Resurgent analysis of singularly perturbed differential systems: exit Stokes, enter Tes.

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Abstract. Singular and singularly perturbed differential systems display a dual divergence-cum-resurgence regime, *'equational'* and *'coequational'*, depending on whether we expand the solution in power series of the time variable or the perturbation parameter. In this survey, we compare the two situations and highlight the main difference: complex valued *Stokes constants* there, discrete valued *tessellation coefficients* here.

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Contents

1. Introduction. Model problem.

The formal solutions of singular differential systems, when expanded in inverse-power series of the 'critical variable' z, tend to exhibit divergence, but of a regular and well-understood type: resummable and resurgent, with a resurgence regime completely governed by the now classical Bridge equation. When one introduces a singular perturbation parameter ϵ and expands the solution in powers of the same, divergence and resurgence still rule the show, but the picture becomes incomparably more complex: the resurgence calls for two new Bridge equations, not one; the familiar Stokes constants make way for the radically different tessellation coefficients; and it takes the operator *scram* to fully unravel the mechanisms responsible for this new level of complexity.[1]

1.1. Model problem. Consider the following paradigmatic instance of a *doubly singular* differential system – a system not only singular in itself (relative to the time variable t) but also singularly perturbed (by a small parameter $\epsilon \sim 0$):

$$0 = \epsilon\, t^2\, \partial_t\, y^i + \lambda_i\, y^i + b^i(t, \epsilon, y^1, \ldots, y^\nu) \quad (1 \leqslant i \leqslant \nu) \quad \begin{cases} t \sim 0 & (\textit{variable}) \\ \epsilon \sim 0 & (\textit{parameter}) \end{cases} \tag{1}$$

It is advisable, both technically and theoretically, to change to the problem's 'critical variables' z and x, i.e. to set $z := 1/t \sim \infty$ and $x := 1/\epsilon \sim \infty$ so as to prepare for working in the conjugate Borel planes ζ and ξ. This leads to the system:

$$\partial_z Y = x\,\Lambda\, Y + B(z, x, Y) \qquad \begin{cases} Y = \{Y^i\}\ ,\ B = \{B^i\}\ ,\ \Lambda = \textit{diag.matr.}\{\lambda_i\} \\ B^i \in \mathbb{C}\{z^{-1}, x^{-1}, Y\} \ \ \textit{or} \ \ \in \mathbb{C}\{z^{-1}, Y\} \end{cases} \tag{2}$$

From the viewpoint of x-resurgence, choosing the series B^i independent of x, i.e. taking them in $\mathbb{C}\{z^{-1}, Y\}$ rather than $\mathbb{C}\{z^{-1}, x^{-1}, Y\}$, makes little difference to the resurgence pattern in the ξ-plane, and none at all to the location of the singularities. So we shall henceforth stick with this simplifying assumption.
To respect homogeneity, we may re-write our system in compact form:

$$\partial_z Y^i \;=\; Y^i \left(\lambda_i\, x + \sum_{n_j \geqslant 0\ \textit{if}\ j \neq i}^{1+n_i \geqslant 0} B^i_{\boldsymbol{n}}(z)\, Y^{\boldsymbol{n}}\right) \qquad (1 \leqslant i \leqslant \nu) \tag{3}$$

with coefficients $B^i_{\boldsymbol{n}}(z) \in \mathbb{C}\{z^{-1}\}$ analytic at infinity and x-free.
Let us assume that the multipliers λ_i are neither resonant nor quasi-resonant.[2] To rid the formal solution of ramified terms $z^{\rho_i}\,(\rho_i \notin \mathbb{Z})$, which complicate the formal expansions without adding anything of substance to the Analysis, we also assume $B^i_{\boldsymbol{0}}(z) \equiv 0$. Separating the exponentials from the power series, we get for (3) a

[1] For the sections §2-§8 of the present Survey a more detailed treatment may be found in chap.4 of [5]. However, the material presented here in sections §9-§12 is original.
[2] meaning that the combinations $-\lambda_i + \sum_{n_j \geqslant 0} n_j \lambda$ are all $\neq 0$ and do not approximate 0 abnormally fast (diophantine condition).

formal solution of type[3] :

$$\widetilde{Y}^i(z,x,\boldsymbol{\tau}) \;=\; \widetilde{Y}^i(z,x) + \sum_{n_j\geqslant 0\; if\; j\neq i}^{1+n_i\geqslant 0} \widetilde{Y}^i_{\boldsymbol{n}}(z,x)\;\tau_i \boldsymbol{\tau}^{\boldsymbol{n}}\; e^{(\lambda_i+<\boldsymbol{n},\boldsymbol{\lambda}>)\,zx} \tag{4}$$

1.2. Multiple resurgence. As just pointed out, our formal solution $\widetilde{Y}$, or rather its components $\widetilde{Y}^i_{\boldsymbol{n}}$, can be expanded in power series of z^{-1} or x^{-1}. Both types of expansions are generically divergent yet Borel-summable, but with distinctive *singular points*, *singularities* and *resurgence patterns*. Some form of the Bridge equation applies in both situations, but with distinct index reservoirs $\boldsymbol{\Omega_i}$ and above all with this crucial difference: whereas the ordinary, first-order differential operators $\mathbb{A}_\omega$ that govern the z-resurgence in $\mathbf{BE_1}$ do not depend on z, the differential operators $\mathbb{P}_\omega$ that govern the x-resurgence in $\mathbf{BE_2}$ have coefficients that are themselves divergent-resurgent in x and therefore require a third Bridge equation $\mathbf{BE_3}$ for their description:

Despite these far-going differences, there is bound to be a certain kinship between the two types of resurgence, since in the special case when $B^i_{\boldsymbol{n}}(z) = \beta^i_{\boldsymbol{n}}/z$ with $\beta^i_{\boldsymbol{n}}$ scalar, the variable z and the parameter x coalesce into the block zx. It is this loose kinship, or lax 'duality', that justifies the label *equational* for the z-resurgence and *co-equational* for the x-resurgence.

1.3. The normalisers $\Theta^{\pm 1}$. Rather than handling the general solution $\widetilde{Y}$ of our system, it is often advantageous to work with the information-equivalent but more flexible *normalising* operators $\Theta^{\pm1}$:

$$\begin{cases} \Theta \quad = 1 + \sum^{1\leqslant r}_{i_k,\boldsymbol{n}_k} \qquad e^{|\boldsymbol{u}|xz}\, \widetilde{\mathcal{W}}^{\binom{u_1\;,\ldots,\;u_r}{B^{i_1}_{\boldsymbol{n}_1}\;,\ldots,\;B^{i_r}_{\boldsymbol{n}_r}}}(z,x)\;\; \mathbb{D}^{i_r}_{\boldsymbol{n}_r}\ldots\mathbb{D}^{i_1}_{\boldsymbol{n}_1} \\ \Theta^{-1} \quad = 1 + \sum^{1\leqslant r}_{i_k,\boldsymbol{n}_k} (-1)^r e^{|\boldsymbol{u}|xz}\, \widetilde{\mathcal{W}}^{\binom{u_1\;,\ldots,\;u_r}{B^{i_1}_{\boldsymbol{n}_1}\;,\ldots,\;B^{i_r}_{\boldsymbol{n}_r}}}(z,x)\;\; \mathbb{D}^{i_1}_{\boldsymbol{n}_1}\ldots\mathbb{D}^{i_r}_{\boldsymbol{n}_r} \end{cases} \tag{5}$$

with $u_k :=< \boldsymbol{n}_k,\boldsymbol{\lambda} >$, $\mathbb{D}^{i_k}_{\boldsymbol{n}_k} := \boldsymbol{\tau}^{\boldsymbol{n}_k}\,\tau^{i_k}\partial_{\tau_{i_k}}$, $1 \leqslant i_k \leqslant \nu$, $\boldsymbol{\tau}^{\boldsymbol{n}}_k\,\tau_{i_k} \in \boldsymbol{\tau}^{\mathbb{N}}$ and a symmetral (see §2.2) mould $\widetilde{\mathcal{W}}^{\bullet}$ inductively defined by $\widetilde{\mathcal{W}}^{\varnothing} = 1$ and

$$\partial_z\Big(e^{|\boldsymbol{u}|xz}\,\widetilde{\mathcal{W}}^{\binom{u_1\;,\ldots,\;u_r}{B^{i_1}_{\boldsymbol{n}_1}\;,\ldots,\;B^{i_r}_{\boldsymbol{n}_r}}}(z,x)\Big) = -e^{|\boldsymbol{u}|xz}\,\widetilde{\mathcal{W}}^{\binom{u_1\;,\ldots,\;u_{r-1}}{B^{i_1}_{\boldsymbol{n}_1}\;,\ldots,\;B^{i_{r-1}}_{\boldsymbol{n}_{r-1}}}}(z,x)\,B^{i_r}_{\boldsymbol{n}_r}(z) \tag{6}$$

The operators Θ and Θ^{-1} are (mutually inverse) formal automorphisms

$$\Theta^{\pm1}\Big(\widetilde{\varphi}_1(\boldsymbol{\tau}).\widetilde{\varphi}_2(\boldsymbol{\tau})\Big) \equiv \Big(\Theta^{\pm1}\widetilde{\varphi}_1(\boldsymbol{\tau})\Big)\Big(\Theta^{\pm1}\widetilde{\varphi}_2(\boldsymbol{\tau})\Big) \qquad (\widetilde{\varphi}_i \in \mathbb{C}[[\boldsymbol{\tau}]] = \mathbb{C}[[\tau_1,...,\tau_\nu]])$$

Moreover, they exchange the general solution $\widetilde{Y}$ of our system (3) and the elementary general solution Y_{nor} of the corresponding (linear) normal system:

$$\begin{cases} \partial_z Y^i_{\mathrm{nor}} = \lambda_i\, x\, Y^i_{nor} & ; \;\; Y_{\mathrm{nor}}(z,x,\boldsymbol{\tau}) = \tau_i\; e^{\lambda_i\,x\,z} \quad (1\leqslant i\leqslant \nu) \\ \Theta\,\widetilde{Y}^i(z,x,\boldsymbol{\tau}) \equiv Y^i_{\mathrm{nor}}(z,x,\boldsymbol{\tau}) & ; \;\; \Theta^{-1}\,Y^i_{\mathrm{nor}}(z,x,\boldsymbol{\tau}) \equiv \widetilde{Y}^i(z,x,\boldsymbol{\tau}) \end{cases} \tag{7}$$

[3]The tildas, as usual in resurgence theory, signal formalness. They are often omitted when the very context implies formalness.

1.4. Elementary multilinear inputs: biresurgent monomials. In the above expansions of $\Theta^{\pm}$, the *sensitive* (i.e. generically divergent) ingredients are *symmetral* monomials $\widetilde{\mathcal{W}}^{\bullet}(z,x)$ carrying a two-tier indexation $\binom{u_i}{B^i_{\boldsymbol{n_i}}} = \binom{u_i}{b_i}$ with scalar 'frequencies' $u_i \in \mathbb{C}$ and germs $b_i(z) \in \mathbb{C}\{z^{-1}\}$ holomorphic at $z = \infty$. Removing the exponential factors, the induction rule (6) can be rewritten as

$$(\partial_z + |\boldsymbol{u}|\, x)\ \mathcal{W}^{\binom{u_1\ ,\dots,\ u_r}{b_1\ ,\dots,\ b_r}}(z,x) \quad = \quad -\mathcal{W}^{\binom{u_1\ ,\dots,\ u_{r-1}}{b_1\ ,\dots,\ b_{r-1}}}(z,x)\, b_r(z) \tag{8}$$

with *biresurgent monomials* $\mathcal{W}^{\bullet}(z,x)$ (- separately resurgent in z and x -) that hold the key to everything.

Equational resurgence: The z-Borel tranform turns the induction (8) into (9)

$$\begin{cases} \mathcal{B}_z\ :\ z^{-n} \mapsto \frac{\zeta^{n-1}}{(n-1)!} \quad , \quad b(z) \mapsto \widehat{b}(\zeta) \quad , \quad \mathcal{W}^{\bullet}(z,x) \mapsto \widehat{\mathcal{W}}^{\bullet}(\zeta,x) \\ \widehat{\mathcal{W}}^{\binom{u_1\ ,\dots,\ u_r}{b_1\ ,\dots,\ b_r}}(\zeta,x) = \frac{1}{\zeta-|\boldsymbol{u}|\,x} \int_0^{\zeta} \widehat{\mathcal{W}}^{\binom{u_1\ ,\dots,\ u_{r-1}}{b_1\ ,\dots,\ \hat{b}_{r-1}}}(\zeta_1,x)\, b_r(\zeta-\zeta_1)\, dz_1 \end{cases} \tag{9}$$

and readily yields all the information we need: location of singularities, Stokes constants, pattern of z-resurgence, etc.
Coequational resurgence: The x-Borel tranform turns the induction (8) into (10)

$$\begin{cases} \mathcal{B}_x\ :\ x^{-n} \mapsto \frac{\xi^{n-1}}{(n-1)!} \quad , \quad \mathcal{W}^{\bullet}(z,x) \mapsto \mathcal{B}_x\mathcal{W}^{\bullet}(z,\xi) \\ (\partial_z + |\boldsymbol{u}|\,\partial_\xi)\ \mathcal{B}_x\mathcal{W}^{\binom{u_1\ ,\dots,\ u_r}{b_1\ ,\dots,\ b_r}}(z,\xi) = -\mathcal{B}_x\mathcal{W}^{\binom{u_1\ ,\dots,\ u_{r-1}}{b_1\ ,\dots,\ b_{r-1}}}(z,\xi)\, b_r(z) \end{cases} \tag{10}$$

with $\mathcal{B}_x\mathcal{W}^{\binom{u_1\ ,\dots,\ u_r}{b_1\ ,\dots,\ b_r}}(z,0) = 0\ \forall r \geqslant 2$. For $r = 1$, $\mathcal{B}_x\mathcal{W}^{\binom{u_1}{b_1}}(z,\xi) = -\frac{1}{u_1}\, b_1(z-\frac{\xi}{u_1})$ but no such simplistic formula can be expected for $\mathcal{B}_x\mathcal{W}^{\bullet}(z,\xi)$ when $r \geqslant 2$, and we must then resort to the *weighted convolution weco*, to be introduced in §3.1.

2. Reminders on resurgence, moulds, and hyperlogarithms.

2.1. Reminders about resurgence.
Resurgent functions. They exist simultaneously:
(i) in the *formal model*, usually as divergent power series $\widetilde{\varphi}(z) := \sum c_n z^{-n}$
(ii) in the *convolution model*, as Borel transforms $\widehat{\varphi}(\zeta) := \sum c_n \zeta^{n-1}/(n-1)!$ convergent at $\zeta = 0$, with endless (usually highly ramified) analytic continuation and at most exponential growth at infinity.
(iii) in the *geometric model*, as sectorial germs $\varphi_\theta(z) = \int_{\arg\zeta=\theta} \widehat{\varphi}(\zeta) e^{-z\zeta} d\zeta$.
Alien derivations. The linear operators $\widehat{\Delta}_\omega$ $(\omega \in \widetilde{\mathbb{C} - \{0\}})$ act in this way

$$\widehat{\Delta}_\omega\, \widehat{\varphi}\,(\zeta) := \sum_{\epsilon_i=\pm} \frac{\epsilon_r}{2\pi i} \frac{p!\, q!}{r!}\, \widehat{\varphi}^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{\omega_1\ ,\dots,\ \omega_r}}(\zeta+\omega)\ \ \textit{with}\ \ \omega_r := \omega\ \ \textit{and}\ \ \begin{cases} p := \sum_{\epsilon_i=+}^{1\leqslant i\leqslant r-1} 1 \\ q := \sum_{\epsilon_i=-}^{1\leqslant i\leqslant r-1} 1 \end{cases}$$

in the convolution model. The finite sum on the right-hand side is first defined for small $\zeta \in [0,\omega]$, and then analytically continued in the large. Here $\omega_1, \omega_2 \ldots$ denotes the sequence of singular points lying between 0 and $\omega_r := \omega$, and $\widehat{\varphi}^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{\omega_1\ ,\dots,\ \omega_r}}$ denotes the corresponding determination of $\widehat{\varphi}$. The operators $\widehat{\Delta}_\omega$ and their pull-backs Δ_ω in the multiplicative models (formal/geometric) are *derivations*:

$$\widehat{\Delta}_\omega\,(\widehat{\varphi}_1 * \widehat{\varphi}_2) \equiv (\widehat{\Delta}_\omega\,\widehat{\varphi}_1) * \widehat{\varphi}_2 + \widehat{\varphi}_1 * (\widehat{\Delta}_\omega\,\widehat{\varphi}_2) \quad , \quad \Delta_\omega\,(\varphi_1\varphi_2) \equiv (\Delta_\omega\,\varphi_1)\varphi_2 + \varphi_1(\Delta_\omega\,\varphi_2)$$

The related *'invariant derivations'* $\boldsymbol{\Delta}_\omega := e^{-\omega z}\Delta_\omega$ verify $[\partial_z, \boldsymbol{\Delta}_\omega] = 0$. Lastly, the axis-crossing automophisms $\mathbb{R}_\theta$ and full-turn rotators $\mathbb{R}_{[\theta,\theta+2\pi[}$ are defined by:
$\mathbb{R}_\theta := \exp(2\pi i \sum_{\arg\omega=\theta} \boldsymbol{\Delta}_\omega) \quad , \quad \mathbb{R}_{[\theta,\theta+2\pi[} := \prod_{\theta\leqslant\theta_j<\theta+w\pi} \mathbb{R}_{\theta_j} \quad (\theta_j \downarrow)$
with the factors $\mathbb{R}_{\theta_j}$ arranged according to decreasing values of θ_j.
Active alien algebras. The full Lie algebra *ALIEN* generated by all $\boldsymbol{\Delta}_\omega$ is free, but its concrete incarnations, the *active alien algebras* $ALIEN_{\mathcal{A}}$, tend on the contrary to be isomorphic to algebras of ordinary differential operators. Here, $\mathcal{A}$ denotes any algebra of resurgent functions; $\mathbb{I}_{\mathcal{A}}$ the bilateral ideal of *ALIEN* that annihilates all elements of $\mathcal{A}$ and their alien derivatives; and $ALIEN_{\mathcal{A}} := ALIEN/\mathbb{I}_{\mathcal{A}}$.

