Eupolars and their bialternality grid.

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Abstract : *This monograph is almost entirely devoted to the flexion structure generated by a flexion unit $\mathfrak{E}$ or the conjugate unit $\mathfrak{O}$, with special emphasis on the polar specialisation of the units ("eupolar structure").*
(i) We first state and prove the main facts (some of them new) about the central pairs of bisymmetrals $\mathrm{pal}^\bullet/\mathrm{pil}^\bullet$ *and* $\mathrm{par}^\bullet/\mathrm{pir}^\bullet$ *and their even/odd factors, by relating these to four remarkable series of alternals* $\{\mathfrak{re}^\bullet_r\}$, $\{\mathfrak{le}^\bullet_r\}$, $\{\mathfrak{he}^\bullet_r\}$, $\{\mathfrak{ke}^\bullet_{2r}\}$, *and that too in a way that treats the swappees* $\mathrm{pal}^\bullet$ *and* $\mathrm{pil}^\bullet$ *(resp.* $\mathrm{par}^\bullet$ *and* $\mathrm{pir}^\bullet$*) as they should be treated, i.e. on a strictly equal footing.*
(ii) Next, we derive from the central bisymmetrals two series of bialternals, distinct yet partially (and rather mysteriously) related.
(iii) Then, as a first step towards a complete description of the eupolar structure, we introduce the notion of bialternality grid and present some facts and conjectures suggested by our (still ongoing) computations.
(iv)Lastly, two complementary sections have been added, to show which features of the eupolar structure survive, change form or altogether diappear when one moves on to the next two cases in order of importance: eutrigonometric and polynomial.

Contents

1 Prefatory remarks. Dilators and their uses.

§1-1. Preamble.

We assume some familiarity with [E1] or [E3], though the main definitions have been recalled towards the end, in the appendix §17. In the main, the present paper concerns itself with the simplest, most basic flexion structure, namely the multialgebra-cum-multigroup $Flex(\mathfrak{E})$ generated by a single flexion unit $\mathfrak{E}$, and the companion structure $Flex(\mathfrak{O})$ generated by the conjugate unit $\mathfrak{O}$. Under the polar specialisation $(\mathfrak{E}, \mathfrak{O}) \mapsto (Pi, Pa)$, this becomes the *eupolar structure*, seemingly much simpler than the general eumonogenous structure[1] but in fact isomorphic to it. Eupolars can therefore serve as a

[1] meaning the structure generated under all *flexion operations* by a given *flexion unit.* Monogenous structures generated by an arbitrary element of $BIMU_1$ are of course more complex. For two equivalent characterisations of *flexion units*, in particular Pa and Pi, see §17.12 below. As for the (unary or binary) *flexion operations* allowed in the generative

prop for the intuition as well as a vehicle for simple proofs.

Within its self-assigned limits (eupolars and monogenous flexion structures) our paper deals with two sorts of questions – some clearly and provenly essential, others at first sight gratuitous but, we suspect, potentially of equal relevance. Let us explain.

The *essential part* revolves around the eupolar bisymmetral pair $pal^\bullet/pil^\bullet$ and its mirror image, the somewhat less important bisymmetrals $par^\bullet/pir^\bullet$. The first pair is doubly relevant to multizeta theory: *firstly*, because, together with its trigonometric counterpart $tal^\bullet/til^\bullet$, it goes into the making of the first factor $Zag_{\mathrm{I}}^\bullet/Zig_{\mathrm{I}}^\bullet$ in the classical trifactorisation of the fundamental bimould $Zag^\bullet/Zig^\bullet$ that "carries all multizetas"; and *secondly* because it enters into the construction of the so-called *singulators*, themselves key to the study of the canonical multizeta irreducibles.

The pair $pal^\bullet/pil^\bullet$, as also $par^\bullet/pir^\bullet$, had already been dealt with in our previous papers, but somewhat desultorily, on a piecemeal basis. So a unified treatment, complete with motivations, definitions, characterisations and proofs, was long overdue. The sections §2-§8 offer just such a treatment and, as is so often the case, systematisation brings its own rewards. Thus we exhibit two series, unsurpassed for simplicity, of alternals $\{\mathfrak{le}_r^\bullet\}$ and $\{\mathfrak{re}_r^\bullet\}$, and show that they are connected respectively to $pal^\bullet$ and $pil^\bullet$, as the ingredients of the *mu*-dilator $dupal^\bullet$ of $pal^\bullet$ and the *gari*-dilator $dipil^\bullet$ of $pil^\bullet$. This is a deeply satisfying state of affairs: it not only restores the symmetry (somewhat impaired in the previous approaches) between the co-equal swappees $pal^\bullet$ and $pil^\bullet$ but also leads to a simple proof of their bisymmetrality – of all extant proofs, the shortest. Nor do the pleasant surprises stop there. We introduce two additional series of alternals $\{\mathfrak{he}_r^\bullet\}$ and $\{\mathfrak{ke}_{2r}^\bullet\}$, less elementary than the first pair but still capable of a simple, transparent description, and show that these, too, are closely related to $ripal^\bullet$ (the *gari*-inverse of $pal^\bullet$) and its *even* factor $ripal_{\mathrm{ev}}^\bullet$. It is truly gratifying to see that our four elementary or semi-elementary series of alternals (so far the only of their kind, i.e. the only ones known to admit a simple description) turn out to be, each in its own way, intimately interwoven with the central bisymmetrals.

The paper's second part, from section §9 onwards, deals with the eupolar structure *per se*, without immediate applications in mind. The main challenge here is to generate, describe, and classify all *regular*, i.e. *neg*-invariant bisymmetrals and bialternals. Now, unlike the central bisymmetrals $pal^\bullet/pil^\bullet$ and $par^\bullet/pir^\bullet$, which are *irregular* (in the sense of being invariant under nei-

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process, they can all be constructed from the four elementary flexions $\lceil, \rfloor, \lfloor, \rceil$ in proper association. They include all operations listed in §17.2-§17.5 with the sole exceptions of *swap* and *pus* (*push* is allowed).

ther *neg* nor *pari* but only under the product $pari \circ neg$), the *regular* bisymmetrals $Sa^\bullet/Si^\bullet$ (as elements of *GARI*) correspond one-to-one to the *regular* bialternals (as elements of *ARI*) via the exponentiation *expari* from ARI to $GARI$[2]. So the attention now shifts to the bialternals which, living as they do in an algebra, are much easier to handle than the bisymmetrals. Starting from the two central-irregular pairs $pal^\bullet/pil^\bullet$ and $par^\bullet/pir^\bullet$, we describe two distinct procedures for producing two infinite series of bialternals, which in turn generate two distinct bialternal subalgebras of *ARI*. These two subalgebras do not coincide but partly overlap – though how far is yet unclear. Nor do we know whether, between themselves, they generate *all* bialternals.

This ignorance is galling. It is true that at the moment the polar bialternals, unlike the central bisymmetrals,[3] have no known applications to multizeta algebra. But this may change. It would indeed be strange if the eupolar structure, even in its most recondite aspects, did not have some bearing on the study on multizetas. On the contrary, there is every reason to believe, and past experience strongly suggests, that most difficulties, irregularities or anomalies besetting multizeta theory[4] originate in the eupolar domain which, being itself purely *singular*, holds the key to all the 'singularity' scattered over the wider flexion field. Be that as it may, and all applications aside, the eupolar structure is a fascinating subject in its own right and deserves to be studied for its own sake.

So how are we to advance our knowledge of polar bialternals? Paradoxically, by widening the search: instead of obsessing about the sole bialternals and the spaces $ARI_r^{\underline{\mathrm{al}}/\underline{\mathrm{al}}} = ARI_r^{(1,1)}$ spanned by them, we may relax the notion and consider the larger spaces $ARI_r^{(d_1,d_2)}$ spanned by all eupolars of a (suitably defined) bialternality codegree (d_1, d_2). The new approach embraces all eupolars, since for (d_1, d_2) large enough[5] $ARI_r^{(d_1,d_2)}$ coincides with the whole of ARI. Moreover, the dimensions

$$Bial_r^{d_1,d_2} := \dim(ARI_r^{(d_1,d_2)})$$

or rather the differences

$$bial_r^{d_1,d_2} := Bial_r^{d_1,d_2} - Bial_r^{d_1-1,d_2} - Bial_r^{d_1,d_2-1} + Bial_r^{d_1-1,d_2-1}$$

[2]The much simpler correspondance between *GARI*-elements and their various dilators, though extremely useful, does not respect *double symmetries*, but merely turns *symmetrality* into *alternality*.

[3]and, of course, unlike the polynomial bialternals!

[4]like, for example, the existence of the exceptional, polynomial-valued bialternals $carma^\bullet/carmi^\bullet$. See E1 and E2.

[5]$d_1 + d_2 > r$ suffices.

which constitute the entries of the so-called *bialternality grid*, seem to follow a remarkable pattern. In particular, when we add the quite natural requirement of *push*-invariance, every second grid entry vanishes, leading to the so-called *bialternality chessboard*.

The corresponding computations, however, are extremely complex and progress only haltingly. At the moment we are stuck at length $r = 8$: enough to discern the outlines of a tantalising pattern; not enough to see the full picture emerge. The investigation goes on but it may be quite some time before the next batch of data arrives.[6] So, rather than delay indefinitely the paper's publication, we have chosen to post this first draft, with its still incomplete section §12. We mean to update it regularly as the computations progress.

The present update (May 2014) already contains two sizeable additions: section §15, which shows what sort of changes the bialternality grid and chessboard undergo when we move on to polynomial-valued bimoulds; and section §16, which (pending a systematic treatment in [E4]) sketches the sort of complications attendant upon the passage from polar to trigonometric bisymmetrals. We wind up with section §17, which recalls the main definitions about flexion theory, and section §18, which gives short Maple programs for generating some of the main objects discussed in the paper. Lastly, numerous illustrative Tables have been posted on our homepage.[7]

§1-2. Conceptual vs mechanical proofs. The priorities of exploration.

The sheer profusion of formulae in flexion theory makes it strictly impossible to write down regular proofs for each one of them. Clearly, identities involving such key bimoulds as $pal^\bullet/pil^\bullet$ deserve to be established with care, to do justice to the centrality and flagship quality of these objects. But what about the common run of flexion formulae? For them, it would be nice (time-saving and reassuring) to be able to fall back on a

Mechanical truth criterion *(conjectural):*
Any bimould-valued flexion identity of the form

$$\mathcal{R}^\bullet(F_1, ..., F_p; A_1^\bullet, ..., A_q^\bullet) \equiv 0 \quad \textit{with} \;\; F_i \in \text{FLEXIONS} \;,\; A_j^\bullet \in \text{BIMU} \qquad (1)$$

[6] With many flexion operations, especially when working in algebras, it does not take much computational power to reach even length $r = 20$. With others, such as inflected group inversion, inflected exponentiation or, like in the present instance, when it comes to expressing that a bimould has a given bialternality codegree, difficulties arise much earlier.

[7] at http://www.math.u-psud.fr/∼ecalle/flexion.html

of total depth d

$$d = \mathrm{depth}(\mathcal{R}^\bullet) := \sum_i \mathrm{depth}(F_i) + \sum_j \mathrm{depth}(A_j^\bullet) \qquad (2)$$

is automatically true for all lengths r as soon as it holds identically for all arguments $A_j^\bullet$ and all lengths $r \leq d+1$.

This of course would require that we properly define the partial depths in formula (2).

The *depth* of 'products' F_i (associative or pre-Lie) would be 1; that of 'alternate' operations (commutators, Lie brackets etc) would be 2; and that of complex operations like the *singulators* would probably have to be 3 or 4.

The *depth* of the arguments $A_j^\bullet$ would be 1 when $A_j^\bullet$ is allowed to range unrestrained over $BIMU$; or 2 if when $A_j^\bullet$ ranges over the set of all bimoulds with a *simple symmetry*; or again 3 or 4 if when it ranges over all bimoulds with a *regular* double symmetry.

Though the existence of some such truth criterion would seem almost certain, none has been established as yet. On the other hand, in the identities commonly encountered in flexion theory the total depth d, summarily assessed along the above lines, rarely exceeds 6 or 7. So we may make safety doubly or trebly safe by verifying our identities up to the length $2d$ or $3d$ instead of $d+1$, which remains well within the range of the computationally feasible, and if the identities pass the test, confidently assume their validity.

But there is a catch here: in many important instances the arguments $A_j^\bullet$ do not range over a vast enough domain of *BIMU*. For instance, the *irregular* (though central!) bisymmetrals $pal^\bullet/pil^\bullet$ are fairly 'isolated' creatures, unlike the *regular*[8] (though less central!) bisymmetrals $Sa^\bullet/Si^\bullet$. For the likes $pal^\bullet/pil^\bullet$ or $par^\bullet/pir^\bullet$, therefore, no 'mechanical truth criterion' would work, and there is no way we can dispense with regular proofs here.

That said, *careful consolidation*, essential in the central, vital parts of an evolving theory, is one thing, and *unfettered exploration*, normal and legitimate at the fringes of the theory, is another. Each has its own logic, norms, and imperatives, and it would be foolish to mix up the two.

§1-3. Lie or pre-Lie brackets and group laws. Anti-actions.

This first paragraph is there simply to dispel possible misconceptions about the flexion *laws*, the corresponding *anti-actions*, and the impact on these of the basic involution *swap*, which is the very glue of *dimorphy*.

[8] i.e. *neg*-invariant

First, we have the overarching structure AXI/GAXI, whose elements are bimould pairs $\mathcal{A}^\bullet = (\mathcal{A}^\bullet_L, \mathcal{A}^\bullet_R)$. Then we have the unary structures (seven in number, up to isomorphism) consisting of simple bimoulds $A^\bullet$ and corresponding to as many substructures of AXI/GAXI, each one of which is defined by an involutive linkage $\mathcal{A}^\bullet_R \equiv h.\mathcal{A}^\bullet_L$ between left and right components (the number of suitable involutions h is of course very limited).

Let A∫I/GA∫I be such a unary structure[9]; let I∫A/GI∫A be the mirror structure under *swap*; and let h_1, h_2, h_3, h_4 be the four corresponding involutions:

$$\begin{array}{rclcrcl} \mathrm{a{\textstyle\int}i} & \longrightarrow & h_1 & ; & \mathrm{i{\textstyle\int}a} & \longrightarrow & h_2 \\ \mathrm{ga{\textstyle\int}i} & \longrightarrow & h_3 & ; & \mathrm{gi{\textstyle\int}a} & \longrightarrow & h_4 \end{array}$$

The *laws* are simply derived from the overstructure AXI/GAXI:

$$\begin{array}{rclcrcl} \mathrm{prea{\textstyle\int}i}(A^\bullet, B^\bullet) & = & \mathrm{preaxi}(\mathcal{A}^\bullet_1, \mathcal{B}^\bullet_1) & ; & \mathrm{prei{\textstyle\int}a}(A^\bullet, B^\bullet) & = & \mathrm{preaxi}(\mathcal{A}^\bullet_2, \mathcal{B}^\bullet_2) \\ \mathrm{a{\textstyle\int}i}(A^\bullet, B^\bullet) & = & \mathrm{axi}(\mathcal{A}_1, \mathcal{B}_1) & ; & \mathrm{i{\textstyle\int}a}(A^\bullet, B^\bullet) & = & \mathrm{axi}(\mathcal{A}_2, \mathcal{B}_2) \\ \mathrm{ga{\textstyle\int}i}(A^\bullet, B^\bullet) & = & \mathrm{gaxi}(\mathcal{A}^\bullet_3, \mathcal{B}^\bullet_3) & ; & \mathrm{gi{\textstyle\int}a}(A^\bullet, B^\bullet) & = & \mathrm{gaxi}(\mathcal{A}^\bullet_4, \mathcal{B}^\bullet_4) \end{array}$$

with

$$\begin{array}{lcll} \mathcal{A}^\bullet_{i,L} := A^\bullet & ; & \mathcal{A}^\bullet_{i,R} := h_i.A^\bullet & \qquad (\forall i \in \{1,2,3,4\}) \\ \mathcal{B}^\bullet_{i,L} := B^\bullet & ; & \mathcal{B}^\bullet_{i,R} := h_i.A^\bullet & \qquad (\forall i \in \{1,2,3,4\}) \end{array}$$

The *anti-actions* also are similarly defined:

$$\begin{array}{rclcrcl} \mathrm{a{\textstyle\int}it}(A^\bullet) & = & \mathrm{axit}(\mathcal{A}^\bullet_1) & ; & \mathrm{i{\textstyle\int}at}(A^\bullet) & = & \mathrm{axit}(\mathcal{A}^\bullet_2) \\ \mathrm{ga{\textstyle\int}it}(A^\bullet) & = & \mathrm{gaxit}(\mathcal{A}^\bullet_3) & ; & \mathrm{gi{\textstyle\int}at}(A^\bullet) & = & \mathrm{gaxit}(\mathcal{A}^\bullet_4) \end{array}$$

but whereas under the vowel swap $a \leftrightarrow i$ the three types of laws (pre-Lie, Lie, or associative) transmute into one another:

$$\begin{array}{rcl} \mathrm{prei{\textstyle\int}a}(A^\bullet, B^\bullet) & = & \mathrm{swap.prea{\textstyle\int}i}(\mathrm{swap}.A^\bullet, \mathrm{swap}.B^\bullet) \\ \mathrm{i{\textstyle\int}a}(A^\bullet, B^\bullet) & = & \mathrm{swap.a{\textstyle\int}i}(\mathrm{swap}.A^\bullet, \mathrm{swap}.B^\bullet) \\ \mathrm{gi{\textstyle\int}a}(A^\bullet, B^\bullet) & = & \mathrm{swap.ga{\textstyle\int}i}(\mathrm{swap}.A^\bullet, \mathrm{swap}.B^\bullet) \end{array}$$

the corresponding anti-actions *do not* relate in this way

$$\begin{array}{rcl} \mathrm{i{\textstyle\int}at}(A^\bullet) & \neq & \mathrm{swap.a{\textstyle\int}it}(\mathrm{swap}.A^\bullet).\mathrm{swap} \\ \mathrm{gi{\textstyle\int}at}(A^\bullet) & \neq & \mathrm{swap.ga{\textstyle\int}it}(\mathrm{swap}.A^\bullet).\mathrm{swap} \end{array}$$

and clearly *cannot*, since the right-hand sides (above) fail to define a *mu*-derivation resp. a *mu*-isomorphism.

[9]with the unusual mid-letter ∫ (pronounced *sh*) suggesting generality.

Nonetheless, the *laws* may be expressed in terms of the *anti-actions*. Thus for the first law we have:

$$\begin{aligned}
\mathrm{prea\!\int\! i}(A^\bullet,B^\bullet) &= \mathrm{a\!\int\! it}(B^\bullet).A^\bullet + \mathrm{mu}(A^\bullet,B^\bullet)\\
\mathrm{a\!\int\! i}(A^\bullet,B^\bullet) &= \mathrm{prea\!\int\! i}(A^\bullet,B^\bullet) - \mathrm{prea\!\int\! i}(B^\bullet,A^\bullet)\\
&= \mathrm{a\!\int\! it}(B^\bullet).A^\bullet - \mathrm{a\!\int\! it}(A^\bullet).B^\bullet + \mathrm{lu}(A^\bullet,B^\bullet)\\
\mathrm{ga\!\int\! i}(A^\bullet,B^\bullet) &= \mathrm{mu}(\mathrm{ga\!\int\! it}(B^\bullet).A^\bullet,B^\bullet)
\end{aligned}$$

Of course, the same identities hold with "$a\!\int\! i$" changed everywhere to "$i\!\int\! a$".

§1-4. Left-right separation.

The phenomenon is summed up by the following identities, which speak for themselves:

$$\begin{aligned}
\mathrm{axit}(\mathcal{A}^\bullet) &= \mathrm{amit}(\mathcal{A}^\bullet_L) + \mathrm{anit}(\mathcal{A}^\bullet_R) &(3)\\
\mathrm{gaxit}(\mathcal{A}^\bullet) &= \mathrm{gamit}(\mathcal{A}^\bullet_L)\,.\,\mathrm{ganit}\big((\mathrm{gamit}(\mathcal{A}^\bullet_L))^{-1}\,\mathcal{A}^\bullet_R\big) &(4)\\
&= \mathrm{ganit}(\mathcal{A}^\bullet_R)\,.\,\mathrm{gamit}\big((\mathrm{ganit}(\mathcal{A}^\bullet_R))^{-1}\,\mathcal{A}^\bullet_L\big) &(5)
\end{aligned}$$

The last two identities are easier to check in the following, equivalent form:

$$\mathrm{gamit}(A^\bullet).\mathrm{ganit}(B^\bullet) = \mathrm{gaxit}(\mathcal{C}^\bullet) \quad \textit{with} \quad \mathcal{C}^\bullet_L := A^\bullet\ ,\ \mathcal{C}^\bullet_R := \mathrm{gamit}(A^\bullet).B^\bullet \quad (6)$$
$$\mathrm{ganit}(A^\bullet).\mathrm{gamit}(B^\bullet) = \mathrm{gaxit}(\mathcal{D}^\bullet) \quad \textit{with} \quad \mathcal{D}^\bullet_L := \mathrm{ganit}(A^\bullet).B^\bullet\ ,\ \mathcal{D}^\bullet_R := A^\bullet \quad (7)$$

§1-5. Closure under the basic involution *swap* .

There exist many "closure identities", which essentially reduce $i\!\int\! a$ / $gi\!\int\! a$ to $a\!\int\! i$ / $ga\!\int\! i$. We mention the only one that we shall really require:

$$\mathrm{gira}(A^\bullet,B^\bullet) \equiv \mathrm{ganit}(\mathrm{rash}.B^\bullet).\mathrm{gari}(A^\bullet,\mathrm{ras}.B^\bullet) \qquad (8)$$

with

$$\begin{aligned}
\mathrm{rash}.B^\bullet &:= \mathrm{mu}(\mathrm{push.swap.invmu.swap}.B^\bullet,B^\bullet) &(9)\\
\mathrm{ras}.B^\bullet &:= \mathrm{invgari.swap.invgari.swap}.B^\bullet &(10)
\end{aligned}$$

§1-6. The monogenous algebra $Flex(\mathfrak{E})$. Basis and projectors.

The monogenous algebra $Flex(\mathfrak{E}) = \oplus Flex_r(\mathfrak{E})$ was constructed in [E3] §3-§4, along with the standard basis $\{\mathfrak{e}^\bullet_t\} \sim \{\mathfrak{e}^\bullet_{\underline{t}}\}$ of $Flex_r(\mathfrak{E})$. That standard basis has cardinality $(2r)!/(r!\,(r+1)!)$ and admits a natural indexation either by r-node binary trees $\boldsymbol{t}$ or by some special r-term sequences $\underline{\boldsymbol{t}}$ that stand in one-to-one correspondance with these trees. The basis elements are defined inductively:

$$\begin{aligned} \mathfrak{e}^{\bullet}_{\boldsymbol{t}} &:= \operatorname{amnit}(\mathfrak{e}^{\bullet}_{\boldsymbol{t_1}}, \mathfrak{e}^{\bullet}_{\boldsymbol{t_2}}).\mathfrak{E}^{\bullet} \qquad \Longleftrightarrow & (11)\\ \mathfrak{e}^{\boldsymbol{w}}_{\boldsymbol{t}} &:= \mathfrak{e}^{\boldsymbol{w^1}\rfloor}_{\boldsymbol{t_1}}\, \mathfrak{E}^{\lceil w_i \rceil}\, \mathfrak{e}^{\lfloor \boldsymbol{w^2}}_{\boldsymbol{t_2}} \quad \textit{with} \quad \boldsymbol{w} = \boldsymbol{w^1}.w_i.\boldsymbol{w^2} \quad \textit{and} \quad r_1+r_2 = r-1 \end{aligned}$$

and the corresponding inductions for trees and sequences go like this:

$$(\boldsymbol{t_1}, \boldsymbol{t_2}) \;\mapsto\; \boldsymbol{t} := \{\boldsymbol{t_1} \leftarrow \bullet \rightarrow \boldsymbol{t_2}\} \tag{12}$$

$$(\underline{\boldsymbol{t_1}}, \underline{\boldsymbol{t_2}}) \;\mapsto\; \underline{\boldsymbol{t}} := [\,\underline{\boldsymbol{t_1}},\, r_1+1,\, \underline{\boldsymbol{t_2}}^{(r_1+1)}\,] \tag{13}$$

Here, $\{\boldsymbol{t_1} \leftarrow \bullet \rightarrow \boldsymbol{t_2}\}$ denotes of course the binary tree we get by glueing $\boldsymbol{t_1}$ (resp. $\boldsymbol{t_2}$) to the root-node $\bullet$ as its left (resp. right) branch. On the sequence side, r_1 denotes the length of $\underline{\boldsymbol{t_1}}$ and $\underline{\boldsymbol{t_2}}^{(r_1+1)}$ results from $\underline{\boldsymbol{t_2}}$ by adding r_1+1 to its every element, after which we concatenate everything, thus producing a sequence $\underline{\boldsymbol{t}}$ that is some well-defined permutation of $[1, 2, \ldots, r]$.

What we now need is an algorithm for projecting the general element $X^{\bullet}$ of $Flex_r(\mathfrak{E})$ onto the standard basis. The following formula does just that:

$$X^{\bullet} \equiv \sum_{\boldsymbol{t}} \mathfrak{e}^{\bullet}_{\boldsymbol{t}}\, \mathrm{Res}^{\boldsymbol{t}} X^{\bullet} \;\stackrel{\text{i.e.}}{=}\; \sum \mathfrak{e}^{\bullet}_{[i_1,\ldots,i_r]}\, \mathrm{Res}^{i_1,\ldots,i_r} X^{\bullet} \tag{14}$$

with projectors $Res^{i_1,\ldots,i_r}$ capable of two interpretations:

$$(i) \quad \mathrm{Res}^{i_1,\ldots,i_r} \;:=\; \mathrm{Res}_{u_{i_r}} \ldots \mathrm{Res}_{u_{i_2}} \mathrm{Res}_{u_{i_1}} \tag{15}$$

$$(ii) \quad \mathrm{Res}^{i_1,\ldots,i_r} \;:=\; \mathrm{Res}_{v_{i_1}} .\mathrm{Res}_{v_{i_2}} \ldots \mathrm{Res}_{v_{i_r}} \tag{16}$$

Mark the order inversion from (i) to (ii). To calculate, $Res_{u_i} X^{\bullet}$, we set all variables v_i equal to 0; then take the coefficient of $\mathfrak{E}^{\binom{u_i}{0}}$ minus[10] the coefficient of $\mathfrak{E}^{\binom{-u_i}{0}}$; then set $u_i = 0$. Performing the operation r times, successively with $Res_{u_{i_1}}$, $Res_{u_{i_2}}$ etc, we end up with a scalar that *does not* depend on the particular expression chosen for $X^{\bullet}$ (elements of $Flex_r(\mathfrak{E})$, we recall, admit many different expressions).

To calculate $Res_{v_i} X^{\bullet}$, we go through exactly the same motions, but with the roles of the u_i's and v_i's exchanged and the order of the operations reversed. Once again, the final result does not depend on the expression[11] of $X^{\bullet}$, and coincides with the result of the first procedure.

[10]Of course, flexion units being odd functions of their variable $w_i = \binom{u_i}{v_i}$, we have $\mathfrak{E}^{\binom{u_i}{v_i}} \equiv -\,\mathfrak{E}^{\binom{-u_i}{-v_i}}$, but since complex superpositions of flexion operations are liable to yield either form, both possibilities must be taken into account.

[11]Elements of $Flex(\mathfrak{E})$ can be expressed/expanded in numerous, outwardly distinct ways and, when resulting from a sequence of flexion operations, they usually appear, prior to simplification, in an absurdly complicated shape.

Clearly, in the polar specialisation $\mathfrak{E} = Pa$ (resp. Pi), the operator Res_{u_i} (resp. Res_{v_i}) corresponds to the taking of the residue at $u_i = 0$ (resp. $v_i = 0$).

§1-7. Dilators: what are they, and what are they good for?

Infinitesimal *generators* and *dilators* have this in common that they often permit to rephrase problems about groups as more tractable problems about algebras. But of the two, the dilators are the more useful by far, mainly because they are so much closer, conceptually and computationally, to the group elements from which they derive.

Here is how the inflected dilators $diS^\bullet$ and $daS^\bullet$ and the uninflected dilator $duS^\bullet$ relate to the corresponding group element $S^\bullet$ (henceforth referred to as the *dilatee*):

$$\begin{aligned} \mathrm{der}.S^\bullet &= \mathrm{preari}(S^\bullet, diS^\bullet) && (\mathrm{diS}^\bullet = \text{gari-}\textit{dilator}) && (17)\\ \mathrm{der}.S^\bullet &= \mathrm{preira}(S^\bullet, daS^\bullet) && (\mathrm{daS}^\bullet = \text{gira-}\textit{dilator}) && (18)\\ \mathrm{dur}.S^\bullet &= \mathrm{mu}(S^\bullet, duS^\bullet) && (\mathrm{duS}^\bullet = \text{mu-}\textit{dilator}) && (19) \end{aligned}$$

The three relations are entirely parallel: indeed, the Lie bracket corresponding to *mu* is *lu* and *mu* may (trivially) be regarded as a pre-Lie bracket *prelu* for *lu*. As for the operators *der* and *dur*, they are *mu*-derivations each:

$$\begin{aligned} \mathrm{der}.S^{w_1,\dots,w_r} &:= r\, S^{w_1,\dots,w_r} && (20)\\ \mathrm{dur}.S^{w_1,\dots,w_r} &:= (u_1{+}\dots u_r)\, S^{w_1,\dots,w_r} && (21) \end{aligned}$$

In the context of the monogenous structures $Flex_r(\mathfrak{E})$ the latter derivation *dur* is particularly relevant when $\mathfrak{E} = Pa$ but even then it has the slight drawback of taking us out of $Flex_r(\mathfrak{E})$ into something which, with due quotation marks, might be called "$Flex_r(\mathfrak{E}) \otimes \{I^\bullet\}$", with an elementary $I^\bullet$ that is 1 or 0 according as the length $r(\bullet)$ is 1 or not.[12]

To remedy the non-internal character of *dur*, we must sometimes replace it by *duur*, which is a *bona fide* internal *mu*-derivation of $Flex(\mathfrak{E})$ into itself. Since all elements of $Flex_r(\mathfrak{E})$ may be expressed[13] as a superposition of terms $M_r^\bullet$ of the form

$$M_r^\bullet := \mathrm{amnit}(M_{r_1}^\bullet, M_{r_2}^\bullet).\mathfrak{E}^\bullet \qquad \textit{with} \quad r_1{+}r_2 = r{-}1 \ \textit{and} \ M_{r_i}^\bullet \in Flex_{r_i}(\mathfrak{E})$$

it is enough to say how *duur* acts on these $M_r^\bullet$, and here is how it acts:

$$\mathrm{duur}.M_r^\bullet := \mathrm{mu}(\mathrm{M}_{\mathrm{r_1}}^\bullet, \mathrm{I}^\bullet, \mathrm{M}_{\mathrm{r_2}}^\bullet) \qquad (22)$$

[12] $I^\bullet$ is the unit for mould composition $\circ$ and should be carefully distinguished from the multiplication unit $1^\bullet$ which is 1 or 0 according as the length $r(\bullet)$ is 0 or > 0.

[13] See [*E*3], (3.35).

The corresponding dilator relation then assumes the form

$$\mathrm{duur}.S^\bullet \;=\; \mathrm{mu}(S^\bullet, \mathrm{duur}.\mathit{duu}S^\bullet) \tag{23}$$

or the equivalent form

$$S^\bullet \;=\; \mathrm{muu}(S^\bullet, \mathit{duu}S^\bullet) \tag{24}$$

with *muu* denoting a sort of integration-by-part operator but with the twist that the underlying product *mu* is non-commutative:

$$\mathrm{muu}(A^\bullet, B^\bullet) \overset{\mathit{essentially}}{:=} \mathrm{duur}^{-1}.\mathrm{mu}(A^\bullet, \mathrm{duur}.B^\bullet) \tag{25}$$

or more rigorously:

$$\mathrm{muu}(A^\bullet, B^\bullet) := \mathrm{amnit}(\mathrm{mu}(A^\bullet, B_1^\bullet), B_2^\bullet).\mathfrak{E}^\bullet \quad \mathit{if} \quad B^\bullet = \mathrm{amnit}(B_1^\bullet, B_2^\bullet).\mathfrak{E}^\bullet$$

§1-8. Relations between inflected and non-inflected dilators.

For any $S^\bullet$ such that $S^\emptyset = 1$, the inflected dilators $\mathit{di}S^\bullet$, $\mathit{da}S^\bullet$ and the non-inflected dilator $\mathit{du}S^\bullet$ relate according to:

$$\mathrm{der.duS^\bullet} - \mathrm{dur.diS^\bullet} + \mathrm{lu}(\mathrm{diS^\bullet}, \mathrm{duS^\bullet}) - \mathrm{arit}(\mathrm{diS^\bullet}).\mathrm{duS^\bullet} \;=\; 0 \tag{26}$$

$$\mathrm{der.duS^\bullet} - \mathrm{dur.daS^\bullet} + \mathrm{lu}(\mathrm{daS^\bullet}, \mathrm{duS^\bullet}) - \mathrm{irat}(\mathrm{daS^\bullet}).\mathrm{duS^\bullet} \;=\; 0 \tag{27}$$

The shortest way to prove (26), (27) is to rewrite the dilator identities (17), (18), (19) as follows

$$\mathrm{D_1}.S^\bullet = \mathrm{mu}(\mathrm{S^\bullet}, \mathrm{diS^\bullet}) \quad \mathit{with} \quad \mathrm{D_1} := \mathrm{der} - \mathrm{arit}(\mathrm{diS^\bullet}) \tag{28}$$

$$\mathrm{D_2}.S^\bullet = \mathrm{mu}(\mathrm{S^\bullet}, \mathrm{daS^\bullet}) \quad \mathit{with} \quad \mathrm{D_2} := \mathrm{der} - \mathrm{irat}(\mathrm{daS^\bullet}) \tag{29}$$

$$\mathrm{D_3}.S^\bullet = \mathrm{mu}(\mathrm{S^\bullet}, \mathrm{duS^\bullet}) \quad \mathit{with} \quad \mathrm{D_3} := \mathrm{dur} \tag{30}$$

and to observe that since the derivation *dur* commutes with all three derivations *der*, arit(diS$^\bullet$), irat(daS$^\bullet$), we have:

$$[\mathrm{D_1}, \mathrm{D_3}] = [\mathrm{D_2}, \mathrm{D_3}] = 0 \qquad (\,\mathit{but}\;\; [\mathrm{D_1}, \mathrm{D_2}] \neq 0\,) \tag{31}$$

To establish (27), which we shall require in the sequel, we apply the commutator $[D_2, D_3]$ to $S^\bullet$. We get successively:

$$0 = \mathrm{D_2.D_3.S^\bullet} - \mathrm{D_3.D_2.S^\bullet}$$

$$0 = \mathrm{D_2.mu}(\mathrm{S^\bullet}, \mathrm{duS^\bullet}) - \mathrm{D_3.mu}(\mathrm{S^\bullet}, \mathrm{daS^\bullet})$$

$$0 = \mathrm{mu}(\mathrm{D_2.S^\bullet}, \mathrm{duS^\bullet}) + \mathrm{mu}(\mathrm{S^\bullet}, \mathrm{D_2.duS^\bullet}) - \mathrm{mu}(\mathrm{D_3.S^\bullet}, \mathrm{daS^\bullet}) - \mathrm{mu}(\mathrm{S^\bullet}, \mathrm{D_3.daS^\bullet})$$

$$0 = \mathrm{mu}(\mathrm{S^\bullet}, \mathrm{daS^\bullet}, \mathrm{duS^\bullet}) + \mathrm{mu}(\mathrm{S^\bullet}, \mathrm{D_2.duS^\bullet}) - \mathrm{mu}(\mathrm{S^\bullet}, \mathrm{duS^\bullet}, \mathrm{daS^\bullet}) - \mathrm{mu}(\mathrm{S^\bullet}, \mathrm{D_3.daS^\bullet})$$

Since we assumed $S^{\emptyset}=1$, our $S^\bullet$ is *mu*-invertible. So we may *mu*-divide the last identity by $S^\bullet$ on the left, and what we are left with is exactly the sought-after identity (27). The proof of (26) is entirely analogous.

We may note that since the relations (26) and (27) are of the form

$$r(\boldsymbol{w}).\mathrm{duS}^{\boldsymbol{w}} = \|\boldsymbol{u}\|.\mathrm{diS}^{\boldsymbol{w}} + \textit{earlier terms} \tag{32}$$
$$r(\boldsymbol{w}).\mathrm{duS}^{\boldsymbol{w}} = \|\boldsymbol{u}\|.\mathrm{daS}^{\boldsymbol{w}} + \textit{earlier terms} \tag{33}$$

they clearly determine $diS^\bullet$ and $daS^\bullet$ in terms of $duS^\bullet$ and *vice versa*.

We may also observe that since *prelu* := *mu* is, trivially, a pre-Lie law for the Lie law *lu*, the relation (26), (27) can be rewritten in the following, particularly harmonious form:

$$\mathrm{dur.diS}^\bullet + \mathrm{prelu}(\mathrm{duS}^\bullet,\mathrm{diS}^\bullet) \;=\; \mathrm{der.duS}^\bullet + \mathrm{preari}(\mathrm{diS}^\bullet,\mathrm{duS}^\bullet) \tag{34}$$
$$\mathrm{dur.daS}^\bullet + \mathrm{prelu}(\mathrm{duS}^\bullet,\mathrm{daS}^\bullet) \;=\; \mathrm{der.duS}^\bullet + \mathrm{preira}(\mathrm{daS}^\bullet,\mathrm{duS}^\bullet) \tag{35}$$

Furthermore, although there exists no simple direct relation between the inflected dilators $diS^\bullet$ and $daS^\bullet$, there exists, interestingly, an indirect one, via the non-inflected $duS^\bullet$.

§1-9. Dilatees in terms of the dilators.

One goes from a *mu*-dilator $duS^\bullet$ or $duuS^\bullet$ to the source element $S^\bullet$ (the "dilatee") via the identities:

$$S^{\boldsymbol{w}} \;=\; 1^{\boldsymbol{w}} + \sum_{\boldsymbol{w^1}\ldots\boldsymbol{w^s}=\boldsymbol{w}} \mathrm{Paj}^{|\boldsymbol{u^1}|,\ldots,|\boldsymbol{u^s}|}\,\mathrm{duS}^{\boldsymbol{w^1}}\ldots\mathrm{duS}^{\boldsymbol{w^s}} \tag{36}$$
$$S^\bullet \;=\; 1^\bullet + \sum_{r_1+\ldots r_s=r(\bullet)} \overrightarrow{\mathrm{muu}}\,(\mathrm{duuS}^\bullet_{r_1},\ldots,\mathrm{duuS}^\bullet_{r_s}) \tag{37}$$

with a symmetral mould $Paj^\bullet$ defined by:

$$\mathrm{Paj}^{x_1,\ldots,x_r} := \prod_{1\le i\le r}\frac{1}{x_1+\cdots+x_i} \tag{38}$$

Similarly, one goes from a *gari*-dilator $diS^\bullet$ to the source $S^\bullet$ via the identity:

$$S^\bullet \;=\; \sum_{r_1+\ldots r_s=r(\bullet)} Paj^{r_1,\ldots,r_s}\,\overrightarrow{\mathrm{preari}}\,(diS^\bullet_{r_1},\ldots,diS^\bullet_{r_s}) \tag{39}$$

with the same auxiliary mould $Paj^\bullet$ but differently indexed.

An analogous formula expresses the product $T^\bullet = gari(R^\bullet, S^\bullet)$ in terms of the dilators:[14]

$$T^\bullet = R^\bullet + S^\bullet + \sum_{r_0 + \ldots r_s = r(\bullet)} Paj^{r_1,\ldots,r_s} \overrightarrow{\text{preari}} \, (R^\bullet_{r_0}, diS^\bullet_{r_1}, \ldots, diS^\bullet_{r_s}) \tag{40}$$

Mark the absence of r_0 in $Paj^{r_1,\ldots,r_s}$.

We may also, and often must, express the operators $garit(S^\bullet)$ and $adari(S^\bullet)$ in terms of $diS^\bullet$:

$$\text{garit}(S^\bullet) = \text{id} + \sum_{\boldsymbol{r_1 + \ldots r_s = r(\bullet)}} Paj^{r_1,\ldots,r_s} \text{arit}(diS^\bullet_{r_s}), \ldots \text{arit}(diS^\bullet_{r_1}) \tag{41}$$

$$\text{adari}(S^\bullet) = \text{id} + \sum_{\boldsymbol{r_1 + \ldots r_s = r(\bullet)}} Paj^{r_1,\ldots,r_s} \underline{\text{ari}}(diS^\bullet_{r_1}), \ldots \underline{\text{ari}}(diS^\bullet_{r_s}) \tag{42}$$

where $\underline{ari}$ denote the adjoint action of ARI on itself.[15] The indexation of the operators $\underline{ari}(diS^\bullet_{r_i})$ and $arit(diS^\bullet_{r_i})$ goes in opposite directions, but this should not come as a surprise, since *adari* defines an *action* (of *GARI* on *ARI*) and *garit* an *anti-action* (of *GARI* on *BIMU*).

§1-10. Some other dilator identities.

How does the *gari*-product affect dilators? Like this:

$$T^\bullet = \text{gari}(R^\bullet, S^\bullet) \quad \Longrightarrow \tag{43}$$

$$diT^\bullet = diS^\bullet + \text{adari}(S^\bullet)^{-1}.diR^\bullet \tag{44}$$

Since according to (42) $\text{adari}(S^\bullet)^{\pm 1}$ can also be expressed in terms of $diS^\bullet$, the above identity amounts to a sort of Campbell-Hausdorff formula for the composition of *gari*-dilators. In the same vein, we must mention the conversion formulae between

(i) the dilator $diS^\bullet$ of $S^\bullet$.

(ii) the dilator $diriS^\bullet$ of $riS^\bullet := invgari(S^\bullet)$

(iii) the infinitesimal generator $liS^\bullet := logari(S^\bullet)$.

The conversion $diS^\bullet \leftrightarrow diriS^\bullet$ is via the involutive formula:

$$\begin{aligned} diriS^\bullet &= \sum_{1 \le s} \sum_{\boldsymbol{w^1 \ldots w^s = w}} \text{Japaj}^{r(\boldsymbol{w^1}),\ldots,r(\boldsymbol{w^s})} \, \overrightarrow{\text{preari}} \, (diS^{\boldsymbol{w^1}}, \ldots, diS^{\boldsymbol{w^s}}) \\ &= \sum_{1 \le s} \frac{1}{s} \sum_{\boldsymbol{w^1 \ldots w^s = w}} \text{Japaj}^{r(\boldsymbol{w^1}),\ldots,r(\boldsymbol{w^s})} \, \overrightarrow{\text{ari}} \, (diS^{\boldsymbol{w^1}}, \ldots, diS^{\boldsymbol{w^s}}) \end{aligned} \tag{45}$$

[14] Of course, on the right-hand side of (40), we must substitute for $S^\bullet$ the expansion (39) and do likewise with $T^\bullet$.

[15] i.e. $\underline{ari}(A^\bullet).B^\bullet \equiv ari(A^\bullet, B^\bullet)$.

with an alternal mould $Japaj^\bullet := Compo(Ja^\bullet, Paj^\bullet)$ defined as $Paj^\bullet$ precomposed by the elementary mould $Ja^{x_1,\dots,x_r} := (-1)^r\, x_1$. Thus we get:

$$\mathrm{Japaj}_1^x = 1\,;\ \mathrm{Japaj}^{x_1,x_2} = \frac{x_1-x_2}{x_1\,x_2}\,;\ \mathrm{Japaj}^{x_1,x_2,x_3} = \frac{x_1x_3-x_1^2+x_2^2-x_3^2}{x_1x_3(x_1+x_2)(x_2+x_3)}\ \mathit{etc}$$

The conversion $liS^\bullet \to diS^\bullet$ is via an even simpler formula:

$$\begin{aligned} diS^\bullet &= \sum_{1\le s}\ \sum_{\boldsymbol{w^1}\ldots\boldsymbol{w^s}\,=\,\boldsymbol{w}} \mathrm{Bin}^{\,r(\boldsymbol{w^1}),\dots,r(\boldsymbol{w^s})}\ \overrightarrow{\mathrm{preari}}\,(liS^{\boldsymbol{w^1}},\dots,liS^{\boldsymbol{w^s}}) \\ &= \sum_{1\le s}\frac{1}{s}\sum_{\boldsymbol{w^1}\ldots\boldsymbol{w^s}\,=\,\boldsymbol{w}} \mathrm{Bin}^{\,r(\boldsymbol{w^1}),\dots,r(\boldsymbol{w^s})}\ \overrightarrow{\mathrm{ari}}\,(liS^{\boldsymbol{w^1}},\dots,liS^{\boldsymbol{w^s}}) \qquad (46) \end{aligned}$$

with an elementary alternal mould $Bin^\bullet$ defined by:

$$Bin^{x_1,\dots,x_r} := \frac{1}{r}\sum_{1\le j\le r}\frac{x_j}{(j-1)!(r-j)!} \qquad (47)$$

§1-11. Internals and externals.

A bimould $A^\bullet$ is said to be *internal* if, for all r, it verifies two dual properties, which in *short* notation read:

$$\begin{aligned} \{u_1+\dots u_r\neq 0\} &\implies \{A^{\binom{u_1\ ,\dots,\ u_r}{v_1\ ,\dots,\ v_r}}\equiv 0\} \qquad (48)\\ \{v_i-v_i'=\mathrm{const}\,;\ \forall i\} &\implies \{A^{\binom{u_1\ ,\dots,\ u_r}{v_1\ ,\dots,\ v_r}}\equiv A^{\binom{u_1\ ,\dots,\ u_r}{v_1'\ ,\dots,\ v_r'}}\} \qquad (49) \end{aligned}$$

and in *long* notation assume the more natural form:

$$\begin{aligned} \{u_0\neq 0\} &\implies \{A^{\left(\left[{u_0\atop v_0}\right],\,{u_1\ ,\dots,\ u_r\atop v_1\ ,\dots,\ v_r}\right)}\equiv 0\} \qquad (50)\\ \{\forall\, v_0\,,\,\forall\, v_0'\} &\implies \{A^{\left(\left(\left[{u_0\atop v_0}\right],\,{u_1\ ,\dots,\ u_r\atop v_1\ ,\dots,\ v_r}\right)\right.}\equiv A^{\left(\left(\left[{u_0\atop v_0'}\right],\,{u_1\ ,\dots,\ u_r\atop v_1\ ,\dots,\ v_r}\right)\right.}\} \qquad (51) \end{aligned}$$

Internals constitute an ideal ARI_{intern} of ARI resp. a normal subgroup $GARI_{intern}$ of $GARI$. The elements of the corresponding quotients are referred to as *externals*:

$$\begin{aligned} \mathrm{ARI_{extern}} &:= \mathrm{ARI/ARI_{intern}} \qquad (52)\\ \mathrm{GARI_{extern}} &:= \mathrm{GARI/GARI_{intern}} \qquad (53) \end{aligned}$$

Moreover, when restricted to internals, the *ari* bracket reduces, up to order, to the simpler *lu* bracket, and the *gari* product, again up to order, reduces to the *mu* product:

$$\mathrm{ari}(A^\bullet, B^\bullet) \equiv \mathrm{lu}(B^\bullet, A^\bullet) \quad , \quad \forall A^\bullet, B^\bullet \in \mathrm{ARI}_{\mathrm{intern}} \tag{54}$$
$$\mathrm{gari}(A^\bullet, B^\bullet) \equiv \mathrm{mu}(B^\bullet, A^\bullet) \quad , \quad \forall A^\bullet, B^\bullet \in \mathrm{GARI}_{\mathrm{intern}} \tag{55}$$

Lastly, we have two useful identities governing the action of *internal* bimoulds on *general* ones:

$$\mathrm{arit}(A^\bullet).B^\bullet \equiv \mathrm{lu}(A^\bullet, B^\bullet) \quad ; \quad \forall\, A^\bullet \in \mathrm{ARI}_{\mathrm{intern}}, \forall\, B^\bullet \in \mathrm{ARI} \tag{56}$$
$$\mathrm{garit}(A^\bullet).B^\bullet \equiv \mathrm{mu}(A^\bullet, B^\bullet) \quad ; \quad \forall\, A^\bullet \in \mathrm{GARI}_{\mathrm{intern}}, \forall\, B^\bullet \in \mathrm{GARI} \tag{57}$$

and two anologous identites for the action of *general* bimoulds on *internals*:

$$\mathrm{arit}(B^\bullet).A^\bullet \equiv \mathrm{ari}(A^\bullet, B^\bullet) \quad ; \quad \forall\, A^\bullet \in \mathrm{ARI}_{\mathrm{intern}}, \forall\, B^\bullet \in \mathrm{ARI} \tag{58}$$
$$\mathrm{garit}(B^\bullet).A^\bullet \equiv \mathrm{gari}(A^\bullet, B^\bullet) \quad ; \quad \forall\, A^\bullet \in \mathrm{GARI}_{\mathrm{intern}}, \forall\, B^\bullet \in \mathrm{GARI} \tag{59}$$

Pay attention to the order of the terms, and observe that any bimould, acting on an internal, produces an internal:

$$\mathrm{arit}(\mathrm{ARI})\,.\,\mathrm{ARI}_{\mathrm{intern}} \;\subset\; \mathrm{ARI}_{\mathrm{intern}} \tag{60}$$
$$\mathrm{garit}(\mathrm{GARI})\,.\,\mathrm{GARI}_{\mathrm{intern}} \;\subset\; \mathrm{GARI}_{\mathrm{intern}} \tag{61}$$

§1-12. Short guide to the nomenclature.

Elements of *Flex*$(\mathfrak{E})$ or *Flex*$(\mathfrak{O})$ are always denoted by a short letter combination in Gothic fonts, with $\mathfrak{e}$ or $\mathfrak{o}$ as root vowels. The exchange $\mathfrak{e} \leftrightarrow \mathfrak{o}$ reflects the involution *syap*[16] while vowel change plus the *Umlaut* double dot ($\mathfrak{e} \to \ddot{\mathfrak{o}}$ or $\mathfrak{o} \to \ddot{\mathfrak{e}}$) is expressive of the involution *swap*[17]

In the polar specialisations, for reasons we cannot go into here, the conventions have to be slightly different: the root vowel here is a (resp. i) for elements of *Flex*(Pa) (resp. *Flex*(Pi)) but the exchange $a \leftrightarrow i$ under conservation of the consonental skeleton usually reflects the *swap* transform: thus $pal^\bullet \leftrightarrow pil^\bullet$ and $par^\bullet \leftrightarrow pir^\bullet$. To express the *syap* transform, on the other hand, we usually change the final consonant plus of course the root vowel:

[16] which is a rigorous isomorphism for all flexion operations.

[17] which respects few operations, but with an all-important exception: when acting on *regular* (i.e. *neg*-invariant) bialternals or bisymmetrals, *swap* commutes respectively with *ari* or *gari*.

thus $pal^\bullet \leftrightarrow pir^\bullet$ and $pil^\bullet \leftrightarrow par^\bullet$. Since *swap* and *syap* thankfully commute, this leads to no major inconsistencies.

Lastly, inversion under the group laws, whether in the 'Gothic' or 'Roman' context, is usually denoted by a prefix reminiscent of the law: *ri* for *gari*, *ra* for *gira*, *mu* for *mu*. The same applies for the dilators, which take the prefix *di*, *da*, *du* depending on the parent group.

