Singular parameters: coequational resurgence, autark functions and flexion operations.

Jean Ecalle

Contents

1 Singular parameters: the model problem.

Consider the following paradigmatic instance of a *doubly singular* differential system — a system not only singular in itself (i.e. relative to the time variable t) but also singularly perturbed (by a small parameter ϵ):

$$\begin{aligned} 0 &= \epsilon\, t^2\, \partial_t\, y^i + \lambda_i\, y^i + b^i(t,\epsilon,y^1,\dots,y^\nu) \qquad (1 \le i \le \nu) \\ t &\sim 0 \quad (\mathit{variable}) \\ \epsilon &\sim 0 \quad (\mathit{parameter}) \end{aligned} \tag{1.1}$$

It is technically more convenient to set $z := 1/t \sim \infty$ and $x := 1/\epsilon \sim \infty$, so as to prepare for working in the conjugate Borel planes ζ and ξ. This leads to the system:

$$\begin{aligned} \partial_z Y &= x\,\Lambda\, Y + B(z,x,Y) \qquad \mathit{with} \\ & Y = (Y^i)\ ,\ B = (B^i)\ ,\ \Lambda = \mathit{diag.matr.}(\lambda_i) \\ & B^i \in \mathbb{C}\{z^{-1},x^{-1},Y^1,\dots,Y^\nu\} \ \ \mathit{or} \ \ \in \mathbb{C}\{z^{-1},Y^1,\dots,Y^\nu\} \end{aligned} \tag{1.2}$$

From the viewpoint of x-resurgence, choosing the series B^i independent of x, i.e. taking them in $\mathbb{C}\{z^{-1}, Y\}$ rather than $\mathbb{C}\{z^{-1}, x^{-1}, Y\}$, makes little difference to the resurgence pattern in the ξ-plane, and none at all to the location of the singularities. So we shall henforth stick with this simplifying assumption.
To respect homogeneity, we may re-write our system thus:

$$\partial_z Y^i \;=\; x\,\lambda_i\,Y^i + \sum_{n_j\geq 0}^{j\neq i}\sum_{n_i\geq -1} B^i_{n_1,\ldots,n_\nu}(z)\,Y^i\prod (Y^j)^{n_j} \quad (1\leq i\leq \nu) \tag{1.3}$$

or in compact form:

$$\partial_z Y^i \;=\; Y^i\,\Big(\lambda_i\,x + \sum_{n_j\geq 0}^{j\neq i}\sum_{n_i\geq -1} B^i_{\boldsymbol{n}}(z)\,Y^{\boldsymbol{n}}\Big) \qquad (1\leq i\leq \nu) \tag{1.4}$$

with coefficients $B^i_{\boldsymbol{n}}(z)\in\mathbb{C}\{z^{-1}\}$ analytic at infinity and x-free.
The general solution, with its full set $\{s_1,\ldots,s_\nu\}$ of integration parameters, may be formally expanded in powers of either z^{-1} or x^{-1}:

$$\tilde{Y} = \tilde{Y}(z,x,\boldsymbol{s}) \in \mathbb{C}[[z^{-1}\ \mathit{or}\ x^{-1}]]\otimes\mathbb{C}\{s_1 z^{\rho_1}e^{\lambda_1 zx},\ldots,s_\nu z^{\rho_\nu}e^{\lambda_\nu zx}\} \tag{1.5}$$

Separating the exponentials from the power series, we get:

$$\tilde{Y}^i(z,x,\boldsymbol{s}) \;=\; \tilde{Y}^i(z,x) + \sum_{n_j\geq 0}^{j\neq i}\sum_{n_i\geq -1} \tilde{Y}^i_{\boldsymbol{n}}(z,x)\; s_i\boldsymbol{s}^{\boldsymbol{n}}\; e^{(\lambda_i + <\boldsymbol{n},\boldsymbol{\lambda}>)\,zx} \tag{1.6}$$

As just pointed out, the formal solution $\tilde{Y}$, or rather its components $\tilde{Y}^i_{\boldsymbol{n}}$, can be expanded in power series of z^{-1} or x^{-1}. Both types of expansions are generically divergent yet Borel-summable, but with distinctive *singular points*, *singularities* and *resurgence patterns*. Some form of the Bridge equation holds sway in both cases, but with distinct index reservoirs $\boldsymbol{\Omega_i}$ and above all with this crucial difference: whereas the ordinary, first-order differential operators $\mathbb{A}_\omega$ that govern the z-resurgence in $\mathbf{BE_1}$ do not depend on z, the differential operators $\mathbb{P}_\omega$ that govern the x-resurgence in $\mathbf{BE_2}$, have coefficients that are themselves divergent-resurgent in x and require a third Bridge equation $\mathbf{BE_3}$ for their description.

Equational resurgence: $\tilde{Y} = \tilde{Y}(z,x,\boldsymbol{s})$ (expanded in z^{-1} with x fixed)

$$\mathbf{BE_1}\ :\qquad \boldsymbol{\Delta}_{\omega_0}\tilde{Y} = \mathbb{A}_{\omega_0}\tilde{Y} \qquad \forall\ \omega_0\in\Omega_1 \tag{1.7}$$

Co-equational resurgence: $\tilde{Y} = \tilde{Y}(z, x, \boldsymbol{s})$ (expanded in x^{-1} with z fixed)

$$\mathbf{BE_2} \ : \qquad \mathbb{\Delta}_{\omega_0}\tilde{Y} = \tilde{\mathbb{P}}_{\omega_0}\,\tilde{Y} \qquad \forall\ \omega_0 \in \Omega_2 \tag{1.8}$$

$$\mathbf{BE_3} \ : \qquad \mathbb{\Delta}_{\omega_0}\tilde{\mathbb{P}}_{\omega_1} = F_{\omega_0,\omega_1}(\{\tilde{\mathbb{P}}_{\omega_j}\}) \qquad \forall\ \omega_0 \in \Omega_3 \tag{1.9}$$

Despite these far-going differences, there is bound to be a certain kinship between the two types of resurgence, since in the special case when $B^i_{\boldsymbol{n}}(z) = \beta^i_{\boldsymbol{n}}/z$ with $\beta^i_{\boldsymbol{n}}$ scalar, due to the underlying homogeneousness, z and x coalesce and both the z- and x-expansion assume the form:

$$\tilde{Y}^i(z, x, \boldsymbol{s}) \;=\; \tilde{Y}^i(z\,x) + \sum_{n_j \geq 0}^{j \neq i} \sum_{n_i \geq -1} \tilde{Y}^i_{\boldsymbol{n}}(z\,x)\; s_i \boldsymbol{s}^{\boldsymbol{n}}\; e^{(\lambda_i + <\boldsymbol{n},\boldsymbol{\lambda}>)\,zx} \tag{1.10}$$

with $\tilde{Y}^i(t)$ and $\tilde{Y}^i_{\boldsymbol{n}}(t) \in \mathbb{C}[[t^{-1}]]$.

It is this loose kinship or duality that justifies the labels of *equational* for the z-resurgence (z being the variable with respect to which we differentiate in the system (1.2)) and *co-equational* for the z-resurgence. *Equational resurgence* is by far the simpler of the two, since the general shape of $\mathbf{BE_1}$ with its operators $\mathbb{A}_\omega$ and their indices ω, can be inferred from purely formal considerations, directly from the differential system (1.2). Equations $\mathbf{BE_2}$ and $\mathbf{BE_3}$ with their index reservoirs $\boldsymbol{\Omega_2}$, $\boldsymbol{\Omega_3}$, are harder to derive, yet there too we are fortunate in having a general machinery, with a strong agebraic-combinatorial flavour, that addresses the general case.

2 Multiplication interfering with convolution. Hyperlogarithms.

The z-resurgence ("equational"), which manifests in the dual ζ-plane, turns out to be totally independent of what singularities the coefficients $B^i_{\boldsymbol{n}}(z)$ of the system S may or may not possess: the depend only on its "multipliers" λ_i. The x-resurgence ("co-equational"), however, which manifests in the dual ξ-plane, depends on both the multipliers λ_i *and* the singularities of the $B^i_{\boldsymbol{n}}(z)$, which live directly in the z-plane, *at* or *over* some points α_j. So what we are facing here is an unusual interference of two structures:
(i) the *multiplicative* structure, which leaves the singularities in place,
(ii) the *convolutive* structure, which *adds* singularities, in the sense that:
(singularity over ω_1)$*$(singularity over ω_2)$\Rightarrow$ (singularities over $\omega_1 + \omega_2$).

The ideal tool for understanding this mixed structure is the *hyperlogarithms*, with their *two encodings*, their stability under *two products* and *two* *sets of exotic derivations* and, not least, their 'denseness': any given resurgent function in the Borel plane is the limit, uniformly on any compact set of its Riemann surface, of a suitable series of hyper logarithms.

Hyperlogarithms in the $\boldsymbol{\alpha}$ and $\boldsymbol{\omega}$-encodings:

$$\widehat{\mathcal{V}}^{[\alpha_1,\ldots,\alpha_r]}(\tau) := \int_0^{\tau} \frac{d\tau_r}{\tau_r-\alpha_r}\cdots\int_0^{\tau_3}\frac{d\tau_2}{\tau_2-\alpha_r}\int_0^{\tau_2}\frac{d\tau_1}{\tau_1-\alpha_r} \tag{2.1}$$

$$\widehat{\mathcal{V}}^{\omega_1,\ldots,\omega_r}(\tau) \equiv \widehat{\mathcal{V}}^{[\alpha_1,\ldots,\alpha_r]}(\tau) \quad \textit{with} \quad \alpha_i \equiv \omega_1+\ldots+\omega_i \quad (\forall i) \tag{2.2}$$

Functional dimorphy:

$$(\widehat{\mathcal{V}}^{[\boldsymbol{\alpha}']} . \widehat{\mathcal{V}}^{[\boldsymbol{\alpha}'']})(\tau) \equiv \sum_{\boldsymbol{\alpha}\in \mathrm{sha}(\boldsymbol{\alpha}',\boldsymbol{\alpha}'')} \widehat{\mathcal{V}}^{[\boldsymbol{\alpha}]}(\tau) \tag{2.3}$$

$$(\widehat{\mathcal{V}}^{\boldsymbol{\omega}'} \mathbin{\widehat{*}} \widehat{\mathcal{V}}^{\boldsymbol{\omega}''})(\tau) \equiv \sum_{\boldsymbol{\omega}\in \mathrm{sha}(\boldsymbol{\omega}',\boldsymbol{\omega}'')} \widehat{\mathcal{V}}^{\boldsymbol{\omega}}(\tau) \tag{2.4}$$

Remark 1: Here $\widehat{*}$ stands for the convolution

$$(\widehat{\phi}_1 \mathbin{\widehat{*}} \widehat{\varphi}_2)(\tau) := \int_0^{\tau} \widehat{\varphi}_1(\tau-\tau_2)\, d\,\widehat{\varphi}_2(\tau_2) \tag{2.5}$$

whose unit (namely $\widehat{\varphi}\,(\tau)\equiv 1$) coincides with the unit of point-wise multiplication – a definite advantage in this context. To fall back on the more familiar convolution $\hat{*}$ (whose unit is the dirac at 0):

$$(\hat{\varphi}_1 \mathbin{\hat{*}} \hat{\varphi}_2)(\tau) := \int_0^{\tau} \widehat{\varphi}_1(\tau-\tau_2)\ \ \widehat{\varphi}_2(\tau_2)\, d\tau_2 \tag{2.6}$$

it is enough to change $\widehat{\mathcal{V}}^{\boldsymbol{\omega}}(\tau)$ into $\hat{\mathcal{V}}^{\boldsymbol{\omega}}(\tau) := \partial_t\, \widehat{\mathcal{V}}^{\boldsymbol{\omega}}(\tau)$.

Remark 2: When some α's or, equivalently, some ω-sums vanish, the definition (2.1) remains in force, but the conversion rule (2.2) has to be slightly modified. Indeed, in the extreme case when all α's and therefore all ω's vanish, to ensure the double *symmetrality*, the definitions have to be:

$$\widehat{\mathcal{V}}^{[\overbrace{0,\ldots,0}^{r\text{ times}}]}(\tau) = \frac{(\log\tau)^r}{r!} \qquad (\boldsymbol{\alpha}\text{-}\textit{encoding}) \tag{2.7}$$

$$\widehat{\mathcal{V}}^{\overbrace{0,\ldots,0}^{r\text{ times}}}(\tau) = \left[\partial_\sigma^r \frac{\tau^\sigma}{\Gamma(1+\tau)}\right]_{\sigma=0} = \frac{(\log\tau)^r}{r!} + \ldots \qquad (\boldsymbol{\omega}\text{-}\textit{encoding}) \tag{2.8}$$

with the dots in (2.8) standing for a polynomial in $\log \tau$ of degree $< r$.

Hyperlogarithms are stable under the action of two systems of *exotic* derivations : *foreign* and *alien.*

Foreign derivations ∇_{α_0} **(**$\alpha_0 \in \mathbb{C}_\bullet$**).**
Each ∇_{α_0} is a linear operator that measures singularities over α_0, without return to the origin. Together, they generate a free Lie algebra of *derivations* with respect to point-wise multiplication:

$$\nabla_{\alpha_0}(\widehat{\varphi}_1 \,.\, \widehat{\varphi}_2) \;\equiv\; (\nabla_{\alpha_0}\ \widehat{\varphi}_1)\,.\ \widehat{\varphi}_2 + \widehat{\varphi}_1\ .(\nabla_{\alpha_0}\ \widehat{\varphi}_2) \tag{2.9}$$

The ∇_{α_0} act only on functions $\widehat{\varphi}$ with (isolated) logarithmic singularities, i.e. such that *over* each point τ_0, each determination of $\widehat{\varphi}\ (\tau_0+\tau)$ be a germ in $\mathbb{C}\{\tau\}$ or $\mathbb{C}\{\tau\} \otimes \mathbb{C}[\log \tau]$ or $\mathbb{C}\{\tau\} \otimes \mathbb{C}\{\log \tau\}_{subexp}$.

Here, $\mathbb{C}\{t\}_{subexp}$ denotes the space of entire functions of t with at most subexponential growth at infintity. This condition makes it possible to separate powers of τ from elements of $\mathbb{C}\{\log \tau\}_{subexp}$.

