The flexion structure and dimorphy: flexion units, singulators, generators, and the enumeration of multizeta irreducibles.

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Abstract: *We present a self-contained survey of the flexion structure and its core* ARI//GARI*. We explain why this pair algebra//group is uniquely suited to the generation, manipulation, description and illumination of double symmetries, and therefore conducive to an in-depth understanding of arithmetical dimorphy. Special emphasis is laid on the monogenous algebras generated by flexion units, their special bimoulds, and the corresponding singulators. We then attempt a broad-brush overview of the whole question of canonical irreducibles and introduce the promising subject of perinomal algebra. As a recreational aside, we also state, justify, and computationally check a refinement of the standard conjectures about the enumeration of multizeta irreducibles.*

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1 Introduction and reminders.

1.1 Multizetas and dimorphy.

Let us take as our starting point *arithmetical dimorphy*, which in its purest form manifests in the ring of *multizetas*. Some extremely important $\mathbb{Q}$-rings of transcendental numbers happen to be *dimorphic*, i.e. to possess two *natural* $\mathbb{Q}$-prebases[1] $\{\alpha_m\}$, $\{\beta_n\}$ with a simple *conversion rule* and two independent *multiplication tables*, all of which involve only rational coefficients and finite sums:

$$\alpha_m = \sum\nolimits^* H_m^n\ \beta_n \quad , \quad \beta_n = \sum\nolimits^* K_n^m\ \alpha_m \qquad (H_m^n\,,\, K_n^m \in \mathbb{Q})$$

$$\alpha_{m_1}\,\alpha_{m_2} = \sum\nolimits^* A^{m_3}_{m_1,m_2}\ \alpha_{m_3} \quad , \quad \beta_{n_1}\,\beta_{n_2} = \sum\nolimits^* B^{n_3}_{n_1,n_2}\ \beta_{n_3} \quad (A^{n_3}_{n_1,n_2}\,,\, B^{n_3}_{n_1,n_2} \in \mathbb{Q})$$

The simplest, most basic of all such rings is $\mathbb{Z}$eta, which is not only *multiplicatively* generated but also *linearly* spanned by the so-called *multizetas.*[2]

In the *first prebasis*, the multizetas are given by polylogarithmic integrals:

$$\mathrm{Wa}_*^{\alpha_1,\dots,\alpha_l} := (-1)^{l_0} \int_0^1 \frac{dt_l}{(\alpha_l - t_l)} \dots \int_0^{t_3} \frac{dt_2}{(\alpha_2 - t_2)} \int_0^{t_2} \frac{dt_1}{(\alpha_1 - t_1)} \qquad (1.1)$$

[1] with some natural countable indexation $\{m\},\{n\}$, not necessarily on $\mathbb{N}$ or $\mathbb{Z}$. We recall that a set $\{\alpha_m\}$ is a $\mathbb{Q}$-prebasis (or 'spanning subset') of a $\mathbb{Q}$-ring $\mathbb{D}$ if any $\alpha \in \mathbb{D}$ is expressible as a finite linear combination of the α_m's with rational coefficients. But the α_m's need not be $\mathbb{Q}$-independent. When they are, we say that $\{\alpha_m\}$ is a $\mathbb{Q}$-basis.
[2] also known as MZV, short for *multiple zeta values.*

with indices α_j that are either 0 or unit roots, and $l_0 := \sum_{\alpha_i=0} 1$.

In the *second prebasis*, multizetas are expressed as "harmonic sums":

$$\mathrm{Ze}_*^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{s_1\ ,\dots,\ s_r}} := \sum_{n_1>\dots>n_r>0} n_1^{-s_1}\dots n_r^{-s_r}\ e_1^{-n_1}\dots e_r^{-n_r} \tag{1.2}$$

with $s_j \in \mathbb{N}^*$ and unit roots $e_j := \exp(2\pi i\epsilon_j)$ with 'logarithms' $\epsilon_j \in \mathbb{Q}/\mathbb{Z}$.

The stars $*$ means that the integrals or sums are provisionally assumed to be convergent or semi-convergent: for $Wa_*^{\boldsymbol{\alpha}}$ this means that $\alpha_1 \neq 0$ and $\alpha_l \neq 1$, and for $Ze_*^{\binom{\boldsymbol{\epsilon}}{\mathbf{s}}}$ this means that $\binom{\epsilon_1}{s_1} \neq \binom{0}{1}$ i.e. $\binom{e_1}{s_1} \neq \binom{1}{1}$.

The corresponding moulds $Wa_*^\bullet$ and $Ze_*^\bullet$ turn out to be respectively *symmetral* and *symmetrel*:[3]

$$\mathrm{Wa}_*^{\boldsymbol{\alpha^1}}\ \mathrm{Wa}_*^{\boldsymbol{\alpha^2}} = \sum_{\boldsymbol{\alpha}\in \mathrm{sha}(\boldsymbol{\alpha^1},\boldsymbol{\alpha^2})} \mathrm{Wa}_*^{\boldsymbol{\alpha}} \qquad \forall\boldsymbol{\alpha^1},\forall\boldsymbol{\alpha^2} \tag{1.3}$$

$$\mathrm{Ze}_*^{\binom{\boldsymbol{\epsilon^1}}{\mathbf{s^1}}}\ \mathrm{Ze}_*^{\binom{\boldsymbol{\epsilon^2}}{\mathbf{s^2}}} = \sum_{\binom{\boldsymbol{\epsilon}}{\mathbf{s}}\in \mathrm{she}(\binom{\boldsymbol{\epsilon^1}}{\mathbf{s^1}},\binom{\boldsymbol{\epsilon^2}}{\mathbf{s^2}}))} \mathrm{Ze}_*^{\binom{\boldsymbol{\epsilon}}{\mathbf{s}}} \qquad \forall\binom{\boldsymbol{\epsilon^1}}{\mathbf{s^1}},\forall\binom{\boldsymbol{\epsilon^2}}{\mathbf{s^2}} \tag{1.4}$$

These are the so-called *quadratic relations*, which express multizeta dimorphy. As for the conversion rule, it reads:[4]

$$\mathrm{Wa}_*{}^{e_1,0^{[s_1-1]},\dots,e_r,0^{[s_r-1]}} := \mathrm{Ze}_*{}^{\binom{\epsilon_r\ ,\ \epsilon_{r-1:r}\ ,\dots,\ \epsilon_{1:2}}{s_r\ ,\ s_{r-1}\ \ ,\dots,\ s_1}} \tag{1.5}$$

$$\mathrm{Ze}_*{}^{\binom{\epsilon_1\ ,\ \epsilon_2\ ,\dots,\ \epsilon_r}{s_1\ ,\ s_2\ ,\dots,\ s_r}} =: \mathrm{Wa}_*{}^{e_1\dots e_r,0^{[s_r-1]},\dots,e_1e_2,0^{[s_2-1]},e_1,0^{[s_1-1]}} \tag{1.6}$$

with $0^{[k]}$ denoting a subsequence of k zeros.

There happen to be unique extensions $Wa_*^\bullet \to Wa^\bullet$ and $Ze_*^\bullet \to Ze^\bullet$ that cover the divergent cases and keep our moulds symmetral or symmetrel while conforming to the 'initial conditions' $Wa^0 = Wa^1 = 0$ and $Ze^{\binom{0}{1}} = 0$. The only price to pay will be a slight modification of the conversion rule: see §1.2 *infra*.

Basic gradations/filtrations.
Four parameters dominate the discussion:
– the *weight* $s := \sum s_i$ (in the $Ze^\bullet$-encoding) or $s := l$ (in the $Wa^\bullet$-encoding)
– the *length* or *"depth"* r := number of ϵ_i's or s_i's or non-zero α_i's.

[3] As usual, $sha(\boldsymbol{\omega'},\boldsymbol{\omega''})$ denotes the set of all simple shufflings of the sequences $\boldsymbol{\omega'},\boldsymbol{\omega''}$, whereas in $she(\boldsymbol{\omega'},\boldsymbol{\omega''})$ we allow (any number of) order-compatible contractions $\omega'_i + \omega''_j$.

[4] with the usual shorthand for differences: $\epsilon_{i:j} := \epsilon_i - \epsilon_j$.

– the *degree* $d := s - r$ = number of zero α_i's in the $Wa^\bullet$-encoding.[5]
– the *"coloration"* $p :=$ smallest p such that all root-related ϵ_i be in $\frac{1}{p}\mathbb{Z}/\mathbb{Z}$.
Only the weight s defines an (additive and multiplicative) gradation; the other parameters merely induce filtrations.

1.2 From scalars to generating series.

The natural encodings $Wa^\bullet$ and $Ze^\bullet$ being unwieldy and too heterogeneous in their indexations, we must replace them by suitable *generating series*, so chosen as to preserve the simplicity of the two quadratic relations and of the conversion rule. This essentially *imposes* the following definitions:[6]

$$\mathrm{Zag}^{\binom{u_1\ ,\dots,\ u_r}{\epsilon_1\ ,\dots,\ \epsilon_r}} := \sum_{1\le s_j} \mathrm{Wa}^{e_1,0^{[s_1-1]},\dots,e_r,0^{[s_r-1]}}\ u_1^{s_1-1}u_{1,2}^{s_2-1}\dots u_{1\dots r}^{s_r-1} \quad (1.7)$$

$$\mathrm{Zig}^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{v_1\ ,\dots,\ v_r}} := \sum_{1\le s_j} \mathrm{Ze}^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{s_1\ ,\dots,\ s_r}}\ v_1^{s_1-1}\dots v_r^{s_r-1} \quad (1.8)$$

The first series $Zag^\bullet$, via its Taylor coefficients, gives rise to yet another $\mathbb{Q}$-prebasis $\{Za^\bullet\}$ for the $\mathbb{Q}$-ring of multizetas. The mould $Za^\bullet$ is symmetral like $Wa^\bullet$ but quite distinct from it and much closer, in form and indexation, to the symmetrel mould $Ze^\bullet$:

$$\mathrm{Zag}^{\binom{u_1\ ,\dots,\ u_r}{\epsilon_1\ ,\dots,\ \epsilon_r}} =: \sum_{1\le s_j} \mathrm{Za}^{\binom{s_1\ ,\dots,\ s_r}{\epsilon_1\ ,\dots,\ \epsilon_r}}\ u_1^{s_1-1}\dots u_r^{s_r-1} \quad (1.9)$$

These power series are actually convergent: they define *generating functions*[7] that are meromorphic, with multiple poles at simple locations. These functions, in turn, verify simple difference equations, and admit an elementary mould factorisation (mark the exchange in the positions of *do* and *co*):

$$\mathrm{Zag}^\bullet := \lim_{k\to\infty} \mathrm{Zag}_k^\bullet = \lim_{k\to\infty} (\mathrm{doZag}_k^\bullet \times \mathrm{coZag}_k^\bullet) \quad (1.10)$$

$$\mathrm{Zig}^\bullet := \lim_{k\to\infty} \mathrm{Zig}_k^\bullet = \lim_{k\to\infty} (\mathrm{coZig}_k^\bullet \times \mathrm{doZig}_k^\bullet) \quad (1.11)$$

with dominant parts $doZag^\bullet / doZig^\bullet$ that carry the $\boldsymbol{u}/\boldsymbol{v}$-dependence[8]:

$$\mathrm{doZag}_k^{\binom{u_1\ ,\dots,\ u_r}{\epsilon_1\ ,\dots,\ \epsilon_r}} := \sum_{1\le m_i\le k} e_1^{-m_1}\dots e_r^{-m_r} P(m_1-u_1)P(m_{1,2}-u_{1,2})\dots P(m_{1..r}-u_{1..r}) \quad (1.12)$$

$$\mathrm{doZig}_k^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{v_1\ ,\dots,\ v_r}} := \sum_{k\ge n_1>n_2>\dots n_r\ge 1} e_1^{-n_1}\dots e_r^{-n_r}\ P(n_1-v_1)P(n_2-v_2)\dots P(n_r-v_r) \quad (1.13)$$

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[5] d is called *degree*, because under the correspondence *scalars* → *generating series*, the multizetas become coefficients of monomials of total degree d. See (2.19),(2.23).

[6] with the usual abbreviations: $u_{i,j} = u_i + u_j$, $u_{i,j,k} = u_i + u_j + u_k$ etc.

[7] still denoted by the same symbols.

[8] with the usual abbreviations $m_{i,j} := m_i + m_j$, $m_{i,j,k} := m_i + m_j + m_k$ etc.

and corrective parts $coZag^\bullet / coZig^\bullet$ that reduce to constants:

$$\mathrm{coZag}_k^{\binom{u_1\,,\dots,\,u_r}{0\,,\dots,\,0}} := (-1)^r \sum_{1\leq m_i\leq k} P(m_1)P(m_{1,2})...P(m_{1...r}) \tag{1.14}$$

$$\mathrm{coZig}_k^{\binom{0\,,\dots,\,0}{v_1\,,\dots,\,v_r}} := (-1)^r \sum_{k\geq n_1\geq n_2\geq ...n_r\geq 1} \mu^{n_1,\dots,n_r}\, P(n_1)P(n_2)...P(n_r) \tag{1.15}$$

$$\mathrm{coZag}_k^{\binom{u_1\,,\dots,\,u_r}{\epsilon_1\,,\dots,\,\epsilon_r}} := 0 \quad \mathit{if} \quad (\epsilon_1,...,\epsilon_r)\neq(0,...,0) \tag{1.16}$$

$$\mathrm{coZig}_k^{\binom{\epsilon_1\,,\dots,\,\epsilon_r}{v_1\,,\dots,\,v_r}} := 0 \quad \mathit{if} \quad (\epsilon_1,...,\epsilon_r)\neq(0,...,0) \tag{1.17}$$

with $P(t) := 1/t$ (here and throughout) and with $\mu^{n_1,n_2,..,n_r} := \frac{1}{r_1!\,r_2!..r_l!}$ if the non-increasing sequence $(n_1,...,n_r)$ attains r_1 times its highest value, r_2 times its second highest value, etc.

Setting $Mini_k^\bullet := Zig_k^\bullet\|_{\mathrm{v}=0}$ we find :[9]

$$\mathrm{Mini}_k^{\binom{0\,,\dots,\,0}{v_1\,,\dots,\,v_r}} := \sum_{\left[\substack{2\leq r_1\leq r_2...\leq r_l\\ r_1+r_2+...r_l=r}\right]}^{\left[\substack{1\leq l\leq r/2\\ 1\leq n_i\leq k}\right]} (-1)^{(r-l)}\mu^{r_1,\dots,r_l}\,\frac{(P(n_1))^{r_1}}{r_1}...\frac{(P(n_l))^{r_l}}{r_l} \tag{1.18}$$

$$\mathrm{Mini}_k^{\binom{\epsilon_1\,,\dots,\,\epsilon_r}{v_1\,,\dots,\,v_r}} := 0 \quad \mathit{if} \quad (\epsilon_1,...,\epsilon_r)\neq(0,...,0) \tag{1.19}$$

Let us now compare the bimoulds $\overline{C}_k^\bullet$ and $\underline{C}_k^\bullet$ thus defined:

$$\mathrm{swap.Zag}_k^\bullet = \overbrace{\mathrm{swap.coZag}_k^\bullet} \times \overbrace{\mathrm{swap.doZag}_k^\bullet} = \overline{A}_k^\bullet \times \overline{B}_k^\bullet =: \overline{C}_k^\bullet \tag{1.20}$$

$$(\mathrm{Mini}_k^\bullet)^{-1} \times \mathrm{Zig}_k^\bullet = \overbrace{(\mathrm{Mini}_k^\bullet)^{-1} \times \mathrm{coZig}_k^\bullet} \times \overbrace{\mathrm{doZig}_k^\bullet} = \underline{A}_k^\bullet \times \underline{B}_k^\bullet =: \underline{C}_k^\bullet \tag{1.21}$$

with $\times$ standing for ordinary mould or bimould multiplication[10]; with $(\mathrm{Mini}_k^\bullet)^{-1}$ denoting the multiplicative inverse of $(\mathrm{Mini}_k^\bullet)$; and with the involution *swap* defined as in (2.9) *infra*. Here, the $\boldsymbol{v}$-dependent factors $\overline{B}_k^{\binom{\epsilon}{v}}$ and $\underline{B}_k^{\binom{\epsilon}{v}}$ are both given by the finite sum

$$\sum e_1^{-n_1}\dots e_r^{-n_r}\,P(n_1-v_1)\dots P(n_r-v_r) \tag{1.22}$$

[9] if we had no factor $\mu^{n_1,\dots,n_r}$ in (1.18), we would have $Zig_k^\bullet\|_{\mathrm{v}=0} = 0$ and therefore no $Mini_k^\bullet$ terms. But the mould $Zig_k^\bullet$ would fail to be *symmetril*, as required. Herein lies the origin of the corrective terms in the conversion rule.

[10] In the case of bimoulds, $\times$ is often noted *mu* the better to distinguish it from the various other *flexion* products.

with summation respectively over the domains $\overline{\mathcal{B}}_{r,k}$ and $\underline{\mathcal{B}}_{r,k}$

$$\begin{aligned}\overline{\mathcal{B}}_{r,k} &:= \{k \geq n_r \geq 1, 2\,k \geq n_{r-1} > n_r, \ldots, (r-1)\,k \geq n_2 > n_3, r\,k \geq n_1 > n_2\}\\ \underline{\mathcal{B}}_{r,k} &:= \{k \geq n_1 > n_2 > \ldots n_{r-1} > n_r \geq 1\}\end{aligned}$$

Likewise, the $\boldsymbol{v}$-independent factors $\overline{A}_k^{\binom{\boldsymbol{\epsilon}}{\boldsymbol{v}}}$ and $\underline{A}_k^{\binom{\boldsymbol{\epsilon}}{\boldsymbol{v}}}$ vanish unless $\boldsymbol{\epsilon} = 0$, in which case they are both given by the finite sum

$$\sum (-1)^r \, P(n_1) \ldots P(n_r) \tag{1.23}$$

with summation respectively over the domains $\overline{\mathcal{A}}_{r,k}$ and $\underline{\mathcal{A}}_{r,k}$

$$\begin{aligned}\overline{\mathcal{A}}_{r,k} &:= \{k \geq n_1 \geq 1, 2\,k \geq n_2 \geq n_1, \ldots, (r-1)\,k \geq n_{r-1} \geq n_{r-2}, r\,k \geq n_r \geq n_{r-1}\}\\ \underline{\mathcal{A}}_{r,k} &:= \{k \geq n_r \geq n_{r-1} \geq \ldots n_2 \geq n_1 \geq 1\}\end{aligned}$$

It easily follows from the above that for any compact $K \subset \mathbb{C}^r$ and k large enough, the difference $\overline{C}_k^{\binom{\boldsymbol{\epsilon}}{\boldsymbol{v}}} - \underline{C}_k^{\binom{\boldsymbol{\epsilon}}{\boldsymbol{v}}}$ is holomorphic on K, and that there exists a constant c_K such that:

$$\|\overline{C}_k^{\binom{\boldsymbol{\epsilon}}{\boldsymbol{v}}} - \underline{C}_k^{\binom{\boldsymbol{\epsilon}}{\boldsymbol{v}}}\| \leq (c_K)^r \, \frac{(\log k)^{r-1}}{k} \quad (\boldsymbol{v} \in K \text{ , } k \textit{ large}) \tag{1.24}$$

Summing up, we have an exact equivalence between old and new symmetries:[11]

$$\{\mathrm{Wa}^\bullet \ \textit{symmetral}\} \iff \{\mathrm{Zag}^\bullet \ \textit{symmetral}\} \tag{1.25}$$
$$\{\mathrm{Ze}^\bullet \ \textit{symmetrel}\} \iff \{\mathrm{Zig}^\bullet \ \textit{symmetril}\} \tag{1.26}$$

and the old conversion rule for scalar multizetas [12] becomes:

$$\mathrm{Zig}^\bullet = \mathrm{Mini}^\bullet \times \mathrm{swap(Zag)}^\bullet \tag{1.27}$$
$$\big(\iff \mathrm{swap}(\mathrm{Zig}^\bullet) = \mathrm{Zag}^\bullet \times \mathrm{Mana}^\bullet \big) \tag{1.28}$$

with elementary moulds $\mathit{Mana}^\bullet / \mathit{Mini}^\bullet := \lim_{k\to\infty} \mathit{Mana}_k^\bullet / \mathit{Mini}_k^\bullet$ whose only non-zero components:

$$\mathrm{Mana}^{\binom{u_1\ ,\ldots,\ u_r}{0\ ,\ldots,\ 0}} \equiv \mathrm{Mini}^{\binom{0\ ,\ldots,\ 0}{v_1\ ,\ldots,\ v_r}} \equiv \mathrm{Mono}_r \tag{1.29}$$

[11] *Symmetrility* is precisely defined in §3.5. Roughly, it mirrors *symmetrelity*, but with all contractions $M^{(\ldots,\omega_i'+\omega_j'',\ldots)}$ systematically replaced by $M^{\binom{\ldots,\ u_i'+u_j''\ ,\ldots}{\ldots,\ \ v_i'\ \ ,\ldots}} P(v_i' - v_j'') + M^{\binom{\ldots,\ u_i'+u_j''\ ,\ldots}{\ldots,\ \ v_j''\ \ ,\ldots}} P(v_j'' - v_i')$.

[12] namely, some modified form of the rules (2.16),(2.17), which apply in the *convergent* case.

due to (1.18), may be expressed in terms of monozetas:

$$1+\sum_{r\geq 2} \text{Mono}_r\, t^r := \exp\Big(\sum_{s\geq 2}(-1)^{s-1}\zeta(s)\,\frac{t^s}{s}\Big) \tag{1.30}$$

To these relations one must add the so-called *self-consistency* relations:

$$\text{Zag}^{\binom{u_1\ ,\dots,\ u_r}{q\,\epsilon_1\ ,\dots,\ q\,\epsilon_r}} \equiv \sum_{q\epsilon_i^*=q\epsilon_i} \text{Zag}^{\binom{q\,u_1\ ,\dots,\ q\,u_r}{\epsilon_1^*\ ,\dots,\ \epsilon_r^*}} \quad \forall q|p\,, \forall u_i\in\mathbb{C}\,, \forall \epsilon_i, \epsilon_i^*\in \frac{1}{p}\mathbb{Z}/\mathbb{Z} \tag{1.31}$$

which merely reflect trivial identities between unit roots of order p.

1.3 ARI//GARI and its dimorphic substructures.

What is required at this point is an algebraic apparatus capable of accommodating Janus-like objects like $Zag^\bullet/Zig^\bullet$, i.e. an apparatus with operations that not only respect double symmetries and reproduce them under composition, but also construct them from scratch, i.e. from a few simple generators.

Such a machinery is at hand: it is the *flexion structure*, which arose in the early 90s in the context of *singularity analysis*, more precisely in the investigation of *parametric* or *"co-equational" resurgence*. Its objects are *bimoulds*, i.e. moulds $M^\bullet$ of the form

$$M^\bullet \in \text{BIMU} \quad \Longleftrightarrow \quad M^\bullet = \{M^{w_1,\dots,w_r} = M^{\binom{u_1\ \dots\ u_r}{v_1\ \dots\ v_r}}\} \tag{1.32}$$

with a double-layered indexation $w_i = \binom{u_i}{v_i}$. What makes these $M^\bullet$ into bimoulds, however, is not so much their double indexation as the very specific manner in which upper and lower indices transform and interact: all bimould operations can be expressed in terms of four elementary *flexions* that go by pairs, $\rceil$ with $\lfloor$ and $\rfloor$ with $\lceil$, and have the effect of *adding together* several consecutive u_i and of *pairwise subtracting* several v_i, and that too in such a way as to conserve the scalar product $<\boldsymbol{u},\boldsymbol{v}>:=\sum u_i v_i$ and the symplectic form $d\boldsymbol{w} := \sum du_i \wedge dv_i$. Lastly, central to the flexion structure is a basic involution *swap* which acts on *BIMU* by turning the u_i's into differences of v_j's, and the v_i's into sums of u_j's (see §2.1 below).

The *flexion structure*, to put it loosely but tellingly, is the sum total of all interesting operations and structures that can be constructed on *BIMU* by deftly combining the four elementary flexions. It turns out that these *interesting structures* consist, up to isomorphism, of
– seven + one Lie groups
– seven + one Lie algebras (each with its pre-Lie structure)
– seven + one pre-Lie algebras.

In the three series, there exist exactly two triplets of type *group//algebra//superalgebra*, which "respect dimorphy", namely *GARI//ARI//SUARI* and *GALI//ALI//SUALI*.

Moreover, when restricted to dimorphic bimoulds (i.e. bimoulds displaying a double symmetry), these two triplets actually coincide, thus sparing us the agony of choosing between them.

1.4 Flexion units, singulators, double symmetries.

To understand dimorphy, and in particular to decompose the pair $Zag^\bullet/Zig^\bullet$ into the elementary building blocks capable of yielding the *multizeta irreducibles*, we require bimoulds $M^\bullet$ which combine three properties that do not sit well together:
— $M^\bullet$ must possess a given symmetry, say alternal or symmetral
— swap.$M^\bullet$ must possess its own symmetry, which usually coincides with that of $M^\bullet$ or a variant thereof
— $M^\bullet$ and swap.$M^\bullet$ must be *entire*, i.e. for a given length r their dependence on the complex indices (the u_i's in the case of $M^\bullet$ and the v_i's in the case of swap.$M^\bullet$) must be polynomial or holomorphic or a power series. That precludes, in particular, singularities at the origin.

The strange thing, however, is that in order to come to grips with *"entire dimorphy"* in the above sense, we cannot avoid making repeated use of bimoulds that are dimorphic alright, but with abundant *poles* at the origin. We must then get rid of these poles by subtracting suitable bimoulds, with exactly the same singular part, but without destroying the double symmetry. The only way to pull this off is by using very specific operators, the so-called *singulators*, whose basic ingredients are quite special dimorphic bimoulds, which:
— possess poles at the origin
— lack the crucial parity property which most other dimorphic bimoulds possess and which ensures their stability under the *ARI* or *GARI* operations.
— are constructed from very elementary functions $\mathfrak{E}^{w_1} = \mathfrak{E}^{\binom{u_1}{v_1}}$, the so-called *flexion units*, of which there exist about a dozen. These *units* are odd in w_1 and verify an elementary functional equation, the *tripartite relation*, which is the most basic relation expressible in terms of flexions.

1.5 Enumeration of multizeta irreducibles.

The $\mathbb{Q}$-ring $\mathbb{Z}eta$ of formal multizetas (i.e. of multizeta symbols subject only to the two *quadratic relations* (1.3),(1.4)) is known to be a polynomial ring, freely generated by a countable set of so-called *irreducibles*.[13] Hence the question: how many irreducibles (let us call that number $D_{d,r}$) must one pick in each cell of degree d and length r to get a complete and free system of irreducibles? The so-called BK-conjectures,[14] which were formulated in 1996 (they applied to the *genuine* rather than *formal* multizetas, and resulted from purely numerical tests) suggest a startlingly complicated formula for $D_{d,r}$ but no plausible rationale for its strange form. Soon after that, we published in [E2] a convincing explanation for the formula, which however went largely unnoticed. We therefore return to the question in §5 and §7 in much greater detail. We actually enunciate four new conjectures which considerably improve on the original BK-formula, and in §8 we report on formal computations carried out by S. Carr to test these strengthened conjectures. But the key lies in the theoretical explanation: in our approach, the irreducibles correspond one-to-one to *polynomial bialternal* bimoulds, of which there exist two series: the regular and utterly simple $ekma^\bullet$ on the one hand, and the exceptional, highly intricate $carma^\bullet$ on the other. We explain in detail the mechanism responsible for the creation of these exceptional generators. That mechanism crucially involves the singulators mentioned in the preceding section.

1.6 Canonical irreducibles and perinomal algebra.

In §6 and §9 we move from the (d,r)-gradation to the more natural s-gradation, s being the weight. In that new setting, the irreducibles correspond to entire bimoulds which are no longer alternal/alternal (or *bialternal* for short) but alternal/alternil and which for that reason never reduce to a single component, as bialternals do. That may seem a complication, and it is, but it also brings a drastic simplification in its wake: instead of the dual system of generators $\{ekma^\bullet_d, carma^\bullet_{d,k}\}$ for the algebra $ALAL \subset ARI^{\mathrm{al/al}}_{ent}$ of entire bialternals, we now have a single system, either $\{lama^\bullet_s\}$ or $\{loma^\bullet_s\}$[15], of generators for the algebra $ALIL \subset ARI^{\mathrm{al/il}}_{ent}$ of all entire bimoulds of alternal/alternil type, with a transparent indexation by all odd weights $s = 3, 5, 7$

[13]This fact is almost implicit in the (right) formalism. Indeed, with the notations of §9, the general bisymmetral, entire bimould $zag^\bullet$ factors as $zag^\bullet = gari\big(Zag_I, expari(ma^\bullet)\big)$ with $ma^\bullet = \sum_S \rho_S\, ma^\bullet_S$ denoting the general element of $ALIL$. Thus, to any linear basis $\{ma^\bullet_S\}$ of $ALIL$, there corresponds one-to-one a set $\{\rho_S\}$ of irreducibles, with the same countable indexation S, and a transparent formula for expressing the multizetas in terms of these irreducibles. A written exposition, resting on very similar ideas but couched in a quite different formalism, may be found in G. Racinet, *Doubles mélanges des polylogarithmes multiples aux racines de l'unité* , Publ. Math. IHES, 2002.

[14]see [B] and §8.4.

[15]these are closely related variants.

etc. Like *carma*$^\bullet$, but to an even greater extent, *lama*$^\bullet$ and *loma*$^\bullet$ depend for their construction on the repeated use of singulators, with parasitical poles being alternately *produced* and then *destroyed*. In §6.7 and §9 we also introduce a third system of generators for *ALIL*, namely $\{{}^n luma^\bullet\}$, with indices n now running through $\mathbb{N}^*$ and with *functional simplicity*[16] replacing *arithmetical simplicity*[17] as guiding principle. Just like with *lama*$^\bullet$ and *loma*$^\bullet$, the singulators are key to the construction of *luma*$^\bullet$, but under a quite different mechanism, which involves infinitely many (interrelated) linear representations of $Sl_r(\mathbb{Z})$. This is a whole new field unto itself, and a fascinating one at that, which we call *perinomal algebra*, and of which we try to give a foretaste.

1.7 Purpose of the present survey.

A four-volume series (on the flexion structure and its applications) is 'in the works', but as often happens with fast-evolving subjects, centrifugal temptations are hard to resist, centripetal discipline difficult to maintain, and the whole bloated project shows more signs of expanding and mutating than of converging. To remedy this, we intend to post some of the accumulated material (including a library of Maple programmes for *ARI//GARI* calculations) online, on our Web-page, before the end of 2010. But we feel that a compact Survey like the present one might also serve a purpose – not least that of fixing notations and nomenclature.[18]

Some of the subject-matter laid out here is fairly old – going back eight years in some cases – but unpublished for the most part.[19] There are novelties, too, the main one being perhaps the systematic use of *flexion units* as a means of introducing order into the theory's bewildering plethora of notions and objects: operations, symmetries, structures (algebras, groups) and substructures, bimoulds, bimould identities etc.

'The' flexion unit $\mathfrak{E}^\bullet$ is an unspecified function $\mathfrak{E}^{w_1}$ that is odd in $w_1 := \binom{u_1}{v_1}$ and verifies a bilinear, three-term relation[20] – the so-called *tripartite relation*. From $\mathfrak{E}^\bullet$ one then constructs a whole string of objects (bimoulds, symmetries, subalgebras of *ARI*, subgroups of *GARI*, etc) which, despite

[16] the components $luma^{\boldsymbol{w}}$ are meromorphic functions with simple poles away from the origin.

[17] the components $lama^{\boldsymbol{w}}$ and $loma^{\boldsymbol{w}}$ have rational coefficients with "manageable" denominators.

[18] which up till now were still fluctuating from context to context in our various papers. Working out a coherent standardisation was, strangely, the hardest part in producing this survey.

[19] although much of it was circulated as private notes and e-files, or taught at Orsay in two DEA courses.

[20] involving the product $\mathfrak{E}^{w_1}\,\mathfrak{E}^{w_2}$ and two flexions thereof.

their considerable complexity, owe all their properties to the *tripartite relation* verified by the seed-unit $\mathfrak{E}^\bullet$. As it happens, $\mathfrak{E}^\bullet$ is capable of a dozen or so *distinct realisations* as a concrete function of w_1, each of which automatically induces a realisation of the whole string of satellite objects (bimoulds, symmetries, etc). The total effect is thus a drastic and welcome *'division by twelve'* of the flexion jungle.

Throughout, there is as much emphasis on the apparatus – the flexion structure and its special bimoulds – as on the applications to multizeta theory. We wind up with a sketch of perinomal algebra, in the hope of stimulating interest in this brand-new subject and of paving the way for a collective programme of exploration,[21] to start hopefully in the course of 2011.

One last word of caution: throughout this paper, the somewhat contentious word *canonical* is never used as a substitute for *unique* (when meaning *unique*, we say *unique*) but as a pointer to the existence, within a class of seemingly undistinguishable objects (like the many conceivable systems of *multizeta irreducibles*) of genuinely privileged representatives. To single out these representatives, esthetic considerations are *unavoidable*, with the residual (often minimal) fuzziness that this entails. But the subjectivity that attaches to the notion in no way detracts from its importance. Quite the opposite, in fact.

2 Basic dimorphic algebras.

2.1 Basic operations.

Elementary flexions.
In addition to ordinary, non-commutative mould multiplication *mu* (or $\times$):

$$A^\bullet = B^\bullet \times C^\bullet = \mathrm{mu}(B^\bullet, C^\bullet) \quad \Longleftrightarrow \quad A^{\boldsymbol{w}} = \sum_{\boldsymbol{w^1}.\boldsymbol{w^2}=\boldsymbol{w}}^{r(\boldsymbol{w^1}),r(\boldsymbol{w^2})\geq 0} B^{\boldsymbol{w^1}}\, C^{\boldsymbol{w^2}} \tag{2.1}$$

and its inverse *invmu*:

$$(\mathrm{invmu}.A)^{\boldsymbol{w}} = \sum_{1\leq s\leq r(\boldsymbol{w})} (-1)^s \sum_{\boldsymbol{w^1}\ldots\boldsymbol{w^s} = \boldsymbol{w}} A^{\boldsymbol{w^1}} \ldots A^{\boldsymbol{w^s}} \qquad (\boldsymbol{w^i} \neq \emptyset) \tag{2.2}$$

the bimoulds $A^\bullet$ in $BIMU = \oplus_{0\leq r} BIMU_r$ (see (1.32))[22] can be subjected to a host of specific operations, all constructed from four elementary *flex-*

[21] Vast, multi-facetted, and very demanding in terms of computation, this field calls, or rather cries, for sustained teamwork.

[22] $BIMU_r$ of course regroups all bimoulds whose components of length other than r vanish. These are often dubbed "length-r bimoulds" for short.

ions $\lfloor, \rceil, \lceil, \rfloor$ that are always defined relative to a given factorisation of the total sequence $\boldsymbol{w}$. The way the flexions act is apparent from the following examples:

$$\begin{array}{llll}
\boldsymbol{w}=\boldsymbol{a}.\boldsymbol{b} & \boldsymbol{a} \;= \binom{u_1,\,u_2,\,u_3}{v_1,\,v_2,\,v_3} & \boldsymbol{b}=\binom{u_4,\,u_5,\,u_6}{v_4,\,v_5,\,v_6} & \\
\Longrightarrow & \boldsymbol{a}\rfloor = \binom{u_1,\;\;u_2,\;\;u_3}{v_{1:4},\,v_{2:4},\,v_{3:4}} & \lceil\boldsymbol{b}=\binom{u_{1234},\,u_5,\,u_6}{v_4,\;\;v_5,\,v_6} & \\
\boldsymbol{w}=\boldsymbol{b}.\boldsymbol{c} & \boldsymbol{b} \;= \binom{u_1,\,u_2,\,u_3}{v_1,\,v_2,\,v_3} & \boldsymbol{c}=\binom{u_4,\,u_5,\,u_6}{v_4,\,v_5,\,v_6} & \\
\Longrightarrow & \boldsymbol{b}\rceil = \binom{u_1,\,u_2,\,u_{3456}}{v_1,\,v_2,\;\;v_3} & \lfloor\boldsymbol{c}=\binom{u_4,\;\;u_5,\;\;u_6}{v_{4:3},\,v_{5:3},\,v_{6:3}} & \\
\boldsymbol{w}=\boldsymbol{a}.\boldsymbol{b}.\boldsymbol{c} & \boldsymbol{a} \;= \binom{u_1,\,u_2,\,u_3}{v_1,\,v_2,\,v_3} & \boldsymbol{b} \;=\binom{u_4,\,u_5,\,u_6}{v_4,\,v_5,\,v_6} & \boldsymbol{c}=\binom{u_7,\,u_8,\,u_9}{v_7,\,v_8,\,v_9} \\
\Longrightarrow & \boldsymbol{a}\rfloor = \binom{u_1,\;\;u_2,\;\;u_3}{v_{1:4},\,v_{2:4},\,v_{3:4}} & \lceil\boldsymbol{b}\rceil =\binom{u_{1234},\,u_5,\,u_{6789}}{v_4,\;\;v_5,\;\;v_6} & \lfloor\boldsymbol{c}=\binom{u_7,\;\;u_8,\;\;u_9}{v_{7:6},\,v_{8:6},\,v_{9:6}}
\end{array}$$

with the usual short-hand: $u_{i,\dots,j} := u_i+\dots+u_j$ and $v_{i:j} := v_i-v_j$. Here and throughout the sequel, we use boldface (with upper indexation) to denote sequences ($\boldsymbol{w}, \boldsymbol{w^i}, \boldsymbol{w^j}$ etc), and ordinary characters (with lower indexation) to denote single sequence elements (w_i, w_j etc), or sometimes sequences of length $r(\boldsymbol{w}) = 1$. Of course, the 'product' $\boldsymbol{w^1.w^2}$ denotes the concatenation of the two factor sequences.