2.2. Reminders about moulds. Moulds $M^\bullet$ depend on index sequences $\bullet$. Put another way, they are functions of a *variable number of variables.* There exist about a dozen main *symmetry types* for moulds. Chief amongst these are:

$$\begin{array}{lcccll} A^\bullet \textit{ alternal} & \Leftrightarrow & 0 & \equiv & \sum_{\boldsymbol{\omega}\in\mathrm{sha}(\boldsymbol{\omega}',\boldsymbol{\omega}'')} A^{\boldsymbol{\omega}} & \forall \boldsymbol{\omega}',\boldsymbol{\omega}'' \\ S^\bullet \textit{ symmetral} & \Leftrightarrow & S^{\boldsymbol{\omega}'}S^{\boldsymbol{\omega}''} & \equiv & \sum_{\boldsymbol{\omega}\in\mathrm{sha}(\boldsymbol{\omega}',\boldsymbol{\omega}'')} S^{\boldsymbol{\omega}} & \forall \boldsymbol{\omega}',\boldsymbol{\omega}'' \end{array}$$

where $sha(\boldsymbol{\omega}',\boldsymbol{\omega}'')$ denotes the set of all shufflings of the sequences $\boldsymbol{\omega}'$,$\boldsymbol{\omega}'$. Moulds can be subjected to various operations, chiefly multiplication and composition :

$$\begin{cases} \textit{multiplication}: & C^\bullet = A^\bullet \times B^\bullet \Leftrightarrow C^{\boldsymbol{u}} = \sum^{\boldsymbol{u}\,=\,\boldsymbol{u}'\boldsymbol{u}''} A^{\boldsymbol{u}'}B^{\boldsymbol{u}''} \\ \textit{composition}: & C^\bullet = A^\bullet \circ B^\bullet \Leftrightarrow C^{\boldsymbol{u}} = \sum_{1\leqslant s}^{\boldsymbol{u}\,=\,\boldsymbol{u}^1..\boldsymbol{u}^s} A^{|\boldsymbol{u}^1|,..,|\boldsymbol{u}^s|}B^{\boldsymbol{u}^1}..B^{\boldsymbol{u}^s} \\ \textit{Lie bracket}: & \mathrm{lu}(A^\bullet,B^\bullet) := A^\bullet \times B^\bullet - B^\bullet \times A^\bullet \end{cases}$$

The units for mould *multiplication* resp. *composition* are $1^\bullet$ and $Id^\bullet$:

$$1^\varnothing \equiv 1;\ \ 1^{u_1,\dots u_r} \equiv 0 \ \ \textit{if} \ \ r \neq 0 \ \ , \ \ \mathrm{Id}^{u_1} \equiv 1;\ \ \mathrm{Id}^{u_1,\dots u_r} \equiv 0 \ \ \textit{if} \ \ r \neq 1$$

Multiplication respects symmetrality; composition and *lu* respect alternality.

2.3. Hyperlogarithmic monomials and monics. We require hyperlogarithmic *resurgence monomials*, i.e. resurgent functions as elementary as possible, yet capable of approximating all others. We also require their resurgence constants, or hyperlogarithmic *monics*, to approximate all Stokes constants. Moreover, coequational resurgence makes simultaneous use of multiplication, which keeps singularities in place, and convolution, which 'adds' them. This forces us to juggle two indexations:

- *incremental*, with sequences $(\omega_1,\dots,\omega_r)$ $\qquad(\omega_i = \alpha_i - \alpha_{i-1})$
- *positional*, with sequences $[\alpha_1,\dots,\alpha_r]$ $\qquad(\alpha_i = \omega_1 + \dots + \omega_i)$

The ∂-friendly monomials $\widehat{\mathcal{V}}^\bullet$ and their monics $V^\bullet$ are thus defined:

$$\widehat{\mathcal{V}}^{\,\omega_1,\dots,\omega_r}(\zeta) \equiv \widehat{\mathcal{V}}^{\,[\alpha_1,\dots,\alpha_r]}(\zeta) := \int_0^{\zeta} \frac{d\zeta_r}{\zeta_r-\alpha_r} \dots \int_0^{\zeta_3} \frac{d\zeta_2}{\zeta_2-\alpha_2} \int_0^{\zeta_2} \frac{d\zeta_1}{\zeta_1-\alpha_1}\ (\alpha_1 \neq 0) \qquad (11)$$

$$\widehat{\mathcal{V}}^\bullet(\zeta) := \partial_\zeta\ \widehat{\mathcal{V}}^\bullet(\zeta) \quad ; \quad \widetilde{\mathcal{V}}^\bullet(z) := \textit{Borel pull-back of } \widehat{\mathcal{V}}^\bullet(\zeta) \qquad (12)$$

$$\Delta_{\omega_0}\widetilde{\mathcal{V}}^{\boldsymbol{\omega}}(z) = \sum V^{\boldsymbol{\omega}'}\ \widetilde{\mathcal{V}}^{\boldsymbol{\omega}''}(z) \quad \textit{with} \quad \boldsymbol{\omega}'\boldsymbol{\omega}'' = \boldsymbol{\omega} \ \textit{ and } \ |\boldsymbol{\omega}'| = \omega_0 \qquad (13)$$

$V^\bullet$ is alternal, while $\widehat{\mathcal{V}}^\bullet(\zeta)$ and $\widehat{\mathcal{V}}^\bullet(\zeta)$ verify the symmetrality relations:
$(\ \widehat{\mathcal{V}}^{\,[\boldsymbol{\alpha}']}.\ \widehat{\mathcal{V}}^{\,[\boldsymbol{\alpha}'']})(\zeta) \equiv \sum_{\boldsymbol{\alpha}\in \mathrm{sha}(\boldsymbol{\alpha}',\boldsymbol{\alpha}'')} \widehat{\mathcal{V}}^{\,[\boldsymbol{\alpha}]}(\zeta)\ ;\ (\widehat{\mathcal{V}}^{\boldsymbol{\omega}'} * \widehat{\mathcal{V}}^{\boldsymbol{\omega}''})(\zeta) \equiv \sum_{\boldsymbol{\omega}\in \mathrm{sha}(\boldsymbol{\omega}',\boldsymbol{\omega}'')} \widehat{\mathcal{V}}^{\boldsymbol{\omega}}(\zeta)$

We will repeatedly require the *partial derivatives* of the monics $V^\bullet$ and their monomials $\widetilde{\mathcal{V}}^\bullet(z)$. Dropping the tilda for convenience, we get for $r=1$: $\omega_1\partial_{\omega_1}V^{\omega_1} = 0$ and $\omega_1\partial_{\omega_1}\mathcal{V}^{\omega_1}(z) = z\partial_z\mathcal{V}^{\omega_1}(z) = -1-\omega_1\, z\,\mathcal{V}^{\omega_1}(z)$ and for $r \geqslant 2$:

$$\begin{cases} \omega_1\partial_{\omega_1}V^{\omega_1,\dots,\omega_r} = -V^{\omega_1+\omega_2,\dots,\omega_r} \\ \omega_j\partial_{\omega_j}V^{\omega_1,\dots,\omega_r} = +V^{\omega_1,\dots,\omega_{j-1}+\omega_j,\dots,\omega_r} - V^{\omega_1,\dots,\omega_j+\omega_{j+1},\dots,\omega_r} \\ \omega_r\partial_{\omega_r}V^{\omega_1,\dots,\omega_r} = +V^{\omega_1,\dots,\omega_{r-1}+\omega_r} \end{cases} \qquad (14)$$

$$\begin{cases} \omega_1(\partial_{\omega_1}+z)\,\mathcal{V}^{\omega_1,\dots,\omega_r}(z) = -\mathcal{V}^{\omega_1+\omega_2,\dots,\omega_r}(z) \\ \omega_j(\partial_{\omega_j}+z)\,\mathcal{V}^{\omega_1,\dots,\omega_r}(z) = +\mathcal{V}^{\omega_1,\dots,\omega_{j-1}+\omega_j,\dots,\omega_r}(z) - \mathcal{V}^{\omega_1,\dots,\omega_j+\omega_{j+1},\dots,\omega_r}(z) \\ \omega_r(\partial_{\omega_r}+z)\,\mathcal{V}^{\omega_1,\dots,\omega_r}(z) = +\mathcal{V}^{\omega_1,\dots,\omega_{r-1}+\omega_r}(z) - \mathcal{V}^{\omega_1,\dots,\omega_{r-1}}(z) \\ z(\partial_z+|\boldsymbol{\omega}|)\,\mathcal{V}^{\omega_1,\dots,\omega_r}(z) = -\mathcal{V}^{\omega_1,\dots,\omega_{r-1}}(z) \end{cases} \qquad (15)$$

Transition equations for the monics. The monics $V^\bullet$ are uniform analytic functions of their indices on a number of domains of $\mathbb{C}^r$, but they undergo discontinuous changes of determination from domain to domain according to the formula:

$$D_{\frac{\omega_1+\dots+\omega_i}{\omega_{i+1}+\dots+\omega_r}} V^{\omega_1,\dots,\omega_r} \equiv 2\pi i\, V^{\omega_1,\dots,\omega_i}\, V^{\omega_{i+1},\dots,\omega_r} \qquad (16)$$

with jump operators $D_x\, F(x) := \lim_{\epsilon\to 0}(F(x+i\,\epsilon) - F(x-i\,\epsilon))\ \ (t,\epsilon \in \mathbb{R}^+)$.

3. Weighted products.

3.1. The weighted convolution *weco*.

Proposition 3.1. *For $u_i \in \mathbb{C}$ and $\widehat{c}_i(\xi) \in \mathbb{C}\{\xi\}$, the following integrals*

$$\mathrm{weco}^{\binom{u_1}{\hat{c}_1}}(\xi) \;=\; \frac{1}{u_1}\,\widehat{c}_1(\frac{\xi}{u_1}) \tag{17}$$

$$\mathrm{weco}^{\binom{u_1\ ,\ u_2}{\hat{c}_1\ ,\ \hat{c}_2}}(\xi) \;=\; \int_0^{\theta_*} \widehat{c}_2(\xi_2)\,d\xi_2\,\widehat{c}_1(\xi_1)\frac{1}{u_1}\ \textit{with} \begin{cases} u_1\,\xi_1 + u_2\,\xi_2 = \xi \\ \theta_* := \xi\,(u_1+u_2)^{-1} \end{cases} \tag{18}$$

....

$$\mathrm{weco}^{\binom{u_1\ ,\dots,\ u_r}{\hat{c}_1\ ,\dots,\ \hat{c}_r}}(\xi) \;=\; \begin{cases} \int_0^{\theta_{r+1}} \widehat{c}_r(\xi_r)\,d\xi_r \int_{\xi_r}^{\theta_r} \widehat{c}_{r-1}(\xi_{r-1})\,d\xi_{r-1}\dots \\ \dots\int_{\xi_4}^{\theta_4} \widehat{c}_3(\xi_3)\,d\xi_3 \int_{\xi_3}^{\theta_3} \widehat{c}_2(\xi_2)\,d\xi_2\,\widehat{c}_1(\xi_1)\frac{1}{u_1} \end{cases} \tag{19}$$

$$\textit{with}\qquad \begin{cases} u_1\,\xi_1 + \dots + u_r\,\xi_r = \xi \\ \theta_i := (\xi - (u_i\,\xi_i + \dots + u_r\,\xi_r))(u_1+\dots+u_{i-1})^{-1} \\ \theta_{r+1} := \xi\,(u_1+\dots+u_r)^{-1} \end{cases}$$

unambiguously define germs $\mathrm{weco}^{\binom{u_1\ ,\dots,\ u_r}{c_1\ ,\dots,\ c_r}}(\xi) \in \mathbb{C}\{\xi\}$ *provided* $u_1+\dots+u_i \neq 0$. *The mould* $\mathrm{weco}^{\bullet}(\xi)$ *is symmetral relative to the ordinary (i.e. non-weighted) convolution product in* ξ.

The symmetrality property, not immediately obvious from the above formulae, will result from *weco* being the Borel image of a weighted multiplication *wemu*.

3.2. The eighted multiplication *wemu*.

Proposition 3.2. *The weighted multiplication* wemu *acts in the multiplicative plane on analytic germs at* ∞*:* $(c_1(x),\dots,c_r(x)) \in \mathbb{C}\{x^{-1}\}^r \mapsto \mathrm{wemu}^{\binom{u_1\ ,\dots,\ u_r}{c_1\ ,\dots,\ c_r}}(x) \in \mathbb{C}\{x^{-1}\}$. *When* $u_1+\dots+u_i \neq 0$ *and* $v_i \neq v_{i+1}$*, it is defined by the integrals*

$$\mathrm{wemu}^{\binom{u_1\ ,\dots,\ u_r}{c_1\ ,\dots,\ c_r}}(x) \;:=\; \frac{1}{(2\pi i)^r}\oint_{\Gamma_i} \mathrm{Sa}^{\binom{u_1\ ,\dots,\ u_r}{x_1\ ,\dots,\ x_r}}(x)\ c_1(x_1)\dots c_r(x_r)\ dx_1\dots dx_r \tag{20}$$

with a symmetral kernel $\mathrm{Sa}^{\binom{u_1\ ,\dots,\ u_r}{x_1\ ,\dots,\ x_r}}(x) = \prod_{i=1}^{i=r}\frac{1}{(u_1+\dots+u_i)\,x-(x_1+\dots+x_i)}$ *and integration along loops* Γ_i *large enough to fall within the domains of definition of the integrands* c_i. *The variable* x *itself must be chosen large enough for* $\mathrm{Sa}^{\bullet}(x)$ *to remain pole-free while the integration variables* x_i *run through these loops* Γ_i. *The resulting mould* $\mathrm{wemu}^{\bullet}(x)$ *is symmetral relative to ordinary multiplication.*

We clearly have *weighted distributivity* of the x-differentiation and x-shift relative to the weighted multiplications:

$$\begin{cases} \partial\,\mathrm{wemu}^{\binom{u_1\ ,\dots,\ u_r}{c_1\ ,\dots,\ c_r}}(x) \equiv \sum_{1\leqslant i\leqslant r} u_i\,\mathrm{wemu}^{\binom{u_1\ ,\dots,\ u_i\ ,\dots,\ u_r}{c_1\ ,\dots,\ \partial\,c_i\ ,\dots,\ c_r}}(x) \qquad (\partial := \partial_x) \\ \underline{\tau}\,\mathrm{wemu}^{\binom{u_1\ ,\dots,\ u_r}{c_1\ ,\dots,\ c_r}}(x) \equiv \mathrm{wemu}^{\binom{u_1\ ,\dots,\ u_r}{\underline{\tau}_1\,c_1\ ,\dots,\ \underline{\tau}_r\,c_r}}(x) \qquad \big(\underline{\tau} := e^{\tau\,\partial}, \underline{\tau}_i := e^{u_i\,\tau\,\partial}\big) \end{cases} \tag{21}$$

Proposition 3.3. *Just as ordinary convolution is the Borel image of ordinary multiplication, the weighted convolution* weco *is the Borel images of the weighted multiplication* wemu*:*

$$\begin{cases} c_1(x),\ldots,c_r(x) & \overset{\text{Borel}}{\longrightarrow} \quad \widehat{c}_1(\xi),\ldots,\widehat{c}_r(\xi) \\ \text{wemu}^{\left(\begin{smallmatrix} u_1 &,\ldots, & u_r \\ c_1 &,\ldots, & c_r \end{smallmatrix}\right)}(x) & \overset{\text{Borel}}{\longrightarrow} \quad \text{weco}^{\left(\begin{smallmatrix} u_1 &,\ldots, & u_r \\ \widehat{c}_1 &,\ldots & \widehat{c}_r \end{smallmatrix}\right)}(\xi) \end{cases} \tag{22}$$

3.3. Link with the biresurgent monomials.

Proposition 3.4. *The biresurgent monomials $\mathcal{W}^\bullet(z,x)$ of (8) and their Borel transforms $x \to \xi$ can be expressed in terms of weighted products:*

$$\begin{cases} \mathcal{B}_x \mathcal{W}^{\left(\begin{smallmatrix} u_1 &,\ldots, & u_r \\ b_1 &,\ldots, & b_r \end{smallmatrix}\right)}(z,\xi) = \text{weco}^{\left(\begin{smallmatrix} u_1 &,\ldots, & u_r \\ \widehat{c}_1 &,\ldots, & \widehat{c}_r \end{smallmatrix}\right)}(\xi) & \textit{with} \quad \widehat{c}_i(\xi) := -b_i(z-\xi) \\ \mathcal{W}^{\left(\begin{smallmatrix} u_1 &,\ldots, & u_r \\ b_1 &,\ldots, & b_r \end{smallmatrix}\right)}(z,x) = \text{wemu}^{\left(\begin{smallmatrix} u_1 &,\ldots, & u_r \\ c_1 &,\ldots, & c_r \end{smallmatrix}\right)}(\xi) & \textit{with} \quad c_i(x) := \int_{-i\infty}^{+i\infty} \widehat{c}_i(\xi)\, e^{-x\xi} d\xi \end{cases} \tag{23}$$

with z chosen close enough to ∞ for the inputs $\widehat{c}_i(\xi)$ to be regular at $\xi = 0$.

