf{n}\,\in\,\mathrm{sha}(\mathbf{n^1},\tilde{\mathbf{n}}^{\mathbf{2}})} kes^{\mathbf{n}} && (313)
\end{aligned}
$$

Only the second relation calls for comments. For convenience it is written in mould form, and the sum ranges over all shuffles $\mathbf{n}$ of $\mathbf{n^1}$ and of the *reverse* $\tilde{\mathbf{n}}^{\mathbf{2}}$ of $\mathbf{n^2}$. The integer r_2 is of course the length of $\mathbf{n^2}$.

Properties of kes . Here are the main ones:

$$
\begin{aligned}
\mathbf{kes}\,(\mathrm{id}_l) &= \frac{1}{1+l}=\frac{1}{r} && (314)\\
\mathbf{kes}\,(\tau) \quad &\textit{is} \quad \textit{maximal for}\ \ \tau=\mathrm{id}_l && (315)\\
\mathbf{kes}\,(\tau) &= \mathbf{kes}\,(\tau^{-1}) \quad \textit{(parity)} && (316)\\
\mathbf{kes}\,(\tau) &= \mathbf{kes}\,(\tau^{\star}) \quad \textit{with}\ \tau^{\star}=\mathrm{rev}_l\circ\tau\circ\mathrm{rev}_l \ \ \textit{(symmetry)} && (317)\\
\sum_{\tau\in\mathbb{S}_l}\mathbf{kes}\,(\tau) &= \frac{l!\,l!}{(2\,l)!} && (318)\\
\sum_{\tau\in\mathbb{S}_l}\epsilon_\tau\,\mathbf{kes}\,(\tau) &= \frac{(l/2)!\,(l/2)!}{l!}\quad \textit{for}\ l\ \textit{even} && (319)\\
\sum_{\tau\in\mathbb{S}_l}\epsilon_\tau\,\mathbf{kes}\,(\tau) &= \frac{((l-1)/2)!\,((l-1)/2)!}{2\quad(l-1)!}\quad \textit{for}\ l\ \textit{odd} && (320)\\
\mathbf{kes} \quad &\textit{has} \quad \textit{an inverse}\ \mathbf{hes}\ \ \textit{in the group algebra}\ \mathbb{A}(\mathbb{S}_l) && (321)
\end{aligned}
$$

Properties of hes. We have:

$$\mathbf{hes}\,(\tau) \quad \in \quad \{0, 2, 2^1, \ldots, 2^l\} \;\; \textit{if} \;\; \tau \in \mathbb{S}_l \tag{322}$$

with the actual values given by a simple rule. That rule is best described by deriving **hes** from a more general, *real-indexed* and *flat* (i.e. locally constant) mould $hes^{\bullet}$. The link is simply:

$$\mathbf{hes}\,(\tau) = hes^{\tau(1),\ldots,\tau(l)} \quad \textit{if} \quad \tau \in \mathbb{S}_l \tag{323}$$

and $hes^{\bullet}$ is defined by the following induction:

$$hes^{\boldsymbol{\omega}} \;\; = \;\; cohes^{\boldsymbol{\omega^1}}\; hes^{\boldsymbol{\omega^2}}\; cohes^{\boldsymbol{\omega^3}} \tag{324}$$

Here $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_l)$ is any sequence of l distinct real number. The sequence $\boldsymbol{\omega^1}$ (resp $\boldsymbol{\omega^3}$) is obtained from $\boldsymbol{\omega}$ by retaining only the terms ω_i such that $\omega_i < \omega_1$ (resp $\omega_i > \omega_l$). The mid-sequence $\boldsymbol{\omega^2}$ is obtained from $\boldsymbol{\omega}$ by retaining only the terms ω_i such that $\omega_1 < \omega_i < \omega_l$ as well as the term ω_1^- immediately inferior to ω_1 (if it exists) and the term ω_l^+ immediately superior to ω_l (if it exists). Some of the factor sequences $\boldsymbol{\omega^i}$ may reduce to the empty sequence $\emptyset$, but the above relations amount to a true induction since in all cases $\textit{length}\,(\boldsymbol{\omega^2}) \leq \textit{length}\,(\boldsymbol{\omega}) - 2$.
To complete the induction rules we must set:

$$hes^{\emptyset} \;\; = \;\; 1 \tag{325}$$
$$cohes^{\emptyset} \;\; = \;\; 2 \quad \textit{and for} \quad \boldsymbol{\omega} \neq \emptyset \;: \tag{326}$$
$$cohes^{\boldsymbol{\omega}} \;\; = \;\; 1 \quad \textit{if } \boldsymbol{\omega} \textit{ is an increasing sequence} \tag{327}$$
$$cohes^{\boldsymbol{\omega}} \;\; = \;\; 0 \quad \textit{otherwise} \tag{328}$$

Remarks about the proofs: Though less than two page long, the proof has to be skipped in this expository paper. Let us just point out the reason for the occurrence of powers of 2 in **hes**. They stem from the standard scalar products of Lie elements of the form $[..[e_{\sigma(1)}, e_{\sigma(2)}], \ldots, e_{\sigma(k)}]$ which happen to be exact powers of 2.

5.19 σ-functions with treble stability.

Stability under $*, \times, \circ$.
To conclude this unashamedly 'botanical' chapter in character, we give two instances of σ-function that display a treble stability:
(i) stability under the convolution product $*$.
(ii) stability of the associated mould under mould multiplication $\times$.

(iii) stability of the associated mould under mould composition $\circ$.
Of course, all three stabilities are completely independent.[65]

The 'uniform' mould family.
The following moulds are associated with the so-called *uniform average* of resurgent theory. Setting $remu_a^\bullet = tu_{-a}^\bullet$ as in §2.3 and $namu_a^\bullet = sad_a^\bullet$ as in §2.4 we have :

$$remu_a^\bullet \times remu_b^\bullet \equiv remu_{a+b}^\bullet \qquad \forall a,b \in \mathbb{C} \qquad (329)$$
$$remu_a^\bullet \circ remu_b^\bullet \equiv remu_{a\,b}^\bullet \qquad \forall a,b \in \mathbb{C} \qquad (330)$$
$$\mathbf{namu}_a^\bullet * \mathbf{namu}_b^\bullet \equiv \mathbf{namu}_{a\,b}^\bullet \qquad \forall a,b \in \mathbb{C} \qquad (331)$$

The proofs are quite short. Far more interesting is the next example.

The 'organic' mould family.
The mould $remo_a^\bullet$ and the closely related mould $romo_a^\bullet$ were defined in §2.7. They are essentially the 'lateral moulds' (see §4.10) associated with the important 'organic average' which is central to resurgence theory. Built from these one-parameter moulds, we have the two-parameter mould $somo_{a,b}^\bullet$, also defined in §2.7, and its unexpected closure properties under mould multiplication and mould composition (see §2.7). But on top of these, we have also stability under convolution. Indeed, along with these *'lateral'* moulds there goes a *'neutral'* mould $namo_a^\bullet$, whose associated σ-function $\mathbf{namo}_a$ turns out to essentially coincide with the σ-function **hok** already encountered in connection with the family $\{\mathbf{has}, \mathbf{kas}, \mathbf{hak}\}$. Indeed, it can be shown that :

$$\mathbf{namo}_a(\sigma) \equiv a^r\, \mathbf{hok}_{\frac{1}{a}}(\sigma) \qquad \forall \sigma \in \mathbb{S}_r \ , \ \forall a \in \mathbb{C} \qquad (332)$$

The closure under convolution follows at once :

$$\mathbf{namo}_a^\bullet * \mathbf{namo}_b^\bullet \equiv \mathbf{namo}_{a\,b}^\bullet \qquad \forall a,b \in \mathbb{C} \qquad (333)$$

6 Conclusion and complements.

6.1 Unique status of arborification-coarborification among all fusion-fission transforms.

In the introduction, we pointed out the effectiveness of the arborification-coarborification transform in *analysis*. In §4 we backed up this claim with a