Short and long indexations on bimoulds.
For bimoulds $M^\bullet \in BIMU_r$ it is sometimes convenient to switch from the usual *short indexation* (with r indices w_i's) to a more homogeneous *long indexation* (with a redundant initial w_0 which gets bracketted for distinctiveness). The correspondence goes like this:

$$M^{\binom{u_1\ ,\dots,\ u_r}{v_1\ ,\dots,\ v_r}} \cong M^{\binom{[u_0^*],\ u_1^*\ ,\dots,\ u_r^*}{[v_0^*],\ v_1^*\ ,\dots,\ v_r^*}} \tag{2.3}$$

with the dual conditions on upper and lower indices:

$$\begin{array}{llll}
u_0^* = -u_{1\dots r} := -(u_1+\dots+u_r) & , \quad u_i^* & = u_i & \forall i \geq 1 \\
v_0^* \quad \textit{arbitrary} & , \quad v_i^* - v_0^* & = v_i & \forall i \geq 1
\end{array}$$

and of course $\sum_{1\leq i\leq r} u_i v_i \equiv \sum_{0\leq i\leq r} u_i^* v_i^*$.

Unary operations.

The following linear transformations on *BIMU* are of constant use:[23]

$$B^\bullet = \mathrm{minu}.A^\bullet \;\Rightarrow\; B^{w_1,\dots,w_r} = -\,A^{w_1,\dots,w_r} \tag{2.4}$$

$$B^\bullet = \mathrm{pari}.A^\bullet \;\Rightarrow\; B^{w_1,\dots,w_r} = (-1)^r\,A^{w_1,\dots,w_r} \tag{2.5}$$

$$B^\bullet = \mathrm{anti}.A^\bullet \;\Rightarrow\; B^{w_1,\dots,w_r} = A^{w_r,\dots,w_1} \tag{2.6}$$

$$B^\bullet = \mathrm{mantar}.A^\bullet \;\Rightarrow\; B^{w_1,\dots,w_r} = (-1)^{r-1}\,A^{w_r,\dots,w_1} \tag{2.7}$$

$$B^\bullet = \mathrm{neg}.A^\bullet \;\Rightarrow\; B^{w_1,\dots,w_r} = A^{-w_1,\dots,-w_r} \tag{2.8}$$

$$B^\bullet = \mathrm{swap}.A^\bullet \;\Rightarrow\; B^{\binom{u_1\ ,\dots,\ u_r}{v_1\ ,\dots,\ v_r}} = A^{\binom{v_r\ \ ,\dots,\ v_{3:4}\ ,\ v_{2:3},\ v_{1:2}}{u_{1..r}\ ,\dots,\ u_{123},\ u_{12},\ u_1}} \tag{2.9}$$

$$B^\bullet = \mathrm{pus}.A^\bullet \;\Rightarrow\; B^{\binom{u_1\ ,\dots,\ u_r}{v_1\ ,\dots,\ v_r}} = A^{\binom{u_r,\ u_1,\ u_2\ ,\dots,\ u_{r-1}}{v_r,\ v_1,\ v_2\ ,\dots,\ v_{r-1}}} \tag{2.10}$$

$$B^\bullet = \mathrm{push}.A^\bullet \;\Rightarrow\; B^{\binom{u_1\ ,\dots,\ u_r}{v_1\ ,\dots,\ v_r}} = A^{\binom{-u_{1\dots r},\ u_1\ ,\ u_2\ ,\dots,\ u_{r-1}}{-v_r\ ,\ v_{1:r},\ v_{2:r}\ ,\dots,\ v_{r-1:r}}} \tag{2.11}$$

All are involutions, save for *pus* and *push*, whose restrictions to each $BIMU_r$ reduce to circular permutations of order r resp. $r+1$:[24]

$$\mathrm{push} = \mathrm{neg.anti.swap.anti.swap} \tag{2.12}$$

$$\mathrm{leng}_r = \mathrm{push}^{r+1}.\mathrm{leng}_r = \mathrm{pus}^r.\mathrm{leng}_r \tag{2.13}$$

with $leng_r$ standing for the natural projection of *BIMU* onto $BIMU_r$.

Inflected derivations and automorphisms of *BIMU*.
Let $BIMU_*$ resp. $BIMU^*$ denote the subset of all bimoulds $M^\bullet$ such that $M^\emptyset = 0$ resp. $M^\emptyset = 1$. To each pair $\mathcal{A}^\bullet = (\mathcal{A}_L^\bullet, \mathcal{A}_R^\bullet) \in BIMU_* \times BIMU_*$ resp. $BIMU^* \times BIMU^*$ we attach two remarkable operators:

$$\mathrm{axit}(\mathcal{A}^\bullet) \in \mathrm{Der}(BIMU) \qquad \textit{resp.} \qquad \mathrm{gaxit}(\mathcal{A}^\bullet) \in \mathrm{Aut}(BIMU)$$

whose action on *BIMU* is given by:[25]

$$N^\bullet = \mathrm{axit}(\mathcal{A}^\bullet).M^\bullet \Leftrightarrow N^{\boldsymbol{w}} = \sum\nolimits^1 M^{\boldsymbol{a}\lceil\boldsymbol{c}} \mathcal{A}_L^{\boldsymbol{b}\rfloor} + \sum\nolimits^2 M^{\boldsymbol{a}\rfloor\boldsymbol{c}} \mathcal{A}_R^{\lfloor\boldsymbol{b}} \tag{2.14}$$

$$N^\bullet = \mathrm{gaxit}(\mathcal{A}^\bullet).M^\bullet \Leftrightarrow N^{\boldsymbol{w}} = \sum\nolimits^3 M^{\lceil\boldsymbol{b^1}\rceil\cdots\lceil\boldsymbol{b^s}\rceil} \mathcal{A}_L^{\boldsymbol{a^1}\rfloor}\dots\mathcal{A}_L^{\boldsymbol{a^s}\rfloor} \mathcal{A}_R^{\lfloor\boldsymbol{c^1}}\dots\mathcal{A}_R^{\lfloor\boldsymbol{c^s}} \tag{2.15}$$

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[23] The reason for dignifying the humble sign change in (2.4) with the special name *minu* is that *minu* enters the definition of scores of operators acting on various algebras: the rule for forming the corresponding operators that act on the corresponding groups, is then simply to change the trivial, linear *minu*, which commutes with everybody, into the non-trivial, non-linear *invmu*, which commutes with practically nobody (see (2.2)). To keep the minus sign instead of *minu* (especially when it occurs twice and so cancels out) would be a sure recipe for getting the transposition wrong.

[24] *pus* resp. *push* is a circular permutation in the *short* resp. *long* indexation of bimoulds. Indeed: $(push.M)^{[w_0],w_1,\dots,w_r} = M^{[w_r],w_0,\dots,w_{r-1}}$.

[25] The sum $\sum^1$ resp. $\sum^2$ extends to all sequence factorisations $\boldsymbol{w} = \boldsymbol{a.b.c}$ with $\boldsymbol{b} \neq \boldsymbol{\emptyset}$, $\boldsymbol{c} \neq \boldsymbol{\emptyset}$ resp. $\boldsymbol{a} \neq \boldsymbol{\emptyset}$, $\boldsymbol{b} \neq \boldsymbol{\emptyset}$. The sum $\sum^3$ extends to all factorisations $\boldsymbol{w} = \boldsymbol{a^1.b^1.c^1.a^2.b^2.c^2...a^s.b^s.c^s}$ such that $s \geq 1$, $\boldsymbol{b^i} \neq \boldsymbol{\emptyset}$, $\boldsymbol{c^i.a^{i+1}} \neq \boldsymbol{\emptyset}\ \forall i$. Note that the extreme factor sequences $\boldsymbol{a^1}$ and $\boldsymbol{c^s}$ may be $\boldsymbol{\emptyset}$.

and verifies the identities:

$$\mathrm{axit}(\mathcal{A}^\bullet).\mathrm{mu}(M_1^\bullet, M_2^\bullet) \equiv \mathrm{mu}(\mathrm{axit}(\mathcal{A}^\bullet).M_1^\bullet, M_2^\bullet) + \mathrm{mu}(M_1^\bullet, \mathrm{axit}(\mathcal{A}^\bullet).M_2^\bullet) \tag{2.16}$$
$$\mathrm{gaxit}(\mathcal{A}^\bullet).\mathrm{mu}(M_1^\bullet, M_2^\bullet) \equiv \mathrm{mu}(g\mathrm{axit}(\mathcal{A}^\bullet).M_1^\bullet, \mathrm{gaxit}(\mathcal{A}^\bullet).M_2^\bullet) \tag{2.17}$$

The *BIMU*-derivations *axit* are stable under the Lie bracket for operators. More precisely, the identity holds:

$$[\mathrm{axit}(\mathcal{B}^\bullet), \mathrm{axit}(\mathcal{A}^\bullet)] = \mathrm{axit}(C^\bullet) \quad \textit{with} \quad \mathcal{C}^\bullet = \mathrm{axi}(\mathcal{A}^\bullet, \mathcal{B}^\bullet) \tag{2.18}$$

relative to a Lie law *axi* on $BIMU_* \times BIMU_*$ given by:

$$\mathcal{C}_L^\bullet \;:=\; \mathrm{axit}(\mathcal{B}^\bullet).\mathcal{A}_L^\bullet - \mathrm{axit}(\mathcal{A}^\bullet).\mathcal{B}_L^\bullet + \mathrm{lu}(\mathcal{A}_L^\bullet, \mathcal{B}_L^\bullet) \tag{2.19}$$
$$\mathcal{C}_R^\bullet \;:=\; \mathrm{axit}(\mathcal{B}^\bullet).\mathcal{A}_R^\bullet - \mathrm{axit}(\mathcal{A}^\bullet).\mathcal{B}_R^\bullet - \mathrm{lu}(\mathcal{A}_R^\bullet, \mathcal{B}_R^\bullet) \tag{2.20}$$

Here, *lu* denotes the standard (non-inflected) Lie law on *BIMU*:

$$\mathrm{lu}(A^\bullet, B^\bullet) := \mathrm{mu}(A^\bullet, B^\bullet) - \mathrm{mu}(B^\bullet, A^\bullet) \tag{2.21}$$

Let *AXI* denote the Lie algebra consisting of all pairs $\mathcal{A}^\bullet \in BIMU_* \times BIMU_*$ under this law *axi*.

Likewise, the *BIMU*-automorphisms *gaxit* are stable under operator composition. More precisely:

$$\mathrm{gaxit}(\mathcal{B}^\bullet).\mathrm{gaxit}(\mathcal{A}^\bullet) = \mathrm{gaxit}(\mathcal{C}^\bullet) \quad \textit{with} \quad \mathcal{C}^\bullet = \mathrm{gaxi}(\mathcal{A}^\bullet, \mathcal{B}^\bullet) \tag{2.22}$$

relative to a law *gaxi* on $BIMU^* \times BIMU^*$ given by:

$$\mathcal{C}_L^\bullet \;:=\; \mathrm{mu}(\mathrm{gaxit}(\mathcal{B}^\bullet).\mathcal{A}_L^\bullet, \mathcal{B}_L^\bullet) \tag{2.23}$$
$$\mathcal{C}_R^\bullet \;:=\; \mathrm{mu}(\mathcal{B}_R^\bullet, \mathrm{gaxit}(\mathcal{B}^\bullet).\mathcal{A}_R^\bullet) \tag{2.24}$$

Let *GAXI* denote the Lie group consisting of all pairs $\mathcal{A}^\bullet \in BIMU^* \times BIMU^*$ under this law *gaxi*. This group *GAXI* clearly admits *AXI* as its Lie algebra.

The mixed operations *amnit* = *anmit*:
For $\mathcal{A}^\bullet := (A^\bullet, 0^\bullet)$ and $\mathcal{B}^\bullet := (0^\bullet, B^\bullet)$ the operators $axit(\mathcal{A}^\bullet)$ and $axit(\mathcal{B}^\bullet)$ reduce to $amit(A^\bullet)$ and $anit(B^\bullet)$ respectively (see (2.32) and (2.33) *infra*) and the identity (2.18) becomes:

$$\mathrm{amnit}(A^\bullet, B^\bullet) \equiv \mathrm{anmit}(A^\bullet, B^\bullet) \qquad (\forall A^\bullet, B^\bullet \in \mathrm{BIMU}_*) \tag{2.25}$$

with

$$\mathrm{amnit}(A^\bullet, B^\bullet) \;:=\; \mathrm{amit}(A^\bullet).\mathrm{anit}(B^\bullet) - \mathrm{anit}(\mathrm{amit}(A^\bullet).B^\bullet) \tag{2.26}$$
$$\mathrm{anmit}(A^\bullet, B^\bullet) \;:=\; \mathrm{anit}(B^\bullet).\mathrm{amit}(A^\bullet) - \mathrm{amit}(\mathrm{anit}(B^\bullet).A^\bullet) \tag{2.27}$$

When one of the two arguments $(A^\bullet, B^\bullet)$ vanishes, the definitions reduce to:

$$\text{amnit}(A^\bullet, 0^\bullet) = \text{anmit}(A^\bullet, 0^\bullet) := \text{amit}(A^\bullet) \tag{2.28}$$

$$\text{amnit}(0^\bullet, B^\bullet) = \text{anmit}(0^\bullet, B^\bullet) = \text{anit}(B^\bullet) \tag{2.29}$$

Moreover, when *amnit* operates on a one-component bimould $M^\bullet \in BIMU_1$ (such as the *flexion units* $\mathfrak{E}^\bullet$, see §3.1 and §3.3 *infra*), its action drastically simplifies:

$$N^\bullet := \text{amnit}(A^\bullet, B^\bullet).M^\bullet \equiv \text{anmit}(A^\bullet, B^\bullet).M^\bullet \Leftrightarrow N^{\boldsymbol{w}} := \sum_{\boldsymbol{a}\, w_i \boldsymbol{b} = \boldsymbol{w}} A^{\boldsymbol{a}\rfloor} M^{\lceil w_i \rceil} B^{\lfloor \boldsymbol{b}} \tag{2.30}$$

Unary substructures.
We have two obvious subalgebras//subgroups of *AXI//GAXI*, answering to the conditions:

$$\text{AMI} \subset \text{AXI} : \ \mathcal{A}_R^\bullet = 0^\bullet \quad , \quad \text{GAMI} \subset \text{GAXI} : \ \mathcal{A}_R^\bullet = 1^\bullet$$
$$\text{ANI} \subset \text{AXI} : \ \mathcal{A}_L^\bullet = 0^\bullet \quad , \quad \text{GANI} \subset \text{GAXI} : \ \mathcal{A}_L^\bullet = 1^\bullet$$

but we are more interested in the *mixed* unary substructures, consisting of elements of the form:

$$\mathcal{A}^\bullet = (\mathcal{A}_L^\bullet, \mathcal{A}_R^\bullet) \quad \textit{with} \quad \mathcal{A}_R^\bullet \equiv \text{h}(\mathcal{A}_L^\bullet) \quad \textit{and } \text{h} \textit{ a fixed involution} \tag{2.31}$$

with everything expressible in terms of the left element $\mathcal{A}_L^\bullet$ of the pair $\mathcal{A}^\bullet$. There exist, up to isomorphism, exactly seven such mixed unary substructures:

algebra	h	*swap*	algebra	h
....	....	....	....	....
ARI	*minu*	↔	IRA	*minu.push*
ALI	*anti.pari*	↔	ILA	*anti.pari.neg*
ALA	$anti.pari.neg_u$	↔	ALA	$anti.pari.neg_u$
ILI	$anti.pari.neg_v$	↔	ILI	$anti.pari.neg_v$
AWI	*anti.neg*	↔	IWA	*anti*
AWA	$anti.neg_u$	↔	AWA	$anti.neg_u$
IWI	$anti.neg_v$	↔	IWI	$anti.neg_v$

group	h	*swap*	group	h
....	....	....	....	....
GARI	*invmu*	↔	GIRA	*push.swap.invmu.swap*
GALI	*anti.pari*	↔	GILA	*anti.pari.neg*
GALA	$anti.pari.neg_u$	↔	GALA	$anti.pari.neg_u$
GILI	$anti.pari.neg_v$	↔	GILI	$anti.pari.neg_v$
GAWI	*anti.neg*	↔	GIWA	*anti*
GAWA	$anti.neg_u$	↔	GAWA	$anti.neg_u$
GIWI	$anti.neg_v$	↔	GIWI	$anti.neg_v$

Each algebra in the first table (e.g. *ARI*) is of course *the* Lie algebra of the like-named group (e.g. *GARI*). Conversely, each Lie group in the second table is essentially determined by its eponymous Lie algebra *and* the condition of left-linearity.[26]

Dimorphic substructures.
Among all seven pairs of substructures, only two respect dimorphy, namely *ARI//GARI* and *ALI//GALI*. Moreover, when restricted to dimorphic objects, they actually coincide:

$$\mathrm{ARI}^{\underline{\mathrm{al}}/\underline{\mathrm{al}}} = \mathrm{ALI}^{\underline{\mathrm{al}}/\underline{\mathrm{al}}} \quad \textit{with} \quad \{\underline{\mathrm{al}}/\underline{\mathrm{al}}\} = \{\text{alternal/alternal } \textit{and} \text{ even}\}$$
$$\mathrm{GARI}^{\underline{\mathrm{as}}/\underline{\mathrm{as}}} = \mathrm{GALI}^{\underline{\mathrm{as}}/\underline{\mathrm{as}}} \quad \textit{with} \quad \{\underline{\mathrm{as}}/\underline{\mathrm{as}}\} = \{\text{symmetral/symmetral } \textit{and} \text{ even}\}$$

We shall henceforth work with the pair *ARI//GARI*, whose definition involves a simpler involution h (it dispenses with the sequence inversion *anti*: see above table).

2.2 The algebra *ARI* and its group *GARI*.

Basic anti-actions.
The proper way to proceed is to define the anti-actions (on *BIMU*, with its uninflected product *mu* and bracket *lu*) first of the lateral pairs *AMI//GAMI*, *ANI//GANI* and then of the mixed pair *ARI//GARI*:

$$N^\bullet = \mathrm{amit}(A^\bullet).M^\bullet \;\Leftrightarrow\; N^{\boldsymbol{w}} = \sum\nolimits^1 M^{\boldsymbol{a}\lceil\boldsymbol{c}} A^{\boldsymbol{b}\rfloor} \tag{2.32}$$

$$N^\bullet = \mathrm{anit}(A^\bullet).M^\bullet \;\Leftrightarrow\; N^{\boldsymbol{w}} = \sum\nolimits^2 M^{\boldsymbol{a}\rfloor\boldsymbol{c}} A^{\lfloor\boldsymbol{b}} \tag{2.33}$$

$$N^\bullet = \mathrm{arit}(A^\bullet).M^\bullet \;\Leftrightarrow\; N^{\boldsymbol{w}} = \sum\nolimits^1 M^{\boldsymbol{a}\lceil\boldsymbol{c}} A^{\boldsymbol{b}\rfloor} - \sum\nolimits^2 M^{\boldsymbol{a}\rfloor\boldsymbol{c}} A^{\lfloor\boldsymbol{b}} \tag{2.34}$$

with sums $\sum^1$ (resp. $\sum^2$) ranging over all sequence factorisations $\boldsymbol{w} = \boldsymbol{abc}$ such that $\boldsymbol{b} \neq \emptyset, \boldsymbol{c} \neq \emptyset$ (resp. $\boldsymbol{a} \neq \emptyset, \boldsymbol{b} \neq \emptyset$).

$$N^\bullet = \mathrm{gamit}(A^\bullet).M^\bullet \;\Leftrightarrow\; N^{\boldsymbol{w}} = \sum\nolimits^1 M^{\lceil\boldsymbol{b^1}\ldots\lceil\boldsymbol{b^s}} A^{\boldsymbol{a^1}\rfloor} \ldots A^{\boldsymbol{a^s}\rfloor} \tag{2.35}$$

$$N^\bullet = \mathrm{ganit}(A^\bullet).M^\bullet \;\Leftrightarrow\; N^{\boldsymbol{w}} = \sum\nolimits^2 M^{\boldsymbol{b^1}\rceil\ldots\boldsymbol{b^s}\rceil} A^{\lfloor\boldsymbol{c^1}} \ldots A^{\lfloor\boldsymbol{c^s}} \tag{2.36}$$

$$N^\bullet = \mathrm{garit}(A^\bullet).M^\bullet \;\Leftrightarrow\; N^{\boldsymbol{w}} = \sum\nolimits^3 M^{\lceil\boldsymbol{b^1}\rceil\ldots\lceil\boldsymbol{b^s}\rceil} A^{\boldsymbol{a^1}\rfloor} \ldots A^{\boldsymbol{a^s}\rfloor} A_*^{\lfloor\boldsymbol{c^1}} \ldots A_*^{\lfloor\boldsymbol{c^s}} \tag{2.37}$$

[26]meaning that the group operation (like $A^\bullet, B^\bullet \mapsto gari(A^\bullet, B^\bullet)$ in our example) is linear in $A^\bullet$ but highly non-linear in $B^\bullet$.

with $A_*^\bullet := \mathrm{invmu}(A^\bullet)$ and with sums $\sum^1, \sum^2, \sum^3$ ranging respectively over all sequence factorisations of the form :

$$\begin{array}{lcll} \boldsymbol{w} & = & \boldsymbol{a^1 b^1 \ldots a^s b^s} & (s \geq 1 \quad , \quad \textit{only } \boldsymbol{a^1} \textit{ may be } \boldsymbol{\emptyset}) \\ \boldsymbol{w} & = & \boldsymbol{b^1 c^1 \ldots b^s c^s} & (s \geq 1 \quad , \quad \textit{only } \boldsymbol{c^s} \textit{ may be } \boldsymbol{\emptyset}) \\ \boldsymbol{w} & = & \boldsymbol{a^1 b^1 c^1 \ldots a^s b^s c^s} & (s \geq 1 \quad , \quad \textit{with } \boldsymbol{b^i} \neq \boldsymbol{\emptyset} \textit{ and } \boldsymbol{c^i a^{i+1}} \neq \boldsymbol{\emptyset}) \end{array}$$

More precisely, in $\sum^3$ two *inner* neighbour factors $\boldsymbol{c^i}$ and $\boldsymbol{a^{i+1}}$ may vanish separately but not simultaneously, whereas the *outer* factors $\boldsymbol{a^1}$ and $\boldsymbol{c^s}$ may of course vanish separately or even simultaneously.

Lie brackets and group laws.
We can now concisely express the Lie algebra brackets *ami, ani, ari* and the group products *gami, gani, gari* :

$$\begin{aligned} \mathrm{ami}(A^\bullet, B^\bullet) &:= \mathrm{amit}(B^\bullet).A^\bullet - \mathrm{amit}(A^\bullet).B^\bullet + \mathrm{lu}(A^\bullet, B^\bullet) && (2.38) \\ \mathrm{ani}(A^\bullet, B^\bullet) &:= \mathrm{anit}(B^\bullet).A^\bullet - \mathrm{anit}(A^\bullet).B^\bullet - \mathrm{lu}(A^\bullet, B^\bullet) && (2.39) \\ \mathrm{ari}(A^\bullet, B^\bullet) &:= \mathrm{arit}(B^\bullet).A^\bullet - \mathrm{arit}(A^\bullet).B^\bullet + \mathrm{lu}(A^\bullet, B^\bullet) && (2.40) \end{aligned}$$

$$\begin{aligned} \mathrm{gami}(A^\bullet, B^\bullet) &:= \mathrm{mu}(\mathrm{gamit}(B^\bullet).A^\bullet), B^\bullet) && (2.41) \\ \mathrm{gani}(A^\bullet, B^\bullet) &:= \mathrm{mu}(B^\bullet, \mathrm{ganit}(B^\bullet).A^\bullet)) && (2.42) \\ \mathrm{gari}(A^\bullet, B^\bullet) &:= \mathrm{mu}(\mathrm{garit}(B^\bullet).A^\bullet), B^\bullet) && (2.43) \end{aligned}$$

Pre-Lie products ('pre-brackets').
Parallel with the three Lie brackets, we have three pre-Lie brackets:

$$\begin{aligned} \mathrm{preami}(A^\bullet, B^\bullet) &:= \mathrm{amit}(B^\bullet).A^\bullet + \mathrm{mu}(A^\bullet, B^\bullet) && (2.44) \\ \mathrm{preani}(A^\bullet, B^\bullet) &:= \mathrm{anit}(B^\bullet).A^\bullet - \mathrm{mu}(A^\bullet, B^\bullet) \quad (\textit{sign!}) && (2.45) \\ \mathrm{preari}(A^\bullet, B^\bullet) &:= \mathrm{arit}(B^\bullet).A^\bullet + \mathrm{mu}(A^\bullet, B^\bullet) && (2.46) \end{aligned}$$

with the usual relations:

$$\begin{aligned} \mathrm{ari}(A^\bullet, B^\bullet) &\equiv \mathrm{preari}(A^\bullet, B^\bullet) - \mathrm{preari}(B^\bullet, A^\bullet) && (2.47) \\ \mathrm{assopreari}(A^\bullet, B^\bullet, C^\bullet) &\equiv \mathrm{assopreari}(A^\bullet, C^\bullet, B^\bullet) && (2.48) \end{aligned}$$

with *assopreari* denoting the *associator*[27] of the pre-bracket *preari.* The same holds of course for *ami* and *ani.*

[27] Here, the associator *assobin* of a binary operation *bin* is straightforwardly defined as $assobin(a, b, c) := bin(bin(a, b), c) - bin(a, bin(b, c))$. Nothing to do with the Drinfeld associators of the sequel!

Exponentiation from *ARI* to *GARI*.
Provided we properly define the multiple pre-Lie brackets, i.e. from left to right:

$$\mathrm{preari}(A_1^\bullet,\dots,A_s^\bullet)=\mathrm{preari}(\mathrm{preari}(A_1^\bullet,\dots,A_{s-1}^\bullet),A_s^\bullet) \qquad (2.49)$$

we have a simple expression for the exponential mapping from a Lie algebra to its group. Thus, the exponential *expari* : *ARI* → *GARI* can be expressed as a series of pre-brackets:

$$\mathrm{expari}(A^\bullet)=\sum_{0\le n}\frac{1}{n!}\,\mathrm{preari}(\overbrace{A^\bullet,\dots,A^\bullet}^{n\ times})=1^\bullet+\sum_{0<n}\frac{1}{n!}\,\mathrm{preari}(...) \qquad (2.50)$$

or, what amounts to the same, as a mixed *mu+arit*-expansion:

$$\mathrm{expari}(A^\bullet)=1^\bullet+\sum_{1\le r\,,\,1\le n_i}\mathrm{Ex}^{n_1,\dots,n_r}\mathrm{mu}(A_{n_1}^\bullet,\dots,A_{n_r}^\bullet) \qquad (2.51)$$

with $A_n^\bullet := \big(arit(A^\bullet)\big)^{n-1}.A^\bullet$ and with the symmetral mould $Ex^\bullet$:

$$\mathrm{Ex}^{n_1,\dots,n_r}:=\frac{1}{(n_1-1)!}\,\frac{1}{(n_2-1)!}\cdots\frac{1}{(n_r-1)!}\,\frac{1}{n_{1\dots r}\,n_{2\dots r}\dots n_r} \qquad (2.52)$$

The operation from *GARI* to *ARI* that inverses *expari* shall be denoted as *logari*. It, too, can be expressed as a series of multiple *pre-ari* products, but in a much less straightforward manner than (2.50).

For any *alternal* mould $L^\bullet$ we also have the identities:

$$\sum_{\sigma\subset\mathfrak{S}(r)}L^{\omega_{\sigma(1)},\dots,\omega_{\sigma(r)}}\mathrm{preari}(A_{\sigma(1)}^\bullet,\dots,A_{\sigma(r)}^\bullet)\ \equiv$$
$$\frac{1}{r}\sum_{\sigma\subset\mathfrak{S}(r)}L^{\omega_{\sigma(1)},\dots,\omega_{\sigma(r)}}\mathrm{ari}(A_{\sigma(1)}^\bullet,\dots,A_{\sigma(r)}^\bullet) \qquad (\forall A_1^\bullet,\dots,A_r^\bullet) \quad (2.53)$$

which actually characterise *preari*.

Adjoint actions.
We shall require the adjoint actions, *adgari* and *adari*, of *GARI* on *GARI* and *ARI* respectively. The definitions are straightforward:

$$\mathrm{adgari}(A^\bullet).B^\bullet:=\mathrm{gari}(A^\bullet,B^\bullet,\mathrm{invgari}.A^\bullet)\quad(A^\bullet,B^\bullet\in\mathrm{GARI}) \qquad (2.54)$$

$$\mathrm{adari}(A^\bullet).B^\bullet:=\mathrm{logari}(\mathrm{adgari}(A^\bullet).\mathrm{expari}(B^\bullet)) \qquad (2.55)$$
$$:=\mathrm{fragari}(\mathrm{preari}(A^\bullet,B^\bullet),A^\bullet)\quad(A^\bullet\in\mathrm{GARI},B^\bullet\in\mathrm{ARI}) \qquad (2.56)$$

except for definition (2.56), which results from (2.55) and (2.43) and uses the *pre-ari* product [28] defined as in (2.46) *supra* and the *gari*-quotient[29] defined as in (2.60) *infra*.

Definition (2.56) has over the equivalent definition (2.55) the advantage of bringing out the $B^\bullet$-linearity of $adari(A^\bullet).B^\bullet$ and of leading to much simpler calculations.[30]

The centers of ARI and $GARI$.
The sets *Center(ARI)* resp. *Center(GARI)* consist of all bimoulds $M^\bullet$ that verify
(i) $M^\emptyset = 0$ resp. $M^\emptyset = 1$
(ii) $M^{\binom{u_1\,,\dots,\,u_r}{0\,,\dots,\,0}} = m_r \in \mathbb{C} \qquad \forall u_i$
(iii) $M^{\binom{u_1\,,\dots,\,u_r}{v_1\,,\dots,\,v_r}} = 0 \quad$ unless $\quad 0 = v_1 = \cdots = v_r$

Moreover, in view of (2.43), *gari*-multiplication by a central element $C^\bullet$ amounts to ordinary post-multiplication by that same $C^\bullet$:

$$\mathrm{gari}(C^\bullet, A^\bullet) \equiv \mathrm{gari}(A^\bullet, C^\bullet) \equiv \mathrm{mu}(A^\bullet, C^\bullet) \quad (C^\bullet \in \mathrm{Center(GARI)}) \tag{2.57}$$

Relatedness of the four main group inversions.
Lastly, we may note that the inversions relative to the four group laws *mu, gari, gami, gani* are not totally unrelated, but verify the rather unexpected identity:

$$\mathrm{invmu} = \mathrm{invgari.invgami.invgani} = \mathrm{invgani.invgami.invgari} \tag{2.58}$$

In fact, the group generated by these four involutions is isomorphic to the group with presentation $< a, b, c, d > /\{a^2, b^2, c^2, d^2, abcd\}$.

Complexity of the flexion operations.
Compared with the uninflected mould operations, the flexion operations on

[28] Properly speaking, *preari* applies only to elements $M^\bullet$ of ARI, i.e. such that $M^\emptyset = 0$. Here, however, only $B^\bullet$ is in ARI, whilst $A^\bullet$ is in $GARI$ and therefore $A^\emptyset = 1$. But this is no obstacle to applying the rule (2.46).

[29] Properly speaking, *fragari* applies only to arguments $S_1^\bullet, S_2^\bullet$ in $GARI$, i.e. such that $S_i^\emptyset = 1$. Here, however, only $S_2^\bullet := A^\bullet$ is in $GARI$, whilst $S_1^\bullet := preari(A^\bullet, B^\bullet)$ is in ARI and therefore $S_1^\emptyset = 0$. But this is no obstacle to applying the rule:

$$\mathrm{fragari}(S_1^\bullet, S_2^\bullet) := \mathrm{mu}(\mathrm{garit}(S_2^\bullet)^{-1}.S_1^\bullet, \mathrm{invgari}.S_2^\bullet) = \mathrm{mu}(\mathrm{garit}(\mathrm{invgari}.S_2^\bullet).S_1^\bullet, \mathrm{invgari}.S_2^\bullet)$$

[30] Despite the spontaneous occurence of the *pre-ari* product in (2.56), it should be noted that $adari(A^\bullet)$ is an automorphisms of ARI but *not* of $PREARI$.

bimoulds tend to be staggeringly complex. Here is the natural complexity ranking for some of the main *unary* operations:

$$invgami \sim invgani \ll invgari \ll logari \ll expari$$

and here is the number of summands involved[31] in $invgari(A^\bullet)$ or $expari(A^\bullet)$ as the length r increases:

length r	1	2	3	4	5	6	7	8	...
#(*invgari*)	1	4	20	112	672	4224	27459	183040	...
#(*expari*)	1	4	21	126	818	5594	39693	289510	...

Fortunately, the whole field is so strongly and harmoniously structured, and offers so many props to intuition, that this underlying complexity remains manageable. While formal computation is often indispensable at the exploratory stage, the patterns and properties that it brings to light tend to yield rather readily to rigorous proof.

2.3 Action of the basic involution *swap*.

Dimorphy is a property that bears on a bimould and its *swappee*. However, even the group product most respectful of dimorphy, i.e. *gari*, does not commute with the involution *swap*. But if we set

$$\mathrm{gira}(A^\bullet, B^\bullet) := \mathrm{swap.gari}(\mathrm{swap}.A^\bullet, \mathrm{swap}.B^\bullet) \tag{2.59}$$

$$\mathrm{fragari}(A^\bullet, B^\bullet) := \mathrm{gari}(A^\bullet, \mathrm{invgari}.B^\bullet) \tag{2.60}$$

$$\mathrm{fragira}(A^\bullet, B^\bullet) := \mathrm{gira}(A^\bullet, \mathrm{invgira}.B^\bullet) \tag{2.61}$$

the operation *gari//gira* and *fragari//fragira*, though distinct, can be expressed in terms of each other

$$\mathrm{gira}(A^\bullet, B^\bullet) \equiv \mathrm{ganit}(\mathrm{rash}.B^\bullet).\mathrm{gari}(A^\bullet, \mathrm{ras}.B^\bullet) \tag{2.62}$$

$$\mathrm{gari}(A^\bullet, B^\bullet) \equiv \mathrm{ganit}(\mathrm{rish}.B^\bullet).\mathrm{gira}(A^\bullet, \mathrm{ris}.B^\bullet) \tag{2.63}$$

$$\mathrm{fragira}(A^\bullet, B^\bullet) \equiv \mathrm{ganit}(\mathrm{crash}.B^\bullet).\mathrm{fragari}(A^\bullet, B^\bullet) \tag{2.64}$$

$$\mathrm{fragari}(A^\bullet, B^\bullet) \equiv \mathrm{ganit}(\mathrm{crish}.B^\bullet).\mathrm{fragira}(A^\bullet, B^\bullet) \tag{2.65}$$

via the anti-action $\mathrm{ganit}(B_*^\bullet)$ and with inputs $B_*^\bullet$ related to $B^\bullet$ through one of the following, highly non-linear operations

$$\mathrm{ras}.B^\bullet := \mathrm{invgari.swap.invgari.swap}.B^\bullet \tag{2.66}$$

$$\mathrm{rash}.B^\bullet := \mathrm{mu}(\mathrm{push.swap.invmu.swap}.B^\bullet, B^\bullet) \tag{2.67}$$

$$\mathrm{crash}.B^\bullet := \mathrm{rash.swap.invgari.swap}.B^\bullet \tag{2.68}$$

[31]each of these *inflected* summands, taken in isolation, is fairly complex!

$$\begin{aligned} \text{ris} &:= \text{ras}^{-1} = \text{swap.invgari.swap.invgari} && (2.69)\\ \text{rish} &:= \text{invgani.rash.ris} && (2.70)\\ \text{crish} &:= \text{invgani.crash} = \text{rish.invgari} && (2.71) \end{aligned}$$

2.4 Straight symmetries and subsymmetries.

• **alternality and symmetrality**.
Like a mould, a bimould $A^\bullet$ is said to be *alternal* (resp. *symmetral*) if it verifies

$$\sum_{\boldsymbol{w}\in \mathrm{sha}(\boldsymbol{w'},\boldsymbol{w''})} A^{\boldsymbol{w}} \equiv 0 \quad \big(\mathit{resp.}\ \ \equiv A^{\boldsymbol{w'}}A^{\boldsymbol{w''}}\big) \qquad \forall \boldsymbol{w'}\neq\emptyset\ ,\ \forall \boldsymbol{w''}\neq\emptyset \tag{2.72}$$

with $\boldsymbol{w}$ running through the set $sha(\boldsymbol{w'},\boldsymbol{w''})$ of all shufflings of $\boldsymbol{w'}$ and $\boldsymbol{w''}$.

• **{alternal} $\Longrightarrow$ {mantar-invariant, pus-neutral}.**
Alternality implies *mantar*-invariance, with $mantar = minu.pari.anti$ defined as in (2.7).
It also implies *pus*-neutrality, which means this:

$$\big(\sum_{1\leq l\leq r(\bullet)} \mathrm{pus}^l\big).A^\bullet \equiv 0 \qquad i.e. \qquad \sum_{\boldsymbol{w'}\overset{\mathrm{circ}}{\sim}\boldsymbol{w}} A^{\boldsymbol{w'}} \equiv 0 \qquad (\mathit{if}\, r(\boldsymbol{w})\geq 2) \tag{2.73}$$

• **{symmetral} $\Longrightarrow$ {gantar-invariant, gus-neutral}.**
Symmetrality implies likewise *gantar*-invariance, with

$$\text{gantar} := \text{invmu.anti.pari} \tag{2.74}$$

as well as *gus*-neutrality, which means $\big(\sum_{1\leq l\leq r(\bullet)}\mathrm{pus}^l\big).\mathrm{logmu}.A^\bullet \equiv 0$ i.e.

$$\sum_{1\leq k\leq r(\boldsymbol{w})} (-1)^{k-1} \sum_{\boldsymbol{w^1}\ldots\boldsymbol{w^k}\overset{\mathrm{circ}}{\sim}\boldsymbol{w}} A^{\boldsymbol{w^1}}\ldots A^{\boldsymbol{w^k}} \equiv 0 \qquad (\mathit{if}\, r(\boldsymbol{w})\geq 2) \tag{2.75}$$

• **{bialternal} $\overset{\mathrm{ess}^{\mathrm{ly}}}{\Longrightarrow}$ {neg-invariant, push-invariant}.**
Bialternality implies not only invariance under *neg.push* but also separate *neg*-invariance and *push*-invariance for any $A^\bullet \in BIMU_r$ but the implication holds only if $r > 1$, since on $BIMU_1$ we have *neg=push*. So *neg.push=id*, meaning that there is no constraint at all on elements of $BIMU_1$. But we must nonetheless impose *neg*-invariance on $BIMU_1$ (or what amounts to the

same, *push*-invariance) to ensure the stability of bialternals under the *ari*-bracket: see §2.7.