4. The scramble transform.

4.1. The ordinary scramble. The scramble is a bimould transform of type:

$$\text{scram} : M^\bullet \mapsto SM^\bullet \quad \textit{with} \quad SM^{\boldsymbol{w}} = \sum_{\boldsymbol{w}'} \boldsymbol{\lambda}^{\boldsymbol{w}}_{\boldsymbol{w}'} M^{\boldsymbol{w}'} \quad \textit{and} \quad \begin{cases} \boldsymbol{w} = \left(\begin{smallmatrix} u_1 &,\ldots, & u_r \\ v_1 &,\ldots, & v_r \end{smallmatrix}\right) \\ \boldsymbol{w}' = \left(\begin{smallmatrix} u'_1 &,\ldots, & u'_r \\ v'_1 &,\ldots, & v'_r \end{smallmatrix}\right) \end{cases} \tag{24}$$

with coefficients $\boldsymbol{\lambda}^{\boldsymbol{w}}_{\boldsymbol{w}'} = \pm 1$. The new indices u'_i either reduce to some original u_j or to a gapless sum of such u_j's, while all new indices v'_i either reduce to some original v_j or to a pairwise difference of (not necessarily consecutive) v_j's. Moreover, the 'scalar product' is preserved: $\sum u_i\, v_i = \sum u'_i\, v'_i$. These, incidentally, are standard features of the so-called *flexion structure*, as is the use of shorthand notations for partial sums and pairwise differences: $u_{i,\ldots,j} := u_1 + \ldots + u_j$, $v_{i:j} := v_i - v_j$.
To actually define the expansion (24) we proceed by induction on r and make use of the index removal operators cutfi^{w_0} and cutla^{w_0} (*fi* for *first*, *la* for *last*):

$$\begin{cases} (\text{cutfi}^{w_0} M)^{w_1,\ldots,w_r} & = M^{w_2,\ldots,w_r} \quad \textit{if} \quad w_0 = w_1 \quad \textit{and} \quad 0 \quad \textit{otherwise} \\ (\text{cutla}^{w_0} M)^{w_1,\ldots,w_r} & = M^{w_1,\ldots,w_{r-1}} \quad \textit{if} \quad w_0 = w_r \quad \textit{and} \quad 0 \quad \textit{otherwise} \end{cases} \tag{25}$$

We have the choice between two very dissimilar, yet equivalent inductions:

Forward induction: Let $SM^\bullet := \text{scram}\, M^\bullet$ and $\boldsymbol{w} = \left(\begin{smallmatrix} u_1 &,\ldots, & u_r \\ v_1 &,\ldots, & v_r \end{smallmatrix}\right)$. We start the induction by imposing $SM^{w_1} := M^{w_1}$, and for $r \geqslant 2$ by imposing $\text{cutla}^{w_0}_M\, SM^{\boldsymbol{w}} \equiv 0$ except for w_0 of the form $\left(\begin{smallmatrix} u_r \\ v_r \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} u_i \\ v_i - v_{i+1} \end{smallmatrix}\right)$,$\left(\begin{smallmatrix} u_i \\ v_i - v_{i-1} \end{smallmatrix}\right)$, in which case we set:

$$\left(\text{cutla}^{\left(\begin{smallmatrix} u_r \\ v_r \end{smallmatrix}\right)}_M SM\right)^{\left(\begin{smallmatrix} u_1,\ldots,u_r \\ v_1,\ldots,v_r \end{smallmatrix}\right)} = +SM^{\left(\begin{smallmatrix} u_1,\ldots,u_{r-1} \\ v_1,\ldots,v_{r-1} \end{smallmatrix}\right)} \tag{26}$$

$$\left(\text{cutla}^{\left(\begin{smallmatrix} u_i \\ v_i - v_{i+1} \end{smallmatrix}\right)}_M SM\right)^{\left(\begin{smallmatrix} u_1,\ldots,u_r \\ v_1,\ldots,v_r \end{smallmatrix}\right)} = +SM^{\left(\begin{smallmatrix} u_1 &,\ldots, & u_i + u_{i+1} &,\ldots, & u_r \\ v_1 &,\ldots, & v_{i+1} &,\ldots, & v_r \end{smallmatrix}\right)} \quad (1 \leqslant i < r) \tag{27}$$

$$\left(\text{cutla}^{\left(\begin{smallmatrix} u_i \\ v_i - v_{i-1} \end{smallmatrix}\right)}_M SM\right)^{\left(\begin{smallmatrix} u_1,\ldots,u_r \\ v_1,\ldots,v_r \end{smallmatrix}\right)} = -SM^{\left(\begin{smallmatrix} u_1 &,\ldots, & u_{i-1} + u_i &,\ldots, & u_r \\ v_1 &,\ldots, & v_{i-1} &,\ldots, & v_r \end{smallmatrix}\right)} \quad (1 < i \leqslant r) \tag{28}$$

The lower index M in $cutla_M^{w_0}$ signals that this operator is made to act, not on $SM^\bullet$, but linearly on the various $M^\bullet$-summands of the expansion (24).

Backward induction: This time, we impose $\mathrm{cutfi}_M^{w_0}\, SM^{\boldsymbol{w}} \equiv 0$ except for w_0 of the form $\left(\begin{smallmatrix} u_1+\ldots+u_j \\ v_i \end{smallmatrix}\right)$ with $i \leqslant j \leqslant r$, in which case we set:

$$\left(\mathrm{cutfi}_M^{\left(\begin{smallmatrix} u_1+\ldots+u_j \\ v_i \end{smallmatrix}\right)} SM\right)^{\boldsymbol{w}} = \mathrm{symlin}\left(SM_{v_i}^{\dot{\boldsymbol{w}}}, {}^{iv}SM_{v_i}^{\ddot{\boldsymbol{w}}}, SM^{\vec{\boldsymbol{w}}}\right) \tag{29}$$

with $\dot{\boldsymbol{w}} = \left(\begin{smallmatrix} u_1,\ldots,u_{i-1} \\ v_1,\ldots,v_{i-1} \end{smallmatrix}\right)$, $\ddot{\boldsymbol{w}} = \left(\begin{smallmatrix} u_{i+1},\ldots,u_j \\ v_{i+1},\ldots,v_j \end{smallmatrix}\right)$, $\vec{\boldsymbol{w}} = \left(\begin{smallmatrix} u_{j+1},\ldots,u_r \\ v_{j+1},\ldots,v_r \end{smallmatrix}\right)$ and

$${}^{iv}SM^{w_1,\ldots,w_r} := (-1)^r\, SM^{w_r,\ldots,w_1} \quad , \quad SM_{v_0}^{\left(\begin{smallmatrix} u_1 & ,\ldots, & u_r \\ v_1 & ,\ldots, & v_r \end{smallmatrix}\right)} := SM^{\left(\begin{smallmatrix} u_1 & ,\ldots, & u_r \\ v_1-v_0 & ,\ldots, & v_r-v_0 \end{smallmatrix}\right)}$$

and *concat* ('concatenation') and *symlin* ('symmetral linearisation') so defined: $\mathrm{concat}(SM^{\boldsymbol{w}^1}, SM^{\boldsymbol{w}^2}) := SM^{\boldsymbol{w}^1\,\boldsymbol{w}^2}$, $\mathrm{symlin}(SM^{\boldsymbol{w}^1}, SM^{\boldsymbol{w}^2}) := \sum_{\boldsymbol{w}\in\mathrm{sha}(\boldsymbol{w}^1,\,\boldsymbol{w}^2)} SM^{\boldsymbol{w}}$ The relation $S^{\boldsymbol{\omega}^1} S^{\boldsymbol{\omega}^2} \equiv \sum_{\boldsymbol{\omega}\in\mathrm{sha}(\boldsymbol{\omega}^1,\,\boldsymbol{\omega}^2)} S^{\boldsymbol{\omega}}$ characterises symmetrality. But in the backward induction the rule (29) *always* applies, even for $SM^\bullet$ non-symmetral. Both inductions are equivalent due to the commutation $[\mathrm{cutfi}_M^{w_1}, \mathrm{cutla}_M^{w_2}] \equiv 0$ and *scram* respect the basic symmetries: $\begin{cases} \{A^\bullet \textit{ alternal}\} & \Rightarrow \{\mathrm{scram}.A^\bullet \textit{ alternal}\} \\ \{S^\bullet \textit{ symmetral}\} & \Rightarrow \{\mathrm{scram}.S^\bullet \textit{ symmetral}\} \end{cases}$

Analytical expression: The backward induction makes it clear that $scram\,A^{w_1,\ldots,w_r}$ involves $r!! := 1.3.5\ldots(2.r-1)$ summands. Thus, for $r = 1, 2, 3$, we find:

$$\begin{aligned}
(\mathrm{scram}\, M)^{\left(\begin{smallmatrix} u_1 \\ v_1 \end{smallmatrix}\right)} &= M^{\left(\begin{smallmatrix} u_1 \\ v_1 \end{smallmatrix}\right)} \\
(\mathrm{scram}\, M)^{\left(\begin{smallmatrix} u_1 & , & u_2 \\ v_1 & , & v_2 \end{smallmatrix}\right)} &= M^{\left(\begin{smallmatrix} u_1, & u_2 \\ v_1, & v_2 \end{smallmatrix}\right)} + M^{\left(\begin{smallmatrix} u_{1,2}, & u_1 \\ v_2 & , v_{1:2} \end{smallmatrix}\right)} - M^{\left(\begin{smallmatrix} u_{1,2}, & u_2 \\ v_1 & , v_{2:1} \end{smallmatrix}\right)} \\
(\mathrm{scram}\, M)^{\left(\begin{smallmatrix} u_1, u_2, u_3 \\ v_1, v_2, v_3 \end{smallmatrix}\right)} &= +M^{\left(\begin{smallmatrix} u_1, & u_2, & u_3 \\ v_1, & v_2, & v_3 \end{smallmatrix}\right)} + M^{\left(\begin{smallmatrix} u_1, & u_{2,3}, & u_2 \\ v_1, & v_3 & , v_{2:3} \end{smallmatrix}\right)} - M^{\left(\begin{smallmatrix} u_1, & u_{2,3}, & u_3 \\ v_1, & v_2 & , v_{3:2} \end{smallmatrix}\right)} \\
&\quad + M^{\left(\begin{smallmatrix} u_{1,2}, & u_1 & , u_3 \\ v_2 & , v_{1:2}, & v_3 \end{smallmatrix}\right)} - M^{\left(\begin{smallmatrix} u_{1,2}, & u_2 & , u_3 \\ v_1 & , v_{2:1}, & v_3 \end{smallmatrix}\right)} \\
&\quad + M^{\left(\begin{smallmatrix} u_{1,2}, & u_3, & u_1 \\ v_2 & , v_3, & v_{1:2} \end{smallmatrix}\right)} - M^{\left(\begin{smallmatrix} u_{1,2}, & u_3, & u_2 \\ v_1 & , v_3, & v_{2:1} \end{smallmatrix}\right)} \\
&\quad + M^{\left(\begin{smallmatrix} u_{1,2,3}, & u_{2,3}, & u_3 \\ v_1 & , v_{2:1}, & v_{3:2} \end{smallmatrix}\right)} - M^{\left(\begin{smallmatrix} u_{1,2,3}, & u_{2,3}, & u_2 \\ v_1 & , v_{3:1}, & v_{2:3} \end{smallmatrix}\right)} + M^{\left(\begin{smallmatrix} u_{1,2,3}, & u_3 & , u_2 \\ v_1 & , v_{3:1}, & v_{2:1} \end{smallmatrix}\right)} \\
&\quad - M^{\left(\begin{smallmatrix} u_{1,2,3}, & u_1 & , u_3 \\ v_2 & , v_{1:2}, & v_{3:2} \end{smallmatrix}\right)} - M^{\left(\begin{smallmatrix} u_{1,2,3}, & u_3 & , u_1 \\ v_2 & , v_{3:2}, & v_{1:2} \end{smallmatrix}\right)} \\
&\quad + M^{\left(\begin{smallmatrix} u_{1,2,3}, & u_1 & , u_2 \\ v_3 & , v_{1:3}, & v_{2:3} \end{smallmatrix}\right)} - M^{\left(\begin{smallmatrix} u_{1,2,3}, & u_{1,2}, & u_2 \\ v_3 & , v_{1:3}, & v_{2:1} \end{smallmatrix}\right)} + M^{\left(\begin{smallmatrix} u_{1,2,3}, & u_{1,2}, & u_1 \\ v_3 & , v_{2:3}, & v_{1:2} \end{smallmatrix}\right)}
\end{aligned}$$

4.2. The v-augmented scramble. It turns ordinary bimoulds $M^{\boldsymbol{w}}$ into v-augmented bimoulds $SM^{\underline{\boldsymbol{w}}}$ with $\underline{w}_i = \left(\begin{smallmatrix} u_i \\ \underline{v}_i \end{smallmatrix}\right)$ and lower indices $\underline{v}_i$ that are sequences. Mark the abbreviations: $\underline{v}_i = (v_i, v_i', \ldots, v_i^\ddagger, v_i^\dagger)$, $\underline{v}_i^* = (v_i, v_i', \ldots, v_i^\ddagger)$, ${}^*\underline{v}_i = (v_i', \ldots, v_i^\ddagger, v_i^\dagger)$

Forward induction. For r=1 and $\underline{w}_1 = \left(\begin{smallmatrix} u_1 \\ \underline{v}_1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} u_1 \\ v_1, v_1', v_1'' \ldots, v_1^\ddagger, v_1^\dagger \end{smallmatrix}\right)$ we start the induction by setting: $SM^{\left(\begin{smallmatrix} u_1 \\ \underline{v}_1 \end{smallmatrix}\right)} := M^{\left(\begin{smallmatrix} u_1, & u_1 & , & u_1 & , & u_1 & ,\ldots, & u_1 \\ v_1 & , v_1'-v_1 & , & v_1''-v_1' & , & v_1'''-v_1'' & ,\ldots, & v_i^\dagger - v_i^\ddagger \end{smallmatrix}\right)}$ To proceed, we distinguish four cases, depending on the nature of the last index w_0 of the sequences $\boldsymbol{w}$ in the various summands $M^{\boldsymbol{w}}$ occuring in the expansion of $SM^{\underline{\boldsymbol{w}}}$:

$$\begin{cases} w_0 &= \binom{u_r}{v_r} & \textit{with} \ \ \#(\underline{v}_r) = 1 \ \ \textit{and} \ \ r = \#(\underline{\boldsymbol{w}}) \\ w_0 &= \binom{u_i}{v_i^\dagger - v_i^\ddagger} & \textit{with} \ \ \#(\underline{v}_i) \geqslant 2 \\ w_0 &= \binom{u_i}{v_i^\dagger - v_{i+1}^\dagger} & \textit{with} \ \ i < r = \#(\underline{\boldsymbol{w}}) \\ w_0 &= \binom{u_i}{v_i^\dagger - v_{i-1}^\dagger} & \textit{with} \ \ 1 < i \end{cases} \tag{30}$$

The linear operators $\mathrm{cutla}_M^{w_0}$ are defined as in §4.1. They act by removing the last index of $M^{\boldsymbol{w}}$ (not of $SM^{\underline{\boldsymbol{w}}}$!) if that last index happens to be w_0, and by annihilating $M^{\boldsymbol{w}}$ otherwise. We set:

$$\begin{cases} \mathrm{cutla}_M^{\binom{u_r}{\underline{v}_r^\dagger}} \, SM^{\underline{w}_1,\dots,\underline{w}_r} &= +SM^{\underline{w}_1,\dots,\underline{w}_{r-1}} \\ \mathrm{cutla}_M^{\binom{u_i}{v_i^\dagger - v_i^\ddagger}} \, SM^{\underline{w}_1,\dots,\underline{w}_r} &= +SM^{\underline{w}_1,\dots,\underline{w}_i^*,\dots,\underline{w}_r} \quad \textit{with} \ \ \underline{w}_i^* = \binom{u_i}{\underline{v}_i^*} \\ \mathrm{cutla}_M^{\binom{u_i}{v_i^\dagger - v_{i+1}^\dagger}} \, SM^{\underline{w}_1,\dots,\underline{w}_r} &= +\sum_{\underline{w}_{i,i+1}^+ \in W_{i,i+1}^+} SM^{\underline{w}_1,\dots,\underline{w}_{i,i+1}^+,\dots,\underline{w}_r} \\ \mathrm{cutla}_M^{\binom{u_i}{v_i^\dagger - v_{i-1}^\dagger}} \, SM^{\underline{w}_1,\dots,\underline{w}_r} &= -\sum_{\underline{w}_{i-1,i}^- \in W_{i-1,i}^-} SM^{\underline{w}_1,\dots,\underline{w}_{i-1,i}^-,\dots,\underline{w}_r} \end{cases}$$

with indices $\underline{w}_{i,i+1}^+$ and $\underline{w}_{i-1,i}^-$ running through the sets

$$\begin{cases} W_{i,i+1}^+ &:= \bigcup_{\underline{v}^*_{i,i+1} \in \mathrm{sha}(\underline{v}_i^*, \underline{v}^*_{i+1})} \left\{ \binom{u_i + u_{i+1}}{\underline{v}^*_{i,i+1}, v^\dagger_{i+1}} \right\} \\ W_{i-1,i}^- &:= \bigcup_{\underline{v}^*_{i-1,i} \in \mathrm{sha}(\underline{v}^*_{i-1}, \underline{v}_i^*)} \left\{ \binom{u_{i-1} + u_i}{\underline{v}^*_{i-1,i}, v^\dagger_{i-1}} \right\} \end{cases} \tag{31}$$

Backward induction. The only operators $\mathrm{cutfi}_M^{w_0}$ acting non-trivially on the $SM^{\underline{\boldsymbol{w}}}$ (viewed as a sum of $M^{\boldsymbol{w}}$ summands) have indices w_0 of the form $\binom{u_1+\ldots+u_j}{v_i}$, where v_i is the first element of some $\underline{v}_i$ with $1 \leqslant i \leqslant j$. The corresponding rule reads:

$$\mathrm{cutfi}_M^{\binom{u_1+\ldots+u_j}{v_i}} SM^{\underline{\boldsymbol{w}}} = \mathrm{symlin}\Big(\mathrm{concat}\big(\mathrm{symlin}(SM_{v_i}^{\underline{\dot{\boldsymbol{w}}}}, {}^{iv}SM_{v_i}^{\underline{\ddot{\boldsymbol{w}}}}), {}^{\sharp}SM_{v_i}^{\underline{w}_i}\big), SM^{\underline{\vec{\boldsymbol{w}}}}\Big)$$

with $\underline{\boldsymbol{w}} := (\underline{w}_1, \ldots, \underline{w}_r)$, $\underline{\dot{\boldsymbol{w}}} := (\underline{w}_1, \ldots, \underline{w}_{i-1})$, $\underline{\ddot{\boldsymbol{w}}} := (\underline{w}_{i+1}, \ldots, \underline{w}_j)$, $\underline{\vec{\boldsymbol{w}}} := (\underline{w}_{j+1}, \ldots, \underline{w}_r)$ and the following conventions:

$$\begin{cases} SM_{v_0}^{\binom{u_1 \ ,\ldots\ldots,\ u_r}{\underline{v}_1 \ ,\ldots\ldots,\ \underline{v}_r}} := SM^{\binom{u_1 \ ,\ldots\ldots,\ u_r}{\underline{v}_1 - v_0 \ ,\ldots\ldots,\ \underline{v}_r - v_0}} \\ {}^{iv}SM_{v_0}^{\binom{u_1 \ ,\ldots\ldots,\ u_r}{\underline{v}_1 \ ,\ldots\ldots,\ \underline{v}_r}} := (-1)^r SM^{\binom{u_r \ ,\ldots\ldots,\ u_1}{\underline{v}_r - v_0 \ ,\ldots\ldots,\ \underline{v}_1 - v_0}} \\ {}^{\sharp}SM_{v_0}^{\binom{u_i}{v_i, v_i', v_i'', v_i''' \ldots}} :={}^{\sharp} SM^{\binom{u_i}{v_i' - v_0, v_i'' - v_0, v_i''' - v_0 \ldots}} \quad (v_i \ \ \textit{gets removed}) \\ \underline{v}_i - v_0 := (v_i - v_0, v_i' - v_0, v_i'' - v_0 \ldots) \qquad \textit{if} \qquad \underline{v}_i := (v_i, v_i', v_i'' \ldots) \end{cases}$$