[65]the first one is at constant length r , the others mix up various lengths.

string of applications . In §2 and §3 we examined the *combinatorial* mechanisms behind the method's success, and the reasons for its superiority over other, *a priori* equally attractive fusion-fission transforms. In the last section, §5, this unique status received a further boost, and that too from an unexpected quarter : *algebra.*

To take stock, let us briefly retrace our main steps in §5. Starting from a series[66] of mutually inverse matrices $(H^\bullet_\bullet, K^\bullet_\bullet)$, which arise naturally when investigating arborification in a free associative context, we have successively encountered all the objects which grace the following table :

$$\begin{array}{ccccc}
(H^\bullet_\bullet , K^\bullet_\bullet) & & (has^\bullet_t \,, kas^\bullet_t) & \overset{shift.}{\Longrightarrow} & (has^\bullet_{t,s} \,, kas^\bullet_{t,s}) \\
\Downarrow & & {\scriptstyle mould.} \Uparrow & & \\
(\mathbf{has} \,, \mathbf{kas}) & \overset{twist}{\Longrightarrow} & (\mathbf{has}_t \,, \mathbf{kas}_t) & \overset{connect.}{\Longrightarrow} & (\mathbf{hak}_{t_1,t_2} \,, \mathbf{h\ddot{a}k}_{t_1,t_2}) \\
\Downarrow & & {\scriptstyle arith.} \Uparrow {\scriptstyle T=Ta_t} & & \Downarrow \\
q\text{-}analogue & & (\mathbf{haus}_T, \mathbf{kaus}_T) & & q\text{-}analogue \\
\Downarrow & {\scriptstyle geom.} \swarrow\!\!\swarrow {\scriptstyle T=Tu_x} & {\scriptstyle bigeom.} \Downarrow {\scriptstyle T=Tu_{x,t}} & & \Downarrow \\
(\mathbf{hus}_x \,, \mathbf{kus}_x) & \overset{twist}{\Longrightarrow} & (\mathbf{hus}_{x,\,t}, \mathbf{kus}_{x,\,t}) & \overset{connect.}{\Longrightarrow} & (\mathbf{huk}_{x;\,t_1,t_2} \,, \mathbf{h\ddot{u}k}_{x;\,t_1,t_2}) \\
 & & {\scriptstyle mould.} \Downarrow & & \\
 & & (hus^\bullet_{x,\,t} \,, kus^\bullet_{x,\,t}) & \overset{shift.}{\Longrightarrow} & (hus^\bullet_{x,\,t,\,s} \,, kus^\bullet_{x,\,t,\,s})
\end{array}$$

These *haukian* objects, some of them moulds, the others σ-functions, turned out to possess no end of unexpected properties :

a) the σ-functions go in pairs of mutually inverse [67] elements, with both terms admitting numerous symmetries, possessing quite explicit expressions, notably simple denominators, and also presenting a tendency towards maximal factorisation – all of which is quite uncommon for mutually inverse σ-functions.[68]

b) unlike σ-functions 'picked at haphazard', ours possess natural extensions to integer-indexed, rational valued *moulds*, the only restriction being that the indices have to be pair-wise distinct.

---

[66] these square matrices of order $r!$ are defined for all r.

[67] in the convolution algebras $\mathbb{A}(\mathbb{S}_r)$ of the symmetric groups $\mathbb{S}_r$.

[68] indeed, inversion in the algebras $\mathbb{A}(\mathbb{S}_r)$ tends to produce huge denominators.

c) the moulds so produced, in turn, display precise symmetries (either *symmetrality* or, more rarely, *alternality*), which may be common enough in "natural moulds", but rather surprising in the present instance[69]

d) there is a tantalising connection between these *"haukian"* moulds and the moulds of the *"organic family"*, which have a quite distinct origin.

But now comes the crux: although the entire construction, starting from the matrix pair $(H^\bullet_\bullet, K^\bullet_\bullet)$ down to the whole set of characters in the above Table, can be duplicated for any other fusion-fission transform, relative to any type of *order* (all partial orders, laminescent orders, arborescent orders of binary, or ternary type, etc etc) none of these parallel constructions [70] retains any of the rich structure or endearing simplicity which is the hallmark of the *haukian* family. Although, at the moment, these curious *haukian* properties seem to have no direct relevance to arborification-coarborification *as a tool for convergence-restoration in analysis*, they certainly enhance its uniqueness status. Even if devoid of deeper meaning, this 'agreement' between analysis and algebra[71] which we observe here is very gratifying.

6.2 Local-analyticity, free-analyticity, alien-analyticity.

$\mathbb{C}[[x_1, \ldots, x_\nu]]$ resp. $\mathbb{C}\{x_1, \ldots, x_\nu\}$ are well-established notations for the ring of all formal, resp. local-analytic[72] power series in the ν commuting indeterminates x_i and with coefficients in $\mathbb{C}$. Going over to non-commuting indeterminates X_i, the question arises: What could be the natural counterpart $\mathbb{C}\{\{X_1, \ldots, X_\nu\}\}$ of $\mathbb{C}\{x_1, \ldots, x_\nu\}$? And how could we characterise its elements:

$$SS \;=\; \sum_{0 \le r \le \infty} \; \sum_{i_k \in \{1, \ldots, \nu\}} A^{i_1, i_2, \ldots, i_r} \, X_{i_r} \ldots X_{i_2} X_{i_1} \qquad (A^\bullet \in \mathbb{C}) \qquad (334)$$

preferably in terms of bounds on $A^\bullet$? That of course will depend on which future 'specialisations' we have in mind for our indeterminates X_i.

$\mathbf{S_1}$: *finite-dimensional specialisations*, e.g. in spaces $\mathrm{End}(V)$ of endomorphisms of μ-dimensional vector spaces V, with μ finite but otherwise unre-

[69] at any rate, these mould symmetries are not a simple rephrasing, nor even a consequence, of the σ-function symmetries.

[70] as far as we could see. We did explore quite a few options.

[71] a similar 'convergence' is also a feature of *resurgence theory* which, despite having its moorings in analysis, often tastes like pure algebra.

[72] i.e. with non-zero convergence radius.

lated to ν.

$\mathbf{S_2}$: *infinite-dimensional specialisations*, e.g. in the spaces $\mathrm{Der}(\mathbb{C}\{x_1,..,x_\mu\})$ of ordinary derivations of the ring of convergent power series of μ variables.[73] For definiteness, let us restrict ourselves to specialisations $X_i \mapsto spe(X_i)$ that are homogeneous and degree-increasing :

$$\mathrm{spe}(X_i)\ :\qquad x^m\,\mathbb{C} \longrightarrow x^{m+d_i}\,\mathbb{C} \qquad (d_i \in \mathbb{N}^\mu\ ,\ \forall m \in \mathbb{N}^\mu) \tag{335}$$

$\mathbf{S_3}$: *alien specialisations*, i.e. incarnations in the space *ALIEN* of alien derivations of some space of resurgent functions. Here again, assume for definiteness that $spe(X_i)$ specialises to *homogeneous* alien derivations. [74]

So, against this backdrop of possible specialisations, let us weigh the merits of the three types of majorisations on $A^\bullet$ which naturally spring to mind. They are :

$$\mathbf{M_1}\ :\qquad |A^{i_1,\dots,i_r}| \ \leq\ c_0\ c_1^r \tag{336}$$

$$\mathbf{M_2}\ :\qquad |A^{i_1,\dots,i_r}| \ \leq\ c_0\ c_1^r\ \frac{1}{r!} \tag{337}$$

$$\mathbf{M_3}\ :\qquad |A^{(i_1,\dots,i_r)^\prec}| \ \leq\ c_0\ c_1^r \qquad \textit{with} \qquad A^\prec := \sum_{\prec\leq\bullet} A^\bullet \tag{338}$$

for some finite positive constants $c_0 = c_0(SS), c_1 = c_1(SS)$ and with, on the third line, the usual convention of *straight* arborification (see §1.2).