2 Polar alternals: the series $\{\mathfrak{re}^\bullet_r\}, \{\mathfrak{le}^\bullet_r\}$ and $\{\mathfrak{he}^\bullet_r\}, \{\mathfrak{ke}^\bullet_{2r}\}$.

We shall construct in $Flex(\mathfrak{E})$ two elementary and two semi-elementary series of alternals by giving in each case a direct description side by side with an inductive definition.

§2-1. The first alternal series $\{\mathfrak{re}^\bullet_r\}$.

The inductive definition, which immediately implies alternality, reads:

$$\mathfrak{re}^\bullet_1 := \mathfrak{E}^\bullet \quad ; \quad \mathfrak{re}_{\mathfrak{r}}{}^\bullet := \mathrm{arit}(\mathfrak{re}^\bullet_{r-1})\, \mathfrak{E}^\bullet \qquad (\forall r \geq 2) \tag{62}$$

To get a direct definition-description of $\mathfrak{re}^\bullet_r$, we may proceed like this. For any sign sequence $\boldsymbol{\epsilon} = \{\epsilon_1, \ldots, \epsilon_{r-1}\}$, we define the decreasing sets $J_i(\boldsymbol{\epsilon})$ by setting $J_1(\boldsymbol{\epsilon}) := [1, 2, \ldots, r]$ and, for $1 < i \leq r$, by taking $J_i(\boldsymbol{\epsilon})$ to be $J_{i-1}(\boldsymbol{\epsilon})$ deprived of its largest (resp. smallest) element if $\epsilon_{i-1} = +$ (resp -). Then:

$$\mathfrak{re}_r^{w_1,\ldots,w_r} := \sum_{\epsilon_1,..,\epsilon_{r-1} \in \{+,-\}} \epsilon_1 \ldots \epsilon_{r-1} \prod_{i=1}^{i=r} \mathfrak{E}^{\binom{u_i^*(\epsilon)}{u_i^*(\epsilon)}} \tag{63}$$

with indices $u_i^*(\boldsymbol{\epsilon}), v_i^*(\boldsymbol{\epsilon})$ defined by the dual conditions:

$$u_i^*(\boldsymbol{\epsilon}) \;:=\; \sum u_j \quad \textit{with } j \textit{ running through } J_i(\boldsymbol{\epsilon}) \tag{64}$$

$$v_i^*(\boldsymbol{\epsilon}) \;:=\; v_{j'} - v_{j''} \quad \textit{with } \; j' \in J_i(\boldsymbol{\epsilon}) - J_{i+1}(\boldsymbol{\epsilon}) \;,\; j'' \in J_{i-1}(\boldsymbol{\epsilon}) - J_i(\boldsymbol{\epsilon}) \tag{65}$$

Of course, for $i = 1$ we must set $v_{j''} = 0$.

Alternatively, one may say that, when projected onto the standard basis $\{e^\bullet_{\boldsymbol{t}}\}$ of $Flex(\mathfrak{E})$, the alternal $\mathfrak{re}^\bullet_r$ takes the coefficient $(-1)^k$ when $\boldsymbol{t}$ is a one-branch tree with k right-leaning slopes, and the coefficient 0 whenever $\boldsymbol{t}$ has more than one branch.

The most outstanding property of the alternals $\mathfrak{re}^\bullet_r$ is their self-reproduction à la Witt under the *ari* bracket:

$$\mathrm{ari}(\mathfrak{re}^\bullet_{r_1}, \mathfrak{re}^\bullet_{r_2}) = (r_1 - r_2)\; \mathfrak{re}^\bullet_{r_1+r_2} \tag{66}$$

§2-2. The second alternal series $\{\mathfrak{le}^\bullet_r\}$.

Here the direct definition reads:

$$\mathfrak{le}_r^{w_1,\dots,w_r} \quad := \quad \sum_{1\le i\le r} (-1)^{i-1} \frac{(r-1)!}{(i-1)!(r-i)!} \;\; \mathfrak{E}^{\binom{u_1+\dots+u_r}{v_i}} \prod_{j\neq i} \mathfrak{E}^{\binom{u_j}{v_j-v_i}} \tag{67}$$

Alternality is nearly obvious on this definitious. It is even more obvious for the closely related bimoulds $\mathfrak{len}^\bullet_r$:

$$\mathfrak{len}_r^{w_1,\dots,w_r} \quad := \quad \sum_{1\le i\le r} (-1)^{i-1} \frac{(r-1)!}{(i-1)!(r-i)!} \;\; \mathrm{I}^{\binom{u_i}{v_i}} \prod_{j\neq i} \mathfrak{E}^{\binom{u_j}{v_j}} \tag{68}$$

Clearly $\mathfrak{len}^\bullet_r = \mathrm{duur}.\mathfrak{le}^\bullet_r$, since we have on the one hand

$$\mathfrak{le}^\bullet_r \quad = \quad \sum_{1\le i\le r} (-1)^{i-1} \frac{(r-1)!}{(i-1)!(r-i)!} \;\; \mathrm{amnit}\big(\mathrm{mu_{i-1}}(\mathfrak{E}^\bullet), \mathrm{mu_{r-i}}(\mathfrak{E}^\bullet)\big).\, \mathfrak{E}^\bullet$$

and on the other

$$\mathfrak{len}^\bullet_r \quad = \quad \sum_{1\le i\le r} (-1)^{i-1} \frac{(r-1)!}{(i-1)!(r-i)!} \;\; \mathrm{mu}\big(\mathrm{mu_{i-1}}(\mathfrak{E}^\bullet), \mathrm{I}^\bullet, \mathrm{mu_{r-i}}(\mathfrak{E}^\bullet)\big)$$

which again implies:

$$\mathfrak{len}^\bullet_r \quad = \quad \vec{\mathrm{lu}}(\mathrm{I}^\bullet, \overbrace{\mathfrak{E}^\bullet, \dots, \mathfrak{E}^\bullet}^{(r-1)\, times}) \tag{69}$$

This last expression (69) ensures the alternality of $\mathfrak{len}^\bullet_r$ and the earlier identity $\mathfrak{len}^\bullet_r = \mathrm{duur}.\mathfrak{le}^\bullet_r$ carries alternality back to $\mathfrak{le}^\bullet_r$.

§2-3. The third alternal series $\{\mathfrak{he}^\bullet_r\}$.

We begin here with the direct, descriptive definition, which relies on the standard basis $\{e^\bullet_{\boldsymbol{t}}\}$ of $Flex(\mathfrak{E})$. The coefficients $he(\boldsymbol{t})$ of $\mathfrak{he}^\bullet_r$ in that basis are not going to depend on the full structure of the indexing binary trees $\boldsymbol{t}$ but only on a four-parameter 'abstract', $slant(\boldsymbol{t})$, which gives the numbers p_1, p_2 (resp. q_1, q_2) of left-leaning (resp. right-leaning) slopes in the two branches issueing from the tree's root node. Clearly, $p_1+p_2+q_1+q_2 = r-1$, and the inductive calculation of $slant(\boldsymbol{t})$ goes like this. If $\mathfrak{e}^\bullet_{\boldsymbol{t}} = amnit(\mathfrak{e}^\bullet_{\boldsymbol{t}'}, \mathfrak{e}^\bullet_{\boldsymbol{t}''}).\mathfrak{E}^\bullet$ with $slant(\boldsymbol{t}') = \left[\begin{smallmatrix} p'_1 \\ q'_1 \end{smallmatrix}\middle| \begin{smallmatrix} p'_2 \\ q'_2 \end{smallmatrix}\right]$ and $slant(\boldsymbol{t}'') = \left[\begin{smallmatrix} p''_1 \\ q''_1 \end{smallmatrix}\middle| \begin{smallmatrix} p''_2 \\ q''_2 \end{smallmatrix}\right]$, then

$$slant(\boldsymbol{t}) \;=\; \left[\begin{matrix} 1+p'_1+p'_2 \\ q'_1+q'_2 \end{matrix}\middle| \begin{matrix} p''_1+p''_2 \\ 1+q''_1+q''_2 \end{matrix}\right] \quad \text{\textit{if}} \quad \boldsymbol{t}',\boldsymbol{t}'' \neq \emptyset \tag{70}$$

$$slant(\boldsymbol{t}) \;=\; \left[\begin{matrix} 1+p'_1+p'_2 \\ q'_1+q'_2 \end{matrix}\middle| \begin{matrix} 0 \\ 0 \end{matrix}\right] \quad \text{\textit{if}} \quad \boldsymbol{t}'' = \emptyset \tag{71}$$

$$slant(\boldsymbol{t}) \;=\; \left[\begin{matrix} 0 \\ 0 \end{matrix}\middle| \begin{matrix} p''_1+p''_2 \\ 1+q''_1+q''_2 \end{matrix}\right] \quad \text{\textit{if}} \quad \boldsymbol{t}' = \emptyset \tag{72}$$

We can now define $\mathfrak{e}^\bullet_{\boldsymbol{t}}$:

$$\mathfrak{he}^\bullet_r \;=\; \sum_{r(\bullet)=r} \mathrm{he}(\boldsymbol{t})\, \mathfrak{e}^\bullet_{\boldsymbol{t}} \tag{73}$$

through coefficients $he(\boldsymbol{t}) = he^{\left[\begin{smallmatrix} p_1 \\ q_1 \end{smallmatrix}\middle| \begin{smallmatrix} p_2 \\ q_2 \end{smallmatrix}\right]}$ that depend only on $slant(\boldsymbol{t})$:

$$\mathrm{he}^{\left[\begin{smallmatrix} p_1 \\ q_1 \end{smallmatrix}\middle| \begin{smallmatrix} p_2 \\ q_2 \end{smallmatrix}\right]} \;=\; (-1)^{q_{12}-1}\frac{(p_{12})!(q_{12})!}{(p_{12}+q_{12})!}\ \det\left[\begin{matrix} p_1 \\ 1+q_1 \end{matrix}\middle| \begin{matrix} 1+p_2 \\ q_2 \end{matrix}\right] \tag{74}$$

with the usual abbreviations $p_{12} := p_1+p_2$, $q_{12} := q_1+q_2$.

The invariance, implied by alternality, of the $\mathfrak{he}^\bullet$ under

$$mantir := minu.anti.pari = -anti.pari$$

is immediate since it amounts to

$$\mathrm{he}^{\left[\begin{smallmatrix} p_1 \\ q_1 \end{smallmatrix}\middle| \begin{smallmatrix} p_2 \\ q_2 \end{smallmatrix}\right]} \equiv (-1)^{p_1+p_2+q_1+q_2}\ \mathrm{he}^{\left[\begin{smallmatrix} q_2 \\ p_2 \end{smallmatrix}\middle| \begin{smallmatrix} q_1 \\ p_1 \end{smallmatrix}\right]}$$

but the full alternality is less obvious. It may be derived from the following identities. Indeed, setting

$$\mathfrak{He}^\bullet := \sum_{1\leq r} \frac{1}{r\,(r+1)}\mathfrak{he}^\bullet_r \quad ; \quad \mathfrak{R}\ddot{\mathfrak{e}}^\bullet := \sum_{1\leq r} \frac{1}{r\,(r+1)}\mathfrak{r}\ddot{\mathfrak{e}}^\bullet_r \tag{75}$$

with $\mathfrak{r}\ddot{\mathfrak{e}}^\bullet_r := swap.\mathfrak{ro}^\bullet_r$ for $\mathfrak{ro}^\bullet_r := syap.\mathfrak{re}^\bullet_r$,[18] and introducing two elementary, mutually *gani*-inverse bimoulds $\mathfrak{se}^\bullet$, $\mathfrak{nise}^\bullet$:

$$\mathfrak{se}^{w_1,\dots,w_r} \;:=\; \mathfrak{E}^{w_1}\dots\mathfrak{E}^{w_r} \qquad (\mathfrak{se}^{\emptyset} := 1) \tag{76}$$

$$\mathfrak{nise}^{w_1,\dots,w_r} \;:=\; \mathfrak{E}^{\left(\begin{smallmatrix} u_1 \\ v_{1:2} \end{smallmatrix}\right)}\mathfrak{E}^{\left(\begin{smallmatrix} u_{12} \\ v_{2:3} \end{smallmatrix}\right)}\dots\mathfrak{E}^{\left(\begin{smallmatrix} u_{1\dots r} \\ v_r \end{smallmatrix}\right)} \qquad (\mathfrak{nise}^{\emptyset} := 1) \tag{77}$$

18 $\mathfrak{ro}^\bullet_r := syap.\mathfrak{re}^\bullet_r$ simply says that $\mathfrak{ro}^\bullet_r$ is constructed from $\mathfrak{O}$ exactly as $\mathfrak{re}^\bullet_r$ was constructed from $\mathfrak{E}$.

we can check (see (245)-(246)) either of the two equivalent identities:

$$\mathfrak{He}^\bullet = \mathrm{ganit}(\mathfrak{nise}^\bullet).\,\mathfrak{R\ddot{e}}^\bullet \tag{78}$$

$$\mathfrak{R\ddot{e}}^\bullet = \mathrm{ganit}(\mathfrak{se}^\bullet).\,\mathfrak{He}^\bullet \tag{79}$$

Since $\mathfrak{R\ddot{e}}^\bullet$ is elementarily $\mathfrak{E}^\bullet$-alternal and since the mutually inverse operators $ganit(\mathfrak{se}^\bullet)$ and $ganit(\mathfrak{nise}^\bullet)$ can be shown, almost as elementarily, to exchange $\mathfrak{E}^\bullet$-alternality and plain alternality

$$\begin{array}{rcccc} ganit(\mathfrak{se}^\bullet) & : & \text{alternal} & \longrightarrow & \mathfrak{E}\text{-alternal} \\ ganit(\mathfrak{nise}^\bullet) & : & \mathfrak{E}\text{-alternal} & \longrightarrow & \text{alternal} \end{array}$$

we conclude that $\mathfrak{He}^\bullet$ is indeed alternal. The hard part in all this is to establish (79) or, preferably, (78). See the remarks in §4, towards the end of the second bisymmetrality proof. But if we do not want to bother with the messy combinatorics involved, we may simply take (78) as definition of $\mathfrak{He}^\bullet$ and $\mathfrak{he}^\bullet_r$. This route is calculation-free and automatically ensures the alternality of $\mathfrak{he}^\bullet_r$.

§2-4. The fourth alternal series $\{\mathfrak{ke}^\bullet_{2r_*}\}$.

These new alternals are defined only for even lengths $r = 2r_*$. Like for the preceding series, we begin with a direct, descriptive definition by projection on the standard basis of $Flex(\mathfrak{E})$. Here too, the coefficients do not depend on the full structure of the indexing binary tree $\boldsymbol{t}$ but on a four-parameter 'abstract', $stack(\boldsymbol{t})$, which gives the numbers m_1, m_2 (resp. n_1, n_2) of end-nodes (resp. non end-nodes) carried by the two branches issueing from the root-node. Like in the previous case, we have $m_1+m_2+n_1+n_2 = r-1$ but, unlike in the previous case, there now exist obvious inequalities between the m_i's and the n_i's. As a result, for any given (even) length r, the number of distinct *stacks* will be less than that of of distinct *slants*.

The inductive definition of $stack(\boldsymbol{t})$ goes like this. If $\mathfrak{e}^\bullet_{\boldsymbol{t}} = amnit(\mathfrak{e}^\bullet_{\boldsymbol{t'}}, \mathfrak{e}^\bullet_{\boldsymbol{t''}}).\mathfrak{E}^\bullet$ with $stack(\boldsymbol{t'}) = \left[{m'_1 \atop n'_1}\middle|{m'_2 \atop n'_2}\right]$ and $stack(\boldsymbol{t''}) = \left[{m''_1 \atop n''_1}\middle|{m''_2 \atop n''_2}\right]$, then

$$stack(\boldsymbol{t}) = \left[\begin{array}{c|c} m'_1+m'_2 & p''_1+p''_2 \\ 1+n'_1+n'_2 & 1+q''_1+q''_2 \end{array}\right] \quad if \quad \boldsymbol{t'},\boldsymbol{t''} \neq \emptyset \tag{80}$$

$$stack(\boldsymbol{t}) = \left[\begin{array}{c|c} m'_1+m'_2 & 0 \\ 1+n'_1+n'_2 & 0 \end{array}\right] \quad if \quad \boldsymbol{t''} = \emptyset \tag{81}$$

$$stack(\boldsymbol{t}) = \left[\begin{array}{c|c} 0 & m''_1+m''_2 \\ 0 & 1+n''_1+n''_2 \end{array}\right] \quad if \quad \boldsymbol{t'} = \emptyset \tag{82}$$

We are now in a position to define $\mathfrak{ke}^\bullet_{2r_*}$

$$\mathfrak{ke}^\bullet_{2r_*} = \sum_{r(\boldsymbol{t})=2r_*(even)} \mathrm{ke}(\boldsymbol{t})\,\mathfrak{e}^\bullet_{\boldsymbol{t}} \tag{83}$$

through coefficients $ke(\boldsymbol{t}) = ke^{\left[{m_1 \atop n_1}\middle|{m_2 \atop n_2}\right]}$ that depend only on $stack(\boldsymbol{t})$:

$$\mathrm{ke}^{\left[{m_1 \atop n_1}\middle|{m_2 \atop n_2}\right]} \; = \; (-2)^{m_{12}-1}(m_{12}-1)!\frac{(n_{12}-m_{12})!!}{(n_{12}+m_{12}-2)!!} \; \det \begin{bmatrix} m_1 & m_2 \\ 1+n_1 & 1+n_2 \end{bmatrix} \tag{84}$$

with the usual abbreviations $m_{12} := m_1+m_2$, $n_{12} := n_1+n_2$ and with the *odd* or *double factorial*[19]:

$$n!! := 1.3.5\ldots(n-2).n = \frac{(n+1)!}{((n+1)/2)!}\, 2^{-(n+1)/2} \qquad (\forall\, n \ odd) \tag{85}$$

The above definition of $\mathfrak{ke}^\bullet_{2r_*}$ is concise enough, and striking too, but one thing it leaves in the dark[20] is the alternality of $\mathfrak{ke}^\bullet_{2r_*}$. One way (and as far as we know, the only way) round this difficulty is to relate $\{\mathfrak{ke}^\bullet_{2r_*}\}$ to $\{\mathfrak{he}^\bullet_{r}\}$. To this end, we set:

$$\mathfrak{He}^\bullet \;\; := \;\; \sum_{1\le r} \frac{1}{r\,(r+1)}\; \mathfrak{he}^\bullet_r \tag{86}$$

$$\mathfrak{He}^\bullet_{\mathrm{ev}} \;\; := \;\; \sum_{1\le r_*} \frac{1}{2r_*\,(2r_*+1)}\; \mathfrak{he}^\bullet_{2r} \tag{87}$$

$$\mathfrak{Ke}^\bullet = \mathfrak{Ke}^\bullet_{\mathrm{ev}} \;\; := \;\; \sum_{1\le r_*} \frac{2^{-2r_*+1}}{(2r_*+1)(2r_*-1)}\; \mathfrak{ke}^\bullet_{2r_*} \tag{88}$$

and we introduce the elementary operator $\mathcal{P}$ (adjoint action on ARI):

$$\mathcal{P}.M^\bullet := \frac{1}{2}\,\mathrm{ari}(\mathfrak{E}^\bullet, \mathrm{M}^\bullet) \tag{89}$$

The thing is now to establish the identity:

$$\mathfrak{Ke}^\bullet_{\mathrm{ev}} \;\; := \;\; -\frac{1}{2}\,\mathfrak{E}^\bullet + \exp(\mathcal{P})\,.\,\mathfrak{He}^\bullet \tag{90}$$

or the equivalent but computationally more economical identity, which involves half as many terms

$$\mathfrak{Ke}^\bullet_{\mathrm{ev}} \;\; := \;\; \cosh(\mathcal{P})^{-1}\,.\,\mathfrak{He}^\bullet_{\mathrm{ev}} \tag{91}$$

[19]This makes sense since the terms in the double factorials, namely $n_{12}+m_{12}-2$ and $n_{12}-m_{12}$, are always odd. The term $m_{12}-1$ may be even or odd, but that is no problem, as it sits in a simple factorial.

[20]apart of course from the obvious relation $anti.\mathfrak{ker}^\bullet_{2r_*} \equiv -\mathfrak{ker}^\bullet_{2r_*}$, which is necessary but far from sufficient for alternality.

and may be derived by inverting (90) to

$$\mathfrak{He}^\bullet \;:=\; \exp(-\mathcal{P})\,.\,(\frac{1}{2}\,\mathfrak{E}^\bullet + \mathfrak{Ke}^\bullet_{\rm ev}) \;\equiv\; \exp(-\mathcal{P})\,.\,\mathfrak{Ke}^\bullet_{\rm ev} \tag{92}$$

then parifying (92) to

$$\mathfrak{He}^\bullet_{\rm ev} \;:=\; \cosh(\mathcal{P})\,.\,\mathfrak{Ke}^\bullet_{\rm ev} \tag{93}$$

and lastly inverting (93) back to (91).

For ways of establishing (90) we refer to the paragraph *"properties of ripal$^\bullet_{\rm ev}$"* (see §4.7 below). But here again, if we are loath to go through the tedium of establishing (90) or (91) straight from the beautiful descriptive definition (83), we may forgo that direct definition and simply take (91) as *the* definition of $\mathfrak{ke}_{2r_*}$. This is sufficient for all practical purposes and it gives us the alternality of $\mathfrak{ke}_{2r_*}$ without our having to fire a single shot.

Remark: parity separation in $\{\mathfrak{he}^\bullet_r\}$.

From (90) and (91) we derive, after elimination of $\mathfrak{Ke}^\bullet_{\rm ev}$, an interesting way of expressing the odd-length components $\mathfrak{he}^\bullet_{2r_*+1}$ in terms of the even-length components. Indeed, setting:

$$\mathfrak{He}^\bullet \;=\; \mathfrak{He}^\bullet_{\rm ev} + \mathfrak{He}^\bullet_{\rm od} = \sum_{r\,\rm even} \frac{1}{r\,(r+1)}\,\mathfrak{he}^\bullet_r + \sum_{r\,\rm odd} \frac{1}{r\,(r+1)}\,\mathfrak{he}^\bullet_r \tag{94}$$

we get:

$$\mathfrak{He}^\bullet_{\rm od} \;=\; = \frac{1}{2}\mathfrak{E}^\bullet + \tanh(\mathcal{P}).\mathfrak{He}^\bullet_{\rm ev} \tag{95}$$

Of course, $\exp(\mathcal{P})$, $\cosh(\mathcal{P})$, $\tanh(\mathcal{P})$ etc should be interpreted as power series of the operator $\mathcal{P}$.

§2-5. Tables for length $r = 4$: the elementary alternals.

basis element	$\mathfrak{re}_4^{\boldsymbol{w}}$	$\mathfrak{le}_4^{\boldsymbol{w}}$
$\mathfrak{e}^{w_1,w_2,w_3,w_4}_{[1,2,3,4]} = \mathfrak{E}^{\binom{u_{1234}}{v_4}}\mathfrak{E}^{\binom{u_{123}}{v_{3:4}}}\mathfrak{E}^{\binom{u_{12}}{v_{2:3}}}\mathfrak{E}^{\binom{u_1}{v_{1:2}}}$	1	-1
$\mathfrak{e}^{w_1,w_2,w_3,w_4}_{[2,1,3,4]} = \mathfrak{E}^{\binom{u_{1234}}{v_4}}\mathfrak{E}^{\binom{u_{123}}{v_{3:4}}}\mathfrak{E}^{\binom{u_{12}}{v_{1:3}}}\mathfrak{E}^{\binom{u_2}{v_{2:1}}}$	-1	-1
$\mathfrak{e}^{w_1,w_2,w_3,w_4}_{[1,3,2,4]} = \mathfrak{E}^{\binom{u_{1234}}{v_4}}\mathfrak{E}^{\binom{u_{123}}{v_{2:4}}}\mathfrak{E}^{\binom{u_1}{v_{1:2}}}\mathfrak{E}^{\binom{u_3}{v_{3:2}}}$	0	-1
$\mathfrak{e}^{w_1,w_2,w_3,w_4}_{[2,3,1,4]} = \mathfrak{E}^{\binom{u_{1234}}{v_4}}\mathfrak{E}^{\binom{u_{123}}{v_{1:4}}}\mathfrak{E}^{\binom{u_{23}}{v_{3:1}}}\mathfrak{E}^{\binom{u_2}{v_{2:3}}}$	-1	-1
$\mathfrak{e}^{w_1,w_2,w_3,w_4}_{[3,2,1,4]} = \mathfrak{E}^{\binom{u_{1234}}{v_4}}\mathfrak{E}^{\binom{u_{123}}{v_{1:4}}}\mathfrak{E}^{\binom{u_{23}}{v_{2:1}}}\mathfrak{E}^{\binom{u_3}{v_{3:2}}}$	1	-1
$\mathfrak{e}^{w_1,w_2,w_3,w_4}_{[1,2,4,3]} = \mathfrak{E}^{\binom{u_{1234}}{v_3}}\mathfrak{E}^{\binom{u_{12}}{v_{2:3}}}\mathfrak{E}^{\binom{u_1}{v_{1:2}}}\mathfrak{E}^{\binom{u_4}{v_{4:3}}}$	0	3
$\mathfrak{e}^{w_1,w_2,w_3,w_4}_{[2,1,4,3]} = \mathfrak{E}^{\binom{u_{1234}}{v_3}}\mathfrak{E}^{\binom{u_{12}}{v_{1:3}}}\mathfrak{E}^{\binom{u_2}{v_{2:1}}}\mathfrak{E}^{\binom{u_4}{v_{4:3}}}$	0	3
$\mathfrak{e}^{w_1,w_2,w_3,w_4}_{[1,3,4,2]} = \mathfrak{E}^{\binom{u_{1234}}{v_2}}\mathfrak{E}^{\binom{u_1}{v_{1:2}}}\mathfrak{E}^{\binom{u_{34}}{v_{4:2}}}\mathfrak{E}^{\binom{u_3}{v_{3:4}}}$	0	-3
$\mathfrak{e}^{w_1,w_2,w_3,w_4}_{[1,4,3,2]} = \mathfrak{E}^{\binom{u_{1234}}{v_2}}\mathfrak{E}^{\binom{u_1}{v_{1:2}}}\mathfrak{E}^{\binom{u_{34}}{v_{3:2}}}\mathfrak{E}^{\binom{u_4}{v_{4:3}}}$	0	-3
$\mathfrak{e}^{w_1,w_2,w_3,w_4}_{[2,3,4,1]} = \mathfrak{E}^{\binom{u_{1234}}{v_1}}\mathfrak{E}^{\binom{u_{234}}{v_{4:1}}}\mathfrak{E}^{\binom{u_{23}}{v_{3:4}}}\mathfrak{E}^{\binom{u_2}{v_{2:3}}}$	-1	1
$\mathfrak{e}^{w_1,w_2,w_3,w_4}_{[3,2,4,1]} = \mathfrak{E}^{\binom{u_{1234}}{v_1}}\mathfrak{E}^{\binom{u_{234}}{v_{4:1}}}\mathfrak{E}^{\binom{u_{23}}{v_{2:4}}}\mathfrak{E}^{\binom{u_3}{v_{3:2}}}$	1	1
$\mathfrak{e}^{w_1,w_2,w_3,w_4}_{[2,4,3,1]} = \mathfrak{E}^{\binom{u_{1234}}{v_1}}\mathfrak{E}^{\binom{u_{234}}{v_{3:1}}}\mathfrak{E}^{\binom{u_2}{v_{2:3}}}\mathfrak{E}^{\binom{u_4}{v_{4:3}}}$	0	1
$\mathfrak{e}^{w_1,w_2,w_3,w_4}_{[3,4,2,1]} = \mathfrak{E}^{\binom{u_{1234}}{v_1}}\mathfrak{E}^{\binom{u_{234}}{v_{2:1}}}\mathfrak{E}^{\binom{u_{34}}{v_{4:2}}}\mathfrak{E}^{\binom{u_3}{v_{3:4}}}$	1	1
$\mathfrak{e}^{w_1,w_2,w_3,w_4}_{[4,3,2,1]} = \mathfrak{E}^{\binom{u_{1234}}{v_1}}\mathfrak{E}^{\binom{u_{234}}{v_{2:1}}}\mathfrak{E}^{\binom{u_{34}}{v_{3:2}}}\mathfrak{E}^{\binom{u_4}{v_{4:3}}}$	-1	1

Tables for length $r = 4$: the semi-elementary alternals.

basis element	slant	$\mathfrak{he}^{\boldsymbol{w}}_4$	stack	$\mathfrak{ke}^{\boldsymbol{w}}_4$
$\mathfrak{e}^{w_1,w_2,w_3,w_4}_{[1,2,3,4]} = \mathfrak{E}^{\binom{u_{1234}}{v_4}}\mathfrak{E}^{\binom{u_{123}}{v_{3:4}}}\mathfrak{E}^{\binom{u_{12}}{v_{2:3}}}\mathfrak{E}^{\binom{u_1}{v_{1:2}}}$	$\left[\begin{array}{c\|c}3&0\\0&0\end{array}\right]$	1	$\left[\begin{array}{c\|c}1&0\\2&0\end{array}\right]$	1
$\mathfrak{e}^{w_1,w_2,w_3,w_4}_{[2,1,3,4]} = \mathfrak{E}^{\binom{u_{1234}}{v_4}}\mathfrak{E}^{\binom{u_{123}}{v_{3:4}}}\mathfrak{E}^{\binom{u_{12}}{v_{1:3}}}\mathfrak{E}^{\binom{u_2}{v_{2:1}}}$	$\left[\begin{array}{c\|c}2&0\\1&0\end{array}\right]$	$-2/3$	$\left[\begin{array}{c\|c}1&0\\2&0\end{array}\right]$	1
$\mathfrak{e}^{w_1,w_2,w_3,w_4}_{[1,3,2,4]} = \mathfrak{E}^{\binom{u_{1234}}{v_4}}\mathfrak{E}^{\binom{u_{123}}{v_{2:4}}}\mathfrak{E}^{\binom{u_1}{v_{1:2}}}\mathfrak{E}^{\binom{u_3}{v_{3:2}}}$	$\left[\begin{array}{c\|c}2&0\\1&0\end{array}\right]$	$-2/3$	$\left[\begin{array}{c\|c}2&0\\1&0\end{array}\right]$	-4
$\mathfrak{e}^{w_1,w_2,w_3,w_4}_{[2,3,1,4]} = \mathfrak{E}^{\binom{u_{1234}}{v_4}}\mathfrak{E}^{\binom{u_{123}}{v_{1:4}}}\mathfrak{E}^{\binom{u_{23}}{v_{3:1}}}\mathfrak{E}^{\binom{u_2}{v_{2:3}}}$	$\left[\begin{array}{c\|c}2&0\\1&0\end{array}\right]$	$-2/3$	$\left[\begin{array}{c\|c}1&0\\2&0\end{array}\right]$	1
$\mathfrak{e}^{w_1,w_2,w_3,w_4}_{[3,2,1,4]} = \mathfrak{E}^{\binom{u_{1234}}{v_4}}\mathfrak{E}^{\binom{u_{123}}{v_{1:4}}}\mathfrak{E}^{\binom{u_{23}}{v_{2:1}}}\mathfrak{E}^{\binom{u_3}{v_{3:2}}}$	$\left[\begin{array}{c\|c}1&0\\2&0\end{array}\right]$	1	$\left[\begin{array}{c\|c}1&0\\2&0\end{array}\right]$	1
$\mathfrak{e}^{w_1,w_2,w_3,w_4}_{[1,2,4,3]} = \mathfrak{E}^{\binom{u_{1234}}{v_3}}\mathfrak{E}^{\binom{u_{12}}{v_{2:3}}}\mathfrak{E}^{\binom{u_1}{v_{1:2}}}\mathfrak{E}^{\binom{u_4}{v_{4:3}}}$	$\left[\begin{array}{c\|c}2&0\\0&1\end{array}\right]$	$1/3$	$\left[\begin{array}{c\|c}1&1\\1&0\end{array}\right]$	2
$\mathfrak{e}^{w_1,w_2,w_3,w_4}_{[2,1,4,3]} = \mathfrak{E}^{\binom{u_{1234}}{v_3}}\mathfrak{E}^{\binom{u_{12}}{v_{1:3}}}\mathfrak{E}^{\binom{u_2}{v_{2:1}}}\mathfrak{E}^{\binom{u_4}{v_{4:3}}}$	$\left[\begin{array}{c\|c}1&0\\1&1\end{array}\right]$	$1/3$	$\left[\begin{array}{c\|c}1&1\\1&0\end{array}\right]$	2
$\mathfrak{e}^{w_1,w_2,w_3,w_4}_{[1,3,4,2]} = \mathfrak{E}^{\binom{u_{1234}}{v_2}}\mathfrak{E}^{\binom{u_1}{v_{1:2}}}\mathfrak{E}^{\binom{u_{34}}{v_{4:2}}}\mathfrak{E}^{\binom{u_3}{v_{3:4}}}$	$\left[\begin{array}{c\|c}1&1\\0&1\end{array}\right]$	$-1/3$	$\left[\begin{array}{c\|c}1&1\\0&1\end{array}\right]$	-2
$\mathfrak{e}^{w_1,w_2,w_3,w_4}_{[1,4,3,2]} = \mathfrak{E}^{\binom{u_{1234}}{v_2}}\mathfrak{E}^{\binom{u_1}{v_{1:2}}}\mathfrak{E}^{\binom{u_{34}}{v_{3:2}}}\mathfrak{E}^{\binom{u_4}{v_{4:3}}}$	$\left[\begin{array}{c\|c}1&0\\0&2\end{array}\right]$	$-1/3$	$\left[\begin{array}{c\|c}1&1\\0&1\end{array}\right]$	-2
$\mathfrak{e}^{w_1,w_2,w_3,w_4}_{[2,3,4,1]} = \mathfrak{E}^{\binom{u_{1234}}{v_1}}\mathfrak{E}^{\binom{u_{234}}{v_{4:1}}}\mathfrak{E}^{\binom{u_{23}}{v_{3:4}}}\mathfrak{E}^{\binom{u_2}{v_{2:3}}}$	$\left[\begin{array}{c\|c}0&2\\0&1\end{array}\right]$	-1	$\left[\begin{array}{c\|c}0&1\\0&2\end{array}\right]$	-1
$\mathfrak{e}^{w_1,w_2,w_3,w_4}_{[3,2,4,1]} = \mathfrak{E}^{\binom{u_{1234}}{v_1}}\mathfrak{E}^{\binom{u_{234}}{v_{4:1}}}\mathfrak{E}^{\binom{u_{23}}{v_{2:4}}}\mathfrak{E}^{\binom{u_3}{v_{3:2}}}$	$\left[\begin{array}{c\|c}0&1\\0&2\end{array}\right]$	$2/3$	$\left[\begin{array}{c\|c}0&1\\0&2\end{array}\right]$	-1
$\mathfrak{e}^{w_1,w_2,w_3,w_4}_{[2,4,3,1]} = \mathfrak{E}^{\binom{u_{1234}}{v_1}}\mathfrak{E}^{\binom{u_{234}}{v_{3:1}}}\mathfrak{E}^{\binom{u_2}{v_{2:3}}}\mathfrak{E}^{\binom{u_4}{v_{4:3}}}$	$\left[\begin{array}{c\|c}0&1\\0&2\end{array}\right]$	$2/3$	$\left[\begin{array}{c\|c}0&2\\0&1\end{array}\right]$	4
$\mathfrak{e}^{w_1,w_2,w_3,w_4}_{[3,4,2,1]} = \mathfrak{E}^{\binom{u_{1234}}{v_1}}\mathfrak{E}^{\binom{u_{234}}{v_{2:1}}}\mathfrak{E}^{\binom{u_{34}}{v_{4:2}}}\mathfrak{E}^{\binom{u_3}{v_{3:4}}}$	$\left[\begin{array}{c\|c}0&1\\0&2\end{array}\right]$	$2/3$	$\left[\begin{array}{c\|c}0&1\\0&2\end{array}\right]$	-1
$\mathfrak{e}^{w_1,w_2,w_3,w_4}_{[4,3,2,1]} = \mathfrak{E}^{\binom{u_{1234}}{v_1}}\mathfrak{E}^{\binom{u_{234}}{v_{2:1}}}\mathfrak{E}^{\binom{u_{34}}{v_{3:2}}}\mathfrak{E}^{\binom{u_4}{v_{4:3}}}$	$\left[\begin{array}{c\|c}0&0\\0&3\end{array}\right]$	-1	$\left[\begin{array}{c\|c}0&1\\0&2\end{array}\right]$	-1

....

3 Polar bisymmetrals: main statements.

For perspective, let us start with a synoptic table of our central bimoulds:

$$\begin{array}{ccccccc}
\mathfrak{ess}^\bullet & \overset{swap}{\leftrightarrow} & \ddot{\mathfrak{o}}\mathfrak{ss}^\bullet & (\mathfrak{E}\mapsto \mathrm{Pi}) & \mathrm{pil}^\bullet & \overset{swap}{\leftrightarrow} & \mathrm{pal}^\bullet \\
syap \updownarrow & & syap \updownarrow & \overset{polar\ specialisation}{\Longrightarrow} & syap \updownarrow & & syap \updownarrow \\
\mathfrak{oss}^\bullet & \overset{swap}{\leftrightarrow} & \ddot{\mathfrak{e}}\mathfrak{ss}^\bullet & (\mathfrak{O}\mapsto \mathrm{Pa}) & \mathrm{par}^\bullet & \overset{swap}{\leftrightarrow} & \mathrm{pir}^\bullet
\end{array}$$

We take our stand on the self-reproduction property (66) of the alternals $\mathfrak{re}^\bullet_r$ under the *ari* bracket, which is entirely analogous to the behaviour of the monomials x^{r+1} under the bracket $\{\phi,\psi\} := \phi'\psi - \phi\psi'$. As a consequence, the Lie algebra isomorphism induced by $x^{r+1} \mapsto \mathfrak{re}^\bullet_r$ extends to an isomorphism of the group of formal identity-tangent mappings $f := x \mapsto x + \sum a_r\, x^{r+1}$ into the group $GARI_{re}$ consisting of bimoulds of the form $S^\bullet := expari(\sum \gamma_r\, \mathfrak{re}^\bullet_r)$. All elements of $GARI_{re}$ are automatically symmetral.

Proposition 3.1 (Direct bisymmetral: definition)
The source mapping $f : x \mapsto 1 - e^{-x} = x - 1/2\, x^2 + \ldots$ *has for images in* $GARI_{\mathfrak{re}}$ *resp.* $GARI_{\mathfrak{ro}}$ *bimoulds denoted by* $\mathfrak{ess}^\bullet$ *resp.* $\mathfrak{oss}^\bullet$. *They are automatically symmetral, but their swappees* $\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet$ *resp.* $\ddot{\mathfrak{e}}\mathfrak{ss}^\bullet$ *are also symmetral. The same-vowelled bimoulds* $\mathfrak{ess}$ *and* $\ddot{\mathfrak{e}}\mathfrak{ss}$ *(and by way of consequence* $\mathfrak{oss}$ *and* $\ddot{\mathfrak{o}}\mathfrak{ss}$*) coincide up to length* $r = 3$ *inclusively but differ ever after. Under the polar specialisation* $(\mathfrak{O}, \mathfrak{E}) \mapsto (\mathrm{Pa}, \mathrm{Pi})$ *our universal bimoulds specialise to:*

$$(\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet, \mathfrak{ess}^\bullet) \;\mapsto\; (\mathrm{pal}^\bullet, \mathrm{pil}^\bullet) \tag{96}$$
$$(\mathfrak{oss}^\bullet, \ddot{\mathfrak{e}}\mathfrak{ss}^\bullet) \;\mapsto\; (\mathrm{par}^\bullet, \mathrm{pir}^\bullet) \tag{97}$$

At this point, the reader may well ask: why, among all identity-tangent mappings f, single out precisely $f : x \mapsto 1 - e^{-x}$? The short answer is: because only this choice and no other[21] ensures that the separator $gepar(\mathfrak{ess}^\bullet)$ be symmetral (see (109)) below), which in turn is a necessary condition for $\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet$ (not $\mathfrak{ess}^\bullet$!) to be symmetral. The condition, however, is not sufficient, and the full bisymmetrality proofs (two of them), as indeed all the other proofs backing up this section's statements, shall be given in §4.

Proposition 3.2 (Direct bisymmetral: characterisation)
The bimould $\mathrm{pal}^\bullet$ *has only poles of the form* $P(u_i)$ *or* $P(u_1 + \ldots + u_{2i})$. *Equivalently, its swappee* $\mathrm{pil}^\bullet$, *or rather* $\mathrm{anti.pil}^\bullet$, *has only poles of the form*[22] $P(v_i - v_{i-1})$ *or* $P(v_{2i})$. *This pole pattern characterises* $\mathrm{pal}^\bullet/\mathrm{pil}^\bullet$ *among all other polar bisymmetrals.*

Proposition 3.3 (Inverse bisymmetral: properties)
The gari-*inverses (prefix "*ri*") of the bisymmetrals are automatically* symmetral, *but they are not* bisymmetral, *meaning that their swappees, which may also be viewed as* gira-*inverses (prefix "*ra*") are not exactly symmetral, but rather* $\mathfrak{E}$*-symmetral or* $\mathfrak{O}$*-symmetral, depending of course on the root vowel. Thus side by side with the straight symmetries*

$$\mathfrak{riess}^\bullet = \mathrm{invgari}(\mathfrak{ess}^\bullet) \quad \textit{and} \quad \mathfrak{ri}\ddot{\mathfrak{e}}\mathfrak{ss}^\bullet = \mathrm{invgari}(\ddot{\mathfrak{e}}\mathfrak{ss}^\bullet) \;\in\; \textit{symmetral} \tag{98}$$
$$\mathfrak{riess}^\bullet = \mathrm{invgari}(\mathfrak{ess}^\bullet) \quad \textit{and} \quad \mathfrak{ri}\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet = \mathrm{invgari}(\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet) \;\in\; \textit{symmetral} \tag{99}$$

[21]that is, up to a rescaling $f \mapsto f_c$ with $f_c : x \mapsto c^{-1} f(c\,x)$. But the applications we have in mind, as well as intrinsic considerations, dictate that we take $c = 1$.

[22]for $i = 1$, "$P(v_1 - v_0)$" of course reduces to $P(v_1)$.

we have the tweaked symmetries

$$\begin{aligned}
\mathfrak{raess}^\bullet &= \mathrm{invgira}(\mathfrak{ess}^\bullet) &&= \mathrm{swap}(\mathfrak{ri\ddot{o}ss}^\bullet) &&\in \mathfrak{E}\text{-}\mathit{symmetral} && (100)\\
\mathfrak{ra\ddot{e}ss}^\bullet &= \mathrm{invgira}(\mathfrak{\ddot{e}ss}^\bullet) &&= \mathrm{swap}(\mathfrak{rioss}^\bullet) &&\in \mathfrak{E}\text{-}\mathit{symmetral} && (101)\\
\mathfrak{raoss}^\bullet &= \mathrm{invgira}(\mathfrak{oss}^\bullet) &&= \mathrm{swap}(\mathfrak{ri\ddot{e}ss}^\bullet) &&\in \mathfrak{O}\text{-}\mathit{symmetral} && (102)\\
\mathfrak{ra\ddot{o}ss}^\bullet &= \mathrm{invgira}(\mathfrak{\ddot{o}ss}^\bullet) &&= \mathrm{swap}(\mathfrak{riess}^\bullet) &&\in \mathfrak{O}\text{-}\mathit{symmetral} && (103)
\end{aligned}$$

In the polar specialisation $(\mathfrak{O},\mathfrak{E}) \mapsto (\mathrm{Pa},\mathrm{Pi})$ *this becomes*

$$\begin{aligned}
&\mathrm{ripal}^\bullet \;,\; \mathrm{ripar}^\bullet \;,\; \mathrm{ripil}^\bullet \;,\; \mathrm{ripir}^\bullet \;, && \in \;\mathit{symmetral} && (104)\\
&\mathrm{rapil}^\bullet = \mathrm{swap.ripal}^\bullet \;,\; \mathrm{rapir}^\bullet = \mathrm{swap.ripar}^\bullet && \in \;\mathit{symmetril} && (105)\\
&\mathrm{rapal}^\bullet = \mathrm{swap.ripil}^\bullet \;,\; \mathrm{rapar}^\bullet = \mathrm{swap.ripir}^\bullet && \in \;\mathit{symmetrul} && (106)
\end{aligned}$$

We now recall the definition of the two separators[23] *gepar* and *hepar*

$$\begin{aligned}
\mathrm{gepar}.S^\bullet &:= \mathrm{mu}(\mathrm{anti.swap.S}^\bullet, \mathrm{swap.S}^\bullet) && (107)\\
\mathrm{hepar}.S^\bullet &:= \sum_{1\le k\le r(\bullet)} \mathrm{pus}^{\mathrm{k}}.\mathrm{logmu.swap.S}^\bullet && (108)
\end{aligned}$$

Proposition 3.4 (Direct bisymmetral: separators) .
The separation identities read

$$\begin{aligned}
\mathrm{gepar}.\mathfrak{ess}^\bullet &:= \mathrm{mu}(\mathrm{anti}.\mathfrak{\ddot{o}ss}^\bullet, \mathfrak{\ddot{o}ss}^\bullet) &&= \mathrm{expmu}(-\mathfrak{O}^\bullet) && (109)\\
\mathrm{hepar}.\mathfrak{ess}^\bullet &:= \textstyle\sum_{1\le k\le r(\bullet)} \mathrm{pus}^{\mathrm{k}}.\mathrm{logmu}.\mathfrak{\ddot{o}ss}^\bullet &&= -\frac{1}{2}\,\mathfrak{O}^\bullet && (110)
\end{aligned}$$

with their obvious analogues under the exchange $\mathfrak{e} \leftrightarrow \mathfrak{o}$.

Proposition 3.5 (Inverse bisymmetral: separators)
The separation identities read

$$\begin{aligned}
\mathrm{gepar}.\mathfrak{riess}^\bullet &:= \mathrm{mu}(\mathrm{anti}.\mathfrak{ra\ddot{o}ss}^\bullet, \mathfrak{ra\ddot{o}ss}^\bullet) &&= 1^\bullet + \sum_{r\ge 1} \mathrm{mu}_r(\mathfrak{O}^\bullet) && (111)\\
\mathrm{hepar}.\mathfrak{riess}^\bullet &:= \textstyle\sum_{1\le k\le r(\bullet)} \mathrm{pus}^{\mathrm{k}}.\mathrm{logmu}.\mathfrak{ra\ddot{o}ss}^\bullet &&= \frac{1}{2}\sum_{r\ge 1} \mathrm{mu}_r(\mathfrak{O}^\bullet) && (112)
\end{aligned}$$

They possess obvious analogues under the exchange $\mathfrak{e} \leftrightarrow \mathfrak{o}$. *Here* $\mathrm{mu}_r(\mathfrak{O}^\bullet)$ *stands, as usual, for the* r*-th* mu*-power of* $\mathfrak{O}$.

[23]so-called because, acting on elements $S^\bullet$ of the group $GARI_{\mathfrak{re}}$, they have the virtue of separating (or manifesting, if you prefer) the coefficients a_r of the source mapping f: see the remarks immediately before Proposition 3.1 and also [E3] §4.1.

Proposition 3.6 (Direct bisymmetral: *gari*-dilator)
The identity reads

$$\begin{aligned} \mathrm{der}.\mathfrak{ess}^\bullet &= \mathrm{preari}(\mathfrak{ess}^\bullet, \mathfrak{diess}^\bullet) \qquad \textit{with} && (113)\\ \mathfrak{diess}^\bullet &:= -\sum_{r\geq 1} \frac{1}{(1+r)!}\, \mathfrak{re}_r^\bullet \quad \in \quad \textit{alternal} && (114) \end{aligned}$$

and has an obvious analogue under the exchange $\mathfrak{e} \leftrightarrow \mathfrak{o}$.

Proposition 3.7 (Inverse bisymmetral: *gari*-dilator)
The identities read

$$\begin{aligned} \mathrm{der}.\mathfrak{riess}^\bullet &= \mathrm{preari}(\mathfrak{riess}^\bullet, \mathfrak{diriess}^\bullet) && (115)\\ \mathrm{der}.\mathfrak{ri\ddot{o}ss}^\bullet &= \mathrm{preari}(\mathfrak{ri\ddot{o}ss}^\bullet, \mathfrak{diri\ddot{o}ss}^\bullet) && (116) \end{aligned}$$

with dilators equal to

$$\begin{aligned} \mathfrak{diriess}^\bullet &:= +\sum_{r\geq 1} \frac{1}{r.(1+r)}\, \mathfrak{re}_r^\bullet \quad \in \quad \textit{alternal} && (117)\\ \mathfrak{diri\ddot{o}ss}^\bullet &:= +\sum_{r\geq 1} \frac{1}{r.(1+r)}\, \mathfrak{ho}_r^\bullet \quad \in \quad \textit{alternal} && (118) \end{aligned}$$

and with the semi-elementary alternals $\mathfrak{ho}_r^\bullet$ *defined as in (73) but based on the unit* $\mathfrak{O}$ *instead of* $\mathfrak{E}$.

Proposition 3.8 (Bisymmetral swappee: *mu*-dilator)
The identity reads

$$\begin{aligned} \mathfrak{\ddot{o}ss}^\bullet &= \mathrm{muu}(\mathfrak{\ddot{o}ss}, \mathfrak{duu\ddot{o}ss}) \qquad \textit{with} && (119)\\ \mathfrak{duu\ddot{o}ss}^\bullet &:= +\sum_{r\geq 1} \alpha_r\, \mathfrak{lo}_r^\bullet \quad \in \quad \textit{alternal} && (120) \end{aligned}$$

with muu *defined as in (25) and the elementary alternals* $\mathfrak{lo}_r^\bullet$ *defined as in §2 but with respect to the unit* $\mathfrak{O}$ *instead of* $\mathfrak{E}$. *The coefficients* α_r *are the Bernoulli numbers :*

$$\sum_{r\geq 1} \alpha_r\, t^r := -1 + \frac{t}{e^t - 1} = -\frac{1}{2}\,t + \frac{1}{12}\,t^2 - \frac{1}{720}\,t^4 + \frac{1}{30240}t^6 + \ldots \qquad (121)$$

Under the polar specialisation $\mathfrak{O} \mapsto Pa$, *the above relations assume the simpler form:*

$$\begin{aligned} \mathrm{dur.pal}^\bullet &= \mathrm{mu}.(\mathrm{pal}^\bullet, \mathrm{dupal}^\bullet) && (122)\\ \mathrm{dupal}^\bullet &:= \sum_{r\geq 1} \alpha_r\, \mathrm{lan}_r^\bullet && (123) \end{aligned}$$

relatively to the elementary alternals

$$\mathrm{lan}_r^\bullet \;:=\; \vec{\mathrm{lu}}(\mathrm{I}^\bullet, \overbrace{\mathrm{Pa}^\bullet, ..., \mathrm{Pa}^\bullet}^{r-1\; times}) \tag{124}$$

Before examining the parity properties of our bisymmetrals, a few general considerations are in order. It is clear that any bimould $M^\bullet$ such that $M^\emptyset = 1$ can be uniquely factored as follows

$$M^\bullet \;=\; \mathrm{gari}(M^\bullet_{\mathrm{od}}, M^\bullet_{\mathrm{ev}}) \;=\; \mathrm{mu}(M^\bullet_{\mathrm{odd}}, M^\bullet_{\mathrm{evv}}) \tag{125}$$

or in reverse order

$$M^\bullet \;=\; \mathrm{gari}(M^\bullet_{\mathrm{ev}}, M^\bullet_{\mathrm{od}}) \;=\; \mathrm{mu}(M^\bullet_{\mathrm{evv}}, M^\bullet_{\mathrm{odd}}) \tag{126}$$

with factors that of course differ from (125) to (126) but in both cases satisfy the parity conditions:

$$\begin{array}{lclclcl} \mathrm{pari}.M^\bullet_{\mathrm{ev}} & \equiv & M^\bullet_{\mathrm{ev}} & ; & \mathrm{pari}.M^\bullet_{\mathrm{od}} & \equiv & \mathrm{invgari}.M^\bullet_{\mathrm{od}} \\ \mathrm{pari}.M^\bullet_{\mathrm{evv}} & \equiv & M^\bullet_{\mathrm{evv}} & ; & \mathrm{pari}.M^\bullet_{\mathrm{odd}} & \equiv & \mathrm{invmu}.\,M^\bullet_{\mathrm{odd}} \end{array}$$

With the 'upper' factorisations (125), for example, we find

$$\mathrm{gari}(M^\bullet_{\mathrm{od}}, M^\bullet_{\mathrm{od}}) \;=\; \mathrm{gari}(M^\bullet, \mathrm{pari}.\mathrm{invgari}.M^\bullet) \tag{127}$$

$$\mathrm{mu}(M^\bullet_{\mathrm{odd}}, M^\bullet_{\mathrm{odd}}) \;=\; \mathrm{mu}(M^\bullet, \mathrm{pari}.\mathrm{invmu}.M^\bullet) \tag{128}$$

From there, by square rooting,[24] we go to $M^\bullet_{\mathrm{od}}$ and $M^\bullet_{\mathrm{odd}}$ and thence to $M^\bullet_{\mathrm{ev}}$ and $M^\bullet_{\mathrm{evv}}$.