Alien derivations Δ_{ω_0} **(**$\omega_0 \in \mathbb{C}_\bullet\ ,\ \omega_0 \neq 0$**).**
Each Δ_{ω_0} is a linear operator that measures singularities over ω_0 (it calculates a suitable average of finitely many determinations over ω_0) but *with return to the origin.* Together, they generate a free Lie algebra of *derivations* with respect to convolution:[1]

$$\Delta_{\omega_0}(\widehat{\varphi}_1 \mathbin{\widehat{*}} \widehat{\varphi}_2) \;\equiv\; (\Delta_{\omega_0}\ \widehat{\varphi}_1) \mathbin{\widehat{*}} \widehat{\varphi}_2 + \widehat{\varphi}_1 \mathbin{\widehat{*}} (\Delta_{\omega_0}\, \widehat{\varphi}_2) \tag{2.10}$$

$$\Delta_{\omega_0}(\hat{\varphi}_1 \mathbin{\hat{*}} \hat{\varphi}_2) \;\equiv\; (\Delta_{\omega_0}\ \hat{\varphi}_1) \mathbin{\hat{*}} \hat{\varphi}_2 + \hat{\varphi}_1 \mathbin{\hat{*}} (\Delta_{\omega_0}\, \hat{\varphi}_2) \tag{2.11}$$

The derivations Δ_{ω_0} act only on functions $\widehat{\varphi}$ with isolated singularities[2], but they act on all such functions, without any restriction whatsoever, and can tackle *any* singular behaviour at ω_0. Although the Δ_{ω_0} are clearly far more basic and general than the ∇_{ω_0}, while investigating co-equational resurgence it is advisable to think of these two systems of exotic derivations as being on the same footing.

[1]Depending on which convolution we work with ($\widehat{*}$ or $\hat{*}$), slight modifications are called for in the definition of the operators Δ_{ω_0}.

[2]logarithmic or much more general.

Stability of hyperlogarithms under exotic derivations.

$$\nabla_{\alpha_0}\, \widehat{\mathcal{V}}^{[\bullet]}(t) \;=\; V_{\alpha_0}^{[\bullet]} \times \widehat{\mathcal{V}}^{[\bullet]} \tag{2.12}$$

$$\Delta_{\omega_0}\, \widehat{\mathcal{V}}^{\bullet}(t) \;=\; V_{\omega_0}^{\bullet} \times \widehat{\mathcal{V}}^{\bullet} \tag{2.13}$$

Here, $\times$ denotes non-commutative mould multiplication. The function-valued moulds $\widehat{\mathcal{V}}^{[\bullet]}(t)$ and $\widehat{\mathcal{V}}^{\bullet}(t)$ are *symmetral*, whereas the scalar-valued moulds $V_{\alpha_0}^{[\bullet]}$ and $V_{\omega_0}^{\bullet}$ are *alternal*, meaning that:

$$\sum_{\boldsymbol{\alpha}\in \mathrm{sha}(\boldsymbol{\alpha}',\boldsymbol{\alpha}'')} V^{[\boldsymbol{\alpha}]} \equiv 0 \qquad (\forall \boldsymbol{\alpha}', \boldsymbol{\alpha}'' \neq \boldsymbol{\emptyset}) \tag{2.14}$$

$$\sum_{\boldsymbol{\omega}\in \mathrm{sha}(\boldsymbol{\omega}',\boldsymbol{\omega}'')} V^{\boldsymbol{\omega}} \equiv 0 \qquad (\forall \boldsymbol{\omega}', \boldsymbol{\omega}'' \neq \boldsymbol{\emptyset}) \tag{2.15}$$

It should be noted, though, that $V^{[\boldsymbol{\alpha}]}$ and $V^{\boldsymbol{\omega}}$ are *not* related under the correspondence (2.2).

3 Hyperlogarithmic expansions.

Elementary multilinear inputs:
The basic r-linear inputs (r-linear, that is, in the coefficients B^i_n) that go into the making of the formal solution $\tilde{Y}(z, x, \boldsymbol{s})$ of our prototypal system (1.3) are the following 'monomials' $\mathcal{S}^\bullet$:

$$\mathcal{S}^{B^{i_1}_{\boldsymbol{n_1}},\dots,B^{i_r}_{\boldsymbol{n_r}}}(z,x) \;:=\; \mathcal{D}_r.B^{i_r}_{\boldsymbol{n_r}}\dots\mathcal{D}_1.B^{i_1}_{\boldsymbol{n_1}} \tag{3.1}$$

$$\textit{with} \qquad \mathcal{D}_i \;:=\; \Big(\partial_z + (u_1{+}...{+}u_i)\,x\Big)^{-1} \quad \textit{and} \quad u_i := <\boldsymbol{n_i}, \boldsymbol{\lambda}>$$

Equational resurgence:
Under the Borel tranform $\mathcal{B}_z\,:\,z^{-n} \mapsto \frac{\zeta^{n-1}}{(n-1)!}$, our monomials $\mathcal{S}^\bullet$ become:

$$(\mathcal{B}_z.\mathcal{S})^{B^{i_1}_{\boldsymbol{n_1}},\dots,B^{i_r}_{\boldsymbol{n_r}}}(\zeta,x) \;:=\; \hat{\mathcal{D}}_r\,.\hat{B}^{i_r}_{\boldsymbol{n_r}}(\zeta)\dots\hat{\mathcal{D}}_1\,.\hat{B}^{i_1}_{\boldsymbol{n_1}}(\zeta) \tag{3.2}$$

$$\textit{with} \qquad \hat{\mathcal{D}}_i \;:=\; \Big(-\zeta + (u_1{+}...{+}u_i)\,x\Big)^{-1} \quad \textit{and} \quad u_i := <\boldsymbol{n_i}, \boldsymbol{\lambda}>$$

This expression readily yields all the information we need: location of singularities, Stokes constants, pattern of z-resurgence etc.

Coequational resurgence:
Under the Borel tranform $\mathcal{B}_x\,:\,x^{-n}\mapsto\frac{\xi^{n-1}}{(n-1)!}$, these selfsame monomials $\mathcal{S}^\bullet$ formally become:

$$(\mathcal{B}_x.\mathcal{S})^{B^{i_1}_{\boldsymbol{n_1}},\dots,B^{i_r}_{\boldsymbol{n_r}}}(z,\xi)\;:=\;\overset{\#}{\mathcal{D}}_r\;.B^{i_r}_{\boldsymbol{n_r}}(z)\cdots\overset{\#}{\mathcal{D}}_1\;.B^{i_1}_{\boldsymbol{n_1}}(z) \tag{3.3}$$
$$\textit{with}\qquad \overset{\#}{\mathcal{D}}_i\;:=\;\Big(\partial_z+(u_1{+}...{+}u_i)\,\partial_\xi\Big)^{-1}\quad\textit{and}\quad u_i:=<\boldsymbol{n_i},\boldsymbol{\lambda}>$$

This amounts to a system of partial differential equations which, in view of the limit conditions for $\xi=0$, admits a unique solution given by the following multiple integral:[3]

$$(\mathcal{B}_x.\mathcal{S})^{B^{i_1}_{\boldsymbol{n_1}},\dots,B^{i_r}_{\boldsymbol{n_r}}}(z,\xi)\;:=\;\int B^{i_1}_{\boldsymbol{n_1}}(t_1)\dots B^{i_r}_{\boldsymbol{n_r}}(t_r)\,dt_1\dots dt_r \tag{3.4}$$

with integration on a horrendous domain:

$$0<t_r<t_{r-1}<\cdots<t_2<t_1$$
$$(u_1+\cdots+u_i)\,t_i+(u_{i+1}t_{i+1}+\cdots+u_r)<\xi\qquad (2\le i\le r)$$
$$u_1t_1+\dots u_rt_r=\xi$$

While this integral representation has its uses for majorising the monomials $(\mathcal{B}_x.\mathcal{S})^\bullet(z,\xi)$, it is pretty hopeless when it comes to locating their singular points (in the ξ-plane) and describing their singular behaviour there.

The way round this difficulty is to decompose each coefficient $B^i_{\boldsymbol{n_i}}$ of "weight" $u_i:=<\boldsymbol{n_i},\boldsymbol{\lambda}>$, as a finite superposition of hyperlogarithms in the $\boldsymbol{\alpha}$-encoding[4]

$$\hat{\mathcal{V}}^{[\boldsymbol{\alpha^i}]}(z)\;=\;\hat{\mathcal{V}}^{[\alpha^i_1,\dots,\alpha^i_{m_i}]}(z) \tag{3.5}$$

or as a uniform limit of such superpositions.

For our purposes, it will be convenient to symbolise the monomials (3.5) along with the component $B^i_{\boldsymbol{n^i}}(z)$ where they originate by a two-tier index W_i comprising, in higher position, the *weight* $u_i:=<\boldsymbol{n_i},\boldsymbol{\lambda}>$ attached to $B^i_{\boldsymbol{n^i}}(z)$ and, in lower position, the sequence $(v_{i,1},v_{i,2}...)$ consisting of the singular points $\alpha_{i,k}$ (at home in the z-plane) *minus* the variable z itself:

$$\{\;B^i_{\boldsymbol{n_i}}\;,\;\hat{\mathcal{V}}^{\alpha_{i,1},\dots,\alpha_{i,m_i}}(z)\;\}\qquad\Longrightarrow \tag{3.6}$$

$$W_i=\binom{u_i}{V_i}\qquad\textit{with}\quad V_i=(v_{i,1},...,v_{i,m_i})=(\alpha_{i,1}-z,\dots,\alpha_{i,m_i}-z) \tag{3.7}$$

[3]at least when all partial sum $u_1{+}...{+}u_i$ are $\neq 0$.

[4]but of type $\widehat{\mathcal{V}}^{[\boldsymbol{\alpha^i}]}(z)$ rather than $\hat{\mathcal{V}}^{[\boldsymbol{\alpha^i}]}(z)$

By multilinearity, it is enough to consider the monomial expressions:

$$(\mathcal{B}_x.\mathcal{S})^{\hat{\mathcal{V}}^{\boldsymbol{\alpha}^1},\ldots,\hat{\mathcal{V}}^{\boldsymbol{\alpha}^r}}(z,\xi) =: \mathcal{T}^{W_1,\ldots,W_r}(z,\xi) \tag{3.8}$$

and these monomials $\mathcal{T}^{\bullet}(z,\xi)$, in turn, car be expressed as finite supperpositions of a (usually very large) number of hypermonomials $\hat{\mathcal{V}}^{\boldsymbol{\omega}}(\xi)$ of the sole variable ξ, with the z-dependence migrating to the upper indices $\boldsymbol{\omega} = (\omega_1,...,\omega_m)$. All these $\boldsymbol{\omega}$'s have the same length m equal to the added multiplicities m_i of the V_i's, and their total number $\mu(\boldsymbol{m})$ depends only on the *signature* $\boldsymbol{m} = (m_1,\ldots,m_r)$ of $\boldsymbol{W}$. Explicitly:

$$\begin{aligned}\mathcal{T}^{\boldsymbol{W}}(z,\xi) &= \mathcal{T}^{W_1,\ldots,W_r}(z,\xi) \equiv \sum_{1\leq j\leq \mu(m_1,\ldots,m_r)} \epsilon_j\, \hat{\mathcal{V}}^{\omega_{j,1},\ldots,\omega_{j,m}}(\xi) \qquad (3.9)\\ \textit{with} \qquad & \epsilon_j = \pm 1 \ ,\ m = \sum m_i \ ,\ m_i := \#(V_i)\end{aligned}$$

We observe that the z-dependence of $\mathcal{T}^{\bullet}(z,\xi)$ is now concentrated, on the right-hand side of (3.9), in the upper indices $\omega_{j,i}$. Indeed, each such ω_i (for simplicity, let us drop the indices j) is bilinear in the *weights* u_i and the z-shifted singularities $v_i := \alpha_i - z$. Moreover, as the induction (see *infra*) that makes explicit the expansion (3.9) of $\mathcal{T}^{\bullet}$ will show, the bilinear expression of ω_i involves only sums of consecutive u_i's multiplied by pair-wise differences of v_i's (which are z-constant) and also, on accasion, single v_j's (which depend on z).This already explains the appearance of a z-dependent index set $\boldsymbol{\Omega_2}$ in the second Bridge equation, and prepares us for the emergence of a z-independent index set $\boldsymbol{\Omega_3}$ in the third Bridge equation.

Backward induction for $\mathcal{T}^{\bullet}$ in the general case :

For r=1 and $W_1 = \binom{u_1}{V_1} = \binom{u_1}{v_{1,1},\ldots,v_{1,m_1}}$ we start the induction by setting:

$$\begin{aligned}\mathcal{T}^{W_1} &:= \hat{\mathcal{V}}^{[u_1\, v_{1,1}\,,\, u_1\, v_{1,2}\,,\ldots,\, u_1\, v_{1,m_1}]}\\ &:= \hat{\mathcal{V}}^{u_1\, v_{1,1}\,,\, u_1(v_{1,2}-v_{1,1})\,,\ldots,\, u_1(v_{1,m_1}-v_{1,m_1-1})}\end{aligned}$$

To continue the induction, we must distinguish four types of sequences $\boldsymbol{\omega}$, depending on the nature of their last index ω_m:

Case 1: $\quad \omega_m = u_r\, v_r^{\dagger} = u_r\, v_r \qquad (\textit{if}\ \ \#(V_r) = 1)$

Case 2: $\quad \omega_m = u_i\, (v_i^{\dagger} - v_{i+1}^{\dagger})$

Case 3: $\quad \omega_m = u_i\, (v_i^{\dagger} - v_{i-1}^{\dagger})$

Case 4: $\quad \omega_m = u_i\, (v_i^{\dagger} - v_i^{\ddagger})$

Then we must know how the operators $drop_{\omega_m}$ act on the monomials $\mathcal{T}^{\bullet}$:

$$\textbf{Case 1}: \text{drop}_{\omega_m} \mathcal{T}^{W_1,\ldots,W_r} = +\mathcal{T}^{W_1,\ldots\ldots\ldots,W_{r-1}}$$

$$\textbf{Case 2}: \text{drop}_{\omega_m} \mathcal{T}^{W_1,\ldots,W_r} = +\mathcal{T}^{W_1,\ldots,W^+_{i,i+1},\ldots,W_r} \quad \textit{with } W^+_{i,i+1} = \binom{u_i + u_{i+1}}{V_i^{\dagger} \diamond V_{i+1}^{\dagger}, v_{i+1}^{\dagger}}$$

$$\textbf{Case 3}: \text{drop}_{\omega_m} \mathcal{T}^{W_1,\ldots,W_r} = -\mathcal{T}^{W_1,\ldots,W^-_{i-1,i},\ldots,W_r} \quad \textit{with } W^-_{i-1,i} = \binom{u_{i-1} + u_i}{V_{i-1}^{\dagger} \diamond V_i^{\dagger}, v_{i-1}^{\dagger}}$$

$$\textbf{Case 4}: \text{drop}_{\omega_m} \mathcal{T}^{W_1,\ldots,W_r} = +\mathcal{T}^{W_1,\ldots,W_i^{\dagger},\ldots,W_r} \quad \textit{with } W_i^{\dagger} = \binom{u_i}{V_i^{\dagger}}$$