Condition M_1 is adequate for specialisations of type S_1, but clearly not for those of type S_2, even in the case of a single X_1, and much less for type S_3.

Condition M_2, on the other hand, ensures the convergence of specialisations S_2 and S_3, but is unnecessarily stringent.

The "proper" condition would seem to be the one involving arborification, namely M_3. As we saw, it implies the convergence of all specialisations S_2, and it does so at a much lesser cost[75]– in fact, at a *minimal* cost. Moreover, the space $\mathbb{C}\{\{X_1,\dots,X_\nu\}\}$ of all *SS* subject to M_3 [76] enjoys all the stability

[73]here again, μ is unrelated to ν, and can be any finite number.

[74]i.e. alien derivations of a given frequency ω, like Δ_ω or $[..[\Delta_{\omega_1},\Delta_{\omega_2}]\dots\Delta_{\omega_r}]$ with $\sum\omega_i = \omega$, but no *superpositions* corresponding to different ω's.

[75]in the uninteresting case of a single variable X_i, where non-commutativity doesn't come into play, M_3 is readily seen to coincide with M_2, but for several variables it is considerably weaker.

[76]with constants that depend on *SS*.

properties that one may wish for, e.g. under multiplication and substitution. We then speak of *free-analyticity.*

Condition M_3 also happens to be the *weakest* condition that guarantees the convergence of specialisations of type S_3. Dually, it is the *strongest* condition to be verified by the *displayed* and *restricted forms* of natural resurgent functions. We speak in that context of *alien-analyticity.*

7 Tables.

7.1 The σ-functions $\mathbf{has}, \mathbf{kas}$.

To handle integers only, we set : $\mathbf{has}^*(\sigma) := \frac{r!\,(1+r)!}{2^r} \quad \mathbf{has}(\sigma) \quad \forall \sigma \in \mathbb{S}_r$.

σ	$\mathbf{has}^*$	$\mathbf{kas}$	σ	$\mathbf{has}^*$	$\mathbf{kas}$	σ	$\mathbf{has}^*$	$\mathbf{kas}$
$[1]$	1	1	$[1,2,3]$	6	7	$[2,3,1]$	2	-1
$[1,2]$	2	2	$[1,3,2]$	4	-4	$[3,1,2]$	2	-1
$[2,1]$	1	-1	$[2,1,3]$	3	-2	$[3,2,1]$	1	2
$[1,2,3,4]$	24	38	$[2,3,1,4]$	8	-2	$[3,4,1,2]$	4	-3
$[1,2,4,3]$	18	-22	$[2,3,4,1]$	6	-2	$[3,4,2,1]$	2	4
$[1,3,2,4]$	16	-12	$[2,4,1,3]$	6	0	$[4,1,2,3]$	6	-2
$[1,3,4,2]$	12	-7	$[2,4,3,1]$	4	1	$[4,1,3,2]$	4	1
$[1,4,2,3]$	12	-7	$[3,1,2,4]$	8	-2	$[4,2,1,3]$	3	1
$[1,4,3,2]$	8	14	$[3,1,4,2]$	6	0	$[4,2,3,1]$	2	2
$[2,1,3,4]$	12	-12	$[3,2,1,4]$	4	4	$[4,3,1,2]$	2	4
$[2,1,4,3]$	9	9	$[3,2,4,1]$	3	1	$[4,3,2,1]$	1	-7

σ	**has***	**kas**	σ	**has***	**kas**	σ	**has***	**kas**
$[1,2,3,4,5]$	120	296	$[2,4,5,1,3]$	18	0	$[4,2,3,1,5]$	10	4
$[1,2,3,5,4]$	96	-172	$[2,4,5,3,1]$	12	1	$[4,2,3,5,1]$	8	1
$[1,2,4,3,5]$	90	-94	$[2,5,1,3,4]$	24	0	$[4,2,5,1,3]$	9	0
$[1,2,4,5,3]$	72	-57	$[2,5,1,4,3]$	18	0	$[4,2,5,3,1]$	6	0
$[1,2,5,3,4]$	72	-57	$[2,5,3,1,4]$	16	0	$[4,3,1,2,5]$	10	8
$[1,2,5,4,3]$	54	114	$[2,5,3,4,1]$	12	1	$[4,3,1,5,2]$	8	0
$[1,3,2,4,5]$	80	-104	$[2,5,4,1,3]$	12	0	$[4,3,2,1,5]$	5	-14
$[1,3,2,5,4]$	64	79	$[2,5,4,3,1]$	8	-2	$[4,3,2,5,1]$	4	-2
$[1,3,4,2,5]$	60	-19	$[3,1,2,4,5]$	40	-19	$[4,3,5,1,2]$	6	6
$[1,3,4,5,2]$	48	-22	$[3,1,2,5,4]$	32	16	$[4,3,5,2,1]$	3	-7
$[1,3,5,2,4]$	48	0	$[3,1,4,2,5]$	30	0	$[4,5,1,2,3]$	12	-12
$[1,3,5,4,2]$	36	11	$[3,1,4,5,2]$	24	0	$[4,5,1,3,2]$	8	6
$[1,4,2,3,5]$	60	-19	$[3,1,5,2,4]$	24	0	$[4,5,2,1,3]$	6	6
$[1,4,2,5,3]$	48	0	$[3,1,5,4,2]$	18	0	$[4,5,2,3,1]$	4	9
$[1,4,3,2,5]$	40	38	$[3,2,1,4,5]$	20	38	$[4,5,3,1,2]$	4	12
$[1,4,3,5,2]$	32	11	$[3,2,1,5,4]$	16	-32	$[4,5,3,2,1]$	2	-22
$[1,4,5,2,3]$	36	-33	$[3,2,4,1,5]$	15	2	$[5,1,2,3,4]$	24	-7
$[1,4,5,3,2]$	24	44	$[3,2,4,5,1]$	12	4	$[5,1,2,4,3]$	18	4
$[1,5,2,3,4]$	48	-22	$[3,2,5,1,4]$	12	0	$[5,1,3,2,4]$	16	2
$[1,5,2,4,3]$	36	11	$[3,2,5,4,1]$	9	-3	$[5,1,3,4,2]$	12	1
$[1,5,3,2,4]$	32	11	$[3,4,1,2,5]$	20	-6	$[5,1,4,2,3]$	12	1
$[1,5,3,4,2]$	24	22	$[3,4,1,5,2]$	16	0	$[5,1,4,3,2]$	8	-2
$[1,5,4,2,3]$	24	44	$[3,4,2,1,5]$	10	8	$[5,2,1,3,4]$	12	4
$[1,5,4,3,2]$	16	-77	$[3,4,2,5,1]$	8	1	$[5,2,1,4,3]$	9	-3
$[2,1,3,4,5]$	60	-94	$[3,4,5,1,2]$	12	-12	$[5,2,3,1,4]$	8	1
$[2,1,3,5,4]$	48	52	$[3,4,5,2,1]$	6	14	$[5,2,3,4,1]$	6	4
$[2,1,4,3,5]$	45	38	$[3,5,1,2,4]$	16	0	$[5,2,4,1,3]$	6	0
$[2,1,4,5,3]$	36	26	$[3,5,1,4,2]$	12	0	$[5,2,4,3,1]$	4	-2
$[2,1,5,3,4]$	36	26	$[3,5,2,1,4]$	8	0	$[5,3,1,2,4]$	8	1
$[2,1,5,4,3]$	27	-52	$[3,5,2,4,1]$	6	0	$[5,3,1,4,2]$	6	0
$[2,3,1,4,5]$	40	-19	$[3,5,4,1,2]$	8	6	$[5,3,2,1,4]$	4	-2
$[2,3,1,5,4]$	32	16	$[3,5,4,2,1]$	4	-7	$[5,3,2,4,1]$	3	-2
$[2,3,4,1,5]$	30	-4	$[4,1,2,3,5]$	30	-4	$[5,3,4,1,2]$	4	9
$[2,3,4,5,1]$	24	-7	$[4,1,2,5,3]$	24	0	$[5,3,4,2,1]$	2	-12
$[2,3,5,1,4]$	24	0	$[4,1,3,2,5]$	20	2	$[5,4,1,2,3]$	6	14
$[2,3,5,4,1]$	18	4	$[4,1,3,5,2]$	16	0	$[5,4,1,3,2]$	4	-7
$[2,4,1,3,5]$	30	0	$[4,1,5,2,3]$	18	0	$[5,4,2,1,3]$	3	-7
$[2,4,1,5,3]$	24	0	$[4,1,5,3,2]$	12	0	$[5,4,2,3,1]$	2	-12
$[2,4,3,1,5]$	20	2	$[4,2,1,3,5]$	15	2	$[5,4,3,1,2]$	2	-22
$[2,4,3,5,1]$	16	2	$[4,2,1,5,3]$	12	0	$[5,4,3,2,1]$	1	38