- **{bisymmetral} $\stackrel{\text{ess}^{\text{ly}}}{\Longrightarrow}$ {neg-invariant, gush-invariant}.**

Bisymmetrality implies not only invariance under *neg.gush*, with

$$\text{gush} := \text{neg.gantar.swap.gantar.swap} \tag{2.76}$$

but also separate *neg*-invariance and *gush*-invariance, but only if we assume *neg*-invariance for the component of length 1. If we do not make that assumption, every bisymmetral bimould in *GARI* splits into two bisymmetral factors: a regular right factor (invariant under *neg*) and an irregular left factor (invariant under *pari.neg*)

Let us now examine the *stable* combinations of alternality or 'subalternality' (resp. symmetrality or 'subsymmetrality'), i.e. the combinations that are preserved under at least *some* flexion operations and give rise to interesting algebras or groups.

Primary and secondary subalgebras and subgroups.
Broadly speaking, simple symmetries or subsymmetries (i.e. those that bear only on bimoulds or their swappees but not both) tend to be stable under a vast range of binary operations, both uninflected (like the *lu*-bracket or the *mu*-product) or inflected (like *ari/gari* or *ali/gali*). The corresponding algebras or groups are called *primary*. On the other hand, double symmetries or subsymmetries (i.e. those that bear simultaneously on bimoulds *and* their swappees) are only stable – when at all – under (suitable) inflected operations. We speak in this case of *secondary* algebras or groups.

"Finitary" and "infinitary" constraints.
Another important distinction lies in the character – "finitary" or otherwise – of the contraints corresponding to each set of symmetries of subsymmetries. These constraints always assume the form

$$0 \;=\; \sum_{\tau} \epsilon(\tau)\, M^{\tau(\boldsymbol{w})} + \sum_{\sigma} \epsilon(\sigma,\boldsymbol{w})\, M^{\sigma(\boldsymbol{w})} \tag{2.77}$$

$$\textit{with} \qquad \boldsymbol{w} = (w_1,\dots,w_r);\;\; \epsilon(\tau)\in\mathbb{Z},\;\; \epsilon(\sigma,\boldsymbol{w})\in\mathbb{C},\;\; \tau\in \mathrm{Gl}_r(\mathbb{Z}) \tag{2.78}$$

with a first sum involving a finite number of sequences $\tau(\boldsymbol{w})$ (resp. $\sigma(\boldsymbol{w})$) that are linearly dependent on $\boldsymbol{w}$ and of equal (resp. lesser) length. What really matters is the subgroup $<\tau>_r$ of $\mathrm{Gl}_r(\mathbb{Z})$ generated by the τ in the first sum and unambiguously determined (up to isomorphism) by the constraints.

When $<\tau>_r$ is finite[32] we speak of *finitary* constraints. The corresponding algebras or groups are always easy to investigate; the algebras in particular split into 'cells', or component subspaces in $BIMU_r$, whose dimensions are readily calculated by using standard invariant theory. When $<\tau>_r$ is infinite [33] things can of course get much trickier, but the important point to note is this: whereas simple symmetries (like alternality) are always *finitary*, and full double symmetries (like bialternality) always infinitary, there exists a very useful intermediary class – that namely of *finitary double symmetries.* The prototypal case is the (*ari*-stable) combination of alternality and *push*-invariance.[34]

We can now proceed to catalogue all the *basic* symmetry-induced algebras and groups – *basic* in the sense that all others can be derived from them by intersection.

Throughout, we adopt the following convenient notations. For any set $E \subset BIMU$:
(i) E^h or $E^{h/*}$ denotes the subset of all bimoulds $M^\bullet$ with the property h
(ii) $E^{h/k}$ denotes the subset of all bimoulds such that $M^\bullet$ has the property h and $swap.M^\bullet$ has the property k.
(iii) if h or k is a unary operation, the property in question should be taken to mean h- or k-invariance
(iv) $\overline{pusnu}$ or $\overline{gusnu}$ denote *pus*- or *gus*-neutrality (see §2.4)
(v) the underlining (as in $\underline{al}/\underline{al}$ or $\underline{as}/\underline{as}$) always signals the *parity condition* for the length-1 component
(vi) boldface **ARI** or **GARI** is used to distinguish the few *infinitary* subalgebras or subgroups of *ARI* or *GARI.*

The only *infinitary* algebras are:

$$\mathbf{ARI}^{\underline{\mathrm{al}}/\underline{\mathrm{al}}} \;\;,\;\; \mathbf{ARI}^{\overline{\mathrm{pusnu}}/\overline{\mathrm{pusnu}}} \;\;,\;\; \mathbf{ARI}^{\overline{\mathrm{pusnu}}/\overline{\mathrm{pusnu}}}_{\mathrm{mantar}/.} := \mathbf{ARI}^{\overline{\mathrm{pusnu}}/\overline{\mathrm{pusnu}}} \cap \mathrm{ARI}^{\mathrm{mantar}/.}$$

As for the intersection $\mathbf{ARI}^{\overline{\mathrm{pusnu}}/\overline{\mathrm{pusnu}}} \cap \mathrm{ARI}^{\mathrm{push}}$, it can be shown to coincide with $\mathbf{ARI}^{\underline{\mathrm{al}}/\underline{\mathrm{al}}}$ deprived of its length-one component. The same pattern holds the groups.

[32]like with the alternality constraints, in which case $<\tau>_r \sim \mathfrak{S}_r$.

[33]like with the bialternality constraints, in which case $<\tau>_r$ is generated by two distinct finite subgroups of $\mathrm{Gl}_r(\mathbb{Z})$, which we may denote as $\mathfrak{S}_r$ and $swap.\mathfrak{S}_r.swap$.

[34]That combination is indeed a double symmetry, since a bimould's *push*-invariance is a consequence of its *and* its swappee's alternality or at least *mantar*-invariance.

2.5 Main subalgebras.

$la^\bullet$		$li^\bullet := \mathrm{swap}(la^\bullet)$		*subalgebra*
....	..	....	...	....
push-*invariant*	$\Leftrightarrow$	push-*invariant*	...	ARI^{push}
pus-*neutral*		....	...	$ARI^{pusnu/*}$
....		pus-*neutral* (*strictly*)	...	$ARI^{*/\overline{pusnu}}$
pus-*neutral* (*strictly*)		pus-*neutral* (*strictly*)	...	$\mathbf{ARI}^{\overline{pusnu}/\overline{pusnu}}$
push-*neutral*	$\Leftrightarrow$	push-*neutral*	...	*unstable*
pus-*invariant*		....	...	*unstable*
....		pus-*invariant*	...	*unstable*
mantar-*invariant*		....	...	$ARI^{mantar/*}$
....		mantar-*invariant*	...	*unstable*
mantar-*invariant*		mantar-*invariant*	...	*unstable*
mantar-*invariant*		mantar-*invariant*	neg	$ARI^{\underline{mantar}/\underline{mantar}}$
push-*invariant*		mantar-*invariant*	...	$ARI^{push/mantar}$
mantar-*invariant*		push-*invariant*	...	$ARI^{mantar/push}$
alternal		....	...	$ARI^{al/*}$
....		*alternal*	...	*unstable*
alternal		*alternal*	...	*unstable*
alternal		*alternal*	neg	$\mathbf{ARI}^{\underline{al}/\underline{al}}$
alternal		mantar-*invariant*	...	*unstable*
alternal		mantar-*invariant*	neg	$ARI^{\underline{al}/\underline{mantar}}$
alternal		push-*invariant*	...	$ARI^{al/push}$
mantar-*invariant*		*alternal*	...	*unstable*
mantar-*invariant*		*alternal*	neg	$ARI^{\underline{mantar}/\underline{al}}$
push-*invariant*		*alternal*	...	$ARI^{push/al}$

2.6 Main subgroups.

$ga^\bullet$		$gi^\bullet := swap(ga^\bullet)$		*subgroup*
....	..	....	...	....
gush-*invariant*	$\Leftrightarrow$	gush-*invariant*	...	$GARI^{gush}$
gus-*neutral*		....	...	$GARI^{gusnu/*}$
....		gus-*neutral* (*strictly*)	...	$GARI^{*/\overline{gusnu}}$
gus-*neutral* (*strictly*)		gus-*neutral* (*strictly*)	...	$\mathbf{GARI}^{\overline{gusnu}/\overline{gusnu}}$
gush-*neutral*	$\Leftrightarrow$	gush-*neutral*	...	*unstable*
gus-*invariant*		....	...	*unstable*
....		gus-*invariant*	...	*unstable*
gantar-*invariant*		....	...	$GARI^{gantar/*}$
....		gantar-*invariant*	...	*unstable*
gantar-*invariant*		gantar-*invariant*	...	*unstable*
gantar-*invariant*		gantar-*invariant*	neg	$GARI^{\underline{gantar}/\underline{gantar}}$
gush-*invariant*		gantar-*invariant*	...	$GARI^{gush/gantar}$
gantar-*invariant*		gush-*invariant*	...	$GARI^{gantar/gush}$
alternal		....	...	$GARI^{as/*}$
....		*symmetral*	...	*unstable*
symmetral		*symmetral*	...	*unstable*
symmetral		*symmetral*	neg	$\mathbf{GARI}^{\underline{as}/\underline{as}}$
symmetral		gantar-*invariant*	...	*unstable*
symmetral		gantar-*invariant*	neg	$GARI^{\underline{as}/\underline{gantar}}$
symmetral		gush-*invariant*	...	$GARI^{as/gush}$
gantar-*invariant*		*symmetral*	...	*unstable*
gantar-*invariant*		*symmetral*	neg	$GARI^{\underline{gantar}/\underline{as}}$
gush-*invariant*		*symmetral*	...	$GARI^{gush/as}$

2.7 The dimorphic algebra $ARI^{\underline{al}/\underline{al}} \subset ARI^{al/al}$.

The space $ARI^{\underline{al}/\underline{al}}$ of *bialternal* and *even* bimoulds is a subalgebra of ARI. The total space $ARI^{al/al}$ of *all* bialternals is only marginally larger, since

$$\mathrm{ARI}^{\mathrm{al/al}} = \mathrm{ARI}^{\dot{\mathrm{a}}\mathrm{l}/\dot{\mathrm{a}}\mathrm{l}} \oplus \mathrm{ARI}^{\underline{\mathrm{al}}/\underline{\mathrm{al}}} \tag{2.79}$$

with a complement space $ARI^{\dot{a}l/\dot{a}l} := BIMU_{1,odd}$ that simply consists of all *odd* bimoulds with a single non-zero component of length 1. The total space

$ARI^{al/al}$ is not an algebra, but there is some additional structure on it, in the form of a bilinear mapping *oddari* of $ARI^{\dot{a}l/\dot{a}l}$ into $ARI^{\underline{al}/\underline{al}}$:

$$\text{oddari}: \quad (\text{ARI}^{\dot{\text{a}}\text{l}/\dot{\text{a}}\text{l}}, \text{ARI}^{\dot{\text{a}}\text{l}/\dot{\text{a}}\text{l}}) \longrightarrow \text{ARI}^{\underline{\text{al}}/\underline{\text{al}}} \qquad (\text{oddari} \neq \text{ari}) \tag{2.80}$$

with

$$\begin{aligned} C^\bullet &= \text{oddari}(A^\bullet.B^\bullet) \quad \Longrightarrow \\ C^{\binom{u_1\,,\,u_2}{v_1\,,\,v_2}} &:= +A^{\binom{u_1}{v_1}}B^{\binom{u_2}{v_2}} + A^{\binom{-u_1-u_2}{-v_2}}B^{\binom{u_1}{v_1-v_2}} + A^{\binom{u_2}{v_2-v_1}}B^{\binom{-u_1-u_2}{-v_1}} \\ &\quad -B^{\binom{u_1}{v_1}}A^{\binom{u_2}{v_2}} - B^{\binom{-u_1-u_2}{-v_2}}A^{\binom{u_1}{v_1-v_2}} - B^{\binom{u_2}{v_2-v_1}}A^{\binom{-u_1-u_2}{-v_1}} \end{aligned} \tag{2.81}$$

Remark. Although *swap* doesn't act as an automorphism on *ARI*, it does on $ARI^{\underline{\text{al}}/\underline{\text{al}}}$, essentially because all elements of $ARI^{\underline{\text{al}}/\underline{\text{al}}}$ are *push* invariant.

2.8 The dimorphic group $GARI^{\underline{as}/\underline{as}} \subset GARI^{as/as}$.

The set $GARI^{\underline{al}/\underline{al}}$ of *bisymmetral* and *even* bimoulds is a subgroup of *GARI*. The total set $GARI^{as/as}$ of *all* bisymmetrals is only marginally larger, since we have the factorisation

$$\text{GARI}^{\text{as}/\text{as}} = \text{gari}(\text{GARI}^{\dot{\text{a}}\text{s}/\dot{\text{a}}\text{s}}, \text{GARI}^{\underline{\text{as}}/\underline{\text{as}}}) \tag{2.82}$$

$$\text{GARI}^{\dot{\text{a}}\text{s}/\dot{\text{a}}\text{s}} = \bigcup_{\mathfrak{E}} \mathfrak{ess}^\bullet_{\mathfrak{E}} \qquad (\mathfrak{E} = \textit{flexion unit}\ ,\ \mathfrak{ess}^\bullet_{\mathfrak{E}}\ \textit{bisymmetral}) \tag{2.83}$$

with a left factor $\text{GARI}^{\dot{a}s/\dot{a}s}$ consisting of bisymmetral bimoulds that are invariant under *pari.neg* (rather than *neg*) and correspond one-to-one to very special bimoulds of $BIMU_1$, the so-called *flexion units* (see §3.2 and §3.5). Of course, the union $\bigcup_{\mathfrak{E}^\bullet}$ extends to the *vanishing* unit $\mathfrak{E}^\bullet = 0^\bullet$, to which there corresponds $\mathfrak{ess}_{\mathfrak{E}} = id_{GARI}$. The total set $GARI^{as/as}$ is not a group, but the above decomposition makes it clear that it is stable under postcomposition by $GARI^{\underline{\text{as}}/\underline{\text{as}}}$:

$$\text{gari}(\text{GARI}^{\text{as}/\text{as}}, \text{GARI}^{\underline{\text{as}}/\underline{\text{as}}}) = \text{GARI}^{\text{as}/\text{as}} \tag{2.84}$$

Remark. Although *swap* doesn't act as an automorphism on *GARI*, it does on $GARI^{\underline{\text{as}}/\underline{\text{as}}}$, essentially because all elements of $GARI^{\underline{\text{as}}/\underline{\text{as}}}$ are *gush* invariant. In fact, for $B^\bullet$ in $GARI^{\underline{\text{as}}/\underline{\text{as}}}$, formula (2.62) reads $gira(A^\bullet, B^\bullet) = gari(A^\bullet, B^\bullet)$ since in that case $rash(B^\bullet) = 1^\bullet$ and $ras(B^\bullet) = B^\bullet$.

3 Flexion units and twisted symmetries.

3.1 The free monogenous flexion algebra $Flex(\mathfrak{E})$.

To any $\mathfrak{E}^\bullet \in BIMU_1$ of a given parity type $\binom{s_1}{s_2}$, i.e. such that

$$\mathfrak{E}^{\binom{\epsilon\, u_1}{\eta\, v_1}} \equiv \epsilon^{s_1}\, \eta^{s_2}\, \mathfrak{E}^{\binom{u_1}{v_1}} \qquad \textit{with } s_1, s_2 \in \{0,1\}\ ;\ \epsilon, \eta \in \{+,-\}\ ;\ \forall u_1, v_1 \quad (3.1)$$

let us attach the space $Flex(\mathfrak{E})$ of all bimoulds generated by $\mathfrak{E}^\bullet$ under *all* flexion operations, unary or binary[35]. $Flex(\mathfrak{E})$ thus contains subalgebras not just of *ARI* but of all 7+1 distinct flexion algebras, and subgroups not just of *GARI* but of all 7+1 distinct flexion groups. Moreover, for truly random generators $\mathfrak{E}^\bullet$, all realisations $Flex(\mathfrak{E})$ are clearly isomorphic: they depend only on the parity type $\binom{s_1}{s_2}$. Lastly, for all four parity types, we have the same universal decomposition of $Flex(\mathfrak{E})$ into cells $Flex_r(\mathfrak{E}) \subset BIMU_r$ whose dimensions are as follows:

$$\mathrm{Flex}(\mathfrak{E}) \;=\; \bigoplus_{r\geq 0} \mathrm{Flex}_r(\mathfrak{E}) \quad \textit{with} \quad \dim(\mathrm{Flex}_r(\mathfrak{E})) = \frac{(3\,r)!}{r!\,(2\,r+1)!} \quad (3.2)$$

The reason is that $Flex_r(\mathfrak{E})$ can be *freely* generated by just two operations, namely *mu* and *amnit*:

$$A_i^\bullet \in Flex_{r_i}(\mathfrak{E}) \implies \mathrm{mu}(A_1,\dots,A_s) \in Flex_{r_1+..r_s}(\mathfrak{E}) \quad (3.3)$$

$$A_i^\bullet \in Flex_{r_i}(\mathfrak{E}) \implies \mathrm{amnit}(A_1,A_2).\mathfrak{E}^\bullet \in Flex_{1+r_1+r_2}(\mathfrak{E}) \quad (3.4)$$

As a consequence, each cell $Flex_r(\mathfrak{E})$ can be shown to possess four natural bases of exactly the required cardinality, namely $\{\mathfrak{e}^\bullet_{\boldsymbol{t}}\} \sim \{\mathfrak{e}^\bullet_{\boldsymbol{p}}\} \sim \{\mathfrak{e}^\bullet_{\boldsymbol{o}}\} \sim \{\mathfrak{e}^\bullet_{\boldsymbol{g}}\}$. Theses bases are actually *one*, and merely differ by the indexation:
1) $\boldsymbol{t}$ runs through all r-node ternary trees.
2) $\boldsymbol{p}$ runs through all r-fold arborescent parenthesisings.
3) $\boldsymbol{o}$ runs through all arborescent, coherent orders on $\{1,\dots,r\}$.
4) $\boldsymbol{g}$ runs through all pairs $\boldsymbol{g} = (\boldsymbol{ga}, \boldsymbol{gi})$ of r-edged, non-overlapping graphs.

The basis $\{\mathfrak{e}^\bullet_{\boldsymbol{t}}\}$.
The free generation of $Flex_r(\mathfrak{E})$ under the operations (3.3) and (3.4) produces an indexation by trees $\boldsymbol{\theta}$ of a definite sort which, though not ternary, stand in one-to-one correspondence with ternary trees $\boldsymbol{t}$. We need not bother with that here.

[35]other than *swap*, which exchanges the u_i's and v_i's, and *pus* (see (2.10)) which, we recall, doesn't qualify as a proper flexion operation. But *push* is allowed, as well as all algebra and group operations.

The basis $\{\mathfrak{e}^\bullet_{\boldsymbol{g}}\}$.
We fix r and puncture the unit circle at all points Si_k and Sa_k of the form

$$Si_k := \exp\big(2\pi i \frac{k}{r+1}\big) \quad , \quad Sa_k := \exp\big(2\pi i \frac{k+\frac{1}{2}}{r+1}\big) \qquad (k \in \mathbb{Z}/(r+1)\mathbb{Z})$$

Let $\boldsymbol{G}_r$ be the set of all $\frac{(3\,r)!}{r!\,(2\,r+1)!}$ pairs $\boldsymbol{g} = (\boldsymbol{ga}, \boldsymbol{gi})$ such that :
(i) $\boldsymbol{ga}$ is a connected graph with vertices at each Sa_j and with exactly r straight, non-intersecting edges.
(ii) $\boldsymbol{gi}$ is a connected graph with vertices at each Si_j and with exactly r straight, non-intersecting edges.
(iii) $\boldsymbol{ga}$ and $\boldsymbol{gi}$ are 'orthogonal' in the sense that each edge of one intersects exactly one edge of the other.[36]
To each such $\boldsymbol{g} = (\boldsymbol{ga}, \boldsymbol{gi})$ we attach the bimould $\mathfrak{e}^\bullet_{\boldsymbol{g}} \in Flex_r(\mathfrak{E})$ defined by

$$\mathfrak{e}_{\boldsymbol{g}}^{\binom{u_1\,,\ldots,\,u_r}{v_1\,,\ldots,\,v_r}} := \prod_{x\in \boldsymbol{ga}\cap\boldsymbol{gi}} \mathfrak{E}^{\binom{u(x)}{v(x)}} \qquad (\textit{exactly } r \textit{ factors}) \tag{3.5}$$

with

$$\begin{aligned} u(x) &:= \sum_{[\mathrm{Si}_0<\mathrm{Sa}_{m_1}<\mathrm{Sa}_n<\mathrm{Sa}_{m_2}]^{\mathrm{circ}}} u_n \qquad (\textit{with } 1\leq n\leq r)\\ v(x) &:= v_{n_2}-v_{n_1} \qquad (n_2\neq 0\,;\, v_{n_1}=0 \textit{ if } n_1=0) \end{aligned}$$

with Sa_{m_1}, Sa_{m_2} (resp. Si_{n_1}, Si_{n_2}) denoting the end-points of the edge of $\boldsymbol{ga}$ (resp. $\boldsymbol{gi}$) going through x and with the indexation order so chosen as to ensure

$$[\mathrm{Si}_0 < \mathrm{Sa}_{m_1} < \mathrm{Sa}_{m_2}]^{\mathrm{circ}} \quad \textit{and} \quad [\mathrm{Si}_{n_1} < \mathrm{Sa}_{m_1} < \mathrm{Si}_{n_2} < \mathrm{Sa}_{m_2}]^{\mathrm{circ}}$$

The basis $\{\mathfrak{e}^\bullet_{\boldsymbol{o}}\}$.
A partial order $\boldsymbol{o}$ on $\{1,\ldots,r\}$ is *arborescent* if each i in $\{1,\ldots,r\}$ has at most one direct $\boldsymbol{o}$-antecedent i_-, and it is *coherent* if the following implication (which involves both the natural order $\leq$ and the $\boldsymbol{o}$-order $\preceq$) holds:

$$\{i_1 \leq i_2 \leq i_3 \ , \ i \preceq i_1 \ , \ i \preceq i_3\} \Longrightarrow \{i \preceq i_2\} \tag{3.6}$$

This amounts to saying that the set of all j such that $i \preceq j$ has to be an *interval* $i^- \leq j \leq i^+$ for the natural order. The basis elements are then

[36]Each $\boldsymbol{ga}$ verifying (i) has *one* orthogonal $\boldsymbol{gi}$ verifying (ii) and *vice versa.* We are told that these objects are known as *non-crossing trees* in combinatorics.

defined as follows

$$\mathfrak{e}_{\boldsymbol{o}}^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ v_1 & ,\dots, & v_r \end{smallmatrix}\right)} := \prod_{1\leq i\leq r} \mathfrak{E}^{\left(\begin{smallmatrix} u(i) \\ v(i) \end{smallmatrix}\right)} \quad \textit{with} \quad u(i) := \sum_{i\preceq j} u_j = \sum_{j=i^-}^{j=i^+} u_j \ \ , \ \ v(i) := v_i - v_{i_-}$$

If i has no $\boldsymbol{o}$-antecedent i_- we must of course set $v(i) := v_i$.

The basis $\{\mathfrak{e}^\bullet_{\boldsymbol{p}}\}$.
The set $\mathcal{P}_r$ of all r-fold arborescent parenthesisings may be visualised as consisting of non-commutative words $\boldsymbol{p}$ made up of r letters a ("opening parentheses"), r letters b ("inter-parenthesis content") and r letters c ("closing parentheses"). These words, in turn, are defined by a simple induction: each non-prime $\boldsymbol{p}$ admits a unique factorisation into prime factors $\boldsymbol{p}_i$, and each prime $\boldsymbol{p}$ admits a unique expression of the form

$$\boldsymbol{p} = a.\boldsymbol{p}_1.b.\boldsymbol{p}_2.c \qquad\qquad (\boldsymbol{p}_1, \boldsymbol{p}_2 \in \mathcal{P}) \tag{3.7}$$

with factors $\boldsymbol{p}_1, \boldsymbol{p}_2$ that need not be prime, and one of which may be empty.[37]
Thus $\mathcal{P}_1 = \{abc\}$, $\mathcal{P}_2 = \{aabcbc, ababcc, abcabc\}$, etc.

To define the correspondance between the $\boldsymbol{p}$- and $\boldsymbol{o}$-indexations, we assimilate each i in $\{1, \dots, r\}$ to the i-th letter b in the words $\boldsymbol{p} \in \mathcal{P}_r$ and set

$$h(i) := \alpha - \gamma = \gamma' - \alpha' \tag{3.8}$$

if that i-th letter b is preceded in $\boldsymbol{p}$ by α letters a and γ letters c or, what amounts to the same, followed by α' letters a and γ' letters c. We then define the order $\boldsymbol{o}$ on $\{1, \dots, r\}$ by decreeing that $i \prec j$ iff $h(i) < h(j)$ and $h(i) < h(k)$ for all k between i and j.[38]

3.2 Flexion units.

As it happens, the most useful monogenous algebras $Flex(\mathfrak{E})$ are not those spawned by 'random' generators $\mathfrak{E}$ but on the contrary by very special ones - the so-called *flexion units.*

Exact flexion units. The tripartite relation.
A *flexion unit* is a bimould $\mathfrak{E}^\bullet \in BIMU_1$ that is *odd* in w_1 and verifies the

[37] or even both, if $\boldsymbol{p} \in \mathcal{P}_1$.

[38] As a consequence, if the i-th and j-th letters b fall into distinct prime factors of $\boldsymbol{p}$, then i and j are non-comparable.

tripartite relation below. More precisely:

$$\mathfrak{E}^{-w_1} \equiv -\mathfrak{E}^{w_1} \quad , \quad \mathfrak{E}^{w_1}\,\mathfrak{E}^{w_2} \equiv \mathfrak{E}^{w_1\rfloor}\,\mathfrak{E}^{\lceil w_2} + \mathfrak{E}^{w_1\rceil}\,\mathfrak{E}^{\lfloor w_2} \qquad i.e$$

$$\mathfrak{E}^{\binom{-u_1}{-v_1}} \equiv -\mathfrak{E}^{\binom{u_1}{v_1}} \quad , \quad \mathfrak{E}^{\binom{u_1}{v_1}}\mathfrak{E}^{\binom{u_2}{v_2}} \equiv \mathfrak{E}^{\binom{u_1}{v_{1:2}}}\mathfrak{E}^{\binom{u_{12}}{v_2}} + \mathfrak{E}^{\binom{u_{12}}{v_1}}\mathfrak{E}^{\binom{u_2}{v_{2:1}}} \tag{3.9}$$

In view of the imparity of $\mathfrak{E}^\bullet$ the tripartite identity may also be written in more symmetric form:

$$\mathfrak{E}^{\binom{u_1}{v_{1:0}}}\mathfrak{E}^{\binom{u_2}{v_{2:0}}}+\mathfrak{E}^{\binom{u_2}{v_{2:1}}}\mathfrak{E}^{\binom{u_0}{v_{0:1}}}+\mathfrak{E}^{\binom{u_0}{v_{0:2}}}\mathfrak{E}^{\binom{u_1}{v_{1:2}}} \equiv 0 \quad \forall u_i, \forall v_i \;\; with \;\; u_0+u_1+u_2 = 0$$

Another way of characterising flexion units is via the *push*-neutrality of their powers $mu^n(\mathfrak{E}^\bullet)$. Indeed, if we set:

$$\mathrm{mu}^n(\mathfrak{E}^\bullet) = \mathrm{mu}(\overbrace{\mathfrak{E}^\bullet, \ldots, \mathfrak{E}^\bullet}^{n\ times}) \tag{3.10}$$

then $\mathfrak{E}$ is a flexion unit *iff* $mu^1(\mathfrak{E}^\bullet)$ and $mu^2(\mathfrak{E}^\bullet)$ are *push*-neutral, in which case it can be shown that *all* powers $mu^n(\mathfrak{E}^\bullet)$ are automatically *push*-neutral:

$$\Big\{\mathfrak{E} \textit{ is a flexion unit}\Big\} \Leftrightarrow \Big\{\big(\sum_{0\le k\le n} \mathrm{push}^k\big).\mathrm{mu}^n(\mathfrak{E}^\bullet) = 0 \;\; , \;\; \forall n \in \mathbb{N}^*\Big\} \tag{3.11}$$

If two units $\mathfrak{E}^\bullet$ and $\mathfrak{O}^\bullet$ are *constant* respectively in v_1 and u_1 , then the sum $\mathfrak{E}^\bullet + \mathfrak{O}^\bullet$ is also a unit.

Lastly, if $\mathfrak{E}^\bullet$ is a unit, then for each $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ the relation

$$\mathfrak{E}^{\binom{u_1}{v_1}}_{[\alpha,\beta,\gamma,\delta]} := \delta\, e^{\gamma\, u_1\, v_1}\, \mathfrak{E}^{\binom{u_1/\alpha}{v_1/\beta}} \tag{3.12}$$

defines a new unit $\mathfrak{E}^\bullet_{[\alpha,\beta,\gamma,\delta]}$.

Conjugate units:
If $\mathfrak{E}^\bullet$ is a unit, then the relation $\mathfrak{O}^{\binom{u_1}{v_1}} := \mathfrak{E}^{\binom{v_1}{u_1}}$ define another unit $\mathfrak{O}^\bullet$ – the so-called *conjugate* of $\mathfrak{E}^\bullet$. Indeed, setting $(u_1, u_2) := (v_1', v_2'-v_1')$, $(v_1, v_2) := (u_1'+u_2', u_2')$, then using the *imparity* of $\mathfrak{E}^\bullet$ and re-ordering the terms, we find that (3.9) becomes:

$$\mathfrak{O}^{\binom{u_1'}{v_1'}}\mathfrak{O}^{\binom{u_2'}{v_2'}} \equiv \mathfrak{O}^{\binom{u_1'}{v_{1:2}'}}\mathfrak{O}^{\binom{u_{12}'}{v_2'}} + \mathfrak{O}^{\binom{u_{12}'}{v_1'}}\mathfrak{O}^{\binom{u_2'}{v_{2:1}'}} \qquad with \quad \mathfrak{O}^{\binom{u_1}{v_1}} := \mathfrak{E}^{\binom{v_1}{u_1}}$$

i.e. conserves its form.

Let us now mention the most useful flexion units, some *exact* and others only *approximate*. Throughout the sequel, we shall set:

$$P(t) := \frac{1}{t} \quad , \quad Q(t) := \frac{1}{\tan(t)} \quad , \quad Q_c(t) := \frac{c}{\tan(c\,t)} \tag{3.13}$$

Polar units:
They consist purely of poles at the origin:

$$\mathrm{Pa}^{w_1} = P(u_1) \tag{3.14}$$
$$\mathrm{Pi}^{w_1} = P(v_1) \tag{3.15}$$
$$\mathrm{Pai}^{w_1}_{\alpha,\beta} = P(\frac{u_1}{\alpha}) + P(\frac{v_1}{\beta}) = \frac{\alpha}{u_1} + \frac{\beta}{v_1} \tag{3.16}$$

$Pa^{\bullet}, Pi^{\bullet}, Pai^{\bullet}_{\alpha,\beta}$ are *exact* units.

Trigonometric units:
They are 'periodised' variants of the polar units:

$$\mathrm{Qa}^{w_1}_c = Q_c(u_1) = \frac{c}{\tan(c\,u_1)} \tag{3.17}$$
$$\mathrm{Qi}^{w_1}_c = Q_c(v_1) = \frac{c}{\tan(c\,v_1)} \tag{3.18}$$
$$\mathrm{Qai}^{w_1}_{c,\alpha,\beta} = Q_c(\frac{u_1}{\alpha}) + Q_c(\frac{v_1}{\beta}) = \frac{c}{\tan(\frac{c\,u_1}{\alpha})} + \frac{c}{\tan(\frac{c\,v_1}{\beta})} \tag{3.19}$$
$$\mathrm{Qaih}^{w_1}_{c,\alpha,\beta} = Q_c(\frac{u_1}{\alpha}) - Q_{c\,i}(\frac{v_1}{\beta}) = \frac{c}{\tan(\frac{c\,u_1}{\alpha})} - \frac{c}{\tanh(\frac{c\,v_1}{\beta})} \tag{3.20}$$

$Qa^{\bullet}_c$, $Qi^{\bullet}_c$ are *approximate* units but $Qai^{\bullet}_{c,\alpha,\beta}$, $Qaih^{\bullet}_{c,\alpha,\beta}$ are *exact*.

Elliptic units (after C. Brembilla):
Let $\sigma(z\,;\,g_2,g_3)$ be the classical Weierstrass sigma function:

$$\sigma(z\,;\,g_2,g_3) = z - \frac{g_2}{2^4.3.5}\,z^5 - \frac{g_3}{2^3.3.5.7}\,z^7 + \mathcal{O}(z^9) \qquad \textit{with}$$
$$\sigma(z\,;\,g_2,g_3) \equiv -\sigma(-z\,;\,g_2,g_3) \equiv t\,\sigma(z\,t^{-1}\,;\,g_2\,t^4, g_3\,t^6) \qquad (\forall t)$$

Then for all $g_2, g_3, \alpha, \beta, \gamma, \delta \in \mathbb{C}$ $(\alpha\beta \neq 0)$, the relation

$$\mathfrak{E}^{\binom{u_1}{v_1}}_{g_2,g_3} := \frac{\sigma(u_1+v_1\,;\,g_2,\,g_3)}{\sigma(u_1\,;\,g_2,\,g_3)\;\sigma(v_1\,;\,g_2,\,g_3)} \tag{3.21}$$

defines a two-parameter family of exact flexion units, which in turn, under the standard parameter saturation of (3.12), give rise to:

$$\mathfrak{E}^{\binom{u_1}{v_1}}_{g_2,g_3,\alpha,\beta,\gamma,\delta} \;:=\; \delta\, e^{\gamma\, u_1\, v_1}\; \mathfrak{E}^{\binom{u_1/\alpha}{v_1/\beta}}_{g_2,g_3} \tag{3.22}$$

$$\mathfrak{E}^{\bullet}_{g_2,g_3,\alpha,\beta,\gamma,\delta} \;\equiv\; \mathfrak{E}^{\bullet}_{g_2\, t^4\,,g_3\, t^6\,,\alpha\, t\,,\beta\, t\,,\gamma\,,\delta\, t^{-1}} \qquad (\forall t) \tag{3.23}$$

This six-parameter, five-dimensional complex variety of flexion units contains all previously listed *exact units* (polar or trigonometric) as limit cases. In fact, it would seem (the matter is still under investigation) that it exhausts *all* flexion units meromorphic in both u_1 and v_1.

We must now examine further units, exact or approximate, that fail to be meromorphic in one of these variables, or both.

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Bitrigonometric units:
$Qaa_c^{w_1}$ (resp. $Qii_c^{w_1}$) is defined for $u_1 \in \mathbb{C}$ and $v_1 \in \mathbb{Q}/\mathbb{Z}$ (resp. vice versa):

$$\mathrm{Qaa}_c^{\binom{u_1}{v_1}} := \sum_{n_1\in\mathbb{Z}} \frac{c\, e^{-2\,\pi i n_1 v_1}}{\pi n_1 + c u_1} = \sum_{1\le n_1\le \mathrm{den}(v_1)} \frac{c\, e^{-2\pi i n_1 v_1}}{\mathrm{den}(v_1)}\; Q_c\Big(\frac{\pi\, n_1 + c\, u_1}{\mathrm{den}(v_1)}\Big) \tag{3.24}$$

$$\mathrm{Qii}_c^{\binom{u_1}{v_1}} := \sum_{n_1\in\mathbb{Z}} \frac{c\, e^{-2\,\pi i n_1 u_1}}{\pi n_1 + c v_1} = \sum_{1\le n_1\le \mathrm{den}(u_1)} \frac{c\, e^{-2\pi i n_1 u_1}}{\mathrm{den}(u_1)}\; Q_c\Big(\frac{\pi\, n_1 + c\, v_1}{\mathrm{den}(u_1)}\Big) = \mathrm{Qaa}_c^{\binom{v_1}{u_1}}$$

with *den* denoting the denominator (of a rational number). $Qaa_c^{\bullet}$ and $Qii_c^{\bullet}$ are both *approximate* units (see (3.30),(3.31) below).

Flat units:
Let σ be the sign function on $\mathbb{R}$, i.e. $\sigma(\mathbb{R}^{\pm}) = \pm 1$ and $\sigma(0) = 0$. Then set:

$$\mathrm{Sa}^{w_1} = \sigma(u_1) \quad , \quad \mathrm{Si}^{w_1} = \sigma(v_1) \quad , \quad \mathrm{Sai}^{w_1} = \sigma(u_1) + \sigma(v_1) \tag{3.25}$$

$Sa^{\bullet}$, $Si^{\bullet}$ are *approximate* units but $Sai^{\bullet}$ is *exact.*[39]

Mixed units:

$$\mathrm{Qas}^{w_1}_{c,\pm} = \mathrm{Q}_c(u_1) \pm c\, i\, \sigma(v_1) \quad , \quad \mathrm{Qis}^{w_1}_{c,\pm} = \mathrm{Q}_c(v_1) \pm c\, i\, \sigma(u_1) \tag{3.26}$$

$Qas^{\bullet}_{c,\pm}$, $Qis^{\bullet}_{c,\pm}$ are *exact* units.