4.3. Weighted convolution with polar or hyperlogarithmic inputs.

Proposition 4.1. *The weighted convolution of simple poles* $\pi_i(\xi) = (\xi - \alpha_i)^{-1}$ *coincides with the x-Borel transform* $\widehat{\mathcal{S}}^{\bullet}(\xi)$ *of* $\mathcal{S}^{\boldsymbol{w}}(x)$ *for indices* $w_i = \binom{u_i}{\alpha_i}$. *Similarly,* *the bi-resurgent monomials $\mathcal{W}^\bullet(z,x)$ of (8) with polar inputs $b_i(z) := (z-\alpha_i)^{-1}$, coincide with $\mathcal{S}^{\boldsymbol{w}}(x)$ for indices $w_i = \binom{u_i}{z-\alpha_i}$. In other words:*

$$\mathrm{weco}^{\binom{u_1\ ,\ldots,\ u_r}{\pi_1\ ,\ldots,\ \pi_r}}(\xi) \;=\; \widehat{\mathcal{S}}^{\binom{u_1\ ,\ldots,\ u_r}{\alpha_1\ ,\ldots,\ \alpha_r}}(\xi) \qquad \textit{with} \quad \pi_i(\xi) = \frac{1}{\xi-\alpha_i} \tag{32}$$

$$\mathcal{W}^{\binom{u_1\ ,\ldots,\ u_r}{b_1\ ,\ldots,\ b_r}}(z,x) \;=\; \mathcal{S}^{\binom{u_1\ ,\ldots,\ u_r}{z-\alpha_1\ ,\ldots,\ z-\alpha_r}}(x) \quad \textit{with} \quad b_i(z) = \frac{1}{z-\alpha_i} \tag{33}$$

Proposition 4.2. *The weighted convolution of hyperlogarithmic functions $\pi_i(\xi) = \widehat{\mathcal{V}}^{[\underline{\alpha}_i]}(\xi)$ coincides with the x-Borel transform $\widehat{\mathcal{S}}^{\underline{\boldsymbol{w}}}(\xi)$ of $\mathcal{S}^{\underline{\boldsymbol{w}}}(x)$ for indices $\underline{w}_i = \binom{u_i}{\underline{\alpha}_i}$. Similarly, the bi-resurgent monomials $\mathcal{W}^\bullet(z,x)$ of (8) with inputs $b_i(z) = \widehat{\mathcal{V}}^{[\underline{\alpha}_i]}(z)$, coincides with a finite sum of $\mathcal{S}^{\underline{\boldsymbol{w}}}(x)$ with indices $\underline{w}_i = \binom{u_i}{z-\underline{\alpha}''_i}$.*

$$\mathrm{weco}^{\binom{u_1\ ,\ldots,\ u_r}{\pi_1\ ,\ldots,\ \pi_r}}(\xi) \;=\; \widehat{\mathcal{S}}^{\binom{u_1\ ,\ldots,\ u_r}{\underline{\alpha}_1\ ,\ldots,\ \underline{\alpha}_r}}(\xi) \qquad \textit{with} \quad \pi_i(\xi) = \widehat{\mathcal{V}}^{[\alpha_i,\alpha'_i,\ldots]}(\xi) \tag{34}$$

$$\mathcal{W}^{\binom{u_1\ ,\ldots,\ u_r}{b_1\ ,\ldots,\ b_r}}(z,x) \;=\; \sum_{\underline{\alpha}'_i\underline{\alpha}''_i=\underline{\alpha}_i} c^{\underline{\alpha}'_1,\ldots,\underline{\alpha}'_r}(z)\,\mathcal{S}^{\binom{u_1\ ,\ldots,\ u_r}{z-\underline{\alpha}''_1\ ,\ldots,\ z-\underline{\alpha}''_r}}(x) \tag{35}$$

The above statements assume $u_1+\ldots+u_i \neq 0$ and, failing that hypothesis, call for minor modifications. Their proof reduces to showing, based on the identities (14), that the expressions found for the biresurgent monomials verify the required differential properties in x, z, and all indices. Lastly, (35) relies on decompositions $\widehat{\mathcal{V}}^{\underline{\alpha}}(z-\xi) = \sum_{\underline{\alpha}'\underline{\alpha}''=\underline{\alpha}} c^{\underline{\alpha}'}(z)\widehat{\mathcal{V}}^{\underline{\alpha}''}(\xi)$ with coefficients $c^{\underline{\alpha}'}(z)$ independent of ξ.

5. Hyperlogarithmic monomials under alien differentiation.

How do we calculate the alien derivatives of the monomials $\mathcal{S}^\bullet(x)$? In a sense, we already 'know' the answer: having expanded $\mathcal{S}^\bullet(x)$ into finite sums of hyperlogarithms $\mathcal{V}^\bullet(x)$ and possessing with formula (12) a prescription for alien-differentiating each $\mathcal{V}^\bullet(x)$, we have all it takes to calculate $\Delta_{\omega_0}\mathcal{S}^\bullet(x)$. In practice, however, we require explicit and compact formulae covering each one of the many possible situations. This is the object of the present section.

5.1. The ordinary monomials $\mathcal{S}^{\boldsymbol{w}}(x)$.

Proposition 5.1 (Alien derivatives of $\mathcal{S}^{\boldsymbol{w}}(x)$). *The only alien derivations Δ_{ω_0} acting effectively on a given monomial $\mathcal{S}^{\boldsymbol{w}}(x) = \mathcal{S}^{\binom{u_1\ ,\ldots,u_*\ ,\ldots,u_r}{v_1\ ,\ldots,v_*\ ,\ldots,v_r}}$ correspond either to simple indices ω_0 of the form (36) or to composite ones of the form (37) :*

$$\omega_0 \;=\; |\boldsymbol{u}|\,v_* \quad \textit{with} \quad \boldsymbol{w} = \dot{\boldsymbol{w}}.w_*.\ddot{\boldsymbol{w}}.\vec{\boldsymbol{w}} \ \ , \ \ |\boldsymbol{u}| = |\dot{\boldsymbol{u}}| + u_* + |\ddot{\boldsymbol{u}}| \tag{36}$$

$$\omega_0 \;=\; |\boldsymbol{u}^{\boldsymbol{1}}|\,v_{1,*} + \ldots + |\boldsymbol{u}^{\boldsymbol{s}}|\,v_{s,*} \quad \textit{with} \quad \begin{cases} \boldsymbol{w} = \dot{\boldsymbol{w}}^{\boldsymbol{1}}.w_{1*}.\ddot{\boldsymbol{w}}^{\boldsymbol{1}}\ldots\dot{\boldsymbol{w}}^{\boldsymbol{s}}.w_{s*}.\ddot{\boldsymbol{w}}^{\boldsymbol{s}}.\vec{\boldsymbol{w}} \\ |\boldsymbol{u}^{\boldsymbol{i}}| = |\dot{\boldsymbol{u}}^{\boldsymbol{i}}| + u_{i*} + |\ddot{\boldsymbol{u}}^{\boldsymbol{i}}| \end{cases} \tag{37}$$

For a simple index ω_0, the operator Δ_{ω_0} acts as follows:

$$\Delta_{\omega_0}\,\mathcal{S}^{\boldsymbol{w}}(x) \quad = \quad \mathcal{T}^{\dot{\boldsymbol{w}};\ddot{\boldsymbol{w}}}_{v_*}(x)\,\mathcal{S}^{\vec{\boldsymbol{w}}}(x) \begin{cases} \mathcal{T}^{\dot{\boldsymbol{w}};\ddot{\boldsymbol{w}}} := \mathcal{S}^{\dot{\boldsymbol{w}}}\;{}^{iv}\mathcal{S}^{\ddot{\boldsymbol{w}}} \quad , \quad \mathcal{T}^{\dot{\boldsymbol{w}};\ddot{\boldsymbol{w}}}_{v_*} := \mathcal{S}^{\dot{\boldsymbol{w}}}_{v_*}\;{}^{iv}\mathcal{S}^{\ddot{\boldsymbol{w}}}_{v_*} \\ {}^{iv}\mathcal{S}^{w_1,\dots,w_r} = (-1)^r\;\mathcal{S}^{w_r,\dots,w_1} \\ \mathcal{S}_{v_*}^{\binom{u_1\;,\dots,\;u_r}{v_1\;,\dots,\;v_r}} := \mathcal{S}^{\binom{u_1\quad,\dots,\quad u_r}{v_1-v_*\;,\dots,\;v_r-v_*}} \end{cases} \tag{38}$$

For a composite ω_0, a new ingredient comes in: the tessellation bimould $\mathrm{tes}^\bullet$*, defined as the scramble of the mould $V^\bullet$ or rather its bimould extension $\underline{V}^\bullet$:*

$$\Delta_{\omega_0}\,\mathcal{S}^{\boldsymbol{w}}(x) \quad = \quad \mathrm{tes}^{\binom{|\boldsymbol{u^1}|\;,\dots,\;|\boldsymbol{u^s}|}{v_1\;,\dots,\;v_s}}\;\mathcal{T}^{\dot{\boldsymbol{w}}^{\boldsymbol{1}};\ddot{\boldsymbol{w}}^{\boldsymbol{1}}}_{v_{1*}}(x)\dots\mathcal{T}^{\dot{\boldsymbol{w}}^{\boldsymbol{s}};\ddot{\boldsymbol{w}}^{\boldsymbol{s}}}_{v_{s*}}(x)\,\mathcal{S}^{\vec{\boldsymbol{w}}}(x) \tag{39}$$

with $V^\bullet$ as in (12), $\underline{V}^{\binom{u_1\;,\dots,\;u_r}{v_1\;,\dots,\;v_r}} := V^{u_1\,v_1,\dots,u_r\,v_r}$, and $\mathrm{tes}^\bullet := \mathrm{scram}.\underline{V}^\bullet$.

5.2. The v-augmented monomials $\mathcal{S}^{\underline{\boldsymbol{w}}}(x)$. To enunciate suitably compact statements, we need the following:

Definition 5.1. *Let v_* be some element (- first, middle, last -) of some lower index $\underline{v}_*$ inside a sequence $\underline{\boldsymbol{w}} = \binom{u_1\;,\dots,\;u_*\;,\dots,\;u_r}{\underline{v}_1\;,\dots,\;\underline{v}_*\;,\dots,\;\underline{v}_r}$. A $\underline{v}_*$-splitting of $\underline{\boldsymbol{w}}$ is a joint factorisation of all $\underline{v}_i$ such that* $\begin{cases} \underline{v}_i \;\;=\; (\underline{v}'_i, \underline{v}''_i) \;\; \text{if} \;\; \underline{v}_i \neq \underline{v}_* \;\; (\text{only } \underline{v}''_i \text{ may be } \varnothing) \\ \underline{v}_* \;=\; (\underline{v}'_*, v_*, \underline{v}''_*) \;\; (\text{both } \underline{v}'_* \text{ and } \underline{v}''_* \text{ may be } \varnothing) \end{cases}$
To each $\underline{v}_*$-splitting we associate (i) a non-ordered sequence $\{\underline{\boldsymbol{v}}'\}$ consisting of ordered sequences $\underline{v}'_i$ (ii) two ordered sequences $\underline{\dot{\boldsymbol{w}}}''$ and $\underline{\ddot{\boldsymbol{w}}}''$ (iii) a lone index $\underline{w}''_*$ (that may be empty). They are defined in this way:

$$\begin{cases} \{\underline{\boldsymbol{v}}'\} \;\; := \;\; \{\underline{v}'_1; \underline{v}'_2; \dots; \underline{v}'_*; \dots; \underline{v}'_{r-1}; \underline{v}'_r\} \\ \underline{\dot{\boldsymbol{w}}}'' \;\; := \;\; (\underline{w}''_1, \dots, \underline{w}''_i, \dots) = \binom{u_1\;,\dots,\;u_i\;,\dots}{\underline{v}''_1\;,\dots,\;\underline{v}''_i\;,\dots} \quad \text{with } \underline{w}_i \text{ earlier than } \underline{w}_* \\ \underline{\ddot{\boldsymbol{w}}}'' \;\; := \;\; (\dots, \underline{w}''_i, \dots, \underline{w}_r) = \binom{\dots,\;u_i\;,\dots,\;u_r}{\dots,\;\underline{v}''_i\;,\dots,\;\underline{v}''_r} \quad \text{with } \underline{w}_i \text{ later than } \underline{w}_* \\ \underline{w}''_* \;\; := \;\; \binom{u_*}{\underline{v}''_*} \qquad (\underline{w}''_* := \varnothing \;\text{ if }\; \underline{v}''_* := \varnothing) \end{cases}$$

Proposition 5.2 (Alien derivatives of $\mathcal{S}^{\underline{\boldsymbol{w}}}(x)$). *Once again, the only alien derivations Δ_{ω_0} acting effectively on a general monomial $\mathcal{S}^{\underline{\boldsymbol{w}}}(x) = \mathcal{S}^{\binom{u_1\;,\dots,\;u_*\;,\dots,\;u_r}{\underline{v}_1\;,\dots,\;\underline{v}_*\;,\dots,\;\underline{v}_r}}$ correspond to indices ω_0 either simple (40) or composite (41):*

$$\omega_0 = |\boldsymbol{u}|\,v_* \qquad \text{with} \quad \underline{\boldsymbol{w}} = \underline{\dot{\boldsymbol{w}}}.\underline{w}_*.\underline{\ddot{\boldsymbol{w}}}.\underline{\vec{\boldsymbol{w}}}\;,\; |\boldsymbol{u}| = |\dot{\boldsymbol{u}}| + u_* + |\ddot{\boldsymbol{u}}| \tag{40}$$

$$\omega_0 = \sum_{1\leqslant i\leqslant s} |\boldsymbol{u^i}|\,v_{i*} \quad \text{with} \quad \begin{cases} \underline{\boldsymbol{w}} = \underline{\dot{\boldsymbol{w}}^{\boldsymbol{1}}}.\underline{w}_{1*}.\underline{\ddot{\boldsymbol{w}}^{\boldsymbol{1}}}\dots\underline{\dot{\boldsymbol{w}}^{\boldsymbol{s}}}.\underline{w}_{s*}.\underline{\ddot{\boldsymbol{w}}^{\boldsymbol{s}}}.\underline{\vec{\boldsymbol{w}}} \\ |\boldsymbol{u^i}| = |\dot{\boldsymbol{u}}^{\boldsymbol{i}}| + u_{i*} + |\ddot{\boldsymbol{u}}^{\boldsymbol{i}}| \end{cases} \tag{41}$$

but with v_* *(resp.* v_{i*}*) now denoting some element*[4] *of the sequence* $\underline{v}_*$ *(resp.* $\underline{v}_{i*}$*). For a simple* ω_0*, the action of* Δ_{ω_0} *involves the so-called* texture mould $\mathrm{tex}^{\bullet}$*:*

$$\Delta_{\omega_0}\, \mathcal{S}^{\underline{\boldsymbol{w}}}(x) \quad = \quad \sum_{v_*\text{-}split} \mathrm{tex}^{\{\underline{\boldsymbol{v}}'\}}_{v_*}\, \mathcal{T}_{v_*}^{\underline{\dot{\boldsymbol{w}}}'', \underline{w}''^{\sharp}_{*}, \underline{\ddot{\boldsymbol{w}}}''}(x)\, \mathcal{S}^{\vec{\underline{\boldsymbol{w}}}}(x) \tag{42}$$

$$\text{with} \qquad \begin{cases} \mathcal{T}^{\underline{\boldsymbol{w}}^{\boldsymbol{1}}, \underline{w}^{\sharp}_{*}, \underline{\boldsymbol{w}}^{\boldsymbol{2}}} := \mathrm{concat}\Big(\mathrm{symlin}(\mathcal{S}^{\underline{\boldsymbol{w}}^{\boldsymbol{1}}}, {}^{iv}\mathcal{S}^{\underline{\boldsymbol{w}}^{\boldsymbol{2}}}), \mathcal{S}^{\underline{w}_*}\Big) \\ \mathcal{T}_{v_*}^{\underline{\boldsymbol{w}}^{\boldsymbol{1}}, \underline{w}^{\sharp}_{*}, \underline{\boldsymbol{w}}^{\boldsymbol{2}}} := \mathrm{concat}\Big(\mathrm{symlin}(\mathcal{S}_{v_*}^{\underline{\boldsymbol{w}}^{\boldsymbol{1}}}, {}^{iv}\mathcal{S}_{v_*}^{\underline{\boldsymbol{w}}^{\boldsymbol{2}}}), \mathcal{S}_{v_*}^{\underline{w}_*}\Big) \\ \mathrm{tex}_{v_*}^{\{\underline{v}_1;\dots;\underline{v}_s\}} := \sum_{\underline{v}\in\mathrm{sha}(\underline{v}_1;\dots;\underline{v}_s)} V^{[\underline{v}, v_*]} \end{cases}$$

For a composite index ω_0*, the action involves a v-augmented tessellator* $\mathrm{vtes}^{\underline{\bullet}}$*:*

$$\Delta_{\omega_0}\, \mathcal{S}^{\underline{\boldsymbol{w}}}(x) = \sum_{v_*\text{-}splits} \mathrm{vtes}^{\binom{|\boldsymbol{u}^{\boldsymbol{1}}|}{\underline{v}'_1, v_{1*}}\, \substack{,\dots, \\ ,\dots,}\, \binom{|\boldsymbol{u}^{\boldsymbol{s}}|}{\underline{v}'_s, v_{s*}}} \Big(\prod_{j=1}^{j=s} \mathcal{T}_{\underline{v}'_j}^{\underline{\dot{\boldsymbol{w}}}^{j''}, \underline{w}''^{\sharp}_{j*}, \underline{\ddot{\boldsymbol{w}}}^{j''}}(x)\Big)\, \mathcal{S}^{\vec{\underline{\boldsymbol{w}}}}(x) \tag{43}$$

with $\mathrm{vtes}^{\bullet} := \mathrm{vscram}.\underline{\mathrm{V}}^{\bullet}$ *(see §3.7). The sum (42) extends to all* v_**-splittings of* $(\underline{\dot{\boldsymbol{w}}}, \underline{w}_*, \underline{\ddot{\boldsymbol{w}}})$*, and the sum (43) to all* v_**-splittings of* $(\underline{\dot{\boldsymbol{w}}}^{\boldsymbol{i}}, \underline{w}_{i*}, \underline{\ddot{\boldsymbol{w}}}^{\boldsymbol{i}})$*.*

6. The tessellation coefficients.

Since the tessellator $tes^{\boldsymbol{w}} := (scram.\underline{V})^{\boldsymbol{w}}$, its v-augmented variant $vtes^{\underline{\boldsymbol{w}}} := (vscram.\underline{V})^{\underline{\boldsymbol{w}}}$, and the closely related $tes^{\underline{\boldsymbol{w}}}$, though defined in terms of hyperlogarithms $V^{\boldsymbol{\omega}}$, turn out to possess remarkable properties of local constancy in their indices, and since both encapsulate some sort of 'universal geometry' that governs co-equational resurgence, we must pause to take a closer look at them.