7.2 The σ-functions $\mathbf{has}, \mathbf{kas}$ with a twist parameter.

$$\mathbf{has}^*_t(\sigma) := r!\,(t+2)(t+3)\ldots(t+r+1)\,\mathbf{has}_t(\sigma) \qquad \forall\sigma \in \mathbb{S}_r \quad (339)$$

$$\mathbf{kas}^*_t(\sigma) := r!\ \mathbf{has}_t(\sigma) \qquad \forall\sigma \in \mathbb{S}_r \quad (340)$$

σ	$\mathbf{has}_t^*$	$\mathbf{kas}_t^*$
$[1]$	$(t+2)$	1
$[1,2]$	$(t+2)(t+4)$	$(t+4)$
$[2,1]$	$(t+2)(t+2)$	$-(t+2)$
$[1,2,3]$	$(t+2)(t+4)(t+6)$	$(t+6)(2t+7)$
$[1,3,2]$	$(t+2)(t+4)(t+4)$	$-(t+8)(t+3)$
$[2,1,3]$	$(t+2)(t+2)(t+6)$	$-(t+6)(t+2)$
$[2,3,1]$	$(t+2)(t+2)(t+4)$	$-(t+3)(t+2)$
$[3,1,2]$	$(t+2)(t+2)(t+4)$	$-(t+3)(t+2)$
$[3,2,1]$	$(t+2)(t+2)(t+2)$	$2(t+3)(t+2)$
$[1,2,3,4]$	$(t+2)(t+4)(t+6)(t+8)$	$(t+8)(7t^2+57t+114)$
$[1,2,4,3]$	$(t+2)(t+4)(t+6)(t+6)$	$-2(t+4)(2t^2+25t+66)$
$[1,3,2,4]$	$(t+2)(t+4)(t+4)(t+8)$	$-2(t+8)(t+6)(t+3)$
$[1,3,4,2]$	$(t+2)(t+4)(t+4)(t+6)$	$-(t+14)(t+4)(t+3)$
$[1,4,2,3]$	$(t+2)(t+4)(t+4)(t+6)$	$-(t+14)(t+4)(t+3)$
$[1,4,3,2]$	$(t+2)(t+4)(t+4)(t+4)$	$2(t+14)(t+4)(t+3)$
$[2,1,3,4]$	$(t+2)(t+2)(t+6)(t+8)$	$-2(2t+9)(t+8)(t+2)$
$[2,1,4,3]$	$(t+2)(t+2)(t+6)(t+6)$	$3(t+9)(t+4)(t+2)$
$[2,3,1,4]$	$(t+2)(t+2)(t+4)(t+8)$	$-(t+8)(t+3)(t+2)$
$[2,3,4,1]$	$(t+2)(t+2)(t+4)(t+6)$	$-2(t+4)(t+3)(t+2)$
$[2,4,1,3]$	$(t+2)(t+2)(t+4)(t+6)$	0
$[2,4,3,1]$	$(t+2)(t+2)(t+4)(t+4)$	$(t+4)(t+3)(t+2)$
$[3,1,2,4]$	$(t+2)(t+2)(t+4)(t+8)$	$-(t+8)(t+3)(t+2)$
$[3,1,4,2]$	$(t+2)(t+2)(t+4)(t+6)$	0
$[3,2,1,4]$	$(t+2)(t+2)(t+2)(t+8)$	$2(t+8)(t+3)(t+2)$
$[3,2,4,1]$	$(t+2)(t+2)(t+2)(t+6)$	$(t+4)(t+3)(t+2)$
$[3,4,1,2]$	$(t+2)(t+2)(t+4)(t+4)$	$-3(t+4)(t+3)(t+2)$
$[3,4,2,1]$	$(t+2)(t+2)(t+2)(t+4)$	$4(t+4)(t+3)(t+2)$
$[4,1,2,3]$	$(t+2)(t+2)(t+4)(t+6)$	$-2(t+4)(t+3)(t+2)$
$[4,1,3,2]$	$(t+2)(t+2)(t+4)(t+4)$	$(t+4)(t+3)(t+2)$
$[4,2,1,3]$	$(t+2)(t+2)(t+2)(t+6)$	$(t+4)(t+3)(t+2)$
$[4,2,3,1]$	$(t+2)(t+2)(t+2)(t+4)$	$2(t+4)(t+3)(t+2)$
$[4,3,1,2]$	$(t+2)(t+2)(t+2)(t+4)$	$4(t+4)(t+3)(t+2)$
$[4,3,2,1]$	$(t+2)(t+2)(t+2)(t+2)$	$-7(t+4)(t+3)(t+2)$

7.3 The σ-functions $\mathbf{has}, \mathbf{kas}$ with twist and shift.

We set :

$$\mathbf{has}^*_{t,s}(\sigma) := \frac{(2\,r)!}{2^r}(t+s+2)\ldots(t+s+r+1)\;\mathbf{has}_{t,s}(\sigma) \qquad \forall \sigma \in \mathbb{S}_r \quad (341)$$

$$\mathbf{kas}^*_{t,s}(\sigma) := r!\;\mathbf{has}_{t,s}(\sigma) \qquad \forall \sigma \in \mathbb{S}_r \quad (342)$$

σ	$\mathbf{has}^*_{t,s}$
$[1]$	$t+s+2$
$[1,2]$	$3\,t^2+6\,t\,s+3\,s^2+18\,t+17\,s+24$
$[2,1]$	$3\,t^2+6\,t\,s+3\,s^2+12\,t+13\,s+12$
$[1,2,3]$	$15\,(t+s+4)\,(t^2+2\,t\,s+s^2+8\,t+7s+12)$
$[1,3,2]$	$15\,t^3+45\,t^2\,s+45\,t\,s^2+15\,s^3+150\,t^2+300\,t\,s+147\,s^2+480\,t+468\,s+480$
$[2,1,3]$	$15\,t^3+45\,t^2\,s+45\,t\,s^2+15\,s^3+150\,t^2+285\,t\,s+141\,s^2+420\,t+414\,s+360$
$[2,3,1]$	$15\,t^3+45\,t^2\,s+45\,t\,s^2+15\,s^3+120\,t^2+255\,t\,s+129\,s^2+300\,t+336\,s+240$
$[3,1,2]$	$15\,t^3+45\,t^2\,s+45\,t\,s^2+15\,s^3+120\,t^2+240\,t\,s+123\,s^2+300\,t+312\,s+240$
$[3,2,1]$	$15\,(t+s+2)\,(t^2+2\,t\,s+s^2+4\,t+5\,s+4)$