[39]when viewed as a distribution or as an almost-everywhere defined function on $\mathbb{R}$. But when viewed as a function on $\mathbb{Z}$, it becomes an approximate unit.

"False" units:

$$\mathrm{Qi}_{c,\pm}^{w_1} = Qi_c^{w_1} \pm c\,i = c\,Q(c\,v_1) \pm c\,i = \pm\,2\,c\,i\frac{e^{\pm\,2\,c\,i\,v_1}}{e^{\pm\,2\,c\,i\,v_1}-1} \tag{3.27}$$

$Qi^{\bullet}_{c,+}$ and $Qi^{\bullet}_{c,-}$ verify the exact *tripartite relation* but not the *imparity* condition.[40]

Approximate flexion units. Tweaking the tripartite relation.
The approximate flexion units listed above verify *tweaked* variants of the tripartite relation:

$$\mathrm{Qa}_c^{w_1}\,\mathrm{Qa}_c^{w_2} \equiv \mathrm{Qa}_c^{w_1\rfloor}\,\mathrm{Qa}_c^{\lceil w_2} + \mathrm{Qa}_c^{w_1\rceil}\,\mathrm{Qa}_c^{\lfloor w_2} + c^2 \tag{3.28}$$

$$\mathrm{Qi}_c^{w_1}\,\mathrm{Qi}_c^{w_2} \equiv \mathrm{Qi}_c^{w_1\rfloor}\,\mathrm{Qi}_c^{\lceil w_2} + \mathrm{Qi}_c^{w_1\rceil}\,\mathrm{Qi}_c^{\lfloor w_2} - c^2 \tag{3.29}$$

$$\mathrm{Qaa}_c^{w_1}\,\mathrm{Qaa}_c^{w_2} \equiv \mathrm{Qaa}_c^{w_1\rfloor}\,\mathrm{Qaa}_c^{\lceil w_2} + \mathrm{Qaa}_c^{w_1\rceil}\,\mathrm{Qaa}_c^{\lfloor w_2} + c^2\,\delta(v_1)\,\delta(v_2) \tag{3.30}$$

$$\mathrm{Qii}_c^{w_1}\,\mathrm{Qii}_c^{w_2} \equiv \mathrm{Qii}_c^{w_1\rfloor}\,\mathrm{Qii}_c^{\lceil w_2} + \mathrm{Qii}_c^{w_1\rceil}\,\mathrm{Qii}_c^{\lfloor w_2} - c^2\,\delta(u_1)\,\delta(u_2) \tag{3.31}$$

$$\mathrm{Sa}^{w_1}\,\mathrm{Sa}^{w_2} \equiv \mathrm{Sa}^{w_1\rfloor}\,\mathrm{Sa}^{\lceil w_2} + \mathrm{Sa}^{w_1\rceil}\,\mathrm{Sa}^{\lfloor w_2} - 1 + \delta(u_1)\,\delta(u_2) \tag{3.32}$$

$$\mathrm{Si}^{w_1}\,\mathrm{Si}^{w_2} \equiv \mathrm{Si}^{w_1\rfloor}\,\mathrm{Si}^{\lceil w_2} + \mathrm{Si}^{w_1\rceil}\,\mathrm{Si}^{\lfloor w_2} + 1 - \delta(v_1)\,\delta(v_2) \tag{3.33}$$

In the last four relations, $\delta(t) := 1$ if $t = 0$ and $\delta(t) := 0$ otherwise.

3.3 Unit-generated algebras $Flex(\mathfrak{E})$.

For an *exact* flexion unit $\mathfrak{E}^\bullet$ the monogenous flexion algebra $Flex(\mathfrak{E})$, also known as *eumonogeneous*[41] algebra, is richer in interesting bimoulds, though much smaller in size than in the case of a random generator $\mathfrak{E}^\bullet$. The total algebra $Flex(\mathfrak{E})$ can still, as in §3.1, be freely-canonically generated, but under the sole operation *amnit* and *without* mould multiplication *mu*. In other words, we retain only the steps (3.4) and forego the steps (3.3). As a consequence, $Flex(\mathfrak{E})$ decomposes into cells $Flex_r(\mathfrak{E}) \subset BIMU_r$ whose dimensions are given by the Catalan numbers and whose inductive construction goes like

[40]In terms of applications, the failure of imparity has more disruptive consequences than the failure to verify the exact *tripartite equation*, because it means that $\mathfrak{E}$ has no proper conjugate $\mathfrak{O}$, which in turn prevents it from serving as building block for dimorphic bimoulds such as $\mathfrak{ess}^\bullet$ etc.

[41]with *eu* standing for *good.* For the polar resp. trigonometric specialisations of the unit, $Flex(\mathfrak{E})$ is known as the *eupolar* resp. *eutrigonometric* algebra. In the eutrigonometric case, though, the basis elements are more numerous than in the eupolar case, and *amnit* is no longer sufficient to generate everything. See the last table in §12.1.

this :

$$\mathrm{Flex}(\mathfrak{E}) \;=\; \bigoplus_{r\geq 0} \mathrm{Flex}_r(\mathfrak{E}) \quad \textit{with} \quad \dim(\mathrm{Flex}_r(\mathfrak{E})) = \frac{(2\,r)!}{r!\,(r+1)!} \tag{3.34}$$

$$\mathrm{Flex}_r(\mathfrak{E}) \;=\; \bigoplus_{\substack{r_1+r_2=r-1\\ r_1,r_2\geq 0}} \mathrm{amnit}(\mathrm{Flex}_{r_1}(\mathfrak{E}), \mathrm{Flex}_{r_2}(\mathfrak{E}))\,.\,\mathfrak{E}^{\bullet} \tag{3.35}$$

The new basis $\{\mathfrak{e}^{\bullet}_{\boldsymbol{t}}\}$.
It follows from (3.35) that $Flex_r(\mathfrak{E})$ has a natural basis $\{\mathfrak{e}^{\bullet}_{\boldsymbol{t}}\}$ indexed by all r-node binary trees $\boldsymbol{t}$. The construction is by induction on r :

$$\mathfrak{e}^{\bullet}_{\boldsymbol{t}} = \mathrm{amnit}(\mathfrak{e}^{\bullet}_{\boldsymbol{t_1}}, \mathfrak{e}^{\bullet}_{\boldsymbol{t_2}})\,.\,\mathfrak{E}^{\bullet} = \mathrm{anmit}(\mathfrak{e}^{\bullet}_{\boldsymbol{t_1}}, \mathfrak{e}^{\bullet}_{\boldsymbol{t_2}})\,.\,\mathfrak{E}^{\bullet} \tag{3.36}$$

where $\boldsymbol{t_1}, \boldsymbol{t_2}$ denote the left and right subtrees (one of them possibly empty) attached to the root of the binary tree $\boldsymbol{t}$.

This new basis $\{\mathfrak{e}^{\bullet}_{\boldsymbol{t}}\}$ is a natural subset of the analogous basis of §3.1, which was indexed by *ternary* trees.

The new basis $\{\mathfrak{e}^{\bullet}_{\boldsymbol{g}}\}$.
It coincides with the analogous system in §3.1, but restricted to the pairs $\boldsymbol{g} = (\boldsymbol{ga}, \boldsymbol{gi})$ meeting either of these two equivalent conditions:
(i) the graph $\boldsymbol{ga}$ has no pair of edges issuing from the same vertex and containing Si_0 in the angle so defined.
(ii) the graph $\boldsymbol{gi}$ has no pair of edges with end-points (Si_p, Si_k), (Si_{k+1}, Si_q) disposed in the circular order $0 \leq p < k < k{+}1 < q \leq r{+}1$.

The new basis $\{\mathfrak{e}^{\bullet}_{\boldsymbol{o}}\}$.
It coincides with its prototype in of §3.1, but under restriction to the *separative* orders $\boldsymbol{o}$, i.e. to orders such that:

$$\{i - j = 1\} \Longrightarrow \{i \preceq j\} \;\textit{or}\; \{j \preceq i\} \tag{3.37}$$

In other words, elements that are *consecutive* in the natural order must be *comparable* in the $\boldsymbol{o}$-order. This implies that $\boldsymbol{o}$ has a *smallest* element. It also implies that if i, j are not $\boldsymbol{o}$-comparable, then the intervals $[i^-, i^+]$ and $[j^-, j^+]$ cannot be contiguous (which justifies calling the order $\boldsymbol{o}$ *"separative"*).

The new basis $\{\mathfrak{e}^{\bullet}_{\boldsymbol{p}}\}$.
It coincides with the analogous system in §3.1, but restricted to the words $\boldsymbol{p}$ constructed from the sole induction rule (3.7), without recourse to word concatenation. These less numerous $\boldsymbol{p}$ are necessarily prime, and can be compactly represented by sequences $\boldsymbol{h} = [h(1), \ldots, h(r)]$, with $h(i)$ denoting the height of the i-th letter b in $\boldsymbol{p}$, as defined in (3.8). For the lengths $r \leq 3$

we have thus:

$$\begin{array}{lcl} \mathcal{H}_1 = \big\{[1]\big\} & \longleftrightarrow & \mathcal{P}_1 = \big\{abc\big\} \\ \mathcal{H}_2 = \big\{[1,2],[2,1]\big\} & \longleftrightarrow & \mathcal{P}_2 = \big\{ababcc,\, aabcbc\big\} \\ \mathcal{H}_3 = \big\{[1,2,3],[1,3,2],[2,1,2],[2,3,1],[3,2,1]\big\} & \longleftrightarrow & \mathcal{P}_3 = \big\{abababccc, \dots\big\} \end{array}$$

The involution *syap* between conjugate flexion structures.
All monogenous structures $Flex(\mathfrak{E})$ generated by the exact flexion units listed in §3.2 are actually isomorphic. In the case of two conjugate units, the isomorphism becomes an involution, denoted *syap*:

$$\text{syap} \;:\quad \text{Flex}_r(\mathfrak{E}) \leftrightarrow \text{Flex}_r(\mathfrak{O}) \quad , \quad \mathfrak{e}^\bullet_{\boldsymbol{t}} \leftrightarrow \mathfrak{o}^\bullet_{\boldsymbol{t}} \qquad (\mathfrak{E},\mathfrak{O}\ \textit{conjugate}) \tag{3.38}$$

The involution *syap*, being defined only on monogenous structures, is quite distinct from the universal involution *swap*, which applies to the whole of *BIMU*. On the other hand, *syap* is more regular: it commutes with *all* flexion operations, whether unary or binary, whereas *swap* commutes only with a few, such as *ami//gami*.

The involution *sap* on each flexion structure.
Both mappings *swap* and *syap* exchange $Flex(\mathfrak{E})$ and $Flex(\mathfrak{O})$. Since these two involutions actually commute, their product *sap* is also a linear involution, with eigenspaces $\{\pm 1\}$ of approximately equal size :

$$\text{syap} \;:\; \text{Flex}_r(\mathfrak{E}) \leftrightarrow \text{Flex}_r(\mathfrak{O}) \tag{3.39}$$
$$\text{swap} \;:\; \text{Flex}_r(\mathfrak{E}) \leftrightarrow \text{Flex}_r(\mathfrak{O}) \tag{3.40}$$
$$\text{sap} \;:\; \text{Flex}_r(\mathfrak{E}) \leftrightarrow \text{Flex}_r(\mathfrak{E})\ ,\ \text{Flex}_r(\mathfrak{O}) \leftrightarrow \text{Flex}_r(\mathfrak{O}) \tag{3.41}$$
$$\textit{with}\ \ \text{sap} := \text{syap.swap} = \text{swap.syap} \;: \tag{3.42}$$

For r even, the dimensions $d^{\pm}_r$ of *sap*'s eigenspaces of eigenvalues ± 1 are equal, but for r odd d^+_r is slightly larger than d^-_r. In fact, computational evidence supports the following conjectures[42] :

$$d^+_{2r} - d^-_{2r} = 0 \qquad (\forall r) \tag{3.43}$$
$$d^+_{2r+1} - d^-_{2r+1} = \frac{(2\,r)!}{r!(r+1)!} = d^+_r + d^-_r \qquad (\forall r) \tag{3.44}$$

Polar specialisation and graphic interpretation.
In the specal case $(\mathfrak{E}^\bullet,\mathfrak{O}^\bullet) = (Pa^\bullet, Pi^\bullet)$, both the canonical basis and the involution *syap* have a simple interpretation, as shown on the polygonal diagrams in §12.1, with the *dotted* resp. *full* lines representing the variables $\boldsymbol{u}$ resp. $\boldsymbol{v}$.

[42] They have been verified up to $r = 8$.

3.4 Twisted symmetries and subsymmetries in universal mode.

To every exact flexion unit $\mathfrak{E}$ there correspond *twisted* variants of all *straight* symmetries and subsymmetries listed in §2.4. But before defining these, we must introduce two elementary bimoulds $\mathfrak{ez}^\bullet$ and $\underline{\mathfrak{ez}}^\bullet = \mathrm{pari}.\mathfrak{ez}^\bullet$:

$$\mathfrak{ez}^{w_1,\dots,w_r} := \mathfrak{E}^{w_1}\dots\mathfrak{E}^{w_r} \quad , \quad \underline{\mathfrak{ez}}^{w_1,\dots,w_r} := (-1)^r\ \mathfrak{E}^{w_1}\dots\mathfrak{E}^{w_r} \tag{3.45}$$

as well as the *symmetral* bimould $\mathfrak{es}^\bullet := sap.\mathfrak{ez}^\bullet$. (see also (4.70)).

• **$\mathfrak{E}$-alternality and $\mathfrak{E}$-symmetrality**.
The simplest characterisation of the $\mathfrak{E}$-twisted symmetries is by means of the equivalence :

$$\{B^\bullet\ \ \mathfrak{E}\text{-alternal } resp.\ \mathfrak{E}\text{-symmetral}\} \iff \{A^\bullet \text{ alternal } resp. \text{ symmetral}\}$$

with $B^\bullet = \mathrm{ganit}(\mathfrak{ez}^\bullet).A^\bullet$ or $B^\bullet = \mathrm{gamit}(\mathfrak{ez}^\bullet).A^\bullet$, on choice.[43]

As for the analytic expression of the twisted symmetries, it reproduces that of the *straight* symmetries on which they are patterned, except for the systematic occurence of inflected pairs (w_i, w_j), with w_i, w_j not in the same factor sequence. Let us illustrate the $\mathfrak{E}$-alternality (resp. $\mathfrak{E}$-symmetrality) relations for two sequences $\boldsymbol{w}', \boldsymbol{w}''$ first of length 1 :

$$B^{w_1,w_2} + B^{w_2,w_1} + B^{w_1\rceil}\underline{\mathfrak{ez}}^{\lfloor w_2} + B^{\lceil w_2}\underline{\mathfrak{ez}}^{w_1\rfloor} = 0\ (resp.\ B^{w_1}B^{w_2}) \qquad i.e$$
$$B^{\left(\begin{smallmatrix} u_1\ ,\ u_2 \\ v_1\ ,\ v_2\end{smallmatrix}\right)} + B^{\left(\begin{smallmatrix} u_2\ ,\ u_1 \\ v_2\ ,\ v_1\end{smallmatrix}\right)} - B^{\left(\begin{smallmatrix} u_{12} \\ v_1\end{smallmatrix}\right)}\mathfrak{E}^{\left(\begin{smallmatrix} u_2 \\ v_{2:1}\end{smallmatrix}\right)} - B^{\left(\begin{smallmatrix} u_{12} \\ v_2\end{smallmatrix}\right)}\mathfrak{E}^{\left(\begin{smallmatrix} u_1 \\ v_{1:2}\end{smallmatrix}\right)} = 0\ (resp.\ B^{\left(\begin{smallmatrix} u_1 \\ v_1\end{smallmatrix}\right)}B^{\left(\begin{smallmatrix} u_2 \\ v_2\end{smallmatrix}\right)})$$

and then of length 2 :

$$\begin{aligned}
&B^{w_1,w_2,w_3,w_4} + B^{w_1,w_3,w_2,w_4} + B^{w_3,w_1,w_2,w_4} + B^{w_1,w_3,w_4,w_2} + B^{w_3,w_1,w_4,w_2} + B^{w_3,w_4,w_1,w_2}\\
&+B^{w_1\rceil,w_2,w_4}\underline{\mathfrak{ez}}^{\lfloor w_3} + B^{\lceil w_3,w_2,w_4}\underline{\mathfrak{ez}}^{w_1\rfloor} + B^{w_1\rceil,w_4,w_2}\underline{\mathfrak{ez}}^{\lfloor w_3} + B^{\lceil w_3,w_4,w_2}\underline{\mathfrak{ez}}^{w_1\rfloor}\\
&+B^{w_3,w_1\rceil,w_2}\underline{\mathfrak{ez}}^{\lfloor w_4} + B^{w_3,\lceil w_4,w_2}\underline{\mathfrak{ez}}^{w_1\rfloor} + B^{w_1,w_2\rceil,w_4}\underline{\mathfrak{ez}}^{\lfloor w_3} + B^{w_1,\lceil w_3,w_4}\underline{\mathfrak{ez}}^{w_2\rfloor}\\
&+B^{w_1,w_3,w_2\rceil}\underline{\mathfrak{ez}}^{\lfloor w_4} + B^{w_1,w_3,\lceil w_4}\underline{\mathfrak{ez}}^{w_2\rfloor} + B^{w_3,w_1,w_2\rceil}\underline{\mathfrak{ez}}^{\lfloor w_4} + B^{w_3,w_1,\lceil w_4}\underline{\mathfrak{ez}}^{w_2\rfloor}\\
&+B^{w_1\rceil,w_2\rceil}\underline{\mathfrak{ez}}^{\lfloor w_3,\lfloor w_4} + B^{\lceil w_3,w_2\rceil}\underline{\mathfrak{ez}}^{w_1\rfloor,\lfloor w_4} + B^{w_1\rceil,\lceil w_4}\underline{\mathfrak{ez}}^{\lfloor w_3,w_2\rfloor} + B^{\lceil w_3,\lceil w_4}\underline{\mathfrak{ez}}^{w_1\rfloor,w_2\rfloor}\\
&= \ 0 \quad (resp.\, B^{w_1,w_2}B^{w_3,w_4})
\end{aligned}$$

These two examples should suffice to make the pattern clear. Remarkably, when $\mathfrak{E}$ runs through the set of all flexion units, the corresponding

[43] $\mathrm{ganit}(\mathfrak{ez}^\bullet)$ and $\mathrm{gamit}(\mathfrak{ez}^\bullet)$ define two distinct mappings $A^\bullet \mapsto B^\bullet$, but both result in the same transformation of symmetries.

$\mathfrak{E}$-symmetralities essentially exhaust all commutative *flexion products*[44] that may be defined on *BIMU*.

Like their *straight* models, the *twisted* symmetries induce important subsymmetries, which we must now sort out.

- **{$\mathfrak{E}$-alternal} $\Longrightarrow$ {$\mathfrak{E}$-mantar-invariant, $\mathfrak{E}$-pus-neutral}.**

$\mathfrak{E}$-*mantar* is a linear operator conjugate to *mantar*:

$$\mathfrak{E}\text{-mantar} := \text{ganit}(\mathfrak{ez}^\bullet).\text{mantar}.\text{ganit}(\mathfrak{ez}^\bullet)^{-1} \tag{3.46}$$

and with explicit action:

$$((\mathfrak{E}\text{-mantar}).B)^{\boldsymbol{w}} = (-1)^{r-1} \sum_{\prod_i \boldsymbol{a}^i b_i \boldsymbol{c}^i = \tilde{\boldsymbol{w}}} B^{\lceil b_1 \rceil \ldots \lceil b_s \rceil} \prod_i \underline{\mathfrak{ez}}^{\boldsymbol{a}^i \rfloor} \prod_i \underline{\mathfrak{ez}}^{\lfloor \boldsymbol{c}^i} \tag{3.47}$$

(Note that $\tilde{\boldsymbol{w}}$ always denotes the sequence $\boldsymbol{w}$ in reverse order).

$\mathfrak{E}$-*pus*-neutrality also is derived from straight *pus*-neutrality:

$$\Big(\sum_{1 \le l \le r(\bullet)} \text{pus}^l\Big).\text{ganit}(\mathfrak{ez})^{-1}.B^\bullet \equiv 0$$

and admits a simpler direct expression:

$$\sum_{\boldsymbol{w}' \overset{\text{circ}}{\sim} \boldsymbol{w}} B^{\boldsymbol{w}'} + (-1)^{r(\boldsymbol{w})} \sum_{\boldsymbol{a}^i w_i \boldsymbol{b}^i = \boldsymbol{w}} B^{\lceil w_i \rceil} \; \underline{\mathfrak{ez}}^{\boldsymbol{a}^i \rfloor} \; \underline{\mathfrak{ez}}^{\lfloor \boldsymbol{b}^i} \equiv 0 \tag{3.48}$$

- **{$\mathfrak{E}$-symmetral} $\Longrightarrow$ {$\mathfrak{E}$-gantar-invariant, $\mathfrak{E}$-gus-neutral}.**

$\mathfrak{E}$-*gantar* is a non-linear operator conjugate to *gantar*:

$$\begin{aligned}
\mathfrak{E}\text{-gantar} &:= \text{ganit}(\mathfrak{ez}^\bullet).\text{gantar}.\text{ganit}(\mathfrak{ez}^\bullet)^{-1} \\
&= \text{ganit}(\mathfrak{ez}^\bullet).\text{invmu}.\text{anti}.\text{pari}.\text{ganit}(\mathfrak{ez}^\bullet)^{-1} \\
&= \text{ganit}(\mathfrak{ez}^\bullet).\text{invmu}.\text{anti}.\text{pari}.\text{minu}.\text{ganit}(\mathfrak{ez}^\bullet)^{-1}\text{minu} \\
&= \text{invmu}.\text{ganit}(\mathfrak{ez}^\bullet).\text{anti}.\text{pari}.\text{minu}.\text{ganit}(\mathfrak{ez}^\bullet)^{-1}\text{minu} \\
&= \text{invmu}.(\mathfrak{E}\text{-mantar}).\text{minu}
\end{aligned}$$

To establish the above sequence, we used the commutation of *ganit*$(M^\bullet)$ with both *minu* and *invmu*, and the mutual commutation of *minu, anti, pari.*

Using the last identity, we see that the action of $\mathfrak{E}$-*gantar* is given by:

$$((\mathfrak{E}\text{-gantar}).B)^{\boldsymbol{w}} = \sum_{\prod_i \boldsymbol{a}^i b_i \boldsymbol{c}^i = \tilde{\boldsymbol{w}}} \; \sum_{\prod_j \boldsymbol{b}^j = \prod_i \lceil b_i \rceil} (-1)^{r-s} \prod_{1 \le j \le s} B^{\boldsymbol{b}^j} \prod_i \underline{\mathfrak{ez}}^{\boldsymbol{a}^i \rfloor} \prod_i \underline{\mathfrak{ez}}^{\lfloor \boldsymbol{c}^i} \tag{3.49}$$

[44] Provided we include the approximate flexion units, for which the twisted symmetries become more intricate. For the trigonometric case, see §11.4.

$\mathfrak{E}$-*gus*-neutrality also is derived from straight *gus*-neutrality:

$$\Big(\sum_{1\leq l\leq r(\bullet)} \mathrm{gus}^l\Big).\mathrm{ganit}(\mathfrak{ez})^{-1}.B^\bullet \equiv 0$$

and admits a simpler direct expression :

$$\sum_{1\leq s}(-1)^s \sum_{\boldsymbol{w^1}\ldots\boldsymbol{w^s} \overset{\mathbf{circ}}{\sim} \boldsymbol{w}} B^{\boldsymbol{w^1}}\ldots B^{\boldsymbol{w^s}} \equiv (-1)^{r(\boldsymbol{w})} \sum_{\boldsymbol{a^i}w_i\boldsymbol{b^i}=\boldsymbol{w}} B^{\lceil w_i \rceil}\ \underline{\mathfrak{ez}}^{\boldsymbol{a^i}\rfloor}\ \underline{\mathfrak{ez}}^{\lfloor \boldsymbol{b^i}} \tag{3.50}$$

One should take care to interpret the circular sums correctly, i.e. without repetitions. Thus, if $\boldsymbol{w}$ has length 4, on the left-hand side of (3.50) the terms $B^{w_1,w_2}B^{w_3,w_4}$ and $B^{w_2,w_3}B^{w_4,w_1}$ occur *once* rather than *twice*, and the term $B^{w_1}B^{w_2}B^{w_3}B^{w_4}$ also occurs *once*, not *four times.*

- **{alternal//$\mathfrak{O}$-alternal} $\overset{\text{ess}^{\text{ly}}}{\Longrightarrow}$ {$\mathfrak{E}$-neg-invariant, $\mathfrak{E}$-push-invariant}.**

As mentioned in §2.4, bialternality implies invariance not just under *neg-push= mantar.swap.mantar.swap* but also[45] separate invariance under *neg* and *push.* Likewise, given any pair of conjugate flexion units $(\mathfrak{E},\mathfrak{O})$, a bimould $B^\bullet$ of type $\underline{\mathrm{al}}/\underline{\mathfrak{ol}}$ (i.e. alternal and with a $\mathfrak{O}$-alternal swappee) is ipso facto invariant not just under $\mathfrak{E}$-*negpush* but also[46] separately so under $\mathfrak{E}$-*neg* and $\mathfrak{E}$-*push.* The definitions of these operators run parallel to those of the straight case[47] :

$$\mathfrak{E}\text{-negpush} := \mathrm{mantar.swap.}(\mathfrak{E}\text{-mantar}).\mathrm{swap} \tag{3.51}$$

$$\mathfrak{E}\text{-neg} := \mathrm{neg.adari}(\mathfrak{es}^\bullet) = \mathrm{adari}(\mathrm{pari}.\mathfrak{es}^\bullet).\mathrm{neg} \tag{3.52}$$

$$\mathfrak{E}\text{-push} := (\mathfrak{E}\text{-neg}).\mathrm{mantar.swap.}(\mathfrak{E}\text{-mantar}).\mathrm{swap} \tag{3.53}$$

In fact, invariance under $\mathfrak{E}$-*push* is equivalent to invariance under a distinct and simpler operator $\mathfrak{E}$-$push_*$, which is defined as follows :

$$\mathfrak{E}\text{-push}_* := (\mathfrak{E}\text{-ter})^{-1}.\mathrm{push.mantar.}(\mathfrak{E}\text{-ter}).\mathrm{mantar} \tag{3.54}$$

with

$$((\mathfrak{E}\text{-ter}).B^\bullet)^{w_1,\ldots,w_r} := B^{w_1,\ldots,w_r} - B^{w_1,\ldots,w_{r-1}}\ \mathfrak{E}^{w_r} + B^{w_1,\ldots,w_{r-1}\rceil}\ \mathfrak{E}^{\lfloor w_r} \tag{3.55}$$

$$((\mathfrak{E}\text{-ter})^{-1}.B^\bullet)^{w_1,\ldots,w_r} := \sum_{\boldsymbol{a}.\boldsymbol{b}.\boldsymbol{c}\,=\,\boldsymbol{w}=(w_1,\ldots,w_r)} B^{\boldsymbol{a}\rceil}\ \mathfrak{mues}^{\lfloor \boldsymbol{b}}\ \mathfrak{es}^{\boldsymbol{c}} \tag{3.56}$$

---

[45] provided we assume (as assume we must, to ensure *ari*-stability) the component of length 1 to be *even.*

[46] again, assuming parity for the length-1 component.

[47] see (2.12) for *push* and also (4.70) for $\mathfrak{es}^\bullet$.

and with $\mathfrak{mues}^\bullet := \text{invmu}.\mathfrak{es}^\bullet = \text{pari.anti}.\mathfrak{es}^\bullet$ and $\mathfrak{es}^\bullet$ as in (4.70).

The reason for this equivalence is the identity:

$$(\text{id} - \mathfrak{E}\text{-push}_*).B^\bullet \equiv \text{swamu}(\mathfrak{es}^\bullet, (\text{id} - \mathfrak{E}\text{-push}).B^\bullet) \qquad \forall B^\bullet \tag{3.57}$$

with *swamu* defined as the *swap*-conjugate of *mu*.[48]
The notable advantage of $\mathfrak{E}$-push$_*$-invariance over $\mathfrak{E}$-push-invariance is that it leads straightaway to the so-called *senary* relation:[49]

$$(\mathfrak{E}\text{-ter}).B^\bullet = \text{push.mantar}.(\mathfrak{E}\text{-ter}).\text{mantar}.B^\bullet \tag{3.58}$$

which is the simplest way of expressing the $\mathfrak{E}$-*push*-invariance of $B^\bullet$.

- **{symmetral//$\mathfrak{O}$-symmetral} $\overset{\text{ess}^{\text{ly}}}{\Longrightarrow}$ {$\mathfrak{E}$-geg-invariant, $\mathfrak{E}$-gush-invariant}.**

Here, the first induced subsymmetry is the same as above, namely invariance under the linear operator $\mathfrak{E}$-*geg*, defined as $\mathfrak{E}$-*neg* in (3.52) but with *adari* replaced by *adgari*:

$$\mathfrak{E}\text{-geg} := \text{neg.adgari}(\mathfrak{es}^\bullet) = \text{adgari}(\text{pari}.\mathfrak{es}^\bullet).\text{neg} \tag{3.59}$$

The second induced subsymmetry is $\mathfrak{E}$-*gush*-invariance, with:

$$\mathfrak{E}\text{-gush} := (\mathfrak{E}\text{-geg}).\text{gantar.swap}.(\mathfrak{E}\text{-gantar}).\text{swap} \tag{3.60}$$

The only moot point is whether $\mathfrak{E}$-*gush*-invariance is equivalent to invariance under some simpler operator $\mathfrak{E}$-*gush*$_*$ defined along the same lines as (3.54). Even though the existence of a senary relation, or for that matter of a relation of *finite* arity is unlikely, it ought to be possible to improve considerably on $\mathfrak{E}$-*gush*.

3.5 Twisted symmetries and subsymmetries in polar mode.

Let us now restate the above results for the most important unit specialisation, which is the polar specialisation $(\mathfrak{E}^\bullet, \mathfrak{O}^\bullet) = (Pa^\bullet, Pi^\bullet)$. The transposi-

[48] i.e. $\text{swamu}(M_1^\bullet, M_2^\bullet) := \text{swap.mu}(\text{swap}.M_1^\bullet, \text{swap}.M_2^\bullet)$

[49] so-called because it involves only six terms – three on the left-hand side and three on the right.

tion goes like this:

$\mathfrak{E}$-alternal	$\to$	alternul	$(*)$	;	$\mathfrak{O}$-alternal	$\to$	alternil	
$\mathfrak{E}$-symmetral	$\to$	symmetrul	$(*)$	;	$\mathfrak{O}$-symmetral	$\to$	symmetril	
$\mathfrak{E}$-mantar	$\to$	mantur	$(*)$	;	$\mathfrak{O}$-mantar	$\to$	mantir	
$\mathfrak{E}$-gantar	$\to$	gantur	$(*)$	;	$\mathfrak{O}$-gantar	$\to$	gantir	
$\mathfrak{E}$-pus	$\to$	pusu	$(*)$	;	$\mathfrak{O}$-mantar	$\to$	pusi	
$\mathfrak{E}$-gus	$\to$	gusu	$(*)$	;	$\mathfrak{O}$-gus	$\to$	gusi	
$\mathfrak{E}$-push	$\to$	pushu		;	$\mathfrak{O}$-push	$\to$	pushi	$(*)$
$\mathfrak{E}$-gush	$\to$	gushu		;	$\mathfrak{O}$-gush	$\to$	gushi	$(*)$
$\mathfrak{E}$-neg	$\to$	negu		;	$\mathfrak{O}$-neg	$\to$	negi	$(*)$
$\mathfrak{E}$-geg	$\to$	gegu		;	$\mathfrak{O}$-geg	$\to$	gegi	$(*)$
$\mathfrak{E}$-ter	$\to$	teru		;	$\mathfrak{O}$-ter	$\to$	teri	$(*)$

And of course:

$$\begin{aligned}\text{alternal}/\mathfrak{O}\text{-}\mathit{alternal} &\to \text{alternal/alternil}\\ \text{alternal}/\mathfrak{E}\text{-}\mathit{alternal} &\to \text{alternal/alternul}\ \ (*)\end{aligned}$$

In the above tables, the stars $(*)$ accompany all symmetry types that are *incompatible* with *entireness.* For further details, see §4.7.

• **Alternility and symmetrility**.
Let us write down the alternility (resp. symmetrility) relations for two sequences $\boldsymbol{w'}, \boldsymbol{w''}$ first of length (1,1):

$$B^{\binom{u_1,\,u_2}{v_1,\,v_2}} + B^{\binom{u_1,\,u_2}{v_1,\,v_2}} - B^{\binom{u_{12}}{v_1}} P^{v_{2:1}} - B^{\binom{u_{12}}{v_2}} P^{v_{1:2}} = 0 \ \ \Big(\mathit{resp.}\ B^{\binom{u_1}{v_1}} B^{\binom{u_2}{v_2}}\Big)$$

then of length (1,2):

$$\begin{aligned}&B^{\binom{u_1,\,u_2,\,u_3}{v_1,\,v_2,\,v_3}} + B^{\binom{u_2,\,u_1,\,u_3}{v_2,\,v_1,\,v_3}} + B^{\binom{u_2,\,u_3,\,u_1}{v_2,\,v_3,\,v_1}} - B^{\binom{u_{12},\,u_3}{v_1,\,v_3}} P^{v_{2:1}} - B^{\binom{u_{12},\,u_3}{v_2,\,v_3}} P^{v_{1:2}}\\ &-B^{\binom{u_2,\,u_{13}}{v_2,\,v_1}} P^{v_{3:1}} - B^{\binom{u_2,\,u_{13}}{v_2,\,v_3}} P^{v_{1:3}} = 0 \quad \Big(\mathit{resp.}\ B^{\binom{u_1}{v_1}} B^{\binom{u_2,\,u_3}{v_2,\,v_3}}\Big)\end{aligned}$$

and then of length (2,2):

$$\begin{aligned}&B^{\binom{u_1,\,u_2,\,u_3,\,u_4}{v_1,\,v_2,\,v_3,\,v_4}} + B^{\binom{u_1,\,u_3,\,u_2,\,u_4}{v_1,\,v_3,\,v_2,\,v_4}} + B^{\binom{u_3,\,u_1,\,u_2,\,u_4}{v_3,\,v_1,\,v_2,\,v_4}}\\ &+B^{\binom{u_1,\,u_3,\,u_4,\,u_2}{v_1,\,v_3,\,v_4,\,v_2}} + B^{\binom{u_3,\,u_1,\,u_4,\,u_2}{v_3,\,v_1,\,v_4,\,v_2}} + B^{\binom{u_3,\,u_4,\,u_1,\,u_2}{v_3,\,v_4,\,v_1,\,v_2}}\\ &-B^{\binom{u_{13},\,u_2,\,u_4}{v_1,\,v_2,\,v_4}} P^{v_{3:1}} - B^{\binom{u_{13},\,u_2,\,u_4}{v_3,\,v_2,\,v_4}} P^{v_{1:3}} - B^{\binom{u_{13},\,u_4,\,u_2}{v_1,\,v_4,\,v_2}} P^{v_{3:1}} - B^{\binom{u_{13},\,u_4,\,u_2}{v_3,\,v_4,\,v_2}} P^{v_{1:3}}\\ &-B^{\binom{u_{13},\,u_1,\,u_2}{v_3,\,v_1,\,v_2}} P^{v_{4:1}} - B^{\binom{u_{13},\,u_{14},\,u_2}{v_3,\,v_4,\,v_2}} P^{v_{1:4}} - B^{\binom{u_1,\,u_{23},\,u_4}{v_1,\,v_2,\,v_4}} P^{v_{3:2}} - B^{\binom{u_1,\,u_{23},\,u_4}{v_1,\,v_3,\,v_4}} P^{v_{2:3}}\\ &-B^{\binom{u_1,\,u_3,\,u_{24}}{v_1,\,v_3,\,v_2}} P^{v_{4:2}} - B^{\binom{u_1,\,u_3,\,u_{24}}{v_1,\,v_3,\,v_4}} P^{v_{2:4}} - B^{\binom{u_3,\,u_1,\,u_{24}}{v_3,\,v_1,\,v_2}} P^{v_{4:2}} - B^{\binom{u_3,\,u_1,\,u_{24}}{v_3,\,v_1,\,v_4}} P^{v_{2:4}}\\ &+B^{\binom{u_{13},\,u_{24}}{v_1,\,v_2}} P^{v_{3:1}} P^{v_{4:2}} + B^{\binom{u_{13},\,u_{24}}{v_3,\,v_2}} P^{v_{1:3}} P^{v_{4:2}}\\ &+B^{\binom{u_{13},\,u_{24}}{v_1,\,v_4}} P^{v_{3:1}} P^{v_{2:4}} + B^{\binom{u_{13},\,u_{24}}{v_3,\,v_4}} P^{v_{1:3}} P^{v_{2:4}}\\ &= \ 0 \quad \Big(\mathit{resp.}\ \ B^{\binom{u_1,\,u_2}{v_1,\,v_2}} B^{\binom{u_3,\,u_4}{v_3,\,v_4}}\Big)\end{aligned}$$

Here and in all such formulas, we set $P^{v_i} := P(v_i) := 1/v_i$, purely for typographical coherence.