6.1. The ordinary tessellation coefficients $tes^{\bullet}$**.**
Consider sequences $\boldsymbol{w}$ of length r, introduce '*long*' coordinates $(u_i^{\flat}, v_i^{\flat})$ defined by $u_i^{\flat} = u_i\,,\ v_i^{\flat} = v_i\,,\ u_0 := -(u_1 + .. + u_r)\,,\ v_0^{\flat} = 0$ and consider the set of "homographies" $H_{i,j}$ on $\mathbb{C}^{2r}$ defined by:

$$\begin{cases} H_{i,j}(\boldsymbol{w}) & := \; Q_{i,j}(\boldsymbol{w})/(<\boldsymbol{u}, \boldsymbol{v}> - Q_{i,j}(\boldsymbol{w})) \quad (i - j \neq 0; i, j \in \mathbb{Z}_{r+1}) \\ Q_{i,j}(\boldsymbol{w}) & := \; \sum_{\mathrm{circ}(i<q\leqslant j)} u_q^{\flat}\,(v_q^{\flat} - v_i^{\flat}) \qquad\qquad (i, j, q \in \mathbb{Z}_{r+1}) \end{cases} \tag{44}$$

Proposition 6.1 (Local constancy of $tes^{\boldsymbol{w}}$**).** *Outside a finite number of hypersurfaces* $\Im(H_{i,j}(\boldsymbol{w})) = 0$ *of* $\mathbb{C}^{2r}$*, the tessellation coefficients* $\mathrm{tes}^{\boldsymbol{w}} := \sum_{\boldsymbol{w}'} \epsilon^{\boldsymbol{w}}_{\boldsymbol{w}'}\, \underline{V}^{\boldsymbol{w}'}$ *are constant in each upper index* u_i *and each lower index* v_i*.*

Proof: Based on the jump rules (16). Note that(except at depth $r = 1$, where $tes^{w_1} \equiv 1$) the tessellation coefficients are not *globally* constant. Indeed:

Proposition 6.2 (The jump rule for $tes^{\boldsymbol{w}}$**).** *It is only when* $\boldsymbol{w}$ *crosses a hypersurface* $\mathcal{H}^{+}_{i,j} = \{\boldsymbol{w} \in \mathbb{C}^{2r}\,;\, H_{i,j}(\boldsymbol{w}) \in \mathbb{R}^{+}\}$ *that* $\mathrm{tes}^{\boldsymbol{w}}$ *can change its value. Let* $\boldsymbol{w}$ *be any*

[4] *any* element, not necessarily the first or last.

point on $\mathcal{H}^+_{i,j}$ and $\boldsymbol{w}^+, \boldsymbol{w}^-$ two points close by, with $\Im \boldsymbol{w}^+ > 0$, $\Im \boldsymbol{w}^- < 0$. Then

$$\boldsymbol{tes}^{\boldsymbol{w}^+} - \boldsymbol{tes}^{\boldsymbol{w}^-} \;=\; 2\pi i \;\; \boldsymbol{tes}^{\boldsymbol{w}^*}\, \boldsymbol{tes}^{\boldsymbol{w}^{**}} \tag{45}$$

$$\textit{with} \quad \begin{cases} \boldsymbol{w}^* := \left(\begin{smallmatrix} u_{i+1} & ,\ldots, & u_p & ,\ldots, & u_j \\ v_{i+1}-v_i & ,\ldots, & v_p - v_i & ,\ldots, & v_j - v_i \end{smallmatrix}\right) & \left(\mathrm{circ}(i < p \leqslant j) \in \mathbb{Z}_{r+1}\right) \\ \boldsymbol{w}^{**} := \left(\begin{smallmatrix} u_{j+1} & ,\ldots, & u_q & ,\ldots, & u_{i-1} \\ v_{j+1}-v_i & ,\ldots, & v_q - v_i & ,\ldots, & v_{i-1} - v_i \end{smallmatrix}\right) & \left(\mathrm{circ}(j < q < i) \in \mathbb{Z}_{r+1}\right) \end{cases}$$

This begs for an alternative, simpler expression of $tes^{\boldsymbol{w}}$, or rather, to get rid of the $2\pi i$ factors, of the normalised variant $\mathrm{tes}_{\mathrm{nor}}^{w_1,\ldots,w_r} := (2\pi i)^{r-1}\, \mathrm{tes}^{w_1,\ldots,w_r}$.

Proposition 6.3 (Calculating $tes^{\boldsymbol{w}}$**).** *We fix $c \in \mathbb{C}^*$, set $\Re_c : z \in \mathbb{C} \mapsto \Re(c\,z) \in \mathbb{R}$ and define:* $\begin{cases} f_{\boldsymbol{w}}^{\boldsymbol{w}'} := <\boldsymbol{u}', \boldsymbol{v}'><\boldsymbol{u}, \boldsymbol{v}>^{-1}, & g_{\boldsymbol{w}}^{\boldsymbol{w}'} := <\boldsymbol{u}', \Re_\theta \boldsymbol{v}'><\boldsymbol{u}, \Re_\theta \boldsymbol{v}>^{-1} \\ f_{\boldsymbol{w}}^{\boldsymbol{w}''} := <\boldsymbol{u}'', \boldsymbol{v}''><\boldsymbol{u}, \boldsymbol{v}>^{-1}, & g_{\boldsymbol{w}}^{\boldsymbol{w}''} := <\boldsymbol{u}'', \Re_\theta \boldsymbol{v}''><\boldsymbol{u}, \Re_\theta \boldsymbol{v}>^{-1} \end{cases}$

From these scalars we construct the crucial sign factor sig *which takes its values in $\{-1, 0, 1\}$. Here, the abbreviation* si(.) *stands for* $\mathrm{sign}(\Im(.))$.

$$\mathrm{sig}^{\boldsymbol{w}',\boldsymbol{w}''} = \mathrm{sig}_c^{\boldsymbol{w}',\boldsymbol{w}''} := \frac{1}{8} \begin{cases} \left(\mathrm{si}(f_{\boldsymbol{w}}^{\boldsymbol{w}'} - f_{\boldsymbol{w}}^{\boldsymbol{w}''}) - \mathrm{si}(g_{\boldsymbol{w}}^{\boldsymbol{w}'} - g_{\boldsymbol{w}}^{\boldsymbol{w}''})\right) \times \\ \left(1 + \mathrm{si}(f_{\boldsymbol{w}}^{\boldsymbol{w}'}/g_{\boldsymbol{w}}^{\boldsymbol{w}'})\;\; \mathrm{si}(f_{\boldsymbol{w}}^{\boldsymbol{w}'} - g_{\boldsymbol{w}}^{\boldsymbol{w}'})\right) \times \\ \left(1 + \mathrm{si}(f_{\boldsymbol{w}}^{\boldsymbol{w}''}/g_{\boldsymbol{w}}^{\boldsymbol{w}''})\; \mathrm{si}(f_{\boldsymbol{w}}^{\boldsymbol{w}''} - g_{\boldsymbol{w}}^{\boldsymbol{w}''})\right) \end{cases} \tag{46}$$

Next, from the pair $(\boldsymbol{w}', \boldsymbol{w}'')$ we derive a pair $(\boldsymbol{w}^*, \boldsymbol{w}^{**})$ by setting:

$$\boldsymbol{v}^* := \det \begin{pmatrix} \frac{\boldsymbol{v}'}{<\boldsymbol{u},\boldsymbol{v}>} & \frac{\Re_c \boldsymbol{v}'}{<\boldsymbol{u},\Re_c \boldsymbol{v}>} \\ \Im \frac{<\boldsymbol{u}',\boldsymbol{v}'>}{<\boldsymbol{u},\boldsymbol{v}>} & \Im \frac{<\boldsymbol{u}',\Re_c \boldsymbol{v}'>}{<\boldsymbol{u},\Re_c \boldsymbol{v}>} \end{pmatrix} \;,\; \boldsymbol{v}^{**} := \det \begin{pmatrix} \frac{\boldsymbol{v}''}{<\boldsymbol{u},\boldsymbol{v}>} & \frac{\Re_c \boldsymbol{v}''}{<\boldsymbol{u},\Re_c \boldsymbol{v}>} \\ \Im \frac{<\boldsymbol{u}'',\boldsymbol{v}''>}{<\boldsymbol{u},\boldsymbol{v}>} & \Im \frac{<\boldsymbol{u}'',\Re_c \boldsymbol{v}''>}{<\boldsymbol{u},\Re_c \boldsymbol{v}>} \end{pmatrix}$$

From these ingredients, we construct an auxilliary $\mathrm{urtes}_{\mathrm{nor}}^{\bullet}$*, then* $\mathrm{tes}_{\mathrm{nor}}^{\bullet}$ *itself:*

$$\mathrm{urtes}_{\mathrm{nor}}^{\boldsymbol{w}} \;=\; \sum_{\boldsymbol{w}'\boldsymbol{w}''=\boldsymbol{w}} \mathrm{sig}^{\boldsymbol{w}'\boldsymbol{w}''}\; \mathrm{tes}_{\mathrm{nor}}^{\boldsymbol{w}^*}\; \mathrm{tes}_{\mathrm{nor}}^{\boldsymbol{w}^{**}} \qquad \Big((\boldsymbol{w}', \boldsymbol{w}'') \neq (\boldsymbol{w}^*, \boldsymbol{w}^{**})\Big) \tag{47}$$

$$\mathrm{tes}_{\mathrm{nor}}^{\bullet} \;=\; \sum_{0 \leqslant n \leqslant r(\bullet)} \mathrm{push}^n\, \mathrm{urtes}_{\mathrm{nor}}^{\bullet} \qquad (\forall c \in \mathbb{C}^*\;,\; \textit{with}\ \mathrm{push}\ \textit{as in}\ (\textbf{??})) \tag{48}$$

Proposition 6.4 (Main properties of $\mathrm{tes}^\bullet$**).** .

$\boldsymbol{P_1}$*: $tes^\bullet$ is invariant under the involution* swap *and the idempotent* push*:*

$$\mathrm{swap}.A^{\left(\begin{smallmatrix} u_1 & ,\ldots, & u_r \\ v_1 & ,\ldots, & v_r \end{smallmatrix}\right)} \;=\; A^{\left(\begin{smallmatrix} v_r & ,\ldots, & v_3 - v_4 & , & v_2 - v_3 & , & v_1 - v_2 \\ u_1 + \ldots + u_r & ,\ldots, & u_1 + u_2 + u_3 & , & u_1 + u_2 & , & u_1 \end{smallmatrix}\right)} \;\; (\mathrm{swap}^2 = \mathrm{iden}) \tag{49}$$

$$\mathrm{push}.A^{\left(\begin{smallmatrix} u_1 & ,\ldots, & u_r \\ v_1 & ,\ldots, & v_r \end{smallmatrix}\right)} \;=\; A^{\left(\begin{smallmatrix} -u_1 \ldots - u_r & u_1 & , & u_2 & ,\ldots, & u_{r-1} \\ -v_r & v_1 - v_r & , & v_2 - v_r & ,\ldots, & v_{r-1} - v_r \end{smallmatrix}\right)} \;\; (\mathrm{push}^{r+1} = \mathrm{iden}) \tag{50}$$

$\boldsymbol{P_2}$*: the bimould* $\mathrm{tes}^\bullet$ *is bialternal, i.e. alternal and of alternal swappee.*

$\boldsymbol{P_3}$*:* $\mathrm{tes}^\bullet_{nor}$ *assumes all its values in $\mathbb{Z}$ and $|tes^{w_1,\ldots,w_r}| < (r-1)!(r+1)!$ (unsharp)*

$\boldsymbol{P_4}$*: As* r *increases, the set where* $\mathrm{tes}^{\boldsymbol{w}} \neq 0$ *has surprisingly small Lebesgue measure on $\mathbb{S}^{2\,r}$ ($\mathbb{S}$ being the Riemann sphere).*

$\boldsymbol{P_5}$*: in presence of vanishing u_i-sums, local constancy in the v_j's fails.*

$\boldsymbol{P_6}$*: in presence of v_i-repetitions, local constancy in the u_j's fails.*

$\boldsymbol{P_7}$*: in the semi-real case, i.e. when either all u_i's or all v_i's are aligned with the origin, the tessellation coefficients vanish as soon as $2 \leqslant$* r*.*

6.2. The v-augmented tesselation coefficients $vtes^{\underline{\bullet}}$ **and** $tes^{\underline{\bullet}}$**.**

To enunciate the main statement, we require the *lower* (or *positional*) mould composition $\underline{\circ}$, which is what becomes of ordinary mould composition $\circ$ when we switch from the incremental $\omega_1, \omega_2...$ to the positional indexation $\alpha_1, \alpha_2...$, with $\omega_1 = \alpha_1$ and $\omega_i = \alpha_i - \alpha_{i-1}$ for $2 \leqslant i$. Here is the formula:

$$\{A^\bullet = B^\bullet \underline{\circ}\, C^\bullet\} \Longleftrightarrow \Big\{A^{\boldsymbol{\alpha}} = \sum_{\boldsymbol{\alpha}=\boldsymbol{\alpha}^{\mathbf{1}}\alpha_{i_1}\ldots\boldsymbol{\alpha}^{\boldsymbol{s}}\alpha_{i_s}}^{1\leqslant s} B^{\,\alpha_{i_1},\ldots,\alpha_{i_s}} \prod_{1\leqslant k\leqslant s} C_{\alpha_{i_{k-1}}}^{\boldsymbol{\alpha}^{\boldsymbol{k}}\alpha_{i_k}}\Big\} \tag{51}$$

with the notation $C_{\alpha_*}^{\alpha_1,\ldots,\alpha_r} := C^{\alpha_1-\alpha_*,\ldots,\alpha_r-\alpha_*}$ and (since there is no index α_{i_0}) with the convention $C_{\alpha_{i_0}}^{\boldsymbol{\alpha}^{\mathbf{1}},\alpha_1} \equiv C^{\boldsymbol{\alpha}^{\mathbf{1}},\alpha_1}$ for the first term in the product $\prod(...)$. Of course, some of the factor sequences $\boldsymbol{\alpha}^{\boldsymbol{i}}$, even all of them, may be empty. Thus, when reduced to its two 'extreme' terms, (51) reads:

$$A^{\alpha_1,\ldots,\alpha_r} = B^{\alpha_r}C^{\alpha_1,\ldots,\alpha_r} + (\ldots\ldots\ldots) + B^{\alpha_1,\ldots,\alpha_r}C^{\alpha_1}C^{\alpha_2-\alpha_1}\ldots C^{\alpha_r-\alpha_{r-1}}$$

Proposition 6.5. (Local constancy properties of $\boldsymbol{vtes}^{\underline{\boldsymbol{w}}}$ **and** $\boldsymbol{tes}^{\underline{\boldsymbol{w}}}$**.)** *The coefficients* $\mathrm{vtes}^{\underline{\boldsymbol{w}}} := (\mathrm{vscram}\,\underline{\mathrm{V}})^{\underline{\boldsymbol{w}}}$ *are locally constant in each index* u_i *but not in the indices* $v_i, v_i', v_i'' \ldots$ *of the lower* $\underline{v}_i$*. However, they admit a unique decomposition:*

$$\mathrm{vtes}^{\underline{\bullet}} = \mathrm{tes}^{\underline{\bullet}}\ \underline{\circ}\ \mathrm{V}^{[\bullet]} \qquad (\mathrm{V}^{[\bullet]} = \textit{hyperlog. monics in positional notation}) \tag{52}$$

with a second factor $V^{\,[\underline{v}_i]}$ *absorbing the non-elementary part of the* $\underline{v}_i$*-dependence, and a first factor* $\mathrm{tes}^{\underline{\bullet}}$ *locally constant in each* u_i *and each* $v_i, v_i', v_i''...$

7. Weighted products under alien differentiation.

6.1. The second Bridge equation. Purely for notational convenience, we shall state the results in the x-plane, i.e. in terms of *wemu, welu* rather than *weco, welo.*