σ	$\mathbf{kas}^*_{t,s}$
$[1]$	1
$[1,2]$	$t\,s+s^2+t+3\,s+4$
$[2,1]$	$-(s+1)\,(t+s+2)$
$[1,2,3]$	$(t\,s+s^2+t+4\,s+6)\,(t\,s+s^2+2\,t+4\,s+7)$
$[1,3,2]$	$-\frac{1}{2}\,(s+2)\,(t+s+3)\,(t\,s+s^2+t+3\,s+8)$
$[2,1,3]$	$-\frac{1}{2}\,(s+1)\,(t\,s+s^2+2\,t+5\,s+12)\,(t+s+2)$
$[2,3,1]$	$-\frac{1}{2}\,(s+1)\,(s+2)\,(t+s+2)\,(t+s+3)$
$[3,1,2]$	$-\frac{1}{2}\,(s+1)\,(s+2)\,(t+s+2)\,(t+s+3)$
$[3,2,1]$	$(s+1)\,(s+2)\,(t+s+2)\,(t+s+3)$

7.4 The σ-functions $\mathbf{hak}, \mathbf{hok}$.

$$\mathbf{hak}^*_{a,b}(\sigma) := r!\;(a+2)(a+3)\ldots(a+r+1)\;\mathbf{hak}_{a,b}(\sigma) \qquad \forall \sigma \in \mathbb{S}_r \quad (343)$$

$$\mathbf{hok}^*_{b}(\sigma) := r!\;\mathbf{hok}_{b}(\sigma) \qquad \forall \sigma \in \mathbb{S}_r \quad (344)$$

σ	$\mathbf{hak}^*_{a,b}$	$\mathbf{hok}^*_b$
$[1]$	$(a+2)$	1
$[1,2]$	$(a+2)(a+b+6)$	$(1+b)$
$[2,1]$	$(a-b)(a+2)$	$(1-b)$
$[1,2,3]$	$(a+2)(a+b+6)(a+2\,b+12)$	$(1+b)(1+2\,b)$
$[1,3,2]$	$(a+2)(a-b)(a+4)$	$(1-b)$
$[2,1,3]$	$(a-b)(a+2)(a+2\,b+12)$	$(1-b)(1+2\,b)$
$[2,3,1]$	$(a+2)(a-b)(a+2)$	$(1-b)$
$[3,1,2]$	$(a-b)(a+2)(a+2\,b+10)$	$(1-b)(1+2\,b)$
$[3,2,1]$	$(a-b)(a-2\,b-4)(a+2)$	$(1-b)(1-2\,b)$
$[1,2,3,4]$	$(a+2)(a+b+6)(a+2\,b+12)(a+3\,b+20)$	$(1+b)(1+2\,b)(1+3\,b)$
$[1,2,4,3]$	$(a+2)(a+b+6)(a-b)(a+6)$	$(1+b)(1-b)$
$[1,3,2,4]$	$(a+2)(a-b)(a+4)(a+3\,b+20)$	$(1-b)(1+3\,b)$
$[1,3,4,2]$	$(a+2)(a+b+6)(a-b)(a+4)$	$(1+b)(1-b)$
$[1,4,2,3]$	$(a+2)(a-b)(a+4)(a+3\,b+18)$	$(1-b)(1+3\,b)$
$[1,4,3,2]$	$(a+2)(a-b)(a-3\,b-8)(a+4)$	$(1-b)(1-3\,b)$
$[2,1,3,4]$	$(a-b)(a+2)(a+2\,b+12)(a+3\,b+20)$	$(1-b)(1+2\,b)(1+3\,b)$
$[2,1,4,3]$	$(a-b)(a+2)(a-b)(a+6)$	$(1-b)(1-b)$
$[2,3,1,4]$	$(a+2)(a-b)(a+2)(a+3\,b+20)$	$(1-b)(1+3\,b)$
$[2,3,4,1]$	$(a+2)(a+b+6)(a-b)(a+2)$	$(1+b)(1-b)$
$[2,4,1,3]$	$(a+2)(a-b)(a+2)(a+3\,b+18)$	$(1-b)(1+3\,b)$
$[2,4,3,1]$	$(a+2)(a-b)(a-3\,b-8)(a+2)$	$(1-b)(1-3\,b)$
$[3,1,2,4]$	$(a-b)(a+2)(a+2\,b+10)(a+3\,b+20)$	$(1-b)(1+2\,b)(1+3\,b)$
$[3,1,4,2]$	$(a-b)(a+2)(a-b)(a+4)$	$(1-b)(1-b)$
$[3,2,1,4]$	$(a-b)(a-2\,b-4)(a+2)(a+3\,b+20)$	$(1-b)(1-2\,b)(1+3\,b)$
$[3,2,4,1]$	$(a-b)(a+2)(a-b)(a+2)$	$(1-b)(1-b)$
$[3,4,1,2]$	$(a+2)(a-b)(a+2)(a+3\,b+16)$	$(1-b)(1+3\,b)$
$[3,4,2,1]$	$(a+2)(a-b)(a-3\,b-10)(a+2)$	$(1-b)(1-3\,b)$
$[4,1,2,3]$	$(a-b)(a+2)(a+2\,b+10)(a+3\,b+18)$	$(1-b)(1+2\,b)(1+3\,b)$
$[4,1,3,2]$	$(a-b)(a+2)(a-b-2)(a+4)$	$(1-b)(1-b)$
$[4,2,1,3]$	$(a-b)(a-2\,b-4)(a+2)(a+3\,b+18)$	$(1-b)(1-2\,b)(1+3\,b)$
$[4,2,3,1]$	$(a-b)(a+2)(a-b-2)(a+2)$	$(1-b)(1-b)$
$[4,3,1,2]$	$(a-b)(a-2\,b-4)(a+2)(a+3\,b+16)$	$(1-b)(1-2\,b)(1+3\,b)$
$[4,3,2,1]$	$(a-b)(a-2\,b-4)(a-3\,b-10)(a+2)$	$(1-b)(1-2\,b)(1-3\,b)$

7.5 The σ-functions häk, hök.

$$\mathbf{häk}^*_{a,b}(\sigma) := r!(a+2)(a+3)\dots(a+r+1)\,\mathbf{häk}_{a,b}(\sigma) \qquad \forall\sigma\in\mathbb{S}_r \quad (345)$$
$$\mathbf{hök}^*_{b}(\sigma) := r!\,\mathbf{hök}_{b}(\sigma) \qquad \forall\sigma\in\mathbb{S}_r \quad (346)$$