• **{alternil} $\Longrightarrow$ {mantir-invariant, pusi-neutral}**.
For length $r = 1, 2, 3$ the *mantir* operator acts thus:[50]

$$
\begin{aligned}
(\mathrm{mantir}.B)^{\binom{u_1}{v_1}} &= +B^{\binom{u_1}{v_1}}\\
(\mathrm{mantir}.B)^{\binom{u_1,\,u_2}{v_1,\,v_2}} &= -B^{\binom{u_2,\,u_1}{v_2,\,v_1}} + B^{\binom{u_{12}}{v_1}}P^{v_{2:1}} + B^{\binom{u_{12}}{v_2}}P^{v_{1:2}}\\
(\mathrm{mantir}.B)^{\binom{u_1,\,u_2,\,u_3}{v_1,\,v_2,\,v_3}} &= +B^{\binom{u_3,\,u_2,\,u_1}{v_3,\,v_2,\,v_1}}\\
&\quad -B^{\binom{u_{23},\,u_1}{v_3,\,v_1}}P^{v_{2:3}} - B^{\binom{u_{23},\,u_1}{v_2,\,v_1}}P^{v_{3:2}} - B^{\binom{u_3,\,u_{12}}{v_3,\,v_2}}P^{v_{1:2}} - B^{\binom{u_3,\,u_{12}}{v_3,\,v_1}}P^{v_{2:1}}\\
&\quad +B^{\binom{u_{123}}{v_1}}P^{v_{2:1}}P^{v_{3:1}} + B^{\binom{u_{123}}{v_2}}P^{v_{1:2}}P^{v_{3:2}} + B^{\binom{u_{123}}{v_3}}P^{v_{1:3}}P^{v_{2:3}}
\end{aligned}
$$

and *pusi*-neutrality means this:

$$
\begin{aligned}
\sum_{\mathrm{circ}} B^{\binom{u_1,\,u_2}{v_1,\,v_2}} &= +B^{\binom{u_{12}}{v_1}}P^{v_{2:1}} + B^{\binom{u_{12}}{v_2}}P^{v_{1:2}}\\
\sum_{\mathrm{circ}} B^{\binom{u_1,\,u_2,\,u_3}{v_1,\,v_2,\,v_3}} &= +B^{\binom{u_{123}}{v_1}}P^{v_{2:1}}P^{v_{3:1}} + B^{\binom{u_{123}}{v_2}}P^{v_{1:2}}P^{v_{3:2}} + B^{\binom{u_{123}}{v_3}}P^{v_{1:3}}P^{v_{2:3}}
\end{aligned}
$$

• **{symmetril} $\Longrightarrow$ {gantir-invariant, gusi-neutral}**.
For length $r = 1, 2, 3$ the *gantir* operator acts thus:

$$
\begin{aligned}
(\mathrm{gantir}.B)^{\binom{u_1}{v_1}} &= +B^{\binom{u_1}{v_1}}\\
(\mathrm{gantir}.B)^{\binom{u_1,\,u_2}{v_1,\,v_2}} &= -B^{\binom{u_2,\,u_1}{v_2,\,v_1}} + B^{\binom{u_2}{v_2}}B^{\binom{u_1}{v_1}} + B^{\binom{u_{12}}{v_1}}P^{v_{2:1}} + B^{\binom{u_{12}}{v_2}}P^{v_{1:2}}\\
(\mathrm{gantir}.B)^{\binom{u_1,\,u_2,\,u_3}{v_1,\,v_2,\,v_3}} &= +B^{\binom{u_3,\,u_2,\,u_1}{v_3,\,v_2,\,v_1}} + B^{\binom{u_3}{v_3}}B^{\binom{u_2}{v_2}}B^{\binom{u_1}{v_1}} - B^{\binom{u_3\ u_2}{v_3\ v_2}}B^{\binom{u_1}{v_1}} - B^{\binom{u_3}{v_3}}B^{\binom{u_2\ u_1}{v_2\ v_1}}\\
&\quad -B^{\binom{u_{23},\,u_1}{v_3,\,v_1}}P^{v_{2:3}} - B^{\binom{u_{23},\,u_1}{v_2,\,v_1}}P^{v_{3:2}} - B^{\binom{u_3,\,u_{12}}{v_3,\,v_2}}P^{v_{1:2}} - B^{\binom{u_3,\,u_{12}}{v_3,\,v_1}}P^{v_{2:1}}\\
&\quad +B^{\binom{u_{23}}{v_3}}B^{\binom{u_1}{v_1}}P^{v_{2:3}} + B^{\binom{u_{23}}{v_2}}B^{\binom{u_1}{v_1}}P^{v_{3:2}} + B^{\binom{u_3}{v_3}}B^{\binom{u_{12}}{v_2}}P^{v_{1:2}} + B^{\binom{u_3}{v_3}}B^{\binom{u_{12}}{v_1}}P^{v_{2:1}}\\
&\quad +B^{\binom{u_{123}}{v_1}}P^{v_{2:1}}P^{v_{3:1}} + B^{\binom{u_{123}}{v_2}}P^{v_{1:2}}P^{v_{3:2}} + B^{\binom{u_{123}}{v_3}}P^{v_{1:3}}P^{v_{2:3}}
\end{aligned}
$$

As for *gusi*-neutrality, it has the same expression as *pusi*-neutrality, but with left-hand side replaced for $r = 2, 3$, etc, respectively by:

$$B^{\binom{u_1,\,u_2}{v_1,\,v_2}} + B^{\binom{u_2,\,u_1}{v_2,\,v_1}} - B^{\binom{u_1}{v_1}}B^{\binom{u_3}{v_3}}$$

$$B^{\binom{u_1,u_2,u_3}{v_1,v_2,v_3}} + B^{\binom{u_2,u_3,u_1}{v_2,v_3,v_1}} + B^{\binom{u_3,u_1,u_2}{v_3,v_1,v_2}} - B^{\binom{u_1,u_2}{v_1,v_2}}B^{\binom{u_3}{v_3}} - B^{\binom{u_2,u_3}{v_2,v_3}}B^{\binom{u_1}{v_1}} - B^{\binom{u_3,u_1}{v_3,v_1}}B^{\binom{u_2}{v_2}}$$

etc.

• **{alternal//alternil} $\Longrightarrow$ {negu-invariant, pushu-invariant}**.
The first induced subsymmetry here is invariance under *negu*, with

$$\mathrm{negu} := \mathrm{neg}.\mathrm{adari}(\mathrm{paj}^{\bullet}) = \mathrm{adari}(\mathrm{pari}.\mathrm{paj}^{\bullet}).\mathrm{neg} \qquad (3.61)$$

[50] To get the general formula, one simply transposes (3.47).

and with *paj*$^\bullet$ defined as in (4.72). The second induced subsymmetry is invariance under *pushu*,with

$$\text{pushu} := \text{negu.mantar.swap.mantir.swap} \tag{3.62}$$

with *mantar* as in (2.7) and *mantir* as above; and it is in fact equivalent to invariance under the simpler operator *pushu*$_*$:

$$\text{pushu}_* := \text{teru}^{-1}.\text{push.mantar.teru.mantar} \tag{3.63}$$

whose main ingredient is the arity-3 operator *teru* and its inverse:[51]

$$\begin{aligned}
(\text{teru}.B^\bullet)^{w_1,\dots,w_r} \quad &:= \quad B^{w_1,\dots,w_r} - B^{w_1,\dots,w_{r-1}}\,\mathfrak{E}^{w_r} + B^{w_1,\dots,w_{r-1}\rceil}\,Pa^{\lfloor w_r}\\
(\text{teru}^{-1}.B^\bullet)^{w_1,\dots,w_r} \quad &:= \quad \sum_{\boldsymbol{a}.\boldsymbol{b}.\boldsymbol{c}\,=\,\boldsymbol{w}=(w_1,\dots,w_r)} B^{\boldsymbol{a}\rceil}\,\text{mupaj}^{\lfloor \boldsymbol{b}}\,\text{paj}^{\boldsymbol{c}}
\end{aligned}$$

leading to the linear *senary relation*:

$$\text{teru}.B^\bullet = \text{push.mantar.teru.mantar}.B^\bullet \tag{3.64}$$

• **{symmetral//symmetril} $\Longrightarrow$ {negu-invariant, gushu-invariant}.** Here, the first induced subsymmetry is *gegu*-invariance, with *gegu* defined as *negu* in(3.61), but with *adari* replaced by *adgari*:

$$\text{gegu} := \text{neg.adgari}(\text{paj}^\bullet) \;=\; \text{adgari}(\text{pari.paj}^\bullet).\text{neg} \tag{3.65}$$

and the second is *gushu*-invariance, with

$$\text{gushu} := \text{gegu.gantar.swap.gantir.swap} \tag{3.66}$$

with *gantar* as in (2.74) and *gantir* as above.

4 Flexion units and dimorphic bimoulds.

4.1 Remarkable substructures of $Flex(\mathfrak{E})$.

We shall now use the flexion units to construct two objects of pivotal importance: two very special *secondary* or *dimorphic* bimoulds (i.e. bimoulds with a double symmetry) which are, uncharacteristically, invariant under *pari.neg*

[51] The inverse $teru^{-1}$ is not of finite arity, of course, but its main ingredient is the mould *mupaj*$^\bullet$:= *invmu.paj*$^\bullet$ which, due to symmetrality, has the simple form *pari.anti.paj*$^\bullet$.

rather than *neg*, and which, owing to that rare property, will prove helpful
– in bridging the gap between *straight* and *twisted* double symmetries
– in connecting $GARI^{as/as}$ with $GARI^{\underline{as}/\underline{as}}$
– in constructing the *singulators* on which all the deeper results rest.

To do this, however, we must proceed step by step, and begin by constructing some important subspaces of $Flex(\mathfrak{E})$ and some remarkable bimould families like the $\mathfrak{re}^\bullet_r$ which, though not exactly dimorphic, come very close.

The subspaces $Flexinn(\mathfrak{E}) \subset Flexin(\mathfrak{E}) \subset Flex(\mathfrak{E})$**.**
For each integer sequence $\boldsymbol{r} := (r_1, \ldots, r_s)$ let us define inductively the three bimoulds $\mathfrak{me}^\bullet_{\boldsymbol{r}}$, $\mathfrak{ne}^\bullet_{\boldsymbol{r}}$, $\mathfrak{re}^\bullet_{\boldsymbol{r}}$:[52]

$$\begin{array}{llllll}
\mathfrak{me}^\bullet_1 := \mathfrak{E}^\bullet & ; & \mathfrak{me}^\bullet_r := \text{amit}(\mathfrak{me}^\bullet_{r-1}).\mathfrak{E}^\bullet & ; & \mathfrak{me}^\bullet_{r_1,\ldots,r_s} := \text{mu}(\mathfrak{me}^\bullet_{r_1}, \ldots, \mathfrak{me}^\bullet_{r_s}) \\
\mathfrak{ne}^\bullet_1 := \mathfrak{E}^\bullet & ; & \mathfrak{ne}^\bullet_r := \text{anit}(\mathfrak{ne}^\bullet_{r-1}).\mathfrak{E}^\bullet & ; & \mathfrak{ne}^\bullet_{r_1,\ldots,r_s} := \text{mu}(\mathfrak{ne}^\bullet_{r_1}, \ldots, \mathfrak{ne}^\bullet_{r_s}) \\
\mathfrak{re}^\bullet_1 := \mathfrak{E}^\bullet & ; & \mathfrak{re}^\bullet_r := \text{arit}(\mathfrak{re}^\bullet_{r-1}).\mathfrak{E}^\bullet & ; & \mathfrak{re}^\bullet_{r_1,\ldots,r_s} := \text{mu}(\mathfrak{re}^\bullet_{r_1}, \ldots, \mathfrak{re}^\bullet_{r_s})
\end{array}$$

Clearly, $\mathfrak{me}^\bullet_{\boldsymbol{r}}$, $\mathfrak{ne}^\bullet_{\boldsymbol{r}}$, $\mathfrak{re}^\bullet_{\boldsymbol{r}}$ are in $Flex_r(\mathfrak{E})$ with $r := \|\boldsymbol{r}\| = \sum r_i$. In fact, one can show that all three sets: $\{\mathfrak{me}^\bullet_{\boldsymbol{r}}, \|\boldsymbol{r}\| = r\}$, $\{\mathfrak{ne}^\bullet_{\boldsymbol{r}}, \|\boldsymbol{r}\| = r\}$, $\{\mathfrak{re}^\bullet_{\boldsymbol{r}}, \|\boldsymbol{r}\| = r\}$ span one and the same[53] subspace $Flexin_r(\mathfrak{E})$ of $Flex_r(\mathfrak{E})$, with dimension 2^{r-1}.

These three bases of $Flexin_r(\mathfrak{E})$ are connected by six simple matrices (two of them rational-valued, the other four entire-valued). Indeed:

$$\begin{aligned}
\mathfrak{me}^\bullet_{r_0} &= \sum_{1\le s} \sum_{\sum r_i = r_0} (-1)^{s+r} \;\; \mathfrak{ne}^\bullet_{r_1,\ldots,r_s} \\
\mathfrak{ne}^\bullet_{r_0} &= \sum_{1\le s} \sum_{\sum r_i = r_0} (-1)^{s+r} \;\; \mathfrak{me}^\bullet_{r_1,\ldots,r_s} \\
\mathfrak{re}^\bullet_{r_0} &= \sum_{1\le s} \sum_{\sum r_i = r_0} (-1)^{s+1}\, r_s \;\; \mathfrak{me}^\bullet_{r_1,\ldots,r_s} \\
\mathfrak{re}^\bullet_{r_0} &= \sum_{1\le s} \sum_{\sum r_i = r_0} (-1)^{s+r}\, r_1 \;\; \mathfrak{ne}^\bullet_{r_1,\ldots,r_s} \\
\mathfrak{me}^\bullet_{r_0} &= \sum_{1\le s} \sum_{\sum r_i = r_0} \frac{1}{r_1 r_{12} \ldots r_{12\ldots s}} \;\; \mathfrak{re}^\bullet_{r_1,\ldots,r_s} \\
\mathfrak{ne}^\bullet_{r_0} &= \sum_{1\le s} \sum_{\sum r_i = r_0} \frac{(-1)^{s+r}}{r_{12\ldots s} \ldots r_{s-1,s} r_s} \;\; \mathfrak{re}^\bullet_{r_1,\ldots,r_s}
\end{aligned}$$

with $r_{i,j,\ldots}$ or even $r_{ij\ldots}$ standing as usual for $r_i + r_j + \ldots$

---

[52]For their analytical expressions, see §12.2.

[53]This would no longer be the case if $\mathfrak{E}^\bullet$ were not a flexion unit.

If we now denote by $\{r_1,\ldots,r_s\}$ any non-ordered integer set with repetitions allowed (or 'partition', if you prefer) and if we set:[54]

$$\begin{aligned}
\mathfrak{re}^{\bullet}_{\{r_1,\ldots,r_s\}} &= \sum_{\{r'_1,\ldots,r'_s\}=\{r_1,\ldots,r_s\}} \frac{1}{s!}\ \mathrm{preari}(\mathfrak{re}^{\bullet}_{r'_1},\ldots,\mathfrak{re}^{\bullet}_{r'_s})\\
\mathfrak{se}^{\bullet}_{\{r_1,\ldots,r_s\}} &= \sum_{\{r'_1,\ldots,r'_s\}=\{r_1,\ldots,r_s\}} \frac{1}{r'_1 r'_{12}\ldots r'_{12\ldots s}}\ \mathrm{preari}(\mathfrak{re}^{\bullet}_{r'_1},\ldots,\mathfrak{re}^{\bullet}_{r'_s})
\end{aligned}$$

then it can be shown that, despite the very different summation weights, the two sets $\{\mathfrak{re}^{\bullet}_{\{\boldsymbol{r}\}}, \|\boldsymbol{r}\|=r\}$, $\{\mathfrak{se}^{\bullet}_{\{\boldsymbol{r}\}}, \|\boldsymbol{r}\|=r\}$ span one and the same[55] subspace $\mathit{Flexinn}_r(\mathfrak{E})$ of $\mathit{Flexin}_r(\mathfrak{E})$, with dimension $p(r)$ equal to the number of partitions of r. Summing up, we have:

$$\mathit{Flexinn}(\mathfrak{E})=\oplus \mathit{Flexinn}_r(\mathfrak{E})\subset \mathit{Flexin}(\mathfrak{E})=\oplus \mathit{Flexin}_r(\mathfrak{E})\subset \mathit{Flex}(\mathfrak{E})=\oplus \mathit{Flex}_r(\mathfrak{E})$$

$$\dim(\mathrm{Flexinn}_r(\mathfrak{E}))=p(r)\ ;\ \dim(\mathrm{Flexin}_r(\mathfrak{E}))=2^{r-1}\ ;\ \dim(\mathrm{Flex}_r(\mathfrak{E}))=\frac{(2\,r)!}{r!\,(r+1)!}$$

(i) $\mathit{Flex}(\mathfrak{E})$ is stable under all flexion operations.
(ii) $\mathit{Flexin}(\mathfrak{E})$ is stable under *mu*, *lu*, and $\mathit{arit}(\mathfrak{re}^{\bullet}_{r_0})$ $(\forall r_0)$.
(iii) $\mathit{Flexinn}(\mathfrak{E})$ is stable under nothing much, but crucial nonetheless.

Action of $\mathit{arit}(\mathfrak{re}^{\bullet}_r)$ on $\mathit{Flexin}(\mathfrak{E})$.

It is neatly encapsulated in the formulas:

$$\mathrm{arit}(\mathfrak{re}^{\bullet}_q).\mathfrak{me}^{\bullet}_p = \sum_{s\geq 1}\ \sum_{\sum r_i=p+q\,,\,r_1\geq p} (-1)^{1+s}\, r_s\ \mathfrak{me}^{\bullet}_{r_1,\ldots,r_s} \tag{4.1}$$

$$\mathrm{arit}(\mathfrak{re}^{\bullet}_q).\mathfrak{ne}^{\bullet}_p = \sum_{s\geq 1}\ \sum_{\sum r_i=p+q\,,\,r_s\geq p} (-1)^{1+s+q}\, r_1\ \mathfrak{ne}^{\bullet}_{r_1,\ldots,r_s} \tag{4.2}$$

$$\mathrm{arit}(\mathfrak{re}^{\bullet}_q).\mathfrak{re}^{\bullet}_p = p\ \mathfrak{re}^{\bullet}_{p+q}+\sum_{i\leq q}\mathrm{lu}(\mathfrak{re}^{\bullet}_i,\mathfrak{re}^{\bullet}_{p+q-i}) \tag{4.3}$$

$$= p\ \mathfrak{re}^{\bullet}_{p+q}+\sum_{i<p}\mathrm{lu}(\mathfrak{re}^{\bullet}_i,\mathfrak{re}^{\bullet}_{p+q-i}) \tag{4.4}$$

The algebra $ARI_{<\mathfrak{re}>}$ and its group $GARI_{<\mathfrak{se}>}$.
Of the three bases of $\mathit{Flex}(\mathfrak{E})$, the first two are simplest, in the sense that

[54] with the multiple pre-brackets *preari* taken, as usual, *from left to right.*

[55] The simplest way to show that $\{\mathfrak{re}^{\bullet}_{\{\boldsymbol{r}\}}\}$ and $\{\mathfrak{se}^{\bullet}_{\{\boldsymbol{r}\}}\}$ span the same space and to find the conversion rule between the two bases, is to equate the expansions (4.11) and (4.12) for $\mathfrak{Se}^{\bullet}_f$ while expressing the coefficients α_n of the infinitesimal generator and the coefficients γ_n of the infinitesimal dilator in terms of each other.

here we have *atomic* basis elements $\mathfrak{me}^\bullet_{r_1,\dots,r_s}$ or $\mathfrak{ne}^\bullet_{r_1,\dots,r_s}$, i.e. elements that reduce to single products of the form $\mathfrak{E}^{w'_1}\dots\mathfrak{E}^{w'_r}$ for suitably inflected w'_i. With the third basis, on the other hand, we have *molecular* basis elements that can only be expressed as superpositions of at least $\prod r_i$ atoms. But the individual $\mathfrak{re}^\bullet_r$ $(r\in\mathbb{N}^*)$, whose definition we recall:

$$\begin{array}{lllr} \mathfrak{re}_r^{w_1,\dots,w_s} := 0 & & \textit{if } r\neq s & \\ \mathfrak{re}_1^{w_1} := \mathfrak{E}^{w_1} & , \quad \mathfrak{re}^\bullet_r := \mathrm{arit}(\mathfrak{re}^\bullet_{r-1}).\mathfrak{re}^\bullet_1 & \textit{if } r\geq 2 & (4.5) \end{array}$$

more than make up for their 'molecularity' by possessing three essential properties:
(i) The bimoulds $\mathfrak{re}^\bullet_r$ thus defined are alternal.
(ii) When suitably combined, they exhibit traces of dimorphy, since the bimould $\mathfrak{sre}^\bullet$:

$$\mathfrak{sre}^\bullet := \frac{1}{2}\mathfrak{re}^\bullet_1+\frac{1}{6}\mathfrak{re}^\bullet_2+\frac{1}{12}\mathfrak{re}^\bullet_3+\dots=\sum_{r\geq 1}\frac{1}{r(r+1)}\,\mathfrak{re}^\bullet_r \;\in \mathrm{ARI}^{\mathrm{al}/\mathfrak{ol}} \tag{4.6}$$

is not only alternal, but has a $\mathfrak{O}$-alternal swappee $\mathfrak{sr\ddot{o}}^\bullet$.
(iii) But the real importance of the $\mathfrak{re}^\bullet_r$ derives from the remarkable identities:

$$\mathrm{ari}(\mathfrak{re}^\bullet_{r_1},\mathfrak{re}^\bullet_{r_2})=(r_1-r_2)\,\mathfrak{re}^\bullet_{r_1+r_2} \qquad \forall r_1,r_2\geq 1 \tag{4.7}$$

which lead straightaway to the following commutative diagram:

$$\begin{array}{llllll} \mathrm{GIFF}_{<x>} & \overset{\text{isom.}}{\longrightarrow} & \mathrm{GARI}_{<\mathfrak{se}>} \subset \mathrm{GARI}^{\mathrm{as}} & \| & \mathrm{se}_r(x)=\frac{x}{(1-x^r)^{\frac{1}{r}}} \longrightarrow & \mathfrak{se}^\bullet_r \\ \uparrow \exp & & \uparrow \mathrm{expari} & \| & \uparrow \exp & \uparrow \mathrm{expari} \\ \mathrm{DIFF}_{<x>} & \overset{\text{isom.}}{\longrightarrow} & \mathrm{ARI}_{<\mathfrak{re}>} \subset \mathrm{ARI}^{\mathrm{al}} & \| & \mathrm{re}_r(x)=x^{r+1}\partial_x \longrightarrow & \mathfrak{re}^\bullet_r \end{array}$$

Here, $GIFF_{<x>}$ denotes the group of (formal, one-dimensional) identity-tangent mappings of the form:

$$f := x\mapsto x.(1+\sum_{1\leq r}a_r\,x^r) \tag{4.8}$$

and $DIFF_{<x>}$ denotes its infinitesimal algebra, whose elements may be represented as sums $\sum_{1\leq r}a_r\,x^{r+1}\partial_x$, provided we *change the sign* before their natural bracket.

Of course, since the Lie algebra $ARI_{<\mathfrak{re}>}$ contains only *alternal* bimoulds, its exponential, the group $GARI_{<\mathfrak{se}>}$, contains only *symmetral* bimoulds. Moreover, since elements of $ARI_{<\mathfrak{re}>}$ also possess traces of dimorphy, so too will their images in $GARI_{<\mathfrak{se}>}$. In the case of two remarkable bimoulds, $\mathfrak{ess}^\bullet$ and $\mathfrak{esz}^\bullet$ of $GARI_{<\mathfrak{se}>}$, we shall even get exact dimorphy rather than 'traces'.

But rather than jumping ahead, let us first explicate the isomorphisms $f \leftrightarrow \mathfrak{Se}^\bullet_f$ between the classical group $GIFF_{<x>}$ and its counterpart $GARI_{<\mathfrak{se}>}$ in the flexion structure. We begin with the easier direction, i.e. from *flexion* to *classical*.

The isomorphism $GARI_{<\mathfrak{se}>} \to GIFF_{<x>}$ made explicit.
Let $\mathfrak{Se}^\bullet_f$ in $GARI_{<\mathfrak{se}>}$ be the image of some $f(x) := x\,(1 + \sum a_r\, x^r)$ in $GIFF_{<x>}$. How do we read the coefficients a_r directly off the bimould $\mathfrak{Se}^\bullet_f$ itself, without going through the costly operation *logari*? The answer is given by the bilinear operator *gepar*:

$$\text{gepar}.H^\bullet := \text{mu}(\text{anti.swap}.H^\bullet, \text{swap}.H^\bullet) \tag{4.9}$$

and by the formula:

$$(\text{gepar}.\mathfrak{Se}_f)^{w_1,\dots,w_r} \equiv (r{+}1)\, a_r\, \mathfrak{O}^{w_1} \dots \mathfrak{O}^{w_r} \qquad \textit{with } \mathfrak{O} \textit{ conjugate to } \mathfrak{E} \tag{4.10}$$

The isomorphism $GIFF_{<x>} \to GARI_{<\mathfrak{se}>}$ made explicit.
The isomorphism from *classical* to *flexion* is more difficult but also more interesting to unravel. We may of course transit through $DIFF_{<x>}$ and $ARI_{<\mathfrak{re}>}$ in the above diagram, but that involves performing the 'costly' operation *expari* and leads, in the course of the calculations, to rational coefficients with large denominators, which vanish in the end result. Concretely, that means forming the infinitesimal generator f_* of f (see (4.13),(4.15)) and inserting its coefficients ϵ_n into (4.11). Fortunately, there exists a much more direct scheme, which involves only integer coefficients: this time, we form the infinitesimal dilator $f_\#$ of f, which is a far more accessible object than f_* (see (4.14),(4.16)) and inject its coefficients γ_n into (4.12).

$$\mathfrak{Se}^\bullet_f = \sum_{\{\boldsymbol{r}\}} \mathfrak{re}^\bullet_{\{\boldsymbol{r}\}}\, \epsilon_{\{\boldsymbol{r}\}} \qquad \textit{with} \qquad \boldsymbol{\epsilon}_{\{r_1,\dots,r_s\}} := \epsilon_{r_1} \dots \epsilon_{r_s} \tag{4.11}$$

$$\mathfrak{Se}^\bullet_f = \sum_{\{\boldsymbol{r}\}} \mathfrak{se}^\bullet_{\{\boldsymbol{r}\}}\, \gamma_{\{\boldsymbol{r}\}} \qquad \textit{with} \qquad \boldsymbol{\gamma}_{\{r_1,\dots,r_s\}} := \gamma_{r_1} \dots \gamma_{r_s} \tag{4.12}$$

$$f_*(x) = x \sum_{1 \le k} \epsilon_k\, x^k = \text{infinitesimal generator of } f \tag{4.13}$$

$$f_\#(x) = x \sum_{1 \le k} \gamma_k\, x^k = x - \frac{f(x)}{f'(x)} = \text{infinitesimal dilator of } f \tag{4.14}$$

$$\big(\exp(f_*(x)\,\partial_x)\big).\,x \;=\; f(x) \tag{4.15}$$

$$\big(f\,\circ\,(id+\epsilon\,f_\#)\big)(x) \;=\; x+\sum_{1\leq n}(1+\epsilon\,n)\,a_n\,x^{n+1}+\mathcal{O}(\epsilon^2) \tag{4.16}$$

Ultimately, of course, the coefficients γ_n of $f_\#$ have to be expressed in terms of those of f itself. Here, however, we have the choice between the three main representations of $GIFF_{<x>}$:

$$\begin{array}{lllllll} x & \mapsto & f(x) & = & x+\sum_{1\leq n}a_n\,x^{1+n} & & (x\sim 0)\\ y & \mapsto & \underline{f}(y) & = & y+\sum_{1\leq n}b_n\,y^{1-n} & = \;1/f(y^{-1}) & (y\sim 0)\\ z & \mapsto & \underline{\underline{f}}(z) & = & z+\sum_{1\leq n}c_n\,e^{n\,z} & = \;\log f(e^z) & (z\sim 0)\end{array}$$

leading for γ_n to three rather similar expressions:

$$\sum\gamma_n\,x^n\equiv\frac{\sum n\,a_n\,x^n}{1{+}\sum(n{+}1)\,a_n\,x^n}\equiv\frac{-\sum n\,b_n\,x^n}{1{-}\sum(n{-}1)\,b_n\,x^n}\equiv\frac{\sum n\,c_n\,x^n}{1{+}\sum n\,c_n\,x^n} \tag{4.17}$$

Under closer examination, it turns out that the coefficients $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ of f, $\underline{f}$, $\underline{\underline{f}}$ are well-suited for expressing $\mathfrak{Se}^\bullet_f$ in the bases $\{\mathfrak{ne}^\bullet_{\boldsymbol{r}}\}$,$\{\mathfrak{me}^\bullet_{\boldsymbol{r}}\}$,$\{\mathfrak{re}^\bullet_{\boldsymbol{r}}\}$ respectively (mark the order!), leading to three expansions:

$$\mathfrak{Se}^\bullet_f=\sum_{\boldsymbol{r}}\mathbf{A}^{\boldsymbol{r}}\,\mathfrak{ne}^\bullet_{\boldsymbol{r}}=\sum_{\boldsymbol{r}}\mathbf{B}^{\boldsymbol{r}}\,\mathfrak{me}^\bullet_{\boldsymbol{r}}=\sum_{\boldsymbol{r}}\mathbf{C}^{\boldsymbol{r}}\,\mathfrak{re}^\bullet_{\boldsymbol{r}} \tag{4.18}$$

To get a complete grip on the situation, we must calculate the moulds $\mathbf{A}^\bullet$, $\mathbf{B}^\bullet$, $\mathbf{C}^\bullet$ in terms of the a_n, b_n, c_n. To this end, we lift the infinitesimal dilation identity (4.16) from $GIFF_{<x>}$ to $GARI_{<\mathfrak{se}>}$. We find:

$$r(\bullet).\,\mathfrak{Se}^\bullet_f=\mathrm{arit}(\mathfrak{Te}^\bullet_f).\mathfrak{Se}^\bullet_f+\mathrm{mu}(\mathfrak{Se}^\bullet_f,\mathfrak{Te}^\bullet_f)\qquad \textit{with}\qquad \mathfrak{Te}^\bullet_f:=\sum_{1\leq r}\gamma_r\,\mathfrak{re}^\bullet_r \tag{4.19}$$

or more compactly:

$$r(\bullet).\,\mathfrak{Se}^\bullet_f=\mathrm{preari}(\mathfrak{Se}^\bullet_f,\mathfrak{Te}^\bullet_f)\qquad \textit{with}\qquad \mathfrak{Te}^\bullet_f:=\sum_{1\leq r}\gamma_r\,\mathfrak{re}^\bullet_r \tag{4.20}$$

In view of the formulas (4.1), (4.2), (4.3) for the action of $arit(\mathfrak{re}^\bullet_r)$ on $Flexin(\mathfrak{E})$, the identity (4.19) immediately translates into these three simple induction rules for the calculation of $\mathbf{A}^\bullet$, $\mathbf{B}^\bullet$, $\mathbf{C}^\bullet$:

$$\|\boldsymbol{r}\|\mathbf{A}^{\boldsymbol{r}} = \sum_{\boldsymbol{r^1 r^2} = \boldsymbol{r}} \mathbf{A}^{\boldsymbol{r^1}} \mathcal{N}_0^{\boldsymbol{r^2}} + \sum_{\boldsymbol{r^1 r^2 r^3} = \boldsymbol{r}} \sum_{1 \le r_0 < \|\boldsymbol{r^2}\|} \mathbf{A}^{\boldsymbol{r^1} r_0 \boldsymbol{r^3}} \mathcal{N}_{r_0}^{\boldsymbol{r^2}} \tag{4.21}$$

$$\|\boldsymbol{r}\|\mathbf{B}^{\boldsymbol{r}} = \sum_{\boldsymbol{r^1 r^2} = \boldsymbol{r}} \mathbf{B}^{\boldsymbol{r^1}} \mathcal{M}_0^{\boldsymbol{r^2}} + \sum_{\boldsymbol{r^1 r^2 r^3} = \boldsymbol{r}} \sum_{1 \le r_0 < \|\boldsymbol{r^2}\|} \mathbf{B}^{\boldsymbol{r^1} r_0 \boldsymbol{r^3}} \mathcal{M}_{r_0}^{\boldsymbol{r^2}} \tag{4.22}$$

$$\|\boldsymbol{r}\|\mathbf{C}^{\boldsymbol{r}} = \sum_{\boldsymbol{r^1 r^2} = \boldsymbol{r}} \mathbf{C}^{\boldsymbol{r^1}} \mathcal{R}_0^{\boldsymbol{r^2}} + \sum_{\boldsymbol{r^1 r^2 r^3} = \boldsymbol{r}} \sum_{1 \le r_0 < \|\boldsymbol{r^2}\|} \mathbf{C}^{\boldsymbol{r^1} r_0 \boldsymbol{r^3}} \mathcal{R}_{r_0}^{\boldsymbol{r^2}} \tag{4.23}$$

The auxiliary moulds $\mathcal{N}^\bullet$, $\mathcal{M}^\bullet$, $\mathcal{R}^\bullet$ are defined as follows:

$$\mathcal{N}_{r_0}^{\boldsymbol{r}} := \gamma_{\|\boldsymbol{r}\|-r_0}(\boldsymbol{a})\ (-1)^{1+s+\|\boldsymbol{r}\|-r_0}\, r_1\, (\Xi_{0<r_0\le r_s} - \Xi_{0=r_0}) \tag{4.24}$$

$$\mathcal{M}_{r_0}^{\boldsymbol{r}} := \gamma_{\|\boldsymbol{r}\|-r_0}(\boldsymbol{b})\ (-1)^{1+s}\, r_s\, (\Xi_{0\le r_0\le r_1}) \tag{4.25}$$

$$\mathcal{R}_{r_0}^{r_1} := \gamma_{\|\boldsymbol{r}\|-r_0}(\boldsymbol{c})\ (\Xi_{0=r_0=r_1} + r_0\, \Xi_{0<r_0<r_1}) \tag{4.26}$$

$$\mathcal{R}_{r_0}^{r_1,r_2} := \gamma_{\|\boldsymbol{r}\|-r_0}(\boldsymbol{c})\ (\Xi_{r_1<r_0\le r_2} - \Xi_{r_2<r_0\le r_1}) \tag{4.27}$$

$$\mathcal{R}_{r_0}^{r_1,\ldots,r_s} := 0 \qquad \textit{if}\ \ s \ge 3 \tag{4.28}$$

(i) with $\boldsymbol{r} := (r_1, \ldots, r_s)$ for any $s, r_i \in \mathbb{N}^*$
(ii) with $\Xi_{\mathcal{S}}$ denoting the characteristic function of any given set $\mathcal{S}$,
(iii) with $\gamma_n(\boldsymbol{a})$, $\gamma_n(\boldsymbol{b})$, $\gamma_n(\boldsymbol{c})$ denoting the coefficients of the infinitesimal dilator $f_\#$ expressed (via the formulas (4.17)) in terms of the coefficients a_i, b_i, c_i respectively.

The main facts here are these:
(i) The moulds $\mathbf{B}^\bullet$ and $\mathbf{A}^\bullet$ are *symmetrel* whereas $\mathbf{C}^\bullet$ is *symmetral.*
(ii) $\mathbf{A}^{\boldsymbol{r}}$, $\mathbf{B}^{\boldsymbol{r}}$, $\mathbf{C}^{\boldsymbol{r}}$ are homogeneous polynomials of total degree $\|\boldsymbol{r}\|$ in the variables a_i, b_i, c_i respectively, but whereas $\mathbf{C}^{\boldsymbol{r}}$ has (predictably) rational coefficients, $\mathbf{A}^{\boldsymbol{r}}$ and $\mathbf{B}^{\boldsymbol{r}}$ have (unexpectedly) entire coefficients.
(iii) These rational (resp. entire) coefficients display remarkable symmetry properties: see (4.31),(4.32) below.