Interpretation:
(i) True to their name, the linear operators $drop_{\omega_0}$ act on the monomials $\hat{\mathcal{V}}^{\boldsymbol{\omega}}$ by annihilation or removal of the last index:
$drop_{\omega_0} \hat{\mathcal{V}}^{\omega_1,\ldots,\omega_{m-1},\omega_m} := \hat{\mathcal{V}}^{\omega_1,\ldots,\omega_{m-1}}$ if $\omega_0 = \omega_m$ and $:= 0$ otherwise.
(ii) For any sequence V_i, $v_i^{\dagger}$ denotes the last element of V_i and $v_i^{\ddagger}$ its last but one element (when it exists).
(iii)The sequence $V_i^{\dagger}$ is simply V_i deprived of its last element $v_i^{\dagger}$.
(iv) $(V_i^{\dagger} \diamond V_{i+1}^{\dagger}, v_{i+1}^{\dagger})$ denotes the *set* of all sequences V_* obtained by shuffling $V_i^{\dagger}$ with $V_{i+1}^{\dagger}$ and then adding, in last position, the last element $v_{i+1}^{\dagger}$ of $V_{i+1}^{\dagger}$. Likewise, $(V_{i-1}^{\dagger} \diamond V_i^{\dagger}, v_{i-1}^{\dagger})$ denotes the *set* of all sequences V_* obtained by shuffling $V_{i-1}^{\dagger}$ with $V_i^{\dagger}$. and then adding, in last position, the last element $v_{i-1}^{\dagger}$ of $V_{i-1}^{\dagger}$.
(v) The symbols $W^+_{i,i+1}$ and $W^-_{i-1,i}$ therefore represent each a collection of double indices W_* of type $\binom{u_i+u_{i+1}}{V_*}$ or $\binom{u_{i-1}+u_i}{V_*}$, and the induction in case 2 and 3 should correspondingly be interpreted as
$\textbf{Case 2}: \text{drop}_{\omega_m} \mathcal{T}^{W_1,\ldots,W_r} = +\sum_{W_* \in W^+_{i,i+1}} \mathcal{T}^{W_1,\ldots,W_*,\ldots,W_r}$
$\textbf{Case 3}: \text{drop}_{\omega_m} \mathcal{T}^{W_1,\ldots,W_r} = -\sum_{W_* \in W^-_{i-1,i}} \mathcal{T}^{W_1,\ldots,W_*,\ldots,W_r}$
(vi) Summig up, the induction means that the monomials $\hat{\mathcal{V}}^{\boldsymbol{\omega}}$ on the right-hand side of (3.9) carry a last index ω_m capable of assuming only four distinct shapes, and react to the action of $drop_{\omega_m}$ according to the four aforementioned rules.

Remark 1: forward-backward asymmetry.
The innocent-looking induction rule applying in case 1 suffices to upset the left-right symmetry: there also exists a *forward induction* for $\mathcal{T}^{\bullet}$, based on operators cut_{ω_1} that remove first indices, but it differs sharply from the backward induction, based on the operators $drop_{\omega_m}$.

Remark 2: relegation of z into the upper indices.
The induction rules 2, 3, 4 involve indices ω_m carrying only $\boldsymbol{v}$-differences $v' - v'' = \alpha' - \alpha''$ wherefrom z is absent, but the rule 1 involves a single

$v = \alpha - z$, with z persisting.

Remark 3: number of hyperlogarithmic summands.
The number $\mu(\boldsymbol{W}) = \mu(m_1, \ldots, m_r)$ of hyperlogarithmic summands in the standard expansion (3.9) of $\mathcal{T}^{\boldsymbol{W}}$ depends only on the multiplicities $m_i := \#(V_i)$. It tends to be huge. Thus:

$$
\begin{array}{lcrcrl}
\mu(\overbrace{1,\ldots,1}^{r \text{ times}}) & = & 1.3.5\ldots(2\,r-1) & = & r!! & \\
\mu(5,5,5) & = & 29\,135\,106 & \sim & 29 & 10^6 \\
\mu(4,5,6) & = & 22\,855\,560 & \sim & 23 & 10^6 \\
\mu(6,5,4) & = & 23\,963\,940 & \sim & 24 & 10^6 \\
\mu(4,4,4,4) & = & 10\,050\,665\,625 & \sim & 10 & 10^9 \\
\mu(1,3,5,7) & = & 349\,098\,750 & \sim & 0.4 & 10^9 \\
\mu(7,5,3,1) & = & 539\,188\,650 & \sim & 0.5 & 10^9 \\
\mu(3,3,3,3,3) & = & 60\,575\,515\,000 & \sim & 60 & 10^9 \\
\mu(1,2,3,4,5) & = & 6\,067\,061\,000 & \sim & 6 & 10^9 \\
\mu(5,4,3,2,1) & = & 9\,641\,071\,440 & \sim & 10 & 10^9
\end{array}
$$

Remark 4: convergence.
This runaway proliferation of summands doesn't prevent the corresponding expansions of $\mathcal{B}_x Y(z,\xi,\boldsymbol{s})$ from converging, after suitable regroupings (of *arborification-coarborification* type) that can be deftly handled by the mould formalism. Besides, convergence in the (ramified) Borel ξ-plane can also be derived directly, by standard PDE methods, or based on the integral representation (3.4). But our concern here is with the precise expression of the singularities (in the ξ-plane) and their exact resurgence equations, and for that there can be no alternative to the present approach since for truly general data W_i no cancellations occur and all the singularities produced by the induction mechanism are truly *there*.

Remark 5: the case of vanishing indices.
In the case of vanishing $\boldsymbol{u}$-sums and in that of vanishing $\boldsymbol{v}$-differences, the above formulas hold without modification, although multiple cancellations tend to drastically reduce the number of hyperlogaritmic summands in the expansion of $\mathcal{T}^{\boldsymbol{W}}$.

Both cases are of frequent occurrence. The reason why $\boldsymbol{u}$-sums often vanish is that (as soon the so-called *corner invariant* $\mathbb{A}_{-\lambda_i}$ is $\neq 0$), there is going to be a 'weight' $-\lambda_i$ effectively present alongside the 'weights' λ_i, $2\,\lambda_i$, $3\,\lambda_i$ etc. The reason why $\boldsymbol{v}$-differences often vanish is that, in non-linear problems, the same singularities $\hat{\mathcal{V}}^{[\boldsymbol{\alpha}]}(z)$ may occur in various components $B^i_{\boldsymbol{n_i}}$, leading to index repetitions within the sequences $\boldsymbol{W}$.

Remark 6: symmetrality and consistency.
Under convolution, both $\mathcal{T}^\bullet$ and $\hat{\mathcal{V}}^\bullet$ behave as *symmetral* moulds. The existence of a unique expansion of $\mathcal{T}^{\boldsymbol{W}}$ into a finite sum of $\hat{\mathcal{V}}^{\boldsymbol{\omega}}$'s leads therefore to a commutative diagram:

$$\begin{array}{ccc}
\mathcal{T}^{\boldsymbol{W}'} * \mathcal{T}^{\boldsymbol{W}''} & \overset{\text{symmetral linearisatiom}}{\longrightarrow} & \sum \mathcal{T}^{\boldsymbol{W}} \\
{\scriptstyle \text{hyperlogarithmic expansion}} \downarrow * \downarrow {\scriptstyle \text{hyperlogarithmic expansion}} & & \downarrow {\scriptstyle \text{hyperlogarithmic expansion}} \\
(\sum \epsilon_{\boldsymbol{\omega}'} \hat{\mathcal{V}}^{\boldsymbol{\omega}'}) * (\sum \epsilon_{\boldsymbol{\omega}''} \hat{\mathcal{V}}^{\boldsymbol{\omega}''}) & \overset{\text{symmetral linearisation}}{\longrightarrow} & \sum \epsilon_{\boldsymbol{\omega}} \hat{\mathcal{V}}^{\boldsymbol{\omega}}
\end{array}$$

The path *expansion then linearisation* always leads to a number of summands $\hat{\mathcal{V}}^{\boldsymbol{\omega}}$ considerably less than the path *linearisation then expansion*, but the latter gives rise to massive (pair-wise) cancellations, ensuring the same end result.

Backward induction for $\mathcal{T}^\bullet$ in the special case of simple V_i's :
When each V_i reduces to a single element v_i, case 4 is ruled out, and cases 1,2,3 simplify to:

Case 1: $\omega_m = u_r\, v_r$
Case 2: $\omega_m = u_i\,(v_i - v_{i+1})$
Case 3: $\omega_m = u_i\,(v_i - v_{i-1})$

Case 1: $\text{drop}_{\omega_m}\, \mathcal{T}^{w_1,\dots,w_r} = +\mathcal{T}^{w_1,\dots\dots\dots\dots,w_{r-1}}$

Case 2: $\text{drop}_{\omega_m}\, \mathcal{T}^{w_1,\dots,w_r} = +\mathcal{T}^{w_1,\dots,w^+_{i,i+1},\dots,w_r}$ *with* $w^+_{i,i+1} = \binom{u_i + u_{i+1}}{v_{i+1}}$

Case 3: $\text{drop}_{\omega_m}\, \mathcal{T}^{w_1,\dots,w_r} = -\mathcal{T}^{w_1,\dots,w^-_{i-1,i},\dots,w_r}$ *with* $w^-_{i-1,i} = \binom{u_{i-1} + u_i}{v_{i-1}}$

4 The scramble transform.

Originally, the scramble transform arose during the search for an algorithmic way of reducing the complex monomials $\mathcal{T}^\bullet$ of (3.9) to sums of simpler monomials $\mathcal{V}^\bullet$, but in the simpler context of meromorphic singularities in the z-plane, so that instead of indices $W_i := \binom{u_i}{V_i}$ we had to handle the simpler $w_i := \binom{u_i}{v_i}$. Our reason for mentioning the scramble here is because that transform has applications that extend far beyond co-equational resurgence. In particular, it led to:
(i) the first systematic use of *flexions* ;
(ii) the first systematic production of *double symmetries.*

Indeed, the *scramble* is a linear transform on bimoulds of *BIMU*:

$$M^{\bullet} \to S^{\bullet} = \mathrm{scram}.M^{\bullet} \quad \textit{with} \quad S^{\boldsymbol{w}} := \sum_{\boldsymbol{w}^*} \epsilon(\boldsymbol{w}, \boldsymbol{w}^*)\; M^{\boldsymbol{w}^*} \tag{4.1}$$

which not only preserves simple symmetries (*alternal* or *symmetral*) but, when applied to *all-even* bimoulds[5] $M^{\bullet}$, turns simple into double symmetries (*alternal* into *biaternal* and *symmetral* into *bisymmetral*).

To define the sums $S^{\boldsymbol{w}}$ in (4.1) we have the choice between a *forward* and *backward* induction, quite dissimilar in outward form but equivalent nonetheless. They involve respectively the 'mutilation' operators *cut* and *drop*:

$$\begin{array}{llll}
(\mathrm{cut}_{w_0} M)^{w_1,\dots,w_r} & := & M^{w_2,\dots,w_r} & \textit{if} \quad w_0 = w_1 \\
& := & 0 & \textit{if} \quad w_0 \neq w_1 \\
(\mathrm{drop}_{w_0} M)^{w_1,\dots,w_r} & := & M^{w_1,\dots,w_{r-1}} & \textit{if} \quad w_0 = w_r \\
& := & 0 & \textit{if} \quad w_0 \neq w_r
\end{array}$$

We get each induction started by setting $S^{w_1} := A^{w_1}$ and then apply the following rules, which rely on the usual flexion notation $\boldsymbol{w}\rceil$, $\boldsymbol{w}\rfloor$, $\lceil\boldsymbol{w}$, $\lfloor\boldsymbol{w}$.

Forward induction rule:
We set $(\mathrm{cut}_{w_0}.S)^{\mathbf{w}} := 0$ unless w_0 be of the form $\lceil w_i \rceil$ with respect to some sequence factorisation $\boldsymbol{w} = \mathbf{a} w_i \boldsymbol{bc}$, in which case we set:

$$(\mathrm{cut}_{\lceil w_i \rceil})^{\boldsymbol{w}} := (-1)^{r(\boldsymbol{b})} \sum_{\boldsymbol{w}' \in \mathrm{sha}\left(\mathbf{a}\rfloor,\, \lfloor\tilde{\mathbf{b}},\, \boldsymbol{c}\right)} S^{\boldsymbol{w}'} \qquad (\textit{if} \;\; \boldsymbol{w} = \boldsymbol{a} w_i \mathbf{b}\boldsymbol{c}) \tag{4.2}$$

with $\tilde{\boldsymbol{b}}$ denoting the sequence $\boldsymbol{b}$ in reverse order. If $A^{\bullet}$ is symmetral, so is $S^{\bullet}$ (see below). In that important case the forward induction rules assumes the much simpler form:

$$(\mathrm{cut}_{\lceil w_i \rceil})^{\boldsymbol{w}} := S^{\boldsymbol{a}}\, (\mathrm{invmu}.S)^{\boldsymbol{b}}\, S^{\boldsymbol{c}} \qquad (\textit{if} \;\; \boldsymbol{w} = \mathbf{a} w_i \boldsymbol{bc}) \tag{4.3}$$

Backward induction rule:
We set $(\mathrm{cut}_{w_0}.S)^{\boldsymbol{w}} := 0$ unless w_0 be of the form $\lfloor w_i$ or $w_i \rfloor$ with respect to some sequence factorisation $\boldsymbol{w} = \boldsymbol{a} w_i \boldsymbol{b}$, in which case we set:

$$(\mathrm{cut}_{\lfloor w_i})^{\boldsymbol{w}} \;:=\; -S^{\mathbf{a}\rceil\,\mathbf{b}} \qquad (\textit{if} \;\; \boldsymbol{w} = \boldsymbol{a} w_i \boldsymbol{b}) \tag{4.4}$$

$$(\mathrm{cut}_{w_i \rfloor})^{\boldsymbol{w}} \;:=\; +S^{\boldsymbol{a}\,\lceil\mathbf{b}} \qquad (\textit{if} \;\; \boldsymbol{w} = \boldsymbol{a} w_i \boldsymbol{b}) \tag{4.5}$$

[5] i.e. when applied to bimoulds $M^{\bullet}$ that are *even* separately in each double index w_i.