Proposition 7.1 (Alien derivatives of *wemu*, **hence** *weco*). *The only alien derivatives* Δ_{ω_0} *acting effectively on* $\mathrm{wemu}^{\binom{u_1\ ,\ldots,\ u_r}{c_1\ ,\ldots,\ c_r}}(x)$ *correspond either to simple* $(s=1)$ *or composite* $(s\geqslant 1)$ *indices* ω_0 *of the form*

$$\omega_0 = |\boldsymbol{u}^{\mathbf{1}}|\, v_{i_1}^1 + \cdots + |\boldsymbol{u}^{\boldsymbol{s}}|\, v_{i_s}^s \quad \textit{with } \boldsymbol{u}^{\mathbf{1}}\,\boldsymbol{u}^{\mathbf{2}}\ldots\boldsymbol{u}^{\boldsymbol{s-1}}\,\boldsymbol{u}^{\boldsymbol{s}}\,\boldsymbol{u}^{*} = \boldsymbol{u} \textit{ and } \binom{u_{i_k}^k}{c_{i_k}^k} \in \binom{\boldsymbol{u}^{\boldsymbol{k}}}{\boldsymbol{c}^{\boldsymbol{k}}}$$

with each $\binom{\boldsymbol{u}^{\boldsymbol{k}}}{\boldsymbol{c}^{\boldsymbol{k}}}$ *re-indexed for convenience as* $\binom{u_1^k,\ldots,u_{r_k}^k}{c_1^k,\ldots,c_{r_k}^k}$*. The formula reads:*

$$\Delta_{\omega_0}\mathrm{wemu}^{\binom{u_1\ ,\ldots,\ u_r}{c_1\ ,\ldots,\ c_r}}(x) = \begin{cases} \sum_{\check{v}_j^k \text{ over } v_{i_k}^k} \mathrm{Tes}^{\binom{|\boldsymbol{u}^{\mathbf{1}}|\qquad ,\ldots,\qquad |\boldsymbol{u}^{\boldsymbol{s}}|}{\check{v}_1^1,..,\check{v}_{r_1}^1\ ,\ldots,\ \check{v}_1^s,..,\check{v}_{r_s}^s}} \times \\ \prod_{1\leqslant k\leqslant s} \mathrm{welu}^{\left(\begin{smallmatrix} u_1^k & ,\ldots, & \left(\begin{smallmatrix} u_{i_k}^k \\ \Delta_{\check{v}_{i_k}^k}\ c_{i_k}^k\end{smallmatrix}\right)^{\sharp} & ,\ldots, & u_{r_k}^k \\ \check{v}_1^k\ c_1^k & ,\ldots, & & ,\ldots, & \check{v}_{r_k}^k\ c_{r_k}^k \end{smallmatrix}\right)}(x) \times \\ \mathrm{wemu}^{\binom{u_1^*\ ,\ldots,\ u_{r_*}^*}{c_1^*\ ,\ldots,\ c_{r_*}^*}}(x) \end{cases} \tag{53}$$

with an alternal mould welu$^\bullet$ *carrying one* $\sharp$*-marked index and defined by:*

$$\mathrm{welu}^{w_1,..,w_k^\sharp,..,w_r} = \mathrm{concat}(\mathrm{symlin}(\mathrm{wemu}^{w_1,..,w_{k-1}}, {}^{iv}\mathrm{wemu}^{w_{k+1},..,w_r}), \mathrm{wemu}^{w_k^\sharp} \quad (54)$$

7.2. The third Bridge equation. Since (54) expresses *welu*$^\bullet$ in terms of *wemu*$^\bullet$, (53) gives a indirect way of alien differentiating *welu*$^\bullet$. But here is a direct formula:

Proposition 7.2 (Alien derivatives of *welu***, hence** *welo***).** *The only alien derivatives* Δ_{ω_0} *acting effectively on* $\mathrm{welu}^{\left(\begin{smallmatrix} u_1 \\ c_1\end{smallmatrix},\dots,\left(\begin{smallmatrix} u_j \\ c_j\end{smallmatrix}\right)^\sharp,\dots,\begin{smallmatrix} u_r \\ c_r\end{smallmatrix}\right)}(x)$ *correspond either to simple (*$s=1$*) or composite (*$s>1$*) indices* ω_0 *of three possible types – initial, final, global:*

$$\omega_0^{ini} = |\boldsymbol{u^1}|\, v^1_{i_1} + \dots + |\boldsymbol{u^s}|\, v^s_{i_s} \ \textit{with} \ \begin{cases} \boldsymbol{u^1} \ \dots \ \boldsymbol{u^s}\, \boldsymbol{u^*} = \boldsymbol{u} \ \ ; \ \ \left(\begin{smallmatrix} u_j \\ c_j\end{smallmatrix}\right)^\sharp \in \left(\begin{smallmatrix} \boldsymbol{u^*} \\ \boldsymbol{c^*}\end{smallmatrix}\right) \\ \Delta_{v^k_{i_k}}\, c^k_{i_k} \neq 0 \ \textit{and} \ \left(\begin{smallmatrix} u^k_{i_k} \\ c^k_{i_k}\end{smallmatrix}\right) \in \left(\begin{smallmatrix} \boldsymbol{u^k} \\ \boldsymbol{c^k}\end{smallmatrix}\right) \end{cases} \quad (55)$$

$$\omega_0^{fin} = |\boldsymbol{u^1}|\, v^1_{i_1} + \dots + |\boldsymbol{u^s}|\, v^s_{i_s} \ \textit{with} \ \begin{cases} {}^*\boldsymbol{u}\, \boldsymbol{u^1} \ \dots \ \boldsymbol{u^s} = \boldsymbol{u} \ \ ; \ \ \left(\begin{smallmatrix} u_j \\ c_j\end{smallmatrix}\right)^\sharp \in \left(\begin{smallmatrix} {}^*\boldsymbol{u} \\ {}^*\boldsymbol{c}\end{smallmatrix}\right) \\ \Delta_{v^k_{i_k}}\, c^k_{i_k} \neq 0 \ \textit{and} \ \left(\begin{smallmatrix} u^k_{i_k} \\ c^k_{i_k}\end{smallmatrix}\right) \in \left(\begin{smallmatrix} \boldsymbol{u^k} \\ \boldsymbol{c^k}\end{smallmatrix}\right) \end{cases} \quad (56)$$

$$\omega_0^{glo} = |\boldsymbol{u^1}|\, v^1_{i_1} + \dots + |\boldsymbol{u^s}|\, v^s_{i_s} \ \textit{with} \ \begin{cases} \boldsymbol{u^1} \ \dots \ \boldsymbol{u^s} = \boldsymbol{u} \\ \Delta_{v^k_{i_k}}\, c^k_{i_k} \neq 0 \ \textit{and} \ \left(\begin{smallmatrix} u^k_{i_k} \\ c^k_{i_k}\end{smallmatrix}\right) \in \left(\begin{smallmatrix} \boldsymbol{u^k} \\ \boldsymbol{c^k}\end{smallmatrix}\right) \end{cases} \quad (57)$$

with each $\left(\begin{smallmatrix} \boldsymbol{u^k} \\ \boldsymbol{c^k}\end{smallmatrix}\right)$ *re-indexed as* $\left(\begin{smallmatrix} u^k_1 & ,\dots, & u^k_{r_k} \\ c^k_1 & ,\dots, & c^k_{r_k}\end{smallmatrix}\right)$*. The alien derivatives are given by:*

$$\Delta_{\omega_0^{ini}} \mathrm{welu}^{\left(\begin{smallmatrix} u_1 & \dots, & (u_j)^\sharp & \dots, & u_r \\ c_1 & ,\dots, & (c_j) & ,\dots, & c_r\end{smallmatrix}\right)}(x) = \begin{cases} + \sum_{\check v^k_j \text{ over } v^k_{i_k}} \mathrm{Tes}^{\left(\begin{smallmatrix} |\boldsymbol{u^1}| & ,\dots, & |\boldsymbol{u^s}| \\ \check v^1_1,\dots,\check v^1_{r_1} & ,\dots, & \check v^s_1,\dots,\check v^s_{r_s}\end{smallmatrix}\right)} \times \\ \prod_{1\leqslant k\leqslant s} \mathrm{welu}^{\left(\begin{smallmatrix} u^k_1 & ,\dots, & \left(\begin{smallmatrix} u^k_{i_k} \\ \Delta_{\check v^k_{i_k}} c^k_{i_k}\end{smallmatrix}\right)^\sharp & ,\dots, & u^k_{r_k} \\ \check v^k_1\, c^k_1 & ,\dots, & & ,\dots, & \check v^k_{r_k}\, c^k_{r_k}\end{smallmatrix}\right)}(x) \times \\ \mathrm{welu}^{\left(\begin{smallmatrix} u^*_1 & \dots, & (u_j)^\sharp & \dots, & u^*_{r_*} \\ c^*_1 & ,\dots, & (c_j) & ,\dots, & c^*_{r_*}\end{smallmatrix}\right)}(x) \end{cases}$$

$$\Delta_{\omega_0^{fin}} \mathrm{welu}^{\left(\begin{smallmatrix} u_1 & \dots, & (u_j)^\sharp & \dots, & u_r \\ c_1 & ,\dots, & (c_j) & ,\dots, & c_r\end{smallmatrix}\right)}(x) = \begin{cases} - \sum_{\check v^k_j \text{ over } v^k_{i_k}} \mathrm{Tes}^{\left(\begin{smallmatrix} |\boldsymbol{u^1}| & ;\dots; & |\boldsymbol{u^s}| \\ \check v^1_1,\dots,\check v^1_{r_1} & ;\dots; & \check v^s_1,\dots,\check v^s_{r_s}\end{smallmatrix}\right)} \times \\ \prod_{1\leqslant k\leqslant s} \mathrm{welu}^{\left(\begin{smallmatrix} u^k_1 & ,\dots, & \left(\begin{smallmatrix} u^k_{i_k} \\ \Delta_{\check v^k_{i_k}} c^k_{i_k}\end{smallmatrix}\right)^\sharp & ,\dots, & u^k_{r_k} \\ \check v^k_1\, c^k_1 & ,\dots, & & ,\dots, & \check v^k_{r_k}\, c^k_{r_k}\end{smallmatrix}\right)}(x) \times \\ \mathrm{welu}^{\left(\begin{smallmatrix} {}^*u_1 & \dots, & (u_j)^\sharp & \dots, & {}^*u_{r_*} \\ {}^*c_1 & ,\dots, & (c_j) & ,\dots, & {}^*c_{r_*}\end{smallmatrix}\right)}(x) \end{cases}$$

$$\Delta_{\omega_0^{glo}} \mathrm{welu}^{\left(\begin{smallmatrix} u_1 & \dots, & (u_j)^\sharp & \dots, & u_r \\ c_1 & ,\dots, & (c_j) & ,\dots, & c_r\end{smallmatrix}\right)}(x) = \begin{cases} + \sum_{\check v^k_j \text{ over } v^k_{i_k}} \mathrm{Tes}^{\left(\begin{smallmatrix} |\boldsymbol{u^1}| & ;\dots; & |\boldsymbol{u^s}| \\ \check v^1_1,\dots,\check v^1_{r_1} & ;\dots; & \check v^s_1,\dots,\check v^s_{r_s}\end{smallmatrix}\right)} \times \\ \prod_{1\leqslant k\leqslant s} \mathrm{welu}^{\left(\begin{smallmatrix} u^k_1 & ,\dots, & \left(\begin{smallmatrix} u^k_{i_k} \\ \Delta_{\check v^k_{i_k}} c^k_{i_k}\end{smallmatrix}\right)^\sharp & ,\dots, & u^k_{r_k} \\ \check v^k_1\, c^k_1 & ,\dots, & & ,\dots, & \check v^k_{r_k}\, c^k_{r_k}\end{smallmatrix}\right)}(x) \end{cases}$$

For meromorphic inputs, *Tes*$^\bullet$ coincides with *tes*$^\bullet$ of §6.1. For general ramified data, it essentially coincides with *tes*$^{\underline{\bullet}}$ of §6.2, although that assumes that we properly define the shifts $\check{v}_j^k$. We cannot enter into these details here, and in any case what really matters is (i) the existence of $Tes^\bullet$ (ii) its alternality (iii) its local constancy (tempered by the caveats $\mathbf{P_5}$, $\mathbf{P_6}$ of Proposition 6.4).

8. The Bridge equations I,II,III.

7.1. Equational resurgence. First Bridge equation. It is the classical identity:

$$\textbf{BE1} \qquad [\boldsymbol{\Delta}_\omega, \Theta^{-1}] = \mathbb{A}_\omega\, \Theta^{-1} \tag{58}$$

with $\boldsymbol{\Delta}_\omega := e^{-\omega z}\, \Delta_\omega$ (z-resurgence) and, due to $W^\bullet$'s alternality:

$$\begin{aligned}
\mathbb{A}_\omega &= -\sum (-1)^r \sum W^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ B^{i_1}_{\boldsymbol{n}_1} & ,\dots, & B^{i_r}_{\boldsymbol{n}_r}\end{smallmatrix}\right)}(x)\ \ \mathbb{D}^{i_1}_{\boldsymbol{n}_1}\mathbb{D}^{i_2}_{\boldsymbol{n}_2}\dots\mathbb{D}^{i_r}_{\boldsymbol{n}_r} \\
&= -\sum \frac{(-1)^r}{r} \sum W^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ B^{i_1}_{\boldsymbol{n}_1} & ,\dots, & B^{i_r}_{\boldsymbol{n}_r}\end{smallmatrix}\right)}(x)\ \ [..[\mathbb{D}^{i_1}_{\boldsymbol{n}_1}, \mathbb{D}^{i_1}_{\boldsymbol{n}_2}]\dots\mathbb{D}^{i_r}_{\boldsymbol{n}_r}]
\end{aligned}$$

Since any two $\mathbb{D}_{\omega_1}$ and $\mathbb{A}_{\omega_2}$ commute, (58) lends itself to indefinite iteration:

$$[\boldsymbol{\Delta}_{\omega_r}\dots[\boldsymbol{\Delta}_{\omega_2}, [\boldsymbol{\Delta}_{\omega_1}, \Theta^{-1}]..] = \mathbb{A}_{\omega_1}\,\mathbb{A}_{\omega_2}\dots\mathbb{A}_{\omega_r}\,\Theta^{-1} \tag{59}$$

8.2. Coequational resurgence. From the molecular to the higher levels. Coequational resurgence already forced us to distinguish two levels of complexity – the *'atomic'* $\mathcal{V}^\bullet$'s and the *'molecular'* $\mathcal{S}^\bullet$'s. It will shortly impose two more:
(*) a *'microscopic'* level, where we deal with derivation operators $\mathbb{Q}_\omega$ obtained by contracting alternal products *welu* with ordinary differential operators. The resulting sums being usually infinite, the gap from *molecular* to *microscopic* is large.[5]
(**) a *'macroscopic'* level, where we deal with new derivation operators $\mathbb{P}_\omega$ obtained by contracting the tessellation mould with the previous $\mathbb{Q}_\omega$. These new sums, too, tend to be infinite, making the gap from *microscopic* to *macroscopic* as large as the earlier ones, although in some relatively rare but important instances the relation between the $\mathbb{Q}_\omega$'s and the $\mathbb{P}_\omega$'s simplifies.

The distance between the $\mathbb{P}_\omega$'s and the $\mathbb{Q}_\omega$'s will be least when the tessellation coefficients $Tes^\bullet$ connecting the two will be simplest. For elementary indices $w_i = \left(\begin{smallmatrix} u_i \\ v_i \end{smallmatrix}\right)$, $Tes^\bullet$ coincides with $tes^\bullet$ and each of these four conditions, when met, tends to simplify the coefficients: (i) no vanishing u_i-sums. (ii) no identical consecutive v_i's. (iii) all u_i are aligned with the origin (iv) all v_i are aligned with the origin

Imposing (i) in our model problem (§1) leads to a general solution Y with components $Y^i_{\boldsymbol{n}}$ that reduce to finite sums of monomials $\mathcal{W}^\bullet(z,x)$.

Imposing (ii) means restricting oneself to the linear case. It leads to interesting results provided we are dealing with a true system, i.e. for $\nu \geqslant 2$.

The conditions (iii) or (iv), are perfectly reasonable. They lead to massive simplifications by ensuring that $\mathrm{tes}^{\boldsymbol{w}} = 0$ for all $\boldsymbol{w}$ of length $r(\boldsymbol{w}) > 1$ that meet the conditions (i) and (ii). For $\boldsymbol{w}$ of length 1 we have of course $tes^{w_1} \equiv 1$.

We should expect, and do in fact get, particularly simple results when the convolands $\widehat{c}_i$ are meromorphic, or hyperlogarithmic, or again when they enjoy

[5] even if the convergence of these sums in the space of resurgent functions is not really an issue.

special closure properties under ω-shifts and Δ_ω-derivations, globally for the same ω's. In any case, since $\widehat{c}_i(\xi) = -b_i(z-\xi)$, it stands to reason that to get full x-resurgence we must assume each $b_i(z)$ to possess endless analytic continuation (on the Riemann sphere, starting from ∞), whereas for z-resurgence it was enough for the $b_i(z)$ to be locally analytic at ∞ (with suitable uniformity conditions in i).

8.3. Coequational resurgenge. Second and third Bridge equations. Let us give some illustrations, mostly in the meromorphic context. To lighten notations, we write the results when our model system (3) reduces to a single (non-linear) equation, i.e. when $\nu = 1$, because in that case the operators $\mathbb{D}^i_{\boldsymbol{n}} = \tau_i \boldsymbol{\tau}^{\boldsymbol{n}} \partial_{\tau_i}$ correspond one-to-one with the weights u and can be re-indexed as $\mathbb{D}_{\|u} = \tau^{n+1}\partial_\tau$. The transposition to the case $\nu > 1$ offers mainly notational complications but deserves special consideration because it allows non-aligned weights $u = <\boldsymbol{\lambda}, \boldsymbol{n}>$.