None of this requires $M^\bullet$ to be symmetral or in $Flex(\mathfrak{E})$. Elements of $Flex(\mathfrak{E})$, though, behave identically under *pari* and *neg*, so that for them the labels *even* and *odd* acquire redoubled significance.

In any case the existence of *even* $\times$ *odd* or *odd* $\times$ *even* factorisations is a universal phenomenon.[25] What distinguishes the bisymmetrals is the existence of *remarkable* and *multiple* factorisations of that sort, with odd factors that tend to be exceedingly simple.

[24]an unambiguous operation, if we impose, as we do, that

$$M^\emptyset = M^\emptyset_{\mathrm{od}} = M^\emptyset_{\mathrm{ev}} = M^\emptyset_{\mathrm{odd}} = M^\emptyset_{\mathrm{evv}} = 1$$

[25]*universal* but by no means *elementary*: it involves square rooting, which in the case of identity-tangent mappings f generically produces divergence (of 'resurgent' type).

Proposition 3.9 (Parity properties)
We have three similar-looking but logically independent identities:

$$\mathfrak{ess}^\bullet = \mathrm{gari}(\mathfrak{ess}^\bullet_{\mathrm{od}}, \mathfrak{ess}^\bullet_{\mathrm{ev}}) \tag{129}$$
$$\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet = \mathrm{gari}(\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_{\mathrm{od}}, \ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_{\mathrm{ev}}) \tag{130}$$
$$\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet = \mathrm{mu}(\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_{\mathrm{evv}}, \ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_{\mathrm{odd}}) \tag{131}$$

with six symmetral factors. Three of these, namely $\mathfrak{ess}^\bullet_{\mathrm{ev}}$, $\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_{\mathrm{ev}}$*, and* $\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_{\mathrm{evv}}$ *are highly non-elementary and "even", i.e. simultaneously invariant under* neg *and* pari*, which implies that they carries only non-vanishing components of even length. The bimoulds in the next triplet,* $\mathfrak{ess}^\bullet_{\mathrm{od}}$*,* $\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_{\mathrm{od}}$ *and* $\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_{\mathrm{odd}}$*, are quite elementary, being given by:*

$$\mathfrak{ess}^\bullet_{\mathrm{od}} = \mathrm{expari}\big(-\frac{1}{2}\,\mathfrak{E}^\bullet\big) \tag{132}$$
$$\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_{\mathrm{od}} = \mathrm{expari}\big(-\frac{1}{2}\,\mathfrak{O}^\bullet\big) \tag{133}$$
$$\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_{\mathrm{odd}} = \mathrm{expmu}\big(-\frac{1}{2}\,\mathfrak{O}^\bullet\big) \tag{134}$$

or more explicitly:

$$\mathfrak{ess}^{w_1,\ldots,w_r}_{\mathrm{od}} = \frac{(-1)^r}{2^r}\,\mathfrak{E}^{\binom{u_1}{v_{1:2}}}\,\mathfrak{E}^{\binom{u_{12}}{v_{2:3}}}\ldots\mathfrak{E}^{\binom{u_{1\ldots r}}{v_r}} \tag{135}$$
$$\ddot{\mathfrak{o}}\mathfrak{ss}^{w_1,\ldots,w_r}_{\mathrm{od}} = \frac{(-1)^r}{2^r}\,\mathfrak{O}^{\binom{u_1}{v_{1:2}}}\,\mathfrak{O}^{\binom{u_{12}}{v_{2:3}}}\ldots\mathfrak{O}^{\binom{u_{1\ldots r}}{v_r}} \tag{136}$$
$$\ddot{\mathfrak{o}}\mathfrak{ss}^{w_1,\ldots,w_r}_{\mathrm{odd}} = \frac{(-1)^r}{2^r}\,\frac{1}{r!}\,\mathfrak{O}^{w_1}\ldots\mathfrak{O}^{w_r} \tag{137}$$

They are also "odd" in the sense of being invertible under pari *or* neg*:*

$$\mathrm{invgari}(\mathfrak{ess}^\bullet_{\mathrm{od}}) = \mathrm{pari}(\mathfrak{ess}^\bullet_{\mathrm{od}}) = \mathrm{neg}(\mathfrak{ess}^\bullet_{\mathrm{od}}) \tag{138}$$
$$\mathrm{invgari}(\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_{\mathrm{od}}) = \mathrm{pari}(\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_{\mathrm{od}}) = \mathrm{neg}(\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_{\mathrm{od}}) \tag{139}$$
$$\mathrm{invmu}(\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_{\mathrm{od}}) = \mathrm{pari}(\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_{\mathrm{od}}) = \mathrm{neg}(\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_{\mathrm{od}}) \tag{140}$$

Three points deserve attention here.

First, note the presence of a factor $\frac{1}{r!}$ in (137) and its absence in the inflected counterparts (135) and (136).

Second, there is no equivalent to (140) on the $\mathfrak{E}$-side, that is to say, no remarkable *mu*-factorisation[26] of $\mathfrak{ess}^\bullet$, whether of type $mu(\mathfrak{ess}^\bullet_{\mathrm{evv}}, \mathfrak{ess}^\bullet_{\mathrm{odd}})$ or of type $mu(\mathfrak{ess}^\bullet_{\mathrm{odd}}, \mathfrak{ess}^\bullet_{\mathrm{evv}})$.

[26] i.e. no factorisation with at least one elementary factor.

Third, while $\mathfrak{ess}^\bullet/\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet$ are *swap*-related, $\mathfrak{ess}^\bullet_{\text{od}}/\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_{\text{od}}$ are *syap*-related and $\mathfrak{ess}^\bullet_{\text{ev}}/\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_{\text{ev}}$ are not related at all (in any simple way). There would be some justification, therefore, for denoting the odd factor $\mathfrak{oss}^\bullet_{\text{ev}}$ rather than $\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_{\text{ev}}$, though in a way that too might be confusing. The truth is that this theory is so replete with symmetries that no nomenclature can possibly do justice to them all.

Proposition 3.10 (Even factors: separators)
The separators of $\mathfrak{ess}_{\text{ev}}$ *are unremarkable*[27] *but those of* $\mathfrak{riess}_{\text{ev}}$ *exactly mirror, up to parity, the formulae for* $\mathfrak{riess}$*:*

$$\text{gepar}.\mathfrak{riess}_{\text{ev}} = 1^\bullet + \sum_{r\geq 1} 4^{-r}\,\text{mu}_{\text{r}}(\mathfrak{O}^\bullet) \tag{141}$$

$$\text{hepar}.\mathfrak{riess}_{\text{ev}} = \sum_{r\geq 1} 4^{-r}\,\text{mu}_{\text{r}}(\mathfrak{O}^\bullet) \tag{142}$$

Proposition 3.11 (Even factors: ***gari*-** **and** ***gira*-dilators.)**
The three identities read

$$\text{der}.\mathfrak{ess}^\bullet_{\text{ev}} = \text{preari}(\mathfrak{ess}^\bullet_{\text{ev}}, \mathfrak{diess}^\bullet_{\text{ev}}) \tag{143}$$

$$\text{der}.\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_{\text{ev}} = \text{preira}(\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_{\text{ev}}, \mathfrak{da}\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_{\text{ev}}) \tag{144}$$

$$\text{der}.\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_{\text{evv}} = \text{preira}(\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_{\text{evv}}, \mathfrak{da}\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_{\text{ev}}) + \frac{1}{2}\,\text{mu}(\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_{\text{evv}}, \mathfrak{coda}\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_{\text{ev}}) \tag{145}$$

with

$$\mathfrak{diess}^\bullet_{\text{ev}} = -\sum_{1\leq r} \frac{1}{(2r+1)!}\,\mathfrak{re}^\bullet_{2r} \tag{146}$$

$$\mathfrak{da}\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_{\text{ev}} = -\sum_{1\leq r} \frac{1}{(2r+1)!}\,\mathfrak{r}\ddot{\mathfrak{o}}^\bullet_{2r} \tag{147}$$

$$\mathfrak{coda}\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_{\text{ev}} = \frac{1}{2}\,\text{expmu}(\mathfrak{O}^\bullet) + \frac{1}{2}\,\text{expmu}(-\mathfrak{O}^\bullet) - 1^\bullet \tag{148}$$

$$= -\mathfrak{da}\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_{\text{ev}} - \text{anti}.\mathfrak{da}\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_{\text{ev}} \tag{149}$$

Warning: the simultaneous occurrence of *ev/evv* in (145) (where $\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_{\text{evv}}$ stands side by side with $\mathfrak{da}\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_{\text{ev}}$ and $\mathfrak{coda}\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_{\text{ev}}$) is no misprint! This awkward jumble in notations is rooted in the nature of our objects and cannot be helped.[28]

[27] The generating functions for gepar($\mathfrak{ess}^\bullet_{\text{ev}}$) and hepar($\mathfrak{ess}^\bullet_{\text{ev}}$) are respectively $\frac{1}{\cosh(x/2)^2}$ and $-\frac{1}{2}\frac{x}{\tanh(x/2)}$.

[28] The only bimould that would deserve the label $\mathfrak{da}\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_{\text{evv}}$ would be the *gira*-dilator of $\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_{\text{evv}}$, characterised by the identity der.$\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_{\text{evv}}$ = preira($\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_{\text{evv}}, \mathfrak{da}\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_{\text{evv}}$). That bimould very much exists, of course, but it is thoroughly uninteresting and we can forget about it.

We may note, besides, that due to (149) the 'jumbled' identity (145) can be rewritten as follows:

$$\mathrm{der}.\mathfrak{\ddot{o}ss}^{\bullet}_{\mathrm{evv}} = \mathrm{irat}(\mathfrak{da\ddot{o}ss}^{\bullet}_{\mathrm{ev}}).\mathfrak{\ddot{o}ss}^{\bullet}_{\mathrm{evv}} + \frac{1}{2}\,\mathrm{mu}(\mathfrak{\ddot{o}ss}^{\bullet}_{\mathrm{evv}}, \mathfrak{da\ddot{o}ss}^{\bullet}_{\mathrm{ev}} - \mathrm{anti}.\mathfrak{da\ddot{o}ss}^{\bullet}_{\mathrm{ev}}) \quad (150)$$

with $id - anti$ rather than $id + anti$ in front of $\mathfrak{da\ddot{o}ss}^{\bullet}_{\mathrm{ev}}$.

Proposition 3.12 (Inverse even factor: *gari*-dilator)
We have two similar looking but logically totally distinct identities

$$\begin{aligned} \mathrm{der}.\mathfrak{riess}^{\bullet}_{\mathrm{ev}} &= \mathrm{preari}(\mathfrak{riess}^{\bullet}, \mathfrak{diriess}^{\bullet}_{\mathrm{ev}}) && (151)\\ \mathrm{der}.\mathfrak{ri\ddot{o}ss}^{\bullet}_{\mathrm{ev}} &= \mathrm{preari}(\mathfrak{ri\ddot{o}ss}^{\bullet}, \mathfrak{diri\ddot{o}ss}^{\bullet}_{\mathrm{ev}}) && (152)\end{aligned}$$

with dilators equal to

$$\begin{aligned} \mathfrak{diriess}^{\bullet}_{\mathrm{ev}} &:= +\sum_{r\geq 1} \frac{2^{1-2r}}{(2r-1).(2r+1)}\, \mathfrak{re}^{\bullet}_{2r} \quad \in \quad alternal && (153)\\ \mathfrak{diri\ddot{o}ss}^{\bullet}_{\mathrm{ev}} &:= +\sum_{r\geq 1} \frac{2^{1-2r}}{(2r-1).(2r+1)}\, \mathfrak{ko}^{\bullet}_{2r} \quad \in \quad alternal && (154)\end{aligned}$$

and with the semi-elementary alternals $\mathfrak{ko}^{\bullet}_{2r}$ defined as in §2 but based on the unit $\mathfrak{O}$ instead of $\mathfrak{E}$.

Proposition 3.13 (Even factors: *mu*-dilators.)
We have two similar looking but logically rather distinct identities

$$\begin{aligned} \mathfrak{\ddot{o}ss}^{\bullet}_{\mathrm{ev}} &= \mathrm{muu}(\mathfrak{\ddot{o}ss}_{\mathrm{ev}}, \mathfrak{duu\ddot{o}ss}_{\mathrm{ev}}) && (155)\\ \mathfrak{\ddot{o}ss}^{\bullet}_{\mathrm{evv}} &= \mathrm{muu}(\mathfrak{\ddot{o}ss}_{\mathrm{evv}}, \mathfrak{duu\ddot{o}ss}_{\mathrm{evv}}) && (156)\end{aligned}$$

$$\begin{aligned} \mathfrak{duu\ddot{o}ss}^{\bullet}_{\mathrm{ev}} &:= +\sum_{r\geq 1} \alpha_{2r}\, \mathfrak{lo}^{\bullet}_{2r} \quad \in \quad alternal && (157)\\ \mathfrak{duu\ddot{o}ss}^{\bullet}_{\mathrm{evv}} &:= +\sum_{r\geq 1} \beta_{2r}\, \mathfrak{lo}^{\bullet}_{2r} \quad \in \quad alternal && (158)\end{aligned}$$

with the bilinear product muu *defined as in (25) and the same elementary alternals $\mathfrak{lo}^{\bullet}_{r}$ as above. The coefficients α_{2r} are also the same as in (121) except for the omission of α_1, but (158) involves new coefficients β_{2r} given by*

$$\sum_{r\geq 1} \beta_{2r}\, t^{2r} := \frac{t}{e^{t/2} - e^{-t/2}} - 1 = -\frac{1}{24}\, t^2 + \frac{7}{5760}\, t^4 - \frac{31}{967680}\, t^6 + \dots \quad (159)$$

Under the polar specialisation $\mathfrak{O} \mapsto Pa$ the above relations assume a simpler form, with muu *replaced by the familiar product* mu *:*

$$\mathrm{dur}.\mathrm{pal}^\bullet_{\mathrm{ev}} = \mathrm{mu}.(\mathrm{pal}^\bullet_{\mathrm{ev}}, \mathrm{dupal}^\bullet_{\mathrm{ev}}) \tag{160}$$
$$\mathrm{dur}.\mathrm{pal}^\bullet_{\mathrm{evv}} = \mathrm{mu}.(\mathrm{pal}^\bullet_{\mathrm{evv}}, \mathrm{dupal}^\bullet_{\mathrm{evv}}) \tag{161}$$

and with

$$\mathrm{dupal}^\bullet_{\mathrm{ev}} := \sum_{r_*\geq 1} \alpha_{2r}\ \mathrm{lan}^\bullet_{2r_*} \quad ; \quad \mathrm{dupal}^\bullet_{\mathrm{evv}} := \sum_{r_*\geq 1} \beta_{2r}\ \mathrm{lan}^\bullet_{2r_*} \tag{162}$$

relatively to the same elementary alternals $\mathrm{lan}^\bullet_r$ *as in (124).*

This concludes our list of 'main statements' about the bisymmetrals. For easy reference, we now tabulate the main source functions behind their separators and dilators.

Table 1: ***gari*****-dilators and their coefficients:**

In all the instances encountered in this section (six in all), we list the identity-tangent diffeomorphisms f with their images in $GARI_{\mathfrak{ve}}$ or $GARI_{\mathfrak{vo}}$ for the unit choice $\mathfrak{E}$ or $\mathfrak{O}$ and the corresponding polar specialisations:

$$\{f := x \mapsto x + x \sum a_n\, x^n\} \mapsto \{\mathfrak{fe}^\bullet, \mathfrak{fo}^\bullet\} \ \ \textit{and} \ \ \{\mathrm{fi}^\bullet, \mathrm{fa}^\bullet\} \tag{163}$$

along with the four relevant generating functions:

- $f_0(x) := x^{-1} f_\#(x) = 1 - \frac{f(x)}{x\, f'(x)}$: carries the coefficients of the *gari*-dilators.
- $f_1(x) := f'(x)$: carries the coefficients of the first separator *gepar.*
- $f_2(x) := \frac{1}{2}\, x\, \frac{f''(x)}{f'(x)}$: carries the coefficients of the second separator *hepar.*
- $f_3(x) := \frac{f'''(x)}{f'(x)} - \frac{3}{2}\big(\frac{f''(x)}{f'(x)}\big)^2$ = Schwarzian of f: ought to carry the coefficients of a conjectural third separator (still unknown).

Instance 1 : $\{f(x) = 1 - e^{-x}\} \ \mapsto\ \{\mathfrak{ess}^\bullet, \mathfrak{oss}^\bullet\}$ *and* $\{\mathrm{pil}^\bullet, \mathrm{pal}^\bullet\}$

$$f_0(x) \;=\; \frac{1+x-\exp(x)}{x} = \sum_{1\leq r} \frac{-1}{(r+1)!}\,x^r \tag{164}$$

$$f_1(x) \;=\; \exp(-x) = 1 + \sum_{1\leq r} \frac{(-1)^r}{r!}\,x^r \tag{165}$$

$$f_2(x) \;=\; -\frac{1}{2}\,x \tag{166}$$

$$f_3(x) \;=\; -\frac{1}{2} \tag{167}$$

Instance 2 : $\{f(x) = \frac{x}{1+\frac{1}{2}\,x}\} \;\mapsto\; \{\mathfrak{ess}^{\bullet}_{\mathrm{od}}, \mathfrak{oss}^{\bullet}_{\mathrm{od}}\}$ *and* $\{\mathrm{pil}^{\bullet}_{\mathrm{od}}, \mathrm{pal}^{\bullet}_{\mathrm{od}}\}$

$$f_0(x) = \; -\frac{1}{2}\,x \tag{168}$$

$$f_1(x) = \; = \frac{1}{(1+\frac{1}{2}\,x)^2} \tag{169}$$

$$f_2(x) = \; = -\frac{x}{2}\,\frac{1}{(1+\frac{1}{2}\,x)} \tag{170}$$

$$f_3(x) = \; = 0 \tag{171}$$

Instance 3 : $\{f(x) = 2\tanh(\frac{x}{2})\} \;\mapsto\; \{\mathfrak{ess}^{\bullet}_{\mathrm{ev}}, \mathfrak{oss}^{\bullet}_{\mathrm{ev}}\}$ *and* $\{\mathrm{pil}^{\bullet}_{\mathrm{ev}}, \mathrm{pal}^{\bullet}_{\mathrm{ev}}\}$

$$f_0(x) \;=\; 1 - \frac{\sinh(x)}{x} = \sum_{1\leq r_*} \frac{-1}{(2r_*+1)!}\,x^{2r_*} \tag{172}$$

$$f_1(x) \;=\; \left(\cosh(\frac{x}{2})\right)^{-2} = 1 - \frac{1}{4}\,x^2 + \frac{1}{24}\,x^4 - \frac{17}{2880}\,x^6 + \frac{31}{40320}\,x^8 + \ldots \tag{173}$$

$$f_2(x) \;=\; -\frac{x}{2}\,\tanh(\frac{x}{2}) = -\frac{1}{4}\,x^2 + \frac{1}{48}\,x^4 - \frac{1}{480}\,x^6 + \frac{17}{80640}\,x^8 + \ldots \tag{174}$$

$$f_3(x) \;=\; -\frac{1}{2} \tag{175}$$

Instance 4 : $\{f(x) = \log(\frac{1}{1-x})\} \;\mapsto\; \{\mathfrak{riess}^{\bullet}, \mathfrak{rioss}^{\bullet}\}$ *and* $\{\mathrm{ripil}^{\bullet}, \mathrm{ripal}^{\bullet}\}$

$$
\begin{aligned}
f_0(x) &= 1+\frac{(1-x)}{x}\,\log(1-x) = \sum_{1\leq r}\frac{1}{r\,(r+1)}\,x^r && (176)\\
f_1(x) &= \frac{1}{(1-x)} && (177)\\
f_2(x) &= \frac{x}{2}\,\frac{1}{(1-x)} && (178)\\
f_3(x) &= \frac{1}{2}\,\frac{1}{(1-x)^2} && (179)
\end{aligned}
$$

$$
\textbf{Instance 5}:\ \{f(x)=\frac{1}{1-\frac{1}{2}x}\}\ \mapsto\ \{\mathfrak{riess}^{\bullet}_{\mathrm{od}},\mathfrak{rioss}^{\bullet}_{\mathrm{od}}\}\ \ \textit{and}\ \ \{\mathrm{ripil}^{\bullet}_{\mathrm{od}},\mathrm{ripal}^{\bullet}_{\mathrm{od}}\}
$$

$$
\begin{aligned}
f_0(x) &= \frac{1}{2}\,x && (180)\\
f_1(x) &= \frac{1}{(1-\frac{1}{2}\,x)^2} && (181)\\
f_2(x) &= \frac{x}{2}\,\frac{1}{(1-\frac{1}{2}\,x)} && (182)\\
f_3(x) &= 0 && (183)
\end{aligned}
$$

$$
\textbf{Instance 6}:\ \{f(x)=2\,\mathrm{arctanh}(\frac{x}{2})\}\mapsto\{\mathfrak{riess}^{\bullet}_{\mathrm{ev}},\mathfrak{rioss}^{\bullet}_{\mathrm{ev}}\}\ \textit{and}\ \{\mathrm{ripil}^{\bullet}_{\mathrm{ev}},\mathrm{ripal}^{\bullet}_{\mathrm{ev}}\}
$$

$$
\begin{aligned}
f_0(x) &= 1+(\frac{1}{x}-\frac{x}{4})\log\big(\frac{1-\frac{1}{2}\,x}{1+\frac{1}{2}\,x}\big) = x\sum_{1\leq r_*}\frac{2^{1-2\,r_*}}{(2r_*-1)(2r_*+1)}\,x^{2r*} && (184)\\
f_1(x) &= \frac{1}{1-\frac{1}{4}\,x^2} && (185)\\
f_2(x) &= \frac{x^2}{4}\,\frac{1}{(1-\frac{1}{4}\,x^2)} && (186)\\
f_3(x) &= \frac{1}{2}\,\frac{1}{(1-\frac{1}{4}\,x^2)^2} && (187)
\end{aligned}
$$

Table 2: ***mu*-dilators and their coefficients:**

The swappees $\{\ddot{\mathfrak{oss}}^\bullet, \ddot{\mathfrak{ess}}^\bullet, pal^\bullet, pir^\bullet\}$ possess simple *mu*-dilators whose coefficients admit the following generating function:

$$\frac{t}{e^t-1}-1=-\frac{1}{2}\,t+\frac{1}{12}\,t^2-\frac{1}{720}\,t^4+\frac{1}{30240}\,t^6-\frac{1}{120960}\,t^8+\ldots \tag{188}$$

The even *gari*-factors $\{\ddot{\mathfrak{oss}}^\bullet_{\mathrm{ev}}, \ddot{\mathfrak{ess}}^\bullet_{\mathrm{ev}}, pal^\bullet_{\mathrm{ev}}, pir^\bullet_{\mathrm{ev}}\}$ of these swappees possess simple *mu*-dilators whose coefficients admit the same generating function, minus the first exceptional odd term:

$$\frac{t}{e^t-1}-1+\frac{1}{2}\,t=\frac{1}{12}\,t^2-\frac{1}{720}\,t^4+\frac{1}{30240}\,t^6-\frac{1}{120960}\,t^8+\ldots \tag{189}$$

Their even *mu*-factors $\{\ddot{\mathfrak{oss}}^\bullet_{\mathrm{evv}}, \ddot{\mathfrak{ess}}^\bullet_{\mathrm{evv}}, pal^\bullet_{\mathrm{evv}}, pir^\bullet_{\mathrm{evv}}\}$ also possess simple *mu*-dilators but with coefficients admitting a rather distinct generating function:

$$\frac{t}{e^{t/2}-e^{-t/2}}-1=-\frac{1}{24}\,t^2+\frac{7}{5760}\,t^4-\frac{31}{967680}\,t^6+\frac{127}{15482880}\,t^8+\ldots \tag{190}$$

4 Polar bisymmetrals: proofs.

We shall work mostly with the natural polar specialition $(\mathfrak{E},\mathfrak{O})\mapsto(Pi,Pa)$.

§4-1. Separators of $pil^\bullet$ and $ripil^\bullet$.

All separator identities in §3 result from the general statement:

If $\mathrm{fi}^\bullet$ *is the image in the group* $\mathrm{GARI}_{\mathfrak{ce}}$ *of the identity-tangent mapping* $f: x\mapsto x+\sum_{1\le r}a_r\,x^{r+1}$, *then its two separators are of the form*

$$\begin{aligned}\mathrm{gepar.fi}^{w_1,\ldots,w_r} &= a^*_r\,\mathrm{Pa}^{w_1}\ldots\mathrm{Pa}^{w_r}\quad \textit{with}\quad a^*_r=(r+1)\,a_r &(191)\\ \mathrm{hepar.fi}^{w_1,\ldots,w_r} &= a^{**}_r\,\mathrm{Pa}^{w_1}\ldots\mathrm{Pa}^{w_r}\quad \textit{with}\quad \sum_{1\le r}a^{**}_r x^r:=\frac{x}{2}\,\frac{f''(x)}{f'(x)} &(192)\end{aligned}$$

To prove (191) we note that the bimould $fi^\bullet$, being the image of f, has a *gari*-dilator of the form:

$$\mathrm{der.fi}^\bullet = \mathrm{preari}(\mathrm{fi}^\bullet,\mathrm{difi}^\bullet)\qquad \textit{with}\qquad \mathrm{difi}^\bullet=\sum_{1\le r}\alpha_r\,\mathrm{ri}^\bullet_r \tag{193}$$

so that its swappee $fa^\bullet$ has a *gira*-dilator of the form:

$$\mathrm{der.fa}^\bullet = \mathrm{preira}(\mathrm{fa}^\bullet,\mathrm{dafa}^\bullet)\qquad \textit{with}\qquad \mathrm{dafa}^\bullet=\sum_{1\le r}\alpha_r\,\mathrm{sra}^\bullet_r \tag{194}$$

with $sra^\bullet_r := swap.ri^\bullet_r$ and with identical coefficients α_r given by

$$1-\frac{f(x)}{x\,f'(x)}=\sum_{1\leq r}\alpha_r\,x^r \tag{195}$$

Due to the very special form of $sra^\bullet_r$ and $anti.sra^\bullet_r$:

$$\mathrm{anti.sra}^{\mathrm{w}_1,\dots,\mathrm{w}_r}=P(u_1+\dots u_r)\sum_{1\leq i\leq r} i\prod_{j\neq i}P(u_j) \tag{196}$$

the pre-bracket *preira* in (194) may be replaced by *preiwa*, which becomes:

$$\mathrm{der.fa}^\bullet \;=\; \mathrm{preiwa}(\mathrm{fa}^\bullet,\mathrm{dafa}^\bullet) \;=\; \mathrm{iwat}(\mathrm{dafa}^\bullet).\mathrm{fa}^\bullet+\mathrm{mu}(\mathrm{fa}^\bullet,\mathrm{dafa}^\bullet) \tag{197}$$

Setting $gefa^\bullet := mu(anti.fa^\bullet, fa^\bullet)$ and applying the *mu*-derivation *der* to both sides, we find, in view of (197) and $anti.iwat(sra^\bullet) = iwat(sra^\bullet).anti$:

$$\mathrm{der.gefa}^\bullet \;=\; \mathrm{iwat}(\mathrm{dafa}^\bullet).\mathrm{gefa}^\bullet+\mathrm{mu}(\mathrm{gefa}^\bullet,\mathrm{dafa}^\bullet)+\mathrm{mu}(\mathrm{anti.dafa}^\bullet,\mathrm{gefa}^\bullet) \tag{198}$$

Using the elementary identities

$$\mathrm{sra}^\bullet_\mathrm{r}+\mathrm{anti.sra}^\bullet_\mathrm{r} \;=\; (r{+}1).\,\mathrm{mu}_\mathrm{r}(\mathrm{Pa}^\bullet) \tag{199}$$

and

$$\begin{aligned}\mathrm{irat}(\mathrm{sra}^\bullet_\mathrm{p}).\mathrm{mu}_\mathrm{q}(\mathrm{Pa}^\bullet) \;&=\; \mathrm{iwat}(\mathrm{sra}^\bullet_\mathrm{p}).\mathrm{mu}_\mathrm{q}(\mathrm{Pa}^\bullet)\\ &=\; -(p{-}q{+}1)\,.\,\mathrm{mu}_{\mathrm{p+q}}(\mathrm{Pa}^\bullet)\\ &\quad\; +\mathrm{mu}(\mathrm{sra}^\bullet_\mathrm{p},\mathrm{mu}_\mathrm{q}(\mathrm{P}^\bullet))\\ &\quad\; +\mathrm{mu}(\mathrm{mu}_\mathrm{q}(\mathrm{P}^\bullet),\mathrm{anti.sra}^\bullet_\mathrm{p})\end{aligned} \tag{200}$$

it is but a short step fom (198) to (191).

The proof for *hepar* runs along similar lines but is more intricate. Since we do not really require the result in the sequel, let us just mention the key step in the argument. Let $\underline{\boldsymbol{r}}=\{r_1,\dots,r_s\}$ denote any non-ordered sequence of s positive integers, and let $fa^\bullet_{\underline{\boldsymbol{r}}}$ resp. $lofa^\bullet_{\underline{\boldsymbol{r}}}$ denote the part of $fa^\bullet$ resp. $lofa^\bullet$ that is multilinear in $sra^\bullet_{r_1}, \dots, sra^\bullet_{r_s}$. Applying the rules of §1-9 we find:

$$\mathrm{fa}^\bullet_{\underline{\boldsymbol{r}}} \;=\; a_{r_1}\dots a_{r_s}\sum_{\sigma\in\mathfrak{S}(s)}\mathrm{Paj}^{r_{\sigma(1)},\dots,r_{\sigma(s)}}\;\overrightarrow{\mathrm{preira}}\,(\mathrm{sra}^\bullet_{r_{\sigma(1)}},\dots,\mathrm{sra}^\bullet_{r_{\sigma(s)}}) \tag{201}$$

$$\mathrm{lofa}^\bullet_{\underline{\boldsymbol{r}}} \;=\; \sum_{1\leq m\leq s}\frac{(-1)^{m-1}}{m}\sum_{\underline{\mathbf{r}}^1\dots\underline{\mathbf{r}}^m=\underline{\mathbf{r}}}\mathrm{mu}(\mathrm{fa}^\bullet_{\underline{\mathbf{r}}^1},\dots,\mathrm{fa}^\bullet_{\underline{\mathbf{r}}^m}) \tag{202}$$

Next, consider

$$\mathrm{rofa}^\bullet_{\underline{\mathbf{r}}} \;=\; a_{r_1}...a_{r_s}\sum_{\sigma\in\mathfrak{S}(s)}\mathrm{Paj}^{\,r_{\sigma(1)},\dots,r_{\sigma(s)}}\;\mathrm{irat}(\mathrm{sra}^\bullet_{r_{\sigma(r)}})\dots\mathrm{irat}(\mathrm{sra}^\bullet_{r_{\sigma(2)}}).\mathrm{sra}^\bullet_{r_{\sigma(1)}} \quad (203)$$

Although $rofa^\bullet_{\underline{\mathbf{r}}}$ has a much simpler (less composite) definition than $lofa^\bullet_{\underline{\mathbf{r}}}$ and actually differs from it as soon as $r \geq 2$, one can nonetheless show that after *pus*-averaging the two expressions do coincide:

$$\sum_{1\leq k\leq|\boldsymbol{r}|}\mathrm{pus}^k.\mathrm{lofa}^\bullet_{\underline{\mathbf{r}}} \equiv \sum_{1\leq k\leq|\boldsymbol{r}|}\mathrm{pus}^k.\mathrm{rofa}^\bullet_{\underline{\mathbf{r}}} \quad (204)$$

§4-2. Shape of the *gari*-dilators of $pil^\bullet$ and $ripil^\bullet$.

This is a standard application of the correspondance $f \mapsto f_{\#}$. See the Table 1 at the end of the preceding section, where $f_0(x) \equiv f_{\#}(x)/x$. See also §4 in [$E3$], from (4.11) through (4.17).

§4-3. Bisymmetrality of $pal^\bullet/pil^\bullet$: first proof.

This proof strives to be even-handed, in the spirit of dimorphy: it treats $pal^\bullet$ and $pil^\bullet$ in exactly the same way, by relating each to its dilator. So, rather than defining $pil^\bullet$ from its source mapping f as in Proposition 3.1, we adopt the following, strictly equivalent definition, polar-transposed from Proposition 3.6 and based on the *gari*-dilator $dipil^\bullet$:

$$\begin{aligned} \mathrm{der.pil}^\bullet &= \mathrm{preari}(\mathrm{pil}^\bullet, \mathrm{dipil}^\bullet) \\ \textit{with} \qquad \mathrm{dipil}^\bullet &:= -\sum_{1\leq r}\frac{1}{(r+1)!}\;\mathrm{ri}^\bullet_r \end{aligned} \quad (205)$$

The alternals $ri^\bullet_r$ are of course the specialisation of $\mathfrak{re}^\bullet_r$ under $\mathfrak{E} \mapsto Pi$.

We then consider a bimould $pal^\bullet$ defined, *not as the swappee* of $pil^\bullet$, but directly and independently, via the *mu*-dilator $dupal^\bullet$:

$$\begin{aligned} \mathrm{dur.pal}^\bullet &= \mathrm{mu}(\mathrm{pal}^\bullet, \mathrm{dupal}^\bullet) \\ \textit{with} \qquad \mathrm{dupal}^\bullet &:= \sum_{1\leq r}\alpha_r\;\mathrm{lan}^\bullet_r \qquad \big(\alpha_r \;\; \textit{as in } (121)\big) \end{aligned} \quad (206)$$

with the same Bernoulli coefficients α_r as in Proposition 3.8 and with $lan^\bullet_r$

being the specialisation of $\mathfrak{len}^\bullet_r$ under $\mathfrak{E} \mapsto Pa$. See §2. Quite explicitely:

$$\begin{aligned}
\mathrm{lan}^\bullet_r &= \sum_{1\le i\le r} (-1)^{i-1} \frac{(r-1)!}{(i-1)!(r-i)!}\ \mathrm{mu}\big(\mathrm{mu_{i-1}}(\mathrm{Pa}^\bullet), \mathrm{I}^\bullet, \mathrm{mu_{r-i}}(\mathrm{Pa}^\bullet)\big) \\
&= \vec{\mathrm{lu}}(\mathrm{I}^\bullet, \overbrace{\mathrm{Pa}^\bullet, ..., \mathrm{Pa}^\bullet}^{(r-1)\, times}) \qquad (207)
\end{aligned}$$

Both dilators *dipil*$^\bullet$ and *dupal*$^\bullet$ being alternal, it immediately follows that *pil*$^\bullet$ and *pal*$^\bullet$ are symmetral: this is obvious from the inversion formulae (36) and (39) and from the symmetrality of the mould $Paj^\bullet$ common to both.

So everything now reduces to showing that *pal*$^\bullet$ is actually the swappee of *pil*$^\bullet$ or, what amounts to the same, that the system (206) that defines *pal*$^\bullet$ is equivalent to the system

$$\begin{aligned}
\mathrm{der.pal}^\bullet &= \mathrm{preira}(\mathrm{pal}^\bullet, \mathrm{dapal}^\bullet) \\
&= \mathrm{irat}(\mathrm{dapal}^\bullet).\mathrm{pal}^\bullet + \mathrm{mu}(\mathrm{pal}^\bullet, \mathrm{dapal}^\bullet) \qquad (208) \\
\textit{with} \qquad \mathrm{dapal}^\bullet &:= -\sum_{1\le r} \frac{1}{(r+1)!}\ \mathrm{sra}^\bullet_\mathrm{r} \qquad \big(\mathrm{sra}^\bullet_\mathrm{r} := \mathrm{swap.ri}^\bullet_\mathrm{r}\big)
\end{aligned}$$

deduced under the *swap* transform from the system (205) that defines *pil*$^\bullet$.

Before taking that one last step, let us recall the universal relation (27) between the *gira*-dilator *daS*$^\bullet$ and the *mu*-dilator *duS*$^\bullet$ of a given $S^\bullet$:

$$\mathrm{der.duS}^\bullet - \mathrm{dur.daS}^\bullet + \mathrm{lu}(\mathrm{daS}^\bullet, \mathrm{duS}^\bullet) - \mathrm{irat}(\mathrm{daS}^\bullet).\mathrm{duS}^\bullet = 0$$

Specialising the triplet $\{S^\bullet, daS^\bullet, duS^\bullet\}$ to the triplet $\{pal^\bullet, dapal^\bullet, dupal^\bullet\}$, we get:

$$\mathrm{der.dupal}^\bullet - \mathrm{dur.dapal}^\bullet + \mathrm{lu}(\mathrm{dapal}^\bullet, \mathrm{dupal}^\bullet) - \mathrm{irat}(\mathrm{dapal}^\bullet).\mathrm{dupal}^\bullet = 0 \quad (209)$$

which, as observed in the universal case (cf §1), determines *dapal*$^\bullet$ in terms of *dupal*$^\bullet$ and *vice versa*.

Now, this appealingly symmetrical and winningly simple relation (209) involves only elementary monomials $Pa(.)$ and readily follows from the basic identities (199), (200) and (207).

This establishes beyond cavil that the symmetral bimould *pil*$^\bullet$ as defined by (205) and the equally symmetral bimould *pal*$^\bullet$ as defined by (206) are *mutual swappees.*

Remark: This last identity (209) is totally *rigid* in the sense that if we tinker with the common coefficients $-1/(r+1)!$ of *dipil*$^\bullet$ and *dapal*$^\bullet$, there is no way we can adjust the coefficients α_r of $dupal^\bullet$ to salvage (209). This rigidity will stand us in good stead in [E4] for unravelling the structure of the trigonometric bisymmetrals $tal^\bullet/til^\bullet$. For a foretaste, see §17 *infra.*

§4-4. Bisymmetrality of $pal^\bullet/pil^\bullet$: second proof.

This alternative proof is more roundabout[29] but makes up for it by yielding valuable extra information. We now starts from $pil^\bullet$ and its *gari*-inverse $ripil^\bullet$, which are automatically symmetral by construction. The challenge is to show that $pal^\bullet$ (now defined derivatively, as the swappee of $pil^\bullet$) is also symmetral or, what amounts to the same but turns out to be easier, that its *gari*-inverse $ripal^\bullet$ is symmetral. The key here is to compare $ripal^\bullet$ with the swappee $rapal^\bullet$ of $ripil^\bullet$, which may be also be viewed as the *gira*-inverse of $pal^\bullet$ (hence the prefix *"ra"*). According to (10) $ripal^\bullet$ is also the *ras*-transform of $rapal^\bullet$:

$$\mathrm{ripal}^\bullet = \mathrm{ras.rapal}^\bullet := \mathrm{invgari.swap.invgari.swap.rapal}^\bullet \tag{210}$$

The following picture sums up the situation:

$$\begin{array}{ccccc} & \mathrm{pal}^\bullet & \overset{swap}{\longleftrightarrow} & \mathrm{pil}^\bullet & \\ invgari & \updownarrow & & \updownarrow & invgari \\ & \mathrm{ripal}^\bullet & & \mathrm{ripil}^\bullet & \\ ras & \uparrow & \swarrow_{swap}\nearrow & & \\ & \mathrm{rapal}^\bullet & & & \end{array}$$

In view of (9) we also have:

$$\begin{aligned} \mathrm{rash.rapal}^\bullet &= \mathrm{mu}(\mathrm{corapal}^\bullet, \mathrm{rapal}^\bullet) \qquad \textit{with} && (211)\\ \mathrm{corapal}^\bullet &= \mathrm{push.swap.invmu.swap.rapal}^\bullet && (212) \end{aligned}$$

Replacing *push* by its definition (439) in (212) and using the fact that $ripil^\bullet$, being symmetral, is *mu*-invertible under *pari.anti*, we get successively:

$$\begin{aligned} \mathrm{corapal}^\bullet &= \mathrm{neg.anti.swap.anti.swap.swap.invmu.swap.rapal}^\bullet && (213)\\ &= \mathrm{neg.anti.swap.anti.invmu.ripil}^\bullet && (214)\\ &= \mathrm{neg.anti.swap.anti.anti.pari.ripil}^\bullet && (215)\\ &= \mathrm{neg.anti.swap.pari.ripil}^\bullet && (216)\\ &= \mathrm{anti.swap.neg.pari.ripil}^\bullet && (217)\\ &= \mathrm{anti.swap.ripil}^\bullet && (218)\\ &= \mathrm{anti.rapal}^\bullet && (219) \end{aligned}$$

[29] Before starting, the reader may have a look at the overall logical scheme as pictured at the end of the paragraph §4-4.

So we end up with

$$\begin{aligned}
\mathrm{corapal}^\bullet &= \mathrm{mu}(\mathrm{anti.rapal}^\bullet, \mathrm{rapal}^\bullet) && (220)\\
&= \mathrm{gepar}(\mathrm{ripil}^\bullet) && (221)\\
&= \mathrm{pac}^\bullet \qquad (\textit{due to } (111)) && (222)
\end{aligned}$$

with an elementary *pac*$^\bullet$ that admits an equally elementary *gani*-inverse *nipac*$^\bullet$:

$$\begin{aligned}
\mathrm{pac}^{w_1,\ldots,w_r} &= \prod_{1\leq i\leq r} P(u_i) && (223)\\
\mathrm{nipac}^{w_1,\ldots,w_r} &= (-1)^r \prod_{1\leq i\leq r} P(u_i{+}\ldots{+}u_r) && (224)\\
\mathrm{gani}(\mathrm{pac}^\bullet, \mathrm{nipac}^\bullet) &= 1^\bullet && (225)
\end{aligned}$$

Thus, in view of (8), we go from *ripal*$^\bullet$ to *rapal*$^\bullet$ and back via the relations

$$\mathrm{ganit}(\mathrm{pac}^\bullet).\mathrm{ripal}^\bullet = \mathrm{rapal}^\bullet \qquad (226)$$
$$\mathrm{ganit}(\mathrm{nipac}^\bullet).\mathrm{rapal}^\bullet = \mathrm{ripal}^\bullet \qquad (227)$$

Now, it is an easy matter to ckeck[30] that

$$\begin{aligned}
\mathrm{ganit}(\mathrm{pac}^\bullet) &: \mathit{altern}\boldsymbol{al}//\mathit{symmetr}\boldsymbol{al} \longrightarrow \mathit{altern}\boldsymbol{ul}//\mathit{symmetr}\boldsymbol{ul} && (228)\\
\mathrm{ganit}(\mathrm{nipac}^\bullet) &: \mathit{altern}\boldsymbol{ul}//\mathit{symmetr}\boldsymbol{ul} \longrightarrow \mathit{altern}\boldsymbol{al}//\mathit{symmetr}\boldsymbol{al} && (229)
\end{aligned}$$

Let us now write down the dilator identity for *ripil*$^\bullet$ (see (151)-(153)) and the logically equivalent identity for the swappee *rapal*$^\bullet$:

$$\mathrm{der.ripil}^\bullet = \mathrm{preari}(\mathrm{ripil}^\bullet, \mathrm{diripil}^\bullet) \quad \textit{with} \;\; \mathrm{diripil}^\bullet = \sum_{1\leq r} \frac{1}{r.(r{+}1)} \mathrm{ri}_r^\bullet \qquad (230)$$

$$\mathrm{der.rapal}^\bullet = \mathrm{preira}(\mathrm{rapal}^\bullet, \mathrm{darapal}^\bullet) \quad \textit{with} \;\; \mathrm{darapal}^\bullet = \sum_{1\leq r} \frac{1}{r.(r{+}1)} \mathrm{sra}_r^\bullet \qquad (231)$$

As usual, $sra_r^\bullet := swap.ri_r^\bullet$. More explicitely:

$$\mathrm{sra}_r^{w_1,\ldots,w_r} = \frac{\sum (r{+}1{-}i)\, u_i}{u_1 \ldots u_r (u_1{+}...u_r)} \qquad (232)$$

[30]especially in the form (228). For details about the 'twisted symmetries' *alternil/symmetril* and *alternul/symmetrul* , see [E3], §3.5.

From that we infer the shuffle identity:

$$\sum_{\boldsymbol{w}\,\in\,\mathrm{sha}(\boldsymbol{w^1},\boldsymbol{w^2})} \mathrm{esra}_r^{\boldsymbol{w}} \;\equiv\; \mathrm{esra}_{\mathrm{r}_1}^{\boldsymbol{w^1}}\,\mathrm{expa}_{\mathrm{r}_2}^{\boldsymbol{w^2}} + \mathrm{expa}_{\mathrm{r}_1}^{\boldsymbol{w^1}}\,\mathrm{esra}_{\mathrm{r}_2}^{\boldsymbol{w^2}} \qquad \textit{with} \tag{233}$$

$$\mathrm{esra}_r^{\bullet} \;:=\; \frac{1}{(r+1)!}\,\mathrm{dur.sra}_{\mathrm{r}}^{\bullet} \tag{234}$$

$$\mathrm{expa}_r^{\bullet} \;:=\; \mathrm{expmu}(\mathrm{Pa}^{\bullet}) \tag{235}$$

which in turn easily implies that the dilator *darapal*$^\bullet$, as given by (239), is altern*ul*.[31] Now, if from "*darapal*$^\bullet$ $\in$ altern*ul*" we could directly deduce "*rapal*$^\bullet$ $\in$ symmetr*ul*", life would be easy: we could, applying (227) and (229), immediately conclude that *ripal*$^\bullet$ and therefore *pal*$^\bullet$ are symmetral, and be done with it. Unfortunately, we cannot [32] – at least not directly – and must take the detour through the dilators *darapal*$^\bullet$ and *diripal*$^\bullet$.

So our goal now is to go from the proven identity (231) to an identity of the form:

$$\begin{aligned} \mathrm{der.ripal}^{\bullet} &= \mathrm{preari}(\mathrm{ripal}^{\bullet},\mathrm{diripal}^{\bullet}) \qquad \textit{with} \\ \mathrm{diripal}^{\bullet} &:= \mathrm{ganit}(\mathrm{nipac}^{\bullet}).\mathrm{darapal}^{\bullet} \end{aligned} \tag{236}$$

and from there to the identity:

$$\mathrm{der.ripal}^{\bullet} = \mathrm{preari}(\mathrm{ripal}^{\bullet},\mathrm{diripal}^{\bullet}) \quad \textit{with} \;\; \mathrm{diripal}^{\bullet} = \sum_{1\le r}\frac{1}{r.(r+1)}\mathrm{ha}_r^{\bullet} \tag{237}$$

To deal with the first step, let us parse the identities (231) and (236) respectively as $A_1 + A_2 = 0$ and $B_1 + B_2 = 0$ with

$$A_1 := \big(-\mathrm{der} + \mathrm{irat}(\mathrm{darapal}^{\bullet})\big).\mathrm{rapal}^{\bullet} \qquad A_2 := \mathrm{mu}(\mathrm{rapal}^{\bullet},\mathrm{darapal}^{\bullet}) \tag{238}$$

$$B_1 := \big(-\mathrm{der} + \mathrm{arit}(\mathrm{diripal}^{\bullet})\big).\mathrm{ripal}^{\bullet} \qquad B_2 := \mathrm{mu}(\mathrm{ripal}^{\bullet},\mathrm{diripal}^{\bullet}) \tag{239}$$

and then check that:

$$\mathrm{ganit}(\mathrm{nipac}^{\bullet}).A_1 \;=\; B_1 \tag{240}$$

$$\mathrm{ganit}(\mathrm{nipac}^{\bullet}).A_2 \;=\; B_2 \tag{241}$$

[31] This fact is already mentioned in [$E3$], in "universal mode": see (4.6) p 73.

[32] To do that *directly*, we would require the altern*ul*ity of the *gari*-dilator *dirapal*$^\bullet$ of *rapal*$^\bullet$ (not considered here) rather than the altern*ul*ity of its *gira*-dilator *darapal*$^\bullet$ (considered!). Extreme caution is called for here; great care must be taken to distinguish between the various dilators: *diripil*$^\bullet$ (linked to *ripil*), *diripal*$^\bullet$ (linked to *ripal*), and the pair *darapal*$^\bullet$/*dirapal*$^\bullet$ (both linked to *rapal*$^\bullet$, but in different ways). Always pay close attention to the vowels and their placement: no agglutinative language with vocalic alternation could beat flexion theory for fiendish intricacy! But that's no fault of ours. That's just the way things are, and there in no point in carping.

The relation (241) is simply the definition of $\mathit{diripal}^\bullet$: see (236), second line. To prove the non-trivial part, namely

$$\mathrm{ganit}(\mathrm{nipac}^\bullet).A_1 = B_1 \tag{242}$$

we apply to $\mathit{rapal}^\bullet$ both terms of the operator identity

$$\begin{gathered}\mathrm{ganit}(\mathrm{nipac}^\bullet).\big[-\mathrm{der}+\mathrm{irat}(\mathrm{darapal}^\bullet)\big] \equiv \\ \big[-\mathrm{der}+\mathrm{arit}(\mathrm{ganit}(\mathrm{nipac}^\bullet).\mathrm{darapal}^\bullet)\big].\mathrm{ganit}(\mathrm{nipac}^\bullet)\end{gathered} \tag{243}$$

which is easier to check in this equivalent formulation:[33]

$$\begin{gathered}\big[-\mathrm{der}+\mathrm{irat}(\mathrm{darapal}^\bullet)\big].\mathrm{ganit}(\mathrm{pac}^\bullet) \equiv \\ \mathrm{ganit}(\mathrm{pac}^\bullet).\big[-\mathrm{der}+\mathrm{arit}(\mathrm{ganit}(\mathrm{nipac}^\bullet).\mathrm{darapal}^\bullet)\big]\end{gathered} \tag{244}$$

Thus, the *mu*-isomorphism $\mathit{ganit}(\mathit{nipac}^\bullet)$ takes us from (231) to (236), thereby establishing the latter identy, with a dilator $\mathit{diripal}^\bullet$ which, being the image under $\mathit{ganit}(\mathit{nipac}^\bullet)$ of the altern*u*l $\mathit{darapal}^\bullet$, is automatically alternal. This in turn immediately implies that $\mathit{ripal}^\bullet$ and $\mathit{pal}^\bullet$ are symmetral. In also implies, in view of (227), that $\mathit{rapal}^\bullet$ is symmetr*u*l — the very property, recall, that we could not directly derive from "$\mathit{darapal}^\bullet \in$ altern*u*l".