Remark: Extending the *scramble* to ordinary moulds.
We must often let the *scramble* act on moulds $M^\bullet$ by first 'lifting' these into bimoulds $\underline{M}^\bullet$ according to the rule: $\underline{M}^{\binom{u_1\,,\dots,\,u_r}{v_1\,,\dots,\,v_r}} = M^{u_1 v_1+\dots+u_r v_r}$. Of course, the *scramble* of a mould is a bimould – not a mould. Thus, the bimould $\mathcal{T}^\bullet$ of (3.9) is essentially the *scramble* of the mould $\mathcal{V}^\bullet$ of (2.2).

Remark: The general *scramble*.
The scramble also extends to generalized bimoulds $M^\bullet$ carrying an indexation $\boldsymbol{W} = (W_1, \dots, W_r)$ with $W_i = \binom{u_i}{V_i}$ and $\#V_i \geq 1$. Its definition then simply relies on the backward induction of §3. In that case too, there exists an (outwardly non-linear) forward induction, somewhat on the lines of formulae (4.2) and (4.3).

5 The tessellation mould.

The general tesselation mould $tes^{\boldsymbol{W}}$ is key to the third Bridge equation $\mathbf{BE_3}$ in the most general situation, i.e. when the coefficients of the differential system are z-ramified. Let us here describe the special tesselation mould $tes^{\boldsymbol{w}}$, which is enough when dealing with differential systems whose coefficients are z-meromorphic with simple poles.

Let $V^\bullet$ be the classical scalar mould produced under alien derivation from the equally classical resurgent mould $\mathcal{V}^\bullet(z)$:

$$\Delta_{\omega_0} \mathcal{V}^{\boldsymbol{\omega}}(z) = \sum_{\boldsymbol{\omega}=\boldsymbol{\omega}'\boldsymbol{\omega}''}^{\|\boldsymbol{\omega}'\|=\omega_0} V^{\boldsymbol{\omega}'}\, \mathcal{V}^{\boldsymbol{\omega}''}(z) \tag{5.1}$$

$\mathcal{V}^\bullet(z)$ is symmetral; $V^\bullet$ is alternal.

If we now apply the *scramble* transform to the alternal mould $V^\bullet$ (see Remark 2 *supra* about the lift $V^\bullet \mapsto \underline{V}^\bullet$), we get a bialternal bimould $tes^\bullet$:[6]

$$\mathrm{tes}^\bullet = \mathrm{scram}.V^\bullet \qquad \textit{with} \qquad \mathrm{tes}^{\boldsymbol{w}} := \sum_{\boldsymbol{w}^*} \epsilon(\boldsymbol{w}, \boldsymbol{w}^*)\, \underline{V}^{\boldsymbol{w}^*} \tag{5.2}$$

which surprisingly turns out to be piecewise constant in each u_i and v_i, despite being a sum of hyperlogarithmic summands $\underline{V}^{\boldsymbol{w}^*}$. This begs for an alternative, simpler expression of $tes^\bullet$. The following induction formula provides such an elementary alternative:

$$\mathrm{tes}^{\boldsymbol{w}} = \sum_{0\leq n\leq r(\boldsymbol{w})} \mathrm{push}^n \sum_{\boldsymbol{w}'\boldsymbol{w}''=\boldsymbol{w}} \mathrm{sig}^{\boldsymbol{w}'\boldsymbol{w}''}\, \mathrm{tes}^{\boldsymbol{w}^*}\, \mathrm{tes}^{\boldsymbol{w}^{**}} \tag{5.3}$$

[6] Its real place is in resurgence theory – in the description of the "geometry" of *co-equational resurgence*.

Do not mix up the pair $(\boldsymbol{w'},\boldsymbol{w''})$, that simply results from splitting $\boldsymbol{w}$, and the derivative pair $(\boldsymbol{w^*},\boldsymbol{w^{**}})$, whose construction is explained below.
The notations in (5.3) are as follows.
We fix $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ and set $\Re_\theta : z \in \mathbb{C} \mapsto \Re(e^{i\theta}z) \in \mathbb{R}$. Then we define:

$$f_{\boldsymbol{w}}^{\boldsymbol{w}'} := <\boldsymbol{u}',\boldsymbol{v}'><\boldsymbol{u},\boldsymbol{v}>^{-1} \quad , \quad g_{\boldsymbol{w}}^{\boldsymbol{w}'} := <\boldsymbol{u}',\Re_\theta \boldsymbol{v}'><\boldsymbol{u},\Re_\theta \boldsymbol{v}>^{-1} \tag{5.4}$$
$$f_{\boldsymbol{w}}^{\boldsymbol{w}''} := <\boldsymbol{u}'',\boldsymbol{v}''><\boldsymbol{u},\boldsymbol{v}>^{-1} \quad , \quad g_{\boldsymbol{w}}^{\boldsymbol{w}''} := <\boldsymbol{u}'',\Re_\theta \boldsymbol{v}''><\boldsymbol{u},\Re_\theta \boldsymbol{v}>^{-1} \tag{5.5}$$

From these scalars we construct the crucial sign factor *sig* which takes its values in $\{-1,0,1\}$. Here, the abbreviation $si(.)$ stands for $sign(\Im(.))$.

$$\begin{aligned}\mathrm{sig}^{\boldsymbol{w}',\boldsymbol{w}''} = \mathrm{sig}_\theta^{\boldsymbol{w}',\boldsymbol{w}''} := \frac{1}{8}\, &\Big(\mathrm{si}(f_{\boldsymbol{w}}^{\boldsymbol{w}'} - f_{\boldsymbol{w}}^{\boldsymbol{w}''}) - \mathrm{si}(g_{\boldsymbol{w}}^{\boldsymbol{w}'} - g_{\boldsymbol{w}}^{\mathbf{w}''})\Big) \times \\ &\Big(1 + \mathrm{si}(f_{\boldsymbol{w}}^{\boldsymbol{w}'}/g_{\boldsymbol{w}}^{\mathbf{w}'})\ \ \mathrm{si}(f_{\boldsymbol{w}}^{\boldsymbol{w}'} - g_{\boldsymbol{w}}^{\boldsymbol{w}'})\Big) \times \\ &\Big(1 + \mathrm{si}(f_{\boldsymbol{w}}^{\boldsymbol{w}''}/g_{\boldsymbol{w}}^{\boldsymbol{w}''})\ \mathrm{si}(f_{\boldsymbol{w}}^{\boldsymbol{w}''} - g_{\boldsymbol{w}}^{\boldsymbol{w}''})\Big)\end{aligned} \tag{5.6}$$

Lastly, the pair $(\boldsymbol{w}^*,\boldsymbol{w}^{**})$ is constructed from the pair $(\boldsymbol{w}',\boldsymbol{w}'')$ according to:

$$\boldsymbol{u}^* := \boldsymbol{u}' \quad , \quad \boldsymbol{v}^* := \boldsymbol{v}'\ <\boldsymbol{u},\boldsymbol{v}>^{-1}\ \Im g_{\boldsymbol{w}}^{\boldsymbol{w}'}\ -\ \Re_\theta \boldsymbol{v}'\ <\boldsymbol{u},\Re_\theta \boldsymbol{v}>^{-1}\ \Im g_{\mathbf{w}}^{\boldsymbol{w}'} \tag{5.7}$$
$$\boldsymbol{u}^{**} := \boldsymbol{u}'' \quad , \quad \boldsymbol{v}^{**} := \boldsymbol{v}''\ <\boldsymbol{u},\boldsymbol{v}>^{-1}\ \Im g_{\boldsymbol{w}}^{\boldsymbol{w}''}\ -\ \Re_\theta \boldsymbol{v}''\ <\boldsymbol{u},\Re_\theta \boldsymbol{v}>^{-1}\ \Im g_{\boldsymbol{w}}^{\boldsymbol{w}''} \tag{5.8}$$

Remark 1: The above induction for $tes^\bullet$ is elementary in the sense of being non-transcendental: it depends only on the *sign function*. But on the face of it, it looks non-intrinsical. Indeed, the partial sum:

$$\mathrm{urtes}_\theta^{\boldsymbol{w}} := \sum_{\boldsymbol{w}'\boldsymbol{w}''=\boldsymbol{w}} \mathrm{sig}^{\boldsymbol{w}',\boldsymbol{w}''}\mathrm{tes}^{\boldsymbol{w}^*}\mathrm{tes}^{\boldsymbol{w}^{**}} = \sum_{\boldsymbol{w}'\boldsymbol{w}''=\boldsymbol{w}} \mathrm{sig}_\theta^{\boldsymbol{w}',\boldsymbol{w}''}\mathrm{tes}^{\boldsymbol{w}_\theta^*}\mathrm{tes}^{\boldsymbol{w}_\theta^{**}} \tag{5.9}$$

is *polarised*, i.e. θ-dependent. However, its *push*-invariant offshoot :

$$\mathrm{tes}^\bullet := \sum_{0 \le n \le r(\boldsymbol{w})} \mathrm{push}^n\ \mathrm{urtes}_\theta^\bullet \tag{5.10}$$

is duly *unpolarised*. We might of course remove the polarisation in $urtes_\theta^\bullet$ itself by replacing it by this isotropic variant:

$$\mathrm{urtes}_{\mathrm{iso}}^\bullet := \frac{1}{2\,\pi}\int_0^{2\pi} \mathrm{urtes}_\theta^\bullet\ d\theta \tag{5.11}$$

but at the cost of rendering it less elementary, since $urtes_{iso}^\bullet$ would assume its value in $\mathbb{R}$ rather than $\{-1,0,1\}$. It would also depend hyperlogarithmically on its indices, and thus take us back to something rather like formula (5.2), which we wanted to get away from. So the alternative for $tes^\bullet$ is: *either* an intrinsical but heavily transcendental expression *or* an elementary but heavily polarised one!

Remark 2: In the induction (5.9) we might exchange everywhere the role of $\boldsymbol{u}$ and $\boldsymbol{v}$ and still get the correct answer $tes^\bullet$, but via a different polarised intermediary $urtes^\bullet_\theta$. The natural setting for studying $tes^\bullet$ is the *biprojective* space $\mathbb{P}^{r,r}$ equal to $\mathbb{C}^{2r}$ quotiented by the relation $\{\boldsymbol{w}^1 \sim \boldsymbol{w}^2\} \Leftrightarrow \{\boldsymbol{u}^1 = \lambda \boldsymbol{u}^2\,,\, \boldsymbol{v}^1 = \mu\, \boldsymbol{v}^2\ (\lambda, \mu \in \mathbb{C}^*)\}$. But rather than using biprojectivity to get rid of two coordinates (u_i, v_i), it is often useful, on the contrary, to resort to the *augmented* or *long* notation, by *adding* two redundant coordinates (u_0, v_0). The *long* coordinates (u_i^*, v_i^*) relate to the short ones (u_i, v_i) under the rules:

$$u_i = u_i^* \quad , \quad v_i = v_i^* - v_0^* \qquad (1 \le i \le r) \tag{5.12}$$

The *long* u_i^* are constrained by $u_0^* + \cdots + u_r^* = 0$ while the *long* v_i^* are, dually, regarded as defined up to a common additive constant. Thus we have $<u^*, v^*>=<u, v>$.The indices i of the *long* coordinates are viewed as elements of $\mathbb{Z}_{r+1} = \mathbb{Z}/(r+1)\mathbb{Z}$ with the natural circular ordering on number triplets $circ(i_1 < i_2 < i_3)$ that goes with it. Lastly, we require r^2-1 basic "homographies" $H_{i,j}$ on $\mathbb{P}^{r,r}$, defined by:

$$H_{i,j}(\mathbf{w}) \ := \ Q_{i,j}(\mathbf{w})/Q^*_{i,j}(\mathbf{w}) \qquad (i-j \neq 0; i,j \in \mathbb{Z}_{r+1}) \tag{5.13}$$

$$Q_{i,j}(\mathbf{w}) \ := \ \sum_{\mathrm{circ}(j \le q < i)} u_q^*\,(v_q^* - v_j^*) \tag{5.14}$$

$$Q^*_{i,j}(\mathbf{w}) \ := \ \sum_{\mathrm{circ}(i \le q < j)} u_q^*\,(v_q^* - v_j^*) \ \neq \ Q_{j,i}(\mathbf{w}) \tag{5.15}$$

Main properties of $tes^\bullet$.
$\boldsymbol{P_1}$: the bimould $tes^\bullet$ is bialternal, i.e. alternal and of alternal *swappee.*
$\boldsymbol{P_2}$: in fact *swap* $tes^\bullet = tes^\bullet$.
$\boldsymbol{P_3}$: $tes^\bullet$ is *push*-invariant.
$\boldsymbol{P_4}$: $tes^\bullet$ is *pus*-variant, i.e. of zero *pus*-average.
$\boldsymbol{P_5}$: $tes^\bullet$ assumes the sole values -1,0,1.
$\boldsymbol{P_6}$: for r fixed but large, the sets $S_\pm \subset \mathbb{P}^{r,r}$ where $tes^{\boldsymbol{w}}$ is ± 1, have positive but incredibly small Lebesgue measure.
$\boldsymbol{P_7}$: for r fixed, all three sets S_-, S_0, S_+ are path-connected.
$\boldsymbol{P_8}$: for r fixed, the hypersurfaces $\Im(H_{i,j}(\boldsymbol{w})) = 0$ *limit*[7] but do not *separate*[8]

[7]that is to say, the boundaries of these sets lie on the hypersurfaces.

[8]that is to say, none of the three sets can be defined in terms of the sole signs $si(H_{i,j}(\boldsymbol{w})) := sign(\Im(H_{i,j}(\boldsymbol{w})))$, at least for $r \ge 3$. For $r = 1$, $tes^\bullet \equiv 1$ and for $r = 2$, $tes^\bullet = \pm 1$ iff $si(H_{0,1}(\boldsymbol{w})) = si(H_{1,2}(\boldsymbol{w})) = si(H_{2,0}(\boldsymbol{w})) = \pm$ and 0 otherwise.

the sets S_-, S_0, S_+.
$\boldsymbol{P_9}$: $tes^{\boldsymbol{w}} = 0$ whenever $\boldsymbol{w}$ is semi-real, i.e. whenever one of its two components $\boldsymbol{u}$ or $\boldsymbol{v}$ is real. [9]

6 Tables.

For a double sequence $\boldsymbol{W}$ as in (3.9), we set $\boldsymbol{m(W)} := (\#V_1, \ldots, \#V_r)$ as usual. The following table gives, for low signatures $\boldsymbol{m(W)}$, the number $\mu = \mu^+ + \mu^-$ of terms on the right-hand side of (3.9), with $\mu^\pm$ denoting the number of summands preceded by the sign $\pm$.