σ	$\mathbf{häk}^*_{a,b}$	$\mathbf{hök}^*_b$
$[1]$	$(a+2)$	1
$[1,2]$	$(a+2)(a-b)$	$(1-b)$
$[2,1]$	$(a+b+6)(a+2)$	$(1+b)$
$[1,2,3]$	$(a+2)(a-b)(a-2b-4)$	$(1-b)(1-2b)$
$[1,3,2]$	$(a+2)(a+b+8)(a+2)$	$(1+b)$
$[2,1,3]$	$(a+b+6)(a+2)(a-2b-4)$	$(1+b)(1-2b)$
$[2,3,1]$	$(a+4)(a+b+8)(a+2)$	$(1+b)$
$[3,1,2]$	$(a+b+8)(a+2)(a-2b-4)$	$(1+b)(1-2b)$
$[3,2,1]$	$(a+b+8)(a+2b+10)(a+2)$	$(1+b)(1+2b)$
$[1,2,3,4]$	$(a+2)(a-b)(a-2b-4)(a-3b-10)$	$(1-b)(1-2b)(1-3b)$
$[1,2,4,3]$	$(a+2)(a-b)(a+b+10)(a+2)$	$(1-b)(1+b)$
$[1,3,2,4]$	$(a+2)(a+b+8)(a+2)(a-3b-10)$	$(1+b)(1-3b)$
$[1,3,4,2]$	$(a+2)(a-b+2)(a+b+10)(a+2)$	$(1-b)(1+b)$
$[1,4,2,3]$	$(a+2)(a+b+10)(a+2)(a-3b-10)$	$(1+b)(1-3b)$
$[1,4,3,2]$	$(a+2)(a+b+10)(a+3b+16)(a+2)$	$(1+b)(1+3b)$
$[2,1,3,4]$	$(a+b+6)(a+2)(a-2b-4)(a-3b-10)$	$(1+b)(1-2b)(1-3b)$
$[2,1,4,3]$	$(a+b+6)(a+2)(a+b+10)(a+2)$	$(1+b)(1+b)$
$[2,3,1,4]$	$(a+4)(a+b+8)(a+2)(a-3b-10)$	$(1+b)(1-3b)$
$[2,3,4,1]$	$(a+4)(a-b+2)(a+b+10)(a+2)$	$(1-b)(1+b)$
$[2,4,1,3]$	$(a+4)(a+b+10)(a+2)(a-3b-10)$	$(1+b)(1-3b)$
$[2,4,3,1]$	$(a+4)(a+b+10)(a+3b+16)(a+2)$	$(1+b)(1+3b)$
$[3,1,2,4]$	$(a+b+8)(a+2)(a-2b-4)(a-3b-10)$	$(1+b)(1-2b)(1-3b)$
$[3,1,4,2]$	$(a+b+8)(a+2)(a+b+10)(a+2)$	$(1+b)(1+b)$
$[3,2,1,4]$	$(a+b+8)(a+2b+10)(a+2)(a-3b-10)$	$(1+b)(1+2b)(1-3b)$
$[3,2,4,1]$	$(a+b+8)(a+4)(a+b+10)(a+2)$	$(1+b)(1+b)$
$[3,4,1,2]$	$(a+6)(a+b+10)(a+2)(a-3b-10)$	$(1+b)(1-3b)$
$[3,4,2,1]$	$(a+6)(a+b+10)(a+3b+16)(a+2)$	$(1+b)(1+3b)$
$[4,1,2,3]$	$(a+b+10)(a+2)(a-2b-4)(a-3b-10)$	$(1+b)(1-2b)(1-3b)$
$[4,1,3,2]$	$(a+b+10)(a+2)(a+b+10)(a+2)$	$(1+b)(1+b)$
$[4,2,1,3]$	$(a+b+10)(a+2b+10)(a+2)(a-3b-10)$	$(1+b)(1+2b)(1-3b)$
$[4,2,3,1]$	$(a+b+10)(a+4)(a+b+10)(a+2)$	$(1+b)(1+b)$
$[4,3,1,2]$	$(a+b+10)(a+2b+12)(a+2)(a-3b-10)$	$(1+b)(1+2b)(1-3b)$
$[4,3,2,1]$	$(a+b+10)(a+2b+12)(a+3b+16)(a+2)$	$(1+b)(1+2b)(1+3b)$

7.6 The σ-functions $\mathbf{haus}, \mathbf{kaus}$.

We set $T := [t_1, t_2, t_3, \dots]$ and

$$\mathbf{haus}^*_T(\sigma) := t_1\,(t_1{+}t_2)\dots(t_1{+}\cdots{+}t_r)\;\mathbf{haus}_T(\sigma) \qquad \forall \sigma \in \mathbb{S}_r$$

σ	$\mathbf{haus}^*_T$	σ	$\mathbf{haus}^*_T$	σ	$\mathbf{haus}^*_T$
$[1]$	t_1	$[1,2,3]$	$t_1\,t_2\,t_3$	$[2,3,1]$	$t_1^2\,t_2$
$[1,2]$	$t_1\,t_2$	$[1,3,2]$	$t_1\,t_2^2$	$[3,1,2]$	$t_1^2\,t_2$
$[2,1]$	t_1^2	$[2,1,3]$	$t_1^2\,t_3$	$[3,2,1]$	t_1^3
$[1,2,3,4]$	$t_1\,t_2\,t_3\,t_4$	$[2,3,1,4]$	$t_1^2\,t_2\,t_4$	$[3,4,1,2]$	$t_1^2\,t_2^2$
$[1,2,4,3]$	$t_1\,t_2\,t_3^2$	$[2,3,4,1]$	$t_1^2\,t_2\,t_3$	$[3,4,2,1]$	$t_1^3\,t_2$
$[1,3,2,4]$	$t_1\,t_2^2\,t_4$	$[2,4,1,3]$	$t_1^2\,t_2\,t_3$	$[4,1,2,3]$	$t_1^2\,t_2\,t_3$
$[1,3,4,2]$	$t_1\,t_2^2\,t_3$	$[2,4,3,1]$	$t_1^2\,t_2^2$	$[4,1,3,2]$	$t_1^2\,t_2^2$
$[1,4,2,3]$	$t_1\,t_2^2\,t_3$	$[3,1,2,4]$	$t_1^2\,t_2\,t_4$	$[4,2,1,3]$	$t_1^3\,t_3$
$[1,4,3,2]$	$t_1\,t_2^3$	$[3,1,4,2]$	$t_1^2\,t_2\,t_3$	$[4,2,3,1]$	$t_1^3\,t_2$
$[2,1,3,4]$	$t_1^2\,t_3\,t_4$	$[3,2,1,4]$	$t_1^3\,t_4$	$[4,3,1,2]$	$t_1^3\,t_2$
$[2,1,4,3]$	$t_1^2\,t_3^2$	$[3,2,4,1]$	$t_1^3\,t_3$	$[4,3,2,1]$	t_1^4

7.7 The σ-functions $\mathbf{hus}, \mathbf{kus}$.

Reverting to the simple cyclotomic polynomials of §5.12 , we set :

$$\mathbf{hus}^*_x(\sigma) \;:=\; \mathbf{hus}_x(\sigma)\prod_{1\le k\le r}\frac{(1-x^k)}{(1-x)} \qquad \forall \sigma \in \mathbb{S}_r$$

$$\mathbf{kus}^*_x(\sigma) \;:=\; \mathbf{kus}_x(\sigma)\prod_{1\le k\le r}\frac{(1-x)(1-x^{k\,(k-1)})}{(1-x^k)} \qquad \forall \sigma \in \mathbb{S}_r$$

σ	$\mathbf{hus}^*_x$	$\mathbf{kus}^*_x$
$[1]$	1	1
$[1,2]$	x	x
$[2,1]$	1	-1
$[1,2,3]$	x^3	$x^3\,(x^2+1)$
$[1,3,2]$	x^2	$-x^4$
$[2,1,3]$	x^2	$-x^4$
$[2,3,1]$	x	$-x$
$[3,1,2]$	x	$-x$
$[3,2,1]$	1	x^2+1
$[1,2,3,4]$	x^6	$x^6\,(x^8+2\,x^6+x^4+2\,x^2+1)$
$[1,2,4,3]$	x^5	$-x^7\,(x^2+1)\,(x^4+1)$
$[1,3,2,4]$	x^5	$-x^7\,(x^2+1)\,(x^4-x^2+1)$
$[1,3,4,2]$	x^4	$-x^{10}$
$[1,4,2,3]$	x^4	$-x^{10}$
$[1,4,3,2]$	x^3	$x^9\,(x^2+1)$
$[2,1,3,4]$	x^5	$-x^7\,(x^2+1)\,(x^4+1)$
$[2,1,4,3]$	x^4	$x^8\,(x^2-x+1)\,(x^2+x+1)$
$[2,3,1,4]$	x^4	$-x^{10}$
$[2,3,4,1]$	x^3	$-x^3\,(x^2+1)$
$[2,4,1,3]$	x^3	0
$[2,4,3,1]$	x^2	x^4
$[3,1,2,4]$	x^4	$-x^{10}$
$[3,1,4,2]$	x^3	0
$[3,2,1,4]$	x^3	$x^9\,(x^2+1)$
$[3,2,4,1]$	x^2	x^4
$[3,4,1,2]$	x^2	$-x^2\,(x^2-x+1)\,(x^2+x+1)$
$[3,4,2,1]$	x	$x\,(x^2+1)\,(x^4+1)$
$[4,1,2,3]$	x^3	$-x^3\,(x^2+1)$
$[4,1,3,2]$	x^2	x^4
$[4,2,1,3]$	x^2	x^4
$[4,2,3,1]$	x	$x\,(x^2+1)\,(x^4-x^2+1)$
$[4,3,1,2]$	x	$x\,(x^2+1)\,(x^4+1)$
$[4,3,2,1]$	1	$-(x^8+2\,x^6+x^4+2\,x^2+1)$