Here are the first structure polynomials $\mathbf{A}^{\boldsymbol{r}}, \mathbf{B}^{\boldsymbol{r}}, \mathbf{C}^{\boldsymbol{r}}$ up to $\|\boldsymbol{r}\| = 4$:

$$\mathbf{B}^1 = -b_1 \qquad \mathbf{A}^1 = a_1 \qquad \mathbf{C}^1 = c_1$$

$\mathbf{B}^2 = -2\,b_2 + b_1^2$	$\mathbf{A}^2 = -2\,a_2 + a_1^2$	$\mathbf{C}^2 = +c_2$
$\mathbf{B}^{1,1} = +b_2$	$\mathbf{A}^{1,1} = +a_2$	$\mathbf{C}^{1,1} = +\frac{1}{2}\,c_1^2$
$\mathbf{B}^3 = -3b_3 + 3b_1b_2 - b_1^3$	$\mathbf{A}^3 = +3a_3 - 3a_1a_2 + a_1^3$	$\mathbf{C}^3 = +c_3$
$\mathbf{B}^{1,2} = +2b_3$	$\mathbf{A}^{1,2} = -a_3 - a_1a_2 + a_1^3$	$\mathbf{C}^{1,2} = +c_1c_2 - \frac{1}{6}c_1^3$
$\mathbf{B}^{2,1} = +b_3 - b_1b_2$	$\mathbf{A}^{2,1} = -2a_3 + 2a_1a_2 - a_1^3$	$\mathbf{C}^{2,1} = +\frac{1}{6}c_1^3$
$\mathbf{B}^{1,1,1} = -b_3$	$\mathbf{A}^{1,1,1} = +a_3$	$\mathbf{C}^{1,1,1} = +\frac{1}{6}c_1^3$

$$
\begin{array}{llrrrrr}
\mathbf{B}^{4} & = & -4\,b_4 & +4\,b_1\,b_3 & +2\,b_2^2 & -4\,b_1^2\,b_2 & +b_1^4 \\
\mathbf{B}^{1,3} & = & +3\,b_4 & & & & \\
\mathbf{B}^{3,1} & = & +b_4 & -b_1\,b_3 & -2\,b_2^2 & +b_1^2\,b_2 & \\
\mathbf{B}^{2,2} & = & +2\,b_4 & -2\,b_1\,b_3 & +b_2^2 & & \\
\mathbf{B}^{1,1,2} & = & -2\,b_4 & & -b_2^2 & & \\
\mathbf{B}^{1,2,1} & = & -b_4 & & +b_2^2 & & \\
\mathbf{B}^{2,1,1} & = & -b_4 & +b_1\,b_3 & & & \\
\mathbf{B}^{1,1,1,1} & = & +b_4 & & & &
\end{array}
$$

$$
\begin{array}{llrrrrr}
\mathbf{A}^{4} & = & -4\,a_4 & +4\,a_1\,a_3 & +2\,a_2^2 & -4\,a_1^2\,a_2 & +a_1^4 \\
\mathbf{A}^{1,3} & = & +a_4 & +2\,a_1\,a_3 & +a_2^2 & -5\,a_1^2\,a_2 & +2\,a_1^4 \\
\mathbf{A}^{3,1} & = & +3\,a_4 & -3\,a_1\,a_3 & -3\,a_2^2 & +6\,a_1^2\,a_2 & -2\,a_1^4 \\
\mathbf{A}^{2,2} & = & +2\,a_4 & -2\,a_1\,a_3 & +a_2^2 & & \\
\mathbf{A}^{1,1,2} & = & -a_4 & & -a_2^2 & +a_1^2\,a_2 & \\
\mathbf{A}^{1,2,1} & = & -a_4 & -a_1\,a_3 & & +2\,a_1^2\,a_2 & -a_1^4 \\
\mathbf{A}^{2,1,1} & = & -2\,a_4 & +2\,a_1\,a_3 & +a_2^2 & -3\,a_1^2\,a_2 & +a_1^4 \\
\mathbf{A}^{1,1,1,1} & = & +a_4 & & & &
\end{array}
$$

$$
\begin{array}{llrrrrr}
\mathbf{C}^{4} & = & +c_4 & & & & \\
\mathbf{C}^{1,3} & = & & +c_1\,c_3 & +\frac{1}{2}\,c_2^2 & -\frac{1}{2}\,c_1^2\,c_2 & +\frac{1}{24}\,c_1^4 \\
\mathbf{C}^{3,1} & = & & & -\frac{1}{2}\,c_2^2 & +\frac{1}{2}\,c_1^2\,c_2 & -\frac{1}{24}\,c_1^4 \\
\mathbf{C}^{2,2} & = & & & +\frac{1}{2}\,c_2^2 & & \\
\mathbf{C}^{1,1,2} & = & & & & +\frac{1}{2}\,c_1^2\,c_2 & -\frac{1}{8}\,c_1^4 \\
\mathbf{C}^{1,2,1} & = & & & & & +\frac{1}{12}\,c_1^4 \\
\mathbf{C}^{2,1,1} & = & & & & & +\frac{1}{24}\,c_1^4 \\
\mathbf{C}^{1,1,1,1} & = & & & & & +\frac{1}{24}\,c_1^4
\end{array}
$$

For any *unordered* integer sequence $\{\boldsymbol{r}\} := \{r_1, \ldots, r_s\}$, with *repetitions* allowed, we set:

$$
\boldsymbol{a}_{\{\boldsymbol{r}\}} := \prod_i a_{r_i} \quad ; \quad \boldsymbol{b}_{\{\boldsymbol{r}\}} := \prod_i b_{r_i} \quad ; \quad \boldsymbol{c}_{\{\boldsymbol{r}\}} := \prod_i c_{r_i} \tag{4.29}
$$

There exist efficient algorithms for calculating the three series of structure coefficients $A^{\bullet,\{\bullet\}}, B^{\bullet,\{\bullet\}}, C^{\bullet,\{\bullet\}}$ which occur in the above tables:

$$
\mathbf{A}^{\boldsymbol{r}} = \sum_{\{\boldsymbol{r}''\}} A^{\boldsymbol{r},\{\boldsymbol{r}''\}}\, \boldsymbol{a}_{\{\boldsymbol{r}''\}} \quad ; \quad \mathbf{B}^{\boldsymbol{r}} = \sum_{\{\boldsymbol{r}''\}} B^{\boldsymbol{r},\{\boldsymbol{r}''\}}\, \boldsymbol{b}_{\{\boldsymbol{r}''\}} \quad ; \quad \mathbf{C}^{\boldsymbol{r}} = \sum_{\{\boldsymbol{r}''\}} C^{\boldsymbol{r},\{\boldsymbol{r}''\}}\, \boldsymbol{c}_{\{\boldsymbol{r}''\}}
$$

and which encode, each in their way, all the information about the mapping from $GIFF_{<x>}$ to $GARI_{<\mathfrak{se}>}$. These structure coefficients have many properties, some of which are still imperfectly understood. We mention here but two of them. Consider the regularised coefficients $B^{\{\bullet\},\{\bullet\}}$,$A^{\{\bullet\},\{\bullet\}}$ defined by:[56]

$$A^{\{\boldsymbol{r}'\},\{\boldsymbol{r}''\}} = \sum_{\boldsymbol{r}\in\{\boldsymbol{r}'\}} A^{\boldsymbol{r},\{\boldsymbol{r}''\}} \qquad ; \qquad B^{\{\boldsymbol{r}'\},\{\boldsymbol{r}''\}} = \sum_{\boldsymbol{r}\in\{\boldsymbol{r}'\}} B^{\boldsymbol{r},\{\boldsymbol{r}''\}} \tag{4.30}$$

We then have the remarkable symmetry properties:

$$A^{\{\boldsymbol{r}'\},\{\boldsymbol{r}''\}} = A^{\{\boldsymbol{r}''\},\{\boldsymbol{r}'\}} \qquad ; \qquad B^{\{\boldsymbol{r}'\},\{\boldsymbol{r}''\}} = B^{\{\boldsymbol{r}''\},\{\boldsymbol{r}'\}} \tag{4.31}$$

together with the identity:

$$B^{\{\boldsymbol{r}'\},\{\boldsymbol{r}''\}} = (-1)^r \, A^{\{\boldsymbol{r}'\},\{\boldsymbol{r}''\}} \qquad \textit{with} \qquad r := \sum r'_i = \sum r''_i \tag{4.32}$$

The following tables give $A^{\{\boldsymbol{r}'\},\{\boldsymbol{r}''\}}$ up to $r = 6$. The entries left vacant correspond to zeros.

	2	**1²**
2	−2	+1
1²	+1	

	3	**1.2**	**1³**
3	+3	−3	+1
1.2	−3	+1	
1³	+1		

	4	**1.3**	**2²**	**1².2**	**1⁴**
4	−4	+4	+2	−4	+1
1.3	+4	−1	−2	+1	
2²	+2	−2	+1		
1².2	−4	+1			
1⁴	+1				

	5	**1.4**	**2.3**	**1².3**	**1.2²**	**1³.2**	**1⁵**
5	+5	−5	−5	+5	+5	−5	+1
1.4	−5	+1	+5	−1	−3	+1	
2.3	−5	+5	−1	−2	+1		
1².3	+5	−1	−2	+1			
1.2²	+5	−3	+1				
1³.2	−5	+1					
1⁵	+1						

[56] the two sums in (4.30) range over all ordered sequences $\boldsymbol{r}$ that coincide, up to order, with the unordered sets $\{\boldsymbol{r}'\}$.

	6	**1.5**	**2.4**	$\mathbf{3^2}$	$\mathbf{1^2.4}$	**1.2.3**	$\mathbf{1^3.3}$	$\mathbf{2^3}$	$\mathbf{1^2.2^2}$	$\mathbf{1^4.2}$	$\mathbf{1^6}$
6	−6	+6	+6	+3	−6	−12	+6	−2	+9	−6	+1
1.5	+6	−1	−6	−3	+1	+7	−1	+2	−4	+1	
2.4	+6	−6	+2	−3	+2	+4	−2	−2	+1		
3.3	+3	−3	−3	+3	+3	−3	0	+1			
$\mathbf{1^2.4}$	−6	+1	+2	+3	−1	−3	+1				
1.2.3	−12	+7	+4	−3	−3	+1					
$\mathbf{1^3.3}$	+6	−1	−2	0	+1						
$\mathbf{2^3}$	−2	+2	−2	+1							
$\mathbf{1^2.2^2}$	+9	−4	+1								
$\mathbf{1^4.2}$	−6	+1									
$\mathbf{1^6}$	+1										

If now, following (4.30), we set $C^{\{\boldsymbol{r}'\},\{\boldsymbol{r}''\}} = \sum_{\boldsymbol{r}\in\{\boldsymbol{r}'\}} C^{\boldsymbol{r},\{\boldsymbol{r}''\}}$, we are saddled with rational numbers, but the symmetry relation becomes even more striking than with $A^{\{\bullet\},\{\bullet\}}$ and $B^{\{\bullet\},\{\bullet\}}$. Indeed:

$$C^{\{\boldsymbol{r}'\},\{\boldsymbol{r}''\}} = C^{\{\boldsymbol{r}''\},\{\boldsymbol{r}'\}} = 0 \quad \textit{if} \quad \{\boldsymbol{r}'\} \neq \{\boldsymbol{r}''\} \tag{4.33}$$

$$C^{\{\boldsymbol{r}\},\{\boldsymbol{r}\}} = \frac{C_{r_1}^{s_1}}{s_1!}\,\frac{C_{r_2}^{s_2}}{s_2!}\ldots \quad \textit{if} \quad \{\boldsymbol{r}\} = \{\overbrace{r_1,\ldots,r_1}^{s_1\ times}, \overbrace{r_2,\ldots,r_2}^{s_2\ times}, \ldots\} \tag{4.34}$$

4.2 The secondary bimoulds $\mathfrak{ess}^\bullet$ and $\mathfrak{esz}^\bullet$.

Dimorphic elements of $GARI_{<\mathfrak{se}>}$.
We are now, at long last, in a position to construct the two main dimorphic bimoulds $\mathfrak{ess}^\bullet_\sigma$ and $\mathfrak{esz}^\bullet_\sigma$ of $GARI_{<\mathfrak{se}>}$, simply by taking the images of two well-chosen elements f_σ and g_σ of $GIFF_{<x>}$. In the last section, we mentioned the *economical* way of taking such images, without transiting through the algebras. Here, for the sake of expediency, we plump for the *theoretical* way, via the infinitesimal generators:

$$\begin{array}{lcl|lcl}
\mathrm{f}_\sigma(x) & \longrightarrow & \mathfrak{ess}^\bullet_\sigma & \mathrm{g}_\sigma(x) & \longrightarrow & \mathfrak{esz}^\bullet_\sigma \\
\uparrow \exp & & \uparrow \mathrm{expari} & \uparrow \exp & & \uparrow \mathrm{expari} \\
\mathrm{f}_{*\sigma}(x) & \longrightarrow & \mathfrak{less}^\bullet_\sigma & \mathrm{g}_{*\sigma}(x) & \longrightarrow & \mathfrak{lesz}^\bullet_\sigma
\end{array}$$

The above diagram immediately translates into the formulas:

$$\mathfrak{ess}^\bullet_\sigma := \mathrm{expari}\big(\textstyle\sum_{r\geq 1} \sigma^r\, \epsilon_r\, \mathfrak{re}^\bullet_r\big) \longleftrightarrow f_\sigma(x) := \dfrac{1-e^{-\sigma x}}{\sigma} \tag{4.35}$$

$$\mathfrak{esz}^\bullet_\sigma := \mathrm{expari}\big(\textstyle\sum_{r\geq 1} \eta_{\sigma,r}\, \mathfrak{re}^\bullet_r\big) \longleftrightarrow g_\sigma(x) := \dfrac{1-(1-x)^{1-2\sigma}}{1-2\sigma} \tag{4.36}$$

with rational coefficients ϵ_r and $\eta_{\sigma,r}$ determined by:

$$\Big(\exp\big((\sum_{r\geq 1}\sigma^r \epsilon_r\, x^r).x.\partial_x\big)\Big).x \;=\; f_\sigma(x) = x\,(1+\sum_{r\geq 1}\sigma^r c_r\, x^r) = x-\frac{\sigma}{2}x^2+\dots \tag{4.37}$$

$$\Big(\exp\big((\sum_{r\geq 1}\eta_{\sigma,r}\, x^r).x.\partial_x\big)\Big).x \;=\; g_\sigma(x) = x\,(1+\sum_{r\geq 1}d_{\sigma,r}\; x^r) = x+\sigma\, x^2+\dots \tag{4.38}$$

Thus :

$$\epsilon_1 = -\frac{1}{2}\ ,\ \epsilon_2 = -\frac{1}{12}\ ,\ \epsilon_3 = -\frac{1}{48}\ ,\ \epsilon_4 = -\frac{1}{180}\ ,\ \epsilon_5 = -\frac{11}{8640}\ ,\ \epsilon_6 = -\frac{1}{6720}\dots$$

$$\eta_{\sigma,1} = s\ ,\ \eta_{\sigma,2} = \frac{1}{3}\sigma\,(1-\sigma)\ ,\ \eta_{\sigma,3} = \frac{1}{6}\sigma\,(1-\sigma)^2\ ,\ \eta_{\sigma,4} = \frac{1}{90}\sigma\,(1-\sigma)(3-4\,\sigma)(3-2\,\sigma)...$$

Main property: The bimoulds $\mathfrak{ess}^\bullet_\sigma$ are bisymmetral (i.e. of type as/as) whilst the bimoulds $\mathfrak{esz}^\bullet_\sigma$ are symmetral/$\mathfrak{O}$-symmetral (i.e. of type $as/\mathfrak{os}$). Here, $\mathfrak{O}$ denotes as usual the flexion unit conjugate to $\mathfrak{E}$.

Remark 1: This is a survey, dedicated to *stating* rather than *proving*. However, the double symmetries of $\mathfrak{ess}^\bullet_\sigma$ and $\mathfrak{esz}^\bullet_\sigma$ are so essential that we must pause to justify them. The symmetrality of these two bimoulds is easy enough: it simply results from their being, by construction, elements of $GARI_{<\mathfrak{se}>}$. But what about their *swappees*? The way the operator *gepar* is defined (see (4.1)), it is clear that if $\mathfrak{ess}^\bullet_\sigma$ is to be symmetral, then $gepar.\mathfrak{ess}^\bullet_\sigma$ too has to be symmetral. Similarly, if $\mathfrak{esz}^\bullet_\sigma$ is to be $\mathfrak{O}$-symmetral, then $gepar.\mathfrak{esz}^\bullet_\sigma$ too has to be $\mathfrak{O}$-symmetral. But in view of (4.10) and (4.37),(4.38), we can see that

$$(\mathrm{gepar}.\mathfrak{ess}_\sigma)^{w_1,\dots,w_r} \;=\; \mathcal{S}_{\sigma,r}\,\mathfrak{O}^{w_1}...\mathfrak{O}^{w_r} \tag{4.39}$$

$$(\mathrm{gepar}.\mathfrak{esz}_\sigma)^{w_1,\dots,w_r} \;=\; \mathcal{Z}_{\sigma,r}\,\mathfrak{O}^{w_1}...\mathfrak{O}^{w_r} \tag{4.40}$$

with

$$\mathcal{S}_{\sigma,r} \;=\; (r+1)\,\sigma^r\,c_r = \frac{(-\sigma)^r}{r!} \tag{4.41}$$

$$\mathcal{Z}_{\sigma,r} \;=\; (r+1)\,d_{\sigma,r} = \frac{1}{r!}\prod_{0\leq j\leq r-1}(2\,\sigma+j) \tag{4.42}$$

Now, it is an easy matter to check that the above coefficients $\mathcal{S}_{\sigma,r}$ resp. $\mathcal{Z}_{\sigma,r}$ are *the only ones* that can make the bimoulds defined by the right-hand sides of (4.39) resp. (4.40) *symmetral* resp. $\mathfrak{O}$-*symmetral.* Thus, $gepar.\mathfrak{ess}^\bullet_\sigma$ and $gepar.\mathfrak{esz}^\bullet_\sigma$ do possess the right symmetries, and from there it is but a short step to check that their constituent factors, namely $swap.\mathfrak{ess}^\bullet_\sigma$ and

$anti.swap.\mathfrak{ess}^\bullet_\sigma$ resp. $swap.\mathfrak{esz}^\bullet_\sigma$ and $anti.swap.\mathfrak{esz}^\bullet_\sigma$, also possess the right symmetries: see §11.9 and §11.10.

Remark 2: Whereas the bimoulds $\mathfrak{esz}^\bullet_\sigma$ really differ when σ varies, the bimoulds $\mathfrak{ess}^\bullet_\sigma$ merely undergo dilatation – an elementary transform that commutes with all flexion operations. So all these $\mathfrak{ess}^\bullet_\sigma$ essentially reduce to their prototype $\mathfrak{ess}^\bullet := \mathfrak{ess}^\bullet_1$, which we shall henceforth call *the* bisymmetral element of $Flex(\mathfrak{E})$.

Remark 3: By continuity in σ, we see that $g_{1/2}(x) = -\log(1-x)$. Thus $f_1 \circ g_{1/2} = id$ and therefore $gari(\mathfrak{ess}^\bullet_1, \mathfrak{esz}^\bullet_{1/2}) = id_{GARI} = 1^\bullet$, which shows that invgari.$\mathfrak{ess}^\bullet_1$ and by implication all invgari.$\mathfrak{ess}^\bullet_\sigma$ are *not* bisymmetral.

Remark 4: odd-even factorisations of the bisymmetrals.
The pre-image $f(x) := 1-e^{-x}$ of $\mathfrak{ess}^\bullet$ in $GIFF_{<x>}$ factors as $f = f_\diamond \circ f_{\diamond\diamond}$, with an elementary first factor and a second factor that carries only even-indexed coefficients:

$$f_\diamond(x) \;:=\; \frac{x}{1+\frac{1}{2}x} \;=\; \Big(\exp(-\frac{1}{2}\,x^2\,\partial_x)\Big).\,x \tag{4.43}$$

$$f_{\diamond\diamond}(x) \;:=\; x\,\big(1+\sum_{1\leq n} a^{\diamond\diamond}_{2\,n}\,x^{2\,n}\big) \tag{4.44}$$

For $\mathfrak{ess}^\bullet$ this immediately translates into the factorisation (4.45) in $GARI_{<\mathfrak{se}>}$. For $swap.\mathfrak{ess}^\bullet =: \ddot{\mathfrak{o}}\mathfrak{ss}^\bullet$, it translates, though less immediately, into the factorisation (4.46) in $BIMU$. Mark the order inversion, though, and note that $swap.\mathfrak{ess}^\bullet_* \neq \ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_\star$, $swap.\mathfrak{ess}^\bullet_{**} \neq \ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_{\star\star}$.

$$\mathfrak{ess}^\bullet \;=\; \mathrm{gari}(\mathfrak{ess}^\bullet_*, \mathfrak{ess}^\bullet_{**}) \quad \textit{with}\;\; \mathfrak{ess}^\bullet_*, \mathfrak{ess}^\bullet_{**}\;\textit{symmetral} \tag{4.45}$$

$$\mathrm{swap}.\mathfrak{ess}^\bullet = \ddot{\mathfrak{o}}\mathfrak{ss}^\bullet \;=\; \mathrm{mu}(\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_{\star\star}, \ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_\star) \quad \textit{with}\;\; \ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_\star, \ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_{\star\star}\;\textit{symmetral} \tag{4.46}$$

All four factor bimoulds are symmetral. The single-starred ones are elementary:

$$\mathfrak{ess}^\bullet_* := \mathrm{expari}(-\frac{1}{2}\mathfrak{E}^\bullet) \Rightarrow \mathfrak{ess}_*^{\binom{u_1\,,\dots,\,u_r}{v_1\,,\dots,\,v_r}} = \frac{(-1)^r}{2^r}\,\mathfrak{E}^{\binom{u_1}{v_{1:2}}}\mathfrak{E}^{\binom{u_{12}}{v_{2:3}}}\dots\mathfrak{E}^{\binom{u_{1\dots r}}{v_r}} \tag{4.47}$$

$$\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_\star := \mathrm{expmu}(-\frac{1}{2}\mathfrak{O}^\bullet) \Rightarrow \ddot{\mathfrak{o}}\mathfrak{ss}_\star^{\binom{u_1\,,\dots,\,u_r}{v_1\,,\dots,\,v_r}} = \frac{(-1)^r}{2^r\,r!}\,\mathfrak{O}^{\binom{u_1}{v_1}}\mathfrak{O}^{\binom{u_2}{v_2}}\dots\mathfrak{O}^{\binom{u_r}{v_r}} \tag{4.48}$$

The double-starred factors, though non-elementary, carry only (non-zero) components of *even* length:

$$\mathfrak{ess}^\bullet_*\,,\,\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_\star \;\in\; \mathrm{BIMU}^{\mathrm{as}}_{\mathrm{neg.pari}} \tag{4.49}$$

$$\mathfrak{ess}^\bullet_{**}\,,\,\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_{\star\star} \;\in\; \mathrm{BIMU}^{\mathrm{as}}_{\mathrm{neg}} \cap \mathrm{BIMU}^{\mathrm{as}}_{\mathrm{pari}} \tag{4.50}$$

As a consequence:

$$\text{anti}.\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_\star = \ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_\star \qquad ; \qquad \text{anti}.\mathfrak{oss}^\bullet_{\star\star} = \text{invmu}.\mathfrak{oss}^\bullet_\star \tag{4.51}$$

and therefore:

$$\begin{aligned} \text{gepar}.\mathfrak{ess}^\bullet &= \text{mu}(\text{anti}.\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_\star, \text{anti}.\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_{\star\star}, \ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_{\star\star}, \ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_\star) \\ &= \text{mu}(\mathfrak{oss}^\bullet_\star, \mathfrak{oss}^\bullet_\star) = \text{expmu}(-\mathfrak{O}^\bullet) \end{aligned} \tag{4.52}$$

Remark 5: induction for the calculation of $\mathfrak{ess}, \mathfrak{ess}_{**}$ and $\mathfrak{oss}, \ddot{\mathfrak{o}}\mathfrak{ss}_{\star\star}$.
The source diffeos for $\mathfrak{ess}^\bullet$ and $\mathfrak{ess}^\bullet_{**}$ are f and f, with infinitesimal dilators:

$$f_\#(x) = 1 + x - e^x \qquad ; \qquad f_{\diamond\diamond\#}(x) = x - \cosh(x) \tag{4.53}$$

to which there answer the following elements of $ARI_{<\mathfrak{re}>}$ and $IRA_{<\mathfrak{r}\ddot{\mathfrak{o}}>}$:

$$\begin{aligned} \mathfrak{ett}^\bullet &:= -\textstyle\sum_{1\le n} \frac{1}{(n+1)!}\, \mathfrak{re}^\bullet_n \qquad ; \qquad \mathfrak{ett}^\bullet_{**} := -\textstyle\sum_{1\le n} \frac{1}{(2\,n+1)!}\, \mathfrak{re}^\bullet_{2\,n} \\ \ddot{\mathfrak{o}}\mathfrak{tt}^\bullet &:= -\textstyle\sum_{1\le n} \frac{1}{(n+1)!}\, \mathfrak{r}\ddot{\mathfrak{o}}^\bullet_n \qquad ; \qquad \ddot{\mathfrak{o}}\mathfrak{tt}^\bullet_{\star\star} := -\textstyle\sum_{1\le n} \frac{1}{(2\,n+1)!}\, \mathfrak{r}\ddot{\mathfrak{o}}^\bullet_{2\,n} \end{aligned}$$

which in turn lead to these linear and highly effective inductive formulas[57] for the calculation of our four bimoulds:

$$r(\bullet)\,\mathfrak{ess}^\bullet = \text{preari}(\mathfrak{ess}^\bullet, \mathfrak{ett}^\bullet) \tag{4.54}$$

$$r(\bullet)\,\mathfrak{ess}^\bullet_{**} = \text{preari}(\mathfrak{ess}^\bullet_{**}, \mathfrak{ett}^\bullet_{**}) \tag{4.55}$$

$$r(\bullet)\,\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet = \text{preira}(\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet, \ddot{\mathfrak{o}}\mathfrak{tt}^\bullet) \tag{4.56}$$

$$r(\bullet)\,\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_{\star\star} = \text{preira}(\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_{\star\star}, \ddot{\mathfrak{o}}\mathfrak{tt}^\bullet_{\star\star}) + \frac{1}{2}\,\text{mu}(\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_{\star\star}, \ddot{\mathfrak{o}}\mathfrak{tt}^\bullet_\star) \tag{4.57}$$

with

$$\ddot{\mathfrak{o}}\mathfrak{tt}^\bullet_\star := \text{coshmu}(\mathfrak{O}^\bullet) := \frac{1}{2}\big(\text{expmu}(\mathfrak{O}^\bullet) + \text{expmu}(-\mathfrak{O}^\bullet)\big)$$

In (4.54) and (4.55), *preari* may be replaced by *preali* or *preawi*; and in (4.56) and (4.57), *preira* may be replaced by *preila* or *preiwa*, since the involutions h_1, h_2, h_3 that define the algebras *ARI*, *ALI*, *AWI* (see §2.1 towards the end) have the same effects on the basic alternals $\mathfrak{re}^\bullet_n$:

$$h_1\,\mathfrak{re}^\bullet_n \equiv h_2\,\mathfrak{re}^\bullet_n \equiv h_3\,\mathfrak{re}^\bullet_n \qquad ; \qquad h_1^*\,\mathfrak{r}\ddot{\mathfrak{o}}^\bullet_n \equiv h_2^*\,\mathfrak{r}\ddot{\mathfrak{o}}^\bullet_n \equiv h_3^*\,\mathfrak{r}\ddot{\mathfrak{o}}^\bullet_n \tag{4.58}$$

[57] these are true induction, since the sought-after bimoulds occur only once, with length r, on the left-hand side; and several times on the right-hand side, but with lengths at most $r-1$ (resp. $r-2$) in (4.54), (4.56) (resp. (4.55), (4.57)).

Whatever the pre-bracket chosen, the induction algorithm yields the same result, but expressed in very different bases. For the direct bimoulds, the best choice is *preari* or *preali*[58]; and for the swappees, it is *preiwa*[59] along with the following expression of $\ddot{\mathfrak{ro}}^\bullet$:

$$\ddot{\mathfrak{ro}}^{\binom{u_1\,,\ldots,\,u_r}{v_1\,,\ldots,\,v_r}} = \sum_i (r{+}1-i)\,\mathfrak{O}^{\binom{u_{1\ldots r}}{v_i}} \prod_{j\neq i} \mathfrak{O}^{\binom{u_j}{v_{j:i}}} \tag{4.59}$$

Comparing $\mathfrak{ess}^\bullet$ and $\ddot{\mathfrak{ess}}^\bullet := sap.\mathfrak{ess}^\bullet$:
The bimould $\mathfrak{ess}^\bullet$ belongs to the group $GARI_{<\mathfrak{se}>}$ whereas its image $\ddot{\mathfrak{ess}}^\bullet$ under the involution *sap=swap.syap* belongs to $swap.GARI_{<syap.\mathfrak{se}>}$ i.e. to $swap.GARI_{<\mathfrak{so}>}$, which is not a group – only the *swappee* of one. Nevertheless, $\mathfrak{ess}^\bullet$ and $\ddot{\mathfrak{ess}}^\bullet$ have much in common, since they
– belong both to $Flex(\mathfrak{E})$ and are both bisymmetral, i.e. in $GARI^{as/as}$
– are both invariant under *pari.neg*
– have both the same length-one component: $\mathfrak{ess}^{w_1} = \ddot{\mathfrak{ess}}^{w_1}$.
This is enough for them to be exchanged under *gari*-postcomposition by a bimould $\ddot{\mathfrak{ses}}^\bullet$ that is not only bisymmetral, but also *even*[60], i.e. in $GARI^{\underline{as}/\underline{as}}$. It is therefore the exponential of an element $\ddot{\mathfrak{lel}}^\bullet$ of $ARI^{\underline{al}/\underline{al}}$. In other words:

$$\mathfrak{ess}^\bullet = \mathrm{gari}(\ddot{\mathfrak{ess}}^\bullet, \ddot{\mathfrak{ses}}^\bullet) = \mathrm{gari}(\ddot{\mathfrak{ess}}^\bullet, \mathrm{expari}(\ddot{\mathfrak{lel}}^\bullet)) \tag{4.60}$$

But since both $\ddot{\mathfrak{ses}}^\bullet$ and $\ddot{\mathfrak{lel}}^\bullet$ are invariant under *neg* and *pari.neg*, they are invariant under *pari*. All their non-vanishing components are therefore of *even* length; or more precisely of even length $r \geq 4$, since an initial, length-2 component of $\ddot{\mathfrak{ses}}^\bullet$ would have to be a bialternal element of $Flex_2(\mathfrak{E})$, and no such element exists.

Up to length $r = 14$, the bialternal subalgebra $Flex^{\underline{al}/\underline{al}}(\mathfrak{E})$ of $ARI^{\underline{al}/\underline{al}}$ is *freely* generated by the non-vanishing components of $\ddot{\mathfrak{lel}}^\bullet$, i.e.

$$\ddot{\mathfrak{lel}}^\bullet_4 \;,\; \ddot{\mathfrak{lel}}^\bullet_6 \;,\; \ddot{\mathfrak{lel}}^\bullet_8 \;,\; \ddot{\mathfrak{lel}}^\bullet_{10} \;,\; \ddot{\mathfrak{lel}}^\bullet_{12} \;,\; \ddot{\mathfrak{lel}}^\bullet_{14} \;\ldots \tag{4.61}$$

or, alternatively, by the series of *singulates* $\mathfrak{lel}^\bullet_{2r}$ (see §4.2 below):

$$\mathfrak{lel}^\bullet_{2r} := \mathrm{senk}_{2r}(\mathfrak{ess}^\bullet).\mathfrak{E}^\bullet \qquad (r \geq 2) \tag{4.62}$$

but after 14 this no longer holds. As of now, for large values of r, the exact dimension of $Flex^{\underline{al}/\underline{al}}_{2r}(\mathfrak{E})$ is not known.

[58] since h_1 and h_2, unlike h_3, involve no sign changes.
[59] since h_3^*, unlike h_1^* and h_2^*,, involves no sign changes.
[60] i.e. invariant under *neg* rather than *pari.neg*.

If we now repeat the above construction but with $\mathfrak{E}$ replaced by the conjugate unit $\mathfrak{O}$, identity (4.60) becomes, with self-explanatory notations:

$$\mathfrak{oss}^{\bullet} = \mathrm{gari}(\ddot{\mathfrak{o}}\mathfrak{ss}^{\bullet}, \mathfrak{s}\ddot{\mathfrak{o}}\mathfrak{os}^{\bullet}) = \mathrm{gari}(\ddot{\mathfrak{o}}\mathfrak{ss}^{\bullet}, \mathrm{expari}(\mathfrak{l}\ddot{\mathfrak{o}}\mathfrak{ol}^{\bullet})) \tag{4.63}$$

So far, so predictable. The remarkable thing, however, is that the components $\mathfrak{l}\ddot{\mathfrak{e}}\mathfrak{el}^{\bullet}_{2r}$ and $\mathfrak{l}\ddot{\mathfrak{o}}\mathfrak{ol}^{\bullet}_{2r}$ of the rightmost bimoulds in (4.60) and (4.63) get exchanged, up to sign, under the involutions *swap* and *syap* (see (§3.3)). As a consequence, each one of them is, again up to sign, invariant under the involution *sap*.

Polar and trigonometric specialisations:
Let us now consider the three polar and the three trigonometric specialisations of $\mathfrak{E}^{\bullet}$, along with the corresponding bisymmetrals and their *swappees*:

Flexion units	$\mathfrak{E}^{\bullet}$	:	$\mathrm{Pa}^{\bullet}$	$\mathrm{Pi}^{\bullet}$	$\mathrm{Pai}^{\bullet}_{\alpha,\beta}$	$\mathrm{Qa}^{\bullet}_{c}$	$\mathrm{Qi}^{\bullet}_{c}$	$\mathrm{Qai}^{\bullet}_{c,\alpha,\beta}$
bisymmetrals	$\mathfrak{ess}^{\bullet}$	:	$\mathrm{par}^{\bullet}$	$\mathrm{pil}^{\bullet}$	$\mathrm{pail}^{\bullet}_{\alpha,\beta}$	...	$\mathrm{til}^{\bullet}_{c}$	$\mathrm{tail}^{\bullet}_{c,\alpha,\beta}$
swappees	$\ddot{\mathfrak{o}}\mathfrak{ss}^{\bullet}$	:	$\mathrm{pir}^{\bullet}$	$\mathrm{pal}^{\bullet}$	$\mathrm{pial}^{\bullet}_{\alpha,\beta}$	...	$\mathrm{tal}^{\bullet}_{c}$	$\mathrm{tial}^{\bullet}_{c,\alpha,\beta}$
type as/$\mathfrak{os}$	$\mathfrak{esz}^{\bullet}_{\sigma}$	:	$\mathrm{bar}^{\bullet}_{\sigma}$	$\mathrm{bil}^{\bullet}_{\sigma}$	$\mathrm{bail}^{\bullet}_{c,\alpha,\beta}$	...	...	$\mathrm{dail}^{\bullet}_{\sigma,c,\alpha,\beta}$
swappees	$\ddot{\mathfrak{o}}\mathfrak{sz}^{\bullet}_{\sigma}$	:	$\mathrm{bir}^{\bullet}_{\sigma}$	$\mathrm{bal}^{\bullet}_{\sigma}$	$\mathrm{bial}^{\bullet}_{\sigma,\alpha,\beta}$	...	...	$\mathrm{dial}^{\bullet}_{\sigma,c,\alpha,\beta}$

All these unit specialisations are *exact*, except $Qa^{\bullet}_{c}$, which generates no bisymmetral, and $Qi^{\bullet}_{c}$, which does.[61]

Let D^t be the dilation operator:

$$(D^t.M)^{\binom{u_1\ ,\dots,\ u_r}{v_1\ ,\dots,\ v_r}} := M^{\binom{u_1/t\ ,\dots,\ u_r/t}{v_1/t\ ,\dots,\ v_r/t}}$$

It clearly respects bialternality and bisymmetrality. Due to the general identities:

$$\begin{aligned} &A^{\bullet}\ \boldsymbol{v}\text{-}constant \quad and \quad B^{\bullet}\ \boldsymbol{u}\text{-}constant \quad \Longrightarrow \\ &\mathrm{swap.mu}(A^{\bullet}, B^{\bullet}) \quad \equiv \quad \mathrm{mu}(\mathrm{swap}.B^{\bullet}, \mathrm{swap}.A^{\bullet}) \end{aligned}$$

we can form new bisymmetrals:

$$\begin{aligned} \mathrm{vipail}^{\bullet}_{\alpha,\beta} &:= \mathrm{mu}(D^{\alpha}.\mathrm{pal}^{\bullet}, D^{\beta}.\mathrm{pil}^{\bullet}) \quad \subset \mathrm{GARI}^{\mathrm{as/as}} \\ \mathrm{vipair}^{\bullet}_{\alpha,\beta} &:= \mathrm{mu}(D^{\alpha}.\mathrm{par}^{\bullet}, D^{\beta}.\mathrm{pir}^{\bullet}) \quad \subset \mathrm{GARI}^{\mathrm{as/as}} \\ \mathrm{vitail}^{\bullet}_{c,\alpha,\beta} &:= \mathrm{mu}(D^{\alpha}.\mathrm{tal}^{\bullet}_{c}, D^{\beta}.\mathrm{til}^{\bullet}_{c}) \quad \subset \mathrm{GARI}^{\mathrm{as/as}} \end{aligned}$$

[61] But of course with an elementary corrective factor $mini^{\bullet}_{c} \in center(GARI)$ in the connection formula: $swap.til^{\bullet}_{c} = gari(mana^{\bullet}_{c}, tal_{c}{}^{\bullet}) = gari(tal_{c}{}^{\bullet}, mana^{\bullet}_{c})$.

The next four identities are special cases of (4.60) when $\mathfrak{E}$ specialises respectively to Pa, Pi, $Pai_{\alpha,\beta}$, $Qai_{c,\alpha,\beta}$:

$$
\begin{array}{llllll}
\mathrm{par}^\bullet & \equiv & \mathrm{gari}(\mathrm{pal}^\bullet, \mathrm{lar}^\bullet) & \textit{with} & \mathrm{lar}^\bullet & \subset \mathrm{GARI}^{\underline{\mathrm{as}}/\underline{\mathrm{as}}} \\
\mathrm{pil}^\bullet & \equiv & \mathrm{gari}(\mathrm{pir}^\bullet, \mathrm{ril}^\bullet) & \textit{with} & \mathrm{ril}^\bullet & \subset \mathrm{GARI}^{\underline{\mathrm{as}}/\underline{\mathrm{as}}} \\
\mathrm{pail}^\bullet_{\alpha,\beta} & \equiv & \mathrm{gari}(\mathrm{pial}^\bullet_{\beta,\alpha}, \mathrm{lappil}^\bullet_{\alpha,\beta}) & \textit{with} & \mathrm{lappil}^\bullet_{\alpha,\beta} & \subset \mathrm{GARI}^{\underline{\mathrm{as}}/\underline{\mathrm{as}}} \\
\mathrm{tail}^\bullet_{c,\alpha,\beta} & \equiv & \mathrm{gari}(\mathrm{tial}^\bullet_{c,\beta,\alpha}, \mathrm{lattil}^\bullet_{c,\alpha,\beta}) & \textit{with} & \mathrm{lattil}^\bullet_{c,\alpha,\beta} & \subset \mathrm{GARI}^{\underline{\mathrm{as}}/\underline{\mathrm{as}}} \\
\mathrm{vipail}^\bullet_{\alpha,\beta} & \equiv & \mathrm{gari}(\mathrm{pail}^\bullet_{\alpha,\beta}, \mathrm{paiv}^\bullet_{\alpha,\beta}) & \textit{with} & \mathrm{paiv}^\bullet_{\alpha,\beta} & \subset \mathrm{GARI}^{\underline{\mathrm{as}}/\underline{\mathrm{as}}} \\
\mathrm{vitail}^\bullet_{c,\alpha,\beta} & \equiv & \mathrm{gari}(\mathrm{tail}^\bullet_{c,\alpha,\beta}, \mathrm{taiv}^\bullet_{c,\alpha,\beta}) & \textit{with} & \mathrm{taiv}^\bullet_{c,\alpha,\beta} & \subset \mathrm{GARI}^{\underline{\mathrm{as}}/\underline{\mathrm{as}}}
\end{array}
$$

while the last two identities provide yet other examples of elements of $GARI^{as/as}$ sharing the same first component and related under postcomposition by an element of $GARI^{\underline{as}/\underline{as}}$.