$\boldsymbol{m}$	$\mu = \mu^+ + \mu^-$	$\boldsymbol{m}$	$\mu = \mu^+ + \mu^-$	$\boldsymbol{m}$	$\mu = \mu^+ + \mu^-$
$(1,1)$	$3 = 2+1$	$(1,1,1)$	$15 = 8+7$	$(1,1,1,1)$	$105 = 53+52$
$(1,2)$	$5 = 3+2$	$(1,1,2)$	$35 = 18+17$	$(1,1,1,2)$	$315 = 158+157$
$(2,1)$	$6 = 4+2$	$(1,2,1)$	$42 = 22+20$	$(1,1,2,1)$	$378 = 190+188$
$(1,3)$	$7 = 4+3$	$(2,1,1)$	$45 = 24+21$	$(1,2,1,1)$	$405 = 204+201$
$(2,2)$	$15 = 9+6$	$(1,1,3)$	$63 = 32+31$	$(2,1,1,1)$	$420 = 212+208$
$(3,1)$	$9 = 6+3$	$(1,3,1)$	$81 = 42+39$	$(1,1,1,3)$	$693 = 347+346$
$(1,4)$	$9 = 5+4$	$(3,1,1)$	$90 = 48+42$	$(1,1,3,1)$	$891 = 447+444$
$(2,3)$	$28 = 16+12$	$(1,2,2)$	$135 = 69+66$	$(1,3,1,1)$	$990 = 498+492$
$(3,2)$	$30 = 18+12$	$(2,1,2)$	$140 = 72+68$	$(3,1,1,1)$	$1050 = 530+520$
$(4,1)$	$12 = 8+4$	$(2,2,1)$	$168 = 88+80$		

The following tables give, for elementary signatures $\boldsymbol{m(W)} := (1,1,...)$, the scramble $S^\bullet$ of $M^\bullet$.

$$
\begin{aligned}
S^{\left(\begin{smallmatrix} u_1 \\ v_1 \end{smallmatrix}\right)} &= +M^{\left(\begin{smallmatrix} u_1 \\ v_1 \end{smallmatrix}\right)} \\
S^{\left(\begin{smallmatrix} u_1 & , & u_2 \\ v_1 & , & v_2 \end{smallmatrix}\right)} &= +M^{\left(\begin{smallmatrix} u_1 & , & u_2 \\ v_1 & , & v_2 \end{smallmatrix}\right)} + M^{\left(\begin{smallmatrix} u_{12} & , & u_1 \\ v_2 & , & v_{1:2} \end{smallmatrix}\right)} - M^{\left(\begin{smallmatrix} u_{12} & , & u_2 \\ v_1 & , & v_{2:1} \end{smallmatrix}\right)} \\
S^{\left(\begin{smallmatrix} u_1 & , & u_2 & , & u_3 \\ v_1 & , & v_2 & , & v_3 \end{smallmatrix}\right)} &= +M^{\left(\begin{smallmatrix} u_1 & , & u_2 & , & u_3 \\ v_1 & , & v_2 & , & v_3 \end{smallmatrix}\right)} + M^{\left(\begin{smallmatrix} u_1 & , & u_{23} & , & u_2 \\ v_1 & , & v_3 & , & v_{2:3} \end{smallmatrix}\right)} - M^{\left(\begin{smallmatrix} u_1 & , & u_{23} & , & u_3 \\ v_1 & , & v_2 & , & v_{3:2} \end{smallmatrix}\right)} \\
&\quad +M^{\left(\begin{smallmatrix} u_{12} & , & u_1 & , & u_3 \\ v_2 & , & v_{1:2} & , & v_3 \end{smallmatrix}\right)} - M^{\left(\begin{smallmatrix} u_{12} & , & u_2 & , & u_3 \\ v_1 & , & v_{2:1} & , & v_3 \end{smallmatrix}\right)} \\
&\quad +M^{\left(\begin{smallmatrix} u_{12} & , & u_3 & , & u_1 \\ v_2 & , & v_3 & , & v_{1:2} \end{smallmatrix}\right)} - M^{\left(\begin{smallmatrix} u_{12} & , & u_3 & , & u_2 \\ v_1 & , & v_3 & , & v_{2:1} \end{smallmatrix}\right)} \\
&\quad +M^{\left(\begin{smallmatrix} u_{123} & , & u_{23} & , & u_3 \\ v_1 & , & v_{2:1} & , & v_{3:2} \end{smallmatrix}\right)} - M^{\left(\begin{smallmatrix} u_{123} & , & u_{23} & , & u_2 \\ v_1 & , & v_{3:1} & , & v_{2:3} \end{smallmatrix}\right)} + M^{\left(\begin{smallmatrix} u_{123} & , & u_3 & , & u_2 \\ v_1 & , & v_{3:1} & , & v_{2:1} \end{smallmatrix}\right)} \\
&\quad -M^{\left(\begin{smallmatrix} u_{123} & , & u_1 & , & u_3 \\ v_2 & , & v_{1:2} & , & v_{3:2} \end{smallmatrix}\right)} - M^{\left(\begin{smallmatrix} u_{123} & , & u_3 & , & u_1 \\ v_2 & , & v_{3:2} & , & v_{1:2} \end{smallmatrix}\right)} \\
&\quad +M^{\left(\begin{smallmatrix} u_{123} & , & u_1 & , & u_2 \\ v_3 & , & v_{1:3} & , & v_{2:3} \end{smallmatrix}\right)} - M^{\left(\begin{smallmatrix} u_{123} & , & u_{12} & , & u_2 \\ v_3 & , & v_{1:3} & , & v_{2:3} \end{smallmatrix}\right)} + M^{\left(\begin{smallmatrix} u_{123} & , & u_{12} & , & u_1 \\ v_3 & , & v_{2:3} & , & v_{1:2} \end{smallmatrix}\right)}
\end{aligned}
$$

[9] or purely imaginary, since under biprojectivity this amounts to the same. Of course, $tes^{\boldsymbol{w}}$ vanishes in many more cases. In fact it vanishes most of the time: see $\boldsymbol{P_6}$ above.

$$S^{\left(\begin{smallmatrix} u_1, & u_2, & u_3, & u_4 \\ v_1, & v_2, & v_3, & v_4 \end{smallmatrix}\right)} =$$

$$+M^{\left(\begin{smallmatrix} u_1, & u_2, & u_3, & u_4 \\ v_1, & v_2, & v_3, & v_4 \end{smallmatrix}\right)}$$
$$+M^{\left(\begin{smallmatrix} u_{12}, & u_1, & u_3, & u_4 \\ v_2, & v_{1:2}, & v_3, & v_4 \end{smallmatrix}\right)}$$
$$-M^{\left(\begin{smallmatrix} u_{12}, & u_2, & u_3, & u_4 \\ v_1, & v_{2:1}, & v_3, & v_4 \end{smallmatrix}\right)}$$
$$+M^{\left(\begin{smallmatrix} u_{12}, & u_3, & u_1, & u_4 \\ v_2, & v_3, & v_{1:2}, & v_4 \end{smallmatrix}\right)}$$
$$-M^{\left(\begin{smallmatrix} u_{12}, & u_3, & u_2, & u_4 \\ v_1, & v_3, & v_{2:1}, & v_4 \end{smallmatrix}\right)}$$
$$+M^{\left(\begin{smallmatrix} u_{12}, & u_3, & u_4, & u_1 \\ v_2, & v_3, & v_4, & v_{1:2} \end{smallmatrix}\right)}$$
$$-M^{\left(\begin{smallmatrix} u_{12}, & u_3, & u_4, & u_2 \\ v_1, & v_3, & v_4, & v_{2:1} \end{smallmatrix}\right)}$$
$$+M^{\left(\begin{smallmatrix} u_1, & u_{23}, & u_4, & u_2 \\ v_1, & v_3, & v_4, & v_{2:3} \end{smallmatrix}\right)}$$
$$-M^{\left(\begin{smallmatrix} u_1, & u_{23}, & u_4, & u_3 \\ v_1, & v_2, & v_4, & v_{3:2} \end{smallmatrix}\right)}$$
$$+M^{\left(\begin{smallmatrix} u_1, & u_{23}, & u_2, & u_4 \\ v_1, & v_3, & v_{2:3}, & v_4 \end{smallmatrix}\right)}$$
$$-M^{\left(\begin{smallmatrix} u_1, & u_{23}, & u_3, & u_4 \\ v_1, & v_2, & v_{3:2}, & v_4 \end{smallmatrix}\right)}$$
$$+M^{\left(\begin{smallmatrix} u_1, & u_2, & u_{34}, & u_3 \\ v_1, & v_2, & v_4, & v_{3:4} \end{smallmatrix}\right)}$$
$$-M^{\left(\begin{smallmatrix} u_1, & u_2, & u_{34}, & u_4 \\ v_1, & v_2, & v_3, & v_{4:3} \end{smallmatrix}\right)}$$
$$+M^{\left(\begin{smallmatrix} u_{123}, & u_3, & u_2, & u_4 \\ v_1, & v_{3:1}, & v_{2:1}, & v_4 \end{smallmatrix}\right)}$$
$$+M^{\left(\begin{smallmatrix} u_{123}, & u_1, & u_2, & u_4 \\ v_3, & v_{1:3}, & v_{2:3}, & v_4 \end{smallmatrix}\right)}$$
$$-M^{\left(\begin{smallmatrix} u_{123}, & u_1, & u_3, & u_4 \\ v_2, & v_{1:2}, & v_{3:2}, & v_4 \end{smallmatrix}\right)}$$
$$-M^{\left(\begin{smallmatrix} u_{123}, & u_3, & u_1, & u_4 \\ v_2, & v_{3:2}, & v_{1:2}, & v_4 \end{smallmatrix}\right)}$$
$$+M^{\left(\begin{smallmatrix} u_{123}, & u_3, & u_4, & u_2 \\ v_1, & v_{3:1}, & v_4, & v_{2:1} \end{smallmatrix}\right)}$$
$$+M^{\left(\begin{smallmatrix} u_{123}, & u_1, & u_4, & u_2 \\ v_3, & v_{1:3}, & v_4, & v_{2:3} \end{smallmatrix}\right)}$$
$$-M^{\left(\begin{smallmatrix} u_{123}, & u_1, & u_4, & u_3 \\ v_2, & v_{1:2}, & v_4, & v_{3:2} \end{smallmatrix}\right)}$$
$$-M^{\left(\begin{smallmatrix} u_{123}, & u_3, & u_4, & u_1 \\ v_2, & v_{3:2}, & v_4, & v_{1:2} \end{smallmatrix}\right)}$$
$$+M^{\left(\begin{smallmatrix} u_{123}, & u_4, & u_1, & u_2 \\ v_3, & v_4, & v_{1:3}, & v_{2:3} \end{smallmatrix}\right)}$$
$$+M^{\left(\begin{smallmatrix} u_{123}, & u_4, & u_3, & u_2 \\ v_1, & v_4, & v_{3:1}, & v_{2:1} \end{smallmatrix}\right)}$$
$$-M^{\left(\begin{smallmatrix} u_{123}, & u_4, & u_3, & u_1 \\ v_2, & v_4, & v_{3:2}, & v_{1:2} \end{smallmatrix}\right)}$$
$$-M^{\left(\begin{smallmatrix} u_{123}, & u_4, & u_1, & u_3 \\ v_2, & v_4, & v_{1:2}, & v_{3:2} \end{smallmatrix}\right)}$$
$$+M^{\left(\begin{smallmatrix} u_1, & u_{234}, & u_4, & u_3 \\ v_1, & v_2, & v_{4:2}, & v_{3:2} \end{smallmatrix}\right)}$$
$$+M^{\left(\begin{smallmatrix} u_1, & u_{234}, & u_2, & u_3 \\ v_1, & v_4, & v_{2:4}, & v_{3:4} \end{smallmatrix}\right)}$$
$$-M^{\left(\begin{smallmatrix} u_1, & u_{234}, & u_2, & u_4 \\ v_1, & v_3, & v_{2:3}, & v_{4:3} \end{smallmatrix}\right)}$$
$$-M^{\left(\begin{smallmatrix} u_1, & u_{234}, & u_4, & u_2 \\ v_1, & v_3, & v_{4:3}, & v_{2:3} \end{smallmatrix}\right)}$$
$$+M^{\left(\begin{smallmatrix} u_{12}, & u_{34}, & u_2, & u_4 \\ v_1, & v_3, & v_{2:1}, & v_{4:3} \end{smallmatrix}\right)}$$
$$+M^{\left(\begin{smallmatrix} u_{12}, & u_{34}, & u_3, & u_1 \\ v_2, & v_4, & v_{3:4}, & v_{1:2} \end{smallmatrix}\right)}$$
$$+M^{\left(\begin{smallmatrix} u_{12}, & u_{34}, & u_1, & u_3 \\ v_2, & v_4, & v_{1:2}, & v_{3:4} \end{smallmatrix}\right)}$$
$$+M^{\left(\begin{smallmatrix} u_{12}, & u_{34}, & u_4, & u_2 \\ v_1, & v_3, & v_{4:3}, & v_{2:1} \end{smallmatrix}\right)}$$
$$-M^{\left(\begin{smallmatrix} u_{12}, & u_{34}, & u_2, & u_3 \\ v_1, & v_4, & v_{2:1}, & v_{3:4} \end{smallmatrix}\right)}$$
$$-M^{\left(\begin{smallmatrix} u_{12}, & u_{34}, & u_1, & u_4 \\ v_2, & v_3, & v_{1:2}, & v_{4:3} \end{smallmatrix}\right)}$$