This completes our second, less direct proof of the bisymmetrality of $\mathit{pal}^\bullet/\mathit{pil}^\bullet$. What it doesn't do, though, is prove that our *definitely alternal* bimould $\mathit{diripil}^\bullet$ admits the exact expansion (237), with $\mathit{ha}_r^\bullet$ the polar specialisation of $\mathfrak{he}_r^\bullet$ under $\mathfrak{E} \mapsto Pa$. To rigorously establish this non-essential, but very nice extra bit of information unfortunately requires rather lengthy and tedious, though in a sense elementary calculations. One way to proceed is to start from the expansion (231) of $\mathit{darapal}^\bullet$; to apply $\mathit{ganit}(\mathit{nipac}^\bullet)$ to each $\mathit{sra}_r{}^\bullet$ separately, resulting in a bimould $\mathit{hasra}_r{}^\bullet$ with infinitely many non-vanishing components:

$$\mathrm{hasra}_r^\bullet := \sum_{r \le r_*} \mathrm{hasra}_{r,r_*}^\bullet \quad \textit{with} \quad \mathrm{hasra}_{r,r_*}^\bullet \in \mathrm{BIMU}_{r_*} \tag{245}$$

One may then expand each $\mathit{hasra}_{r,r_*}^\bullet$ in the standard basis of $\mathit{Flex}_{r_*}(Pa)$, where it admits a rather simple, highly lacunary projection; and eventually piece everything together inside the double sum

$$\sum_{1\le r\le r_*} \frac{1}{r.(r+1)}\, \mathrm{hasra}_{r,r_*}^\bullet \equiv \frac{1}{r_*(r_*{+}1)}\, \mathrm{ha}_{r_*}^\bullet \tag{246}$$

[33]These are 'rigid' identities, strictly dependent on the nature of the inputs: if we were to modify the definition of $\mathit{darapal}^\bullet$ by, say, modifying the coefficients of $\mathit{sra}_r^\bullet$ in (231), we would have to simultaneously modify the pair $\mathit{pac}^\bullet$, $\mathit{nipac}^\bullet$ of *gani*-inverse elements.

The combinatorially minded reader may fill in the dots.[34]

To conclude, let us sum up the various steps of the whole argument (– our second bisymmetrality proof –) with the number of stars alongside each arrow reflecting the trickiness of the corresponding implication:

$$\begin{array}{ccc}
\{\mathrm{pil}^\bullet \in \mathit{symmetr}\boldsymbol{al}\} & \Longrightarrow & \{\mathrm{ripil}^\bullet \in \mathit{symmetr}\boldsymbol{al}\} \\
 & & \Downarrow \\
\{\mathrm{darapal}^\bullet \in \mathit{altern}\boldsymbol{ul}\} & \stackrel{*}{\Longleftarrow} & \{\mathrm{diripil}^\bullet \in \mathit{altern}\boldsymbol{al}\} \\
\Downarrow ** & & \\
\{\mathrm{diripal}^\bullet \in \mathit{altern}\boldsymbol{al}\} & \stackrel{***}{\Longrightarrow} & \{\mathrm{diripal}^\bullet = \sum \frac{1}{r.(r+1)} \mathrm{ha}_r^\bullet\} \\
\Downarrow & & \\
\{\mathrm{ripal}^\bullet \in \mathit{symmetr}\boldsymbol{al}\} & \stackrel{*}{\Longrightarrow} & \{\mathrm{rapal}^\bullet \in \mathit{symmetr}\boldsymbol{ul}\} \\
\Downarrow & & \\
\{\mathrm{pal}^\bullet \in \mathit{symmetr}\boldsymbol{al}\} & &
\end{array}$$

§4-6. Even and odd factors of $pal^\bullet/pil^\bullet$.

We must first establish the three factorisations (129), (130), (131). Despite their air of kinship, they are in fact quite distinct, and must be dealt with separately. Under our preferred polar specialisation $(\mathfrak{E}, \mathfrak{O}) \mapsto (Pi, Pa)$ they become respectively:

$$\mathrm{pil}^\bullet = \mathrm{gari}(\mathrm{pil}^\bullet_{\mathrm{od}}, \mathrm{pil}^\bullet_{\mathrm{ev}}) \quad \mathit{with} \quad \mathrm{pil}^\bullet_{\mathrm{od}} = \mathrm{expari}(-\frac{1}{2}\,\mathrm{Pi}^\bullet) \tag{247}$$

$$\mathrm{pal}^\bullet = \mathrm{gari}(\mathrm{pal}^\bullet_{\mathrm{od}}, \mathrm{pal}^\bullet_{\mathrm{ev}}) \quad \mathit{with} \quad \mathrm{pal}^\bullet_{\mathrm{od}} = \mathrm{expari}(-\frac{1}{2}\,\mathrm{Pa}^\bullet) \tag{248}$$

$$\mathrm{pal}^\bullet = \mathrm{mu}(\mathrm{pal}^\bullet_{\mathrm{evv}}, \mathrm{pal}^\bullet_{\mathrm{odd}}) \quad \mathit{with} \quad \mathrm{pal}^\bullet_{\mathrm{odd}} = \mathrm{expmu}(-\frac{1}{2}\,\mathrm{Pa}^\bullet) \tag{249}$$

(i) The first factorisation (247) merely reflects the factorisation $f = f_{\mathrm{od}} \circ f_{\mathrm{ev}}$ of the source diffeomorphisms. Explicitly:

$$f(x) = 1 - e^{-x} \quad ; \quad f_{\mathrm{od}}(x) = \frac{x}{1 - \frac{1}{2}\,x} \quad ; \quad f_{\mathrm{ev}}(x) = 2\,\frac{e^{x/2} - e^{-x/2}}{e^{x/2} + e^{-x/2}} \tag{250}$$

Of course, as a *function*, $f_{\mathrm{ev}}(x)$ is odd and $f_{\mathrm{od}}(x)$ is neither odd nor even, but what matters in this context is that the quotient $f_{\mathrm{ev}}(x)/x$ should carry only

[34] There exist alternative strategies, like applying $ganit(nipac^\bullet)$ to $sra^\bullet_r$ as (indirectly) defined by (231) and summing, not in i and then r as above, but rather in r and then i, but all these approaches seem to lead to calculations of roughly the same complexity and tediousness.

even powers of x and that $f_{\rm od}(\bullet)$ should admit $-f_{\rm od}(-\bullet)$ as its reciprocal mapping.

(ii) The second factorisation (248) is less immediate to derive. We first observe that if we specialise $\mathfrak{E}$ to Pa rather than Pi, we get instead of (247) the following factorisation:

$$\mathrm{par}^\bullet = \mathrm{gari}(\mathrm{par}^\bullet_{\mathrm{od}}, \mathrm{par}^\bullet_{\mathrm{ev}}) \qquad \textit{with} \qquad \mathrm{par}^\bullet_{\mathrm{od}} = \mathrm{expari}(-\frac{1}{2}\,\mathrm{Pa}^\bullet) \tag{251}$$

Anticipating on the key result of §8 below about the canonical factorisation of bisymmetrals, we may note that the two *exceptional* (i.e. non-*neg*-invariant) bisymmetrals $pal^\bullet$ and $par^\bullet$ necessarily coincide up to *gari*-postcomposition by a *regular* (i.e. simultaneously *neg*- and *pari*-invariant) bisymmetral, which we may call $ral^\bullet$, and whose first three components $ral^\bullet_1$, $ral^\bullet_2$, $ral^\bullet_3$, as well as all later components of *odd* length, necessarily vanish. In other words:

$$\mathrm{pal}^\bullet = \mathrm{gari}(\mathrm{par}^\bullet, \mathrm{ral}^\bullet) = \mathrm{gari}(\mathrm{par}^\bullet_{\mathrm{od}}, \mathrm{par}^\bullet_{\mathrm{ev}}, \mathrm{ral}^\bullet) \tag{252}$$

But this is exactly the sought-after factorisation (248), with explicit factors:

$$\mathrm{pal}^\bullet_{\mathrm{od}} \;=\; \mathrm{par}^\bullet_{\mathrm{od}} \;=\; \mathrm{expari}(-\frac{1}{2}\,\mathrm{Pa}^\bullet) \tag{253}$$

$$\mathrm{pal}^\bullet_{\mathrm{ev}} \;=\; \mathrm{gari}(\mathrm{par}^\bullet_{\mathrm{ev}}, \mathrm{ral}^\bullet) \tag{254}$$

(iii) The third factorisation (249) is rather special in being a *mu*-factorisation incongruously arising out of a purely *gari-gira* context.[35] The quickest way to derive it is to assume the (already doubly established) bisymmetrality of $pal^\bullet/pil^\bullet$, then to define the would-be even factor $pal^\bullet_{\mathrm{evv}}$ via the equation (249) in terms of $pal^\bullet$ and $pal^\bullet_{\mathrm{odd}}$; and then to check its evenness. Injecting the factor $pal^\bullet_{\mathrm{evv}}$ so defined into the first separator identity:

$$\mathrm{gepar.pil}^\bullet = \mathrm{mu}(\mathrm{anti.pal}^\bullet, \mathrm{pal}^\bullet) = \mathrm{expmu}(-\mathrm{Pa}^\bullet) \tag{255}$$

we find at once:

$$\mathrm{mu}(\mathrm{anti.pal}^\bullet_{\mathrm{evv}}, \mathrm{pal}^\bullet_{\mathrm{evv}}) \tag{256}$$

and hence

$$\mathrm{invmu.pal}^\bullet_{\mathrm{evv}} = \mathrm{anti.pal}^\bullet_{\mathrm{evv}} \tag{257}$$

But we have defined $pal^\bullet_{\mathrm{evv}}$ as the *mu*-product of $pal^\bullet$, which we have shown to be symmetral, and of $expmu(\frac{1}{2}\,Pa^\bullet)$, also clearly symmetral. So $pal^\bullet_{\mathrm{evv}}$ is itself symmetral, and as such *mu*-invertible under *pari.anti*. Therefore:

$$\mathrm{invmu.pal}^\bullet_{\mathrm{evv}} = \mathrm{pari.anti.pal}^\bullet_{\mathrm{evv}} \tag{258}$$

[35] For a tentative mitigation of this 'incongruity', see §1-11 *supra*.

Comparing (257) and (258), we see that $pal^{\bullet}_{\mathrm{evv}}$ is *pari*-invariant, and so *neg*-invariant as well, and therefore truly *even*.

Properties of $pal^{\bullet}_{\mathrm{ev}}$ and $pal^{\bullet}_{\mathrm{evv}}$.

In our preferred polar specialisation, the identities (143), (144), (145) become

$$\begin{aligned}
\mathrm{der.pil}^{\bullet}_{\mathrm{ev}} &= \mathrm{preari}(\mathrm{pil}^{\bullet}_{\mathrm{ev}}, \mathrm{dipil}^{\bullet}_{\mathrm{ev}}) && (259)\\
\mathrm{der.pal}^{\bullet}_{\mathrm{ev}} &= \mathrm{preira}(\mathrm{pal}^{\bullet}_{\mathrm{ev}}, \mathrm{dapal}^{\bullet}_{\mathrm{ev}}) && (260)\\
\mathrm{der.pal}^{\bullet}_{\mathrm{evv}} &= \mathrm{preira}(\mathrm{pal}^{\bullet}_{\mathrm{evv}}, \mathrm{dapal}^{\bullet}_{\mathrm{ev}}) + \frac{1}{2}\,\mathrm{mu}(\mathrm{pal}^{\bullet}_{\mathrm{evv}}, \mathrm{codapal}^{\bullet}_{\mathrm{ev}}) && (261)
\end{aligned}$$

with the unavoidable *ev/evv* jumble in (261) and with dilators given by

$$\begin{aligned}
\mathrm{dipil}^{\bullet}_{\mathrm{ev}} &:= -\sum_{1\le r} \frac{1}{(2\,r+1)!}\,\mathrm{ri}^{\bullet}_{2r} && (262)\\
\mathrm{dapal}^{\bullet}_{\mathrm{ev}} &:= -\sum_{1\le r} \frac{1}{(2\,r+1)!}\,\mathrm{sra}^{\bullet}_{2r} \qquad (\mathrm{sra}^{\bullet}_{\mathrm{r}} := \mathrm{swap.ri}^{\bullet}_{\mathrm{r}}) && (263)\\
\mathrm{codapal}^{\bullet}_{\mathrm{ev}} &:= \frac{1}{2}\,\mathrm{expmu}(\mathrm{Pa}^{\bullet}) + \frac{1}{2}\,\mathrm{expmu}(-\mathrm{Pa}^{\bullet}) - 1^{\bullet} && (264)\\
&= -\mathrm{dapal}^{\bullet}_{\mathrm{ev}} - \mathrm{anti.dapal}^{\bullet}_{\mathrm{ev}} && (265)
\end{aligned}$$

The identity (259) simply reflects the form of the preimage $f_{\#}$ of the *gari*-dilator. See $f_0 := x^{-1}\, f_{\#}$ in (172):

The identity (260) is the mechanical transposition of (259) under the involution *swap*.

To establish the last identity (261), we must start, not from (260), but from the corresponding relation for $pal^{\bullet}$, which reads

$$\mathrm{der.pal}^{\bullet} = \mathrm{preira}(\mathrm{pal}^{\bullet}, \mathrm{dapal}^{\bullet}) \quad \textit{with} \quad \mathrm{dapal}^{\bullet} := -\sum_{1\le r} \frac{1}{(r+1)!}\,\mathrm{sra}^{\bullet}_{r} \qquad (266)$$

To declumsify our notations, we set:[36]

$$B := -\sum_{r\ even} \frac{1}{(r+1)!}\,\mathrm{sra}^{\bullet}_{r} \quad ; \quad C := -\sum_{r\ odd} \frac{1}{(r+1)!}\,\mathrm{sra}^{\bullet}_{r} \qquad (267)$$

$$A := B + C \quad ; \quad A^{*} := B - C \qquad (268)$$

[36] Note in passing that B is the *gira*-dilator of b, but that C has nothing to do with the *gira*-dilator of c

$$a := \mathrm{pal}^{\bullet} \quad ; \quad b := \mathrm{pal}^{\bullet}_{\mathrm{evv}} \quad ; \quad c := \mathrm{pal}^{\bullet}_{\mathrm{odd}} \tag{269}$$

Further, we shall denote the *mu*-product by a simple dot "." We shall also abbreviate $irat(A)$, $irat(B)$ etc as $\bar{A}$, $\bar{B}$ etc. Lastly, stars in upper (resp. lower) index position shall stand for the involution *pari* (resp. *anti*).

With these compact notations, the relation (266) we want to establish reads

$$\mathcal{R} := -\mathrm{der}(b.c) + \overline{B}b + b.B - \frac{1}{2}\,B - \frac{1}{2}\,B_{*} \equiv 0 \tag{270}$$

Using the fact that $der, \bar{A}, \bar{B}$ etc are *mu*-derivations, we see that $\mathcal{R}$ may be decomposed as

$$\mathcal{R} = \mathcal{R}_1.c^{-1} + \mathcal{R}_1^{*}.c - b.\mathcal{R}_2 - b.\mathcal{R}_2^{*} \tag{271}$$

with

$$\begin{aligned}
\mathcal{R}_1 &:= -\mathrm{der}(b.c) + \bar{A}\,(b.c) + b.c.A &(272)\\
\mathcal{R}_1^{*} &:= -\mathrm{der}(b.c^{-1}) + \bar{A}^{*}\,(b.c^{-1}) + b.c^{-1}.A^{*} &(273)\\
\mathcal{R}_2 &:= (\bar{A}c).c^{-1} + c.A.c^{-1} - \frac{1}{2}\,A + \frac{1}{2}\,A_{*} - \frac{1}{2}\,Pa^{\bullet} &(274)\\
\mathcal{R}_2^{*} &:= (\bar{A}^{*}c^{-1}).c + c^{-1}.A^{*}.c - \frac{1}{2}\,A^{*} + \frac{1}{2}\,A^{*}_{*} + \frac{1}{2}\,Pa^{\bullet} &(275)
\end{aligned}$$

Let us now show that $\mathcal{R}_1 \equiv \mathcal{R}_1^{\bullet} \equiv \mathcal{R}_2 \equiv \mathcal{R}_2^{*} \equiv 0$. The identities $\mathcal{R}_1^{*} \equiv 0$ and $\mathcal{R}_2^{*} \equiv 0$ follow respectively from $\mathcal{R}_1 \equiv 0$ and $\mathcal{R}_2 \equiv 0$ under *pari*, and the identity $\mathcal{R}_1 \equiv 0$ is none other than (266). So the only thing left to check is $\mathcal{R}_2 \equiv 0$. To do this we apply the derivation rule (200) and then the simplification rule (199) to show that in the expression $(\bar{A}c).c^{-1} + c.A.c^{-1}$ all 'intermediary terms', i.e. all terms of the form

$$\mathrm{mu}\big(\mathrm{mu}_{r_1}(\mathrm{Pa}^{\bullet}), \mathrm{sra}^{\bullet}_{r_2}, \mathrm{mu}_{r_3}(\mathrm{Pa}^{\bullet})\big) \quad or \quad \mathrm{mu}\big(\mathrm{mu}_{r_1}(\mathrm{Pa}^{\bullet}), \mathrm{anti.sra}^{\bullet}_{r_2}, \mathrm{mu}_{r_3}(\mathrm{Pa}^{\bullet})\big)$$

with $r_1 \neq 0, r_2 \geq 2, r_3 \neq 0$ disappear, leaving only 'extreme terms' that cancel out with the terms from $-1/2\,A + 1/2\,A^{*}$, plus of course pure *mu*-powers of $Pa^{\bullet}$, which also cancel out. This establishes $\mathcal{R} \equiv 0$.

§4-7. Properties of $ripal^{\bullet}_{\mathrm{ev}}$.

Applying the identity (44) for dilator composition to the factorisation

$$\mathrm{ripal}^{\bullet}_{\mathrm{ev}} = \mathrm{gari}(\mathrm{ripal}^{\bullet}, \mathrm{pal}^{\bullet}_{\mathrm{od}}) \tag{276}$$

we find

$$\mathrm{diripal}^{\bullet}_{\mathrm{ev}} = \mathrm{dipal}^{\bullet}_{\mathrm{od}} + \mathrm{adari}(\mathrm{pal}^{\bullet}_{\mathrm{od}})^{-1}.\,\mathrm{diripal}^{\bullet} \tag{277}$$

But since $\mathit{pal}^\bullet_{\mathrm{od}} = \mathit{expari}(-1/2\ \mathit{Pa}^\bullet)$, this simplifies to

$$\mathrm{diripal}^\bullet_{\mathrm{ev}} = -\frac{1}{2}\,\mathrm{Pa}^\bullet + (\exp\mathcal{P}).\,\mathrm{diripal}^\bullet \tag{278}$$

with $\mathit{diripal}^\bullet$ as in (236) and with the ordinary exponential $\mathit{exp}\mathcal{P}$ of the elementary operator $\mathcal{P}$:

$$\mathcal{P}.M^\bullet := \frac{1}{2}\,\mathrm{ari}(\mathrm{Pa}^\bullet, M^\bullet) \qquad\qquad (\forall M^\bullet \in \mathrm{BIMU}) \tag{279}$$

Being the *gari*-dilator of a symmetral bimould, $\mathit{diripal}^\bullet_{\mathrm{ev}}$ is of course alternal. And since we have shown that $\mathit{pal}^\bullet_{\mathrm{ev}}$ and therefore $\mathit{ripal}^\bullet_{\mathrm{ev}}$ are 'even' (i.e. *pari*-invariant), the same applies for $\mathit{diripal}^\bullet_{\mathrm{ev}}$, so that, as expained in §2 (see (89) and (90)) the relation between $\mathit{diripal}^\bullet$ and $\mathit{diripal}^\bullet_{\mathrm{ev}}$ may be rewritten as

$$\mathrm{diripal}^\bullet_{\mathrm{ev}} = (\cosh\mathcal{P})^{-1}.\frac{1}{2}\,(\mathrm{id} + \mathrm{pari}).\mathrm{diripal}^\bullet \tag{280}$$

which, appearances notwithstanding, is actually simpler than (278), as it involves only even-length components.

In a sense, this is all we need to know. But in order to get the extra information of formula (154) or rather, in our polar specialisation, the explicit expansion of $\mathit{diripal}^\bullet_{\mathrm{ev}}$ in terms of the remarkable alternals $\mathit{ka}^\bullet_{2r}$ (polar-specialised from the $\mathfrak{ke}^\bullet_{2r}$ of §2), we must work harder. Rather than derive the expansion of $\mathit{diripal}^\bullet_{\mathrm{ev}}$ directly[37] from that of $\mathit{diripal}^\bullet$ via (278) or (280), it is more convenient to reproduce the approach of (245) and (246), i.e. to set

$$\mathrm{kasra}^\bullet_r := (\exp\mathcal{P}).\mathrm{ganit}(\mathrm{nipac}^\bullet).\mathrm{sra}^\bullet_{\mathrm{r}} = \sum_{r\le r_*} \mathrm{kasra}_{r,r_*} \quad \big(\mathrm{kasra}_{r,r_*} \in \mathit{BIMU}_{r_*}\big)$$

and then regroup the (highly lacunary) components of r_*:

$$\sum_{1\le r\le r_*} \frac{1}{r.(r+1)}\,\mathrm{kasra}^\bullet_{r,r_*} \equiv \frac{2^{1-r_*}}{(r_*-1).(r_*+1)}\,\mathrm{ka}^\bullet_{r_*} \tag{281}$$

Comparing the components $\mathit{kasra}^\bullet_{r,r_*}$ with the earlier $\mathit{hasra}^\bullet_{r,r_*}$ of (245), one even gets to understand (however dimly) why the relevant tree-combinatorial

[37]The direct method yields only partial but valuable information. Thus, denoting $\mathit{Proj}_1.M^\bullet$ the first coefficient of $M^\bullet$ in the standard eupolar basis, we may establish the identity $\mathit{Proj}_1.\mathcal{P}^{2r_*-r}.\mathit{diripal}^\bullet_r = \frac{(-2)^{r-2r_*}}{r.r+1}\,\frac{(2r_*-2)!}{(r-2)!}$ which leads to $\mathit{Proj}_1.\mathit{diripal}^\bullet_{\mathrm{ev},2r_*} = \frac{2^{1-2r_*}}{(2r_*-1)(2r_*+1)}$ which in turn yields the important normalisation property $\mathit{Proj}_1.\mathit{ka}^\bullet_{2r_*} = 1$

object for calculating the bimould projections in the standard basis $\{\mathfrak{e}^\bullet_{\boldsymbol{t}}\}$ is *slant*$(\boldsymbol{t})$ in the case of $ha^\bullet_r$ and *stack*$(\boldsymbol{t})$ in the case of $ka^\bullet_{2r}$. Still, the calculations are quite lengthy and the whole approach leaves much to be desired. In particular, one would appreciate a more conceptual explanation for the puzzling *slant/stack* dichotomy.

§4-8. Characterisation of $pal^\bullet/pil^\bullet$.

The explicit expansion of $pal^\bullet$ as given in (300) below (as a direct consequence of (122) and (123)) makes it clear that $pal^\bullet$, and therefore $pil^\bullet$ too, possess exactly the pole pattern described in Proposition 3.2. To prove the converse, namely that no other Pi-polar bisymmetral $varpil^\bullet$ can display the same pole pattern, we must use the results of §8 about the standard factorisation of bisymmetrals. In the case when $varpil^\bullet_1 = 0$, we have

$$\mathrm{varpil}^\bullet = \mathrm{expari.bir}^\bullet \qquad \textit{with} \quad \mathrm{bir}^\bullet \in \textit{bialternal} \tag{282}$$

In the case when our first component $varpil^\bullet_1$ is $\neq 1$, it is necessarily of the form $c\,Pi^\bullet$ and, modulo an elementary dilation $varpil^\bullet_r \mapsto \gamma^r\,varpil^\bullet_r$, we may assume $c = -1/2$ and get $varpil^\bullet_1$ and $pil^\bullet_1$ to coincide, thus ensuring (according to §8) the existence of a factorisation:

$$\mathrm{varpil}^\bullet = \mathrm{gari}(\mathrm{pil}^\bullet, \mathrm{expari.bir}^\bullet) \qquad \textit{with} \quad \mathrm{bir}^\bullet \in \textit{bialternal} \tag{283}$$

The thing now is to focus on the first nonzero component $bir^\bullet_{2r}$ $(2r \geq 4)$. It is bound to occur linearily in the expansion of $varpil^\bullet$, whether the latter be of type (282) or (283). Now, $bir^\bullet_{2r}$ cannot be of the form $c\,ri^\bullet_{2r}$, which is simply alternal, not bialternal. But of all *alternals*, let alone *bialternals*, $ri^\bullet_{2r}$ alone possesses precisely the pole structure described in Proposition 3.2 for $pil^\bullet$. This clinches the argument.

5 Polar bisymmetrals: explicit expansions.

§5-1. Explicit expansions for $pil^\bullet$ and $pil^\bullet_{\mathrm{ev}}$.

From the $\{ri^\bullet_r\}$-expansions of $pil^\bullet$'s dilator $dipil^\bullet$ and infinitesimal generator $lipil^\bullet := logari.pil^\bullet$:

$$\mathrm{dipil}^\bullet = \sum_{1\leq r} \tau_r\ \mathrm{ri}^\bullet_r \qquad \textit{with} \quad \tau_r = -\frac{1}{(r+1)!} \tag{284}$$

$$\mathrm{lipil}^\bullet = \sum_{1\leq r} \theta_r\ \mathrm{ri}^\bullet_r \qquad \textit{with} \quad \theta_r = \textit{horrible} \tag{285}$$

we at once derive (see (39) and (478)) two equally valid expansions for $pil^\bullet$ itself, which in their first raw form read:

$$\mathrm{pil}^\bullet = 1^\bullet + \sum_{r_1,\dots,r_s\geq 1}^{s\geq 1} \tau_{r_1}\dots\tau_{r_s}\,\mathrm{Paj}^{r_1,\dots,r_s}\ \overrightarrow{\mathrm{preari}}\,(\mathrm{ri}^\bullet_{r_1},\dots,\mathrm{ri}^\bullet_{r_s}) \tag{286}$$

$$\mathrm{pil}^\bullet = 1^\bullet + \sum_{r_1,\dots,r_s\geq 1}^{s\geq 1} \frac{1}{s!}\,\theta_{r_1}\dots\theta_{r_s}\ \overrightarrow{\mathrm{preari}}\,(\mathrm{ri}^\bullet_{r_1},\dots,\mathrm{ri}^\bullet_{r_s}) \tag{287}$$

The main difference lies of course in the transparency of the τ_r's compared with the complexity of the θ_r's. But quite apart from the nature of their coefficients, the above expansions are unsatisfactory on two further counts: they are *non-unique*[38] and involve multiple pre-Lie brackets, which are complex, *inflected* expressions. So we must hasten to replace them by unique expansions involving simple, uninflected *mu*-products. There are three ways of doing this, based on the elementary series $\{mi^\bullet_r\}$, $\{ni^\bullet_r\}$, $\{ri^\bullet_r\}$ inductively defined as follows:

$$\mathrm{mi}^\bullet_1 := \mathrm{Pi}^\bullet \quad ; \quad \mathrm{mi}^\bullet_{\mathrm{r}} := \mathrm{amit}(\mathrm{mi}^\bullet_{r-1}).\mathrm{Pi}^\bullet \tag{288}$$

$$\mathrm{ni}^\bullet_1 := \mathrm{Pi}^\bullet \quad ; \quad \mathrm{ni}^\bullet_{\mathrm{r}} := \mathrm{anit}(\mathrm{ni}^\bullet_{r-1}).\mathrm{Pi}^\bullet \tag{289}$$

$$\mathrm{ri}^\bullet_1 := \mathrm{Pi}^\bullet \quad ; \quad \mathrm{ri}^\bullet_{\mathrm{r}} := \mathrm{arit}(\mathrm{ri}^\bullet_{r-1}).\mathrm{Pi}^\bullet \tag{290}$$

and behaving as follows under the anti-action *arit*:

$$\mathrm{arit}.(\mathrm{ri}^\bullet_{\mathrm{q}}).\mathrm{mi}^\bullet_{\mathrm{p}} = \sum_{s\geq 1}\ \sum_{r_1\geq p}^{\sum r_i=p+q} (-1)^{1+s}\, r_s\, \mathrm{mu}(\mathrm{mi}^\bullet_{\mathrm{r_1}},\dots,\mathrm{mi}^\bullet_{\mathrm{r_s}}) \tag{291}$$

$$\mathrm{arit}.(\mathrm{ri}^\bullet_{\mathrm{q}}).\mathrm{ni}^\bullet_{\mathrm{p}} = \sum_{s\geq 1}\ \sum_{r_s\geq p}^{\sum r_i=p+q} (-1)^{1+s+q}\, r_1\, \mathrm{mu}(\mathrm{ni}^\bullet_{\mathrm{r_1}},\dots,\mathrm{ni}^\bullet_{\mathrm{r_s}}) \tag{292}$$

$$\mathrm{arit}.(\mathrm{ri}^\bullet_{\mathrm{q}}).\mathrm{ri}^\bullet_{\mathrm{p}} = \mathrm{p}.\mathrm{ri}^\bullet_{\mathrm{p+q}} + \sum_{k\leq q} \mathrm{lu}(\mathrm{ri}^\bullet_{\mathrm{k}},\mathrm{ri}^\bullet_{\mathrm{p+q-k}}) \tag{293}$$

For $s \geq 1$ and $r_1 + \dots + r_s = r$ each of the three sets

$$\left\{\mathrm{mu}(\mathrm{mi}^\bullet_{\mathrm{r_1}},\dots,\mathrm{mi}^\bullet_{\mathrm{r_s}})\right\} \ ; \ \left\{\mathrm{mu}(\mathrm{ni}^\bullet_{\mathrm{r_1}},\dots,\mathrm{ni}^\bullet_{\mathrm{r_s}})\right\} \ ; \ \left\{\mathrm{mu}(\mathrm{ri}^\bullet_{\mathrm{r_1}},\dots,\mathrm{ri}^\bullet_{\mathrm{r_s}})\right\} \tag{294}$$

consists of linearly independent bimoulds that span one and the same subspace $Flexin_r(Pi)$ of $Flex_r(Pi)$. The six conversion rules between the three

[38] Thus we have (286) side by side with (287), all due to the many a priori relations between multiple pre-Lie brackets.

bases are mentioned in [E3] §4.1. Let us recall the most useful:

$$\mathrm{ri}^\bullet_{r_0} = \sum_{1\leq s} \sum_{\sum r_i = r_0} (-1)^{s+1}\, r_s\, \mathrm{mu}(\mathrm{mi}^\bullet_{r_1}, ..., \mathrm{mi}^\bullet_{r_1}) \tag{295}$$

$$\mathrm{ri}^\bullet_{r_0} = \sum_{1\leq s} \sum_{\sum r_i = r_0} (-1)^{s+r}\, r_1\, \mathrm{mu}(\mathrm{ni}^\bullet_{r_1}, ..., \mathrm{ni}^\bullet_{r_1}) \tag{296}$$

The first two bases (294) of $Flexin_r(Pi)$ have the advantage of consisting of 'atoms' (simple strings of inflected units Pi). The ingredients $ri^\bullet_r$ of the third basis are not atomic (it takes at least $r+1$ strings to express them) but they make up for it by being *alternal*.

Now, the above derivation rules (291), (292), (293) together with the two conversion rules (295), (296) make it easy[39] to expand the multiple *preari*-brackets of (284), (285) in each of the three bases (294). In the event we get three alternative expressions:

$$\mathrm{pil}^\bullet = 1^\bullet + \sum_{r_1,...,r_s\geq 1}^{s\geq 1} \mathrm{Mip}^{r_1,...,r_s}\, \mathrm{mu}(\mathrm{mi}^\bullet_{r_1}, ..., \mathrm{mi}^\bullet_{r_s}) \tag{297}$$

$$\mathrm{pil}^\bullet = 1^\bullet + \sum_{r_1,...,r_s\geq 1}^{s\geq 1} \mathrm{Nip}^{r_1,...,r_s}\, \mathrm{mu}(\mathrm{ni}^\bullet_{r_1}, ..., \mathrm{ni}^\bullet_{r_s}) \tag{298}$$

$$\mathrm{pil}^\bullet = 1^\bullet + \sum_{r_1,...,r_s\geq 1}^{s\geq 1} \mathrm{Rip}^{r_1,...,r_s}\, \mathrm{mu}(\mathrm{ri}^\bullet_{r_1}, ..., \mathrm{ri}^\bullet_{r_s}) \tag{299}$$

with three rational-valued moulds $Mip^\bullet$, $Nip^\bullet$, $Rip^\bullet$ defined by simple induction rules (see next paragraph) that dually reflect the rules (288), (289), (290). In accordance with the nature of the three bases (294), $Mip^\bullet$ and $Nip^\bullet$ are symmetrel while $Rip^\bullet$ is symmetral.

The procedure for expandind $pil^\bullet_{\mathrm{ev}}$ is entirely similar: one need only retain the sole even terms $\tau_{2r}\, ri^\bullet_{2r}$ in (284).

§5-2. General inductions for the moulds $Mip^\bullet, Nip^\bullet, Rip^\bullet$.

[39] since $preari(A^\bullet, B^\bullet) = arit(B^\bullet).A^\bullet + mu(A^\bullet, B^\bullet)$

The first induction goes like this:

$$
\begin{aligned}
\mathbf{Mip}^{\emptyset} &:= 1\ ,\ \mathbf{Mip}^{1} := \alpha_1 \\
\mathbf{Mip}^{n_1} &:= \frac{1}{n_1}\mathrm{Mi}_*^{n_1} + \frac{1}{n_1}\sum_{0<n_0<n_1} \mathbf{Mip}^{n_0}\,\mathrm{Mi}_{n_0}^{n_1} \\
\mathbf{Mip}^{\boldsymbol{n}} &:= \frac{1}{|\boldsymbol{n}|}\sum_{\boldsymbol{n^1}.\boldsymbol{n^2}=\boldsymbol{n}} \mathbf{Mip}^{\boldsymbol{n^1}}\,\mathrm{Mi}_*^{\boldsymbol{n^2}} + \frac{1}{|\boldsymbol{n}|}\sum_{\boldsymbol{n^1}.\boldsymbol{n^2}.\boldsymbol{n^3}=\boldsymbol{n}}^{0<n_0\leq \mathrm{first}(\boldsymbol{n^2})} \mathbf{Mip}^{\boldsymbol{n^1},n_0,\boldsymbol{n^3}}\,\mathrm{Mi}_{n_0}^{\boldsymbol{n^2}}
\end{aligned}
$$

with

$$
\begin{aligned}
\mathrm{Mi}_*^{n_1,\ldots,n_r} &:= (-1)^{1+r} n_r\,\alpha_{|\boldsymbol{n}|} \\
\mathrm{Mi}_{n_0}^{n_1,\ldots,n_r} &:= (-1)^{1+r}\, n_r\,\alpha_{|\boldsymbol{n}|-n_0} \quad \textit{if} \quad 0<n_0\leq n_1 \quad (:= 0\ \textit{otherwise})
\end{aligned}
$$

The second induction is essentially the same under the left-right exchange:

$$
\begin{aligned}
\mathbf{Nip}^{\emptyset} &:= 1\ ,\ \mathbf{Nip}^{1} := \alpha_1 \\
\mathbf{Nip}^{n_1} &:= \frac{1}{n_1}\mathrm{Ni}_*^{n_1} + \frac{1}{n_1}\sum_{0<n_0<n_1} \mathbf{Nip}^{n_0}\,\mathrm{Ni}_{n_0}^{n_1} \\
\mathbf{Nip}^{\boldsymbol{n}} &:= \frac{1}{|\boldsymbol{n}|}\sum_{\boldsymbol{n^1}.\boldsymbol{n^2}=\boldsymbol{n}} \mathbf{Nip}^{\boldsymbol{n^1}}\,\mathrm{Ni}_*^{\boldsymbol{n^2}} + \frac{1}{|\boldsymbol{n}|}\sum_{\boldsymbol{n^1}.\boldsymbol{n^2}.\boldsymbol{n^3}=\boldsymbol{n}}^{0<n_0\leq \mathrm{last}(\boldsymbol{n^2})} \mathbf{Nip}^{\boldsymbol{n^1},n_0,\boldsymbol{n^3}}\,\mathrm{Ni}_{n_0}^{\boldsymbol{n^2}}
\end{aligned}
$$

with

$$
\begin{aligned}
\mathrm{Ni}_*^{n_1,\ldots,n_r} &:= (-1)^{r+|\boldsymbol{n}|} n_1\,\alpha_{|\boldsymbol{n}|} \\
\mathrm{Ni}_{n_0}^{n_1,\ldots,n_r} &:= (-1)^{1+r+|\boldsymbol{n}|-n_0}\, n_1\,\alpha_{|\boldsymbol{n}|-n_0} \quad \textit{if} \quad 0<n_0\leq n_r \quad (:= 0\ \textit{otherwise})
\end{aligned}
$$

The third induction involves less terms and is faster to run on a computer (see §18.A *infra*), the reason being that here the bulk of the complexity is absorbed by the 'molecular' $ri_r^{\bullet}$'s that replace the 'atomic' $mi_r^{\bullet}$'s or $ni_r^{\bullet}$'s of the earlier inductions:

$$
\begin{aligned}
\mathbf{Rip}^{\emptyset} &:= 1\ ,\ \mathbf{Rip}^{1} := \alpha_1\ ,\ \ \mathbf{Rip}^{\overbrace{1,\ldots,1}^{r\ times}} := \frac{1}{r!}\,(\alpha_1)^r \\
\mathbf{Rip}^{n_1} &:= \frac{1}{n_1}\alpha_{n_1} + \frac{1}{n_1}\sum_{0<n_0<n_1} \mathbf{Rip}^{n_0}\,\mathrm{Ri}_{n_0}^{n_1} \\
\mathbf{Rip}^{\boldsymbol{n}} &:= \frac{1}{|\boldsymbol{n}|}\,\mathbf{Rip}^{\boldsymbol{n}'}\,\alpha_{n_r} + \frac{1}{|\boldsymbol{n}|}\sum_{\boldsymbol{n^1}.\boldsymbol{n^2}.\boldsymbol{n^3}=\boldsymbol{n}}^{0<n_0<|\boldsymbol{n^2}|} \mathbf{Nip}^{\boldsymbol{n^1},n_0,\boldsymbol{n^3}}\,\mathrm{Ni}_{n_0}^{\boldsymbol{n^2}}
\end{aligned}
$$

with

$$\begin{aligned}
\mathrm{Ri}^{n_1}_{n_0} &:= n_0\,\alpha_{n_1-n_0} \;\; \textit{if} \;\; n_0<n_1 \;\; (:=0 \;\textit{otherwise}) \\
\mathrm{Ri}^{n_1,n2}_{n_0} &:= +\alpha_{n_1+n_1-n_0} \;\; \textit{if} \;\; n_1<n_0\leq n_2 \\
&:= -\alpha_{n_1+n_2-n_0} \;\; \textit{if} \;\; n_2<n_0\leq n_1 \\
&:= 0 \qquad\qquad \textit{otherwise} \\
\mathrm{Ri}^{n_1,\dots,n_r}_{n_0} &:= 0 \qquad\qquad \textit{if} \quad r\geq 3
\end{aligned}$$

S5-3. Explicit expansions for $pal^\bullet$, $pal^\bullet_{\mathrm{ev}}$ and $pal^\bullet_{\mathrm{evv}}$.

We start from the *mu*-dilators $dupal^\bullet$, $dupal^\bullet_{\mathrm{ev}}$, $dupal^\bullet_{\mathrm{evv}}$ as described in §3. Applying the rule (39) we immediately derive these three expansions:

$$\mathrm{pal}^\bullet = 1^\bullet + \sum_{\boldsymbol{w^1}\dots\boldsymbol{w^s}=\bullet}^{r_i\ \textit{even or}\ 1} \alpha_{r_1}\dots\alpha_{r_s}\,\mathrm{Paj}^{|\boldsymbol{u^1}|,\dots,|\boldsymbol{u^s}|}\,\mathrm{mu}(\mathrm{lan}^\bullet_{\mathrm{r_1}},\dots,\mathrm{lan}^\bullet_{\mathrm{r_s}}) \tag{300}$$

$$\mathrm{pal}^\bullet_{\mathrm{ev}} = 1^\bullet + \sum_{\boldsymbol{w^1}\dots\boldsymbol{w^s}=\bullet}^{r_i\ \textit{even}} \alpha_{r_1}\dots\alpha_{r_s}\,\mathrm{Paj}^{|\boldsymbol{u^1}|,\dots,|\boldsymbol{u^s}|}\,\mathrm{mu}(\mathrm{lan}^\bullet_{\mathrm{r_1}},\dots,\mathrm{lan}^\bullet_{\mathrm{r_s}}) \tag{301}$$

$$\mathrm{pal}^\bullet_{\mathrm{evv}} = 1^\bullet + \sum_{\boldsymbol{w^1}\dots\boldsymbol{w^s}=\bullet}^{r_i\ \textit{even}} \beta_{r_1}\dots\beta_{r_s}\,\mathrm{Paj}^{|\boldsymbol{u^1}|,\dots,|\boldsymbol{u^s}|}\,\mathrm{mu}(\mathrm{lan}^\bullet_{\mathrm{r_1}},\dots,\mathrm{lan}^\bullet_{\mathrm{r_s}}) \tag{302}$$

with $r_i = r(\boldsymbol{w^i}) = r(\boldsymbol{u^i})$; with the selfsame Bernoulli-like numbers α_r, β_r as in (121),(159); and with

$$\mathrm{lan_r}^\bullet := \vec{\mathrm{lu}}(\mathrm{I}^\bullet, \overbrace{\mathrm{Pa}^\bullet,\dots,\mathrm{Pa}^\bullet}^{(r-1)\ \textit{times}}) \tag{303}$$

The last two expansions must be preferred to the first, since they involve only *even* terms. Of these two *even* expansions, (302) is again preferrable to (301), since the passage from $pal^\bullet_{\mathrm{evv}}$ to $pal^\bullet$ (*mu*-multiplication) is so much simpler than the passage from $pal^\bullet_{\mathrm{ev}}$ to $pal^\bullet$ (*gari*-multiplication).

But there is still room for improvement. Indeed, (302) is blighted by some redundancy since the summands on the right-hand side are not linearly independent.[40]. To get a true basis, we must introduce bimoulds $Lan^\bullet_{\epsilon_1,\dots,\epsilon_s} \in$

[40] The products $mu(lan^\bullet_{r_1},\dots,lan^\bullet_{r_s})$ are of course linearly independent, but cease to be so when 'precomposed' by $Paj^\bullet$ as in (300), (301), (302).

Flex$_{2s}(Pa)$ inductively defined by

$$\begin{aligned}
\mathrm{Lan}^{w_1,\ldots,w_{2s}}_{\epsilon_1,\ldots,\epsilon_s} &= \mathrm{Lan}^{w_1,\ldots,w_{2s-2}}_{\epsilon_1,\ldots,\epsilon_{s-1}}\,\mathrm{Pan}^{w_1,\ldots,w_{2s}}_{\epsilon_s} \quad \textit{with} \qquad (304)\\
\mathrm{Pan}^{w_1,\ldots,w_{2s}}_{0} &:= \mathrm{P}(u_{2s-1})\,\mathrm{P}(u_{2s})\\
\mathrm{Pan}^{w_1,\ldots,w_{2s}}_{1} &:= \mathrm{P}(u_{2s-1})\,\mathrm{P}(u_1+\ldots+u_{2s})\\
\mathrm{Pan}^{w_1,\ldots,w_{2s}}_{2} &:= \mathrm{P}(u_{2s})\,\mathrm{P}(u_1+\ldots+u_{2s})
\end{aligned}$$

Fixing s and letting each ϵ_i range over $\{0,1,2\}$, *except for the first* ϵ_1 *which is forbidden to be 0*, we get a set of bimoulds $Lan^\bullet_{\epsilon_1,\ldots,\epsilon_s}$ that
(i) are linearly independent
(ii) span the same subspace of *Flex*$_{2s}(Pa)$ as the $Paj^\bullet \circ mu(lan^\bullet_{r_1},\ldots,lan^\bullet_{r_s})$
(iii) permit to express these $Paj^\bullet \circ mu(lan^\bullet_{r_1},\ldots,lan^\bullet_{r_s})$ via a simple rule.

So (302) may be rewritten more economically as

$$\mathrm{pal}^\bullet_{\mathrm{evv}} = 1^\bullet + \sum_{\epsilon_1,\ldots,\epsilon_s\in\{0,1,2\}}^{s\geq 1} \mathrm{Han}^{\epsilon_1,\ldots,\epsilon_s}\,\mathrm{Lan}^\bullet_{\epsilon_1,\ldots,\epsilon_s} \qquad \big(s=\frac{1}{2}\,r(\bullet)\big) \qquad (305)$$

with a rational valued mould $Han^\bullet$ belonging to none of the classical symmetry types but nonetheless calculable by a simple induction.

From $pal^\bullet_{\mathrm{evv}}$ we easily go to $pal^\bullet$, through elementary *mu*-multiplication by the arch-elementary factor $pal^\bullet_{\mathrm{odd}}$, and from there we go to $pil^\bullet$ through the equally elementary involution *swap*. Moreover, of all expansions currently at our disposal, this ultimate expansion (305) for $pal^\bullet_{\mathrm{evv}}$ is clearly optimal, since it involves only $2.3^{r/2-1}$ atomic summands, as compared with the 2^r summands in each of the three expansions (297), (298), (299) for $pil^\bullet$.

Remark: If in (304) we had prohibited for ϵ_1 the value 1 resp. 2 instead of 0, we would still have got two valid bases $Lan^\bullet_{\epsilon_1,\ldots,\epsilon_r}$ and two expansions of the form (303), though with changed moulds $H^\bullet$. There exist yet other bases with the same indexation. These multiple choices, hardly relevant in the eupolar case, acquire real significance in the eutrigonometric case ([E4]) and shall be discussed there.

6 Polar bisymmetrals: seven remarks.

.

Remark 1. Nearly complete restoration of symmetry.

The first proof presented here (in §4) of the bisymmetrality of $pal^\bullet/pil^\bullet$ is definitely shorter than the second one, which in turn is simpler than either of the two proofs sketched in [E3]. As we see it, it has two further merits: it respects the symmetry between the two swappees (unlike the earlier treatments, which gave precedence to $pil^\bullet$ and relegated $pal^\bullet$ to the subordinate status of a derivative object) and it does so in the most satisfactory way that could be dreamt of, by linking $pal^\bullet$ and $pil^\bullet$ separately to the only two completely elementary alternal series that exist in $Flex(\mathfrak{E})$, namely $\{\mathfrak{le}_r^\bullet\}$ and $\{\mathfrak{re}_r^\bullet\}$.

The linkage between each swappee and its alternal series is provided by the notion of *dilator*, but the two dilators in question are rather different: one is geared to the uninflected *mu*-product, the other to the inflected *gari*-product. The two alternal series $\{\mathfrak{le}_r^\bullet\}$ and $\{\mathfrak{re}_r^\bullet\}$ also differ, and in much the same way. We have here, we suggest, the whole essence of dimorphy in a nutshell: a symmetry that is *nearly complete, yet stops just short of being thoroughly, dully, and barrenly complete.* In fact the whole flexion structure – dimorphy's natural framework – is *largely* though not *perfectly* self-dual under *swap.* So is its core *ARI//GARI.* And so is the core's core, consisting of the two pairs $pal^\bullet/pil^\bullet$ and $tal^\bullet/til^\bullet$. Experience shows that such mathematical structures are among the most fecund.

Remark 2. Pervasiveness of parity.

Considerations of parity are paramount in all branches of the theory, not just in the factorisation of the key bimoulds but also when it comes to constructing and describing their length-r components.

Regarding the factorisations, they come in all sorts and shapes. Thus, all three formulae (129), (130), (131) are logically independent, carry unrelated even factors, and involve two distinct group laws, *mu* and *gari.* Nor is the phenomenon restricted to the eupolar context; it extends to such objects as the important bimould $Zag^\bullet$, though with a nuance: unlike eupolar bimoulds, which are automatically invariant under $pari \circ neg$, general bimoulds such as $Zag^\bullet$ react differently to *pari* and *neg*, leading to a more intricate factorisation pattern, with three factors $Zag_I^\bullet$, $Zag_{II}^\bullet$, $Zag_{III}^\bullet$, the first of which again splits into three subfactors.

Regarding the mould components, the even/odd dichotomy makes itself felt in this way: whereas we have to *work* in order to find the even-length components of our bisymmetrals[41], their odd-length components immediately and effortlessly *follow*, and that too under any one of at least four distinct mechanisms.[42] The dichotomy also holds for the components of $Zag^\bullet$ and

[41]This applies for the eutriginometric $tal^\bullet/til^\bullet$ even more than for the eupolar $pal^\bullet/pil^\bullet$.

[42]we can use either the three identities (129), (130), (131) in section §3 or again the

those of each of its three factors. Thus, constructing the even-length components of $Zag^\bullet_I$ or $Zag^\bullet_{II}$ is hard work, while the odd-length components easily follow. With $Zag^\bullet_{III}$, it is exactly the reverse.

Ultimately, the dominance of parity in flexion theory can be traced back to one root cause: the essential parity of bialternals (see §7 *infra*). Germane considerations also explain the existence of a surperalgebra *SUARI* parallel to *ARI* (see [E1], §24, pp 456-459).

Remark 3. Native complexity of bisymmetrals

No bisymmetrality proof for $pal^\bullet/pil^\bullet$ is entirely elementary, even though the first of the two proofs presented here (in §4-3) keeps complications down to a minimum. Bisymmetrality proofs for the trigonometric $tal^\bullet/til^\bullet$ are even longer and harder.

This relative difficulty in proving what is after all the signature property of our two bimould pairs (their birthmark as it were and the one reason behind their ubiquity in multizeta theory) simply reflects the non-trivial nature of these objects – their native and irreducible complexity.

Remark 4. Nature picks exactly the right polar specialisations

Though the two structures $Flex(Pi)$ and $Flex(Pa)$ are strictly isomorphic, the two polar specialisations, when applied to a given element of $Flex(\mathfrak{E})$, often lead to rational functions that differ widely in appearance, complexity, and (rational) degree.

Thus $pal^\bullet/pil^\bullet$ is far simpler than $par^\bullet/pir^\bullet$. Unlike $par^\bullet/pir^\bullet$, it admits a trigonometric counterpart. And unlike $par^\bullet/pir^\bullet$, it spontaneously occurs in the double trifactorisation of $Zag^\bullet/Zig^\bullet$.

Similarly, the alternal series $\{\mathfrak{re}^\bullet_r\}$ is simpler when specialised to $\{ri^\bullet_r\}$ under $\mathfrak{E} \mapsto Pi$ than when specialised to $\{ra^\bullet_r\}$ under $\mathfrak{E} \mapsto Pa$. Conversely, the series $\{\mathfrak{le}^\bullet_r\}$, $\{\mathfrak{he}^\bullet_r\}$, $\{\mathfrak{ke}^\bullet_{2r}\}$ are simpler in their incarnation as $\{la^\bullet_r\}$, $\{ha^\bullet_r\}$, $\{ka^\bullet_{2r}\}$ than as $\{li^\bullet_r\}$, $\{hi^\bullet_r\}$, $\{ki^\bullet_{2r}\}$.

Lastly, as if to complete this picture of harmony, it so happens that it is precisely in their simpler form $\{ri^\bullet_r\}$ and $\{la^\bullet_r\}$, $\{ha^\bullet_r\}$, $\{ka^\bullet_{2r}\}$ that the four alternals series occur in the dilators of $pal^\bullet/pil^\bullet$.

Remark 5. Direct vs inverse bisymmetrals.