Second Bridge equation.

$$(\mathbf{BE2}) \qquad [\mathbb{\Delta}_\omega, \Theta^{-1}] \;=\; \mathbb{P}_\omega\, \Theta^{-1} \tag{60}$$

with $\mathbb{\Delta}_\omega := e^{-\omega x}\Delta_\omega$ (x-resurgence) and:

$$\mathbb{P}_\omega \;:=\; \sum_{\sum u_i(z-\alpha_i)=\omega} \mathrm{Tes}^{\left(\substack{u_1\\ z-\alpha_1}\,\substack{,\dots,\\ ,\dots,}\,\substack{u_r\\ z-\alpha_r}\right)} \mathbb{Q}_{\left[\substack{u_1\\ \alpha_1}\right]} \dots \mathbb{Q}_{\left[\substack{u_r\\ \alpha_r}\right]} \tag{61}$$

$$\mathbb{Q}_{\left[\substack{u_0\\ \alpha_0}\right]} \;:=\; e^{u_0\alpha_0\, x} \sum_{\sum u_i = u_0} \mathrm{welu}^{\left(\substack{u_1\\ \bar{\alpha}_0\cdot c_1}\,\substack{,\dots,\\ ,\dots,}\,\left(\substack{u_i\\ \Delta_{\alpha_0} c_i}\right)^\sharp\,\substack{,\dots,\\ ,\dots,}\,\substack{u_r\\ \bar{\alpha}_0\cdot c_r}\right)} \mathbb{D}_{\|u_1} \dots \mathbb{D}_{\|u_r} \tag{62}$$

Here $Tes^\bullet$ coincides with the elementary $tes^\bullet$.

Third Bridge equation.

$$(\mathbf{BE3}) \qquad \mathbb{\Delta}_\omega \mathbb{Q}_{\left[\substack{u_0\\ \alpha_0}\right]} \;=\; \begin{cases} +\sum_{u_1+u_2=u_0} \mathbb{P}_{\omega,\left[\substack{u_1\\ \alpha_0}\right]}\, \mathbb{Q}_{\left[\substack{u_2\\ \alpha_0}\right]} \\ -\sum_{u_1+u_2=u_0} \mathbb{Q}_{\left[\substack{u_1\\ \alpha_0}\right]}\, \mathbb{P}_{\omega,\left[\substack{u_2\\ \alpha_0}\right]} \end{cases} \tag{63}$$

with

$$\mathbb{P}_{\omega,\left[\substack{u_0\\ \alpha_0}\right]} := \sum_{\sum u_i(\alpha_0-\alpha_i)=\omega}^{\sum u_i = u_0} \mathrm{Tes}^{\left(\substack{u_1\\ \alpha_0-\alpha_1}\,\substack{,\dots,\\ ,\dots,}\,\substack{u_r\\ \alpha_0-\alpha_r}\right)} \mathbb{Q}_{\left[\substack{u_1\\ \alpha_1}\right]} \dots \mathbb{Q}_{\left[\substack{u_r\\ \alpha_r}\right]} \tag{64}$$

Remark 1: With the notations of (64), the operator $\mathbb{P}_\omega$ of **BE2** may be rewritten as $\mathbb{P}_\omega = \sum_u \mathbb{P}_{\omega,\left[\substack{u\\ z}\right]}$. The $\mathbb{P}_\omega$ in **BE2** are locally (though not globally) constant in z, just as the operators $\mathbb{P}_{\omega,\left[\substack{u\\ \alpha_0}\right]}$ in **BE3** are locally (not globally) constant in α_0.

Remark 2: In the important instances when the tessellation coefficients $Tes^{w_1,\dots,w_r}$ turn trivial (i.e. $\equiv 1$ for $r = 1$ and $\equiv 0$ for $r \neq 1$), **BE3** simplifies:

$$(\mathbf{BE3}) \qquad \mathbb{\Delta}_\omega \mathbb{Q}_{\left[\substack{u_0\\ \alpha_0}\right]} \;=\; \sum_{u_1+u_2=u_0}^{u_1(\alpha_0-\alpha_1)=\omega} \left[\mathbb{Q}_{\left[\substack{u_1\\ \alpha_1}\right]}\,,\, \mathbb{Q}_{\left[\substack{u_2\\ \alpha_0}\right]}\right] \tag{65}$$

and one can check the equality of the exponential factors on both sides:

Remark 3. (**BE2**) and (**BE3**) also extend in the opposite direction, when the inputs $\mathrm{b}_i(z)$ (and thus $\widehat{c}_i(\xi)$) are no longer meromorpic, but hyperlogarithmic, or general ramified functions. But we must now switch to a multiple indexation $\alpha_i \to$

$\check{\alpha}_i$ and the third Bridge equation becomes saddled with a third term, corresponding to the case Δ^{glo}_{ω} *welu*$^\bullet$ of Proposition 7.2. We get:

$$\textbf{(BE3)}\qquad \Delta_\omega \mathbb{Q}_{[{u_0 \atop \check{\alpha}_0}]} \;=\; \begin{cases} +\sum_{u_1+u_2=u_0} \mathbb{P}_{\omega,[{u_1 \atop \check{\alpha}_0}]}\, \mathbb{Q}_{[{u_2 \atop \check{v}_0}]} \\ -\sum_{u_1+u_2=u_0} \mathbb{Q}_{[{u_1 \atop \check{\alpha}_0}]}\, \mathbb{P}_{\omega,[{u_2 \atop \check{\alpha}_0}]} \\ +\mathbb{P}_{\omega,[{u_0 \atop \check{\alpha}_0}]} \end{cases} \qquad (66)$$

9. The equational-coequational link at the monomial level.

To elucidate the eq.-coeq. link at the most basic level, let us write down *BE1* and *BE2* for the biresurgent monomials $\mathcal{W}^\bullet(z,x)$ of §1, and compare their resurgence coefficients, respectively $W^\bullet_*(x)$ (entire in x) and $T^\bullet_*(x)$ (resurgent in x).

9.1. Equational resurgence and its entire coefficients $W^\bullet_*(x)$.

$$\mathbf{BE_1}\qquad {}^{z}\!\boldsymbol{\Delta}_{|\boldsymbol{u}|x}\, \widetilde{\mathcal{W}}^{({\boldsymbol{u} \atop \boldsymbol{\alpha}})}(z,x) = e^{-|\boldsymbol{u}|xz}\widetilde{W}^{({\boldsymbol{u} \atop \boldsymbol{\alpha}})}(x) = e^{-|\boldsymbol{u}|xz} e^{<\boldsymbol{u},\boldsymbol{\alpha}>x}\, W_*^{({\boldsymbol{u} \atop \boldsymbol{\alpha}})}(x)$$

To calculate the resurgence constants $W^\bullet_*(x)$ attached to the monomials $\widetilde{\mathcal{W}}_*^{({\boldsymbol{u} \atop \boldsymbol{\alpha}})}(z,x) := e^{-<\boldsymbol{u},\boldsymbol{\alpha}>x}\widetilde{\mathcal{W}}^{({\boldsymbol{u} \atop \boldsymbol{\alpha}})}(z,x)$, we resort to the following decomposition

$$\widetilde{\mathcal{W}}^\bullet_*(z,x) = \big(\mathcal{V}^\bullet(z)\circ W^\bullet_{**}(x)\big)\times \widetilde{\mathcal{W}}^\bullet_{**}(z,x) \quad \begin{cases} \widehat{\mathcal{W}}^\bullet_*(z,x) & z\text{-}resurgent \\ \widehat{\mathcal{W}}^\bullet_{**}(z,x) & z\text{-}holomorphic\ at\ \infty \\ W^\bullet_{**}(x) & x\text{-}entire \end{cases}$$

If, applying (12) and taking into account the z-convergence of $\widetilde{\mathcal{W}}_*(z,x)$ at ∞, we alien-differentiate the above relation, we find that our coefficients $W^\bullet_*(x)$ neatly split into a universal part $V^\bullet$ hyperlogarithmic in $\boldsymbol{u}$ but constant in $\boldsymbol{\alpha}$ and x, and a sensitive part $W^\bullet_{**}(x)$ entire in $\boldsymbol{u}$, $\boldsymbol{\alpha}$, and x. Explicitly:

$$W_*^{({\boldsymbol{u} \atop \boldsymbol{\alpha}})}(x) = W_{**}^{({\boldsymbol{u} \atop \boldsymbol{\alpha}})}(x) + \sum_{\boldsymbol{u}^1..\boldsymbol{u}^s=\boldsymbol{u}}^{2\leqslant s} V^{|\boldsymbol{u}^1|,..,|\boldsymbol{u}^s|}\prod_{1\leqslant i\leqslant s} W_{**}^{({\boldsymbol{u}^i \atop \boldsymbol{\alpha}^i})}(x) = \sum_{\boldsymbol{u}^1..\boldsymbol{u}^s=\boldsymbol{u}}^{1\leqslant s} V^{|\boldsymbol{u}^1|,..,|\boldsymbol{u}^s|}\prod_{1\leqslant i\leqslant s} W_{**}^{({\boldsymbol{u}^i \atop \boldsymbol{\alpha}^i})}(x)$$

The crucial monics $W^\bullet_{**}(x)$, along with the monomials $\widetilde{\mathcal{W}}^\bullet_{**}(z,x)$ or rather their z-Borel transforms $\widehat{\mathcal{W}}^\bullet_{**}(\zeta,x)$, are given by the joint induction:

$$W_{**}^{({u_1 \atop \alpha_1}\,{,\ldots, \atop ,\ldots,}\,{u_r \atop \alpha_r})}(x) = \begin{cases} +\int_0^{|\boldsymbol{u}|} \widehat{\mathcal{W}}_{**}^{({u_1 \atop \alpha_1}\,{,\ldots, \atop ,\ldots,}\,{u_{r-1} \atop \alpha_{r-1}})}(xt_0,x)\, e^{\alpha_r x(|\boldsymbol{u}|-u_r-t_0)}\,dt_0 \\ -\sum_{j=1}^{r-1} W_{**}^{({u_1 \atop \alpha_1}\,{,\ldots, \atop ,\ldots,}\,{u_j \atop \alpha_j})}(x)\int_0^{|\boldsymbol{u}|} \widehat{\mathcal{W}}_{*}^{({u_{j+1} \atop \alpha_{j+1}}\,{,\ldots, \atop ,\ldots,}\,{u_r \atop \alpha_r})}(xt_0,x)\,dt_0 \end{cases}$$

$$\widehat{\mathcal{W}}_{**}^{({u_1 \atop \alpha_1}\,{,\ldots, \atop ,\ldots,}\,{u_r \atop \alpha_r})}(xt,x) = \frac{1}{t-|\boldsymbol{u}|}\begin{cases} +\int_0^{\boldsymbol{t}} \widehat{\mathcal{W}}_{*}^{({u_1 \atop \alpha_1}\,{,\ldots, \atop ,\ldots,}\,{u_{r-1} \atop \alpha_{r-1}})}(xt_0,x)\, e^{\alpha_r x(t-u_r-t_0)}\,dt_0 \\ -\sum_{j=1}^{r-1} W_{**}^{({u_1 \atop \alpha_1}\,{,\ldots, \atop ,\ldots,}\,{u_j \atop \alpha_j})}(x)\int_0^{\boldsymbol{t}} \widehat{\mathcal{W}}_{*}^{({u_{j+1} \atop \alpha_{j+1}}\,{,\ldots, \atop ,\ldots,}\,{u_r \atop \alpha_r})}(xt_0,x)dt_0 \\ -W_{**}^{({u_1 \atop \alpha_1}\,{,\ldots, \atop ,\ldots,}\,{u_r \atop \alpha_r})}(x) \end{cases}$$

Note that in the last equation the sum {... on the right-hand side vanishes for $t=|\boldsymbol{u}|$, so that $\widehat{\mathcal{W}}^\bullet_*(\zeta,x)$ is entire not just in x but also in $\zeta=tx$.

9.2. Coequational resurgence and its resurgent coefficients $T_*^\bullet(x)$.

$$\mathbf{BE_2}\qquad \sum_{w_j\in\boldsymbol{w}} {}^{x}\!\Delta_{|\boldsymbol{u}|v_j}\widetilde{\mathcal{S}}^{\boldsymbol{w}}(x)=\sum_{\boldsymbol{w}'_j w_j \boldsymbol{w}''_j=\boldsymbol{w}} e^{-|\boldsymbol{u}|v_j x}\widetilde{\mathcal{T}}_{v_j}^{\boldsymbol{w}'_j w_j^\sharp \boldsymbol{w}''_j}(x)=e^{-|\boldsymbol{u}|xz}e^{<\boldsymbol{u},\boldsymbol{\alpha}>x}\,\widetilde{T}_*^{\boldsymbol{w}}(x)$$

with $\widetilde{T}_*^{\boldsymbol{w}}(x)=\sum_j \widetilde{\mathcal{T}e}_{v_j}^{\boldsymbol{w}'_j w_j^\sharp \boldsymbol{w}''_j}(x)$ and with $\widetilde{\mathcal{T}e}_{v_j}^{\boldsymbol{w}}(x)$ denoting $\widetilde{\mathcal{T}}_{v_j}^{\boldsymbol{w}}(x)$ multiplied by its natural exponential factor $e^{\sum u_i(v_i-v_j)x}$. Thus, to the expansion $\mathcal{T}^{\boldsymbol{w}}(x)=\sum c_{\boldsymbol{\omega}}^{\boldsymbol{w}}\mathcal{V}^{\boldsymbol{\omega}}(x)$ there corresponds an expansion $\mathcal{T}e^{\boldsymbol{w}}(x)=\sum c_{\boldsymbol{\omega}}^{\boldsymbol{w}}\ \mathcal{V}e^{\boldsymbol{\omega}}(x)$ with $\mathcal{V}e^{\boldsymbol{\omega}}(x):=e^{|\boldsymbol{\omega}|x}\mathcal{V}^{\omega}(x)$. But any Laplace sum $\mathcal{V}e_\theta^\bullet(x)$ admits a decomposition $\mathcal{V}e_\theta^\bullet(x)=V_\theta^\bullet\times\Lambda^\bullet(x)\times\mathcal{V}e_*^\bullet(x)$ with three symmetral mould factors: an hyperlogarithmic $V_\theta^\bullet$ constant in x; an elementary $\Lambda^\bullet(x):=(-\log x)^{r(\bullet)}/r(\bullet)!$; and a mould $\mathcal{V}e_*^\bullet(x)$ that is an entire function of x characterised by

$$\mathcal{V}e_*^{\omega_1,\dots,\omega_r}(x)=\int_0^x\left(\mathcal{V}e_*^{\omega_2,\dots,\omega_r}(t)-\mathcal{V}e_*^{\omega_1,\dots,\omega_{r-1}}(t)\,e^{\omega_r t}\right)\frac{dt}{t}\tag{67}$$

Moreover, one easily checks that the x-rotators $\mathbb{R}$ (see §2.1) leave the x-resurgent $\widetilde{T}_*^\bullet(x)$ invariant. Their Laplace sums are therefore unramified at infinity. So, if we decompose their summands $\widetilde{\mathcal{T}e}_{v_j}^\bullet(x)$ into subsummands $\widetilde{\mathcal{V}e}^\bullet(x)$, and further decompose these into factors $V_\theta^\bullet$, $\Lambda^\bullet(x)$, $\widetilde{\mathcal{V}e}_*^\bullet(x)$ as above, the ramified part $\Lambda^\bullet(x)$ will vanish, leaving only elementary, x-independent hyperlogatithms $V_\theta^\bullet$ and a sensitive part $T_{**}^\bullet(x)$ entire in $\boldsymbol{u}$,$\boldsymbol{v}$ and x. Lastly, although $v_i:=z-\alpha_i$, only the combinations $v_{i:j}:=v_i-v_j$ feature in $T_{**}^\bullet(x)$, making it independent of z.

9.3. The equational-coequational linkage $W_{**}^\bullet(x)\leftrightarrow T_{**}^\bullet(x)$.

Proposition 9.1. *The coefficients $W_*^\bullet(x)$ and $T_*^\bullet(x)$ which respectively govern the z- and x-resurgence are fully determined by their logarithm-free parts $W_{**}^\bullet(x)$ and $T_{**}^\bullet(x)$, and these, despite being expressed by markedly distinct integrals, do in fact coincide: $W_{**}^\bullet(x)\equiv T_{**}^\bullet(x)$.*