$$-M^{\left(\begin{smallmatrix} u_{12}, & u_{34}, & u_4, & u_1 \\ v_2, & v_3, & v_{4:3}, & v_{1:2} \end{smallmatrix}\right)}$$
$$-M^{\left(\begin{smallmatrix} u_{12}, & u_{34}, & u_3, & u_2 \\ v_1, & v_4, & v_{3:4}, & v_{2:1} \end{smallmatrix}\right)}$$
$$+M^{\left(\begin{smallmatrix} u_{12}, & u_1, & u_{34}, & u_3 \\ v_2, & v_{1:2}, & v_4, & v_{3:4} \end{smallmatrix}\right)}$$
$$+M^{\left(\begin{smallmatrix} u_{12}, & u_2, & u_{34}, & u_4 \\ v_1, & v_{2:1}, & v_3, & v_{4:3} \end{smallmatrix}\right)}$$
$$-M^{\left(\begin{smallmatrix} u_{12}, & u_2, & u_{34}, & u_3 \\ v_1, & v_{2:1}, & v_4, & v_{3:4} \end{smallmatrix}\right)}$$
$$-M^{\left(\begin{smallmatrix} u_{12}, & u_1, & u_{34}, & u_4 \\ v_2, & v_{1:2}, & v_3, & v_{4:3} \end{smallmatrix}\right)}$$
$$+M^{\left(\begin{smallmatrix} u_{123}, & u_{12}, & u_1, & u_4 \\ v_3, & v_{2:3}, & v_{1:2}, & v_4 \end{smallmatrix}\right)}$$
$$-M^{\left(\begin{smallmatrix} u_{123}, & u_{12}, & u_2, & u_4 \\ v_3, & v_{1:3}, & v_{2:1}, & v_4 \end{smallmatrix}\right)}$$
$$+M^{\left(\begin{smallmatrix} u_{123}, & u_{12}, & u_4, & u_1 \\ v_3, & v_{2:3}, & v_4, & v_{1:2} \end{smallmatrix}\right)}$$
$$-M^{\left(\begin{smallmatrix} u_{123}, & u_{12}, & u_4, & u_2 \\ v_3, & v_{1:3}, & v_4, & v_{2:1} \end{smallmatrix}\right)}$$
$$+M^{\left(\begin{smallmatrix} u_{123}, & u_{23}, & u_3, & u_4 \\ v_1, & v_{2:1}, & v_{3:2}, & v_4 \end{smallmatrix}\right)}$$
$$-M^{\left(\begin{smallmatrix} u_{123}, & u_{23}, & u_2, & u_4 \\ v_1, & v_{3:1}, & v_{2:3}, & v_4 \end{smallmatrix}\right)}$$
$$+M^{\left(\begin{smallmatrix} u_{123}, & u_{23}, & u_4, & u_3 \\ v_1, & v_{2:1}, & v_4, & v_{3:2} \end{smallmatrix}\right)}$$
$$-M^{\left(\begin{smallmatrix} u_{123}, & u_{23}, & u_4, & u_2 \\ v_1, & v_{3:1}, & v_4, & v_{2:3} \end{smallmatrix}\right)}$$
$$+M^{\left(\begin{smallmatrix} u_{123}, & u_4, & u_{12}, & u_1 \\ v_3, & v_4, & v_{2:3}, & v_{1:2} \end{smallmatrix}\right)}$$
$$-M^{\left(\begin{smallmatrix} u_{123}, & u_4, & u_{12}, & u_2 \\ v_3, & v_4, & v_{1:3}, & v_{2:1} \end{smallmatrix}\right)}$$
$$+M^{\left(\begin{smallmatrix} u_{123}, & u_4, & u_{23}, & u_3 \\ v_1, & v_4, & v_{2:1}, & v_{3:2} \end{smallmatrix}\right)}$$
$$-M^{\left(\begin{smallmatrix} u_{123}, & u_4, & u_{23}, & u_2 \\ v_1, & v_4, & v_{3:1}, & v_{2:3} \end{smallmatrix}\right)}$$
$$+M^{\left(\begin{smallmatrix} u_1, & u_{234}, & u_{23}, & u_2 \\ v_1, & v_4, & v_{3:4}, & v_{2:3} \end{smallmatrix}\right)}$$
$$-M^{\left(\begin{smallmatrix} u_1, & u_{234}, & u_{23}, & u_3 \\ v_1, & v_4, & v_{2:4}, & v_{3:2} \end{smallmatrix}\right)}$$
$$+M^{\left(\begin{smallmatrix} u_1, & u_{234}, & u_{34}, & u_4 \\ v_1, & v_2, & v_{3:2}, & v_{4:3} \end{smallmatrix}\right)}$$
$$-M^{\left(\begin{smallmatrix} u_1, & u_{234}, & u_{34}, & u_3 \\ v_1, & v_2, & v_{4:2}, & v_{3:4} \end{smallmatrix}\right)}$$
$$+M^{\left(\begin{smallmatrix} u_{1234}, & u_1, & u_2, & u_3 \\ v_4, & v_{1:4}, & v_{2:4}, & v_{3:4} \end{smallmatrix}\right)}$$
$$+M^{\left(\begin{smallmatrix} u_{1234}, & u_1, & u_4, & u_3 \\ v_2, & v_{1:2}, & v_{4:2}, & v_{3:2} \end{smallmatrix}\right)}$$
$$+M^{\left(\begin{smallmatrix} u_{1234}, & u_4, & u_1, & u_3 \\ v_2, & v_{4:2}, & v_{1:2}, & v_{3:2} \end{smallmatrix}\right)}$$
$$+M^{\left(\begin{smallmatrix} u_{1234}, & u_4, & u_3, & u_1 \\ v_2, & v_{4:2}, & v_{3:2}, & v_{1:2} \end{smallmatrix}\right)}$$
$$-M^{\left(\begin{smallmatrix} u_{1234}, & u_4, & u_1, & u_2 \\ v_3, & v_{4:3}, & v_{1:3}, & v_{2:3} \end{smallmatrix}\right)}$$
$$-M^{\left(\begin{smallmatrix} u_{1234}, & u_1, & u_4, & u_2 \\ v_3, & v_{1:3}, & v_{4:3}, & v_{2:3} \end{smallmatrix}\right)}$$
$$-M^{\left(\begin{smallmatrix} u_{1234}, & u_4, & u_3, & u_2 \\ v_1, & v_{4:1}, & v_{3:1}, & v_{2:1} \end{smallmatrix}\right)}$$
$$-M^{\left(\begin{smallmatrix} u_{1234}, & u_1, & u_2, & u_4 \\ v_3, & v_{1:3}, & v_{2:3}, & v_{4:3} \end{smallmatrix}\right)}$$
$$+M^{\left(\begin{smallmatrix} u_{1234}, & u_{12}, & u_3, & u_1 \\ v_4, & v_{2:4}, & v_{3:4}, & v_{1:2} \end{smallmatrix}\right)}$$
$$+M^{\left(\begin{smallmatrix} u_{1234}, & u_{12}, & u_1, & u_3 \\ v_4, & v_{2:4}, & v_{1:2}, & v_{3:4} \end{smallmatrix}\right)}$$
$$+M^{\left(\begin{smallmatrix} u_{1234}, & u_{12}, & u_4, & u_2 \\ v_3, & v_{1:3}, & v_{4:3}, & v_{2:1} \end{smallmatrix}\right)}$$
$$+M^{\left(\begin{smallmatrix} u_{1234}, & u_{12}, & u_2, & u_4 \\ v_3, & v_{1:3}, & v_{2:1}, & v_{4:3} \end{smallmatrix}\right)}$$
$$-M^{\left(\begin{smallmatrix} u_{1234}, & u_{12}, & u_3, & u_2 \\ v_4, & v_{1:4}, & v_{3:4}, & v_{2:1} \end{smallmatrix}\right)}$$

$$-M^{\left(\begin{smallmatrix} u_{1234}, & u_{12}, & u_2, & u_3 \\ v_4, & v_{1:4}, & v_{2:1}, & v_{3:4} \end{smallmatrix}\right)}$$
$$-M^{\left(\begin{smallmatrix} u_{1234}, & u_{12}, & u_4, & u_1 \\ v_3, & v_{2:3}, & v_{4:3}, & v_{1:2} \end{smallmatrix}\right)}$$
$$-M^{\left(\begin{smallmatrix} u_{1234}, & u_{12}, & u_1, & u_4 \\ v_3, & v_{2:3}, & v_{1:2}, & v_{4:3} \end{smallmatrix}\right)}$$
$$+M^{\left(\begin{smallmatrix} u_{1234}, & u_{34}, & u_2, & u_3 \\ v_1, & v_{4:1}, & v_{2:1}, & v_{3:4} \end{smallmatrix}\right)}$$
$$+M^{\left(\begin{smallmatrix} u_{1234}, & u_{34}, & u_3, & u_2 \\ v_1, & v_{4:1}, & v_{3:4}, & v_{2:1} \end{smallmatrix}\right)}$$
$$+M^{\left(\begin{smallmatrix} u_{1234}, & u_{34}, & u_1, & u_4 \\ v_2, & v_{3:2}, & v_{1:2}, & v_{4:3} \end{smallmatrix}\right)}$$
$$+M^{\left(\begin{smallmatrix} u_{1234}, & u_{34}, & u_4, & u_1 \\ v_2, & v_{3:2}, & v_{4:3}, & v_{1:2} \end{smallmatrix}\right)}$$
$$-M^{\left(\begin{smallmatrix} u_{1234}, & u_{34}, & u_3, & u_1 \\ v_2, & v_{4:2}, & v_{3:4}, & v_{1:2} \end{smallmatrix}\right)}$$
$$-M^{\left(\begin{smallmatrix} u_{1234}, & u_{34}, & u_1, & u_3 \\ v_2, & v_{4:2}, & v_{1:2}, & v_{3:4} \end{smallmatrix}\right)}$$
$$-M^{\left(\begin{smallmatrix} u_{1234}, & u_{34}, & u_4, & u_2 \\ v_1, & v_{3:1}, & v_{4:3}, & v_{2:1} \end{smallmatrix}\right)}$$
$$-M^{\left(\begin{smallmatrix} u_{1234}, & u_{34}, & u_2, & u_4 \\ v_1, & v_{3:1}, & v_{2:1}, & v_{4:3} \end{smallmatrix}\right)}$$
$$+M^{\left(\begin{smallmatrix} u_{1234}, & u_1, & u_{23}, & u_2 \\ v_4, & v_{1:4}, & v_{3:4}, & v_{2:3} \end{smallmatrix}\right)}$$
$$+M^{\left(\begin{smallmatrix} u_{1234}, & u_4, & u_{23}, & u_2 \\ v_1, & v_{4:1}, & v_{3:1}, & v_{2:3} \end{smallmatrix}\right)}$$
$$-M^{\left(\begin{smallmatrix} u_{1234}, & u_1, & u_{23}, & u_3 \\ v_4, & v_{1:4}, & v_{2:4}, & v_{3:2} \end{smallmatrix}\right)}$$
$$-M^{\left(\begin{smallmatrix} u_{1234}, & u_4, & u_{23}, & u_3 \\ v_1, & v_{4:1}, & v_{2:1}, & v_{3:2} \end{smallmatrix}\right)}$$
$$+M^{\left(\begin{smallmatrix} u_{1234}, & u_4, & u_{12}, & u_2 \\ v_3, & v_{4:3}, & v_{1:3}, & v_{2:1} \end{smallmatrix}\right)}$$
$$-M^{\left(\begin{smallmatrix} u_{1234}, & u_4, & u_{12}, & u_1 \\ v_3, & v_{4:3}, & v_{2:3}, & v_{1:2} \end{smallmatrix}\right)}$$
$$+M^{\left(\begin{smallmatrix} u_{1234}, & u_1, & u_{34}, & u_4 \\ v_2, & v_{1:2}, & v_{3:2}, & v_{4:3} \end{smallmatrix}\right)}$$
$$-M^{\left(\begin{smallmatrix} u_{1234}, & u_1, & u_{34}, & u_3 \\ v_2, & v_{1:2}, & v_{4:2}, & v_{3:4} \end{smallmatrix}\right)}$$
$$+M^{\left(\begin{smallmatrix} u_{1234}, & u_{123}, & u_1, & u_2 \\ v_4, & v_{3:4}, & v_{1:3}, & v_{2:3} \end{smallmatrix}\right)}$$
$$+M^{\left(\begin{smallmatrix} u_{1234}, & u_{123}, & u_3, & u_2 \\ v_4, & v_{1:4}, & v_{3:1}, & v_{2:1} \end{smallmatrix}\right)}$$
$$-M^{\left(\begin{smallmatrix} u_{1234}, & u_{123}, & u_1, & u_3 \\ v_4, & v_{2:4}, & v_{1:2}, & v_{3:2} \end{smallmatrix}\right)}$$
$$-M^{\left(\begin{smallmatrix} u_{1234}, & u_{123}, & u_3, & u_1 \\ v_4, & v_{2:4}, & v_{3:2}, & v_{1:2} \end{smallmatrix}\right)}$$
$$+M^{\left(\begin{smallmatrix} u_{1234}, & u_{234}, & u_2, & u_4 \\ v_1, & v_{3:1}, & v_{2:3}, & v_{4:3} \end{smallmatrix}\right)}$$
$$+M^{\left(\begin{smallmatrix} u_{1234}, & u_{234}, & u_4, & u_2 \\ v_1, & v_{3:1}, & v_{4:3}, & v_{2:3} \end{smallmatrix}\right)}$$
$$-M^{\left(\begin{smallmatrix} u_{1234}, & u_{234}, & u_2, & u_3 \\ v_1, & v_{4:1}, & v_{2:4}, & v_{3:4} \end{smallmatrix}\right)}$$
$$-M^{\left(\begin{smallmatrix} u_{1234}, & u_{234}, & u_4, & u_3 \\ v_1, & v_{2:1}, & v_{4:2}, & v_{3:2} \end{smallmatrix}\right)}$$
$$+M^{\left(\begin{smallmatrix} u_{1234}, & u_{123}, & u_{12}, & u_1 \\ v_4, & v_{3:4}, & v_{2:3}, & v_{1:2} \end{smallmatrix}\right)}$$
$$-M^{\left(\begin{smallmatrix} u_{1234}, & u_{123}, & u_{12}, & u_2 \\ v_4, & v_{3:4}, & v_{1:3}, & v_{2:1} \end{smallmatrix}\right)}$$
$$+M^{\left(\begin{smallmatrix} u_{1234}, & u_{123}, & u_{23}, & u_3 \\ v_4, & v_{1:4}, & v_{2:1}, & v_{3:2} \end{smallmatrix}\right)}$$
$$-M^{\left(\begin{smallmatrix} u_{1234}, & u_{123}, & u_{23}, & u_2 \\ v_4, & v_{1:4}, & v_{3:1}, & v_{2:3} \end{smallmatrix}\right)}$$
$$+M^{\left(\begin{smallmatrix} u_{1234}, & u_{234}, & u_{23}, & u_3 \\ v_1, & v_{4:1}, & v_{2:4}, & v_{3:2} \end{smallmatrix}\right)}$$
$$-M^{\left(\begin{smallmatrix} u_{1234}, & u_{234}, & u_{23}, & u_2 \\ v_1, & v_{4:1}, & v_{3:4}, & v_{2:3} \end{smallmatrix}\right)}$$
$$+M^{\left(\begin{smallmatrix} u_{1234}, & u_{234}, & u_{34}, & u_3 \\ v_1, & v_{2:1}, & v_{4:2}, & v_{3:4} \end{smallmatrix}\right)}$$
$$-M^{\left(\begin{smallmatrix} u_{1234}, & u_{234}, & u_{34}, & u_4 \\ v_1, & v_{2:1}, & v_{3:2}, & v_{4:3} \end{smallmatrix}\right)}$$

The following tables give, for general signatures $\boldsymbol{m(W)} := (m_1, m_2, ...)$, the scramble $S^\bullet$ of $M^\bullet$.