Difference between even and non-even bissymmetrals:
To bring out the sharp difference between *even* and *non-even* bisymmetrals, we introduce two distinct copies $\mathfrak{E}_1$,$\mathfrak{E}_2$ of the universal unit $\mathfrak{E}$, and define their *blend* as follows:

$$
\mathfrak{sse}^\bullet_{1,2} = \mathrm{blend}(\mathfrak{E}^\bullet_1, \mathfrak{E}^\bullet_2) \qquad \Longleftrightarrow
$$

$$
\mathfrak{sse}_{1,2}^{\left(\begin{smallmatrix} u_1, & \dots, & u_r \\ v_1, & \dots, & v_r \end{smallmatrix}\right)} = \mathfrak{E}_1^{\left(\begin{smallmatrix} u_1 \\ v_{1:2} \end{smallmatrix}\right)} \mathfrak{E}_1^{\left(\begin{smallmatrix} u_{12} \\ v_{2:3} \end{smallmatrix}\right)} \mathfrak{E}_1^{\left(\begin{smallmatrix} u_{123} \\ v_{3:4} \end{smallmatrix}\right)} \dots \mathfrak{E}_1^{\left(\begin{smallmatrix} u_{1..r} \\ v_r \end{smallmatrix}\right)} \mathfrak{E}_2^{\left(\begin{smallmatrix} u_1 \\ v_1 \end{smallmatrix}\right)} \mathfrak{E}_2^{\left(\begin{smallmatrix} u_2 \\ v_2 \end{smallmatrix}\right)} \mathfrak{E}_2^{\left(\begin{smallmatrix} u_3 \\ v_3 \end{smallmatrix}\right)} \dots \mathfrak{E}_2^{\left(\begin{smallmatrix} u_r \\ v_r \end{smallmatrix}\right)} \tag{4.64}
$$

The blend $\mathfrak{sse}^\bullet_{1,2}$ is obviously *even*. It is also easily seen to be *symmetral*. In fact, since, up to order, *blend* commutes with *swap*:

$$
\mathrm{swap.blend}(\mathfrak{E}^\bullet_1, \mathfrak{E}^\bullet_2) \;\equiv\; \mathrm{blend}(\mathrm{swap}.\mathfrak{E}^\bullet_2, \mathrm{swap}.\mathfrak{E}^\bullet_1) \tag{4.65}
$$

and since the swappee of an exact flexion unit $\mathfrak{E}$ coincides with the conjugate unit $\mathfrak{O}$, the *blend* is actually *bisymmetral*.

Moreover, we have a remarkable (non-elementary) identity for expressing the *gari*-inverse of the *blend* of two flexion units: it is itself a *blend*, but preceded by *pari* and with the two arguments arguments exchanged. Therefore, under *invgari*, the two entries of (4.65) become:

$$
\mathrm{pari.blend}(\mathfrak{E}^\bullet_2, \mathfrak{E}^\bullet_1) \overset{\mathrm{swap}}{\longleftrightarrow} \mathrm{pari.blend}(\mathrm{swap}.\mathfrak{E}^\bullet_1, \mathrm{swap}.\mathfrak{E}^\bullet_2) \tag{4.66}
$$

and are still connected by *swap*.

As a consequence, for the *even*[62] bisymmetral $\mathfrak{ess}^\bullet_{1,2}$ we have this commu-

[62] i.e. *neg*-invariant.

tative diagram,[63] with self-explanatory notations:

$$\begin{array}{cccccc} (\textit{symmetral}) & \mathfrak{sse}^\bullet_{1,2} & \overset{\text{swap}}{\longleftrightarrow} & \mathfrak{sso}^\bullet_{2,1} & (\textit{symmetral}) \\ & \text{invgari} \updownarrow & & \updownarrow \text{invgari} & \\ (\textit{symmetral}) & \text{pari}.\mathfrak{sse}^\bullet_{2,1} & \overset{\text{swap}}{\longleftrightarrow} & \text{pari}.\mathfrak{sso}^\bullet_{1,2} & (\textit{symmetral}!) \end{array}$$

In sharp contrast, with the *non-even*[64] bisymmetral $\mathfrak{ess}^\bullet$ constructed in (4.35), the diagram's commutativity breaks down:

$$\begin{array}{ccccccc} (\textit{symmetral}) & \mathfrak{ess}^\bullet & \overset{\text{swap}}{\longleftrightarrow} & & & \ddot{\mathfrak{o}}\mathfrak{ss}^\bullet & (\textit{symmetral}) \\ & \text{invgari} \updownarrow & & & & \searrow \text{invgari} & \\ (\textit{symmetral}) & \mathfrak{ess}^\bullet_* & \overset{\text{swap}}{\longleftrightarrow} & \ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_* & \neq & \ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_{**} & (\textit{non symmetral}!) \end{array}$$

4.3 The related primary bimoulds $\mathfrak{es}^\bullet$ and $\mathfrak{ez}^\bullet$.

After constructing the *secondary* bimoulds $\mathfrak{ess}^\bullet, \mathfrak{esz}^\bullet_\sigma$ (non-elementary, with a double symmetry), we must now define the much simpler, yet closely related *primary* bimoulds $\mathfrak{es}^\bullet, \mathfrak{ez}^\bullet$ (elementary, with a single symmetry):

$$\mathfrak{es}^\bullet := \text{expari}(\mathfrak{E}^\bullet) \tag{4.67}$$

$$\mathfrak{ez}^\bullet := \text{invmu}(1^\bullet - \mathfrak{E}^\bullet) \tag{4.68}$$

$$\mathfrak{es}^\bullet \overset{\text{sap}}{\leftrightarrow} \mathfrak{ez}^\bullet \tag{4.69}$$

This leads to the more explicit formulas:[65]

$$\mathfrak{es}^{\binom{u_1,\dots,u_r}{v_1,\dots,v_r}} := \mathfrak{E}^{\binom{u_1}{v_{1:2}}}\mathfrak{E}^{\binom{u_{12}}{v_{2:3}}}\mathfrak{E}^{\binom{u_{123}}{v_{3:4}}}\dots\mathfrak{E}^{\binom{u_{1\dots r}}{v_r}} \quad (\textit{symmetral}) \tag{4.70}$$

$$\mathfrak{ez}^{\binom{u_1,\dots,u_r}{v_1,\dots,v_r}} := \mathfrak{E}^{\binom{u_1}{v_1}}\mathfrak{E}^{\binom{u_2}{v_2}}\mathfrak{E}^{\binom{u_3}{v_3}}\dots\mathfrak{E}^{\binom{u_r}{v_r}} \quad (\mathfrak{E}\textit{-symmetral}) \tag{4.71}$$

The symmetrality of $\mathfrak{es}^\bullet$ resp. $\mathfrak{E}$-symmetrality of $\mathfrak{ez}^\bullet$ relies entirely on $\mathfrak{E}$ being an exact flexion unit, but the definitions also extend, albeit at the cost of significant complications, to approximate units.

Let us now consider the three polar and the three trigonmetric specialisations of $\mathfrak{E}^\bullet$ and the corresponding incarnations of $\mathfrak{es}^\bullet$ and $\mathfrak{ez}^\bullet$:

Flexion units	$\mathfrak{E}^\bullet$	:	$\mathrm{Pa}^\bullet$	$\mathrm{Pi}^\bullet$	$\mathrm{Pai}^\bullet_{\alpha,\beta}$	$\mathrm{Qa}^\bullet_c$	$\mathrm{Qi}^\bullet_c$	$\mathrm{Qai}^\bullet_{c,\alpha,\beta}$
symmetrals	$\mathfrak{es}^\bullet$	:	$\mathrm{paj}^\bullet$	$\mathrm{pij}^\bullet$	$\mathrm{paij}^\bullet_{\alpha,\beta}$	taj_c	$\mathrm{tij}^\bullet_c$	$\mathrm{taij}^\bullet_{c,\alpha,\beta}$
$\mathfrak{E}$-*symmetrals*	$\mathfrak{ez}^\bullet$	:	$\mathrm{pac}^\bullet$	$\mathrm{pic}^\bullet$	$\mathrm{paic}^\bullet_{\alpha,\beta}$	$\mathrm{tac}^\bullet_c$	$\mathrm{tic}^\bullet_c$	$\mathrm{taic}^\bullet_{c,\alpha,\beta}$

[63]we would of course have similarly commutative diagrams (only with less explicit *gari*-inverses) if we replaced $\mathfrak{sse}^\bullet$ by any element of $GARI^{\underline{\text{as}}/\underline{\text{as}}}$, since on that subgroup *swap* acts as an automorphism, just as it does on $ARI^{\underline{\text{al}}/\underline{\text{al}}}$.

[64]more precisely: $\mathfrak{ess}^\bullet$ is *pari.neg*-invariant instead of *neg*-invariant.

[65]To derive (4.70) from (4.67), one must use the fact that $\mathfrak{E}$ is a flexion unit.

The definitions of the new bimoulds are straightforward for the exact units, but less so for the approximate units Qa_c and Qi_c. In those two cases, we mention only the elementary part (mod. c^2), which conforms entirely to the general formulas (4.70) and (4.71), and refer to §3.9 for the corrective terms.

$$\mathrm{paj}^{w_1,\ldots,w_r} := \prod_{1\leq j\leq r} P(u_1+\ldots+u_j) \tag{4.72}$$

$$\mathrm{pij}^{w_1,\ldots,w_r} := \prod_{1\leq j\leq r}^{\mathrm{circ}} P(v_j-v_{j+1}) \tag{4.73}$$

$$\mathrm{paij}_{\alpha,\beta}^{w_1,\ldots,w_r} := \prod_{1\leq j\leq r}^{\mathrm{circ}} \Big(P(\frac{u_1+\ldots+u_j}{\alpha}) + P(\frac{v_j-v_{j+1}}{\beta})\Big) \tag{4.74}$$

$$\mathrm{taj}_c^{w_1,\ldots,w_r} := \prod_{1\leq j\leq r} Q_c(u_1+\ldots+u_j) \qquad (\textit{modulo } c^2) \tag{4.75}$$

$$\mathrm{tij}_c^{w_1,\ldots,w_r} := \prod_{1\leq j\leq r}^{\mathrm{circ}} Q_c(v_j-v_{j+1}) \qquad (\textit{modulo } c^2) \tag{4.76}$$

$$\mathrm{taij}_{c,\alpha,\beta}^{w_1,\ldots,w_r} := \prod_{1\leq j\leq r}^{\mathrm{circ}} \Big(Q_c(\frac{u_1+\ldots+u_j}{\alpha}) + Q_c(\frac{v_j-v_{j+1}}{\beta})\Big) \quad (\textit{exactly}) \tag{4.77}$$

In the above products, *circ* means that the (non-existing) variable v_{r+1} should be construed as $v_0 = 0$ whenever it occurs. No such precaution is required for the following specialisations of $\mathfrak{ez}^\bullet$.

$$\mathrm{pac}^{w_1,\ldots,w_r} := \prod_{1\leq j\leq r} P(u_j) \tag{4.78}$$

$$\mathrm{pic}^{w_1,\ldots,w_r} := \prod_{1\leq j\leq r} P(v_j) \tag{4.79}$$

$$\mathrm{paic}_{\alpha,\beta}^{w_1,\ldots,w_r} := \prod_{1\leq j\leq r} \Big(P(\frac{u_j}{\alpha}) + P(\frac{v_j}{\beta})\Big) \tag{4.80}$$

$$\mathrm{tac}_c^{w_1,\ldots,w_r} := \prod_{1\leq j\leq r} Q_c(u_j) \qquad (\textit{modulo } c^2) \tag{4.81}$$

$$\mathrm{tic}_c^{w_1,\ldots,w_r} := \prod_{1\leq j\leq r} Q_c(v_j) \qquad (\textit{modulo } c^2) \tag{4.82}$$

$$\mathrm{taic}_{c,\alpha,\beta}^{w_1,\ldots,w_r} := \prod_{1\leq j\leq r} \Big(Q_c(\frac{u_j}{\alpha}) + Q_c(\frac{v_j}{\beta})\Big) \qquad (\textit{exactly}) \tag{4.83}$$

4.4 Some basic bimould identities.

Let us list, first in universal mode, the main relations between the primary bimoulds:

$$
\begin{array}{llcll}
\mathrm{invmu}.\mathfrak{es}^\bullet & = \mathrm{pari.anti}.\mathfrak{es}^\bullet & \| & \mathrm{invmu}.\mathfrak{ez}^\bullet & = 1^\bullet - \mathfrak{E}^\bullet \\
\mathrm{invgami}.\mathfrak{es}^\bullet & = \mathrm{pari}.\mathfrak{ez}^\bullet & \| & \mathrm{invgami}.\mathfrak{ez}^\bullet & = \mathrm{pari}.\mathfrak{es}^\bullet \\
\mathrm{invgani}.\mathfrak{es}^\bullet & = \textit{unremarkable} & \| & \mathrm{invgani}.\mathfrak{ez}^\bullet & = \mathrm{pari.anti}.\mathfrak{es}^\bullet \\
\mathrm{invgari}.\mathfrak{es}^\bullet & = \mathrm{pari}.\mathfrak{es}^\bullet & \| & \mathrm{invgari}.\mathfrak{ez}^\bullet & = \textit{unremarkable}
\end{array}
$$

The relations that really matter, however, are the ones linking primary and secondary bimoulds. To state them, we require a highly non-linear operator *slash* which measures, in terms of *GARI*, the *un-evennness* of a bimould:

$$\mathrm{slash}.B^\bullet \;:=\; \mathrm{fragari}(\mathrm{neg}.B^\bullet, B^\bullet) \;=\; \mathrm{gari}(\mathrm{neg}.B^\bullet, \mathrm{invgari}.B^\bullet) \quad (4.84)$$

We can now write down the two secondary-to-primary identities:

$$\mathrm{slash}.\mathfrak{ess}^\bullet \;=\; \mathfrak{es}^\bullet \qquad \textit{with} \quad \mathfrak{ess}^\bullet := \mathfrak{ess}_1^\bullet \quad (4.85)$$

$$\mathrm{sap}.\mathfrak{esz}_0^\bullet \;=\; \mathfrak{ez}^\bullet \qquad \textit{with} \quad \mathrm{sap} := \mathrm{syap.swap} = \mathrm{swap.syap} \quad (4.86)$$

To conclude this section, let us reproduce some of the above identities in the polar and trigonometric specialisations – for definiteness, and also to show which relations survive and which don't when $\mathfrak{E}$ specialises to the *approximate* flexion units like $Qa_c^\bullet$ and $Qi_c^\bullet$.

$$
\begin{array}{lcllcl}
\mathrm{slash.pal}^\bullet & = & \mathrm{paj}^\bullet & , \quad \mathrm{slash.tal}_c^\bullet & = & \mathrm{taj}_c^\bullet \\
\mathrm{slash.pil}^\bullet & = & \mathrm{pij}^\bullet & , \quad \mathrm{slash.til}_c^\bullet & = & \mathrm{tij}_c^\bullet \\
\mathrm{slash.pail}_{\alpha,\beta}^\bullet & = & \mathrm{paij}_{\alpha,\beta}^\bullet & , \quad \mathrm{slash.tail}_{c,\alpha,\beta}^\bullet & = & \mathrm{taij}_{c,\alpha,\beta}^\bullet
\end{array}
$$

$$
\begin{array}{lcllcl}
\mathrm{paj}^\bullet & = & \mathrm{expari.Pa}^\bullet & , \quad \mathrm{taj}_c^\bullet & = & \mathrm{expari.Qa}_c^\bullet \\
\mathrm{pij}^\bullet & = & \mathrm{expari.Pi}^\bullet & , \quad \mathrm{tij}_c^\bullet & \neq & \mathrm{expari.Qi}_c^\bullet \\
\mathrm{paij}_{\alpha,\beta}^\bullet & = & \mathrm{expari.Pai}_{\alpha,\beta}^\bullet & , \quad \mathrm{taij}_{c,\alpha,\beta}^\bullet & = & \mathrm{expari.Qai}_{c,\alpha,\beta}^\bullet
\end{array}
$$

$$
\begin{array}{lclcl}
\mathrm{invgami.paj}^\bullet & \overset{\text{trivially}}{=} & \mathrm{invgani.anti.paj}^\bullet & \equiv & \mathrm{pari.pac}^\bullet \\
\mathrm{invgami.pij}^\bullet & \overset{\text{trivially}}{=} & \mathrm{invgani.anti.pij}^\bullet & \equiv & \mathrm{pari.pic}^\bullet \\
\mathrm{invgami.paij}_{\alpha,\beta}^\bullet & \overset{\text{trivially}}{=} & \mathrm{invgani.anti.paij}_{\alpha,\beta}^\bullet & \equiv & \mathrm{pari.paic}_{\alpha,\beta}^\bullet \\
\mathrm{invgami.taj}_c^\bullet & \overset{\text{trivially}}{=} & \mathrm{invgani.anti.taj}_c^\bullet & \not\equiv & \mathrm{pari.tac}_c^\bullet \\
\mathrm{invgami.tij}_c^\bullet & \overset{\text{trivially}}{=} & \mathrm{invgani.anti.tij}_c^\bullet & \not\equiv & \mathrm{pari.tic}_c^\bullet \\
\mathrm{invgami.taij}_{c,\alpha,\beta}^\bullet & \overset{\text{trivially}}{=} & \mathrm{invgani.anti.taij}_{c,\alpha,\beta}^\bullet & \equiv & \mathrm{pari.taic}_{c,\alpha,\beta}^\bullet
\end{array}
$$

4.5 Trigonometric and bitrigonometric bimoulds.

Correspondence between polar and trigonometric.
Polar bimoulds of a given type may have one trigonometric equivalent, or several, or none. The reverse correspondence, however, is always straightforward: when c goes to 0, (Qa_c, Qi_c) goes to (Pa, Pi) and the various trigonometric bimoulds, whenever they exist, go to their polar namesakes.

Correspondence between trigonometric and bitrigonometric.
The correspondence, here, is always one-to-one. This may come as a surprise, since the bitrigonometric units Qaa_c, Qii_c are far more complex than their trigonometric counterparts Qa_c, Qi_c. To turn a trigonomeric bimould of a given type into a bitrigonometric one of the same type, the recipe is:
– to change Qa_c resp. Qi_c into Qaa_c resp. Qii_c.
– to change $c^{2\,s}$ into $c^{2\,s}\delta(lin_1^{\boldsymbol{w}})\ldots\delta(lin_{2\,s}^{\boldsymbol{w}})$ with discrete diracs δ defined as in §3.2 (see after (3.33)) and with their arguments $lin_j^{\boldsymbol{w}}$ denoting suitable differences of v_i's or sums of u_i's, as the case may be. There are simple rules for picking, in each instance, the right inputs $lin_j^{\boldsymbol{w}}$, which alone preserve the symmetries. We shall see examples in the last para of the present section, when explicating the passage from *trigo* to *bitrigo* for the primary bimoulds.

The secondary bimoulds $tal_c^\bullet/til_c^\bullet$ and $taal_c^\bullet/tiil_c^\bullet$.
Of all the bimoulds constructed so far, these are the most important,[66] but also the most difficult to construct and describe. We can do no more here than state the main facts:
– the secondary bimoulds $\mathfrak{esz}_\sigma^\bullet$ have no trigonometric specialisation, whether under $\mathfrak{E} = Qa_c$ or $\mathfrak{E} = Qi_c$.
– the secondary bimould $\mathfrak{ess}^\bullet$ has no trigonometric specialisation under $\mathfrak{E} = Qa_c$, but it has one under $\mathfrak{E} = Qi_c$, namely $til_c^\bullet$, with $tal_c^\bullet$ as *swappee*.

In other words, while the polar pair $par^\bullet/pir^\bullet$ has no trigonometric, and therefore no bitrigonometric, counterpart, the polar pair $pal^\bullet/pil^\bullet$ does possess exact, though far more complex analogues, namely $tal_c^\bullet/til_c^\bullet$ and $taal_c^\bullet/tiil_c^\bullet$.

For illustration, the pair $taal_c^\bullet/tiil_c^\bullet$ has been tabulated in §12.10 up to length $r = 4$. The simpler pair $tal_c^\bullet/til_c^\bullet$ can be deduced from it, simply by recalibrating the flexion units and by changing all δ's into 1's.

Like $pil^\bullet$ in the polar case, the bisymmetral $til_c^\bullet$ and its *gari*-inverse $ritil_c^\bullet$ possess the important property of *separativity*: under the *gepar* transform[67] they turn into polynomials of c and the $Q_c(u_i)$ (all strict u_i-sums vanish!),

[66] because it is the main part of the first factor $Zag_I^\bullet$ in the trifactorisation of $Zag^\bullet$ and also the main ingredient of the canonical-rational associator.

[67] We recall that $gepar.S^\bullet := mu(anti.swap.S^\bullet, swap.S^\bullet)$.

with a particularly simple expression in the case of $ritil^\bullet_c$:

$$(\mathrm{gepar.til_c})^{w_1,\dots,w_r} = \textit{homog. polynomial in } (c, Q_c(u_1), ..., Q_c(u_r)) \quad (4.87)$$

$$(\mathrm{gepar.ritil_c})^{w_1,\dots,w_r} = \sum_{0\leq s\leq \frac{r}{2}} \frac{(-1)^s c^{2\,s}}{2\,s+1}\ \mathrm{sym}_{r-2\,s}(Q_c(u_1), ..., Q_c(u_r)) \quad (4.88)$$

with $sym_k(x_1,\dots,x_r)$ denoting the k-th symmetric function of the x_i.[68]

The primary bimoulds: trigonometric specialisation.
To explicate the primary bimoulds, we require six series of coefficients that are best defined by their generating series:

$$\begin{aligned}
\alpha(t) &= \arctan(t) & &= \textstyle\sum_{s\geq 0} \alpha_n\, t^{n+1} = t - \frac{1}{3}t^3 + \frac{1}{5}\,t^5 - \frac{1}{7}t^7 \dots\\
\beta(t) &= \tan(t) & &= \textstyle\sum_{s\geq 0} \beta_n\, t^{n+1} = t + \frac{1}{3}t^3 + \frac{2}{15}t^5 + \frac{17}{315}t^7 \dots\\
\hat\alpha(t) &= \tfrac{t}{(1+t^2)^{1/2}} & = t\,(\alpha'(t))^{\frac{1}{2}} &= \textstyle\sum_{s\geq 0} \hat\alpha_n\, t^{n+1} = t - \frac{1}{2}t^3 + \frac{3}{8}t^5 - \frac{5}{16}t^7 \dots\\
\hat\beta(t) &= \tfrac{t}{\cos(t)} & = t\,(\beta'(t))^{\frac{1}{2}} &= \textstyle\sum_{s\geq 0} \hat\beta_n\, t^{n+1} = t + \frac{1}{2}t^3 + \frac{5}{24}t^5 + \frac{61}{720}t^7 \dots\\
\check\alpha(t) &= \tfrac{\arctan(t)}{(1+t^2)^{-1/2}} & = \alpha(t)(\alpha'(t))^{-\frac{1}{2}} &= \textstyle\sum_{s\geq 0} \check\alpha_n\, t^{n+1} = t + \frac{1}{6}t^3 - \frac{11}{120}t^5 + \frac{103}{1680}t^7 \dots\\
\check\beta(t) &= \sin(t) & = \beta(t)(\beta'(t))^{-\frac{1}{2}} &= \textstyle\sum_{s\geq 0} \check\beta_n\, t^{n+1} = t - \frac{1}{6}t^3 + \frac{1}{120}t^5 - \frac{1}{5040}t^7 \dots
\end{aligned}$$

As in the polar case, the basic primary bimoulds $taj^\bullet_c, tij^\bullet_c$ (symmetral) derive from the secondary bimoulds $tal^\bullet_c, til^\bullet_c$ (bisymmetral) under the *slash*-tranform[69] and are best expressed via their *swappees*. To the polar pair $pac^\bullet/pic^\bullet$, however, there now correspond two trigonometric pairs, namely $tac^\bullet_c, tic^\bullet_c$ and the "correction" $tak^\bullet_c, tik^\bullet_c$ which will be needed to reproduce all the exact relations between primary bimoulds that obtained in the polar

[68] sym_0 is $\equiv 1$; sym_1 is the sum; sym_r is the product.
[69] We recall that $slash.S^\bullet := gari(neg.S^\bullet, invgari.S^\bullet)$.

case. Let us begin with the definitions. We have:

$$
\begin{aligned}
\mathrm{swap.taj}_c^{\boldsymbol{w}} &= \sum_{s\geq 0} \sum_{\boldsymbol{w^0} w_{n_1} \boldsymbol{w^1} \ldots w_{n_s} \boldsymbol{w^s} = \boldsymbol{w}} \mathrm{Ji}_*^{\boldsymbol{w^0}}\, \mathrm{Qi}_c^{w_{n_1}}\, \mathrm{Ji}^{\boldsymbol{w^1}} \ldots \mathrm{Qi}_c^{w_{n_s}}\, \mathrm{Ji}^{\boldsymbol{w^s}} \\
\mathrm{swap.tij}_c^{\boldsymbol{w}} &= \sum_{s\geq 0} \hat{\alpha}_{r-s} \sum_{\boldsymbol{w^0} w_{n_1} \boldsymbol{w^1} \ldots w_{n_s} \boldsymbol{w^s} = \boldsymbol{w}} \mathrm{Ja}^{\boldsymbol{w^0}} \mathrm{Qa}_c^{w_{n_1}} \mathrm{Ja}^{\boldsymbol{w^1}} \ldots \mathrm{Qa}_c^{w_{n_s}} \mathrm{Ja}^{\boldsymbol{w^s}} \\
\mathrm{tac}_c^{\boldsymbol{w}} &= \sum_{s\geq 0} \alpha_{r-s} \sum_{\boldsymbol{w^0} w_{n_1} \boldsymbol{w^1} \ldots w_{n_s} \boldsymbol{w^s} = \boldsymbol{w}} \mathrm{Ca}^{\boldsymbol{w^0}} \mathrm{Qa}_c^{w_{n_1}} \mathrm{Ca}^{\boldsymbol{w^1}} \ldots \mathrm{Qa}_c^{w_{n_s}} \mathrm{Ca}^{\boldsymbol{w^s}} \\
\mathrm{tic}_c^{\boldsymbol{w}} &= \sum_{s\geq 0} \sum_{\boldsymbol{w^0} w_{n_1} \boldsymbol{w^1} \ldots w_{n_s} \boldsymbol{w^s} = \boldsymbol{w}} \mathrm{Ci}^{\boldsymbol{w^0}}\, \mathrm{Qi}_c^{w_{n_1}}\, \mathrm{Ci}^{\boldsymbol{w^1}} \ldots \mathrm{Qi}_c^{w_{n_s}}\, \mathrm{Ci}^{\boldsymbol{w^s}} \\
\mathrm{tak}_c^{\boldsymbol{w}} &= \sum_{s\geq 0} \check{\alpha}_{r-s} \sum_{\boldsymbol{w^0} w_{n_1} \boldsymbol{w^1} \ldots w_{n_s} \boldsymbol{w^s} = \boldsymbol{w}} \mathrm{Ka}^{\boldsymbol{w^0}} \mathrm{Qa}_c^{w_{n_1}} \mathrm{Ka}^{\boldsymbol{w^1}} \ldots \mathrm{Qa}_c^{w_{n_s}} \mathrm{Ka}_*^{\boldsymbol{w^s}} \\
\mathrm{tik}_c^{\boldsymbol{w}} &= \check{\beta}_r\, c^r \qquad \textit{if} \;\; \boldsymbol{w} = (w_1, \ldots, w_r)
\end{aligned}
$$

with auxiliary building blocks themselves defined by:

$$
\begin{array}{lllll}
\mathrm{Ja}^{w_1,\ldots,w_r} & = & \mathrm{Ca}^{w_1,\ldots,w_r} & := c^r & (\forall r \geq 0) \\
\mathrm{Ji}^{w_1,\ldots,w_r} & = & \mathrm{Ci}^{w_1,\ldots,w_r} & := c^r\, \beta_r & (\forall r \geq 0) \\
\mathrm{Ji}_*^{w_1,\ldots,w_r} & := & c^r\, \hat{\beta}_r & & (\forall r \geq 0) \\
\mathrm{Ka}^{w_1,\ldots,w_r} & := & c^r & & (\forall r \geq 0) \\
\mathrm{Ka}_*^{w_1,\ldots,w_r} & := & c^r & (\forall r \geq 1) \quad \textit{but} & \mathrm{Ka}_*^{\emptyset} := 0
\end{array}
$$

Here are some of the main trigonometric identities that are exact transpositions of their polar prototypes:

$$
\begin{aligned}
\mathrm{slash.tal}_c^{\bullet} &= \mathrm{taj_c}^{\bullet} && (4.89) \\
\mathrm{slash.til}_c^{\bullet} &= \mathrm{tij_c}^{\bullet} && (4.90) \\
\mathrm{invgani.tac}_c^{\bullet} &= \mathrm{anti.swap.anti.pari.tic}_c^{\bullet} && (4.91) \\
\mathrm{invgani.tic}_c^{\bullet} &= \mathrm{anti.swap.anti.pari.tac}_c^{\bullet} && (4.92) \\
\mathrm{invgami.taj}_c^{\bullet} &= \mathrm{invgani.anti.taj}_c^{\bullet} && (4.93) \\
\mathrm{invgami.tij}_c^{\bullet} &= \mathrm{invgani.anti.tij}_c^{\bullet} && (4.94)
\end{aligned}
$$

And here is an example when polar identities:

$$
\begin{aligned}
\mathrm{invmu.paj}^{\bullet} \overset{\text{trivially}}{=} \mathrm{pari.anti.paj}^{\bullet} &= \mathrm{invgani.pac}^{\bullet} && (4.95) \\
\mathrm{invmu.pij}^{\bullet} \overset{\text{trivially}}{=} \mathrm{pari.anti.pij}^{\bullet} &= \mathrm{invgani.pic}^{\bullet} && (4.96)
\end{aligned}
$$

require a corrective term in the trigonometric transposition:

$$
\begin{aligned}
\mathrm{invmu.taj}_c^{\bullet} \overset{\text{trivially}}{=} \mathrm{pari.anti.taj}_c^{\bullet} &= \mathrm{fragani}(\mathrm{tak}_c^{\bullet}, \mathrm{tac}_c^{\bullet}) && (4.97) \\
\mathrm{invmu.tij}_c^{\bullet} \overset{\text{trivially}}{=} \mathrm{pari.anti.tij}_c^{\bullet} &= \mathrm{fragani}(\mathrm{tik}_c^{\bullet}, \mathrm{tic}_c^{\bullet}) && (4.98)
\end{aligned}
$$

The abbreviation *fragani* denotes of course the *gani*-fraction:

$$\mathrm{fragani}(A^\bullet, B^\bullet) := \mathrm{gani}(A^\bullet, \mathrm{invgani}.B^\bullet)$$

and the relations (4.97), (4.98) basically reflect the functional identities:

$$\hat{\beta} = \check{\alpha} \circ \alpha \qquad ; \qquad \hat{\alpha} = \check{\beta} \circ \beta$$

Here is another example. The important polar identity:

$$\mathrm{pij}^\bullet = \mathrm{expari.Pi}^\bullet$$

doesn't transpose to $\mathrm{tij}_c^\bullet = \mathrm{expari.Qi}_c^\bullet$ but to the variant:

$$\mathrm{tijj}_c^\bullet = \mathrm{expari.Qi}_c^\bullet \qquad (\mathrm{anti.swap.tijj}^\bullet =: \mathrm{astajj})$$

with a bimould $tijj_c^\bullet$ best defined via its *anti.swap*-transform $astajj_c^\bullet$, for which the following remarkable expansion holds:

$$\mathrm{astajj}_c^{w_1,\dots,w_r} = \sum_{0\leq t\leq \frac{r}{2}} (-1)^t\, c^{2\,t} \sum_{\boldsymbol{w^0} w_{n_1} \boldsymbol{w^1} \dots w_{n_s} \boldsymbol{w^s} = \boldsymbol{w}}^{s=r-2\,t} \mathrm{Ta}^{m_1,m_2\dots,m_{2\,t}}\, Qa^{w_{n_1}} \dots Qa^{w_{n_s}}$$

with

$$[m_1, m_2, \dots, m_{2\,t}] := [1, 2, \dots, r] \stackrel{\bullet}{-} [n_1, n_2, \dots, n_s]$$

and

$$\mathrm{Ta}^{m_1,m_2\dots,m_{2\,t}} := \frac{m_1}{m_2}\frac{m_3}{m_4}\dots\frac{m_{2\,t-1}}{m_{2\,t}}$$

Primary bimoulds: bitrigonometric specialisation.
The bimoulds of the preceding para become:

$$\mathrm{swap.taaj}_c^{\boldsymbol{w}} = \sum_{s\geq 0} \sum_{\boldsymbol{w^0} w_{n_1} \boldsymbol{w^1} \dots w_{n_s} \boldsymbol{w^s} = \boldsymbol{w}} \mathrm{Jii}_*^{\boldsymbol{w^0}}\, \mathrm{Qii}_c^{w_{n_1}}\, \mathrm{Jii}^{\boldsymbol{w^1}} \dots \mathrm{Qii}_c^{w_{n_s}}\, \mathrm{Jii}^{\boldsymbol{w^s}}$$

$$\mathrm{swap.tiij}_c^{\boldsymbol{w}} = \sum_{s\geq 0} \hat{\alpha}_{r-s} \sum_{\boldsymbol{w^0} w_{n_1} \boldsymbol{w^1} \dots w_{n_s} \boldsymbol{w^s} = \boldsymbol{w}} \mathrm{Jaa}^{\boldsymbol{w^0}}\, \mathrm{Qaa}_c^{w_{n_1}}\, \mathrm{Jaa}^{\boldsymbol{w^1}} \dots \mathrm{Qaa}_c^{w_{n_s}}\, \mathrm{Jaa}^{\boldsymbol{w^s}}$$

$$\mathrm{taac}_c^{\boldsymbol{w}} = \sum_{s\geq 0} \alpha_{r-s} \sum_{\boldsymbol{w^0} w_{n_1} \boldsymbol{w^1} \dots w_{n_s} \boldsymbol{w^s} = \boldsymbol{w}} \mathrm{Caa}^{\boldsymbol{w^0}}\, \mathrm{Qaa}_c^{w_{n_1}}\, \mathrm{Caa}^{\boldsymbol{w^1}} \dots \mathrm{Qaa}_c^{w_{n_s}}\, \mathrm{Caa}^{\boldsymbol{w^s}}$$

$$\mathrm{tiic}_c^{\boldsymbol{w}} = \sum_{s\geq 0} \sum_{\boldsymbol{w^0} w_{n_1} \boldsymbol{w^1} \dots w_{n_s} \boldsymbol{w^s} = \boldsymbol{w}} \mathrm{Cii}^{\boldsymbol{w^0}}\, \mathrm{Qii}_c^{w_{n_1}}\, \mathrm{Cii}^{\boldsymbol{w^1}} \dots \mathrm{Qii}_c^{w_{n_s}}\, \mathrm{Cii}^{\boldsymbol{w^s}}$$

$$\mathrm{taak}_c^{\boldsymbol{w}} = \sum_{s\geq 0} \check{\alpha}_{r-s} \sum_{\boldsymbol{w^0} w_{n_1} \boldsymbol{w^1} \dots w_{n_s} \boldsymbol{w^s} = \boldsymbol{w}} \mathrm{Kaa}^{\boldsymbol{w^0}}\, \mathrm{Qaa}_c^{w_{n_1}}\, \mathrm{Kaa}^{\boldsymbol{w^1}} \dots \mathrm{Qaa}_c^{w_{n_s}}\, \mathrm{Kaa}_*^{\boldsymbol{w^s}}$$

$$\mathrm{tiik}_c^{\boldsymbol{w}} = \check{\beta}_r\, c^r\, \delta(u_1) \dots \delta(u_r) \qquad \textit{if} \;\; \boldsymbol{w} = (w_1, \dots, w_r)$$

with elementary building blocks defined by:

$$\begin{array}{lclll}
\mathrm{Caa}^{w_1,\ldots,w_r} & = & \mathrm{Jaa}^{w_1,\ldots,w_r} & := \quad c^r\,\delta(v_1)\ldots\delta(v_r) & (\forall r\geq 0)\\
\mathrm{Cii}^{w_1,\ldots,w_r} & = & \mathrm{Jii}^{w_1,\ldots,w_r} & := \beta_r\,c^r\,\delta(u_1)\ldots\delta(u_r) & (\forall r\geq 0)\\
\mathrm{Jii}^{\emptyset}:=0 & , & \mathrm{Jii}_*^{w_1,\ldots,w_r} & := \hat{\beta}_r\,c^r\,\delta(u_1)\ldots\delta(u_r) & (\forall r\geq 1)\\
\mathrm{Kaa}^{\emptyset}:=1 & , & \mathrm{Kaa}^{w_1,\ldots,w_r} & := \quad c^r\,\delta(v_1)\ldots\delta(v_r) & (\forall r\geq 1)\\
\mathrm{Kaa}_*^{\emptyset}:=0 & , & \mathrm{Kaa}_*^{w_1,\ldots,w_r} & := \quad c^r\,\delta(v_1)\ldots\delta(v_r) & (\forall r\geq 1)
\end{array}$$

Remark 1: though there is one and only one 'proper' way of 'filling in' the trigonometric formulas with δ's to get the bitrigonometric equivalents, the procedure is non-trivial. Indeed, the arguments inside the δ's are not always single u_i's or v_i's but often non-trivial sums or differences.[70]

Remark 2: The even-odd factorisations (4.45),(4.46) have their exact counterpart here. Thus, in trigonometric mode:

$$til^\bullet = \mathrm{gari}(til_*^\bullet, til_{**}^\bullet) \qquad \textit{with } til_*^\bullet, til_{**}^\bullet \textit{ symmetral} \tag{4.99}$$

$$tal^\bullet = \mathrm{mu}(tal_{\star\star}^\bullet, tal_\star^\bullet) \qquad \textit{with } tal_\star^\bullet, tal_{\star\star}^\bullet \textit{ symmetral} \tag{4.100}$$

with elementary factors $tal_\star^\bullet$, $til_*^\bullet$ alongside non-elementary factors $tal_{\star\star}^\bullet$, $til_{**}^\bullet$ that carry only even-lengthed components.