In some ways (e.g. with regard to their separators and dilators) the

'secondary-to-primary' identity (4.85) in [E3].

gari-inverses of bisymmetrals are better-behaved than the originals. This fact, already noticeable with eupolars, becomes particularly striking in the eutrigonometric case: compare for example the transparent right-hand side of (4.88) in [E3] with that of (4.87), for which no simple closed formula exists.

But the main difference is one of 'universality': whereas $pal^\bullet/pil^\bullet$ and $par^\bullet/pir^\bullet$ and indeed all 'intermediate' bisymmetrals[43] have different *gepar*-separators, the separators of the *gari*-inverses $ripal^\bullet/ripil^\bullet$ and $ripar^\bullet/ripir^\bullet$ (and of all other exceptional, non *neg*-invariant bisymmetrals) do coincide.[44]

Lastly, we may note that in the applications to multizeta algebra it is the *inverse* polar bisymmetrals $ripal^\bullet/ripil^\bullet$ and the *direct* trigonometric bisymmetrals $tal^\bullet/til^\bullet$ that matter most.

Remark 6. Coexistence of inflected and non-inflected opeations.

Quite often, when comparing flexion formulae,[45] one is struck by a recurrent anomaly: that of complex inflected operations like *gari*, *expari* etc inexplicably morphing into non-inflected ones like *mu*, *expmu* etc. While there is no neat, sweeping reason for this stealthy tendency towards 'desinflexion', but only case to case explanations, one may still point to the existence of a large ideal ARI_{intern} of ARI and of a large normal subgroup $GARI_{intern}$ of $GARI$ where *ari* and *gari* reduce to *lu* and *mu* (but with the order of the arguments reversed). See §1-11 *supra*.

Remark 7. The trigonometric bisymmetral $tal^\bullet/til^\bullet$.

The 'trigonometric specialisation'

$$(\mathfrak{E},\mathfrak{O}) \mapsto (\mathrm{Qi}_c, \mathrm{Qa}_c) \quad \textit{with} \quad \mathrm{Qi}_c^{w_1} := \frac{c}{\tan(c\,v_1)}\ ;\ \mathrm{Qa}_c^{w_1} := \frac{c}{\tan(c\,u_1)} \tag{306}$$

is no proper specialisation, since $\mathrm{Qi}_c^\bullet$ and $\mathrm{Qa}_c^\bullet$ are only approximate units, due to the corrective terms $\pm c^2$ in the identities (3.28) and (3.29) of [E3]. See also §17-12 *infra*. One should therefore be prepared for serious complications when going from $pal^\bullet/pil^\bullet$ to the trigonometric equivalent $tal^\bullet/til^\bullet$, and in that respect the trigonometric bisymmetrals do not disappoint. A long monograph [E5] will be devoted to them and their natural environment, the structures $Flex(Qi_c)$ and $Flex(Qa_c)$, which are not isomorphic to the polar prototypes nor indeed to each other.

[43] of type $gari(pal^\bullet, expari(bal^\bullet))$ with $bal^\bullet$ any bialternal.

[44] This is not always an asset: it is sometimes useful to have simple criteria that tell the canonical from the non-canonical bisymmetrals.

[45] for example (247), (248), (249).

We shall be content here with a few hints, to highlight the key steps in the transition from *eupolar* to *eutrigometric*. The formula (113) linking $pil^{\bullet}$ to its *gari*-dilator $dipil^{\bullet}$ survives unchanged (as to its general form). The link between $pal^{\bullet}$ to its *mu*-dilator $dupal^{\bullet}$ also survives, especially regarding the even factors, though not exactly in the 'differential' form (119) but rather in the 'integral' form (300), with the auxiliary mould $Paj^{\bullet}$ replaced, unsurprisingly, by a more complex $Taj^{\bullet}$. But the main change is this: while the polar dilators had their components $dipil^{\bullet}_r$ resp. $dupal^{\bullet}_r$ simply proportional to $ri^{\bullet}_r$ resp. $la^{\bullet}_r$ (or rather $lan^{\bullet}_r$), the trigonmetric dilator components $ditil^{\bullet}_r$ and $dutal^{\bullet}_r$ take their values in two $\delta(r)$-dimensional spaces of alternals, with a fast (faster than polynomially) increasing $\delta(r)$. So now at each (even) step we have to determine not one, but $\delta(r)$ rational coefficients on both sides, and to understand the *affine* (or *linear*, modulo the 'earlier' coefficients) correspondance between the two sets. The alternal series $\{ha_r\}$ and $\{ka_{2r}\}$ also survive (with single components morphing into linear spaces) and so does their connection with the even factors of the inverse bisymmetrals. Altogether, although almost every single statement of §3 has its counterpart in the new setting, we experience a steep increase in difficulty, resulting in an even more diverse and interesting situation.

7 Essential parity of bialternals.

This section is devoted to establishing the decomposition[46]

$$\mathrm{ARI}^{\mathrm{al/al}} = \mathrm{ARI}^{\dot{\mathrm{al}}/\dot{\mathrm{al}}} \oplus \mathrm{ARI}^{\underline{\mathrm{al}}/\underline{\mathrm{al}}} \tag{307}$$

of the space $ARI^{al/al}$ of all bialternals into:
(i) a large, regular part $ARI^{\underline{al}/\underline{al}}$, consisting of *even* bimoulds and stable under the *ari*-bracket.
(ii) a small, exceptional part $ARI^{\dot{al}/\dot{al}} := BIMU_1^{\mathrm{odd}}$, consisting of *odd* bimoulds of length one and endowed with a bilinear mapping *oddari* into $ARI^{\underline{al}/\underline{al}}$.

Everything rests on the following statement.

Proposition 7.1 (Parity of bialternals).
Any nonzero bialternal bimould $A^{\bullet}$ purely of length $r > 1$ is neg*-invariant or, if you prefer, an even function of its double index sequence: $A^{\boldsymbol{w}} \equiv A^{-\boldsymbol{w}}$.*

[46]See [E3] §2.7

Proof: Alternality implies invariance under *mantar* $:= -anti.pari$. Bialternality, therefore, implies invariance under *neg.push*, with:

$$\begin{aligned}\text{neg.push} &:= \text{mantar.swap.mantar.swap}\\ &= \text{anti.swap.anti.swap}\end{aligned}$$

The *push* operator, we recall, is idempotent of order $r+1$ when acting on $BIMU_r$, i.e. on bimoulds of length r.

Let us assume that $A^{\boldsymbol{w}}$ is odd in $\boldsymbol{w}$, and show that this implies $A^{\boldsymbol{w}} \equiv 0$.

For an *even* length r, this follows at once from the *neg.push*-invariance:

$$A^{\boldsymbol{w}} = (\text{neg.push})^{r+1}.A^{\boldsymbol{w}} = \text{neg}^{r+1}.\text{push}^{r+1}.A^{\boldsymbol{w}} = \text{neg}.A^{\boldsymbol{w}} = -A^{\boldsymbol{w}} \qquad (308)$$

For an *odd* length, the argument is more roundabout. Note first that for $A^{\boldsymbol{w}}$, which we assumed to be odd in $\boldsymbol{w}$, invariance under *neg.push* amounts to invariance under *-push*. Here again, it turns out that the absence of non-trivial solution does not require the full bialternality of $A^\bullet$, but only its alternality and invariance under *-push*. So let us prove this stronger statement:

Lemma 7.1 (Alternality and *push*-invariance).
No nonzero bimould $A^\bullet$ purely of length $r > 1$ can be simultaneously alternal and invariant under $-$push.

Proof: Here again, the statement is obvious for r even. So let us consider an odd length of the form $r = 2t+1 \geq 3$.

Since we shall subject $A^{\boldsymbol{w}}$ to two linear operators, *pus* and *push*, respectively of order r and $r+1$ when restricted to $BIMU_r$, and since *pus* (resp. *push*) reduces to a circular permutation in the *'short'* (resp *'long'*) bimould notation, we shall make use of both. Let us recall the conversion rule:

$$A^{[w_0^*],w_1^*,\dots,w_r^*} \;\; (\textit{long}) \longleftrightarrow A^{w_1,\dots,w_r} \;\; (\textit{short}) \qquad (309)$$

with the dual conditions on upper and lower indices:

$$\begin{array}{llcll} u_0^* = -(u_1 + \dots u_r) & , & u_i^* & = & u_i \quad \forall i \geq 1 \\ v_0^* \quad \textit{arbitrary} & , & v_i^* - v_0^* & = & v_j \quad \forall i \geq 1 \end{array}$$

To show that $A^\bullet = 0$, we start with the elementary alternality relation:

$$0 = \sum_{\boldsymbol{w} \in \text{sha}(\boldsymbol{w}', \boldsymbol{w}'')} A^{\boldsymbol{w}} \quad \textit{with } \boldsymbol{w}' = (w_1, \dots, w_{2t}) \textit{ and } \boldsymbol{w}'' = (w_{2t+1}) \qquad (310)$$

which reads:

$$0 = \sum_{1 \le j \le 2\,t+1} A^{\overline{w_1,\ldots,w_{j-1}},w_{2t+1},\overline{w_j,\ldots,w_{2t}}} \tag{311}$$

Due to the invariance of $A^\bullet$ under *-push*, this may be rewritten as:

$$0 = \sum_{1 \le j \le 2\,t+1} (-1)^j (\mathrm{push}^{\mathrm{j}}.A)^{\overline{w_1,\ldots,w_{j-1}},w_{2t+1},\overline{w_j,\ldots,w_{2t}}} \tag{312}$$

In the 'long' notation (of greater relevance here) this becomes:

$$0 \;=\; \sum_{1 \le j \le 2\,t+1} (-1)^j (\mathrm{push}^{\mathrm{j}}.A)^{[w_0],\overline{w_1,\ldots,w_{j-1}},w_{2t+1},\overline{w_j,\ldots,w_{2t}}} \tag{313}$$

$$\;=\; \sum_{1 \le j \le 2\,t+1} (-1)^j A^{[w_{2t+1}],\overline{w_j,\ldots,w_{2t}},w_0,\overline{w_1,\ldots,w_{j-1}}} \tag{314}$$

Under the exchange $w_0 \leftrightarrow w_{2t+1}$, the last identity becomes:

$$0 = \sum_{1 \le j \le 2\,t+1} (-1)^j A^{[w_0],\overline{w_j,\ldots,w_{2t}},w_{2t+1},\overline{w_1,\ldots,w_{j-1}}} = \sum_{1 \le j \le 2\,t+1} (-1)^j A^{[w_0],\overline{w_j,\ldots,w_{2t+1}},\overline{w_1,\ldots,w_{j-1}}}$$

Or again, reverting to the short notation:

$$0 = \sum_{1 \le j \le 2\,t+1} (-1)^j A^{\overline{w_j,\ldots,w_{2t+1}},\overline{w_1,\ldots,w_{j-1}}} \tag{315}$$

On the other hand, alternality implies *pus*-neutrality[47] $\sum pus^j A^\bullet \equiv 0$, which reads:

$$0 = \sum_{1 \le j \le 2\,t+1} A^{\overline{w_j,\ldots,w_{2t+1}},\overline{w_1,\ldots,w_{j-1}}} \tag{316}$$

From (315) and (316) we get by addition:

$$0 = \sum_{0 \le k \le t} A^{\overline{w_{2k+1},\ldots,w_{2t+1}},\overline{w_1,\ldots,w_{2k}}} \tag{317}$$

and by subtraction:

$$0 = \sum_{1 \le k \le t} A^{\overline{w_{2k},\ldots,w_{2t+1}},\overline{w_1,\ldots,w_{2k-1}}} \tag{318}$$

Under the change $(w_2, w_3, \ldots, w_{2t+1}, w_1) \to (w_1, w_2, \ldots, w_{2t+1})$, (318) becomes:

$$0 = \sum_{1 \le k \le t} A^{\overline{w_{2k+1},\ldots,w_{2t+1}},\overline{w_1,\ldots,w_{2k}}} \tag{319}$$

Subtracting (319) from (317), we end up with $A^{w_1,\ldots,w_r} \equiv 0$. $\Box$.

[47] See [E3], §2.4. For a proof, see below, §3.

8 Standard factorisation of bisymmetrals.

This section is devoted to establishing the factorisation[48]:

$$\mathrm{GARI}^{\mathrm{as/as}} = \mathrm{gari}\big(\mathrm{GARI}^{\dot{\mathrm{a}}\mathrm{s}/\dot{\mathrm{a}}\mathrm{s}}, \mathrm{GARI}^{\underline{\mathrm{as}}/\underline{\mathrm{as}}}\big) \tag{320}$$

of the set $GARI^{as/as}$ of all bisymmetrals into
(i) a large, regular factor $GARI^{\underline{as}/\underline{as}}$ consisting of *even* bimoulds[49] and stable under the *gari* product
(ii) a small, exceptional factor $GARI^{\dot{a}s/\dot{a}s}$ consisting of special bimoulds derived from so-called *flexion units* and with components that are alternately *odd/even*, i.e. invariant under *pari.neg* rather than *neg*.

The proof rests on the construction and properties of the special bisymmetrals $\mathfrak{ess}^\bullet$ and $\mathfrak{oss}^\bullet$ (see Proposition 3.1, *supra*) and on the following statement:

Proposition 8.1 (Factorisation of bisymmetrals).
Any bisymmetral pair of swappees $\mathrm{Sa}^\bullet /\!\!/ \mathrm{Si}^\bullet$ *simultaneously factor as*

$$\mathrm{Sa}^\bullet \;=\; \mathrm{gari}(\mathrm{Sal}^\bullet, \mathrm{Sar}^\bullet) \;=\; \mathrm{gira}(\mathrm{Sal}^\bullet, \mathrm{Sar}^\bullet) \tag{321}$$

$$\mathrm{Si}^\bullet \;=\; \mathrm{gari}(\mathrm{Sil}^\bullet, \mathrm{Sir}^\bullet) \;=\; \mathrm{gira}(\mathrm{Sil}^\bullet, \mathrm{Sir}^\bullet) \tag{322}$$

(i) with $\mathrm{Si}^\bullet = \mathrm{swap.Sa}^\bullet$, $\mathrm{Sil}^\bullet = \mathrm{swap.Sal}^\bullet$, $\mathrm{Sir}^\bullet = \mathrm{swap.Sar}^\bullet$
(ii) with bisymmetral right factors that are at once neg- *and* gush-*invariant*[50]
(iii) with bisymmetral left factors that are at once pari.neg- *and* pari.gush-*invariant.*
In other words:

$$\mathrm{Sar}^\bullet, \mathrm{Sir}^\bullet \;\in\; \mathrm{GARI}^{\mathrm{as/as}}_{neg} = \mathrm{GARI}^{\mathrm{as/as}}_{gush} =: \mathrm{GARI}^{\underline{\mathrm{as}}/\underline{\mathrm{as}}} \tag{323}$$

$$\mathrm{Sal}^\bullet, \mathrm{Sil}^\bullet \;\in\; \mathrm{GARI}^{\mathrm{as/as}}_{pari.neg} = \mathrm{GARI}^{\mathrm{as/as}}_{pari.gush} \tag{324}$$

The above decompositions are not unique, but two of them stand out, namely the one in which

$$\mathrm{Sal}^\bullet = \mathfrak{ess}^\bullet \quad \textit{with} \; -\frac{1}{2}\,\mathfrak{E}^{w_1} = \mathrm{Sal}^{w_1} = \frac{1}{2}(\mathrm{Sa}^{w_1} - \mathrm{Sa}^{-w_1}) \tag{325}$$

and the one in which

$$\mathrm{Sil}^\bullet = \mathfrak{oss}^\bullet \quad \textit{with} \; -\frac{1}{2}\,\mathfrak{O}^{w_1} = \mathrm{Sil}^{w_1} = \frac{1}{2}(\mathrm{Si}^{w_1} - \mathrm{Si}^{-w_1}) \tag{326}$$

[48] See [E3], §2.8.

[49] they are *even* functions of their multiindex $\boldsymbol{w}$, but may possess non-vanishing components of any length, *even* or *odd*.

[50] We recall that $gush := neg.gantar.swap.gantar.swap$ with $gantar := invmu.anti.pari$.

These 'co-canonical' decompositions involve two conjugate flexion units $\mathfrak{E}$ *and* $\mathfrak{O}$ *and, though distinct, easily translate into one another under the classical relation*[51] *between* $\mathfrak{ess}^\bullet$ *and* $\mathfrak{oss}^\bullet$.

Proof: It rests on the Proposition 7.1 of the preceding section, in conjunction with the two following lemmas.

Lemma 8.1 (First components of bisymmetrals).
If the length-one component Sal^{w_1} *of a bisymmetral bimould* $\mathrm{Sal}^\bullet$ *is an even function of* $w_1 = \binom{u_1}{v_1}$*, it may be anything, but if it is an odd function, it is necessarily a flexion unit.*

Proof: Let u_0, u_1, u_2 be constrained by $u_0 + u_1 + u_2 = 0$ and let v_0, v_1, v_2 be defined up to a common additive constant. At length 2, the unique symmetrality relation for $Sal^\bullet$ may be written thus:

$$\mathrm{Sal}^{\left(\begin{smallmatrix} u_1 & , & u_2 \\ v_{1:0} & , & v_{2:0} \end{smallmatrix}\right)} + \mathrm{Sal}^{\left(\begin{smallmatrix} u_2 & , & u_1 \\ v_{2:0} & , & v_{1:0} \end{smallmatrix}\right)} \equiv \mathrm{Sal}^{\left(\begin{smallmatrix} u_1 \\ v_{1:0} \end{smallmatrix}\right)} \mathrm{Sal}^{\left(\begin{smallmatrix} u_2 \\ v_{2:0} \end{smallmatrix}\right)} \tag{327}$$

Due to Sal^{w_1} being odd, this yields:

$$\mathrm{Sal}^{\left(\begin{smallmatrix} -u_1 & , & -u_2 \\ -v_{1:0} & , & -v_{2:0} \end{smallmatrix}\right)} + \mathrm{Sal}^{\left(\begin{smallmatrix} -u_2 & , & -u_1 \\ -v_{2:0} & , & -v_{1:0} \end{smallmatrix}\right)} \equiv \mathrm{Sal}^{\left(\begin{smallmatrix} u_1 \\ v_{1:0} \end{smallmatrix}\right)} \mathrm{Sal}^{\left(\begin{smallmatrix} u_2 \\ v_{2:0} \end{smallmatrix}\right)} \tag{328}$$

Likewise, the unique symmetrality relation for $Sal^\bullet$ may be written as:

$$\mathrm{Sil}^{\left(\begin{smallmatrix} -v_{0:2} & , & v_{1:2} \\ -u_0 & , & u_1 \end{smallmatrix}\right)} + \mathrm{Sil}^{\left(\begin{smallmatrix} v_{1:2} & , & -v_{0:2} \\ u_1 & , & -u_0 \end{smallmatrix}\right)} \equiv \mathrm{Sil}^{\left(\begin{smallmatrix} v_{1:2} \\ u_1 \end{smallmatrix}\right)} \mathrm{Sil}^{\left(\begin{smallmatrix} -v_{0:2} \\ -u_0 \end{smallmatrix}\right)}$$

In the u_i-variables, this translates into:

$$\mathrm{Sal}^{\left(\begin{smallmatrix} u_1 & , & -u_{0,1} \\ v_{1:0} & , & -v_{0:2} \end{smallmatrix}\right)} + \mathrm{Sal}^{\left(\begin{smallmatrix} -u_0 & , & u_{0,1} \\ -v_{0:1} & , & v_{1:2} \end{smallmatrix}\right)} \equiv \mathrm{Sal}^{\left(\begin{smallmatrix} u_1 \\ v_{1:2} \end{smallmatrix}\right)} \mathrm{Sal}^{\left(\begin{smallmatrix} -u_0 \\ -v_{0:2} \end{smallmatrix}\right)}$$

or again, due to imparity and to $\sum u_i = 0$:

$$\mathrm{Sal}^{\left(\begin{smallmatrix} u_1 & , & u_2 \\ v_{1:0} & , & v_{2:0} \end{smallmatrix}\right)} + \mathrm{Sal}^{\left(\begin{smallmatrix} -u_0 & , & -u_2 \\ -v_{0:1} & , & -v_{2:1} \end{smallmatrix}\right)} \equiv -\mathrm{Sal}^{\left(\begin{smallmatrix} u_1 \\ v_{1:2} \end{smallmatrix}\right)} \mathrm{Sal}^{\left(\begin{smallmatrix} u_0 \\ v_{0:2} \end{smallmatrix}\right)} \tag{329}$$

Let E_1 be the identity obtained by adding the three circular permutations of (327) and (328), and E_2 the identity obtained by adding the six permutations, circular or anticircular, of (329). The left-hand sides of E_1 and E_2 clearly coincide, while their right-hand sides coincide only up to the sign. Equating these right-hand sides, we find:

$$4\left(\mathrm{Sal}^{\left(\begin{smallmatrix} u_1 \\ v_{1:0} \end{smallmatrix}\right)} \mathrm{Sal}^{\left(\begin{smallmatrix} u_2 \\ v_{2:0} \end{smallmatrix}\right)} + \mathrm{Sal}^{\left(\begin{smallmatrix} u_2 \\ v_{2:1} \end{smallmatrix}\right)} \mathrm{Sal}^{\left(\begin{smallmatrix} u_0 \\ v_{0:1} \end{smallmatrix}\right)} + \mathrm{Sal}^{\left(\begin{smallmatrix} u_0 \\ v_{0:2} \end{smallmatrix}\right)} \mathrm{Sal}^{\left(\begin{smallmatrix} u_1 \\ v_{1:2} \end{smallmatrix}\right)}\right) \equiv 0 \tag{330}$$

[51] See §9 *infra* or formula (4.63) in §4.2 of [E3].

which is precisely the symmetrical characterisation of a *flexion unit*. $\square$.

Remark 1: On the face of it, the requirement that the length-1 component be a flexion unit is merely a necessary condition for the existence of a bisymmetral 'continuation' at all lengths. However, the theory of unit-generated bisymmetrals $\mathfrak{ess}^\bullet$ shows this condition to be (miraculously) sufficient.[52] This is probably the best *a posteriori* justification for singling out this notion of *flexion unit*, though by no means the only one.

Remark 2: Had we assumed $Sal^\bullet$ to be even, we would have found no constraints at all on the length-1 component – which was only to be expected, since the *ari*-exponential of that length-1 component is automatically in $GARI^{\underline{\mathrm{as}}/\underline{\mathrm{as}}}$.

Remark 3: One should not be too exercised over the presence of the factor 4 in (330), but rather observe that it vanishes after the change $Sal^{w_1} = -\frac{1}{2}\mathfrak{E}^{w_1}$ which, as it happens, the construction of $\mathfrak{ess}^\bullet$ quite naturally imposes.

Lemma 8.2 (General and even bisymmetrals).
Though not a group, the set $\mathrm{GARI}^{\mathrm{as/as}}$ *of all bialternals is stable under both* gari- *and* gira-*postcomposition by the group* $\mathrm{GARI}^{\underline{\mathrm{as}}/\underline{\mathrm{as}}}$ *of even bisymmetrals, and the identity holds:*

$$\mathrm{gari}(\mathrm{S}_1^\bullet, \mathrm{S}_2^\bullet) \equiv \mathrm{gira}(\mathrm{S}_1^\bullet, \mathrm{S}_2^\bullet) \in \mathrm{as/as} \qquad (\forall \mathrm{S}_1^\bullet \in \mathrm{as/as}\,,\ \forall \mathrm{S}_2^\bullet \in \underline{\mathrm{as}}/\underline{\mathrm{as}}) \tag{331}$$

Proof: Here *gira* stands for the pull-back of *gari* under the basic involution *swap*. Both group laws are related as follows[53]:

$$\mathrm{gira}(S_1^\bullet, S_2^\bullet) = \mathrm{ganit}(\mathrm{rash}.S_2^\bullet).\mathrm{gari}(S_1^\bullet, \mathrm{ras}.S_2^\bullet) \tag{332}$$

with non-linear operators *ras, rash* defined by:

$$\mathrm{ras}.S_2^\bullet \ = \ \mathrm{invgari.swap.invgari.swap}.S_2^\bullet \tag{333}$$
$$\mathrm{rash}.S_2^\bullet \ = \ \mathrm{mu}(\mathrm{push.swap.invmu.swap}.S_2^\bullet, S_2^\bullet) \tag{334}$$

But since in Lemma 8.2 the right factor $S_2^\bullet$ is in $GARI^{\underline{\mathrm{as}}/\underline{\mathrm{as}}}$ and since *gari* and *gira* coincide on $GARI^{\underline{\mathrm{as}}/\underline{\mathrm{as}}}$ (even as *ari* and *ira* coincide on $ARI^{\underline{\mathrm{al}}/\underline{\mathrm{al}}}$), this implies:

$$\mathrm{ras}.S_2^\bullet = \mathrm{invgari.invgira}.S_2^\bullet = S_2^\bullet \tag{335}$$

[52] See §3-§4 *supra*.

[53] see §1-5 *supra* or [E3], §2.3. This universal identity holds for *any* factors $S_1^\bullet, S_2^\bullet$.

Likewise, any bimould of $\underline{\text{as}}/\underline{\text{as}}$ type is automatically *gush*-invariant (even as any bimould of $\underline{\text{al}}/\underline{\text{al}}$ type is automatically *push*-invariant). See [E3], §2.4. This in turn implies:

$$\mathrm{rash}.S_2^\bullet = 1^\bullet \quad \textit{and} \quad \mathrm{ganit}(\mathrm{rash}.S_2^\bullet) = \mathrm{id} \tag{336}$$

and establishes (331). $\square$.

Remark 4. Thus $S_2^\bullet$ is the only factor that really matters when comparing $gari(S_1^\bullet, S_2^\bullet)$ and $gira(S_1^\bullet, S_2^\bullet)$. This is less surprising than may appear at first sight, since the *gari* and *gira* products are linear in the *left* factor and violently non-linear in the *right* factor.

We can now return to the proof of Proposition 8.1. To define our left factor $Sal^\bullet$ we set:

$$\mathrm{Sal}_r^\bullet := \mathfrak{ess}^\bullet \quad \textit{with} \quad -\frac{1}{2}\mathfrak{E}^{w_1} := \frac{1}{2}(\mathrm{Sa}^{w_1} - \mathrm{Sa}^{-w_1}) \tag{337}$$

By the general theory of §3-§4 *supra*, this left factor is not just bisymmetral, but also invariant under *pari.neg*. Let us now address the construction of the right factor $Sar^\bullet$. For each r, we can construct bimould pairs $(Sa_r^\bullet, sar_r^\bullet)$ by the following induction. For $r = 1$ we set:

$$\mathrm{Sa}_1^\bullet \;:=\; \mathrm{Sa}^\bullet \tag{338}$$

$$\mathrm{sar}_1^\bullet \;:=\; \frac{1}{2}(\mathrm{Sa}^{w_1} + \mathrm{Sa}^{-w_1}) \tag{339}$$

and for $r > 1$ we set:

$$\mathrm{Sa}_r^\bullet \;:=\; \mathrm{gari}\big(\mathrm{Sa}^\bullet, \mathrm{expari}(-\mathrm{sar}_1^\bullet), \ldots, \mathrm{expari}(-\mathrm{sar}_{r-1}^\bullet)\big) \tag{340}$$

$$\mathrm{sar}_r^{w_1,\ldots,w_r} \;:=\; \mathrm{Sa}_r^{w_1,\ldots,w_r} - \mathrm{Sal}^{w_1,\ldots,w_r} \tag{341}$$

$$\mathrm{sar}_r^{w_1,\ldots,w_k} \;:=\; 0 \quad \textit{if} \quad k \neq r \tag{342}$$

Clearly:

$$\mathrm{sar}_r^\bullet \in \mathrm{BIMU}_r \quad \textit{and} \quad \mathrm{Sa}_r^\bullet \equiv \mathrm{Sal}^\bullet \quad \mathrm{mod} \;\oplus_{r \le r'} \mathrm{BIMU}_{r'}$$

Let us now check that
(i) each ${Sa_k}^\bullet$ is in $GARI^{\mathrm{as/as}}$;
(ii) each ${sar_k}^\bullet$ is in $ARI^{\underline{\mathrm{as}}/\underline{\mathrm{as}}}$;
(iii) and therefore each $expar(\pm\,\mathrm{sar}_k^\bullet)$ is in $GARI^{\underline{\mathrm{as}}/\underline{\mathrm{as}}}$.

This obviously holds for $k = 1$. If it holds for all $k < r$, then by Lemma 2.1 $Sa_k{}^\bullet$ is also in $GARI^{\mathrm{as/as}}$, as the *gari*-product of a bimould of type as/as by a string of several bimoulds of type $\underline{as}/\underline{as}$. As for $sar_r{}^\bullet$, it is defined as the difference of length-r components of two bisymmetral bimoulds, $\mathrm{Sa}^\bullet_r$ and $\mathrm{Sal}^\bullet$, whose earlier components coincide. It is therefore not just of type al/al (bialternal) but also, by Lemma 7.1 in the preceding section, of type $\underline{al}/\underline{al}$ (bialternal *and* even), and its *ari*-exponential is automatically $\underline{as}/\underline{as}$.

Summing up, we arrive at a factorisation of the announced type (321), with a left factor defined by (337) and a right factor defined by

$$\mathrm{Sar}^\bullet = \lim_{r\to\infty} \mathrm{gari}\big(\mathrm{expari}(\mathrm{sar}^\bullet_r), \ldots, \mathrm{expari}(\mathrm{sar}^\bullet_1)\big) \tag{343}$$

The swappee factorisations (322) immediately follow, again under (332). $\Box$

9 Polar bialternals: first main source.

After our in-depth study of the central but exceptional (i.e. non *neg*-invariant) bisymmetrals, we can now turn to our first instance of regular (i.e. *neg*-invariant) bisymmetrals, and thence to the corresponding (automatically regular) bialternals.

Applying the general results of Proposition 8.1 about the standard factorisation $gari(Sal^\bullet, Sar^\bullet)$ of bisymmetrals and bearing in mind that in the eupolar context the right factor $Sar^\bullet$, due to homogeneousness, is not only *neg*- but also *pari*-invariant, we arrive at the following picture:

$$\begin{array}{ccccc}
\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet & = & \mathrm{gari}(\mathfrak{oss}^\bullet, \mathfrak{so}\ddot{\mathfrak{o}}\mathfrak{s}^\bullet) & = & \mathrm{gari}(\mathfrak{oss}^\bullet, \mathrm{expari}(\mathfrak{lo}\ddot{\mathfrak{o}}\mathfrak{l}^\bullet)) \\
{\scriptstyle swap}\updownarrow & & {\scriptstyle swap}\updownarrow & & {\scriptstyle swap}\updownarrow \\
\mathfrak{ess}^\bullet & = & \mathrm{gari}(\ddot{\mathfrak{e}}\mathfrak{ss}^\bullet, \mathfrak{s}\ddot{\mathfrak{e}}\mathfrak{es}^\bullet) & = & \mathrm{gari}(\ddot{\mathfrak{e}}\mathfrak{ss}^\bullet, \mathrm{expari}(\mathfrak{l}\ddot{\mathfrak{e}}\mathfrak{el}^\bullet)) \\
{\scriptstyle syap}\updownarrow & & {\scriptstyle syap}\updownarrow & & {\scriptstyle syap}\updownarrow \\
\mathfrak{oss}^\bullet & = & \mathrm{gari}(\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet, \mathfrak{s}\ddot{\mathfrak{o}}\mathfrak{os}^\bullet) & = & \mathrm{gari}(\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet, \mathrm{expari}(\mathfrak{l}\ddot{\mathfrak{o}}\mathfrak{ol}^\bullet)) \\
{\scriptstyle swap}\updownarrow & & {\scriptstyle swap}\updownarrow & & {\scriptstyle swap}\updownarrow \\
\ddot{\mathfrak{e}}\mathfrak{ss}^\bullet & = & \mathrm{gari}(\mathfrak{ess}^\bullet, \mathfrak{se}\ddot{\mathfrak{e}}\mathfrak{s}^\bullet) & = & \mathrm{gari}(\mathfrak{ess}^\bullet, \mathrm{expari}(\mathfrak{le}\ddot{\mathfrak{e}}\mathfrak{l}^\bullet))
\end{array}$$

As second *gari*-factors we have here regular bisymmetrals $\mathfrak{se}\ddot{\mathfrak{e}}\mathfrak{s}^\bullet$ etc that are themselves exponentials of regular bialternals $\mathfrak{le}\ddot{\mathfrak{e}}\mathfrak{l}^\bullet$ etc. Both carry only even-length components, with a vanishing length-2 component.[54] Moreover, since the involution *sap* (product of *swap* and *syap*, in whichever order) turns $\mathfrak{se}\ddot{\mathfrak{e}}\mathfrak{s}^\bullet$ and $\mathfrak{so}\ddot{\mathfrak{o}}\mathfrak{s}^\bullet$ into their *gari*-inverses, we clearly have

$$\begin{array}{ccccccc}
\mathrm{sap}.\mathfrak{le}\ddot{\mathfrak{e}}\mathfrak{l}^\bullet & = & -\mathfrak{le}\ddot{\mathfrak{e}}\mathfrak{l}^\bullet & = & \mathfrak{l}\ddot{\mathfrak{e}}\mathfrak{el}^\bullet & = & -\mathrm{sap}.\mathfrak{l}\ddot{\mathfrak{e}}\mathfrak{el}^\bullet \\
\mathrm{sap}.\mathfrak{lo}\ddot{\mathfrak{o}}\mathfrak{l}^\bullet & = & -\mathfrak{lo}\ddot{\mathfrak{o}}\mathfrak{l}^\bullet & = & \mathfrak{l}\ddot{\mathfrak{o}}\mathfrak{ol}^\bullet & = & -\mathrm{sap}.\mathfrak{l}\ddot{\mathfrak{o}}\mathfrak{ol}^\bullet
\end{array}$$

54 See Proposition 3.1.

In the polar specialisation, the picture becomes:

$$\begin{array}{ccccc}
\mathrm{pal}^\bullet & = & \mathrm{gari}(\mathrm{par}^\bullet,\mathrm{ral}^\bullet) & = & \mathrm{gari}(\mathrm{par}^\bullet,\mathrm{expari}(\mathrm{liral}^\bullet))\\
{\scriptstyle swap}\updownarrow & & {\scriptstyle swap}\updownarrow & & {\scriptstyle swap}\updownarrow\\
\mathrm{pil}^\bullet & = & \mathrm{gari}(\mathrm{pir}^\bullet,\mathrm{ril}^\bullet) & = & \mathrm{gari}(\mathrm{pir}^\bullet,\mathrm{expari}(\mathrm{liril}^\bullet))\\
{\scriptstyle syap}\updownarrow & & {\scriptstyle syap}\updownarrow & & {\scriptstyle syap}\updownarrow\\
\mathrm{par}^\bullet & = & \mathrm{gari}(\mathrm{pal}^\bullet,\mathrm{lar}^\bullet) & = & \mathrm{gari}(\mathrm{pal}^\bullet,\mathrm{expari}(\mathrm{lilar}^\bullet))\\
{\scriptstyle swap}\updownarrow & & {\scriptstyle swap}\updownarrow & & {\scriptstyle swap}\updownarrow\\
\mathrm{pir}^\bullet & = & \mathrm{gari}(\mathrm{pil}^\bullet,\mathrm{lir}^\bullet) & = & \mathrm{gari}(\mathrm{pil}^\bullet,\mathrm{expari}(\mathrm{lilir}^\bullet))
\end{array}$$

with

$$\mathrm{gari}(\mathrm{lar}^\bullet,\mathrm{ral}^\bullet)=\mathrm{gari}(\mathrm{lir}^\bullet,\mathrm{ril}^\bullet)=1^\bullet \tag{344}$$

and

$$\mathrm{lilar}^\bullet=-\mathrm{liral}^\bullet \quad ; \quad \mathrm{lilir}^\bullet=-\mathrm{liril}^\bullet \tag{345}$$

To construct our first series of bialternals, we now have the choice between the components of infinitesimal generators such as $\mathit{lilir}^\bullet$ or those of dilators such as $\mathit{dilir}^\bullet$ or $\mathit{diril}^\bullet$. Past experience suggests that the latter are to be preferred, and anyway the three systems $\{\mathit{lilir}^\bullet_{2r}\}$, $\{\mathit{dilir}^\bullet_{2r}\}$, $\{\mathit{diril}^\bullet_{2r}\}$ generate exactly the same bialternal subalgebra of *ARI*.

So, forgetting about $\mathit{lilir}^\bullet$, let us look at the dilators $\mathit{dilir}^\bullet$ and $\mathit{diril}^\bullet$ to decide which is simpler. Starting from the factorisations

$$\mathrm{lir}^\bullet=\mathrm{gari}(\mathrm{ripil}^\bullet,\mathrm{pir}^\bullet) \qquad ; \qquad \mathrm{ril}^\bullet=\mathrm{gari}(\mathrm{ripir}^\bullet,\mathrm{pil}^\bullet) \tag{346}$$

or the more economical factorisations

$$\mathrm{lir}^\bullet=\mathrm{gari}(\mathrm{ripil}^\bullet_{\mathrm{ev}},\mathrm{pir}^\bullet_{\mathrm{ev}}) \qquad ; \qquad \mathrm{ril}^\bullet=\mathrm{gari}(\mathrm{ripir}^\bullet_{\mathrm{ev}},\mathrm{pil}^\bullet_{\mathrm{ev}}) \tag{347}$$

and applying the rule (44) for dilator composition, we find respectively

$$\mathrm{dilir}^\bullet=\mathrm{adari}(\mathrm{ripir}^\bullet).(\mathrm{diripil}^\bullet-\mathrm{diripir}^\bullet) \tag{348}$$
$$\mathrm{diril}^\bullet=\mathrm{adari}(\mathrm{ripil}^\bullet).(\mathrm{diripir}^\bullet-\mathrm{diripil}^\bullet) \tag{349}$$

and

$$\mathrm{dilir}^\bullet=\mathrm{adari}(\mathrm{ripir}^\bullet_{\mathrm{ev}}).(\mathrm{diripil}^\bullet_{\mathrm{ev}}-\mathrm{diripir}^\bullet_{\mathrm{ev}}) \tag{350}$$
$$\mathrm{diril}^\bullet=\mathrm{adari}(\mathrm{ripil}^\bullet_{\mathrm{ev}}).(\mathrm{diripir}^\bullet_{\mathrm{ev}}-\mathrm{diripil}^\bullet_{\mathrm{ev}}) \tag{351}$$

The identities (348) and (349) are unnecessarily wasteful, since they draw on all components, even and odd, of the central bisymmetrals to calculate the components $\mathit{dilir}^\bullet_{2r}$ and $\mathit{diril}^\bullet_{2r}$, all even, of the bialternals. And of the two remaining identities, (351) is better than (350) since it involves, via the *adari* action, the bimould $ripil^\bullet_{\mathrm{ev}}$, which is much simpler than $ripir^\bullet_{\mathrm{ev}}$.[55]

We have thus got hold of our first series of bialternals $\{diril^\bullet_{2r}\,;\, r \geq 2\}$ along with a probably optimal algorithm for their calculation. Indeed, using formula (42) and the key results (153) and (154) of §3, we can make the terms on the right-hand side of (351) wholly explicit. For the bimould part we get an expansion in terms of elementary alternals:

$$\mathrm{diripir}^\bullet_{\mathrm{ev}} - \mathrm{diripil}^\bullet_{\mathrm{ev}} = \sum_{1\geq r} \frac{2^{1-2r}}{(2r-1)\,(2r+1)} (\mathrm{ki}^\bullet_{2r} - \mathrm{ri}^\bullet_{2r})$$

and for the operator part we have an equally simple expansion:

$$\mathrm{adari}(\mathrm{ripil}^\bullet_{\mathrm{ev}}) = \mathrm{id} + \sum \mathrm{Paj}^{2r_1,\ldots,2r_s} \left[\prod_{j=1}^{j=s} \frac{2^{1-2r_j}}{(2r_j-1)(2r_j+1)}\right] \underline{\mathrm{ari}}(\mathrm{ri}^\bullet_{2r_1})\ldots\underline{\mathrm{ari}}(\mathrm{ri}^\bullet_{2r_s})$$

10 Polar bialternals: second main source.

§10-1. Abstract singulators.

To begin with we must recall the construction of the 'abstract' singulator *senk* that to any bisymmetral $\mathfrak{ess}^\bullet$ associates (non-linearly) a linear operator

$$\mathrm{senk}(\mathfrak{ess}^\bullet) = \sum_{1\leq r} \mathrm{senk}_r(\mathfrak{ess}^\bullet) \tag{352}$$

whose 'components' $\mathrm{senk}_r(\mathfrak{ess}^\bullet)$ have the astonishing property of turning any length-1 bimould into a bialternal bimould of length r. That, however, comes at a price: every second time the bialternal so produced is identically 0. More precisely:

$$\mathrm{senk}_{2\mathrm{r}}(\mathfrak{ess}^\bullet) \;:\; \mathrm{BIMU}_1^{\mathrm{even}} \longrightarrow 0^\bullet \tag{353}$$

$$\mathrm{senk}_{2\mathrm{r}}(\mathfrak{ess}^\bullet) \;:\; \mathrm{BIMU}_1^{\mathrm{odd}} \longrightarrow \mathrm{BIMU}_{2\mathrm{r}}^{\underline{\mathrm{al}}/\underline{\mathrm{al}}} \tag{354}$$

$$\mathrm{senk}_{2\mathrm{r}-1}(\mathfrak{ess}^\bullet) \;:\; \mathrm{BIMU}_1^{\mathrm{even}} \longrightarrow \mathrm{BIMU}_{2\mathrm{r}-1}^{\underline{\mathrm{al}}/\underline{\mathrm{al}}} \tag{355}$$

$$\mathrm{senk}_{2\mathrm{r}-1}(\mathfrak{ess}^\bullet) \;:\; \mathrm{BIMU}_1^{\mathrm{odd}} \longrightarrow 0^\bullet \tag{356}$$

[55] In fact, $diril^\bullet$ is not just *simpler to calculate* than $dilir^\bullet$; it is also *simpler* in itself, in its coefficient structure, as can be seen from the extensive tables referred to in §18 and posted on our Webpage.

Before constructing *senk*, let us recall the definition of *mut* (anti-action of *BIMU* on itself) and *adari* (action of *GARI* on *ARI*):

$$\begin{aligned}
\mathrm{mut}(\mathrm{B}^\bullet).\mathrm{A}^\bullet &:= \mathrm{mu}(\mathrm{invmu}(\mathrm{B}^\bullet), \mathrm{A}^\bullet, \mathrm{B}^\bullet) && (357)\\
\mathrm{adari}(\mathrm{B}^\bullet).\mathrm{A}^\bullet &:= \mathrm{logari}(\mathrm{gari}(\mathrm{B}^\bullet, \mathrm{expari}(\mathrm{A}^\bullet), \mathrm{invgari}(\mathrm{B}^\bullet))) && (358)\\
&= \mathrm{gari}(\mathrm{preari}(\mathrm{B}^\bullet, \mathrm{A}^\bullet), \mathrm{invgari}(\mathrm{B}^\bullet)) && (359)
\end{aligned}$$

We also require elementary operators that render any bimould *neg*- or *push*-invariant:

$$\begin{aligned}
\mathrm{neginvar} &:= \mathrm{id} + \mathrm{neg} && (360)\\
\mathrm{pushinvar} &:= \sum_{0\leq r} (\mathrm{id} + \mathrm{push} + \mathrm{push}^2 + ... + \mathrm{push}^{\mathrm{r}}).\mathrm{leng}_{\mathrm{r}} && (361)
\end{aligned}$$

We can now enunciate the two equivalent definitions of *senk* :

$$\begin{aligned}
\mathrm{senk}(\mathfrak{ess}^\bullet).\mathrm{S}^\bullet &:= \frac{1}{2}\,\mathrm{neginvar}.\,\big(\mathrm{adari}(\mathfrak{ess}^\bullet)\big)^{-1}.\,\mathrm{mut}(\mathfrak{es}^\bullet).\,\mathrm{S}^\bullet && (362)\\
&= \frac{1}{2}\,\mathrm{pushinvar}.\mathrm{mut}(\mathrm{neg}.\mathfrak{ess}^\bullet).\mathrm{garit}(\mathfrak{ess}^\bullet).\mathrm{S}^\bullet && (363)
\end{aligned}$$

The 'components' $\mathrm{senk}_{\mathrm{r}}(\mathfrak{ess}^\bullet)$ are of course defined in the only possible way:

$$\mathrm{senk}_{\mathrm{r}}(\mathfrak{ess}^\bullet).\mathrm{S}^\bullet := \mathrm{leng}_{\mathrm{r}}\,.\,\mathrm{senk}(\mathfrak{ess}^\bullet).\,\mathrm{S}^\bullet \qquad (364)$$

with $leng_r$ denoting the natural projection of *BIMU* onto $BIMU_r$.

The magic properties of *senk* result from its remarkable behaviour under the *swap* transform:[56]

$$\begin{aligned}
\mathrm{swap}.\mathrm{senk}(\mathfrak{ess}^\bullet).\mathrm{S}^\bullet &:= \mathrm{senk}(\mathrm{pari}.\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet).\mathrm{swap}.\mathrm{S}^\bullet && (365)\\
\mathrm{swap}.\mathrm{senk}_{\mathrm{r}}(\mathfrak{ess}^\bullet).\mathrm{S}^\bullet &:= (-1)^{r-1}\,\mathrm{senk}_{\mathrm{r}}(\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet).\mathrm{swap}.\mathrm{S}^\bullet && (366)
\end{aligned}$$

§10-2. The polar singulators *slank* and *srank*.

Subsituting $pil^\bullet$ or $pir^\bullet$ for $\mathfrak{ess}^\bullet$ in *senk*, we get two operators *slink* and

[56] The $(-1)^{r-1}$ in (366) is no misprint: the operator $senk_r(\mathfrak{ess}^\bullet)$ involves various products of components $\mathfrak{ess}^\bullet_{r_i}$ and for each such product the total length $\sum r_i$ is $r-1$, not r.

srink:[57]

$$\mathrm{slink.S}^\bullet \;:=\; \frac{1}{2}\,\mathrm{neginvar}\,.\,\big(\mathrm{adari}(\mathrm{pil}^\bullet)\big)^{-1}.\,\mathrm{mut}(\mathrm{pil}^\bullet).\,\mathrm{S}^\bullet \tag{367}$$

$$\;=\; \frac{1}{2}\,\mathrm{pushinvar}\,.\,\mathrm{mut}(\mathrm{neg.pil}^\bullet)\,.\,\mathrm{garit}(\mathrm{pil}^\bullet)\,.\,\mathrm{S}^\bullet \tag{368}$$

$$\mathrm{srink.S}^\bullet \;:=\; \frac{1}{2}\,\mathrm{neginvar}\,.\,\big(\mathrm{adari}(\mathrm{pir}^\bullet)\big)^{-1}.\,\mathrm{mut}(\mathrm{pir}^\bullet).\,\mathrm{S}^\bullet \tag{369}$$

$$\;=\; \frac{1}{2}\,\mathrm{pushinvar}\,.\,\mathrm{mut}(\mathrm{neg.pir}^\bullet)\,.\,\mathrm{garit}(\mathrm{pir}^\bullet)\,.\,\mathrm{S}^\bullet \tag{370}$$

whose 'components' $slink_r$ and $srink_r$ turn *arbitrary, entire-valued* length-1 bimoulds into *bialternal, singular-valued* length-r bimoulds. This property makes $slink_r$ and $srink_r$ extremely useful in multizeta algebra, in the back-and-forth known as *singularisation-desingularisation.*

§10-3. The second series of bialternals.

Our aim here, however, is different: we want to produce eupolar bialternals, i.e. bialternal elements of $Flex_r(Pi)$. Here, the 'singuland' (i.e. that on which the singulator acts) can only be $Pi^\bullet$, and so, in view of (353)-(356), the 'singulate' (i.e. the bialternal fruit of the operation) *can* and in fact *will* be nonzero only in the situation (354). So we have no choice but to set

$$\mathrm{visli}^\bullet_{2r} := \mathrm{slink_{2r}.Pi}^\bullet \tag{371}$$

$$\mathrm{visri}^\bullet_{2r} := \mathrm{srink_{2r}.Pi}^\bullet \tag{372}$$

§10-4. Relations between the two series of bialternals.

Like with the two *equivalent* systems $\{diril^\bullet_{2r}\}$ and $\{dilir^\bullet_{2r}\}$ of the preceding section, it is easy to show that the new systems $\{visli^\bullet_{2r}\}$ and $\{visri^\bullet_{2r}\}$ are also *equivalent*, in the sense of generating one and the same bialternal subalgebra of *ARI*. So we shall retain only $\{visli^\bullet_{2r}\}$, since it can be shown to be simpler than $\{visri^\bullet_{2r}\}$, much as $\{diril^\bullet_{2r}\}$ was simpler than $\{dilir^\bullet_{2r}\}$.

The only questions left are these:

(i) how do the systems $\{diril^\bullet_{2r}\}$ and $\{visli^\bullet_{2r}\}$ compare?

(ii) do they, together, generate all eupolar bialternals?

The answer to the second question is probably *no*, but this is no more than a hunch. The answer to the first question is not clear either: up to length

[57] In view of (365), subsituting $pal^\bullet$ or $par^\bullet$ for $\mathfrak{ess}^\bullet$ in *senk* would produce nothing new. It would just yield (up to sign) the *swap* transforms of *slink* and *srink*.

10, the two systems are equivalent; at length 12 they produce a distinct generator each; but at length 14 they do not. And what happens thereafter is anybody's guess.

11 Polar algebra and subalgebras.

Warning: from here on the exposition becomes less systematic and the paper takes a more exploratory turn. It mixes proof-backed statements, conjectures, and mere 'observed facts', while making clear in each case which is which.

The six main subspaces of $Flex(\mathfrak{E})$ are:[58]

$\mathrm{Flex}^{\mathrm{sap}}(\mathfrak{E})$, *consisting of all* sap-*invariant bimoulds.*
$\mathrm{Flex}^{\overline{\mathrm{pus}}}(\mathfrak{E})$, *consisting of all* pus-*variant bimoulds.*
$\mathrm{Flex}^{\mathrm{push}}(\mathfrak{E})$, *consisting of all* p*ush-invariant bimoulds.*
$\mathrm{Flex}^{\mathrm{al}}(\mathfrak{E})$, *consisting of all alternal bimoulds.*
$\mathrm{Flex}^{\mathrm{al/push}}(\mathfrak{E})$, *consisting of all alternal and* p*ush-invariant bimoulds.*
$\mathrm{Flex}^{\underline{\mathrm{al}}/\underline{\mathrm{al}}}(\mathfrak{E})$, *consisting of all bialternal bimoulds.*

All these subspaces except the first (*sap*-invariants) are stable under *ari* and define as many subalgebras. On the other hand, only the fourth (alternals) is stable under *lu*. This again shows how much more flexible, versatile and interesting the flexion operations are. Remarkably, neither the *pus*-invariant subspace $Flex_r^{pus}$ nor the *push*-variant subspace $Flex_r^{\overline{push}}$ are stable under *ari*, let alone *lu*.[59]

Here is a table with the dimensions, up to $r = 14$, of the length-r com-

[58]Recall that $sap := swap.syap = syap.swap$ and that a bimould $A^\bullet$ in $BIMU_r$ is said to be *pus*-variant iff $(id + pus + pus^2 + ...pus^{r-1}).A^\bullet = 0$.