$$\boldsymbol{m} := (1,2) \quad , \quad V_1 = (v_1) \quad , \quad V_2 = (v_2, v_2')$$

$$\begin{aligned}
S^{\left(\begin{smallmatrix} u_1 , & u_2 \\ V_1 , & V_2 \end{smallmatrix}\right)} = \; & +M^{\left(\begin{smallmatrix} u_1 , & u_2 , & u_2 \\ v_1 , & v_2 , & v_{2':2} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{12} , & u_2 , & u_2 \\ v_1 , & v_{2:1} , & v_{2':2} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{12} , & u_1 , & u_2 \\ v_2 , & v_{1:2} , & v_{2':2} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{12} , & u_{12} , & u_2 \\ v_2 , & v_{1:2} , & v_{2':1} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{12} , & u_{12} , & u_1 \\ v_2 , & v_{2':2} , & v_{1:2'} \end{smallmatrix}\right)}
\end{aligned}$$

$$\boldsymbol{m} := (2,1) \quad , \quad V_1 = (v_1, v_1') \quad , \quad V_2 = (v_2)$$

$$\begin{aligned}
S^{\left(\begin{smallmatrix} u_1 , & u_2 \\ V_1 , & V_2 \end{smallmatrix}\right)} = \; & +M^{\left(\begin{smallmatrix} u_1 , & u_1 , & u_2 \\ v_1 , & v_{1':1} , & v_2 \end{smallmatrix}\right)} && +M^{\left(\begin{smallmatrix} u_{12} , & u_{12} , & u_1 \\ v_1 , & v_{2:1} , & v_{1':2} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_1 , & u_2 , & u_1 \\ v_1 , & v_2 , & v_{1':1} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{12} , & u_2 , & u_1 \\ v_1 , & v_{2:1} , & v_{1':1} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{12} , & u_1 , & u_1 \\ v_2 , & v_{1:2} , & v_{1':1} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{12} , & u_{12} , & u_2 \\ v_1 , & v_{1':1} , & v_{2:1'} \end{smallmatrix}\right)}
\end{aligned}$$

$$\boldsymbol{m} := (1,3) \quad , \quad V_1 = (v_1) \quad , \quad V_2 = (v_2, v_2', v_2'')$$

$$\begin{aligned}
S^{\left(\begin{smallmatrix} u_1 , & u_2 \\ V_1 , & V_2 \end{smallmatrix}\right)} = \; & +M^{\left(\begin{smallmatrix} u_1 , & u_2 , & u_2 , & u_2 \\ v_1 , & v_2 , & v_{2':2} , & v_{2'':2'} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{12} , & u_2 , & u_2 , & u_2 \\ v_1 , & v_{2:1} , & v_{2':2} , & v_{2'':2'} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{12} , & u_{12} , & u_{12} , & u_1 \\ v_2 , & v_{2':2} , & v_{2'':2'} , & v_{1:2''} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{12} , & u_{12} , & u_2 , & u_2 \\ v_2 , & v_{1:2} , & v_{2':1} , & v_{2'':2'} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{12} , & u_{12} , & u_1 , & u_2 \\ v_2 , & v_{2':2} , & v_{1:2'} , & v_{2'':2'} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{12} , & u_{12} , & u_{12} , & u_2 \\ v_2 , & v_{2':2} , & v_{1:2'} , & v_{2'':1} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{12} , & u_1 , & u_2 , & u_2 \\ v_2 , & v_{1:2} , & v_{2':2} , & v_{2'':2'} \end{smallmatrix}\right)}
\end{aligned}$$

$$\boldsymbol{m} := (2,2) \quad , \quad V_1 = (v_1, v_1') \quad , \quad V_2 = (v_2, v_2')$$

$$\begin{aligned}
S^{\left(\begin{smallmatrix} u_1 , & u_2 \\ V_1 , & V_2 \end{smallmatrix}\right)} = \; & +M^{\left(\begin{smallmatrix} u_1 , & u_2 , & u_2 , & u_1 \\ v_1 , & v_2 , & v_{2':2} , & v_{1':1} \end{smallmatrix}\right)} && +M^{\left(\begin{smallmatrix} u_{12} , & u_{12} , & u_{12} , & u_1 \\ v_2 , & v_{1:2} , & v_{2':1} , & v_{1':2'} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_1 , & u_2 , & u_1 , & u_2 \\ v_1 , & v_2 , & v_{1':1} , & v_{2':2} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{12} , & u_2 , & u_1 , & u_2 \\ v_1 , & v_{2:1} , & v_{1':1} , & v_{2':2} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_1 , & u_1 , & u_2 , & u_2 \\ v_1 , & v_{1':1} , & v_2 , & v_{2':2} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{12} , & u_{12} , & u_2 , & u_2 \\ v_1 , & v_{1':1} , & v_{2:1'} , & v_{2':2} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{12} , & u_{12} , & u_1 , & u_2 \\ v_1 , & v_{2:1} , & v_{1':2} , & v_{2':2} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{12} , & u_{12} , & u_{12} , & u_2 \\ v_2 , & v_{1:2} , & v_{1':1} , & v_{2':1'} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{12} , & u_{12} , & u_{12} , & u_1 \\ v_1 , & v_{2:1} , & v_{2':2} , & v_{1':2'} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{12} , & u_{12} , & u_{12} , & u_2 \\ v_1 , & v_{2:1} , & v_{1':2} , & v_{2':1'} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{12} , & u_{12} , & u_1 , & u_1 \\ v_2 , & v_{2':2} , & v_{1:2'} , & v_{1':1} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{12} , & u_{12} , & u_2 , & u_1 \\ v_2 , & v_{1:2} , & v_{2':1} , & v_{1':1} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{12} , & u_1 , & u_2 , & u_1 \\ v_2 , & v_{1:2} , & v_{2':2} , & v_{1':1} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{12} , & u_2 , & u_2 , & u_1 \\ v_1 , & v_{2:1} , & v_{2':2} , & v_{1':1} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{12} , & u_1 , & u_1 , & u_2 \\ v_2 , & v_{1:2} , & v_{1':1} , & v_{2':2} \end{smallmatrix}\right)}
\end{aligned}$$

$$\boldsymbol{m} := (3,1) \quad , \quad V_1 = (v_1, v_1', v_1'') \quad , \quad V_2 = (v2)$$

$$\begin{aligned}
S^{\left(\begin{smallmatrix} u_1 , u_2 \\ V_1 , V_2 \end{smallmatrix}\right)} = \; & +M^{\left(\begin{smallmatrix} u_1 , u_1 , u_1 , u_2 \\ v_1 , v_{1':1} , v_{1'':1'} , v_2 \end{smallmatrix}\right)} && +M^{\left(\begin{smallmatrix} u_1 , u_1 , u_2 , u_1 \\ v_1 , v_{1':1} , v_2 , v_{1'':1'} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_1 , u_2 , u_1 , u_1 \\ v_1 , v_2 , v_{1':1} , v_{1'':1'} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{12} , u_{12} , u_{12} , u_2 \\ v_1 , v_{1':1} , v_{1'':1'} , v_{2:1''} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{12} , u_{12} , u_{12} , u_1 \\ v_1 , v_{1':1} , v_{2:1'} , v_{1'':2} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{12} , u_{12} , u_2 , u_1 \\ v_1 , v_{1':1} , v_{2:1'} , v_{1'':1'} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{12} , u_{12} , u_1 , u_1 \\ v_1 , v_{2:1} , v_{1':2} , v_{1'':1'} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{12} , u_2 , u_1 , u_1 \\ v_1 , v_{2:1} , v_{1':1} , v_{1'':1'} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{12} , u_1 , u_1 , u_1 \\ v_2 , v_{1:2} , v_{1':1} , v_{1'':1'} \end{smallmatrix}\right)}
\end{aligned}$$

$$\boldsymbol{m} := (1,1,2) \quad , \quad V_1 = (v_1) \quad , \quad V_2 = (v_2) \quad , \quad V_3 = (v_3, v_3')$$

$$\begin{aligned}
S^{\left(\begin{smallmatrix} u_1 , u_2 , u_3 \\ V_1 , V_2 , V_3 \end{smallmatrix}\right)} = \; & +M^{\left(\begin{smallmatrix} u_1 , u_2 , u_3 , u_3 \\ v_1 , v_2 , v_3 , v_{3':3} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{12} , u_2 , u_3 , u_3 \\ v_1 , v_{2:1} , v_3 , v_{3':3} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{12} , u_3 , u_1 , u_3 \\ v_2 , v_3 , v_{1:2} , v_{3':3} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{12} , u_3 , u_3 , u_2 \\ v_1 , v_3 , v_{3':3} , v_{2:1} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{12} , u_1 , u_3 , u_3 \\ v_2 , v_{1:2} , v_3 , v_{3':3} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_1 , u_{23} , u_3 , u_3 \\ v_1 , v_2 , v_{3:2} , v_{3':3} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{12} , u_3 , u_3 , u_1 \\ v_2 , v_3 , v_{3':3} , v_{1:2} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{12} , u_3 , u_2 , u_3 \\ v_1 , v_3 , v_{2:1} , v_{3':3} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_1 , u_{23} , u_2 , u_3 \\ v_1 , v_3 , v_{2:3} , v_{3':3} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_1 , u_{23} , u_{23} , u_3 \\ v_1 , v_3 , v_{2:3} , v_{3':2} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{123} , u_3 , u_3 , u_2 \\ v_1 , v_{3:1} , v_{3':3} , v_{2:1} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{123} , u_1 , u_3 , u_3 \\ v_2 , v_{1:2} , v_{3:2} , v_{3':3} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{123} , u_3 , u_2 , u_3 \\ v_1 , v_{3:1} , v_{2:1} , v_{3':3} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{123} , u_3 , u_3 , u_1 \\ v_2 , v_{3:2} , v_{3':3} , v_{1:2} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_1 , u_{23} , u_{23} , u_2 \\ v_1 , v_3 , v_{3':3} , v_{2:3'} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{123} , u_3 , u_1 , u_3 \\ v_2 , v_{3:2} , v_{1:2} , v_{3':3} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{123} , u_{123} , u_3 , u_2 \\ v_3 , v_{1:3} , v_{3':1} , v_{2:1} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{123} , u_{12} , u_2 , u_3 \\ v_3 , v_{1:3} , v_{2:1} , v_{3':3} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{123} , u_{12} , u_1 , u_3 \\ v_3 , v_{2:3} , v_{1:2} , v_{3':3} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{123} , u_{12} , u_3 , u_2 \\ v_3 , v_{1:3} , v_{3':3} , v_{2:1} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{123} , u_{23} , u_3 , u_3 \\ v_1 , v_{2:1} , v_{3:2} , v_{3':3} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{123} , u_{123} , u_{12} , u_2 \\ v_3 , v_{3':3} , v_{1:3'} , v_{2:1} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{123} , u_1 , u_{23} , u_2 \\ v_3 , v_{1:3} , v_{3':3} , v_{2:3'} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{123} , u_{123} , u_{23} , u_2 \\ v_3 , v_{1:3} , v_{3':1} , v_{2:3'} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{123} , u_{123} , u_{23} , u_3 \\ v_3 , v_{1:3} , v_{2:1} , v_{3':2} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{123} , u_{23} , u_{23} , u_2 \\ v_1 , v_{3:1} , v_{3':3} , v_{2:3'} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{123} , u_{123} , u_1 , u_2 \\ v_3 , v_{3':3} , v_{1:3'} , v_{2:3'} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{123} , u_{123} , u_3 , u_1 \\ v_3 , v_{2:3} , v_{3':2} , v_{1:2} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{123} , u_{12} , u_3 , u_1 \\ v_3 , v_{2:3} , v_{3':3} , v_{1:2} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{123} , u_{123} , u_1 , u_3 \\ v_3 , v_{2:3} , v_{1:2} , v_{3':2} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{123} , u_{123} , u_{12} , u_1 \\ v_3 , v_{3':3} , v_{2:3'} , v_{1:2)} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{123} , u_{23} , u_2 , u_3 \\ v_1 , v_{3:1} , v_{2:3} , v_{3':3} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{123} , u_1 , u_2 , u_3 \\ v_3 , v_{1:3} , v_{2:3} , v_{3':3} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{123} , u_1 , u_{23} , u_3 \\ v_3 , v_{1:3} , v_{2:3} , v_{3':2} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{123} , u_{23} , u_{23} , u_3 \\ v_1 , v_{3:1} , v_{2:3} , v_{3':2} \end{smallmatrix}\right)}
\end{aligned}$$

$$\boldsymbol{m} := (1,2,1) \quad , \quad V_1 = (v_1) \quad , \quad V_2 = (v_2, v_2') \quad , \quad V_3 = (v_3)$$