Thus, leaving aside the trivial identity $W_{**}^{\binom{u_1}{\alpha_1}}\equiv T_{**}^{\binom{u_1}{\alpha_1}}\equiv 1$, we get for $r=2,3$:

$$\begin{aligned}
W_{**}^{\binom{u_1\,,\,u_2}{\alpha_1\,,\,\alpha_2}}(x) &= +\int_0^{u_{1,2}}\frac{(e^{\alpha_1\,(t_1-u_1)x}-1)\,e^{\alpha_2(u_1-t_1)x}}{t_1-u_1}dt_1-\int_0^{u_{1,2}}\frac{(e^{\alpha_2\,(t_2-u_2)x}-1)}{t_2-u_2}dt_2\\
&=\int_{-u_1}^{+u_2}\frac{e^{\alpha_{1:2}\,t\,x}-1}{t}\,dt\\
T_{**}^{\binom{u_1\,,\,u_2}{v_1\,,\,v_2}}(x) &= \int_0^x\left(e^{u_2\,v_{2:1}\,t}-e^{u_1\,v_{1:2}\,t}\right)\frac{dt}{t}
\end{aligned}$$

$$W_{**}^{\left(\begin{smallmatrix} u_1 \,,\, u_2 \,,\, u_3 \\ \alpha_1 \,,\, \alpha_2 \,,\, \alpha_3 \end{smallmatrix}\right)}(x) \;=\; \begin{cases} +\int_0^{u_{1,2,3}} \widehat{\mathcal{W}}_{**}^{\left(\begin{smallmatrix} u_1 \,,\, u_2 \\ \alpha_1 \,,\, \alpha_2 \end{smallmatrix}\right)}(xt_{12},x)\, e^{\alpha_3(u_{1,2}-t_{12})}\, dt_{12} \\ -W_{**}^{\left(\begin{smallmatrix} u_1 \\ \alpha_1 \end{smallmatrix}\right)}(x) \int_0^{u_{1,2,3}} \widehat{\mathcal{W}}_{**}^{\left(\begin{smallmatrix} u_2 \,,\, u_3 \\ \alpha_2 \,,\, \alpha_3 \end{smallmatrix}\right)}(xt_{23},x)dt_{23} \\ -W_{**}^{\left(\begin{smallmatrix} u_1 \,,\, u_2 \\ \alpha_1 \,,\, \alpha_2 \end{smallmatrix}\right)}(x) \int_0^{u_{1,2,3}} \widehat{\mathcal{W}}_{**}^{\left(\begin{smallmatrix} u_3 \\ \alpha_3 \end{smallmatrix}\right)}(xt_3,x)dt_3 \end{cases}$$

$$T_{**}^{\left(\begin{smallmatrix} u_1 \,,\, u_2 \,,\, u_3 \\ \alpha_1 \,,\, \alpha_2 \,,\, \alpha_3 \end{smallmatrix}\right)}(x) \;=\; \begin{cases} +\mathcal{V}e_*^{u_{2,3}v_{2:1},u_3v_{3:2}}(x) - \mathcal{V}e_*^{u_{2,3}v_{3:1},u_2v_{2:3}}(x) + \mathcal{V}e_*^{u_3v_{3:1},u_2v_{2:1}}(x) \\ -\mathcal{V}e_*^{u_1v_{1:2},u_3v_{3:2}}(x) - \mathcal{V}e_*^{u_3v_{3:2},u_1v_{1:2}}(x) \\ +\mathcal{V}e_*^{u_1v_{1:3},u_2v_{2:3}}(x) - \mathcal{V}e_*^{u_{1,2}v_{1:3},u_2v_{2:1}}(x) + \mathcal{V}e_*^{u_{1,2}v_{2:3},u_1v_{1:2}}(x) \end{cases}$$

with the auxiliary integrals:

$$\widehat{\mathcal{W}}_{**}^{\left(\begin{smallmatrix} u_1 \\ \alpha_1 \end{smallmatrix}\right)}(xt,x)) = \frac{e^{\alpha_1\,(t-u_1)\,x}-1}{t-u_1} \quad ; \quad \widehat{\mathcal{W}}_{**}^{\left(\begin{smallmatrix} u_1 \,,\, u_2 \\ \alpha_1 \,,\, \alpha_2 \end{smallmatrix}\right)}(xt,x) = \frac{1}{t-u_{1,2}} \begin{cases} +\int_0^t \frac{(e^{\alpha_1\,(t_-t_1-u_2)x}-1)\,e^{\alpha_2(u_1-t_1)x}}{t_1-u_1}dt_1 \\ -\int_0^t \frac{(e^{\alpha_2\,(t_2-u_2)x}-1)}{t_2-u_2}dt_2 \\ -\int_{-u_1}^{+u_2} \frac{e^{\alpha_{1:2}t_0x}-1}{t_0}dt_0 \end{cases}$$

$$\mathcal{V}e_*^{\omega_1,\omega_2}(x) = \int_0^x \frac{dx_1}{x_1}\int_0^{x_1}(1-e^{\omega_2\,x_2})\frac{dx_2}{x_2} - \int_0^x e^{\omega_2\,x_2}\frac{dx_2}{x_2}\int_0^{x_2}(1-e^{\omega_1\,x_1})\frac{dx_1}{x_1}$$

10. The equational-coequational link at the global level.

We now examine the *global* eq.-coeq. link on a concrete example.[6]

10.1. The time-independent Schrödinger equation with polynomial potential.

$$\frac{\hbar^2}{2\,m}\,\partial_q^2\Psi(q,\hbar) = W(q)\Psi(q,\hbar) \quad \textit{with} \quad W(q) = q^\nu + \alpha_1 q^{\nu-1} + .. + \alpha_\nu \tag{68}$$

Relative to the *critical* variable z and parameter x, (68) becomes (69), then (70), which is a special case of our model problem but with a $(2{+}\nu)$-ramified z:

$$\begin{cases} z = z(q) = \int_0^q \sqrt{W(q_0)}dq_0 \Rightarrow q = q(z) \sim \left(\frac{\nu+2}{2}\right)^{\frac{2}{\nu+2}} z^{\frac{2}{\nu+2}} \quad ., \quad z = \frac{\sqrt{8\,m}}{\hbar} \\ \Psi(q,\hbar) = \psi(z,x) = C_+(x)e^{\frac{1}{2}xz}q'(z)^{\frac{1}{2}}\varphi_+(z,x) + C_-(x)e^{-\frac{1}{2}xz}q'(z)^{\frac{1}{2}}\varphi_-(z,x) \end{cases}$$

$$\partial_z^2\varphi_\pm \pm x\,\partial_z\varphi_\pm = (H^2(z){-}H'(z))\varphi_\pm \;\; \textit{with} \;\; H(z) = \frac{1}{2}\frac{q''(z)}{q'(z)} \sim -\frac{1}{2}\frac{\nu}{\nu{-}2}z^{-1} \tag{69}$$

$$\partial_z Y_\pm \pm xY_\pm = H - HY_\pm^2 \qquad \textit{with} \qquad \frac{1}{2}\frac{q''}{q'} + \frac{\varphi'_\pm}{\varphi_\pm} = \pm\,x\,\frac{Y_\pm}{1-Y_\pm} \tag{70}$$

10.2. Equational resurgence.

$$\mathbf{BE_1} \quad \begin{cases} {}^{(z)}\Delta_{+x_i}\varphi_+(z,x) = A_i(x)\,\varphi_-(z,x) \quad , \qquad (i = 2,4,\ldots,\nu+2) \\ {}^{(z)}\Delta_{-x_i}\varphi_-(z,x) = A_i(x)\,\varphi_+(z,x) \quad , \qquad (i = 1,3,\ldots,\nu+1) \end{cases}$$

with points $\pm x_i$ above $\pm x$ in the $(2{+}\nu)$-ramified Borel ζ-plane.

[6] The results about the z-resurgence are essentially due to Y. Sibuya; those about the x-resurgence were conjectured by A.Voros. Convergence in the ξ-plane and isography were established by us.

10.3. Coequational resurgence.

$$\mathbf{BE_2} \quad \begin{cases} {}^{(x)}\Delta_{\,z+\lambda_j}\varphi_+(z,x) = P_{j,+}(x)\,\varphi_-(z,x)\ , & P_{j,\pm} \in \mathbb{C}[[x^{-1}]] \\ {}^{(x)}\Delta_{-z-\lambda_j}\varphi_-(z,x) = P_{j,-}(x)\,\varphi_+(z,x)\ , & \lambda_j = \int_{\gamma_j}\sqrt{(W(q_0)}dq_0 \end{cases}$$

If ν is even, we make the simplifying assumption $\oint\sqrt{(W(q_0)}dq_0 = 0$ (loop around ∞), so that $\sum_{j\ even}\lambda_j = \sum_{j\ odd}\lambda_j$. The exact shape of the $P_{j,\pm}(x)$ depends on the configuration $\{\lambda_1,..,\lambda_\nu\}$, but they are always rational functions of ν coefficients $E_i(x)$ bound by (71) and verifying (for $W(q)$ close to $q^\nu - 1$) the system BE_3:

$$E_1(x)E_2(x)\dots E_\nu(x) \equiv 1 \quad \Big(j \in \mathbb{Z}/\nu\mathbb{Z}\Big) \tag{71}$$

$$\mathbf{BE3} \ \begin{cases} 2\pi i\,\Delta_{n\lambda_{i:j}}E_k(x) & (k \neq i,j\ \ ,\ \ \lambda_{i:j} := \lambda_i - \lambda_j) \\ 2\pi i\,\Delta_{n\lambda_{i:j}}E_i(x) = +\frac{1}{n}E_i(x)\,(-F_{i:j}(x))^n & n \in \mathbb{Z}^*\ ,\ F_{i:j} := \frac{E_{i+1}E_{i+2}\dots E_{j-1}}{E_{j+1}E_{j+2}\dots E_{i-1}} \end{cases}$$

Setting $\mathbb{R}_{i:j} := \exp(\sum_{\arg\omega = \arg\lambda_{i:j}} 2\pi i\,\boldsymbol{\Delta}_\omega)$, we get the axis crossing identities:

$$\mathbb{R}_{i:j}E_k = E_k \ \mathit{if}\ k \neq i,j \ \ \mathit{and}\ \begin{cases} \mathbb{R}_{i:j}E_i = E_i\,\left(1 + e^{-\lambda_{i:j}x}\,F_{i:j}\right)^{-1} \\ \mathbb{R}_{i:j}E_j = E_j\,\left(1 + e^{-\lambda_{i:j}x}\,F_{i:j}\right) \end{cases} \tag{72}$$

10.4. Isographic invariance. Setting $T_{i:j} := \frac{t_{i+1}t_{i+2}\dots t_{j-1}}{t_{j+1}t_{j+2}\dots t_{i-1}}$, the change:

$$\begin{array}{lllll} t_j & \mapsto & t_j\ \exp\left(x\sum_{0<k<\nu}(-1)^k\,\lambda_{j+k}\right) & \mathit{if} \quad \nu & \mathit{odd} \\ t_j & \mapsto & t_j\ \exp\left(x\sum_{0<k<\nu'}(-1)^{\nu'+k}\frac{k}{\nu'}\,(\lambda_{\nu'+j-k} - \lambda_{\nu'+j+k}\right) & \mathit{if} \quad \nu = 2\,\nu' & \mathit{even} \end{array}$$

implies $T_{i:j} :=\mapsto e^{x\lambda_{i:j}}\ T_{i:j} \mod (t_1...t_\nu - 1)$. In view of BE_3, the mapping:

$$\boldsymbol{\Delta}_{n\lambda_{i:j}} \equiv e^{-n\lambda_{i:j}\,x}\,\Delta_{n\lambda_{i:j}} \quad \mapsto \quad \mathrm{D}_{n;i,j} \mapsto \frac{1}{n}T^n_{i:j}\left(t_i\partial_{t_i} - t_j\partial_{t_j}\right) \quad (i,j \in \mathbb{Z}_\nu, n \in \mathbb{Z})$$

induces an isomorphism of the *active algebra* (see §2.1) $ALIEN_{\mathcal{A}}$ of $\mathcal{A} := \{E_1,..,E_\nu\}$ into the algebra $\mathcal{D}$ generated by the ordinary differential operators $\mathrm{D}_{n;i,j}$.

Proposition 10.1 (Isographic invariance). *All elements of $\mathcal{D} \sim ALIEN_{\mathcal{A}}$ annihilate the 'isographic form' (73) which, even for ν odd, does not depend on* k*:*

$$\varpi^{iso}_\nu := (-1)^{\nu\,k}\sum_{(k<i<j)^{circ}_\nu}\frac{dt_i}{t_i}\wedge\frac{dt_j}{t_j} \quad \forall k \qquad \mod (t_1...t_\nu - 1) \tag{73}$$

10.5. Idempotence of the rotator. Set $\nu' := [\frac{\nu}{2}]$. The one-turn rotator is given by:

$$\mathbb{R} = \mathbb{R}^{**}_{\nu-1}\mathbb{R}^{*}_{\nu-1}\dots\mathbb{R}^{**}_1\mathbb{R}^{*}_1\mathbb{R}^{**}_0\mathbb{R}^{*}_0 \ \mathit{with}\ \begin{cases} \mathbb{R}^*_j := \prod_{1\leqslant k\leqslant\nu'}\mathbb{R}_{j+k:j+1+\nu-k} \\ \mathbb{R}^{**}_j := \prod_{2\leqslant k\leqslant\nu'}\mathbb{R}_{j+k:j+2+\nu-k} \end{cases} \tag{74}$$

Reverting to the more convenient t_i-variables with the corresponding substitution operators $R_{i:j}, R^*_j, R^{**}_j$, and defining $T_{i:j}$ as above, we find:

$$(R_0^* M)(t_1,..,t_\nu) = M(t_1^*,..,x_\nu^*) \quad \textit{with} \quad \begin{cases} t_k^* = t_k\,(1+T_{k:1+\nu-k})^{-1} & \textit{if } 1 \leqslant k \leqslant \nu' \\ t_k^* = t_k\,(1+T_{1+\nu-k}:k) & \textit{if } 1+\nu \leqslant k \leqslant \nu' \end{cases}$$

$$(R_0^{**} M)(t_1,..,t_\nu) = M(t_1^{**},..,x_\nu^{**}) \quad \textit{with} \quad \begin{cases} t_1^{**} = t_1\ ,\ t_{1+\nu'}^{**} = t_{1+\nu'} \\ t_k^{**} = t_k\,(1+T_{k:2+\nu-k})^{-1} & \textit{if } 2 \leqslant k \leqslant \nu' \\ t_k^{**} = t_k\,(1+T_{2+\nu-k}:k) & \textit{if } 2+\nu' \leqslant k \leqslant \nu \end{cases}$$

Let P be the permutation operator $(PM)(t_1,..,t_\nu) = M(t_\nu,t_1,..,t_{\nu-1})$

Proposition 10.2 (Autarky). *The rotator* R *admits a simple factorisation:*

$$R \equiv \big(P^{1-\nu}R_0^{**}R_0^*P^{\nu-1}\big)...\big(P^{-1}R_0^{**}R_0^*P^1\big)\big(R_0^{**}R_0^*\big) \equiv \big(PR_0^{**}R_0^*\big)^\nu \tag{75}$$

and we can show that $\big(PR_0^{**}R_0^*\big)^{\nu+2} = id\ (\forall\nu)$ *and* $\big(PR_0^{**}R_0^*\big)^{\nu'+1} = P^\nu\ (\forall\nu$ *even*$)$. *Thus the rotator* R *is idempotent of order* $\nu+2$*:* $R^{\nu+2} \equiv \big(PR_0^{**}R_0^*\big)^{\nu(\nu+2)} \equiv id$.

10.6. The equational-coequational linkage $\{A_j(x); j = 1..\nu\} \leftrightarrow \{E_j(x); j = 1..2+\nu\}$. The rotator's idempotence shows that *all* Laplace sums $E_j(x)$ of the resurgent coefficients $\widetilde{E}_j(x)$ are $(\nu{+}2)$-ramified at ∞: for x large enough, $E_j(x) = \sum c_{j,n} x^{-\frac{n}{\nu+2}}$. In fact, these $E_j(x)$ stand in birational correspondence with the earlier coefficients $A_j(x)$ which, as off-shoots of the z-resurgence, are automatically entire functions of $x^{\frac{1}{\nu+2}}$. That correspondence depends on the geometric configuration of the λ_j.

11. Isography and autarky.

11.1. Universality of isography. An *active alien algebra* isomorphic to an ordinary differential algebra $\mathcal{D}$ that annihilates an 'isographic' differential 2-form − these are general features of coequational resurgence, which survive even in presence of non-trivial tessellation coefficients. We saw an instance in §10. Here is another:

$$\partial_z Y(z) = x\,Y(z) + B_-(z) + B_+(z)\,Y^2(z) \ \textit{with}\ B_\pm(z) \begin{cases} \textit{meromorphic in } z \\ \textit{analytic at } z=\infty \end{cases} \tag{76}$$

For this Riccati equation with $B_\pm(z) = \sum_{i\in\mathcal{J}} \frac{\beta_i^\pm}{z-\lambda_i}$, the third Bridge equation involves resurgent functions $E_j(x)$ and alien derivations $\Delta_{\lambda_{i:j}}$ (with $\lambda_{i:j} := \lambda_i - \lambda_j$) The corresponding *active algebra* is isomorphic to the algebra $\mathcal{D}$ generated by ordinary derivations $D_{i:j}$which in turn annihilate the isographic form (78):

$$\Delta_{\lambda_{i:j}} \ \mapsto\ D_{i:j} := t_i^*\,t_j\,\partial_{t_j^*} - t_j^{**}\,t_i\,\partial_{t_i^{**}} + \frac{1}{2}t_i^*\,t_j^{**}(\partial_{t_j} - \partial_{t_i}) \tag{77}$$

$$\varpi^{iso} \ := \ \sum_i \frac{1}{t_i}\,dt_i^* \wedge dt_i^{**} \qquad \mathrm{mod}\ t_i^2 - t_i^* t_i^{**} = \mathrm{Const_i} \tag{78}$$

11.3. Autark functions. *Isographic invariance* is intimately bound up with the presence of *idempotent rotators.* Both facts combine to produce so-called *autark* *functions* – i.e., roughly speaking, entire functions whose asymptotic behaviour in the various sectors is fully described by resurgent asymptotic expansions, which in turn generate, under alien differentiation, *closed finite systems.* Despite being 'transcendental', autark functions have therefore a strong algebraic flavour. Their prototype is the inverse gamma function. They are quite common, too: thus, most Stokes constants are autark relative to their various parameters.

12. Conclusion.

At the end of this tour of coequational resurgence, we find a clear four level stratification:

- *The atomic level*, populated by objects such as simple poles or hyperlogarithms.
- *The molecular level*, consisting of huge clusters of atoms, namely the *wemu* and *welu* products, with unsuspected emergent properties.
- *The microscopic level*, consisting of derivation operators $\mathbb{Q}_\omega$, usually infinite chains of molecules contracted by elementary derivation operators.
- *The macroscopic level*, consisting of new derivation operators $\mathbb{P}_\omega$ assembled from the earlier $\mathbb{Q}_\omega$.
- The passage from the atomic to the molecular level is mediated on the Analysis side by *weighted convolution* and on the combinatorial side by the *scrambling transform.*
- The passage from the molecular to the microsopic level is rather mechanical – mere growth by accumulation.
- The passage from the microscopic to the macroscopic level, arguably the most interesting of the three, is mediated by the *tessellation coefficients.* While much is known about them, it would seem that just as much remains to be discovered.
- To ensure equational resurgence, it is enough for the inputs $b_i(z)$ to be holomorphic germs at infinity (and to verify uniform growth bounds).To ensure coequational resuregence, the $b_i(z)$ must *also* be capable of endless analytic continuation.
- Equational resurgence typically involves Stokes constants that are *transcendental*[7] to the inputs $b_i(z)$. Coequational resurgence typically involves Stokes constants that are *immanent*[8] to these inputs. And for unramified (e.g. meromorphic) $b_i(z)$, coequational resurgence dispenses altogether with the continuous-valued Stokes constants, and relies instead on the discrete, integer-valued tessellation coefficients.
- All three *active alien algebras* generated by the Δ_ω occurring in BE_1, BE_2, BE_3 tend to be isomorphic to ordinary differential algebras $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$, but $\mathcal{D}_2$ and $\mathcal{D}_3$, unlike $\mathcal{D}_1$, typically possess the property of *isographic invariance.*

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[7] In the sense that they cannot be detected directly on the germs $b_i(z)$, but only on complex integrals involving the Borel transforms $\widehat{b}_i(\xi)$.

[8] In the sense that they can be detected directly on the $b_i(z)$ and their ramifications.

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