[59]This underscores the 'complementarity' between *pus* (a circular permutation of order r in the *short* notation) and *push* (a circular permutation of order r in the *long* notation).

ponents of these subspaces or subalgebras.

r	Flex_r	$\mathrm{Flex}_r^{\mathrm{sap}}$	$\mathrm{Flex}_r^{\overline{\mathrm{pus}}}$	$\mathrm{Flex}_r^{\mathrm{push}}$	$\mathrm{Flex}_r^{\mathrm{al}}$	$\mathrm{Flex}_r^{\mathrm{al/push}}$	$\mathrm{Flex}_r^{\underline{\mathrm{al}}/\underline{\mathrm{al}}}$
1	1	1	0	0	1	0	0
2	2	1	1	0	1	0	0
3	5	3	3	0	2	0	0
4	14	7	9	2	4	1	1
5	42	22	28	4	9	1	0
6	132	66	90	18	20	4	1
7	429	217	297	48	48	7	0
8	1430	715	1001	156	115	17	1
9	4862	2438	3432	472	286	36	0
10	16796	8398	11934	1526	719	88	2
11	58786	29414	41990	4852	1842	196	0
12	208012	104006	149226	16000	4766	481	≥ 3
13	742900	371516	534888	52940	12486	1148	0
14	2674440	1337220	1931540	178276	32973	2838	≥ 3

All these dimensions have remarkable combinatorial interpretations, mostly in terms of special trees with r or $r-1$ nodes.

- $\dim(\mathit{Flex}_r(\mathfrak{E})) = \frac{(2r)!}{r!\,(r+1)!}$. For two distinct interpretations and the corresponding *bases*, see Remark 1 below.

- $\dim(\mathit{Flex}_r^{\mathit{sap}}(\mathfrak{E})) = \frac{1}{2}\dim(\mathit{Flex}_r^{\mathit{sap}})$ resp. $= \frac{1}{2}\dim(\mathit{Flex}_r) + \dim(\mathit{Flex}_{(r-1)/2})$ if r is *even* resp. *odd*.

- $\dim(\mathrm{Flex}_r^{\overline{\mathrm{pus}}}(\mathfrak{E})) = \frac{3\,(2\,r-2)!}{(r+1)!\,(r-2)!}$. The sequence occurs in the *Online Encyclopedia of Integer Sequences* under *A*000245 with a number of combinatorial interpretations.

- $\dim(\mathrm{Flex}_r^{\mathrm{push}}(\mathfrak{E})) = 2\,\frac{(2\,r)!}{r!\,(r+1)!} - \frac{1}{2\,r+2}\sum_{d\,|\,r+1}\phi(d)\,\frac{((2\,r+2)/d)!}{((r+1)/d)!\,((r+1)/d)!}$. This formula is due to F. Chapoton, who used it to solve a different problem, but with a combinatorial interpretation easily translatable into ours. See [Ch] or item A106520 in the *Online Encyclopedia of Integer Sequences.*

- $\dim(\mathrm{Flex}_r^{\mathrm{al}}(\mathfrak{E}))$ = number $\beta(r)$ of non-ordered[60] rooted trees with r nodes.[61] For numerous alternative interpretations and formulae for inductive calculation, see A000081 in the *Online Encyclopedia of Integer Sequences.* Thus, the generating series $B(x) := \sum_{0<r}\beta(r)\,x^r$ verifies

[60]The relative position of the various branches issueing from a given node is indifferent.
[61]counting the root as a node.

$B(x) = x\exp\big(\sum_{1\leq k}\frac{1}{k}B(x^k)\big)$. For a combinatorial interpretation directly related to our problem, see Remark 2 below.

- $\dim(\mathrm{Flex}_r^{\mathrm{al/push}}(\mathfrak{E}))$. Though there is no known closed formula, this again appears to coincide with a sequence investigated by F. Chapoton (see A098091 in the *Online Encyclopedia of Integer Sequences*) but with a combinatorial interpretation[62] that doesn't make the connection obvious.

- $\dim(\mathrm{Flex}_r^{\underline{\mathrm{al}}/\underline{\mathrm{al}}}(\mathfrak{E}))$ = unknown at the moment for $r \geq 16$. See §10.4.

Remark 1: Bases of $\mathit{Flex}_r(\mathfrak{E})$**.**

As is well known, the Catalan numbers $\dim(\mathit{Flex}_r(\mathfrak{E})) = \frac{(2r)!}{r!\,(r+1)!}$ are capable of two main tree-theoretic interpretations:
(i) as counting the binary trees with r-nodes
(ii) as counting the ordered trees[63] with r-nodes.[64]
There exists a basis $\{\mathfrak{e}_{\boldsymbol{t}}^{\bullet}\}$ naturally indexed by the binary trees $\boldsymbol{t}$: see §1-6. There also exists two bases $\{\mathfrak{em}_{\boldsymbol{t}}^{\bullet}\}$ and $\{\mathfrak{en}_{\boldsymbol{t}}^{\bullet}\}$ indexed by the ordered trees of the second interpretation. Indeed, let $\boldsymbol{t}$ be a s-rooted tree consisting of an ordered system of s one-rooted trees $\boldsymbol{t_j}$; and let $\boldsymbol{t_*}$ be the one-rooted tree that results from attaching each $\boldsymbol{t_j}$ to a common root.[65] The inductive definition then reads:

$$\begin{array}{rclcrcl} \mathfrak{em}_{\boldsymbol{t}}^{\bullet} & := & \mathrm{mu}(\mathfrak{em}_{\boldsymbol{t_1}}^{\bullet},\ldots,\mathfrak{em}_{\boldsymbol{t_s}}^{\bullet}) & ; & \mathfrak{em}_{\boldsymbol{t_*}}^{\bullet} & := & \mathrm{amit}(\mathfrak{em}_{\boldsymbol{t}}^{\bullet}).\mathfrak{E}^{\bullet} \\ \mathfrak{en}_{\boldsymbol{t}}^{\bullet} & := & \mathrm{mu}(\mathfrak{en}_{\boldsymbol{t_1}}^{\bullet},\ldots,\mathfrak{en}_{\boldsymbol{t_s}}^{\bullet}) & ; & \mathfrak{en}_{\boldsymbol{t_*}}^{\bullet} & := & \mathrm{anit}(\mathfrak{en}_{\boldsymbol{t}}^{\bullet}).\mathfrak{E}^{\bullet} \end{array}$$

starting of course from $\mathfrak{em}_{\boldsymbol{t_0}}^{\bullet} = \mathfrak{em}_{\boldsymbol{t_0}}^{\bullet} := \mathfrak{E}^{\bullet}$ for the one-node, one-root tree $\boldsymbol{t_0}$. The two systems $\{\mathfrak{em}_{\boldsymbol{t}}^{\bullet};\, \mathit{nodes}(\boldsymbol{t}) = r\}$ and $\{\mathfrak{en}_{\boldsymbol{t}}^{\bullet};\, \mathit{nodes}(\boldsymbol{t}) = r\}$ are each a basis[66] of $\mathit{Flex}_r(\mathfrak{E})$. However, the system $\{\mathfrak{er}_{\boldsymbol{t}}^{\bullet};\, \mathit{nodes}(\boldsymbol{t}) = r\}$ similarly constructed but with *arit* in place of *amit* or *anit* defines no basis.[67] Worse still, $\mathit{Flex}(\mathfrak{E})$ cannot be generated from $\mathfrak{E}^{\bullet}$ under repeated use of the sole operations *lu* and arit (much less under *lu* and ari).

[62] According to F. Chapotion, these are the *graded dimensions of the spaces of invariant bilinear forms on the free pre-Lie algebra on one generator.*

[63] Several branches may issue from one and the same node, and their planar disposition, from left to right, matters.

[64] Several roots are allowed in these *"trees"*. Some speak of *bushes* or *forests* instead.

[65] distinct from the original roots of each $\boldsymbol{t_j}$.

[66] Note that the systems $\{\mathfrak{em}_{\boldsymbol{t}}^{\bullet}\}$ and $\{\mathfrak{en}_{\boldsymbol{t}}^{\bullet}\}$ are quite distinct from the similar-looking systems in (**??**). The latter span much smaller subspaces.

[67] There appear linear dependence relations between the $\mathfrak{er}_{\boldsymbol{t}}^{\bullet}$ as soon as $r = 5$.

Remark 2: Basis of $Flex_r^{al}(\mathfrak{E})$.

Let $\boldsymbol{\theta} := \{\overline{\boldsymbol{\theta_1}, \ldots, \boldsymbol{\theta_s}}\}$ be the unordered rooted tree obtained by attaching s unordered rooted trees $\boldsymbol{\theta_j}$ to a common root. Then the inductive rule[68]:

$$\mathfrak{err}^\bullet_{\boldsymbol{\theta}} := \sum_{\sigma \in \mathcal{S}_s} \overset{\rightarrow}{\mathrm{lu}} \left(\mathrm{arit}(\mathfrak{err}^\bullet_{\boldsymbol{\theta}_{\sigma(1)}}).\mathfrak{E}^\bullet, \mathfrak{err}^\bullet_{\boldsymbol{\theta}_{\sigma(2)}}, \ldots, \mathfrak{err}^\bullet_{\boldsymbol{\theta}_{\sigma(s)}}\right) \tag{373}$$

produces, for each r, a system $\{\mathfrak{err}^\bullet_{\boldsymbol{\theta}};\, nodes(\boldsymbol{\theta}) = r\}$ consisting of bimoulds that are alternal of length r (obvious); have the right indexation and so too the right cardinality (obvious); are linearly independent (non obvious); and therefore constitute a basis of $Flex_r^{al}(\mathfrak{E})$. This is a rather unusual situation, given that most free Lie algebras[69] possess no privileged natural basis.

12 Interplay of the *lu* and *ari* structures.

(i) As *lu*-algebras, both $Flex^{al}(\mathfrak{E})$ and $Flex(\mathfrak{E})$ are freely generated by a well-defined number of *prime generators* $\mathfrak{ge}^\bullet_{r,i}$ taken in each component space $Flex_r^{al}(\mathfrak{E})$ or $Flex_r(\mathfrak{E})$.

(ii) As *ari*-algebras, both $Flex^{al}(\mathfrak{E})$ and $Flex(\mathfrak{E})$ decompose as

$$\mathrm{Flex}^{\mathrm{al}}(\mathfrak{E}) \;=\; \mathrm{Flex}^{\mathrm{al}}(\mathfrak{re}) \oplus \mathrm{Flex}^{\mathrm{al}}_{\mathrm{free}}(\mathfrak{E}) \tag{374}$$

$$\mathrm{Flex}(\mathfrak{E}) \;=\; \mathrm{Flex}^{\mathrm{al}}(\mathfrak{re}) \oplus \mathrm{Flex}_{\mathrm{free}}(\mathfrak{E}) \tag{375}$$

The elementary subalgebra $Flex^{al}(\mathfrak{re})$ is generated (and spanned) by the self-reproducing alternals $\mathfrak{re}^\bullet_r$. All its components $Flex_r^{al}(\mathfrak{re})$ are one-dimensional. The algebra $Flex^{al}_{free}(\mathfrak{E})$ resp. $Flex_{free}(\mathfrak{E})$ is freely generated by a well-defined number of *primary generators* $\mathfrak{fe}^\bullet_{r,i}$ taken in each $Flex_r^{al}(\mathfrak{E})$ resp. $Flex_r(\mathfrak{E})$, and supplemented by *secondary generators* of the form

$$\overset{\rightarrow}{ari}\,(\mathfrak{fe}^\bullet_{r_0}, \mathfrak{re}^\bullet_{r_1}, \ldots, \mathfrak{re}^\bullet_{r_s}) \quad with \quad r_0{+}r_1{+}\ldots r_s = r \tag{376}$$

with only non-increasing (or non-decreasing, if one so prefers[70]) integer sequences $(r_1, \ldots, r_s)$.

[68] As usual, we get the induction started by setting $\mathfrak{err}^\bullet_{\boldsymbol{\theta_0}} := \mathfrak{E}^\bullet$ for the one-node one-root tree $\boldsymbol{\theta_0}$.

[69] As a *lu*-algebra, $Flex^{al}(\mathfrak{E})$ is free, and very nearly free as an *ari*-algebra. See §12.

[70] Working out the conversion rules between the two systems (376) that correspond to non-increasing or non-decreasing sequences, and finding a compact expression for these rules, is a wholesome exercise on moulds.

The following table carries for each length-r component of $\mathrm{Flex}^{\mathrm{al}}_{\mathrm{free}}(\mathfrak{E})$ resp. $\mathrm{Flex}_{\mathrm{free}}(\mathfrak{E})$:
(i) the total dimension δ_r resp. d_r
(ii) the number δ^*_r resp. d^*_r of primary generators
(iii) the number δ^{**}_r resp. d^{**}_r of *all* generators (primary and secondary)

	$\mathrm{Flex}^{\mathrm{al}}_{\mathrm{r}}$	$\mathrm{Flex}^{\mathrm{al}}_{\mathrm{r}}$	$\mathrm{Flex}^{\mathrm{al}}_{\mathrm{r}}$	$\mathrm{Flex}_{\mathrm{r}}$	$\mathrm{Flex}_{\mathrm{r}}$	$\mathrm{Flex}_{\mathrm{r}}$
r	δ_r	δ^*_r	δ^{**}_r	d_r	d^*_r	d^{**}_r
	...	...	...	...	...	...
1	1	0	0	1	0	0
2	1	0	0	2	1	1
3	2	1	1	5	3	4
4	4	2	3	14	8	13
5	9	4	8	42	20	37
6	20	8	19	132	62	112
7	48	17	44	429	187	335
8	115	41	103	1430	619	1062
9	286	98	242	4862	2049	3432
10	719	250	586	16796	6998	11451
11	1842	631	1437	58786	24186	38944
12	4766	1645	3616	208012	84673	134696
13	12486	4285	9216	742900	299445	471911
14	32973	11338	23884	2674440	1065675	1668516

13 Alternal codegrees and alternality grids.

§13-1. Loose and strict alternality codegrees.

A bimould $A^\bullet \in BIMU_r$ is said to have *loose* alternality codegree d if the identity[71]

$$\sum_{\boldsymbol{w}\,\in\,\mathrm{sha}(\boldsymbol{w^1},\ldots,\boldsymbol{w^{d+1}})} A^{\boldsymbol{w}} = 0 \qquad (\forall \boldsymbol{w}, \forall \boldsymbol{w^i} \neq \emptyset) \tag{377}$$

holds for all systems $\{\boldsymbol{w^1},\ldots,\boldsymbol{w^{d+1}}\}$, and it is said to have *strict* alternality codegree d if the identity does not always hold for $d-1$. Alternality in the

[71] recall that $sha(\boldsymbol{w^1},\ldots,\boldsymbol{w^{d+1}})$ denotes the set of all $\boldsymbol{w}$ that result from *shuffling* the various $\boldsymbol{w^i}$.

usual sense corresponds to $d = 1$. We speak here of *codegrees* rather than *degrees*, because the notion is clearly dual to that of 'differential' degree.[72]

The (strict) codegree behaves additively under 'products' such as *mu* or *preari*, but with a unit drop in the case of 'brackets' like *lu* or *ari*:

$$
\begin{array}{lll}
C^\bullet = \mathrm{mu}(A^\bullet, B^\bullet) & \Longrightarrow & \mathrm{codeg}^{al}(C^\bullet) = \mathrm{codeg}^{al}(A^\bullet) + \mathrm{codeg}^{al}(B^\bullet) \\
C^\bullet = \mathrm{preari}(A^\bullet, B^\bullet) & \Longrightarrow & \mathrm{codeg}^{al}(C^\bullet) = \mathrm{codeg}^{al}(A^\bullet) + \mathrm{codeg}^{al}(B^\bullet) \\
C^\bullet = \mathrm{lu}(A^\bullet, B^\bullet) & \Longrightarrow & \mathrm{codeg}^{al}(C^\bullet) \leq \mathrm{codeg}^{al}(A^\bullet) + \mathrm{codeg}^{al}(B^\bullet) - 1 \\
C^\bullet = \mathrm{ari}(A^\bullet, B^\bullet) & \Longrightarrow & \mathrm{codeg}^{al}(C^\bullet) \leq \mathrm{codeg}^{al}(A^\bullet) + \mathrm{codeg}^{al}(B^\bullet) - 1
\end{array}
$$

§13-2. Filtration of $Flex_r(\mathfrak{E})$.

Consider the filtration

$$Flex_r(\mathfrak{E}) = Flex_r^{(r)}(\mathfrak{E}) \supset Flex_r^{(r-1)}(\mathfrak{E}) \supset \ldots Flex_r^{(2)}(\mathfrak{E}) \supset Flex_r^{(1)}(\mathfrak{E})$$

of $Flex_r(\mathfrak{E})$ into subspaces $Flex_r^{(d)}(\mathfrak{E})$ consisting of all elements of (loose) alternal codegree d. The following (incomplete) table mentions, for each r, the dimensions al_r^d of the corresponding gradation:

$$\mathrm{al}_r^d := \mathrm{Al}_r^d - \mathrm{Al}_r^{d-1} \quad \textit{with} \quad \mathrm{Al}_r^d := \dim(Flex_r^{(d)}(\mathfrak{E}))$$

	d	1	2	3	4	5	6	7	8
r	*total*	...	...	...	...	...	...	...	...
1	1	1							
2	2	1	1						
3	5	2	2	1					
4	14	4	6	3	1				
5	42	9	16	12	4	1			
6	132	20	47	39	20	5	1		
7	429	48	127	141	76	30	6	1	
8	1430	115	?	?	?	130	42	7	1

$$
\begin{array}{lll}
\mathrm{al}_r^{r-0} & = & 1 \\
\mathrm{al}_r^{r-1} & = & r-1 \\
\mathrm{al}_r^{r-2} & = & (r-2)(r-1) \\
\mathrm{al}_r^{r-3} & = & \frac{1}{2}(r-3)(r^2-r-4) \\
\mathrm{al}_r^{r-4} & = & (r-4)\ldots
\end{array}
$$

[72]Think of mould-comould contractions $\sum A^{w_1,\ldots,w_r}\Delta_{w_r}...\Delta_{w_1}$, with inputs Δ_{w_i} freely generating a Lie algebra. Besides, as d increases, $A^\bullet$ becomes 'less alternal', not more. So it would be jarring to speak of alternality *degree* here.

. . . .

8	7	6	5	4	3	2	1	r	1	2	3	4	5	6	7	8
....	....	....	....	....	....	....	....		...	...	...	...	...	...	...	...
							1	$1^{\pm}$	0							
							1	1^{+}	0							
							0	1^{-}	0							
						2	0	$1^{\pm}$	0	0						
						1	0	2^{+}	0	0						
						1	0	2^{-}	0	0						
					2	3	0	$3^{\pm}$	0	0	0					
					1	2	0	3^{+}	0	0	0					
					1	1	0	3^{-}	0	0	0					
				2	6	5	1	$4^{\pm}$	1	1	0	0				
				1	3	3	0	4^{+}	0	1	0	0				
				1	3	2	1	4^{-}	1	0	0	0				
			2	8	23	9	0	$5^{\pm}$	0	2	2	0	0			
			1	4	12	5	0	5^{+}	0	1	1	0	0			
			1	4	11	4	0	5^{-}	0	1	1	0	0			
		2	10	40	68	17	1	$6^{\pm}$	1	5	8	4	0	0		
		1	5	20	32	8	0	6^{+}	0	2	5	2	0	0		
		1	5	20	30	9	1	6^{-}	1	3	3	2	0	0		
	2	12	60	154	186	15	0	$7^{\pm}$	0	4	24	16	4	0	0	
	1	6	30	77	96	7	0	7^{+}	0					0	0	
	1	6	30	77	90	8	0	7^{-}	0					0	0	
2	14	84					1	$8^{\pm}$	1						0	0
1	14	42					0	8^{+}	0						0	0
1	14	42					1	8^{-}	1						0	0

14 Bialternal codegrees and bialternality grids.

§14-1. Bialternal codegree.

The bialternality codegree (*loose* or *strict*) of a bimould is simply its alternality codegree paired with that of its swappee:

$$\mathrm{codeg}^{bial}(A^\bullet) := \big(\mathrm{codeg}^{al}(A^\bullet)\,,\,\mathrm{codeg}^{al}(\mathrm{swap}.A^\bullet)\big) \tag{378}$$

Ordinary bialternality corresponds to codegree (1,1).

We cannot expect the bialternality codegree (or rather its second component) to behave in anything like a predictable manner under *mu* and *lu* nor indeed under *preari* and *ari*, but there an important exception, namely on the subalgebra of *push*-invariant elements[73], where *swap* commutes with *preari* and *ari*. So for *push*-invariant bimoulds we have:

$$C^\bullet = \mathrm{preari}(A^\bullet, B^\bullet) \Longrightarrow \mathrm{codeg}^{bial}(C^\bullet) = \mathrm{codeg}^{bial}(A^\bullet) + \mathrm{codeg}^{bial}(B^\bullet)$$

$$C^\bullet = \mathrm{ari}(A^\bullet, B^\bullet) \Longrightarrow \mathrm{codeg}^{bial}(C^\bullet) \leq \mathrm{codeg}^{bial}(A^\bullet) + \mathrm{codeg}^{bial}(B^\bullet) - (1,1)$$

Here again we have a filtration of $Flex_r(\mathfrak{E})$ into increasing subspaces $Flex_r^{(d_1,d_2)}(\mathfrak{E})$ with the corresponding dimensions

$$Bial_r^{d_1,d_2} := \dim(Flex_r^{(d_1,d_2)}(\mathfrak{E})) \tag{379}$$

and the even more relevant differences

$$bial_r^{d_1,d_2} := Bial_r^{d_1,d_2} - Bial_r^{d_1-1,d_2} - Bial_r^{d_1,d_2-1} + Bial_r^{d_1-1,d_2-1} \tag{380}$$

which serve as entries of the so-called *bialternality grid*.

In fact, we have two such grids: one for the whole of $Flex_r(\mathfrak{E})$ and one for the *push*-invariant subalgebra $Flex_r^{push}(\mathfrak{E})$. The second grid, also called *bialternality chessboard*, is the more important of the two, but in this 'monogenous' or 'eupolar' context both are equally interesting. In particular, both are symmetrical with respect to the main diagonal. This is due to the existence of a second involution *syap*, specific to this case.

But when we leave the 'eupolar' context and move on for example to the important case of polynomial-valued bimoulds, we still have (highly interesting) bialternality grids and chessboards but there is no *syap* anymore and so the property of diagonal symmetry disappears, though traces of it remain.

§14-2. The bialternality grid for general eupolars.

Here are the cases that proved amenable to computation:

$$\begin{array}{c|cc} 2 & 1 & 0 \\ 1 & 0 & 1 \\ \hline & 1 & 2 \end{array} \qquad \begin{array}{c|ccc} 3 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ \hline & 1 & 2 & 3 \end{array}$$

[73] which, remember, contains all bialternals!

4	1	0	0	0
3	2	1	0	0
2	0	5	1	0
1	1	0	2	1
	1	2	3	4

5	1	0	0	0	0
4	4	0	0	0	0
3	1	10	1	0	0
2	3	3	10	0	0
1	0	3	1	4	1
	1	2	3	4	5

6	1	0	0	0	0	0
5	5	0	0	0	0	0
4	4	16	0	0	0	0
3	9	14	16	0	0	0
2	0	17	14	16	0	0
1	1	0	9	4	5	1
	1	2	3	4	5	6

7	1	0	0	0	0	0	0
6	6	0	0	0	0	0	0
5	11	19	0	0	0	0	0
4	24	34	19	0	0	0	0
3	1	64	56	19	0	0	0
2	5	5	64	34	19	0	0
1	0	5	1	24	11	6	1
	1	2	3	4	5	6	7

8	1	0	0	0	0	0	0	0
7	7	0	0	0	0	0	0	0
6	?	?	0	0	0	0	0	0
5	?	?	?	0	0	0	0	0
4	?	?	?	?	0	0	0	0
3	?	?	?	?	?	0	0	0
2	?	?	?	?	?	?	0	0
1	1	?	?	?	?	?	7	1
	1	2	3	4	5	6	7	8

Two features stand out here: strict diagonal symmetry as well as the vanishing of all entries in the north-west triangles. Both are eupolar-specific phenomena, although as *tendencies* both extend, in a much weakened form, to the case of polynomial-valued bimoulds.

§14-3. The bialternality chessboard for *push*-invariant eupolars.

For $r < 4$ all entries are 0. For $4 \leq r \leq 8$, we get:

4	0	0	0	0
3	0	0	0	0
2	0	1	0	0
1	1	0	0	0
	1	2	3	4

5	0	0	0	0	0
4	0	0	0	0	0
3	0	1	0	0	0
2	1	0	1	0	0
1	0	1	0	0	0
	1	2	3	4	5

6	0	0	0	0	0	0
5	0	0	0	0	0	0
4	0	2	0	0	0	0
3	3	0	2	0	0	0
2	0	5	0	2	0	0
1	1	0	3	0	0	0
	1	2	3	4	5	6

	0	0	0	0	0	0	0
	0	0	0	0	0	0	0
	0	2	0	0	0	0	0
	5	0	3	0	0	0	0
	0	12	0	3	0	0	0
	2	0	12	0	2	0	0
	0	2	0	5	0	0	0
	1	2	3	4	5	6	7

8	0	0	0	0	0	0	0	0
7	?	0	0	0	0	0	0	0
6	0	?	0	0	0	0	0	0
5	?	0	?	0	0	0	0	0
4	0	?	0	?	0	0	0	0
3	?	0	?	0	?	0	0	0
2	0	?	0	?	0	?	0	0
1	1	0	?	0	?	0	?	0
	1	2	3	4	5	6	7	8

We observe the vanishing of all entries on the diagonals of equation $d_1 - d_2 - r = odd$ or, what amounts to the same, on the anti-diagonals $r - d_1 - d_2 = odd$. The phenomenon, this time, is not eupolar-specific but quite general and a direct consequence of *push*-invariance. The reasons behind it are explained in the next section, which is devoted to the case of polynomial-valued bimoulds.

15 Introduction to the polynomial chessboard.

The next two section venture beyond the *eupolar* into the *polynomial* and *eutrigonometric* domains, but unsystematically so, mainly with a view to showing which aspects of the eupolar situation survive and which do not. Our first prerequisite for the present survey of the *polynomial* case shall be a series of projectors $altor_{r,j}$ that sharpen the natural *filtration* by the (loose) alternality codegree j into a *gradation* by the (strict) alternality codegree; and our second prerequisite shall be an $\boldsymbol{u}/\boldsymbol{v}$ exchanging involution *srap* capable of taking over *some* of the functions performed by the involution *syap* in the eupolar case.

§15-1. Standard alternality projectors ('alternators').

For each $j \in \{1, ..., r\}$ there exists a unique projector $altor_{r,j}$ that turns any $M^\bullet \in BIMU_r$ into a bimould of (strict) alternality codegree j and enjoys the property that for any symmetral $S^\bullet \in BIMU_r$ the identity holds:

$$\mathrm{altor}_{r,j}.S^\bullet \equiv \frac{1}{j!}\,\mathrm{mu}_j(\mathrm{logmu}.S^\bullet) \equiv \sum_{j\leq n\leq r} (-1)^{n-j}\, s_1(n,j)\,\mathrm{mu}_n(S^\bullet) \tag{381}$$

with $mu_j(S^\bullet)$ standing for the j^{th} *mu*-power of $S^\bullet$ and $s_1(n,j)$ denoting the (signless) Stirling numbers of the first kind:

$$x\,(x+1)\,\ldots\,(x+n-1) \equiv \sum_{0\leq j\leq n} s_1(n,j)\,x^j \tag{382}$$

Analytically, $altor_{r,j}$ is given as a superposition of *permutators*:

$$(\mathrm{altor}_{r,j}.M)^{w_1,...,w_r} = \sum_{\sigma\in\mathfrak{S}_r} \lambda_j^\sigma \;\; M^{w_{\sigma(1)},...,w_{\sigma(r)}} \tag{383}$$

with coefficients $\lambda_j^\sigma = \tilde{\lambda}_j^\sigma/r!$ ($\tilde{\lambda}_j^\sigma \in \mathbb{Z}$) that are easily calculated by
(i) changing $S^\bullet$ to $M^\bullet$ in (381)
(ii) collecting all products $\prod M^{\boldsymbol{w^i}}$ on the right-most side of (381)
(iii) formally subjecting these products to *symmetral linearisation*

$$M^{\boldsymbol{w^1}}\,M^{\boldsymbol{w^2}}\,\ldots\,M^{\boldsymbol{w^s}} \quad\mapsto \sum_{\boldsymbol{w}\,\in\,\mathrm{sha}(\boldsymbol{w^1},\boldsymbol{w^2},...,\boldsymbol{w^s})} M^{\boldsymbol{w}} \tag{384}$$

despite $M^\bullet$ being an arbitrary (not necessarily symmetral) bimould.

Although these projectors $altor_{r,j}$ are not the most 'economical' as far as the number of permutators involved is concerned[74], they have the advantage of being *complementary*

$$\sum_{1\leq j\leq r} \mathrm{altor}_{r,j} = \mathrm{id}_{BIMU_r} \qquad ; \qquad \mathrm{altor}_{r,i}\,.\,\mathrm{altor}_{r,j} = 0 \;(\forall i \neq j) \tag{385}$$

[74]Thus, the most economical projectors onto the subspace of *alternals* involve only 2^{r-1} permutators.

and the further advantage, crucial for the sequel, of commuting not only with *anti* and one another, but also with the natural 'projector' $pushinvar_r := \sum_{0\le k\le r} push^k$ of $BIMU_r$ onto the subspace of *push*-invariant bimoulds:[75]

$$\mathrm{altor}_{r,j}\,.\,\mathrm{anti} \;=\; \mathrm{anti}\,.\,\mathrm{altor}_{r,j} \;=\; (-1)^{r+j}\,\mathrm{altor}_{r,j} \tag{386}$$

$$\mathrm{altor}_{r,j}\,.\,\mathrm{pushinvar}_r \;=\; \mathrm{pushinvar}_r\,.\,\mathrm{altor}_{r,j} \tag{387}$$

We next tabulate the entire coefficients $\tilde\lambda^\sigma_j := r!\,\lambda^\sigma_j$ for the three cases required in the sequel, i.e. for $r \in \{3,4,5\}$. For $r=5$, we mention only the table's first half, since the rest follows under *anti*: see (386) *supra*.[76]

$\{\sigma(1)\ldots\sigma(3)\}$	$\tilde\lambda^\sigma_1$	$\tilde\lambda^\sigma_2$	$\tilde\lambda^\sigma_3$	$\{\sigma(1)\ldots\sigma(3)\}$	$\tilde\lambda^\sigma_1$	$\tilde\lambda^\sigma_2$	$\tilde\lambda^\sigma_3$
$\{1,2,3\}$	2	3	1	$\{2,3,1\}$	-1	0	1
$\{1,3,2\}$	-1	0	1	$\{3,1,2\}$	-1	0	1
$\{2,1,3\}$	-1	0	1	$\{3,2,1\}$	2	-3	1

$\{\sigma(1)\ldots\sigma(4)\}$	$\tilde\lambda^\sigma_1$	$\tilde\lambda^\sigma_2$	$\tilde\lambda^\sigma_3$	$\tilde\lambda^\sigma_4$	$\{\sigma(1)\ldots\sigma(4)\}$	$\tilde\lambda^\sigma_1$	$\tilde\lambda^\sigma_2$	$\tilde\lambda^\sigma_3$	$\tilde\lambda^\sigma_4$
$\{1,2,3,4\}$	6	11	6	1	$\{3,1,2,4\}$	-2	-1	2	1
$\{1,2,4,3\}$	-2	-1	2	1	$\{3,1,4,2\}$	-2	-1	2	1
$\{1,3,2,4\}$	-2	-1	2	1	$\{3,2,1,4\}$	2	-1	-2	1
$\{1,3,4,2\}$	-2	-1	-2	1	$\{3,2,4,1\}$	2	-1	-2	1
$\{1,4,2,3\}$	-2	-1	2	1	$\{3,4,1,2\}$	-2	-1	2	1
$\{1,4,3,2\}$	2	-1	-2	1	$\{3,4,2,1\}$	2	-1	-2	1
$\{2,1,3,4\}$	-2	-1	2	1	$\{4,1,2,3\}$	-2	-1	2	1
$\{2,1,4,3\}$	2	-1	-2	1	$\{4,1,3,2\}$	2	-1	-2	1
$\{2,3,1,4\}$	-2	-1	2	1	$\{4,2,1,3\}$	2	-1	-2	1
$\{2,3,4,1\}$	-2	-1	2	1	$\{4,2,3,1\}$	2	-1	-2	1
$\{2,4,1,3\}$	2	-1	-2	1	$\{4,3,1,2\}$	2	-1	-2	1
$\{2,4,3,1\}$	2	-1	-2	1	$\{4,3,2,1\}$	-6	11	-6	1

[75] The true projector is of course $\frac{1}{r}\,pushinvar_r$ but we dispense with the factor $\frac{1}{r}$ since it would complicate most formulae where *pushinvar* naturally occurs, like those in §10.2.

[76] Note that, generally speaking, λ^σ_j and $\lambda^{\sigma^{-1}}_j$ need not coincide. So the convention adopted for denoting the permutations matters.

$\{\sigma(1)\dots\sigma(5)\}$	$\tilde\lambda_1^\sigma$	$\tilde\lambda_2^\sigma$	$\tilde\lambda_3^\sigma$	$\tilde\lambda_4^\sigma$	$\tilde\lambda_5^\sigma$	$\{\sigma(1)\dots\sigma(5)\}$	$\tilde\lambda_1^\sigma$	$\tilde\lambda_2^\sigma$	$\tilde\lambda_3^\sigma$	$\tilde\lambda_4^\sigma$	$\tilde\lambda_5^\sigma$
$\{1,2,3,4,5\}$	24	50	35	10	1	$\{2,3,1,4,5\}$	−6	−5	5	5	1
$\{1,2,3,5,4\}$	−6	−5	5	5	1	$\{2,3,1,5,4\}$	4	0	−5	0	1
$\{1,2,4,3,5\}$	−6	−5	5	5	1	$\{2,3,4,1,5\}$	−6	−5	5	5	1
$\{1,2,4,5,3\}$	−6	−5	5	5	1	$\{2,3,4,5,1\}$	−6	−5	5	5	1
$\{1,2,5,3,4\}$	−6	−5	5	5	1	$\{2,3,5,1,4\}$	4	0	−5	0	1
$\{1,2,5,4,3\}$	4	0	−5	0	1	$\{2,3,5,4,1\}$	4	0	−5	0	1
$\{1,3,2,4,5\}$	−6	−5	5	5	1	$\{2,4,1,3,5\}$	4	0	−5	0	1
$\{1,3,2,5,4\}$	4	0	−5	0	1	$\{2,4,1,5,3\}$	4	0	−5	0	1
$\{1,3,4,2,5\}$	−6	−5	5	5	1	$\{2,4,3,1,5\}$	4	0	−5	0	1
$\{1,3,4,5,2\}$	−6	−5	5	5	1	$\{2,4,3,5,1\}$	4	0	−5	0	1
$\{1,3,5,2,4\}$	4	0	−5	0	1	$\{2,4,5,1,3\}$	4	0	−5	0	1
$\{1,3,5,4,2\}$	4	0	−5	0	1	$\{2,4,5,3,1\}$	4	0	−5	0	1
$\{1,4,2,3,5\}$	−6	−5	5	5	1	$\{2,5,1,3,4\}$	4	0	−5	0	1
$\{1,4,2,5,3\}$	−6	−5	5	5	1	$\{2,5,1,4,3\}$	−6	5	5	−5	1
$\{1,4,3,2,5\}$	4	0	−5	0	1	$\{2,5,3,1,4\}$	4	0	−5	0	1
$\{1,4,3,5,2\}$	4	0	−5	0	1	$\{2,5,3,4,1\}$	4	0	−5	0	1
$\{1,4,5,2,3\}$	−6	−5	5	5	1	$\{2,5,4,1,3\}$	−6	5	5	−5	1
$\{1,4,5,3,2\}$	4	0	−5	0	1	$\{2,5,4,3,1\}$	−6	5	5	−5	1
$\{1,5,2,3,4\}$	−6	−5	5	5	1	$\{3,1,2,4,5\}$	−6	−5	5	5	1
$\{1,5,2,4,3\}$	4	0	−5	0	1	$\{3,1,2,5,4\}$	4	0	−5	0	1
$\{1,5,3,2,4\}$	4	0	−5	0	1	$\{3,1,4,2,5\}$	−6	−5	5	5	1
$\{1,5,3,4,2\}$	4	0	−5	0	1	$\{3,1,4,5,2\}$	−6	−5	5	5	1
$\{1,5,4,2,3\}$	4	0	−5	0	1	$\{3,1,5,2,4\}$	4	0	−5	0	1
$\{1,5,4,3,2\}$	−6	5	5	−5	1	$\{3,1,5,4,2\}$	4	0	−5	0	1
$\{2,1,3,4,5\}$	−6	−5	5	5	1	$\{3,2,1,4,5\}$	4	0	−5	0	1
$\{2,1,3,5,4\}$	4	0	−5	0	1	$\{3,2,1,5,4\}$	−6	5	5	−5	1
$\{2,1,4,3,5\}$	4	0	−5	0	1	$\{3,2,4,1,5\}$	4	0	−5	0	1
$\{2,1,4,5,3\}$	4	0	−5	0	1	$\{3,2,4,5,1\}$	4	0	−5	0	1
$\{2,1,5,3,4\}$	4	0	−5	0	1	$\{3,2,5,1,4\}$	−6	5	5	−5	1
$\{2,1,5,4,3\}$	−6	5	5	−5	1	$\{3,2,5,4,1\}$	−6	5	5	−5	1

Remark 1: The dimension, noted $\tau(r+1)$ for convenience, of the subspace of permutator superpositions $\mathcal{P}$

$$(\mathcal{P}.M)^{w_1,\dots,w_r} = \sum_{\sigma\in\mathfrak{S}_r} \lambda_j^\sigma \;\; M^{w_{\sigma(1)},\dots,w_{\sigma(r)}} \quad (\,\mathcal{P} : \mathrm{BIMU}_r \to \mathrm{BIMU}_r\,) \tag{388}$$

that commute with the 'projector' $pushinvar_r$ is of course much larger than the number r of alternality projectors $altor_{r,j}$. That dimension admits several combinatorial interpretations[77] and is given by

$$\tau(n) = \frac{1}{n^2} \sum_{d|n} \phi(d)^2 \, (\frac{n}{d})! \; d^{n/d} \tag{389}$$

with Euler's totient function ϕ. The first ten values of $\tau(r{+}1)$ are
1 , 2 , 3 , 8 , 24 , 108, 640, 4492, 36336, 329900.

Remark 2: In the preceding sections, when dealing with the alternality grids or chessboards for eupolars, we made no use of the alternators $altor_{r,j}$ for the simple reason that these projectors do not act internally on $Flex_r(\mathfrak{E})$ as soon as $r \geq 4$.

§15-2. The involution *srap*.

As observed in §13 and §14, it is the existence of an $\boldsymbol{u}/\boldsymbol{v}$ exchanging involution $syap: \; Flex_r(\mathfrak{E}) \leftrightarrow Flex_r(\mathfrak{O})$, respectful of the entire flexion structure and commuting with *swap*, that accounts for the harmony and symmetries that hold sway in the eupolar case. Unfortunately, *syap* does not extend beyond that setting[78]. For general bimoulds, we must make do with a feebler tool – the involution *srap*, which does not respect much of the flexion structure and fails to commute with *swap*, but at least preserves *push*-invariance and the alternality codegrees. Its action, internal on each $BIMU_r$, is given by the formulae:

$$\forall \mathrm{A}^\bullet \in \mathrm{BIMU}_r \quad , \quad \mathrm{srap.A}^{w_1,\dots,w_r} = \mathrm{A}^{w'_1,\dots,w'_r} \qquad \textit{with} \tag{390}$$

$$u'_i := (r+1)\, v_i - (v_1 + \cdots + v_r) \qquad (\forall i \in \{1,...,r\}) \tag{391}$$

$$v'_i := \frac{u_i + (u_1 + \cdots + u_r)}{r+1} \qquad (\forall i \in \{1,...,r\}) \tag{392}$$

The above rules for the change $w_i \mapsto w'_i$ are, needless to say, relative to the *short notation*, but the remarkable thing is that they extend without modification to the *long notation*. Indeed, if we set $u_0 := -u_1 ... - u_r$, $v_0 := 0$ and retain for u'_0, v'_0 the formal definition (391) and (392), we still find $u'_0 := -u'_1 ... - u'_r$, $v'_0 := 0$. Moreover:

[77] the most relevant here being: the number of orbits in the set of circular permutations under cyclic permutations of the elements. See M.J.A Sloane and Simon Plouffe, *A Handbook of Integer Sequences*, Acad. Press, 1995.

[78] It does not even extend to the *eutrigonmetric* setting!

$$\begin{aligned}\mathrm{srap}\,.\,\mathrm{srap} &= \mathrm{id} && (393)\\ \mathrm{srap}\,.\,\mathrm{pushinvar} &= \mathrm{pushinvar}\,.\,\mathrm{srap} && (394)\\ \mathrm{srap}\,.\,\mathrm{altor_j} &= \mathrm{altor_j}\,.\,\mathrm{srap} \qquad (\forall j) && (395)\end{aligned}$$

These are easy identities to verify, but the main property – the preservation of *push*-invariance under *srap* – really results from the double validity of the relations (391) and (392) which, as noted, apply equally in the *short notation* and in the *long one*. The latter, we recall, is the natural framework for the *push*-transform, since it reduces *push* to a circular permutation of order $r+1$.

§15-3. General and *push*-invariant alternality grids.

Let $Al^{[j]}_{r,d}$ resp. $Al^{[[j]]}_{r,d}$ denote the dimension of the subspace of $BIMU_r$ resp. $BIMU^{push}_r$ consisting of bimoulds [79]
(i) constant either in all v_i or in all u_i variables
(ii) polynomial of total degree d in the remaining u_i or v_i variables
(iii) of (loose) coalternality degree j

Next, denote $al^{[j]}_{r,d} := Al^{[j]}_{r,d} - Al^{[j-1]}_{r,d}$ and $al^{[[j]]}_{r,d} := Al^{[[j]]}_{r,d} - Al^{[[j-1]]}_{r,d}$ the dimensions associated with the *gradation* induced by the alternators $altor_{r,j}$.

Obviously, $Al^{[j]}_{r,d}$ and $al^{[j]}_{r,d}$ do not depend on which set of variables we choose to retain - whether the u_i's or the v_i's - since the constraints of j-alternality are the same in both cases. On the other hand, since *push*-invariance affects both sets of variables in quite different ways, we might expect $Al^{[[j]]}_{r,d}$ and $al^{[[j]]}_{r,d}$ to depend on which set we retain. This is not the case, however, since in view of the relations (393), (394), (395), the evolution *srap* exchanges the j-alternal, *push*-invariant, $\boldsymbol{u}$-dependent bimoulds one-to-one with the j-alternal, *push*-invariant, $\boldsymbol{v}$-dependent sort. So our definitions make good sense, and we may consider the generating functions:

$$\begin{aligned}\mathrm{ge}^{[j]}_r(t) &:= \sum_{0\le d} \mathrm{al}^{[j]}_{r,d}\,.\,t^d && (396)\\ \mathrm{ge}^{[[j]]}_r(t) &:= \sum_{0\le d} \mathrm{al}^{[[j]]}_{r,d}\,.\,t^d && (397)\end{aligned}$$

To understand the nature of these generating function, let $BIMU_r(\boldsymbol{u})$ be the space of all $\boldsymbol{u}$-polynomial, $\boldsymbol{v}$-constant bimoulds of length r, and consider:

$$\begin{aligned}\mathrm{BIMU}^{[j]}_r(\boldsymbol{u}) &:= \mathrm{altor}_{r,j}\,.\,\mathrm{BIMU}_r(\boldsymbol{u}) && (398)\\ \mathrm{BIMU}^{[[j]]}_r(\boldsymbol{u}) &:= \mathrm{pushinvar}_r\,.\,\mathrm{altor}_{r,j}\,.\,\mathrm{BIMU}_r(\boldsymbol{u}) && (399)\end{aligned}$$

[79] As usual, $BIMU^{push}_r$ denotes the *push*-invariant subspace of $BIMU_r$.

Now, the analytical constraints expressing j-alternality - alone or in conjunction with *push*-invarisance - are finitary: the underlying transforms in the $\boldsymbol{u}$-variables generate a *finite group*.[80] This circumstance makes it easy to unravel the structure of our two spaces (398) and (399) as finitely generated modules. Explicitely:

$$\begin{aligned} \mathrm{BIMU}_r^{[j]}(\boldsymbol{u}) &:= \mathrm{NU}_r^{[j]}(\boldsymbol{u})\,.\,\mathrm{DU}_{[r]}(\boldsymbol{u}) && (400)\\ \mathrm{BIMU}_r^{[[j]]}(\boldsymbol{u}) &:= \mathrm{NU}_r^{[[j]]}(\boldsymbol{u})\,.\,\mathrm{DU}_{[[r]]}(\boldsymbol{u}) && (401) \end{aligned}$$

where

(i) $\mathrm{DU}_{[r]}(\boldsymbol{u})$ denotes the ring[81] of symmetric polynomials in $u_1,\dots u_r$.

(ii) $\mathrm{DU}_{[[r]]}(\boldsymbol{u})$ denotes the ring[82] of symmetric polynomials in $u_1,\dots u_r$ and $u_0 := -(u_1+\ldots+u_r)$. We may take the elementary symmetric functions of degree 2, 3,..., r+1 as independent generators of $\mathrm{DU}_{[[r]]}(\boldsymbol{u})$.

(iii) $\mathrm{NU}_r^{[j]}(\boldsymbol{u})$ and $\mathrm{NU}_r^{[[j]]}(\boldsymbol{u})$ denote finite-dimensional vector spaces[83] of $\boldsymbol{u}$-polynomials, with equal dimensions:

$$\dim\left(\mathrm{NU}_r^{[j]}(\boldsymbol{u})\right) = \dim\left(\mathrm{NU}_r^{[[j]]}(\boldsymbol{u})\right) = s_1(r,j) \qquad (\textit{with } s_1 \textit{ as in } (382)\,) \tag{402}$$

but with distinct sets of generators. These may be taken of the form

$$\begin{aligned} &\left\{ \;\mathrm{Pa}^{[j]}_{d_1,\dots,d_r} := \mathrm{altor}_{r,j}\,.\,\mathrm{Pa}^{\bullet}_{d_1,\dots,d_r} \;\right\} && (403)\\ &\left\{ \;\mathrm{Pa}^{[[j]]}_{d'_1,\dots,d'_r} := \mathrm{pushinvar}_r\,.\,\mathrm{altor}_{r,j}\,.\,\mathrm{Pa}^{\bullet}_{d'_1,\dots,d'_r} \;\right\} && (404) \end{aligned}$$

for two distinct sets of monomial-valued bimoulds

$$\mathrm{Pa}^{\boldsymbol{w}}_{d_1,\dots,d_r} := u_1^{d_1}\dots u_r^{d_r} \qquad \textit{and} \qquad \mathrm{Pa}^{\boldsymbol{w}}_{d'_1,\dots,d'_r} := u_1^{d'_1}\dots u_r^{d'_r} \tag{405}$$

It follows at once that our generating functions must be of the form

$$\begin{aligned} \mathrm{ge}_r^{[j]}(t) &= \mathrm{ne}_r^{[j]}(t) \prod_{1\le k\le r} (1-t^k)^{-1} \quad \textit{with} \quad \mathrm{ne}_r^{[j]}(t)\in\mathbb{N}[t] && (406)\\ \mathrm{ge}_r^{[[j]]}(t) &= \mathrm{ne}_r^{[[j]]}(t) \prod_{2\le k\le r+1} (1-t^k)^{-1} \quad \textit{with} \quad \mathrm{ne}_r^{[[j]]}(t)\in\mathbb{N}[t] && (407) \end{aligned}$$

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[80] in the sense of [E3], §2.4, p51.

[81] to make $DU_{[r]}(\boldsymbol{u})$ unitary, we add the constant polynomial 1 to its elements.

[82] Here again, we add 1 to $DU_{[[r]]}(\boldsymbol{u})$.

[83] The spaces $NU_r^{[j]}(\boldsymbol{u})$ and $NU_r^{[[j]]}(\boldsymbol{u})$ are defined only modulo multiplication by 'invertible' elements of $DU_{[r]}(\boldsymbol{u})$ and $DU_{[[r]]}(\boldsymbol{u})$ respectively, in all possible ways that leave the products (400) and (401) unchanged.

with the shape of the numerators and denominators dictated by the nature of the spaces NU and DU. Moreover, (402) implies

$$\mathrm{ne}_r^{[j]}(1) = \mathrm{ne}_r^{[[j]]}(1) = s_1(r,j) \qquad (s_1 \textit{ as in } (382)) \tag{408}$$

Let $\mathcal{A}$ be the associative algebra freely generated on $\mathbb{Q}$ by x_1, x_2 and let $\mathcal{A}_{r,d}$ be the subspace (clearly of dimension $(r\dot{)}!/(r!\,d!)$) consisting of all element of patial degrees (r,d) in (x_1,x_2). The coefficients $al_{r,d}^{[j]}$ of $\mathrm{ge}_r^{[j]}(t)$ are easy to calculate since $al_{r,d}^{[1]}$ (and more generally $al_{r,d}^{[j]}$) can be interpreted as the dimension of the space spanned by the Lie elements in $\mathcal{A}_{r,d}$ (or more generally the elements of formal differential degree j). In particular

$$\mathrm{al}_{r,d}^{[1]} = \frac{1}{r+d} \sum_{\delta|r\,,\,\delta|d} \mu(\delta) \frac{((r+d)/\delta)!}{(r/\delta)!((d/\delta)!} \qquad (\mu = M\ddot{o}bius\, function) \tag{409}$$

Similar formulae apply for $al_{r,d}^{[j]}$ (with $j > 1$) and also for $al_{r,d}^{[[j]]}$.

To sum up, in this new context of polynomial-valued bimoulds, knowing the "alternality grid" reduces to knowing the coefficients of the polynomials $ne_r^{[j]}(t)$ and $ne_r^{[[j]]}(t)$. For illustration, we tabulate *infra* the cases $r = 3,4,5$ (the case $r = 2$ being trivial). Since the polynomial $ne_r^{[j]}(t)$ and $ne_r^{[[j]]}(t)$ tend to display "higher than average" factorisability, we also give the corresponding factorisations on $\mathbb{Z}$ and $\mathbb{N}$ (the latter type being more relevant). Lastly, in the cases $r = 3,4$ we also give simple systems of generators for $\mathrm{NU}_r^{[j]}(\boldsymbol{u})$ and $\mathrm{NU}_r^{[[j]]}(\boldsymbol{u})$, in the notations (403),(404),(405).