$$
\begin{aligned}
S^{\left(\begin{smallmatrix} u_1 , & u_2 , & u_3 \\ V_1 , & V_2 , & V_3 \end{smallmatrix}\right)} = \; & +M^{\left(\begin{smallmatrix} u_1 , & u_2 , & u_2 , & u_3 \\ v_1 , & v_2 , & v_{2':2} , & v_3 \end{smallmatrix}\right)} && +M^{\left(\begin{smallmatrix} u_1 , & u_2 , & u_3 , & u_2 \\ v_1 , & v_2 , & v_3 , & v_{2':2} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_1 , & u_{23} , & u_2 , & u_2 \\ v_1 , & v_3 , & v_{2:3} , & v_{2':2} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{12} , & u_2 , & u_2 , & u_3 \\ v_1 , & v_{2:1} , & v_{2':2} , & v_3 \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{12} , & u_1 , & u_3 , & u_2 \\ v_2 , & v_{1:2} , & v_3 , & v_{2':2} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{12} , & u_2 , & u_3 , & u_2 \\ v_1 , & v_{2:1} , & v_3 , & v_{2':2} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_1 , & u_{23} , & u_{23} , & u_2 \\ v_1 , & v_2 , & v_{3:2} , & v_{2':3} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{12} , & u_3 , & u_{12} , & u_2 \\ v_2 , & v_3 , & v_{1:2} , & v_{2':1} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{12} , & u_3 , & u_1 , & u_2 \\ v_2 , & v_3 , & v_{1:2} , & v_{2':2} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_1 , & u_{23} , & u_3 , & u_2 \\ v_1 , & v_2 , & v_{3:2} , & v_{2':2} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{123} , & u_1 , & u_2 , & u_2 \\ v_3 , & v_{1:3} , & v_{2:3} , & v_{2':2} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{12} , & u_{12} , & u_2 , & u_3 \\ v_2 , & v_{1:2} , & v_{2':1} , & v_3 \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{12} , & u_1 , & u_2 , & u_3 \\ v_2 , & v_{1:2} , & v_{2':2} , & v_3 \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{12} , & u_{12} , & u_3 , & u_2 \\ v_2 , & v_{1:2} , & v_3 , & v_{2':1} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{12} , & u_{12} , & u_1 , & u_3 \\ v_2 , & v_{2':2} , & v_{1:2'} , & v_3 \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{12} , & u_3 , & u_2 , & u_2 \\ v_1 , & v_3 , & v_{2:1} , & v_{2':2} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{12} , & u_3 , & u_{12} , & u_1 \\ v_2 , & v_3 , & v_{2':2} , & v_{1:2} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_1 , & u_{23} , & u_{23} , & u_3 \\ v_1 , & v_2 , & v_{2':2} , & v_{3:2'} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{12} , & u_{12} , & u_3 , & u_1 \\ v_2 , & v_{2':2} , & v_3 , & v_{1:2'} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{123} , & u_{23} , & u_2 , & u_2 \\ v_1 , & v_{3:1} , & v_{2:3} , & v_{2':2} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{123} , & u_3 , & u_2 , & u_2 \\ v_1 , & v_{3:1} , & v_{2:1} , & v_{2':2} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{123} , & u_{123} , & u_{23} , & u_2 \\ v_2 , & v_{1:2} , & v_{3:1} , & v_{2':3} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{123} , & u_{23} , & u_3 , & u_2 \\ v_1 , & v_{2:1} , & v_{3:2} , & v_{2':2} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{123} , & u_{12} , & u_{12} , & u_2 \\ v_3 , & v_{2:3} , & v_{1:2} , & v_{2':1} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{123} , & u_{123} , & u_{12} , & u_1 \\ v_2 , & v_{3:2} , & v_{2':3} , & v_{1:2'} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{123} , & u_1 , & u_3 , & u_2 \\ v_2 , & v_{1:2} , & v_{3:2} , & v_{2':2} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{123} , & u_{23} , & u_{23} , & u_3 \\ v_1 , & v_{2:1} , & v_{2':2} , & v_{3:2'} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{123} , & u_{12} , & u_2 , & u_2 \\ v_3 , & v_{1:3} , & v_{2:1} , & v_{2':2} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{123} , & u_{123} , & u_{23} , & u_3 \\ v_2 , & v_{1:2} , & v_{2':1} , & v_{3:2'} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{123} , & u_{123} , & u_3 , & u_1 \\ v_2 , & v_{2':2} , & v_{3:2'} , & v_{1:2'} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{123} , & u_{123} , & u_1 , & u_2 \\ v_2 , & v_{3:2} , & v_{1:3} , & v_{2':3} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{123} , & u_{123} , & u_{12} , & u_2 \\ v_2 , & v_{3:2} , & v_{1:3} , & v_{2':1} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{123} , & u_3 , & u_{12} , & u_2 \\ v_2 , & v_{3:2} , & v_{1:2} , & v_{2':1} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{123} , & u_1 , & u_{23} , & u_3 \\ v_2 , & v_{1:2} , & v_{2':2} , & v_{3:2'} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{123} , & u_{123} , & u_3 , & u_2 \\ v_2 , & v_{1:2} , & v_{3:1} , & v_{2':1} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{123} , & u_{123} , & u_1 , & u_3 \\ v_2 , & v_{2':2} , & v_{1:2'} , & v_{3:2'} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{123} , & u_{12} , & u_{12} , & u_1 \\ v_3 , & v_{2:3} , & v_{2':2} , & v_{1:2'} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{123} , & u_3 , & u_1 , & u_2 \\ v_2 , & v_{3:2} , & v_{1:2)} , & v_{2':2} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{123} , & u_1 , & u_{23} , & u_2 \\ v_2 , & v_{1:2} , & v_{3:2} , & v_{2':3} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{123} , & u_3 , & u_{12} , & u_1 \\ v_2 , & v_{3:2} , & v_{2':2} , & v_{1:2'} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{123} , & u_{12} , & u_1 , & u_2 \\ v_3 , & v_{2:3} , & v_{1:2} , & v_{2':2} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{123} , & u_{23} , & u_{23} , & u_2 \\ v_1 , & v_{2:1} , & v_{3:2} , & v_{2':3} \end{smallmatrix}\right)}
\end{aligned}
$$

$$\boldsymbol{m} := (2,1,1) \quad , \quad V_1 = (v_1, v_1') \quad , \quad V_2 = (v_2) \quad , \quad V_3 = (v_3)$$

$$
\begin{aligned}
S^{\left(\begin{smallmatrix} u_1 , & u_2 , & u_3 \\ V_1 , & V_2 , & V_3 \end{smallmatrix}\right)} = \;
&+M^{\left(\begin{smallmatrix} u_1 , & u_1 , & u_2 , & u_3 \\ v_1 , & v_{1':1} , & v_2 , & v_3 \end{smallmatrix}\right)}
&&+M^{\left(\begin{smallmatrix} u_1 , & u_2 , & u_3 , & u_1 \\ v_1 , & v_2 , & v_3 , & v_{1':1} \end{smallmatrix}\right)} \\
&+M^{\left(\begin{smallmatrix} u_1 , & u_2 , & u_1 , & u_3 \\ v_1 , & v_2 , & v_{1':1} , & v_3 \end{smallmatrix}\right)}
&&-M^{\left(\begin{smallmatrix} u_1 , & u_1 , & u_{23} , & u_3 \\ v_1 , & v_{1':1} , & v_2 , & v_{3:2} \end{smallmatrix}\right)} \\
&+M^{\left(\begin{smallmatrix} u_1 , & u_{23} , & u_1 , & u_2 \\ v_1 , & v_3 , & v_{1':1} , & v_{2:3} \end{smallmatrix}\right)}
&&-M^{\left(\begin{smallmatrix} u_{12} , & u_3 , & u_2 , & u_1 \\ v_1 , & v_3 , & v_{2:1'} , & v_{1':1} \end{smallmatrix}\right)} \\
&+M^{\left(\begin{smallmatrix} u_{12} , & u_3 , & u_1 , & u_1 \\ v_2 , & v_3 , & v_{1:2} , & v_{1':1} \end{smallmatrix}\right)}
&&-M^{\left(\begin{smallmatrix} u_{12} , & u_2 , & u_1 , & u_3 \\ v_1 , & v_{2:1} , & v_{1':1} , & v_3 \end{smallmatrix}\right)} \\
&+M^{\left(\begin{smallmatrix} u_{12} , & u_1 , & u_3 , & u_1 \\ v_2 , & v_{1:2} , & v_3 , & v_{1':1} \end{smallmatrix}\right)}
&&-M^{\left(\begin{smallmatrix} u_1 , & u_{23} , & u_3 , & u_1 \\ v_1 , & v_2 , & v_{3:2} , & v_{1':1} \end{smallmatrix}\right)} \\
&+M^{\left(\begin{smallmatrix} u_1 , & u_1 , & u_{23} , & u_2 \\ v_1 , & v_{1':1} , & v_3 , & v_{2:3} \end{smallmatrix}\right)}
&&-M^{\left(\begin{smallmatrix} u_{12} , & u_2 , & u_3 , & u_1 \\ v_1 , & v_{2:1} , & v_3 , & v_{1':1} \end{smallmatrix}\right)} \\
&+M^{\left(\begin{smallmatrix} u_{12} , & u_1 , & u_1 , & u_3 \\ v_2 , & v_{1:2} , & v_{1':1} , & v_3 \end{smallmatrix}\right)}
&&-M^{\left(\begin{smallmatrix} u_1 , & u_{23} , & u_1 , & u_3 \\ v_1 , & v_2 , & v_{1':1} , & v_{3:2} \end{smallmatrix}\right)} \\
&+M^{\left(\begin{smallmatrix} u_1 , & u_{23} , & u_2 , & u_1 \\ v_1 , & v_3 , & v_{2:3} , & v_{1':1} \end{smallmatrix}\right)}
&&-M^{\left(\begin{smallmatrix} u_{12} , & u_{12} , & u_2 , & u_3 \\ v_1 , & v_{1':1} , & v_{2:1'} , & v_3 \end{smallmatrix}\right)} \\
&+M^{\left(\begin{smallmatrix} u_{12} , & u_{12} , & u_1 , & u_3 \\ v_1 , & v_{2:1} , & v_{1':2} , & v_3 \end{smallmatrix}\right)}
&&-M^{\left(\begin{smallmatrix} u_{12} , & u_3 , & u_{12} , & u_2 \\ v_1 , & v_3 , & v_{1':1} , & v_{2:1'} \end{smallmatrix}\right)} \\
&+M^{\left(\begin{smallmatrix} u_{12} , & u_3 , & u_{12} , & u_1 \\ v_1 , & v_3 , & v_{2:1} , & v_{1':2} \end{smallmatrix}\right)}
&&-M^{\left(\begin{smallmatrix} u_{12} , & u_{12} , & u_3 , & u_2 \\ v_1 , & v_{1':1} , & v_3 , & v_{2:1'} \end{smallmatrix}\right)} \\
&+M^{\left(\begin{smallmatrix} u_{12} , & u_{12} , & u_3 , & u_1 \\ v_1 , & v_{2:1} , & v_3 , & v_{1':2} \end{smallmatrix}\right)}
&&-M^{\left(\begin{smallmatrix} u_{123} , & u_3 , & u_1 , & u_1 \\ v_2 , & v_{3:2} , & v_{1:2} , & v_{1':1} \end{smallmatrix}\right)} \\
&+M^{\left(\begin{smallmatrix} u_{123} , & u_1 , & u_2 , & u_1 \\ v_3 , & v_{1:3} , & v_{2:3} , & v_{1':1} \end{smallmatrix}\right)}
&&-M^{\left(\begin{smallmatrix} u_{123} , & u_1 , & u_3 , & u_1 \\ v_2 , & v_{1:2} , & v_{3:2} , & v_{1':1} \end{smallmatrix}\right)} \\
&+M^{\left(\begin{smallmatrix} u_{123} , & u_1 , & u_1 , & u_2 \\ v_3 , & v_{1:3} , & v_{1':1} , & v_{2:3} \end{smallmatrix}\right)}
&&-M^{\left(\begin{smallmatrix} u_{123} , & u_{23} , & u_2 , & u_1 \\ v_1 , & v_{3:1} , & v_{2:3} , & v_{1':1} \end{smallmatrix}\right)} \\
&+M^{\left(\begin{smallmatrix} u_{123} , & u_3 , & u_2 , & u_1 \\ v_1 , & v_{3:1} , & v_{2:1} , & v_{1':1} \end{smallmatrix}\right)}
&&-M^{\left(\begin{smallmatrix} u_{123} , & u_3 , & u_{12} , & u_1 \\ v_1 , & v_{3:1} , & v_{2:1} , & v_{1':2} \end{smallmatrix}\right)} \\
&+M^{\left(\begin{smallmatrix} u_{123} , & u_{23} , & u_3 , & u_1 \\ v_1 , & v_{2:1} , & v_{3:2} , & v_{1':1} \end{smallmatrix}\right)}
&&-M^{\left(\begin{smallmatrix} u_{123} , & u_{123} , & u_1 , & u_3 \\ v_1 , & v_{2:1} , & v_{1':2} , & v_{3:2} \end{smallmatrix}\right)} \\
&+M^{\left(\begin{smallmatrix} u_{123} , & u_{23} , & u_1 , & u_3 \\ v_1 , & v_{2:1} , & v_{1':1} , & v_{3:2} \end{smallmatrix}\right)}
&&-M^{\left(\begin{smallmatrix} u_{123} , & u_1 , & u_1 , & u_3 \\ v_2 , & v_{1:2} , & v_{1':1} , & v_{3:2} \end{smallmatrix}\right)} \\
&+M^{\left(\begin{smallmatrix} u_{123} , & u_{12} , & u_1 , & u_1 \\ v_3 , & v_{2:3} , & v_{1:2} , & v_{1':1} \end{smallmatrix}\right)}
&&-M^{\left(\begin{smallmatrix} u_{123} , & u_{12} , & u_2 , & u_1 \\ v_3 , & v_{1:3} , & v_{2:1} , & v_{1':1}) \end{smallmatrix}\right)} \\
&+M^{\left(\begin{smallmatrix} u_{123} , & u_{123} , & u_{23} , & u_3 \\ v_1 , & v_{1':1} , & v_{2:1'} , & v_{3:2} \end{smallmatrix}\right)}
&&-M^{\left(\begin{smallmatrix} u_{123} , & u_{23} , & u_1 , & u_2 \\ v_1 , & v_{3:1} , & v_{1':1} , & v_{2:3} \end{smallmatrix}\right)} \\
&+M^{\left(\begin{smallmatrix} u_{123} , & u_{123} , & u_{12} , & u_1 \\ v_1 , & v_{3:1} , & v_{2:3} , & v_{1':2} \end{smallmatrix}\right)}
&&-M^{\left(\begin{smallmatrix} u_{123} , & u_{123} , & u_3 , & u_1 \\ v_1 , & v_{2:1} , & v_{3:2} , & v_{1':2} \end{smallmatrix}\right)} \\
&+M^{\left(\begin{smallmatrix} u_{123} , & u_3 , & u_{12} , & u_2 \\ v_1 , & v_{3:1} , & v_{1':1} , & v_{2:1'} \end{smallmatrix}\right)}
&&-M^{\left(\begin{smallmatrix} u_{123} , & u_{123} , & u_{23} , & u_2 \\ v_1 , & v_{1':1} , & v_{3:1'} , & v_{2:3} \end{smallmatrix}\right)} \\
&+M^{\left(\begin{smallmatrix} u_{123} , & u_{123} , & u_1 , & u_2 \\ v_1 , & v_{3:1} , & v_{1':3} , & v_{2:3} \end{smallmatrix}\right)}
&&-M^{\left(\begin{smallmatrix} u_{123} , & u_{12} , & u_{12} , & u_2 \\ v_3 , & v_{1:3} , & v_{1':1} , & v_{2:1'} \end{smallmatrix}\right)} \\
&+M^{\left(\begin{smallmatrix} u_{123} , & u_{123} , & u_3 , & u_2 \\ v_1 , & v_{1':1} , & v_{3:1'} , & v_{2:1'} \end{smallmatrix}\right)}
&&-M^{\left(\begin{smallmatrix} u_{123} , & u_{123} , & u_{12} , & u_2 \\ v_1 , & v_{3:1} , & v_{1':3} , & v_{2:1'} \end{smallmatrix}\right)} \\
&+M^{\left(\begin{smallmatrix} u_{123} , & u_{12} , & u_{12} , & u_1 \\ v_3 , & v_{1:3} , & v_{2:1} , & v_{1':2} \end{smallmatrix}\right)}
\end{aligned}
$$

....

....