7.8 The σ-functions $\mathbf{hus}, \mathbf{kus}$ with a twist parameter.

We set :

$$\begin{aligned}
\mathbf{hus}^*_{x,t}(\sigma) &:= DH_r(x,\,t)\ \mathbf{hus}_{x,t}(\sigma) && \forall\sigma\in\mathbb{S}_r\\
\mathbf{kus}^*_{x,t}(\sigma) &:= DK_r(x,\,t)\ \mathbf{kus}_{x,t}(\sigma) && \forall\sigma\in\mathbb{S}_r\\
DH_r(x,t) &:= DH_r(x)\ DH^*_r(x,t) = \prod_{1\le k\le r}\frac{(1-x^k)}{(1-x)}\prod_{1\le k\le r}(t^2-x^{k+1})\\
DK_r(x,t) &:= DK_r(x)\ DK^*_r(x,t) = \prod_{2\le k\le r}\frac{(1-x)(1-x^{k(k-1)})}{(1-x^k)}\prod_{1\le k\le r}\frac{(t^{2k}-x^{k(k+1)})}{(t^2-x^{k+1})}
\end{aligned}$$

σ	$\mathbf{hus}^*_{x,t}$	$\mathbf{kus}^*_{x,t}$
$[1]$	$-(x^2-t^2)$	1
$[1,2]$	$(x^4-t^2)\,(x^2-t^2)$	(x^4-t^2)
$[2,1]$	$x\,(x^2-t^2)^2$	$-x\,(x^2-t^2)$
$[1,2,3]$	$-(x^2-t^2)\,(x^4-t^2)\,(x^6-t^2)$	$(x^6-t^2)\,K_{123}$
$[1,3,2]$	$-x\,(x^2-t^2)\,(x^4-t^2)^2$	$-x\,(x^8-t^2)\,(x^6-t^4)$
$[2,1,3]$	$-x\,(x^2-t^2)^2\,(x^6-t^2)$	$-x\,(x^2-t^2)\,(x^{12}-t^4)$
$[2,3,1]$	$-x^2\,(x^2-t^2)^2\,(x^4-t^2)$	$-x^4\,(x^2-t^2)\,(x^6-t^4)$
$[3,1,2]$	$-x^2\,(x^2-t^2)^2\,(x^4-t^2)$	$-x^4\,(x^2-t^2)\,(x^6-t^4)$
$[3,2,1]$	$-x^3\,(x^2-t^2)^3$	$x^3\,(x^2+1)\,(x^2-t^2)\,(x^6-t^4)$

with $K_{123} := x^{10}+x^8-x^6\,t^2+x^4\,t^2-x^2\,t^4-t^4.$

7.9 The σ-functions $\mathbf{huk}, \mathbf{hük}$.

We set :

$$\begin{aligned}
\mathbf{huk}^*_{x;a,b}(\sigma) &:= DHK_r(x;\,a,b)\ \ \mathbf{huk}_{x,t}(\sigma) && \forall\sigma\in\mathbb{S}_r\\
\mathbf{hük}^*_{x;a,b}(\sigma) &:= DHK_r(x;\,a,b)\ \ \mathbf{hük}_{x,t}(\sigma) && \forall\sigma\in\mathbb{S}_r\\
DHK_r(x;\,a,b) &:= DH^*_r(x,\,a)\ DK^*_r(x,\,b) = \prod_{1\le k\le r}(a^2-x^{k+1})\,\frac{(b^{2k}-x^{k(k+1)})}{(b^2-x^{k+1})}
\end{aligned}$$

σ	$\mathbf{huk}^*_{x;\,a,\,b}$	$\mathbf{h\ddot{u}k}^*_{x;\,a\,b}$
$[1]$	(a^2-x^2)	(a^2-x^2)
$[1,2]$	$(a^2-x^2)(a^2\,b^2-x^6)$	$b^2\,x^3\,(a^2-x^2)(a^2\,b^{-2}-1)$
$[2,1]$	$b^2\,x^3\,(a^2-x^2)(a^2\,b^{-2}-1)$	$(a^2-x^2)(a^2\,b^2-x^6)$
$[1,2,3]$	$(a^2-x^2)(a^2\,b^2-x^6)(a^2\,b^4-x^{12})$	$b^6\,x^{11}\,(a^2-x^2)(a^2\,b^{-2}-1)(a^2\,b^{-4}-x^{-4})$
$[1,3,2]$	$b^4\,x^7\,(a^2-x^2)(a^2\,b^{-2}-1)(a^2-x^4)$	$b^2\,x^4\,(a^2-x^2)^2(a^2\,b^2-x^8)$
$[2,1,3]$	$b^2\,x^3\,(a^2-x^2)(a^2\,b^{-2}-1)(a^2\,b^4-x^{12})$	$b^4\,x^8\,(a^2-x^2)(a^2\,b^2-x^6)(a^2\,b^{-4}-x^{-4})$
$[2,3,1]$	$b^4\,x^8\,(a^2-x^2)^2(a^2\,b^{-2}-1)$	$b^2\,x^3\,(a^2-x^2)(a^2-x^4)(a^2\,b^2-x^8)$
$[3,1,2]$	$b^2\,x^4\,(a^2-x^2)(a^2\,b^{-2}-1)(a^2\,b^4-x^{10})$	$b^4\,x^7\,(a^2-x^2)(a^2\,b^2-x^8)(a^2\,b^{-4}-x^{-4})$
$[3,2,1]$	$b^6\,x^{11}\,(a^2-x^2)(a^2\,b^{-2}-1)(a^2\,b^{-4}-x^{-4})$	$(a^2-x^2)(a^2\,b^2-x^8)(a^2\,b^4-x^{10})$

7.10 The σ-functions ke and hes, kes.

We set :

$$\begin{aligned}
&\mathbf{kes}^*(\sigma)\ :=\ \delta_r\,\mathbf{kes}(\sigma)\quad \textit{but}\quad \mathbf{ke}^*(\sigma)\ :=\ \delta_{r-1}\,\mathbf{ke}(\sigma) \qquad\qquad \forall\sigma\in\mathbb{S}_r\\
&\textit{with}\quad \delta_r\ :=\ \frac{(2\,r)!}{r!\,r!}\,\delta^*_r\\
&\textit{and}\quad \delta^*_1=\delta^*_2=\delta^*_3=1,\ \ \delta^*_4=3^2,\ \ \delta^*_5=2^5.3,\ \ \delta^*_6=2^6.3^2.5^2.41
\end{aligned}$$

σ	$\mathbf{ke}^*$	σ	$\mathbf{ke}^*$	σ	$\mathbf{ke}^*$
$[1]$	1	$[1,2,3]$	2	$[2,3,1]$	-1
$[1,2]$	1	$[1,3,2]$	-1	$[3,1,2]$	-1
$[2,1]$	-1	$[2,1,3]$	-1	$[3,2,1]$	2
$[1,2,3,4]$	5	$[2,3,1,4]$	-1	$[3,4,1,2]$	-1
$[1,2,4,3]$	-2	$[2,3,4,1]$	-2	$[3,4,2,1]$	2
$[1,3,2,4]$	-2	$[2,4,1,3]$	0	$[4,1,2,3]$	-2
$[1,3,4,2]$	-1	$[2,4,3,1]$	1	$[4,1,3,2]$	1
$[1,4,2,3]$	-1	$[3,1,2,4]$	-1	$[4,2,1,3]$	1
$[1,4,3,2]$	2	$[3,1,4,2]$	0	$[4,2,3,1]$	2
$[2,1,3,4]$	-2	$[3,2,1,4]$	2	$[4,3,1,2]$	2
$[2,1,4,3]$	1	$[3,2,4,1]$	1	$[4,3,2,1]$	-5