Alternality grid for 3 variables:

$j \setminus d$:	0	1	2	3	4	5	total
1	0	1	1	0	0	0	2
2	0	1	1	1	0	0	3
3	1	0	0	0	0	0	1
total	1	2	2	1	0	0	6
(*push*) 1	0	0	1	0	1	0	2
(*push*) 2	0	0	0	1	1	1	3
(*push*) 3	1	0	0	0	0	0	1
total	1	0	1	1	2	1	6

Alternality numerators for 3 variables:

$$\begin{aligned}
\mathrm{ne}_3^{[1]}(t) &= t\,(1+t)\\
\mathrm{ne}_3^{[2]}(t) &= t\,(1+t+t^2)\\
\mathrm{ne}_3^{[3]}(t) &= 1\\
\mathrm{ne}_3^{[123]}(t) &= (1+t)\,(1+t+t^2)\\
\mathrm{ne}_3^{[[1]]}(t) &= t^2\,(1+t^2)\\
\mathrm{ne}_3^{[[2]]}(t) &= t^3\,(1+t+t^2)\\
\mathrm{ne}_3^{[[3]]}(t) &= 1\\
\mathrm{ne}_3^{[[123]]}(t) &= 1+t^2+t^3+2t^4+t^5\\
&= (1+t+t^2)\,(1-t+t^2+t^3)
\end{aligned}$$

Alternality generators for 3 variables:

$\boldsymbol{d=1:}\ \mathrm{Pa}^{[1]}_{100},\ \mathrm{Pa}^{[2]}_{100}.\quad \boldsymbol{d=2:}\ \mathrm{Pa}^{[1]}_{200},\ \mathrm{Pa}^{[2]}_{200}.\quad \boldsymbol{d=3:}\ \mathrm{Pa}^{[2]}_{210}$

$\boldsymbol{d=2:}\ \mathrm{Pa}^{[[1]]}_{110}.\quad \boldsymbol{d:=3:}\ \mathrm{Pa}^{[[2]]}_{210}.\quad \boldsymbol{d=4:}\ \mathrm{Pa}^{[[1]]}_{211},\ \mathrm{Pa}^{[[2]]}_{310}.\quad \boldsymbol{d=5:}\ \mathrm{Pa}^{[[2]]}_{210}.$

Alternality grid for 4 variables:

$j\setminus d:$	0	1	2	3	4	5	6	7	8	9	10	total
1	0	1	1	2	1	1	0	0	0	0	0	6
2	0	1	3	2	3	1	1	0	0	0	0	11
3	0	1	1	2	1	1	0	0	0	0	0	6
4	1	0	0	0	0	0	0	0	0	0	0	1
total	1	3	5	6	5	3	1	0	0	0	0	24
(push) 1	0	0	0	1	1	2	1	1	0	0	0	6
(push) 2	0	0	1	1	2	2	2	1	1	0	1	11
(push) 3	0	0	0	1	1	2	1	1	0	0	0	6
(push) 4	1	0	0	0	0	0	0	0	0	0	0	1
total	1	0	1	3	4	6	4	3	1	0	1	24

Alternality numerators for 4 variables:

$$\begin{aligned}
\mathrm{ne}_4^{[1]}(t) &= t\,(1+t^2)\,(1+t+t^2)\\
\mathrm{ne}_4^{[2]}(t) &= t\,(1+3t+2t^2+3t^3+t^4+t^5)\\
\mathrm{ne}_4^{[3]}(t) &= t\,(1+t^2)\,(1+t+t^2)\\
\mathrm{ne}_4^{[4]}(t) &= 1\\
\mathrm{ne}_4^{[1234]}(t) &= (1+t)^2\,(1+t^2)\,(1+t+t^2)
\end{aligned}$$

$$\begin{aligned}
\mathrm{ne}_4^{[[1]]}(t) &= t^3\,(1+t^2)\,(1+t+t^2)\\
\mathrm{ne}_4^{[[2]]}(t) &= t^2\,(1+t+2t^2+2t^3+2t^4+t^5+t^6+t^8)\\
\mathrm{ne}_4^{[[3]]}(t) &= t^3\,(1+t^2)\,(1+t+t^2)\\
\mathrm{ne}_4^{[[4]]}(t) &= (1+t^2)\,(1+3t^3+4t^4+3t^5+t^8)\\
\mathrm{ne}_4^{[[1234]]}(t) &= (1+t^2)\,(1+t)^2\,(1+t+t^2)\,(1-3t+5t^2-3t^3+t^4)
\end{aligned}$$

Alternality generators for 4 variables:

$\boldsymbol{d := 1:}$ $\mathrm{Pa}^{[1]}_{0100}$, $\mathrm{Pa}^{[2]}_{0100}$, $\mathrm{Pa}^{[3]}_{0100}$. $\boldsymbol{d := 2:}$ $\mathrm{Pa}^{[1]}_{0200}$, $\mathrm{Pa}^{[2]}_{0200}$, $\mathrm{Pa}^{[2]}_{1100}$, $\mathrm{Pa}^{[[2]]}_{1010}$, $\mathrm{Pa}^{[3]}_{0200}$.

$\boldsymbol{d := 3:}$ $\mathrm{Pa}^{[1]}_{0300}$, $\mathrm{Pa}^{[1]}_{1200}$, $\mathrm{Pa}^{[2]}_{0300}$, $\mathrm{Pa}^{[2]}_{1200}$, $\mathrm{Pa}^{[3]}_{0300}$, $\mathrm{Pa}^{[3]}_{1200}$. $\boldsymbol{d := 4:}$ $\mathrm{Pa}^{[1]}_{1300}$, $\mathrm{Pa}^{[2]}_{1300}$.

$\mathrm{Pa}^{[2]}_{1120}$, $\mathrm{Pa}^{[2]}_{2020}$, $\mathrm{Pa}^{[3]}_{1300}$. $\boldsymbol{d := 5:}$ $\mathrm{Pa}^{[1]}_{2300}$, $\mathrm{Pa}^{[2]}_{2300}$, $\mathrm{Pa}^{[3]}_{2300}$. $\boldsymbol{d := 6:}$ $\mathrm{Pa}^{[2]}_{1230}$.

$\boldsymbol{d := 2:}$ $\mathrm{Pa}^{[[1]]}_{1100}$. $\boldsymbol{d = 3:}$ $\mathrm{Pa}^{[[1]]}_{1002}$, $\mathrm{Pa}^{[[2]]}_{0012}$. $\boldsymbol{d = 4:}$ $\mathrm{Pa}^{[[1]]}_{1012}$, $\mathrm{Pa}^{[[2]]}_{1003}$, $\mathrm{Pa}^{[[2]]}_{1012}$, $\mathrm{Pa}^{[[3]]}_{1003}$.

$\boldsymbol{d = 5:}$ $\mathrm{Pa}^{[[1]]}_{0113}$, $\mathrm{Pa}^{[[1]]}_{0122}$, $\mathrm{Pa}^{[[2]]}_{0113}$, $\mathrm{Pa}^{[[2]]}_{0122}$, $\mathrm{Pa}^{[[3]]}_{0113}$, $\mathrm{Pa}^{[[3]]}_{0122}$.

$\boldsymbol{d = 6:}$ $\mathrm{Pa}^{[[1]]}_{1122}$, $\mathrm{Pa}^{[[2]]}_{1122}$, $\mathrm{Pa}^{[[2]]}_{0222}$, $\mathrm{Pa}^{[[3]]}_{1122}$. $\boldsymbol{d = 7:}$ $\mathrm{Pa}^{[[1]]}_{1123}$, $\mathrm{Pa}^{[[2]]}_{1123}$, $\mathrm{Pa}^{[[3]]}_{1222}$.

$\boldsymbol{d = 8:}$ $\mathrm{Pa}^{[[2]]}_{1123}$. $\boldsymbol{d = 10:}$ $\mathrm{Pa}^{[[2]]}_{1234}$.

Alternality grid for 5 variables:

$j \setminus d:$	1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	total
1	0	1	2	3	4	4	4	3	2	1	0	0	0	0	0	24
2	0	1	3	6	8	10	9	7	4	2	0	0	0	0	0	50
3	0	1	3	4	6	6	6	4	3	1	1	0	0	0	0	35
4	0	1	1	2	2	2	1	1	0	0	0	0	0	0	0	10
5	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
total	1	4	9	15	20	22	20	15	9	4	1	0	0	0	0	120
$(push)$ 1	0	0	1	1	2	1	4	2	4	3	3	1	2	0	0	24
$(push)$ 2	0	0	0	2	2	5	6	8	7	8	5	4	2	1	0	50
$(push)$ 3	0	0	1	1	3	3	5	4	5	3	4	2	2	1	1	35
$(push)$ 4	0	0	0	1	1	2	2	2	1	1	0	0	0	0	0	10
$(push)$ 5	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
total	1	0	2	5	8	11	17	16	17	15	12	7	6	2	1	120

Alternality numerators for 5 variables:

$$
\begin{aligned}
\mathrm{ne}_5^{[1]}(t) &= t\,(1+t)\,(1+t+t^2+t^3)\,(1+t^2+t^4)\\
\mathrm{ne}_5^{[2]}(t) &= t\,(1+t^2)\,(1+2t+2t^2)\,(1+t+t^2+t^3+t^4)\\
\mathrm{ne}_5^{[3]}(t) &= t\,(1+t+t^2+t^3+t^4)\,(1+2t+t^2+2t^3+t^5)\\
\mathrm{ne}_5^{[4]}(t) &= t\,(1+t^2)\,(1+t+t^2+t^3+t^4)\\
\mathrm{ne}_5^{[5]}(t) &= 1\\
\mathrm{ne}_5^{[1..5]}(t) &= (1+t)\,(1+t+t^2)\,(1+t+t^2+t^3)\,(1+t+t^2+t^3+t^4)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{ne}_5^{[[1]]}(t) &= t^2\,(1+t^2)\,(1+t+t^2+3t^4+2t^5+t^6+t^7+2t^8)\\
\mathrm{ne}_5^{[[2]]}(t) &= t^3\,(1+t^2)\,(1+t+t^2+t^3+t^4)\,(2+t^2+t^3+t^4)\\
\mathrm{ne}_5^{[[3]]}(t) &= t^2\,(1+t+t^2+t^3+t^4)\,(1+2t^2+2t^4+t^6+t^8)\\
\mathrm{ne}_5^{[[4]]}(t) &= t^3\,(1+t^2)\,(1+t+t^2+t^3+t^4)\\
\mathrm{ne}_5^{[[5]]}(t) &= 1\\
\mathrm{ne}_5^{[[1..5]]}(t) &= (1{+}t^2)\,(1{+}t^2{+}5t^3{+}7t^4{+}6t^5{+}10t^6{+}10t^7{+}7t^8{+}5t^9{+}5t^{10}{+}2t^{11}{+}t^{12})\\
&= (1{+}t^2)\,(1+t+t^2+t^3+t^4)\,(1-t+t^2+4t^3+2t^4+3t^6+t^7+t^8)
\end{aligned}
$$

§15-4. Bialternality grid and bialternality chessboard.

Let $Bial_{r,d}^{[j_1,j_2]}$ resp. $Bial_{r,d}^{[[j_1,j_2]]}$ denote the dimension of the subspace $BIMU_{r,d}^{[j_1,j_2]} \subset BIMU_r$ resp. $BIMU_{r,d}^{[[j_1,j_2]]} \subset BIMU_r^{push}$ consisting of all

bimoulds [84]
(i) constant in the v_i variables
(ii) polynomial of total degree d in the remaining u_i variables
(iii) of (loose) alternality codegree j_1
(iv) with a swappee of (loose) alternality codegree j_2.

Next, denote

$$\begin{aligned} bial_{r,d}^{[j_1,j_2]} &:= Bial_{r,d}^{[j_1,j_2]} - Bial_{r,d}^{[j_1-1,j_2]} - Bial_{r,d}^{[j_1,j_2-1]} + Bial_{r,d}^{[j_1-1,j_2-1]} \\ bial_{r,d}^{[j_1,j_2]} &:= Bial_{r,d}^{[[j_1,j_2]]} - Bial_{r,d}^{[[j_1-1,j_2]]} - Bial_{r,d}^{[[j_1,j_2-1]]} + Bial_{r,d}^{[[j_1-1,j_2-1]]} \end{aligned}$$

The chessboard phenomenon.

Since the projectors $altor_{r,j_1}$ and $swap.altor_{r,j_2}.swap$ do not commute, there exists no corresponding *gradation* by the pairs $[j_1, j_2]$ or $[[j_1, j_2]]$. In the *push*-invariant case, however, the *filtration* can be refined, leading to the vanishing of all dimensions $bial_{r,d}^{[j_1,j_2]}$ when $d+j_1+j_2$ is *odd*.

Indeed, since $push \equiv neg.anti.swap.anti.swap$ and since *neg* commutes with everything, the involutions *anti* and *swap.anti.swap*, which do not commute on *BIMU*, do so when restricted to the *push*-invariant subspace $BIMU^{push}$, which thus splits into a direct sum of four subspaces

$$\mathrm{BIMU}^{\mathrm{push}} = \bigoplus_{\epsilon_1,\epsilon_1 \in \{\pm\}} \mathcal{P}^{\epsilon_1,\epsilon_2} \,.\, \mathrm{BIMU}^{\mathrm{push}} \tag{410}$$

with the projectors

$$\mathcal{P}^{\epsilon_1,\epsilon_2} := \frac{1}{2}(id + \epsilon_1.anti)\,.\,\frac{1}{2}(id + \epsilon_1.swap.anti.swap) \tag{411}$$

and with each of the four, (ϵ_1, ϵ_2)-indexed component spaces invariant under

$$\epsilon_1 \,.\, \mathrm{anti} \quad , \quad \epsilon_2 \,.\, \mathrm{swap.anti.swap} \quad , \quad \epsilon_1.\epsilon_2.\mathrm{neg} \tag{412}$$

The decomposition (410) applies in particular to $BIMU_{r,d}^{[[j_1,j_2]]}$. But in view of (386), only the component space $\mathcal{P}^{\epsilon_1,\epsilon_2} \,.\, BIMU_{r,d}^{[[j_1,j_2]]}$ with $\epsilon_1 = (-1)^{1+j_1}$ and $\epsilon_2 = (-1)^{1+j_2}$ may contain elements of *strict* bialternality codegree (j_1, j_2). Moreover, due to (412), that component space has to be invariant under $\epsilon_1.\epsilon_2.neg$ and must therefore vanish unless $d + j_1 + j_2$ be *even*. As an immediate consequence, only the dimensions $bial_{r,d}^{[[j_1,j_2]]}$ with $d + j_1 + j_2$ *even* may be nonzero. This is the so-called *chessboard phenomenon*, which we had already observed in §14, in the eupolar setting, where we had $d \equiv -r$ due to

[84] As usual, $BIMU_r^{push}$ denotes the *push*-invariant subspace of $BIMU_r$.

homogeneity.

Generating functions.

As in §15-3, we may still form the generating series:

$$\mathrm{gee}_r^{[j_1,j_2]}(t) \;:=\; \sum_{0\leq d} \mathrm{bial}_r^{[j_1,j_2]} \,.\, t^d \tag{413}$$

$$\mathrm{gee}_r^{[[j_1,j_2]]}(t) \;:=\; \sum_{0\leq d} \mathrm{bial}_r^{[[j_1,j_2]]} \,.\, t^d \tag{414}$$

but new difficulties arise, since the bialternality constraints are no longer *finitary*. One such difficulty is that the decompositions (400) and (401) have no equivalent here. Nonetheless, it would seem that the new generating series are still *rational functions*:

$$\mathrm{gee}_r^{[j_1,j_2]}(t) = \mathrm{nee}_r^{[j_1,j_2]}(t)/\mathrm{dee}_r^{[j_1,j_2]}(t) \quad \textit{with} \quad \mathrm{nee}_r^{[j_1,j_2]}(t)\,,\mathrm{dee}_r^{[j_1,j_2]}(t)\;\in \mathbb{Z}[t]$$

$$\mathrm{gee}_r^{[[j_1,j_2]]}(t) = \mathrm{nee}_r^{[[j_1,j_2]]}(t)/\mathrm{dee}_r^{[[j_1,j_2]]}(t) \quad \textit{with} \quad \mathrm{nee}_r^{[[j_1,j_2]]}(t), \mathrm{dee}_r^{[[j_1,j_2]]}(t) \in \mathbb{Z}[t]$$

with denominators $\mathrm{dee}_r^{[j_1,j_2]}(t)$ resp. $\mathrm{dee}_r^{[[j_1,j_2]]}(t)$ that may still[85] be taken as products of elementary monomials $(1-t^k)$ resp. $(1-t^{2k})$. The numerators $\mathrm{nee}_r^{[j_1,j_2]}(t)$ and $\mathrm{nee}_r^{[[j_1,j_2]]}(t)$ are still polynomial in t, but with fairly high degrees[86] and with a hopeless mixture of positive and negative (integer) coefficients. Moreover, in the *push*-invariant case, due to the *chessboard phenomenon*, $\mathrm{nee}_r^{[[j_1,j_2]]}(t)$ is *even* resp. *odd* in t exactly when j_1+j_2 is *even* resp. *odd*.

We must stress that in all generality, i.e. for all values of the length r, the above statements are still conjectural, unlike the corresponding results of §15.3 relative to the alternality grids. Another difference worth noting is the absence of bases such as (403) and (404) for the $[j_1,j_2]$- or $[[j_1,j_2]]$-alternal subspaces, although the basis to be constructed in §15-5 *infra* may be regarded as a passable substitute.

Bialternality grids.

The ordinary polynomial bialternality grids (i.e. the ones we get without imposing *push*-invariance) do not display the chessboard effect, nor are they symmetric under the exchange $j_1 \leftrightarrow j_2$, and that too from $r=3$ onwards. Their most outstanding (still unproven) features are the *vanishing*

[85] provided we don't insist on *reducing* the rational fonctions $\mathrm{gee}_r^{[j_1,j_2]}(t)$ and $\mathrm{gee}_r^{[[j_1,j_2]]}(t)$.

[86] much higher in any case than those of the earlier $\mathrm{ne}_r^{[j]}(t)$ and $\mathrm{ne}_r^{[[j]]}(t)$.

of all entries $al^{[j_1,j_2]}_{r,d}$ with $j_1 + j_2 \geq d+3$ (for each r and d large enough, i.e. $d \geq d^*(r)$) and, as already pointed out, the *rationality* of the generating functions $gee^{[j_1,j_2]}_{r,d}(t)$.

Bialternality chessboards.

The bialternality chessboard for *push*-invariant bimoulds is elementary for $r = 2$ and symmetric under the exchange $j_1 \leftrightarrow j_2$ up to $r = 4$ but not beyond[87], although the deviations from symmetry remain weak even then[88], much weaker at any rate than with the general grid. Moreover, the rule of the "vanishing south-east triangle" (i.e. $al^{[[j_1,j_2]}_{r,d} \equiv 0$ for $j_1 + j_2 \geq d+2$ and d even or $j_1 + j_2 \geq d+3$ and d odd) now seems to be holding without exceptions and not jut asymptotically, as was the case with the general grid. Let us tabulate the simplest non-elementary cases, i.e. $r = 3$ and $r = 4$.

Bialternality chessboard for 3 variables:

$$\begin{aligned}
\mathrm{gee}_3^{[[1,1]]}(t) &= \frac{t^8 + t^{10} - t^{12}}{(1-t^2)\,(1-t^4)\,(1-t^6)}\\
\mathrm{gee}_3^{[[1,2]]}(t) &= \frac{t^5}{(1-t^2)^2\,(1-t^6)}\\
\mathrm{gee}_3^{[[1,3]]}(t) &= \frac{t^2 + t^4 - t^8 - t^{10} + t^{12}}{(1-t^2)\,(1-t^4)\,(1-t^6)}\\
\mathrm{gee}_3^{[[2,2]]}(t) &= \frac{t^4}{(1-t^2)^2\,(1-t^4)}\\
\mathrm{gee}_3^{[[2,3]]}(t) &= \frac{t^3}{(1-t^2)\,(1-t^4)\,(1-t^6)}\\
\mathrm{gee}_3^{[[3,3]]}(t) &= 1
\end{aligned}$$

[87] This should not come as a great surprise, since the projectors $altor_{r,j_1}$ and $swap.altor_{r,j_1}.swap$ do not commute on $BIMU^{push}_r$ any more than they do on $BIMU_r$. Nor does the involution *srap* (unlike *syap* in the eupolar case) exchange the bialternality types $[j_1, j_2]$ and $[j_2, j_1]$ or $[[j_1, j_2]]$ and $[[j_2, j_1]]$.

[88] They also appear to be limited to the case of *odd* degrees d.

Bialternality chessboard for 4 variables:

$$\mathrm{gee}_4^{[[1,1]]}(t) = \frac{t^8+2t^{12}+t^{14}+t^{16}+2t^{18}+t^{22}-t^{24}}{(1-t^2)\,(1-t^6)\,(1-t^8)\,(1-t^{12})}$$

$$\mathrm{gee}_4^{[[1,2]]}(t) = \frac{t^5}{(1-t^2)^3\,(1-t^6)}$$

$$\mathrm{gee}_4^{[[1,3]]}(t) = \frac{t^4+3t^6+6t^8+11t^{10}+14t^{12}+17t^{14}+17t^{16}+15t^{18}+11t^{20}+7t^{22}+4t^{24}+t^{26}+t^{28}+t^{32}}{(1-t^6)\,(1-t^8)\,(1-t^{10})\,(1-t^{12})}$$

$$\mathrm{gee}_4^{[[1,4]]}(t) = \frac{t^3}{(1-t^2)^2\,(1-t^6)\,(1-t^{10})}$$

$$\mathrm{gee}_4^{[[2,2]]}(t) = \frac{t^4+2t^6+t^8+t^{10}-2t^{12}+t^{14}}{(1-t^2)^2\,(1-t^6)\,(1-t^4)}$$

$$\mathrm{gee}_4^{[[2,3]]}(t) = \frac{t^3+t^9}{(1-t^2)^3\,(1-t^{10})}$$

$$\mathrm{gee}_4^{[[2,4]]}(t) = \frac{t^2+t^4-t^{14}+t^{18}+t^{20}-t^{22}}{(1-t^2)\,(1-t^4)\,(1-t^6)\,(1-t^{10})}$$

$$\mathrm{gee}_4^{[[3,3]]}(t) = \frac{t^8+2t^{12}+t^{14}+t^{16}+2t^{18}+t^{22}-t^{24}}{(1-t^2)\,(1-t^6)\,(1-t^8)\,(1-t^{12})}$$

$$\mathrm{gee}_4^{[[3,4]]}(t) = \ 0$$

$$\mathrm{gee}_4^{[[4,4]]}(t) = \ 1$$

Bialternality chessboard for 5 variables:

The first (mild) deviation from symmetry occurs for degree $d = 9$. Here are the corresponding entries $al_{5,9}^{[[j_1,j_2]]}$, which duly vanish on a south-east triangle:

$$\begin{array}{rrrrr} 0 & 8 & 0 & 14 & 0 \\ 7 & 0 & 31 & 0 & 6 \\ 0 & 30 & 0 & 1 & 0 \\ 15 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 \end{array}$$

§15-5. Example of bialternality basis.

The main hurdle in the investigation of the bialternality grid and chessboard as soon as $r \geq 3$ is of course the *non-finitary* nature of the underlying constraints[89] which precludes the existence of simple projectors and of ele-

[89]this means that the analytic expression of the constraints $[j_1, j_2]$ of $[[j_1, j_2]]$ necessarily

mentary decompositions of type (400), (401). That does not make the situation totally hopeless, though, and fairly explicit bases for each type $[j_1, j_2]$ of $[[j_1, j_2]]$ may still be produced. For illustration, let us examine the simplest non-elementary case, i.e. the space of all bialternals for $r = 3$ (we recall that bialternals are automatically *push*-invariants, as shown in §7, so that the types $[1, 1]$ and $[[1, 1]]$ coincide; we also recall that the situation for $r = 2$ is elementary since in that case the bialternality constraints are *finitary*, with an underlying group isomorphic with $\mathcal{S}_3$).

We start the construction with the elementary bialternals $ekma^\bullet_{d_1}$ for $r = 1$ and $doma^\bullet_{d_2,d_6}$ for $r = 2$:

$$\begin{array}{rcll}
\mathrm{ekma}^{w_1}_{d_1} & := & u_1^{d_1} & (\,d_1|2\,) \\
\mathrm{doma}^{w_1,w_2}_{d_2,d_6} & := & \mathrm{fa}(u_1,u_2)\,\mathrm{ha}(u_1,u_2)^{d_2}\,\mathrm{ga}(u_1,u_2)^{d_6} & (\,d_2|2\,,\;d_6|6\,)
\end{array}$$

with[90]

$$\begin{array}{rcl}
\mathrm{fa}(u_1,u_2) & := & u_1\,u_2\,(u_1-u_2)\,(u_1+u_2)\,(2\,u_1+u_2)\,(2\,u_2+u_1) \\
\mathrm{ga}(u_1,u_2) & := & (u_1+u_2)^2\,u_1^2\,u_2^2 \\
\mathrm{ha}(u_1,u_2) & := & u_1^2+u_1\,u_2+u_2^2
\end{array}$$

We then define length-3 bialternals $toma^\bullet_{d_1,d_2,d_6}$ as simple *ari*-products:

$$\mathrm{toma}^\bullet_{d_1,d_2,d_6} := \mathrm{ari}(\mathrm{ekma}^\bullet_{d_1}\,,\,\mathrm{doma}^\bullet_{d_2,d_6}) \qquad (d_1|2\,,\,d_2|2\,,\,d_6|6) \tag{415}$$

These new bialternals are not linearly independent, since for a given total degree $d = 6 + d_1 + d_2 + d_6$ their number exceeds that of the dimension of all length-3 bialternals. To get a basis, we must of course do more than ensure the right cardinality. Let us first consider the systems $\mathcal{B}^0_d$, $\mathcal{B}^+_d$, $\mathcal{B}^-_d$:

$$\begin{array}{rcl}
\mathcal{B}^0_d & := & \displaystyle\bigcup_{d_2+d_6\,\equiv\,0 \mod 4} \left\{\mathrm{toma}^\bullet_{\mathrm{d}_1,\mathrm{d}_2,\mathrm{d}_6}\right\} \\
\mathcal{B}^+_d & := & \displaystyle\bigcup_{d_2+d_6\,\equiv\,2 \mod 4}^{d2\,\in\,\{0\}} \left\{\mathrm{toma}^\bullet_{\mathrm{d}_1,\mathrm{d}_2,\mathrm{d}_6}\right\} \\
\mathcal{B}^-_d & := & \displaystyle\bigcup_{d_2+d_6\,\equiv\,2 \mod 4}^{d2\,\in\,\{0,2\}} \left\{\mathrm{toma}^\bullet_{\mathrm{d}_1,\mathrm{d}_2,\mathrm{d}_6}\right\}
\end{array}$$

involves a set of linear transforms in the u_1, v_i variables which, though finite, does generate an infinite group (when expressed, via *swap*, relatively to the sole variables u_i).

[90]note that the present (d_2, d_6) indexation for the *doma* generators slightly differs for that of (7.5) in [E3], §7.2, p 120.

with, in all three cases the common, natural conditions[91]

$$d = 6 + d_1 + d_2 + d_6 \quad ; \quad d_1|2\,,\, d_2|2\,,\, d_6|6 \quad ; \quad d_1 > 0\,,\, d_2 \geq 0\,,\, d_6 \geq 0$$

Then the system $\mathcal{B}$ defined by

$$\begin{aligned} \mathcal{B}_d \;&:=\; \mathcal{B}_d^0 \bigcup \mathcal{B}_d^+ \qquad \textit{if} \quad d \equiv 0 \mod 4 \\ \mathcal{B}_d \;&:=\; \mathcal{B}_d^0 \bigcup \mathcal{B}_d^- \qquad \textit{if} \quad d \equiv 2 \mod 4 \end{aligned}$$

has for each d the right cardinality, is linearly independent, and can be shown to constitute a basis for the space of all length-3 bialternals. One way of proving this is to construct similar bases for all the other bialternality types $[j_1, j_2]$, $[[j_1, j_2]]$ and then produce explicit, complementary projectors onto the subspaces spanned by these bases. But we are still far away from a general theory, valid for all values of r.

16 From polar to trigonometric bisymmetrals.

Replacing $P(t) := 1/t$ by $Q(t) := c/\tan(c\,t)$ changes the exact flexion units $Pa^{w_1} := P(u_1)$ and $Pi^{w_1} := P(v_1)$ into the *approximate* units $Qa^{w_1} := Q(u_1)$ and $Qi^{w_1} := Q(v_1)$, and turns the pair of isomorphic eupolar structures $\mathit{Flex}(Pa)$ and $\mathit{Flex}(Pi)$ into the non-isomorphic eutrigonometric structures $\mathit{Flex}(Qa)$ and $\mathit{Flex}(Qi)$, which remain non-isomorphic even after the (natural) extension to $\mathit{Flex}(Qa, c.I)$ and $\mathit{Flex}(Qi, c.I)$. These eutrigonometric structures being central to multizeta algebra[92] we propose to deal with them at length in a special monograph [E4], but here is a sneak preview, mainly to show which features of the eupolar case carry over and which do not.

§16.1. Disappearance of *syap* and consequences.

The involution *slap* disappears, or rather, if we keep the formal definition of *slap*, loses its quality of being a full flexion isomorphism. The reason is that when we substitute Qa resp. Qi for $\mathfrak{E}$ in the classical three-term sum

$$+\mathfrak{E}^{\binom{u_1}{v_1}}\,\mathfrak{E}^{\binom{u_2}{v_2}} - \mathfrak{E}^{\binom{u_{1,2}}{v_1}}\,\mathfrak{E}^{\binom{u_2}{v_{2:1}}} - \mathfrak{E}^{\binom{u_{1,2}}{v_2}}\,\mathfrak{E}^{\binom{u_1}{v_{1:2}}} \tag{416}$$

we get two constant valued elements of $BIMU_2$:

$$Qaa^{w_1,w_2} \equiv c^2 \quad ; \quad Qii^{w_1,w_2} \equiv -c^2 \tag{417}$$

[91]The first condition ensures the right degree, and the condition $d_1 > 0$ is natural, too, since $\mathit{toma}^{\bullet}_{d_1,d_2,d_6} \equiv 0$ when $d_1 = 0$.

[92]especially for constructing the canonical *rational Drinfelt associator*.

instead of getting 0, as with strict flexion units. The complication here has less to do with the sign alternation $\pm c^2$ than with the fact that $Qaa^\bullet$ *ari*-commutes with all elements of its parent structure $Flex(Qa)$, whereas $Qii^\bullet$ does not *ari*-commute with $Flex(Qi)$. For instance, if we *ari*-bracket $\mathfrak{E}^\bullet$ with the length-2 bimould defined by the three-term sum (416), we get the following expression

$$\begin{aligned}
&+\mathfrak{E}^{\binom{u_{1,2,3}}{v_1}}\,\mathfrak{E}^{\binom{u_{2,3}}{v_{3:1}}}\,\mathfrak{E}^{\binom{u_2}{v_{2:3}}} + \mathfrak{E}^{\binom{u_{1,2,3}}{v_3}}\,\mathfrak{E}^{\binom{u_1}{v_{1:3}}}\,\mathfrak{E}^{\binom{u_2}{v_{2:3}}} + \mathfrak{E}^{\binom{u_{1,2}}{v_1}}\,\mathfrak{E}^{\binom{u_2}{v_{2:1}}}\,\mathfrak{E}^{\binom{u_3}{v_3}}\\
&-\mathfrak{E}^{\binom{u_{1,2,3}}{v_3}}\,\mathfrak{E}^{\binom{u_{1,2}}{v_{1:3}}}\,\mathfrak{E}^{\binom{u_2}{v_{2:1}}} - \mathfrak{E}^{\binom{u_{1,2,3}}{v_1}}\,\mathfrak{E}^{\binom{u_2}{v_{2:1}}}\,\mathfrak{E}^{\binom{u_3}{v_{3:1}}} - \mathfrak{E}^{\binom{u_{2,3}}{v_3}}\,\mathfrak{E}^{\binom{u_2}{v_{2:3}}}\,\mathfrak{E}^{\binom{u_1}{v_1}}
\end{aligned}$$

which vanishes under the specialisation $\mathfrak{E} \mapsto Qa$ but not under $\mathfrak{E} \mapsto Qi$.[93]

§16.2. Appearance of a corrective 'central' factor.

Let us now systematically contrapose the main formulae for polar bialternals and bisymmetrals to their trigonometric equivalents.

The central-exceptional bisymmetrals $tal^\bullet /\!\!/ til^\bullet$ (invariant under *neg.pari* but neither *neg* nor *pari*) are still echanged by the involution *swap*, but only modulo *gari*-multiplication by an element $mana^\bullet \in Centre(GARI)$:

$$\begin{aligned}
&swap.pil^\bullet = pal^\bullet \quad \rightsquigarrow\\
&\mathrm{swap.til}^\bullet = \mathrm{gari}(\mathrm{tal}^\bullet, \mathrm{mana}^\bullet) = \mathrm{gari}(\mathrm{mana}^\bullet, \mathrm{tal}^\bullet) = \mathrm{mu}(\mathrm{tal}^\bullet, \mathrm{mana}^\bullet)
\end{aligned} \tag{418}$$

with

$$\mathrm{mana}^{\binom{u_1,\,\dots,\,u_{2r}}{0,\,\dots,\,0}} \equiv \gamma_{2\,r}\,c^{2\,r} \quad ; \quad \mathrm{mana}^{\binom{u_1,\,\dots,\,u_{2r+1}}{0,\,\dots,\,0}} \equiv 0 \qquad (\forall u_i) \tag{419}$$

and

$$1 + \sum_{1\leq r} \gamma_{2r}\,c^{2\,r} = \Big(\frac{\sin c}{c}\Big)^{\frac{1}{2}} = 1 - \frac{1}{12}\,c^2 + \frac{1}{1440}\,c^4 + \dots \tag{420}$$

§16.3. Proliferation of alternals and symmetrals.

In the eupolar case, $Flex_r(Pi)$ contains (for each r and up to scalar multiplication) exactly one alternal without poles of the form $(v_i - v_j)^{-1}$ with $|i-j| > 1$. By contrast, even with this restriction on the poles, $Flex_r(Qi)$ contains a much richer set of alternals. The corresponding *ari*-structure, with

[93]It would vanish under the specialisation $\mathfrak{E} \mapsto Qi \pm c\,I$ which, however, is not acceptable, since $Qi^{w_1} \pm c\,I^{w_1}$ is not odd in w_1.

its ideal of *'internals'* and its quotient of *'externals'* was investigated in [E3], §11.5 relative to a special basis, but there exist other useful bases.

In $Flex(Qa)$, on the other hand, the most relevant alternals are those freely generated by $Qa^\bullet$ and $c\,I^\bullet$ under the uninflected *lu*-bracket. The corresponding algebra $LU(Qa, c\,I)$ belongs to the extension $Flex(Qa, c\,I)$ rather than $Flex(Qa)$ but the subalgebra $LU^+(Qa, c\,I)$ consisting of all alternals *even* in c is embedded in $Flex(Qa)$ itself.

§16.4. New landscape of bialternals and bisymmetrals.

In the eupolar case, the involution *syap*, when applied to the bisymmetral pair $pal^\bullet /\!/ pil^\bullet$, directly produces another central bisymmetral pair $par^\bullet /\!/ pir^\bullet$ (in reverse order!) and indirectly leads to the *regular* (i.e. *neg*-invariant) bisymmetrals $lar^\bullet /\!/ lir^\bullet$ and $ral^\bullet /\!/ ril^\bullet$ that connect these two central pairs by *gari*-postcomposition. The regular bisymmetrals in turn generate a host of bialternals under *logari*. In the eutrigonometric setting, none of these objects survive and we are left with only one central pair $tal^\bullet /\!/ til^\bullet$.

$$pal^\bullet /\!/ pil^\bullet \quad and \quad par^\bullet /\!/ pir^\bullet \quad \rightsquigarrow \quad \mathrm{tal}^\bullet /\!/ \mathrm{til}^\bullet \quad \text{alone} \tag{421}$$

This does not mean, though, that $tal^\bullet /\!/ til^\bullet$ stands completely isolated. Just like $pal^\bullet /\!/ pil^\bullet$, it produces other irregular (i.e. *neg.pari*-invariant) bisymmetrals under postcomposition by regular (i.e. separately *neg*- and *pari*-invariant) bisymmetrals:[94]

$$gari(pal^\bullet, Sa^\bullet) /\!/ gari(pil^\bullet, Si^\bullet) \rightsquigarrow \mathrm{gari}(\mathrm{tal}^\bullet, \mathrm{Za}^\bullet) /\!/ \mathrm{gari}(\mathrm{til}^\bullet, \mathrm{Zi}^\bullet) \quad with \tag{422}$$

$$pal^\bullet /\!/ pil^\bullet \quad and \quad tal^\bullet /\!/ til^\bullet \ \in\ GARI^{\mathrm{as/as}} \quad (neg.pari\text{-}invariant) \tag{423}$$

$$Sa^\bullet /\!/ Si^\bullet \quad and \quad Za^\bullet /\!/ Zi^\bullet \ \in\ GARI^{\underline{\mathrm{as}}/\underline{\mathrm{as}}} \quad (neg\text{- } and\ pari\text{-}invariant) \tag{424}$$

But here again, the parallelism is only approximate: there exists between the two groups of regular bisymmetrals $Sa^\bullet /\!/ Si^\bullet$ and $Za^\bullet /\!/ Zi^\bullet$ a striking disparity, which extends to the corresponding *polar* and *trigonometric* bialternals. In concrete terms:

(i) The first polar (resp. trigonometric) bialternals appear at length $r = 4$ (resp. $r = 8$).

(ii) As already noted, if we ban all poles of type $(v_i - v_j)^{-1}$ with $|i - j| > 1$, we automatically ban all polar bialternals – which fact in turns leads to a neat characterisation of $pal^\bullet /\!/ pil^\bullet$ among all irregular bisymmetrals. On the

[94] Although the swappees of $gari(pal^\bullet, Sa^\bullet)$ and $gari(tal^\bullet, Za^\bullet)$ are a priori $gira(pil^\bullet, Si^\bullet)$ and $gira(til^\bullet, Zi^\bullet)$, the latter *gira*-products actually coincide with *gari*-products since the second factors $Sa^\bullet /\!/ Si^\bullet$ and $Za^\bullet /\!/ Zi^\bullet$ are *regular* bisymmetrals.

other hand, ruling out all poles of the afore-mentioned type still leaves room for a host of trigonometric bialternals - a circumstance which makes it much harder to isolate *the* canonical $tal^\bullet /\!\!/ til^\bullet$.
(iii) Most (all?) polar bialternals seem to have no trigonometic counterpart. This applies in particular to the polar bialternals constructed in §9 (type I) and §10 (type II).
(iv) Conversely, most (all?) trigonometric bialternals seem to have no polar prototype. This applies in particular to the trigonometric bialternals of the form

$$A^{w_1,\dots,w_{2r}} + B^{w_1,\dots,w_{2r}}\, Q(u_1 + \dots u_{2r}) \qquad (2r \geq 8) \tag{425}$$

with

$$A^\bullet \in \mathrm{LU}^+_{2r}(\mathrm{Q}, c.\mathrm{I}) \quad ; \quad B^\bullet \in \mathrm{LU}^-_{2r}(\mathrm{Q}, c.\mathrm{I}).c^{-1} \tag{426}$$

which, from $2r = 8$ onwards, introduce pesky indeterminacies[95] in the construction of the even factors $tal^\bullet_{ev}$ and $tal^\bullet_{evv}$, to be dealt with in the next paragraph.

§16.5. New pattern of even-odd factorisations.

All three *even/odd* factorisations familiar from the polar case survive in the trigonometric setting, but with predictable complications: the *odd* factors become less elementary, while the *even* factors split into *left* and *right* subfactors, marked by indices *lev/rev* or *levv/revv.* Thus:

$$\begin{array}{llllll}
pil^\bullet & = gari(pil^\bullet_{od}, pil^\bullet_{ev}) & \rightsquigarrow & & & \\
\mathrm{til}^\bullet & = \mathrm{gari}(\mathrm{til}^\bullet_{\mathrm{od}}, \mathrm{til}^\bullet_{ev}) & with & \mathrm{til}^\bullet_{\mathrm{ev}} & = \mathrm{mu}(\mathrm{til}^\bullet_{\mathrm{lev}}, \mathrm{til}^\bullet_{\mathrm{rev}}) & (Facto.\, I) \\
pal^\bullet & = gari(pal^\bullet_{od}, pal^\bullet_{ev}) & \rightsquigarrow & & & \\
\mathrm{tal}^\bullet & = \mathrm{gari}(\mathrm{tal}^\bullet_{\mathrm{od}}, \mathrm{tal}^\bullet_{ev}) & with & \mathrm{tal}^\bullet_{\mathrm{ev}} & = \mathrm{mu}(\mathrm{tal}^\bullet_{\mathrm{lev}}, \mathrm{tal}^\bullet_{\mathrm{rev}}) & (Facto.\, II) \\
pal^\bullet & = mu(pal^\bullet_{evv}, pal^\bullet_{odd}) & \rightsquigarrow & & & \\
tal^\bullet & = mu(tal^\bullet_{evv}, tal^\bullet_{odd}) & with & \mathrm{tal}^\bullet_{\mathrm{evv}} & = \mathrm{mu}(\mathrm{tal}^\bullet_{\mathrm{levv}}, \mathrm{tal}^\bullet_{\mathrm{revv}}) & (Facto.\, III)
\end{array}$$

As in the polar case, this leads to a slight awkwardness (which cannot be helped) in the notations, since $til^\bullet_{od}, til^\bullet_{ev}, til^\bullet_{lev}, til^\bullet_{rev}$ stand in no simple relation to $tal^\bullet_{od}, tal^\bullet_{ev}, tal^\bullet_{lev}, tal^\bullet_{rev}$ and in particular *are not* their swappees.

§16.6. Odd factors: less elementary.

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[95] Which, fortunately, *can be* removed. Even if they could not, they would be automatically offset by corrective terms in the $roma^\bullet$ factor of the classical multizeta decomposition $Zag^\bullet := gari(Zag^\bullet_I, Zag^\bullet_{II}, Zag^\bullet_{III})$ and $Zag^\bullet_I := gari(tal^\bullet, ripal^\bullet, roma^\bullet)$.

Let $paj^\bullet$, $pij^\bullet$ denote the elementary polar bimoulds defined in [E3] §4.3 and let $taj^\bullet$, $tij^\bullet$ denote their (still reasonably elementary) trigonometric counterparts: see [E3] §4.3, §4.5.

Further, for any $t \in \mathbb{Q}$ and any bimould $S^\bullet \in GARI$, let $gari_t(S^\bullet)$ denote the *gari*-iterate of order t of $S^\bullet$:

$$\mathrm{gari}_t(S^\bullet) := \mathrm{expari}(t\,\mathrm{logari}(S^\bullet))$$

The first polar-to-trigonometric transpostion involves some complication:

$$\begin{array}{rclcll} pil^\bullet_{od} &=& gari_{-\frac{1}{2}}(pij^\bullet) &=& (-\frac{1}{2})^{r(\bullet)}\, pij^\bullet & \rightsquigarrow \\ \mathrm{til}^\bullet_{\mathrm{od}} &=& \mathrm{gari}_{-\frac{1}{2}}(\mathrm{tij}^\bullet) &\neq& (-\frac{1}{2})^{r(\bullet)}\, \mathrm{tij}^\bullet & \end{array}$$

The inequality on the second line arises from the fact that, unlike in the polar case where we had an exact identity $logari(pij^\bullet) = Pi^\bullet$, in the trigonometric case we only have $logari(tij^\bullet) = Qi^\bullet \mod c^2$. Nonetheless, both $tij^\bullet$ and $gari_{-\frac{1}{2}}(tij^\bullet)$ possess remarkable *gari*-dilators whose components (barring the first one) belong to the internal ideal ARI_{intern}.

The second transpostion is more straightforward:

$$\begin{array}{rclcll} pal^\bullet_{od} &=& gari_{-\frac{1}{2}}(paj^\bullet) &=& (-\frac{1}{2})^{r(\bullet)}\, paj^\bullet & \rightsquigarrow \\ \mathrm{tal}^\bullet_{\mathrm{od}} &=& \mathrm{gari}_{-\frac{1}{2}}(\mathrm{taj}^\bullet) &=& (-\frac{1}{2})^{r(\bullet)}\, \mathrm{taj}^\bullet & \qquad (\textit{exactly}) \end{array}$$

the reason being that in both cases, polar and trigonometric, we now have exact identities:

$$logari(paj^\bullet) = Pa^\bullet \qquad , \qquad logari(taj^\bullet) = Qa^\bullet$$

No such simplification occurs in the third transposition

$$\begin{array}{rcll} pal^\bullet_{odd} &=& expmu(-\frac{1}{2}Pa^\bullet) & \rightsquigarrow \\ \mathrm{tal}^\bullet_{\mathrm{odd}} &\sim& \mathrm{expmu}(-\frac{1}{2}\mathrm{Qa}^\bullet + \mathrm{lutal}^\bullet_{\mathrm{odd}}) & \textit{with} \;\; \mathrm{lutal}^\bullet_{\mathrm{odd}} \in \mathrm{LU}^+(\mathrm{Qa}, c\,\mathrm{I}) \end{array}$$

and we are saddled with a corrective alternal bimould $lutal^\bullet_{\mathrm{odd}}$ with only (nonzero) odd-length components.

§16.7. Even factors: left and right subfactors.

The first even factor splits into a product (in both *GARI* and *MU*) of internal and external subfactors:[96]

$$\mathrm{til}^\bullet_{\mathrm{ev}} = \mathrm{gari}(\mathrm{til}^\bullet_{\mathrm{rev}}, \mathrm{til}^\bullet_{\mathrm{lev}}) = \mathrm{mu}(\mathrm{til}^\bullet_{\mathrm{lev}}, \mathrm{til}^\bullet_{\mathrm{rev}}) \quad (\textit{inversion}!) \tag{427}$$

$$\textit{with} \qquad \mathrm{til}^\bullet_{\mathrm{lev}} \in \mathrm{GARI}_{\mathrm{intern}} \;\; , \;\; \mathrm{til}^\bullet_{\mathrm{rev}} \in \mathrm{GARI}_{\mathrm{extern}} \tag{428}$$

[96] Cf the definitions in §1-11. Note that the identity (427) (middle term = right term) looks much like the identity (55) but works for slighly different reasons, namely because $til^\bullet_{lev}$ is internal and $til^\bullet_{rev}$ is $\boldsymbol{u}$-constant.

but the really interesting part is what happens to the second and third even factors, namely $\mathrm{tal}^\bullet_{ev}$ and $\mathrm{tal}^\bullet_{evv}$. Surprisingly enough, both split in exactly the same way:

$$\begin{array}{rcl} \mathrm{tal}^\bullet_{\mathrm{lev}} &=& \mathrm{Taj}^\bullet \circ \mathrm{dutal}^\bullet_{\mathrm{lev}} \quad ; \quad \mathrm{der.tal}^\bullet_{\mathrm{rev}} = \mathrm{mu}(\mathrm{tal}^\bullet_{\mathrm{rev}}, \mathrm{detal}^\bullet_{\mathrm{rev}}) \\ \mathrm{tal}^\bullet_{\mathrm{levv}} &=& \mathrm{Taj}^\bullet \circ \mathrm{dutal}^\bullet_{\mathrm{levv}} \quad ; \quad \mathrm{der.tal}^\bullet_{\mathrm{revv}} = \mathrm{mu}(\mathrm{tal}^\bullet_{\mathrm{rev}}, \mathrm{detal}^\bullet_{\mathrm{revv}}) \end{array}$$

with right/left subfactors similarly related to alternals of $\mathrm{LU}^{\pm}(\mathrm{Qa}, c.\mathrm{I})$:

$$\begin{array}{rcl} \mathrm{dutal}^\bullet_{\mathrm{lev}} \ \textit{and} \ \mathrm{dutal}^\bullet_{\mathrm{levv}} &\in& \mathrm{LU}^{-}(\mathrm{Qa}), c.\mathrm{I}).c^{-1} \\ \mathrm{detal}^\bullet_{\mathrm{rev}} \ \textit{and} \ \mathrm{detal}^\bullet_{\mathrm{revv}} &\in& \mathrm{LU}^{+}(\mathrm{Qa}), c.\mathrm{I}) \end{array}$$

Here, the alternals $dutal^\bullet_{lev}$, $dutal^\bullet_{levv}$ are rough equivalents of the *mu*-dilators $dupal^\bullet_{ev}$, $dupal^\bullet_{evv}$ familiar from the polar case, and the reverse passage (from the dilators to their sources) is via precomposition by the mould $Taj^\bullet$ (form-identical with $taj^\bullet$, but viewed as a mould rather than a bimould). As for the alternals $detal^\bullet_{rev}$, $detal^\bullet_{revv}$, they have no polar antecedents and are just another, particularly elementary sort of dilators.[97]

§16.8. Practical calculations.

The simplest way to calculate $tal^\bullet/til^\bullet$ and establish bisymmetrality is to adapt the approach of §4.3. But since (122) has no exact trigonometric equivalent, we must replace $dupal^\bullet$ by the *mu*-dilators of $tal^\bullet_{ev}$ or $tal^\bullet_{evv}$, and $dapal^\bullet$ by the swappee of the *gari*-dilators of $til^\bullet$ or $til^\bullet_{ev}$. So we have four options before us, all of which are practicable but none of which can be as straightforward as the polar prototype (209), not least due to the appearance, in the trigonometric case, of left and right subfactors (*"lev/rev"*).

This is the bad news. The good news is that the mere juxtaposition of the last two factorisations "Facto. II" and "Facto. III" of §16.5 already leads to a set of constraints that *very nearly* determine $tal^\bullet/til^\bullet$. This is hugely helpful, since the corresponding calculations essentially take place within the uninflected algebra $LU(Qa, c.I)$. The lengthy and in places very tedious details shall be set forth in [E4].

17 Basic prerequisites.

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[97]Instead of these, we might work with the slightly less simple alternals $logmu(tal^\bullet_{rev})$ and $logmu(tal^\bullet_{revv})$.

§17-1. Elementary flexions.

In addition to ordinary, non-commutative mould multiplication *mu* (or $\times$):

$$A^\bullet = B^\bullet \times C^\bullet = \mathrm{mu}(B^\bullet, C^\bullet) \iff A^{\boldsymbol{w}} = \sum_{\boldsymbol{w^1}.\boldsymbol{w^2}=\boldsymbol{w}}^{r(\boldsymbol{w^1}),r(\boldsymbol{w^2})\geq 0} B^{\boldsymbol{w^1}} C^{\boldsymbol{w^2}} \tag{429}$$

and its inverse *invmu*:

$$(\mathrm{invmu}.A)^{\boldsymbol{w}} = \sum_{1\leq s\leq r(\boldsymbol{w})} (-1)^s \sum_{\boldsymbol{w^1}\ldots\boldsymbol{w^s} = \boldsymbol{w}} A^{\boldsymbol{w^1}} \ldots A^{\boldsymbol{w^s}} \qquad (\boldsymbol{w^i} \neq \emptyset) \tag{430}$$

the bimoulds[98] $A^\bullet$ in $BIMU = \oplus_{0\leq r} BIMU_r$ can be subjected to a host of specific operations, all constructed from four elementary *flexions* $\lfloor, \rceil, \lceil, \rfloor$ that are always defined relative to a given factorisation of the total sequence $\boldsymbol{w}$. The way these flexions act is apparent from the following examples:

$$\begin{array}{llll}
\boldsymbol{w} = \boldsymbol{a}.\boldsymbol{b} & \boldsymbol{a} = \left(\begin{smallmatrix} u_1, & u_2, & u_3 \\ v_1, & v_2, & v_3 \end{smallmatrix}\right) & \boldsymbol{b} = \left(\begin{smallmatrix} u_4, & u_5, & u_6 \\ v_4, & v_5, & v_6 \end{smallmatrix}\right) & \\
\Longrightarrow & \boldsymbol{a}\rfloor = \left(\begin{smallmatrix} u_1, & u_2, & u_3 \\ v_{1:4}, & v_{2:4}, & v_{3:4} \end{smallmatrix}\right) & \lceil\boldsymbol{b} = \left(\begin{smallmatrix} u_{1234}, & u_5, & u_6 \\ v_4, & v_5, & v_6 \end{smallmatrix}\right) & \\
\boldsymbol{w} = \boldsymbol{b}.\boldsymbol{c} & \boldsymbol{b} = \left(\begin{smallmatrix} u_1, & u_2, & u_3 \\ v_1, & v_2, & v_3 \end{smallmatrix}\right) & \boldsymbol{c} = \left(\begin{smallmatrix} u_4, & u_5, & u_6 \\ v_4, & v_5, & v_6 \end{smallmatrix}\right) & \\
\Longrightarrow & \boldsymbol{b}\rceil = \left(\begin{smallmatrix} u_1, & u_2, & u_{3456} \\ v_1, & v_2, & v_3 \end{smallmatrix}\right) & \lfloor\boldsymbol{c} = \left(\begin{smallmatrix} u_4, & u_5, & u_6 \\ v_{4:3}, & v_{5:3}, & v_{6:3} \end{smallmatrix}\right) & \\
\boldsymbol{w} = \boldsymbol{a}.\boldsymbol{b}.\boldsymbol{c} & \boldsymbol{a} = \left(\begin{smallmatrix} u_1, & u_2, & u_3 \\ v_1, & v_2, & v_3 \end{smallmatrix}\right) & \boldsymbol{b} = \left(\begin{smallmatrix} u_4, & u_5, & u_6 \\ v_4, & v_5, & v_6 \end{smallmatrix}\right) & \boldsymbol{c} = \left(\begin{smallmatrix} u_7, & u_8, & u_9 \\ v_7, & v_8, & v_9 \end{smallmatrix}\right) \\
\Longrightarrow & \boldsymbol{a}\rfloor = \left(\begin{smallmatrix} u_1, & u_2, & u_3 \\ v_{1:4}, & v_{2:4}, & v_{3:4} \end{smallmatrix}\right) & \lceil\boldsymbol{b}\rceil = \left(\begin{smallmatrix} u_{1234}, & u_5, & u_{6789} \\ v_4, & v_5, & v_6 \end{smallmatrix}\right) & \lfloor\boldsymbol{c} = \left(\begin{smallmatrix} u_7, & u_8, & u_9 \\ v_{7:6}, & v_{8:6}, & v_{9:6} \end{smallmatrix}\right)
\end{array}$$

with the usual short-hand: $u_{i,\ldots,j} := u_i + \ldots + u_j$ and $v_{i:j} := v_i - v_j$. Here and throughout the sequel, we use boldface (with upper indexation) to denote sequences ($\boldsymbol{w}, \boldsymbol{w^i}, \boldsymbol{w^j}$ etc), and ordinary fonts (with lower indexation) to denote single sequence elements (w_i, w_j etc), or sometimes sequences of length $r(\boldsymbol{w}) = 1$. Of course, the 'product' $\boldsymbol{w^1}.\boldsymbol{w^2}$ denotes the concatenation of the two factor sequences.