σ	**hes**	**kes***	σ	**hes**	**kes***	σ	**hes**	**kes***
$[1]$	1	1	$[1,2,3]$	4	2	$[2,3,1]$	2	-1
$[1,2]$	2	1	$[1,3,2]$	2	-1	$[3,1,2]$	-6	-1
$[2,1]$	-2	-1	$[2,1,3]$	-6	-1	$[3,2,1]$	4	2
$[1,2,3,4]$	2^4	126	$[2,3,1,4]$	2^2	-19	$[3,4,1,2]$	2^1	-9
$[1,2,4,3]$	2^3	-44	$[2,3,4,1]$	2^1	-19	$[3,4,2,1]$	0	16
$[1,3,2,4]$	2^3	-44	$[2,4,1,3]$	2^1	1	$[4,1,2,3]$	2^1	-19
$[1,3,4,2]$	2^2	-19	$[2,4,3,1]$	0	11	$[4,1,3,2]$	0	11
$[1,4,2,3]$	2^2	-19	$[3,1,2,4]$	2^2	-19	$[4,2,1,3]$	0	11
$[1,4,3,2]$	0	36	$[3,1,4,2]$	2^1	1	$[4,2,3,1]$	0	16
$[2,1,3,4]$	2^3	-44	$[3,2,1,4]$	0	36	$[4,3,1,2]$	0	16
$[2,1,4,3]$	2^2	16	$[3,2,4,1]$	0	11	$[4,3,2,1]$	0	-44

σ	**hes**	**kes***	σ	**hes**	**kes***	σ	**hes**	**kes***
[1, 2, 3, 4, 5]	2^5	4032	[2, 4, 5, 1, 3]	2^1	23	[4, 2, 3, 1, 5]	0	344
[1, 2, 3, 5, 4]	2^4	−1284	[2, 4, 5, 3, 1]	0	157	[4, 2, 3, 5, 1]	0	143
[1, 2, 4, 3, 5]	2^4	−1284	[2, 5, 1, 3, 4]	2^1	32	[4, 2, 5, 1, 3]	0	−53
[1, 2, 4, 5, 3]	2^3	−513	[2, 5, 1, 4, 3]	0	−6	[4, 2, 5, 3, 1]	0	11
[1, 2, 5, 3, 4]	2^3	−513	[2, 5, 3, 1, 4]	0	−51	[4, 3, 1, 2, 5]	0	337
[1, 2, 5, 4, 3]	0	940	[2, 5, 3, 4, 1]	0	143	[4, 3, 1, 5, 2]	0	6
[1, 3, 2, 4, 5]	2^4	−1284	[2, 5, 4, 1, 3]	0	6	[4, 3, 2, 1, 5]	0	−940
[1, 3, 2, 5, 4]	2^3	393	[2, 5, 4, 3, 1]	0	−259	[4, 3, 2, 5, 1]	0	−259
[1, 3, 4, 2, 5]	2^3	−513	[3, 1, 2, 4, 5]	2^3	−513	[4, 3, 5, 1, 2]	0	148
[1, 3, 4, 5, 2]	2^2	−438	[3, 1, 2, 5, 4]	2^2	205	[4, 3, 5, 2, 1]	0	−205
[1, 3, 5, 2, 4]	2^2	27	[3, 1, 4, 2, 5]	2^2	27	[4, 5, 1, 2, 3]	2^1	−213
[1, 3, 5, 4, 2]	0	259	[3, 1, 4, 5, 2]	2^1	32	[4, 5, 1, 3, 2]	0	148
[1, 4, 2, 3, 5]	2^3	−513	[3, 1, 5, 2, 4]	2^1	−56	[4, 5, 2, 1, 3]	0	148
[1, 4, 2, 5, 3]	2^2	27	[3, 1, 5, 4, 2]	0	−6	[4, 5, 2, 3, 1]	0	107
[1, 4, 3, 2, 5]	0	940	[3, 2, 1, 4, 5]	0	940	[4, 5, 3, 1, 2]	0	107
[1, 4, 3, 5, 2]	0	259	[3, 2, 1, 5, 4]	0	−337	[4, 5, 3, 2, 1]	0	−393
[1, 4, 5, 2, 3]	2^2	−213	[3, 2, 4, 1, 5]	0	259	[5, 1, 2, 3, 4]	2^1	−513
[1, 4, 5, 3, 2]	0	337	[3, 2, 4, 5, 1]	0	205	[5, 1, 2, 4, 3]	0	205
[1, 5, 2, 3, 4]	2^2	−438	[3, 2, 5, 1, 4]	0	−6	[5, 1, 3, 2, 4]	0	184
[1, 5, 2, 4, 3]	0	259	[3, 2, 5, 4, 1]	0	−148	[5, 1, 3, 4, 2]	0	143
[1, 5, 3, 2, 4]	0	259	[3, 4, 1, 2, 5]	2^2	−213	[5, 1, 4, 2, 3]	0	157
[1, 5, 3, 4, 2]	0	344	[3, 4, 1, 5, 2]	2^1	23	[5, 1, 4, 3, 2]	0	−259
[1, 5, 4, 2, 3]	0	337	[3, 4, 2, 1, 5]	0	337	[5, 2, 1, 3, 4]	0	205
[1, 5, 4, 3, 2]	0	−940	[3, 4, 2, 5, 1]	0	157	[5, 2, 1, 4, 3]	0	−148
[2, 1, 3, 4, 5]	2^4	−1284	[3, 4, 5, 1, 2]	2^1	−213	[5, 2, 3, 1, 4]	0	143
[2, 1, 3, 5, 4]	2^3	421	[3, 4, 5, 2, 1]	0	337	[5, 2, 3, 4, 1]	0	344
[2, 1, 4, 3, 5]	2^3	393	[3, 5, 1, 2, 4]	2^1	23	[5, 2, 4, 1, 3]	0	11
[2, 1, 4, 5, 3]	2^2	205	[3, 5, 1, 4, 2]	0	−53	[5, 2, 4, 3, 1]	0	−184
[2, 1, 5, 3, 4]	2^2	205	[3, 5, 2, 1, 4]	0	6	[5, 3, 1, 2, 4]	0	157
[2, 1, 5, 4, 3]	0	−337	[3, 5, 2, 4, 1]	0	11	[5, 3, 1, 4, 2]	0	11
[2, 3, 1, 4, 5]	2^3	−513	[3, 5, 4, 1, 2]	0	148	[5, 3, 2, 1, 4]	0	−259
[2, 3, 1, 5, 4]	2^2	205	[3, 5, 4, 2, 1]	0	−205	[5, 3, 2, 4, 1]	0	−184
[2, 3, 4, 1, 5]	2^2	−438	[4, 1, 2, 3, 5]	2^2	−438	[5, 3, 4, 1, 2]	0	107
[2, 3, 4, 5, 1]	2^1	−513	[4, 1, 2, 5, 3]	2^1	32	[5, 3, 4, 2, 1]	0	−421
[2, 3, 5, 1, 4]	2^1	32	[4, 1, 3, 2, 5]	0	259	[5, 4, 1, 2, 3]	0	337
[2, 3, 5, 4, 1]	0	205	[4, 1, 3, 5, 2]	0	−51	[5, 4, 1, 3, 2]	0	−205
[2, 4, 1, 3, 5]	2^2	27	[4, 1, 5, 2, 3]	2^1	23	[5, 4, 2, 1, 3]	0	−205
[2, 4, 1, 5, 3]	2^1	−56	[4, 1, 5, 3, 2]	0	6	[5, 4, 2, 3, 1]	0	−421
[2, 4, 3, 1, 5]	0	259	[4, 2, 1, 3, 5]	0	259	[5, 4, 3, 1, 2]	0	−393
[2, 4, 3, 5, 1]	0	184	[4, 2, 1, 5, 3]	0	−6	[5, 4, 3, 2, 1]	0	1284

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