§17-2. Short and long indexations on bimoulds.

For bimoulds $M^\bullet \in BIMU_r$ it is sometimes convenient to switch from the usual *short indexation* (with r indices w_i's) to a more homogeneous *long indexation* (with a redundant initial w_0 that gets bracketed for distinctiveness). The correspondence goes like this:

$$M^{\left(\begin{smallmatrix} u_1 & ,\ldots, & u_r \\ v_1 & ,\ldots, & v_r \end{smallmatrix}\right)} \cong M^{\left(\begin{smallmatrix} [u_0^*], & u_1^* & ,\ldots, & u_r^* \\ [v_0^*], & v_1^* & ,\ldots, & v_r^* \end{smallmatrix}\right)} \tag{431}$$

[98] $BIMU_r$ of course regroups all bimoulds whose components of length other than r vanish. These are often dubbed "length-r bimoulds" for short.

with the dual conditions on upper and lower indices:

$$\begin{array}{llll} u_0^* = -u_{1\dots r} := -(u_1 + \dots + u_r) & , & u_i^* \quad\quad = u_i & \forall i \geq 1 \\ v_0^* \quad \textit{arbitrary} & , & v_i^* - v_0^* = v_i & \forall i \geq 1 \end{array}$$

and of course $\sum_{1 \leq i \leq r} u_i v_i \equiv \sum_{0 \leq i \leq r} u_i^* v_i^*$.

§17-3. Unary operations.

The following linear transformations on *BIMU* are of constant use:

$$B^\bullet = \mathrm{minu}.A^\bullet \;\Rightarrow\; B^{w_1,\dots,w_r} = -\,A^{w_1,\dots,w_r} \tag{432}$$

$$B^\bullet = \mathrm{pari}.A^\bullet \;\Rightarrow\; B^{w_1,\dots,w_r} = (-1)^r\,A^{-w_1,\dots,-w_r} \tag{433}$$

$$B^\bullet = \mathrm{anti}.A^\bullet \;\Rightarrow\; B^{w_1,\dots,w_r} = A^{w_r,\dots,w_1} \tag{434}$$

$$B^\bullet = \mathrm{mantar}.A^\bullet \;\Rightarrow\; B^{w_1,\dots,w_r} = (-1)^{r-1}\,A^{w_r,\dots,w_1} \tag{435}$$

$$B^\bullet = \mathrm{neg}.A^\bullet \;\Rightarrow\; B^{w_1,\dots,w_r} = A^{-w_1,\dots,-w_r} \tag{436}$$

$$B^\bullet = \mathrm{swap}.A^\bullet \;\Rightarrow\; B^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ v_1 & ,\dots, & v_r \end{smallmatrix}\right)} = A^{\left(\begin{smallmatrix} v_r & ,\dots, & v_{3:4} & , v_{2:3}, & v_{1:2} \\ u_{1..r} & ,\dots, & u_{123}, & u_{12}, & u_1 \end{smallmatrix}\right)} \tag{437}$$

$$B^\bullet = \mathrm{pus}.A^\bullet \;\Rightarrow\; B^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ v_1 & ,\dots, & v_r \end{smallmatrix}\right)} = A^{\left(\begin{smallmatrix} u_r, & u_1, & u_2 & ,\dots, & u_{r-1} \\ v_r, & v_1, & v_2 & ,\dots, & v_{r-1} \end{smallmatrix}\right)} \tag{438}$$

$$B^\bullet = \mathrm{push}.A^\bullet \;\Rightarrow\; B^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ v_1 & ,\dots, & v_r \end{smallmatrix}\right)} = A^{\left(\begin{smallmatrix} -u_{1\dots r}, & u_1 & , u_2 & ,\dots, & u_{r-1} \\ -v_r & , v_{1:r}, & v_{2:r} & ,\dots, & v_{r-1:r} \end{smallmatrix}\right)} \tag{439}$$

All are involutions, save for *pus* and *push*, whose restrictions to each $BIMU_r$ reduce to circular permutations of order r resp. $r+1$:[99]

$$\mathrm{push} \;=\; \mathrm{neg.anti.swap.anti.swap} \tag{440}$$

$$\mathrm{leng}_r \;=\; \mathrm{push}^{r+1}.\mathrm{leng}_r \;=\; \mathrm{pus}^r.\mathrm{leng}_r \tag{441}$$

§17-4. Inflected derivations and automorphisms of *BIMU*.

Let $BIMU_*$ resp. $BIMU^*$ denote the subset of all bimoulds $M^\bullet$ such that $M^\emptyset = 0$ resp. $M^\emptyset = 1$. To each pair $\mathcal{A}^\bullet = (\mathcal{A}_L^\bullet, \mathcal{A}_R^\bullet) \in BIMU_* \times BIMU_*$ resp. $BIMU^* \times BIMU^*$ we attach two remarkable operators:

$$\mathrm{axit}(\mathcal{A}^\bullet) \in \mathrm{Der}(BIMU) \qquad \textit{resp.} \qquad \mathrm{gaxit}(\mathcal{A}^\bullet) \in \mathrm{Aut}(BIMU)$$

[99] *pus* resp. *push* is a circular permutation in the *short* resp. *long* indexation of bimoulds. Indeed: $(push.M)^{[w_0],w_1,\dots,w_r} = M^{[w_r],w_0,\dots,w_{r-1}}$.

whose action on *BIMU* is given by:[100]

$$N^\bullet = \mathrm{axit}(\mathcal{A}^\bullet).M^\bullet \Leftrightarrow N^{\boldsymbol{w}} = \sum\nolimits^1 M^{\boldsymbol{a}\lceil\boldsymbol{c}}\mathcal{A}_L^{\boldsymbol{b}\rfloor} + \sum\nolimits^2 M^{\boldsymbol{a}\rfloor\boldsymbol{c}}\mathcal{A}_R^{\lfloor\boldsymbol{b}} \tag{442}$$

$$N^\bullet = \mathrm{gaxit}(\mathcal{A}^\bullet).M^\bullet \Leftrightarrow N^{\boldsymbol{w}} = \sum\nolimits^3 M^{\lceil\boldsymbol{b^1}\rceil\cdots\lceil\boldsymbol{b^s}\rceil}\mathcal{A}_L^{\boldsymbol{a^1}\rfloor}\ldots\mathcal{A}_L^{\boldsymbol{a^s}\rfloor}\mathcal{A}_R^{\lfloor\boldsymbol{c^1}}\ldots\mathcal{A}_R^{\lfloor\boldsymbol{c^s}} \tag{443}$$

and verifies the identities:

$$\mathrm{axit}(\mathcal{A}^\bullet).\mathrm{mu}(M_1^\bullet, M_2^\bullet) \equiv \mathrm{mu}(\mathrm{axit}(\mathcal{A}^\bullet).M_1^\bullet, M_2^\bullet) + \mathrm{mu}(M_1^\bullet, \mathrm{axit}(\mathcal{A}^\bullet).M_2^\bullet) \tag{444}$$

$$\mathrm{gaxit}(\mathcal{A}^\bullet).\mathrm{mu}(M_1^\bullet, M_2^\bullet) \equiv \mathrm{mu}(g\mathrm{axit}(\mathcal{A}^\bullet).M_1^\bullet, \mathrm{gaxit}(\mathcal{A}^\bullet).M_2^\bullet) \tag{445}$$

The *BIMU*-derivations *axit* are stable under the Lie bracket for operators. More precisely, the identity holds:

$$[\mathrm{axit}(\mathcal{B}^\bullet), \mathrm{axit}(\mathcal{A}^\bullet)] = \mathrm{axit}(C^\bullet) \quad \textit{with} \quad \mathcal{C}^\bullet = \mathrm{axi}(\mathcal{A}^\bullet, \mathcal{B}^\bullet) \tag{446}$$

relative to a Lie law *axi* on $BIMU_* \times BIMU_*$ given by:

$$\mathcal{C}_L^\bullet \;:=\; \mathrm{axit}(\mathcal{B}^\bullet).\mathcal{A}_L^\bullet - \mathrm{axit}(\mathcal{A}^\bullet).\mathcal{B}_L^\bullet + \mathrm{lu}(\mathcal{A}_L^\bullet, \mathcal{B}_L^\bullet) \tag{447}$$

$$\mathcal{C}_R^\bullet \;:=\; \mathrm{axit}(\mathcal{B}^\bullet).\mathcal{A}_R^\bullet - \mathrm{axit}(\mathcal{A}^\bullet).\mathcal{B}_R^\bullet - \mathrm{lu}(\mathcal{A}_R^\bullet, \mathcal{B}_R^\bullet) \tag{448}$$

Here, *lu* denotes the standard (non-inflected) Lie law on *BIMU*:

$$\mathrm{lu}(A^\bullet, B^\bullet) := \mathrm{mu}(A^\bullet, B^\bullet) - \mathrm{mu}(B^\bullet, A^\bullet) \tag{449}$$

Let *AXI* denote the Lie algebra consisting of all pairs $\mathcal{A}^\bullet \in BIMU_* \times BIMU_*$ under this law *axi*.

Likewise, the *BIMU*-automorphisms *gaxit* are stable under operator composition. More precisely:

$$\mathrm{gaxit}(\mathcal{B}^\bullet).\mathrm{gaxit}(\mathcal{A}^\bullet) = \mathrm{gaxit}(\mathcal{C}^\bullet) \quad \textit{with} \quad \mathrm{gaxi}(\mathcal{A}^\bullet, \mathcal{B}^\bullet) \tag{450}$$

relative to a law *gaxi* on $BIMU^* \times BIMU^*$ given by:

$$\mathcal{C}_L^\bullet \;:=\; \mathrm{mu}(\mathrm{gaxit}(\mathcal{B}^\bullet).\mathcal{A}_L^\bullet, \mathcal{B}_L^\bullet) \tag{451}$$

$$\mathcal{A}_R^\bullet \;:=\; \mathrm{mu}(\mathcal{B}_R^\bullet, \mathrm{gaxit}(\mathcal{B}^\bullet).\mathcal{A}_R^\bullet) \tag{452}$$

Let *GAXI* denote the Lie group consisting of all pairs $\mathcal{A}^\bullet \in BIMU^* \times BIMU^*$ under this law *gaxi*.

[100] The sum $\sum^1$ resp. $\sum^2$ extends to all sequence factorisations $\boldsymbol{w} = \boldsymbol{a}.\boldsymbol{b}.\boldsymbol{c}$ with $\boldsymbol{b} \neq \emptyset$, $\boldsymbol{c} \neq \emptyset$ resp. $\boldsymbol{a} \neq \emptyset$, $\boldsymbol{b} \neq \emptyset$. The sum $\sum^3$ extends to all factorisations $\boldsymbol{w} = \boldsymbol{a^1}.\boldsymbol{b^1}.\boldsymbol{c^1}.\boldsymbol{a^2}.\boldsymbol{b^2}.\boldsymbol{c^2}...\boldsymbol{a^s}.\boldsymbol{b^s}.\boldsymbol{c^s}$ such that $s \geq 1$, $\boldsymbol{b^i} \neq \emptyset$, $\boldsymbol{c^i}.\boldsymbol{a^{i+1}} \neq \emptyset$ $\forall i$. Note that the extreme factor sequences $\boldsymbol{a^1}$ and $\boldsymbol{c^s}$ may be $\emptyset$.

§17-5. The mixed operations *amnit* = *anmit*:

For $\mathcal{A}^\bullet := (A^\bullet, 0^\bullet)$ and $\mathcal{B}^\bullet := (0^\bullet, B^\bullet)$ the operators $axit(\mathcal{A}^\bullet)$ and $axit(\mathcal{B}^\bullet)$ reduce to $amit(A^\bullet)$ and $anit(B^\bullet)$ respectively, and the identity (446) becomes:

$$\mathrm{amnit}(A^\bullet, B^\bullet) \equiv \mathrm{anmit}(A^\bullet, B^\bullet) \qquad (\forall A^\bullet, B^\bullet \in \mathrm{BIMU}_*) \tag{453}$$

with

$$\mathrm{amnit}(A^\bullet, B^\bullet) \ := \ \mathrm{amit}(A^\bullet).\mathrm{anit}(B^\bullet) - \mathrm{anit}(\mathrm{amit}(A^\bullet).B^\bullet) \tag{454}$$

$$\mathrm{anmit}(A^\bullet, B^\bullet) \ := \ \mathrm{anit}(B^\bullet).\mathrm{amit}(A^\bullet) - \mathrm{amit}(\mathrm{anit}(B^\bullet).A^\bullet) \tag{455}$$

When one of the two arguments $(A^\bullet, B^\bullet)$ vanishes, the definitions reduce to:

$$\mathrm{amnit}(A^\bullet, 0^\bullet) = \mathrm{anmit}(A^\bullet, 0^\bullet) := \mathrm{amit}(A^\bullet) \tag{456}$$

$$\mathrm{amnit}(0^\bullet, B^\bullet) = \mathrm{anmit}(0^\bullet, B^\bullet) = \mathrm{anit}(B^\bullet) \tag{457}$$

Moreover, when *amnit* operates on a length-1 bimould $M^\bullet \in BIMU_1$ (such as a *flexion units* $\mathfrak{E}^\bullet$, see §17-2 *infra*), its action drastically simplifies:

$$N^\bullet := \mathrm{amnit}(A^\bullet, B^\bullet).M^\bullet \equiv \mathrm{anmit}(A^\bullet, B^\bullet).M^\bullet \Leftrightarrow N^{\boldsymbol{w}} := \sum_{\boldsymbol{a}\, w_i \boldsymbol{b} = \boldsymbol{w}} A^{\boldsymbol{a}\rfloor} M^{\lceil w_i \rceil} B^{\lfloor \boldsymbol{b}} \tag{458}$$

§17-6. Unary substructures.

We have two obvious subalgebras//subgroups of *ARI//GARI*, answering to the conditions:

$$\begin{array}{lll} \mathrm{AMI} \subset \mathrm{AXI}: \ \mathcal{A}^\bullet_R = 0^\bullet & , & \mathrm{GAMI} \subset \mathrm{GAXI}: \ \mathcal{A}^\bullet_R = 1^\bullet \\ \mathrm{ANI} \subset \mathrm{AXI}: \ \mathcal{A}^\bullet_L = 0^\bullet & , & \mathrm{GANI} \subset \mathrm{GAXI}: \ \mathcal{A}^\bullet_L = 1^\bullet \end{array}$$

but we are more interested in the *mixed* unary substructures, consisting of elements of the form:

$$\mathcal{A}^\bullet = (\mathcal{A}^\bullet_L, \mathcal{A}^\bullet_R) \quad \textit{with} \quad \mathcal{A}^\bullet_R \equiv \mathrm{h}(\mathcal{A}^\bullet_L) \quad \textit{and } \mathrm{h} \textit{ a fixed involution} \tag{459}$$

with everything expressible in terms of the left element $\mathcal{A}^\bullet_L$ of the pair $\mathcal{A}^\bullet$. There exist, up to isomorphism, exactly seven such mixed unary substructures:

algebra	h	*swap*	algebra	h
....	....	....	....	....
ARI	$minu$	$\leftrightarrow$	IRA	$minu.push$
ALI	$anti.pari$	$\leftrightarrow$	ILA	$anti.pari.neg$
ALA	$anti.pari.neg_u$	$\leftrightarrow$	ALA	$anti.pari.neg_u$
ILI	$anti.pari.neg_v$	$\leftrightarrow$	ILI	$anti.pari.neg_v$
AWI	$anti$	$\leftrightarrow$	IWA	$anti.neg$
AWA	$anti.neg_u$	$\leftrightarrow$	AWA	$anti.neg_u$
IWI	$anti.neg_v$	$\leftrightarrow$	IWI	$anti.neg_v$

group	h	*swap*	group	h
....	....	....	....	....
GARI	$invmu$	$\leftrightarrow$	GIRA	$push.swap.invmu.swap$
GALI	$anti.pari$	$\leftrightarrow$	GILA	$anti.pari.neg$
GALA	$anti.pari.neg_u$	$\leftrightarrow$	GALA	$anti.pari.neg_u$
GILI	$anti.pari.neg_v$	$\leftrightarrow$	GILI	$anti.pari.neg_v$
GAWI	$anti$	$\leftrightarrow$	GIWA	$anti.neg$
GAWA	$anti.neg_u$	$\leftrightarrow$	GAWA	$anti.neg_u$
GIWI	$anti.neg_v$	$\leftrightarrow$	GIWI	$anti.neg_v$

§17-7. Dimorphic substructures.

Among all seven pairs of substructures, only two respect dimorphy, namely *ARI//GARI* and *ALI//GALI*. Moreover, when restricted to dimorphic objects, they actually coincide:

$$\begin{array}{rll} \mathrm{ARI}^{\underline{\mathrm{al}}/\underline{\mathrm{al}}} = \mathrm{ALI}^{\underline{\mathrm{al}}/\underline{\mathrm{al}}} & \textit{with} & \{\underline{\mathrm{al}}/\underline{\mathrm{al}}\} = \{\text{alternal/alternal } \textit{and} \text{ even}\} \\ \mathrm{GARI}^{\underline{\mathrm{as}}/\underline{\mathrm{as}}} = \mathrm{GALI}^{\underline{\mathrm{as}}/\underline{\mathrm{as}}} & \textit{with} & \{\underline{\mathrm{as}}/\underline{\mathrm{as}}\} = \{\text{symmetral/symmetral } \textit{and} \text{ even}\} \end{array}$$

We shall henceforth work with the pair *ARI//GARI*, whose definition involves a simpler involution h (it dispenses with the sequence inversion *anti*: see above table).

§17-8. The algebra *ARI* and its group *GARI*: basic anti-actions

The proper way to proceed is to define the anti-actions (on *BIMU*, with its uninflected product *mu* and bracket *lu*) first of the lateral pairs *AMI//GAMI*, *ANI//GANI* and then of the mixed pair *ARI//GARI*:

$$N^\bullet = \mathrm{amit}(A^\bullet).M^\bullet \;\Leftrightarrow\; N^{\boldsymbol{w}} = \sum\nolimits^1 M^{\boldsymbol{a}\lceil\boldsymbol{c}} A^{\boldsymbol{b}\rfloor} \tag{460}$$

$$N^\bullet = \mathrm{anit}(A^\bullet).M^\bullet \;\Leftrightarrow\; N^{\boldsymbol{w}} = \sum\nolimits^2 M^{\boldsymbol{a}\rfloor\boldsymbol{c}} A^{\lfloor\boldsymbol{b}} \tag{461}$$

$$N^\bullet = \mathrm{arit}(A^\bullet).M^\bullet \;\Leftrightarrow\; N^{\boldsymbol{w}} = \sum\nolimits^1 M^{\boldsymbol{a}\lceil\boldsymbol{c}} A^{\boldsymbol{b}\rfloor} - \sum\nolimits^2 M^{\boldsymbol{a}\rfloor\boldsymbol{c}} A^{\lfloor\boldsymbol{b}} \tag{462}$$

with sums $\sum^1$ (resp. $\sum^2$) ranging over all sequence factorisations $\boldsymbol{w} = \boldsymbol{abc}$ such that $\boldsymbol{b} \neq \emptyset, \boldsymbol{c} \neq \emptyset$ (resp. $\boldsymbol{a} \neq \emptyset, \boldsymbol{b} \neq \emptyset$).

$$N^\bullet = \mathrm{gamit}(A^\bullet).M^\bullet \;\Leftrightarrow\; N^{\boldsymbol{w}} = \sum\nolimits^1 M^{\lceil\boldsymbol{b^1}\dots\lceil\boldsymbol{b^s}} A^{\boldsymbol{a^1}\rfloor}\dots A^{\boldsymbol{a^s}\rfloor} \tag{463}$$

$$N^\bullet = \mathrm{ganit}(A^\bullet).M^\bullet \;\Leftrightarrow\; N^{\boldsymbol{w}} = \sum\nolimits^2 M^{\boldsymbol{b^1}\rceil\dots\boldsymbol{b^s}\rceil} A^{\lfloor\boldsymbol{c^1}}\dots A^{\lfloor\boldsymbol{c^s}} \tag{464}$$

$$N^\bullet = \mathrm{garit}(A^\bullet).M^\bullet \;\Leftrightarrow\; N^{\boldsymbol{w}} = \sum\nolimits^3 M^{\lceil\boldsymbol{b^1}\rceil\dots\lceil\boldsymbol{b^s}\rceil} A^{\boldsymbol{a^1}\rfloor}\dots A^{\boldsymbol{a^s}\rfloor} A_*^{\lfloor\boldsymbol{c^1}}\dots A_*^{\lfloor\boldsymbol{c^s}} \tag{465}$$

with $A_*^\bullet := \mathrm{invmu}(\mathrm{A}^\bullet)$ and with sums $\sum^1, \sum^2, \sum^3$ ranging respectively over all sequence factorisations of the form :

$$\begin{array}{lcll}
\boldsymbol{w} & = & \boldsymbol{a^1 b^1 \dots a^s b^s} & (s \geq 1 \;\;,\;\; \textit{only } \boldsymbol{a^1} \textit{ may be } \emptyset) \\
\boldsymbol{w} & = & \boldsymbol{b^1 c^1 \dots b^s c^s} & (s \geq 1 \;\;,\;\; \textit{only } \boldsymbol{c^s} \textit{ may be } \emptyset) \\
\boldsymbol{w} & = & \boldsymbol{a^1 b^1 c^1 \dots a^s b^s c^s} & (s \geq 1 \;\;,\;\; \textit{with } \boldsymbol{b^i} \neq \emptyset \textit{ and } \boldsymbol{c^i a^{i+1}} \neq \emptyset)
\end{array}$$

More precisely, in $\sum^3$ two *inner* neigbour factors $\boldsymbol{c^i}$ and $\boldsymbol{a^{i+1}}$ may vanish separately but not simultaneously, whereas the *outer* factors $\boldsymbol{a^1}$ and $\boldsymbol{c^s}$ may of course vanish separately or even simultaneously.

§17-9. The algebra *ARI* and its group *GARI*: Lie brackets and group laws.

We can now concisely express the Lie brackets *ami, ani, ari* and the group products *gami, gani, gari* :

$$\mathrm{ami}(A^\bullet, B^\bullet) := \mathrm{amit}(B^\bullet).A^\bullet - \mathrm{amit}(A^\bullet).B^\bullet + \mathrm{lu}(A^\bullet, B^\bullet) \tag{466}$$

$$\mathrm{ani}(A^\bullet, B^\bullet) := \mathrm{anit}(B^\bullet).A^\bullet - \mathrm{anit}(A^\bullet).B^\bullet - \mathrm{lu}(A^\bullet, B^\bullet) \tag{467}$$

$$\mathrm{ari}(A^\bullet, B^\bullet) := \mathrm{arit}(B^\bullet).A^\bullet - \mathrm{arit}(A^\bullet).B^\bullet + \mathrm{lu}(A^\bullet, B^\bullet) \tag{468}$$

$$\mathrm{gami}(A^\bullet, B^\bullet) := \mathrm{mu}(\mathrm{gamit}(B^\bullet).A^\bullet), B^\bullet) \tag{469}$$

$$\mathrm{gani}(A^\bullet, B^\bullet) := \mathrm{mu}(B^\bullet, \mathrm{ganit}(B^\bullet).A^\bullet)) \tag{470}$$

$$\mathrm{gari}(A^\bullet, B^\bullet) := \mathrm{mu}(\mathrm{garit}(B^\bullet).A^\bullet), B^\bullet) \tag{471}$$

§17-10. The algebra ARI and its group $GARI$: pre-Lie brackets.

Parallel with the three Lie brackets, we have three pre-Lie brackets:

$$\begin{aligned}
\mathrm{preami}(A^\bullet, B^\bullet) &:= \mathrm{amit}(B^\bullet).A^\bullet + \mathrm{mu}(A^\bullet, B^\bullet) && (472)\\
\mathrm{preani}(A^\bullet, B^\bullet) &:= \mathrm{anit}(B^\bullet).A^\bullet - \mathrm{mu}(A^\bullet, B^\bullet) \quad (\textit{sign!}) && (473)\\
\mathrm{preari}(A^\bullet, B^\bullet) &:= \mathrm{arit}(B^\bullet).A^\bullet + \mathrm{mu}(A^\bullet, B^\bullet) && (474)
\end{aligned}$$

with the usual relations:

$$\begin{aligned}
\mathrm{ari}(A^\bullet, B^\bullet) &\equiv \mathrm{preari}(A^\bullet, B^\bullet) - \mathrm{preari}(B^\bullet, A^\bullet) && (475)\\
\mathrm{assopreari}(A^\bullet, B^\bullet, C^\bullet) &\equiv \mathrm{assopreari}(A^\bullet, C^\bullet, B^\bullet) && (476)
\end{aligned}$$

with *assopreari* denoting the *associator* of the pre-Lie bracket *preari*. The same holds of course for *ami* and *ani*.

§17-11. Exponentiation from ARI to $GARI$.

Provided we properly define the multiple pre-Lie brackets, i.e. from left to right:

$$\vec{\mathrm{preari}}(A_1^\bullet, \ldots, A_s^\bullet) = \mathrm{preari}(\vec{\mathrm{preari}}(A_1^\bullet, \ldots, A_{s-1}^\bullet), A_s^\bullet) \tag{477}$$

we have a simple expression for the exponential mapping from a Lie algebra to its group. Thus, the exponential *expari* : $ARI \to GARI$ can be expressed as a series of pre-brackets:

$$\mathrm{expari}(A^\bullet) = \sum_n \frac{1}{n!}\, \vec{\mathrm{preari}}(\overbrace{A^\bullet, \ldots, A^\bullet}^{n\ times}) \tag{478}$$

§17-12. Flexion units.

A flexion unit $\mathfrak{E}$ is an element of $BIMU_1$ that verifies identically

$$\begin{aligned}
0 &\equiv \mathfrak{E}^{\binom{u_1}{v_1}} + \mathfrak{E}^{\binom{-u_1}{-v_1}} && (479)\\
0 &\equiv \mathfrak{E}^{\binom{u_1}{v_1}}\mathfrak{E}^{\binom{u_2}{v_2}} - \mathfrak{E}^{\binom{u_{1,2}}{v_1}}\mathfrak{E}^{\binom{u_2}{v_{2:1}}} - \mathfrak{E}^{\binom{u_{1,2}}{v_1}}\mathfrak{E}^{\binom{u_1}{v_{1:2}}} && (480)
\end{aligned}$$

The above identities may be rewritten as

$$0 \equiv \Big(\sum_{0\le n<r} \mathrm{push}^n\Big)\, \mathrm{mu}(\overbrace{\mathfrak{E}^\bullet, \ldots, \mathfrak{E}^\bullet}^{r\ times}) \tag{481}$$

for $r=1$ and 2, but they actually imply (481) for *all* values of r.

The present paper deals mainly with the *polar units Pa*, *Pi*:

$$\mathrm{Pa}^{w_1} := P(u_1) \;=\; \frac{1}{u_1} \quad , \quad \mathrm{Pi}^{w_1} := P(v_1) \;=\; \frac{1}{v_1} \tag{482}$$

and occasionally with the approximate *trigonometric units Qa*, *Qi*:

$$\mathrm{Qa}^{w_1} := Q(u_1) \;=\; \frac{c}{\tan(c\,u_1)} \quad , \quad \mathrm{Qi}^{w_1} := Q(v_1) \;=\; \frac{c}{\tan(c\,v_1)} \tag{483}$$

for which the expression on the right side of (480), instead of vanishing, becomes $\pm\, c^2$.

For a more substantive exposition of the flexion structure, we refer to [E1] *and* [E3].

18 Tables and Maple programs.

§18.A. MAPLE PROGRAMS.

§18.A.1. Standard eupolar bases.

The following commands **sekatal***(r),***sekitil***(r),* **seketel***(r) calculate the standard bases respectively of* $\mathrm{Flex}_r(\mathrm{Pa})$*,* $\mathrm{Flex}_r(\mathrm{Pi})$*,* $\mathrm{Flex}_r(\mathrm{E})$*, with* E *standing for a general flexion unit* $\mathfrak{E}^{\bullet}$*.*

kat:=n $\to$ $(2*n)!/n!/(n+1)!$:

faa :=proc(p,q): proc(X):
subs(seq(u‖(q+1-k)=u‖(q+1-k+p),k=1..q), X) end: end:
fii:=proc(p,q): proc(X): subs(seq(v‖(q+1-k)=v‖(q+1-k+p),k=1..q),
seq(v‖k=v‖k-v‖p,k=p+1..p+q), X) end: end:
fee:=proc(p,q): proc(X): **fii**(p,q)(**faa**(p,q)(X)) end: end:

gaa:=proc(p,q): proc(X): X end: end:
gii:=proc(p,q): proc(X): subs(seq(v‖k=v‖k-v‖p,k=1..p-1),X) end: end:
gee:=gii:

Faa:=proc(p,q): proc(S): [seq(**faa**(p,q)(op(s,S)),s=1..nops(S))] end: end:
Gaa:=proc(p,q): proc(S): [seq(**gaa**(p,q)(op(s,S)),s=1..nops(S))] end: end:

Fii:=proc(p,q): proc(S): [seq(**fii**(p,q)(op(s,S)),s=1..nops(S))] end: end:
Gii:=proc(p,q): proc(S): [seq(**gii**(p,q)(op(s,S)),s=1..nops(S))] end: end:

Fee:=proc(p,q): proc(S): [seq(**fee**(p,q)(op(s,S)),s=1..nops(S))] end: end:
Gee:=proc(p,q): proc(S): [seq(**gee**(p,q)(op(s,S)),s=1..nops(S))] end: end:

Gluu:=proc(S1,S2,S3): seq(seq(op(s1,S1)∗op(s2,S2)∗S3,
s1=1..nops(S1)),s2=1..nops(S2)) end:

kaa:=proc(p,q): P(add(u||k,k=1..p+q)) end:
kii :=proc(p,q): P(v||p) end:
kee:=proc(p,q): E(add(u||k,k=1..p+q))(v||p) end:

sekatal:=proc(r) option remember; if r=0 then [1] elif r=1 then [P(u1)] else
[seq(**Gluu**(**Gaa**(r-k,k)(**sekatal**(r-1-k)),**Faa**(r-k,k)(**sekatal**(k)),**kaa**(r-k,k)),
k=0..r-1)] fi end:

sekitil:=proc(r) option remember; if r=0 then [1] elif r=1 then [P(v1)] else
[seq(**Gluu**(**Gii**(r-k,k)(**sekitil**(r-1-k)),**Fii**(r-k,k)(**sekitil**(k)),**kii**(r-k,k)),
k=0..r-1)] fi end:

seketel:=proc(r) option remember; if r=0 then [1] elif r=1 then [E(u1)(v1)]
else [seq(**Gluu**(**Gee**(r-k,k)(**seketel**(r-1-k)),**Fee**(r-k,k)(**seketel** (k)),**kee**(r-k,k)),
k=0..r-1)] fi end:

§18.A.2. Standard eupolar projectors.

kat:=n→(2∗n)!/n!/(n+1)!:
fe:=proc(n): proc(X): [seq(n+op(k,X),k=1..nops(X))] end: end:
Fe:=proc(n): proc(XX): [seq(fe(n)(op(kk,XX)),kk=1..nops(XX))] end: end:

glu:=proc(X1,X2,X3): [op(X1),op(X2),op(X3)] end:
Glu:=proc(S1,S2,S3): seq(seq(**glu**(op(kk1,S1),op(kk2,S2),X3),
s1=1..nops(S1)),s2=1..nops(S2)) end:

sekat:=proc(r) option remember; if r=0 then [[]] elif r=1 then [[1]] else
[seq(**Glu**(**sekat**(r-1-k) ,**Fe**(r-k)(**sekat**(k)),[r-k]),k=0..r-1)] fi end:

kow:=proc(x): proc(X): subs(x=0, coeff(X,P(x))-coeff(X,P(-x))) end: end:

koka:=proc(r): proc(m): proc(K) option remember;

if r=1 then **kow**(u‖(op(-1,op(m,K))))
elif r>1 then koka(r-1)(m)(K)@kow(u‖(op(-r,op(m,K)))) fi end: end: end:
koki:=proc(r): proc(m): proc(K) option remember;
if r=1 then **kow**(v‖(op(1,op(m,K))))
elif r>1 then koki(r-1)(m)(K)@kow(v‖(op(r,op(m,K)))) fi end: end: end:

kokata:=proc(r,m) option remember; **koka**(r)(m)(**sekat**(r)) end:
kokiti:=proc(r,m) option remember; **koki**(r)(m)(**sekat**(r)) end:

vokata:=proc(r): proc(X) option remember;
[seq(**kokata**(r,m)(X),m=1..kat(r))] end: end:
vokiti:=proc(r): proc(X) option remember;
[seq(**kokiti**(r,m)(X),m=1..kat(r))] end: end:

Comment : **vokata***(r)(X) resp.* **vokata***(r)(X) projects any length-r eupolar X, whatever its expression, onto the standard basis of* $\mathrm{Flex}_r(\mathrm{Pa})$ *resp.* $\mathrm{Flex}_r(\mathrm{Pa})$*)*

§18.A.3. Computation of the *slant*-coefficients for $\{\mathfrak{he}^{\bullet}_r\}$.

ter:=proc(A,B):
H([op(1,op(1,A)) +op(1,op(2,A))+1, op(2,op(1,A)) +op(2,op(2,A))],
[op(1,op(1,B)) +op(1,op(2,B)) , op(2,op(1,B)) +op(2,op(2,B))+1]) end:

Ter:=proc(X,Y): seq(seq(ter(op(x,X),op(y,Y)),
x=1..nops(X)),y=1..nops(Y)) end:

reslant0:=[H([0,-1/2],[0,-1/2])]:
leslant0:=[H([-1/2,0],[-1/2,0])]:

urslant:=proc(r): if r=1 then [H([0,0],[0,0])] elif r>1 then
[**Ter**(**urslant**(r-1),**reslant0**),
seq(**Ter**(**urslant**(r-1-k),**urslant**(k)),k=1..r-2),
Ter(**leslant0**,**urslant**(r-1))] fi end:

karslant:=proc(XX,YY): (-1)^(op(2,XX)+op(2,YY)-1)∗
(op(1,XX)∗op(2,YY)-(1+op(2,XX))∗(1+op(1,YY)))∗
(op(1,XX)+op(1,YY))!∗(op(2,XX)+op(2,YY))!
/(op(1,XX)+op(1,YY)+op(2,XX)+op(2,YY))! end:

seslant:=r $\rightarrow$ subs(H=**karslant**,**urslant**(r)); %; # (*double click*)

§18.A.4. Computation of the *stack*-coefficients for $\{\mathfrak{ke}^{\bullet}_{2r}\}$.

teer:=proc(A,B):
K([op(1,op(1,A))+op(1,op(2,A)), op(2,op(1,A))+op(2,op(2,A))+1],
[op(1,op(1,B))+op(1,op(2,B)), op(2,op(1,B))+op(2,op(2,B))+1]) end:

Teer:=proc(X,Y): seq(seq(**teer**(op(x,X),op(y,Y)),
x=1..nops(X)),y=1..nops(Y)) end:

urstack:=proc(r): if r=0 then [K([0,-1/2],[0,-1/2])]
elif r=1 then [K([1/2,-1/2],[1/2,-1/2])]
else [seq(**Teer**(**urstack**(r-1-k),**urstack**(k)) ,k=0..r-1)] fi end:

fafa:=proc(n): (n+1)!/((n+1)/2)!/2^((n+1)/2) end:

karstack:=proc(X,Y): (-2)^(op(1,X)+op(1,Y)-1)∗(op(1,X)+op(1,Y)-1)!∗
(op(1,X)∗(op(2,Y)+1)-(op(1,Y))∗(op(2,X)+1))∗
fafa(op(2,X)+op(2,Y)-op(1,X)-op(1,Y))/
fafa(op(2,X)+op(2,Y)+op(1,X)+op(1,Y)-2) end:

sestack:=r$\rightarrow$ subs(K=**karstack**,**urstack**(r)); %; # (*double click*)

§18.A.5. Computation of the alternals series $\{\mathfrak{he}^{\bullet}_{r}\}$ and $\{\mathfrak{ke}^{\bullet}_{2r}\}$.

with(linalg);
multiply(**seketel**(r),**seslant**(r)); # (gives $\mathfrak{he}^{\boldsymbol{w}}_{r}$ in the standard basis)
multiply(**seketel**(2r),**seslack**(2r)); # (gives $\mathfrak{ke}^{\boldsymbol{w}}_{2r}$ in the standard basis)

Comment **:** *To compute* $ha^{\bullet}_{r}, ka^{\bullet}_{2r}$ *resp.* $hi^{\bullet}_{r}, ki^{\bullet}_{2r}$*, use the above commands but with* **sekatal***(r) resp.* **sekitil***(r) in place of* **seketel***(r).*

§18.A.6. About the moulds $\mathrm{Mip}^{\bullet}$, $\mathrm{Nip}^{\bullet}$, $\mathrm{Rip}^{\bullet}$.

Comment **:** *The following paragraphs permit the speedy computation of the moulds* $\mathrm{Mip}^{\bullet}$*,* $\mathrm{Nip}^{\bullet}$ *(symmetrel) and* $\mathrm{Rip}^{\bullet}$ *(symmetral) necessary for expanding any given eupolar bimould* $S^{\bullet} \in \mathrm{Flex(Pi)}$ *in the three bases (294) in function of the coefficients* α_r *(here noted* a[r] *for convenience). By suitably specialising* a[r]*, the formulae yield the expansions of the basic bimoulds* $\mathrm{pil}^{\bullet}$*,* $\mathrm{ripil}^{\bullet}$*,* $\mathrm{pil}^{\bullet}_{\mathrm{ev}}$*,* $\mathrm{ripil}^{\bullet}_{\mathrm{ev}}$ *in all three bases (294). Our moulds* $\mathrm{Mip}^{\bullet}$*,* $\mathrm{Nip}^{\bullet}$*,* $\mathrm{Rip}^{\bullet}$

are indexed by positive integer sequences $\boldsymbol{n}$*, which shall be denoted here by* $\mathrm{X} = [\mathrm{n}_1, \ldots, \mathrm{n}_\mathrm{r}]$*. Each mould is dealt with in a separate paragraph, but the two following commands are required in each case:*

deb:=X → [seq(op(k,X),k=1..nops(X)-1)]:
su:=X → add(op(k,X),k=1..nops(X)):

§18.A.7. Computation of the symmetrel mould $\mathrm{Mip}^\bullet$.

Mi := proc(X): proc(p): if p=0 or p > op(+1,X) then 0
else (-1)^(1+nops(X))*op(-1,X)* **a**[su(X)-p] fi end: end:

Mij := proc(X): (-1)^(1+nops(X))*op(-1,X)***a**[su(X)] end:

Mip := proc(X) option remember;
if X=[ ] then 1
elif nops(X)=1 and op(X)=1 then +**a**[1]
elif nops(X)=1 and op(X)>1
then+1/su(X)***Mij**(X)+1/su(X)*add(**Mip**([p])***Mi**(X)(p),
p=1..op(X)-1)
elif nops(X)>1 then
+1/su(X)*add(**Mip**([seq(op(k,X),k=1..i-1)])***Mij**([seq(op(k,X),
k=i..nops(X))]), i=1..nops(X))
+1/su(X)*add(add(add(
Mip([seq(op(k,X),k=1..i-1),p,seq(op(k,X),k=j+1..nops(X))])*
Mi([seq(op(k,X),k=i..j)])(p),
p=1.. min(op(i,X),add(op(k,X),k=i..j)-1)), i=1..j),j=1..nops(X))
fi end:

§18.A.8. Computation of the symmetrel mould $\mathrm{Nip}^\bullet$.

Ni := proc(X): proc(p): if p=0 or p>op(-1,X) then 0
else (-1)^(1+nops(X)+su(X)-p)*op(+1,X)* **a**[su(X)-p] fi end: end:

Nij := proc(X): (-1)^(nops(X)+su(X))*op(+1,X)***a**[su(X)] end:

Nip := proc(X) option remember;
if X=[ ] then 1
elif nops(X)=1 and op(X)=1 then +**a**[1]
elif nops(X)=1 and op(X)>1 then
+1/su(X)***Nij**(X)+1/su(X)*add(**Nip**([p])***Ni**(X)(p),p=1..op(X)-1)
elif nops(X)>1 then

+1/su(X)∗add(**Nip**([seq(op(k,X),k=1..i-1)])∗**Nij**([seq(op(k,X),k=i..nops(X))]),
i=1..nops(X))
+1/su(X)∗add(add(add(
Nip([seq(op(k,X),k=1..i-1),p,seq(op(k,X),k=j+1..nops(X))])∗
Ni([seq(op(k,X),k=i..j)])(p),
p=1.. min(op(j,X),add(op(k,X),k=i..j)-1)), i=1..j),j=1..nops(X))
fi end:

§18.A.9. Computation of the symmetral mould Rip$^\bullet$.

Ri := proc(X): proc(p,q): if p+q<>add(op(k,X),k=1..nops(X)) then 0
elif nops(X)=1 and p<op(1,X) then p∗**a**[q]
elif nops(X)=2 and op(1,X)<= q and q < op(2,X) then +**a**[q]
elif nops(X)=2 and op(2,X)<= q and q < op(1,X) then - **a**[q]
else 0 fi end: end:

Rip := proc(X) option remember;
if nops(X)=1 and op(X)=1 then +**a**[1]
elif nops(X)=1 and op(X)>1 then +1/su(X)∗ **a**[op(-1,X)]
+1/su(X)∗add(rep[p]∗**Ri**(X)(p,op(X)-p),p=1..op(X)-1)
elif nops(X)>1 and {op(X)} = {1} then 1/(nops(X))!∗**a**[1]ˆ(nops(X))
elif nops(X)>1 and {op(X)}<>{1} then
+1/su(X)∗**Rip**([op(deb(X))])∗ **a**[op(-1,X)]
+1/su(X)∗add(add(**Rip**([seq(op(k,X),k=1..i-1),p,seq(op(k,X),k=i+1..nops(X))])∗
Ri([op(i,X)])(p,op(i,X)-p), p=1..op(i,X)-1),i=1..nops(X))
+1/su(X)∗add(add(**Rip**([seq(op(k,X),k=1..i-1),p,seq(op(k,X),k=i+2..nops(X))])∗
Rip([op(i,X),op(i+1,X)])(p,op(i,X)+op(i+1,X)-p),
p=1..op(i,X)+op(i+1,X)-1),i=1..nops(X)-1)
fi end:

A toolkit for handling bisymmetrals and all flexion operations shall soon be posted on our Webpage.

§18.B. GUIDE TO THE ANNEXED TABLES.

About two dozen illustrative Tables have been posted on our Webpage,[101] in pdf format both for direct inspection and for easy copy-pasting. Each file begins with a Maple program capable of generating the file's contents (and much beyond) and then displays the results (usually up to length or $r = 8$

[101] At <http://www.math.u-psud.fr/∼ecalle/publi.html> and <http://www.math.u-psud.fr/∼ecalle/flexion.html>.

or 10 or sometimes 12) either for their illustrative value or to make them available to non-Maple users.

§18.B.1. General tools.

The files $\boldsymbol{a_1, a_2, a_3}$ give the standard bases of the monogenous algebras *Flex*(Pa), *Flex*(Pi), *Flex*$(\mathfrak{E})$. The files $\boldsymbol{a_4, a_5}$ give the coefficients (*"slant"* and *"stack"*) of the alternal series $\{\mathfrak{he}_r^\bullet\}$, $\{\mathfrak{ke}_{2r}^\bullet\}$ in the standard basis.

§18.B.2. Recovering a general bimould from its *gari*-dilators.

Symmetral bimoulds $S^\bullet$ whose *gari*-dilators $diS^\bullet$ are in the "mock-differential algebra", i.e. of the form $diS^\bullet = \sum \alpha_r\, \mathfrak{re}_r^\bullet$, themselves belong to a subalgebra $Flex_{in}(\mathfrak{E})$ much smaller than $Flex(\mathfrak{E})$ and can be expanded along three remarkable bases, smaller and more tractable than the standard basis:

$$\mathfrak{me}^\bullet_{n_1,\dots,n_s} := \mathrm{mu}(\mathfrak{me}^\bullet_{n_1},\dots,\mathfrak{me}^\bullet_{n_s}) \tag{484}$$
$$\mathfrak{ne}^\bullet_{n_1,\dots,n_s} := \mathrm{mu}(\mathfrak{ne}^\bullet_{n_1},\dots,\mathfrak{ne}^\bullet_{n_s}) \tag{485}$$
$$\mathfrak{re}^\bullet_{n_1,\dots,n_s} := \mathrm{mu}(\mathfrak{re}^\bullet_{n_1},\dots,\mathfrak{re}^\bullet_{n_s}) \tag{486}$$

See (294) in §5. Each basis has its own advantages, and the files $\boldsymbol{b_1, b_2, b_3}$ show how to expand $S^\bullet$ in each of them, using only two ingredients: the coefficients α_r of $dilS^\bullet$ and the three universal moulds $Mip^\bullet$, $Nip^\bullet$, $Rip^\bullet$.

§18.B.3. Recovering $pil^\bullet$, $ripil^\bullet$ from their *gari*-dilators.

The files $\boldsymbol{c_1, c_2, c_3}$ and $\boldsymbol{c_4, c_5, c_6}$ apply the above universal expansions to the standard bisymmetral $pil^\bullet$ and its *gari*-inverse $ripil^\bullet$. The corresponding specialisations of $Mip^\bullet$, $Nip^\bullet$, $Rip^\bullet$ (integer-indexed and rational-valued) possess interesting, Bernoulli-like arithmetical properties.

§18.B.4. Recovering $pil^\bullet_{ev}$, $ripil^\bullet_{ev}$ from their *gari*-dilators.

The files $\boldsymbol{d_1, d_2, d_3}$ and $\boldsymbol{d_4, d_5, d_6}$ similarly expand the *even* factors $pil^\bullet_{ev}$, $ripil^\bullet_{ev}$, leading to more economical expansions of our bisymmetrals, while isolating their essential, *even* part.

§18.B.5. Recovering $pal^\bullet$, $pal^\bullet_{ev}$,$pal^\bullet_{evv}$ from their *mu*-dilators .

This is the object of file $\boldsymbol{e_1}$, to be completed by other tables about the mould $Han^\bullet$ occurring in the expansion 303.

§18.B.6. Regular bisymmetrals and associated bialternals.

The file $\boldsymbol{f_1}$ deals with the regular bialternals $lar^\bullet$ and $ral^\bullet$ (which by *gari*-postcomposition link $pal^\bullet$ and $par^\bullet$ to one another: see §9) and gives their expansions along the standard basis (since for them no simpler basis is available). The file $\boldsymbol{f_2}$ provides similar expansions for the dilators $dilar^\bullet$ and $diral^\bullet$ (bialternals of the "first kind": see §9) and file $\boldsymbol{f_3}$ does the same for the singulator-related bimoulds $visla^\bullet$ and $visra^\bullet$ (bialternals of the "second kind": see §10).

§18.B.7. Construction of $tal^\bullet$ and its even/odd factors.

The file $\boldsymbol{g_1}$ deals with the factorisations

$$\mathrm{tal}^\bullet = \mathrm{gari}(\mathrm{tal}_{\mathrm{od}}, \mathrm{tal}_{\mathrm{ev}}) \quad \textit{and} \quad \mathrm{tal}^\bullet_{\mathrm{ev}} = \mathrm{mu}(\mathrm{tal}_{\mathrm{lev}}, \mathrm{tal}_{\mathrm{rev}})$$

and the file $\boldsymbol{g_2}$ deals with the factorisations

$$\mathrm{tal}^\bullet = \mathrm{mu}(\mathrm{tal}_{\mathrm{evv}}, \mathrm{tal}_{\mathrm{odd}}) \quad \textit{and} \quad \mathrm{tal}^\bullet_{\mathrm{evv}} = \mathrm{mu}(\mathrm{tal}_{\mathrm{levv}}, \mathrm{tal}_{\mathrm{revv}})$$

The non-trivial factors are given via their dilators, which in turn are defined through their coefficients in one the two natural bases of $LU(Qa,\ c\,I)$, namely the one that is spanned by the alternals $Qa^\bullet_{n_1,\ldots,n_s}$ so defined:

$$\mathrm{Qa}^\bullet_{n_1,\ldots,n_s} := \vec{\mathrm{lu}}(\mathrm{Qa}^\bullet_{n_1}, \ldots, \mathrm{Qa}^\bullet_{n_s}) \quad \textit{with} \quad \mathrm{Qa}^\bullet_n := \vec{\mathrm{lu}}(c\,\mathrm{I}^\bullet, \overbrace{\mathrm{Qa}^\bullet, \ldots, \mathrm{Qa}^\bullet}^{(n\text{-}1)\ \textit{times}})$$

The other natural basis of $LU(Qa,\ c\,I)$ is spanned by the alternals $\mathrm{Ka}^\bullet_{n_1,\ldots,n_s}$:

$$\mathrm{Ka}^\bullet_{n_1,\ldots,n_s} := \vec{\mathrm{lu}}(\mathrm{Ka}^\bullet_{n_1}, \ldots, \mathrm{Ka}^\bullet_{n_s}) \quad \textit{with} \quad \mathrm{Ka}^\bullet_n := \vec{\mathrm{lu}}(\mathrm{Qa}^\bullet, \overbrace{c\,\mathrm{I}^\bullet, \ldots, c\,\mathrm{I}^\bullet}^{(n\text{-}1)\ \textit{times}})$$

When comparing the expansions of our trigonometric dilators in these two bases, curious - though limited and still poorly understood - duality phenomena become noticeable.

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