4.6 Dimorphic isomorphisms in universal mode.

We can now enunciate the main statement of the whole section, namely that there exists a canonical isomorphism between *straight* dimorphic structures (algebras or groups) and their *twisted* counterparts.[71] But before that, we must begin with the less remarkable isomorphisms which connect *straight* or *twisted* monomorphic structures[72] and exchange only *one* symmetry with another.

All these results are summarised in the following diagrams
– with various groups in the upper lines,
– with various Lie algebras in the lower lines,
– with horizontal arrows that stand for (algebra or group) isomorphisms.
– with vertical arrows representing the natural exponential mapping of each Lie algebra into its group.

[70] As with $taaj_c^\bullet$ and $tiij_c^\bullet$, once we carry out the *swap* transform in the above definitions.

[71] Or, more properly, "half-twisted", since the first symmetry remains straight, and only the second gets twisted.

[72] i.e. subgroups of $MU := \{BIMU^*, mu\}$ or subalgebras of $LU := \{BIMU_*, lu\}$.

Basic diagrams of monomorphic transport.

$$
\begin{array}{ccc|ccc}
\mathrm{MU}^{as} & \xrightarrow{\mathrm{ganit}(\mathfrak{ez}^\bullet)} & \mathrm{MU}^{\mathfrak{es}} & \mathrm{MU}^{as} & \xleftarrow{\mathrm{ganit}(\mathrm{pari.anti}.\mathfrak{es}^\bullet)} & \mathrm{MU}^{\mathfrak{es}} \\
\uparrow \mathrm{expmu} & & \uparrow \mathrm{expmu} & \uparrow \mathrm{expmu} & & \uparrow \mathrm{expmu} \\
\mathrm{LU}^{al} & \xrightarrow{\mathrm{ganit}(\mathfrak{ez}^\bullet)} & \mathrm{LU}^{\mathfrak{el}} & \mathrm{LU}^{al} & \xleftarrow{\mathrm{ganit}(\mathrm{pari.anti}.\mathfrak{es}^\bullet)} & \mathrm{LU}^{\mathfrak{el}}
\end{array}
$$

$$
\begin{array}{ccc|ccc}
\mathrm{MU}^{as} & \xrightarrow{\mathrm{gamit}(\mathfrak{ez}^\bullet)} & \mathrm{MU}^{\mathfrak{es}} & \mathrm{MU}^{as} & \xleftarrow{\mathrm{gamit}(\mathrm{pari}.\mathfrak{es}^\bullet)} & \mathrm{MU}^{\mathfrak{es}} \\
\uparrow \mathrm{expmu} & & \uparrow \mathrm{expmu} & \uparrow \mathrm{expmu} & & \uparrow \mathrm{expmu} \\
\mathrm{LU}^{al} & \xrightarrow{\mathrm{gamit}(\mathfrak{ez}^\bullet)} & \mathrm{LU}^{\mathfrak{el}} & \mathrm{LU}^{al} & \xleftarrow{\mathrm{gamit}(\mathrm{pari}.\mathfrak{es}^\bullet)} & \mathrm{LU}^{\mathfrak{el}}
\end{array}
$$

Basic diagram of dimorphic transport.

$$
\begin{array}{ccc}
\mathrm{GARI}^{\underline{as}/\underline{as}} & \xrightarrow{\mathrm{adgari}(\mathfrak{ess}^\bullet)} & \mathrm{GARI}^{\underline{as}/\underline{\mathfrak{os}}} \\
\mathrm{logari} \downarrow\uparrow \mathrm{expari} & & \mathrm{logari} \downarrow\uparrow \mathrm{expari} \\
\mathrm{ARI}^{\underline{al}/\underline{al}} & \xrightarrow{\mathrm{adari}(\mathfrak{ess}^\bullet)} & \mathrm{ARI}^{\underline{al}/\underline{\mathfrak{ol}}}
\end{array}
$$

Dimorphic subsymmetries.
The subsymmetries listed below are by no means the only ones[73] but they are the ones that matter most and also (whether coincidentally or not) the only ones that are properly dimorphic.[74]

$$
\begin{array}{lcccccc}
A^\bullet \in \mathrm{ARI}^{\underline{al}/\underline{al}} & \Longrightarrow & A^\bullet & = & \mathrm{neg}.A^\bullet & = & \mathrm{push}.A^\bullet \\
A^\bullet \in \mathrm{GARI}^{\underline{as}/\underline{as}} & \Longrightarrow & A^\bullet & = & \mathrm{neg}.A^\bullet & = & \mathrm{gush}.A^\bullet \\
A^\bullet \in \mathrm{ARI}^{\underline{al}/\underline{\mathfrak{ol}}} & \Longrightarrow & A^\bullet & = & \mathfrak{D}\text{-}\mathrm{neg}.A^\bullet & = & \mathfrak{D}\text{-}\mathrm{push}.A^\bullet \\
A^\bullet \in \mathrm{GARI}^{\underline{as}/\underline{\mathfrak{os}}} & \Longrightarrow & A^\bullet & = & \mathfrak{D}\text{-}\mathrm{geg}.\mathrm{A}^\bullet & = & \mathfrak{D}\text{-}\mathrm{gush}.A^\bullet
\end{array}
$$

As noted earlier, $\mathfrak{D}$-*neg*-invariance is expressible in terms of an elementary *primary* bimould $\mathfrak{es}^\bullet := slash.\mathfrak{ess}^\bullet$, and $\mathfrak{D}$-*push*-invariance also is equivalent to the much simpler *senary relation.*

4.7 Dimorphic isomorphisms in polar mode.

Diagrams of monomorphic transport.
For the specialisation $\mathfrak{E} = \mathrm{Pa}$, the first universal diagrams of monomorphic

[73] See §3.4.

[74] in the sense that it takes *two* symmetries, not *one*, to induce them.

transport become :

$$
\begin{array}{ccc|ccc}
\mathrm{MU}^{\mathrm{as}} & \xrightarrow{\mathrm{ganit}(\mathrm{pac}^\bullet)} & \mathrm{MU}^{\mathrm{us}} & \mathrm{MU}^{\mathrm{as}} & \xleftarrow{\mathrm{ganit}(\mathrm{pari.anti.paj}^\bullet)} & \mathrm{MU}^{\mathrm{us}} \\
\uparrow \mathrm{expmu} & & \uparrow \mathrm{expmu} & \uparrow \mathrm{expmu} & & \uparrow \mathrm{expmu} \\
\mathrm{LU}^{\mathrm{al}} & \xrightarrow{\mathrm{ganit}(\mathrm{pac}^\bullet)} & \mathrm{LU}^{\mathrm{ul}} & \mathrm{LU}^{\mathrm{al}} & \xleftarrow{\mathrm{ganit}(\mathrm{pari.anti.paj}^\bullet)} & \mathrm{LU}^{\mathrm{ul}}
\end{array}
$$

For the specialisation $\mathfrak{E} = \mathrm{Pi}$, they become :

$$
\begin{array}{ccc|ccc}
\mathrm{MU}^{\mathrm{as}} & \xrightarrow{\mathrm{ganit}(\mathrm{pic}^\bullet)} & \mathrm{MU}^{\mathrm{is}} & \mathrm{MU}^{\mathrm{as}} & \xleftarrow{\mathrm{ganit}(\mathrm{pari.anti.pij}^\bullet)} & \mathrm{MU}^{\mathrm{is}} \\
\uparrow \mathrm{expmu} & & \uparrow \mathrm{expmu} & \uparrow \mathrm{expmu} & & \uparrow \mathrm{expmu} \\
\mathrm{LU}^{\mathrm{al}} & \xrightarrow{\mathrm{ganit}(\mathrm{pic}^\bullet)} & \mathrm{LU}^{\mathrm{il}} & \mathrm{LU}^{\mathrm{al}} & \xleftarrow{\mathrm{ganit}(\mathrm{pari.anti.pij}^\bullet)} & \mathrm{LU}^{\mathrm{il}}
\end{array}
$$

Diagrams of dimorphic transport.
For the specialisation $(\mathfrak{E}, \mathfrak{O}) = (\mathrm{Pa}, \mathrm{Pi})$, the diagram of dimorphic transport becomes :

$$
\begin{array}{ccc}
\mathrm{GARI}^{\underline{\mathrm{as}}/\underline{\mathrm{as}}} & \xrightarrow{\mathrm{adgari}(\mathrm{pal}^\bullet)} & \mathrm{GARI}^{\underline{\mathrm{as}}/\underline{\mathrm{is}}} \\
\mathrm{logari} \downarrow\uparrow \mathrm{expari} & & \mathrm{logari} \downarrow\uparrow \mathrm{expari} \\
\mathrm{ARI}^{\underline{\mathrm{al}}/\underline{\mathrm{al}}} & \xrightarrow{\mathrm{adari}(\mathrm{pal}^\bullet)} & \mathrm{ARI}^{\underline{\mathrm{al}}/\underline{\mathrm{il}}}
\end{array}
$$

and the dimorphic subsymmetries become:

$$
\begin{array}{lllllll}
A^\bullet \in \mathrm{ARI}^{\underline{\mathrm{al}}/\underline{\mathrm{il}}} & \Longrightarrow & A^\bullet & = & \mathrm{negu}.A^\bullet & = & \mathrm{pushu}.A^\bullet \\
A^\bullet \in \mathrm{GARI}^{\underline{\mathrm{as}}/\underline{\mathrm{is}}} & \Longrightarrow & A^\bullet & = & \mathrm{gegu}.A^\bullet & = & \mathrm{gushu}.A^\bullet
\end{array}
$$

For the 'conjugate' specialisation $(\mathfrak{E}, \mathfrak{O}) = (\mathrm{Pi}, \mathrm{Pa})$, the diagram becomes:

$$
\begin{array}{ccc}
\mathrm{GARI}^{\underline{\mathrm{as}}/\underline{\mathrm{as}}} & \xrightarrow{\mathrm{adgari}(\mathrm{pil}^\bullet)} & \mathrm{GARI}^{\underline{\mathrm{as}}/\underline{\mathrm{us}}} \\
\mathrm{logari} \downarrow\uparrow \mathrm{expari} & & \mathrm{logari} \downarrow\uparrow \mathrm{expari} \\
\mathrm{ARI}^{\underline{\mathrm{al}}/\underline{\mathrm{al}}} & \xrightarrow{\mathrm{adari}(\mathrm{pil}^\bullet)} & \mathrm{ARI}^{\underline{\mathrm{al}}/\underline{\mathrm{ul}}}
\end{array}
$$

and the dimorphic subsymmetries become:

$$
\begin{array}{lllllll}
A^\bullet \in \mathrm{ARI}^{\underline{\mathrm{al}}/\underline{\mathrm{ul}}} & \Longrightarrow & A^\bullet & = & \mathrm{negi}.A^\bullet & = & \mathrm{pushi}.A^\bullet \\
A^\bullet \in \mathrm{GARI}^{\underline{\mathrm{as}}/\underline{\mathrm{us}}} & \Longrightarrow & A^\bullet & = & \mathrm{gegi}.A^\bullet & = & \mathrm{gushi}.A^\bullet
\end{array}
$$

The matter of 'entireness'.
A few comments are in order here, regarding the preservation, or otherwise,

of the *entire* character of bimoulds.[75]
(i) The simple symmetries *al* and *as* are compatible with entireness, and so are the double symmetries $\underline{al}/\underline{al}$ and $\underline{as}/\underline{as}$.
(ii) The twisted symmetries *il* and *is* are compatible with entireness, but *ul* and *us* are not.
(iii) However, even in second monomorphic diagram, when all four structures contain *entire* bimoulds and the isomorphism $ganit(pic^\bullet)$ might conceivably preserve *entireness*, it *does not*. The same holds when $ganit(pic^\bullet)$ is replaced by $gamit(pic^\bullet)$.
(iv) The (important) twisted double symmetries $\underline{al}/\underline{il}$ and $\underline{as}/\underline{is}$ are compatible with *entireness*, but the (less important) double symmetries $\underline{al}/\underline{ul}$ and $\underline{as}/\underline{us}$ are not.
(v) However, even in the first dimorphic diagram, where all four structures do contain *entire* bimoulds and when the isomorphism $adari(pal^\bullet)$ might conceivably preserve *entireness*, it *does not*.
(vi) The dimorphic subsymmetries induced by $\underline{al}/\underline{il}$ and $\underline{as}/\underline{is}$ (i.e. *negu*- and *pushu*-invariance, resp. *gegu*- and *gushu*-invariance), despite the massive involvement of 'poles', are compatible with *entireness*, whereas the dimorphic subsymmetries induced by $\underline{al}/\underline{ul}$ and $\underline{as}/\underline{us}$ (i.e. *negi*- and *pushi*- or *gushi*-invariance), are not. For the first dimorphic subsymmetries (of the *'neg'* sort), both the compatibility and incompatibility may be checked on the formulas:

$$\mathrm{negu}.B^\bullet = \mathrm{neg.adari(paj^\bullet)}.B^\bullet = \mathrm{adari(pari.paj^\bullet).neg}.B^\bullet \tag{4.101}$$
$$\mathrm{negi}.B^\bullet = \mathrm{neg.adari(pij^\bullet)}.B^\bullet = \mathrm{adari(pari.pij^\bullet).neg}.B^\bullet \tag{4.102}$$

For the first dimorphic subsymmetries (of the *'push'* sort), the compatibility resp. incompatibility may be checked on the *senary* relations:

$$\mathrm{teru}.B^\bullet = \mathrm{push.mantar.teru.mantar}.B^\bullet \tag{4.103}$$
$$\mathrm{teri}.B^\bullet = \mathrm{push.mantar.teri.mantar}.B^\bullet \tag{4.104}$$

which express *pushu*- resp. *pushi*-invariance in much simpler form, and involve the elementary, linear operators:

$$C^\bullet = \mathrm{teru}.B^\bullet \Longleftrightarrow C^{w_1,\dots,w_r} = B^{w_1,\dots,w_r} - B^{w_1,\dots,w_{r-1}}\,\mathrm{Pa}^{w_r} + B^{w_1,\dots,w_{r-1}\rceil}\,\mathrm{Pa}^{\lfloor w_r}$$
$$C^\bullet = \mathrm{teri}.B^\bullet \Longleftrightarrow C^{w_1,\dots,w_r} = B^{w_1,\dots,w_r} - B^{w_1,\dots,w_{r-1}}\,\mathrm{Pi}^{w_r} + B^{w_1,\dots,w_{r-1}\rceil}\,\mathrm{Pi}^{\lfloor w_r}$$

[75] i.e. their being *polynomials* or *entire functions* or *formal power series* of their $\boldsymbol{u}$-variables.

The six entire structures.
All the above remarks still hold, mutatis mutandis, when we replace the polar symmetries by their trigonometric counterparts (to be precisely defined in §11.4). Thus, whereas for the six fundamental structures we have the following commutative diagram, with all horizontal arrows denoting either group or algebra isomorphisms:

$$\begin{array}{ccccccc}
\mathrm{GARI}^{\underline{\mathrm{as}}/\underline{\mathrm{as}}} & \overset{\mathrm{adgari(pal^\bullet)}}{\longrightarrow} & \mathrm{GARI}^{\underline{\mathrm{as}}/\underline{\mathrm{is}}} & \overset{\mathrm{adgari(Zag_I{}^\bullet)}}{\longrightarrow} & \mathrm{GARI}^{\underline{\mathrm{as}}/\underline{\mathrm{iis}}} & \overset{\mathrm{adgari(tal^\bullet)}}{\longleftarrow} & \mathrm{GARI}^{\underline{\mathrm{as}}/\underline{\mathrm{as}}} \\
\uparrow \mathrm{expari} & & \uparrow \mathrm{expari} & & \uparrow \mathrm{expari} & & \uparrow \mathrm{expari} \\
\mathrm{ARI}^{\underline{\mathrm{al}}/\underline{\mathrm{al}}} & \overset{\mathrm{adari(pal^\bullet)}}{\longrightarrow} & \mathrm{ARI}^{\underline{\mathrm{al}}/\underline{\mathrm{il}}} & \overset{\mathrm{adari(Zag_I{}^\bullet)}}{\longrightarrow} & \mathrm{ARI}^{\underline{\mathrm{al}}/\underline{\mathrm{iil}}} & \overset{\mathrm{adari(tal^\bullet)}}{\longleftarrow} & \mathrm{ARI}^{\underline{\mathrm{al}}/\underline{\mathrm{al}}}
\end{array}$$

the picture changes when we add the requirement of entireness: the straight and twisted structures are no longer isomorphic[76] and only the middling isomorphisms $adari(Zag_I^\bullet)$ and $adgari(Zag_I^\bullet)$ between the twisted structures (polar and trigonometric) survives, as pictured in the following diagram:

$$\begin{array}{ccccccc}
\mathrm{GARI}^{\underline{\mathrm{as}}/\underline{\mathrm{as}}}_{\mathrm{ent}} & \overset{\mathrm{adgari(pal^\bullet)}}{\not\longrightarrow} & \mathrm{GARI}^{\underline{\mathrm{as}}/\underline{\mathrm{is}}}_{\mathrm{ent}} & \overset{\mathrm{adgari(Zag_I{}^\bullet)}}{\longrightarrow} & \mathrm{GARI}^{\underline{\mathrm{as}}/\underline{\mathrm{iis}}}_{\mathrm{ent}} & \overset{\mathrm{adgari(tal^\bullet)}}{\not\longleftarrow} & \mathrm{GARI}^{\underline{\mathrm{as}}/\underline{\mathrm{as}}}_{\mathrm{ent}} \\
\uparrow \mathrm{expari} & & \uparrow \mathrm{expari} & & \uparrow \mathrm{expari} & & \uparrow \mathrm{expari} \\
\mathrm{ARI}^{\underline{\mathrm{al}}/\underline{\mathrm{al}}}_{\mathrm{ent}} & \overset{\mathrm{adari(pal^\bullet)}}{\not\longrightarrow} & \mathrm{ARI}^{\underline{\mathrm{al}}/\underline{\mathrm{il}}}_{\mathrm{ent}} & \overset{\mathrm{adari(Zag_I{}^\bullet)}}{\longrightarrow} & \mathrm{ARI}^{\underline{\mathrm{al}}/\underline{\mathrm{iil}}}_{\mathrm{ent}} & \overset{\mathrm{adari(tal^\bullet)}}{\not\longleftarrow} & \mathrm{ARI}^{\underline{\mathrm{al}}/\underline{\mathrm{al}}}_{\mathrm{ent}}
\end{array}$$

The six entire and *v*-constant structures.
This applies in particular to the six important substructures below, whose bimoulds
– are power series of the upper indices u_i
– are constant in the lower indices v_i.
Here is the diagram, with self-explanatory notations:

$$\begin{array}{ccccccc}
\mathrm{ASAS} & \overset{\mathrm{adgari(pal^\bullet)}}{\not\longrightarrow} & \mathrm{ASIS} & \overset{\mathrm{adgari(Zag_I{}^\bullet)}}{\longrightarrow} & \mathrm{ASIIS} & \overset{\mathrm{adgari(tal^\bullet)}}{\not\longleftarrow} & \mathrm{ASAS} \\
\uparrow \mathrm{expari} & & \uparrow \mathrm{expari} & & \uparrow \mathrm{expari} & & \uparrow \mathrm{expari} \\
\mathrm{ALAL} & \overset{\mathrm{adari(pal^\bullet)}}{\not\longrightarrow} & \mathrm{ALIL} & \overset{\mathrm{adari(Zag_I{}^\bullet)}}{\longrightarrow} & \mathrm{ALIIL} & \overset{\mathrm{adari(tal^\bullet)}}{\not\longleftarrow} & \mathrm{ALAL}
\end{array}$$

The projector *cut*:

$$(\mathrm{cut}.M)^{\binom{u_1\ ,\dots,\ u_r}{v_1\ ,\dots,\ v_r}} := M^{\binom{u_1\ ,\dots,\ u_r}{0\ ,\dots,\ 0}} \tag{4.105}$$

clearly defines epimorphisms of
$ARI^{\underline{al}/\underline{al}}$, $GARI^{\underline{as}/\underline{as}}$, $ARI^{\underline{al}/\underline{il}}$, $GARI^{\underline{as}/\underline{is}}$, $ARI^{\underline{al}/\underline{iil}}$, $GARI^{\underline{as}/\underline{iis}}$

[76]neither under $adari(pal^\bullet)$, $adari(tal^\bullet)$, nor any conceivable replacement.

respectively onto

ALAL , *ASAS* , *ALIL* , *ASIS* , *ALIIL* , *ASIIS*

Now, all the bimoulds associated with *colourless multizetas*, happen to have lower indices v_i that are all $= 0$ as elements of $\mathbb{Q}/\mathbb{Z}$. We shall take advantage of the above property of *cut* to identify these bimoulds with their *cuttees*, i.e. to view them as $\boldsymbol{v}$-constant.

Central corrections.

For structures with a *twisted* double symmetry, instead of demanding that the exact *swappee* should display the second symmetry, we often relax the condition and simply demand that the *swappee* corrected[77] by a suitable *central element* should display that symmetry. Thus, under these relaxed conditions:

$$\begin{aligned} A^\bullet \in \mathrm{ARI}^{\underline{\mathrm{al}}/\underline{\mathrm{il}}} \quad &\Leftrightarrow \{A^\bullet \in \textit{alternal}; \quad \mathrm{swap}(A^\bullet + C_A^\bullet) \in \textit{alternil}\} \\ S^\bullet \in \mathrm{GARI}^{\underline{\mathrm{al}}/\underline{\mathrm{il}}} \quad &\Leftrightarrow \{S^\bullet \in \textit{symmetral}; \quad \mathrm{swap}(\mathrm{gari}(A^\bullet, C_S^\bullet)) \in \textit{symmetril}\} \end{aligned}$$

with $C_A^\bullet \in \mathrm{Center(ARI)}$ and $C_S^\bullet \in \mathrm{Center(GARI)}$.

The sets thus defined are still algebras or groups, albeit larger ones. In the case of the $\boldsymbol{v}$-constant family *ALIL*, *ASIS*, *ALIIL*, *ASIIS*, we shall *always* assume this relaxed definition for without the central corrections these sets would be *empty*.[78] Besides, the bimoulds $Zag^\bullet$ associated with the (coloured or uncoloured) multizetas also require a *central correction* to display their double symmetry.

5 Singulators, singulands, singulates.

At this point, we already have a valuable tool at our disposal, namely the operator $adari(pal^\bullet)$, which acts as an algebra isomorphism and respects double symmetries. What it doesn't do, though, is respect entireness: when applied to entire bimoulds of type $\underline{al}/\underline{al}$, it produces bimoulds that have the right type, in this case $\underline{al}/\underline{il}$, but with singularities at the origin. To remove these without destroying the double symmetry $\underline{al}/\underline{il}$, we require a universal machinery capable, roughly speaking, of producing all possible singularities of type $\underline{al}/\underline{il}$. Such a machinery is at hand. It consists of *singulators, singulands*, and *singulates.* The *singulators* are quite complex linear operators. The *singulands* are arbitrary entire bimoulds subject only to simple parity

[77] *additively* in the case of algebras; *multiplicatively* in the case of groups.

[78] For the structures *ALAL* and *ASAS*, on the other hand, central corrections are not required. In fact, allowing such corrections makes no difference at all, which again shows that the pairs *ALAL*//*ASAS* and *ALIL*//*ASIS* cannot be isomorphic.

constraints. Lastly, when acting on singulands, the singulators turn them into *singulates*, which are bimoulds of type $\underline{al}/\underline{il}$ and with singularities at the origin that are, so to speak, 'made to order', and capable of neutralising, by subtraction, any given, unwanted singularity of type $\underline{al}/\underline{il}$.

After some heuristics (destined to divest our construction of its 'contrived' character), we shall examine the singulators, first in universal mode, then in the relevant polar specialisation.

5.1 Some heuristics. Double symmetries and imparity.

Analytical definition of *sen*.

Let us first introduce a mapping $sen : (A^\bullet, S^\bullet) \mapsto B^\bullet$ that is:
– linear in $S^\bullet \in BIMU_1$
– quadrilinear in $A^\bullet \in BIMU^*$
– which turns *group-like* properties of $A^\bullet$ into *algebra-like* properties of $B^\bullet$
– whose action strongly depends on the *parity* properties of $A^\bullet, S^\bullet$.
Here goes the definition:

$$B^\bullet = \mathrm{sen}(A^\bullet).S^\bullet \Leftrightarrow 2\,B^{\boldsymbol{w}} = \sum_{w_i \boldsymbol{w^1} \boldsymbol{w^2} w_j \boldsymbol{w^3} \boldsymbol{w^4} \overset{\mathrm{circ}}{=} \boldsymbol{w^*}} A_1^{\boldsymbol{w^1}} A_2^{\boldsymbol{w^2}\rfloor} S^{\lceil w_j \rceil} A_3^{\lfloor \boldsymbol{w^3}} A_4^{\boldsymbol{w^4}} \quad (5.1)$$

$$\textit{with} \qquad \boldsymbol{w^*} = \mathrm{augment}(\boldsymbol{w}) \qquad \textit{and}$$

$$A_1^\bullet = \mathrm{anti}.A^\bullet\,,\ A_2^\bullet = A^\bullet\,,\ A_3^\bullet = \mathrm{pari.anti}.A^\bullet\,,\ A_4^\bullet = \mathrm{pari}.A^\bullet$$

with the *augment* $\boldsymbol{w^*}$ defined in the usual way:

$$\boldsymbol{w} = \binom{u_1\,,\ldots,\,u_r}{v_1\,,\ldots,\,v_r} \qquad \Rightarrow \qquad \boldsymbol{w^*} = \binom{[u_0]\,,\,u_1\,,\ldots,\,u_r}{[v_0]\,,\,v_1\,,\ldots,\,v_r}$$

with the redundant additional component w_0:

$$u_0 := -u_1 - u_2 \cdots - u_r \qquad , \qquad v_0 = 0$$

and with the circular summation rule amounting to the double summation

$$\sum_{0 \le i \le r} \qquad \sum_{w_i \boldsymbol{w^1} \boldsymbol{w^2} w_j \boldsymbol{w^3} \boldsymbol{w^4} = w_i w_{i+1} \ldots w_r w_0 w_1 \ldots w_{i-1}} \quad (5.2)$$

Main properties of *sen*.

Let $B^\bullet := sen(A^\bullet).S^\bullet$.

$\boldsymbol{P_1}$ **:** If r is even and $S^{-w_1} = S^{w_1}$ then $B^{w_1,\ldots,w_r} = 0$.

$\boldsymbol{P_2}$ **:** If r is odd and $S^{-w_1} = -S^{w_1}$ then $B^{w_1,\ldots,w_r} = 0$.

$\boldsymbol{P_3}$ **:** If $neg.A^\bullet = pari.A^\bullet$, then *sen* essentially commutes with *swap*:

$$\begin{aligned} \mathrm{swap.sen}(A^\bullet).S^\bullet &= -\mathrm{pari.sen}(\mathrm{swap}.A^\bullet).\mathrm{swap}.S^\bullet & (5.3)\\ &= +\mathrm{sen}(\mathrm{pari.swap}.A^\bullet).\mathrm{swap}.S^\bullet & (5.4) \end{aligned}$$

$\boldsymbol{P_4}$ **:** If $A^\bullet$ is *gantar*-invariant, then $B^\bullet$ is *mantar*-invariant.[79]

$\boldsymbol{P_5}$ **:** If $A^\bullet$ is symmetral, then $B^\bullet$ is alternal.

$\boldsymbol{P_6}$ **:** If $neg.A^\bullet = pari.A^\bullet$ and $A^\bullet$ is bisymmetral, then $B^\bullet$ is bialternal.

Compact definition of *sen.*
We may note that, for $A^\bullet$ symmetral, the analytical definition (5.1) of $sen(A^\bullet).S^\bullet$ can be rewritten in compact form as:

$$2\,\mathrm{sen}(A^\bullet).S^\bullet = \mathrm{pushinvar.mut}(\mathrm{pari}.A^\bullet).\mathrm{garit}(A^\bullet).S^\bullet \quad (\forall A^\bullet \in as) \qquad (5.5)$$

– with the linear mapping *pushinvar* of $\oplus_r BIMU_r$ onto $\oplus_r BIMU_r^{push}$:

$$\mathrm{pushinvar}.M^\bullet := \sum_{0\le k\le r} \mathrm{push}^k.M^\bullet \quad \text{if} \quad M^\bullet \in \mathrm{BIMU_r}$$

– with the anti-action $mut(A^\bullet)$ of *MU* on *BIMU*:

$$\mathrm{mut}(A^\bullet).M^\bullet := \mathrm{mu}(A_*^\bullet, M^\bullet, A^\bullet) \quad \textit{with} \quad A_*^\bullet = \mathrm{invmu}(A^\bullet) \qquad (5.6)$$

– with the anti-action $garit(A^\bullet)$ of *GARI* on *BIMU*, which is given by 2.37 but simplifies when $M^\bullet$ is of length 1:

$$(\mathrm{garit}(A^\bullet).M)^{\boldsymbol{w}} = \sum_{\boldsymbol{w^1} w_2 \boldsymbol{w^3} = \boldsymbol{w}} A^{\boldsymbol{w^1}\rfloor} M^{\lceil w_2 \rceil} A_*^{\lfloor \boldsymbol{w^3}} \quad \textit{if } M^\bullet \in \mathrm{BIMU_1} \qquad (5.7)$$

(Pay attention to the position of $A_*^\bullet$ on the *left* in the definition of $mut(A^\bullet)$ and on the *right* in that of $garit(A^\bullet)$. Nonetheless, we have *anti-actions* in both cases.)

5.2 Universal singulators $senk(\mathfrak{ess}^\bullet)$ and $seng(\mathfrak{es}^\bullet)$.

Let $\mathfrak{E}$ be the universal (exact) flexion unit, and let $\mathfrak{es}^\bullet$ (resp. $\mathfrak{ess}^\bullet$) be the primary (resp. secondary) bimould attached to $\mathfrak{E}$. Further, let us set:

$$\begin{aligned} \mathrm{neginvar} &:= \mathrm{id} + \mathrm{neg} & (5.8)\\ \mathrm{pushinvar} &:= \sum_{0\le r} (\mathrm{id} + \mathrm{push} + \mathrm{push}^2 + \ldots \mathrm{push}^r).\mathrm{leng}_r & (5.9) \end{aligned}$$

[79] We recall that *mantar* := −*pari.anti* and *gantar* := *invmu.pari.anti* with *invmu* denoting *inversion* with respect to the mould product *mu*.

(with $leng_r$ denoting the projector from $BIMU$ onto $BIMU_r$) and let us define *mut* as in (5.6) above, and *ganit*, *garit*, *adari*[80] as in §2.2.

One can then prove that the following two identities define one and the same operator $senk(\mathfrak{ess}^\bullet)$:

$$
\begin{aligned}
2\,\mathrm{senk}(\mathfrak{ess}^\bullet).S^\bullet &:= \mathrm{neginvar}.(\mathrm{adari}(\mathfrak{ess}^\bullet))^{-1}.\mathrm{mut}(\mathfrak{es}^\bullet).S^\bullet \qquad (5.10)\\
2\,\mathrm{senk}(\mathfrak{ess}^\bullet).S^\bullet &:= \mathrm{pushinvar}.\mathrm{mut}(\mathrm{neg}.\mathfrak{ess}^\bullet).\mathrm{garit}(\mathfrak{ess}^\bullet).S^\bullet \qquad (5.11)
\end{aligned}
$$

and, likewise, that the following two identities define one and the same operator $seng(\mathfrak{es}^\bullet)$:

$$
\begin{aligned}
2\,\mathrm{seng}(\mathfrak{es}^\bullet).S^\bullet &:= \big(\mathrm{id}+\mathrm{neg}.\mathrm{adari}(\mathfrak{es}^\bullet)\big).\mathrm{mut}(\mathfrak{es}^\bullet).S^\bullet \qquad (5.12)\\
2\,\mathrm{seng}(\mathfrak{es}^\bullet).S^\bullet &:= \mathrm{mut}(\mathfrak{es}^\bullet).S^\bullet + \mathrm{garit}(\mathfrak{es}^\bullet).\mathrm{neg}.S^\bullet\\
&\quad -\mathrm{arit}\big(\mathrm{garit}(\mathfrak{es}^\bullet).\mathrm{neg}.S^\bullet\big).\mathrm{logari}(\mathfrak{es}^\bullet) \qquad (5.13)
\end{aligned}
$$

The next identity shows how the two basic singulators $senk(\mathfrak{ess}^\bullet)$ and $seng(\mathfrak{es}^\bullet)$ are related; and the other two describe their near-commutation with the basic involution *swap*.

$$
\begin{aligned}
\mathrm{seng}(\mathfrak{es}^\bullet) &\equiv \mathrm{adari}(\mathfrak{ess}^\bullet).\mathrm{senk}(\mathfrak{ess}^\bullet) \qquad (5.14)\\
\mathrm{swap}.\mathrm{senk}(\mathfrak{ess}^\bullet) &\equiv \mathrm{senk}(\mathrm{neg}.\mathrm{swap}.\mathfrak{ess}^\bullet).\mathrm{swap} \qquad (5.15)\\
\mathrm{swap}.\mathrm{seng}(\mathfrak{es}^\bullet) &\equiv \mathrm{ganit}(\mathrm{syap}.\mathfrak{ez}^\bullet).\mathrm{seng}(\mathrm{syap}.\mathfrak{es}^\bullet).\mathrm{neg}.\mathrm{swap} \qquad (5.16)
\end{aligned}
$$

Thus, basically, under the impact of the involution *swap*, the inner argument of the singulators also undergoes an involution, namely *neg.swap* in the case of *senk*, and *syap* in the case of *seng*.

Without going into tedious details, let us point out that most of the properties listed above follow:
(i) from the the properties of *sen* (see §4.1)
(ii) from the fact that $senk(\mathfrak{ess}^\bullet).S^\bullet$, as defined by (5.11), is none other than $sen(\mathfrak{ess}^\bullet).S^\bullet$, as defined by (5.1) or (5.5).[81]
(iii) from the following identity, valid for any *push*-invariant bimould $M^\bullet$:

$$
\mathrm{swap}.\mathrm{adari}(\mathfrak{ess}^\bullet).M^\bullet \equiv \mathrm{ganit}(\mathrm{syap}.\mathfrak{ez}^\bullet).\mathrm{adari}(\mathrm{swap}.\mathfrak{ess}^\bullet).\mathrm{swap}.M^\bullet \qquad (5.17)
$$

5.3 Properties of the universal singulators.

The singulators $senk(\mathfrak{ess}^\bullet)$ and $seng(\mathfrak{es}^\bullet)$ do not yield remarkable results when acting on general bimoulds of $BIMU$, but they turn bimoulds of $BIMU_1$

[80] *adari* alone is an action; all the others are anti actions.

[81] Hint: use the fact that $\mathfrak{ess}^\bullet$ is on the one hand invariant under *pari.neg* and on the other of alternal (even bialternal) type, so that $invmu.\mathfrak{ess}^\bullet = pari.anti.\mathfrak{ess}^\bullet$.

into dimorphic bimoulds of type $\underline{al}/\underline{al}$ and $\underline{al}/\underline{\mathfrak{ol}}$ respectively. Thus:

$$\mathrm{senk}(\mathfrak{ess}^\bullet).S^\bullet \;\in\; \mathrm{ARI}^{\underline{al}/\underline{al}} \qquad \forall S^\bullet \in \mathrm{BIMU}_1 \tag{5.18}$$

$$\mathrm{seng}(\mathfrak{es}^\bullet).S^\bullet \;\in\; \mathrm{ARI}^{\underline{al}/\underline{\mathfrak{ol}}} \qquad \forall S^\bullet \in \mathrm{BIMU}_1 \tag{5.19}$$

For $senk(\mathfrak{ess}^\bullet)$, this follows from $senk(\mathfrak{ess}^\bullet) = sen(\mathfrak{ess}^\bullet)$ (because $\mathfrak{ess}^\bullet$ is symmetral, indeed bisymmetral) and then from (5.15). For $seng(\mathfrak{es}^\bullet)$, this follows from (5.14) or (5.16), on choice.

These two operators, however, are in a sense too 'global'. To really generate all possible *'dimorphic derivatives'* of bimoulds $S^\bullet$ in $BIMU_1$, we need to split $senk(\mathfrak{ess}^\bullet)$ and $seng(\mathfrak{es}^\bullet)$ into separate components with the help of the projectors $leng_r$ of $BIMU$ onto $BIMU_r$.

$$\mathrm{senk}(\mathfrak{ess}^\bullet) = \sum_{1\le r} \mathrm{senk}_r(\mathfrak{ess}^\bullet) \tag{5.20}$$

$$\mathrm{seng}(\mathfrak{es}^\bullet) = \sum_{1\le r} \mathrm{seng}_r(\mathfrak{ess}^\bullet) \quad (\textit{mark} : \textit{first } \mathfrak{es}^\bullet \text{ , } \textit{then } \mathfrak{ess}^\bullet \text{ !}) \tag{5.21}$$

with

$$\mathrm{senk}_r(\mathfrak{ess}^\bullet) := \mathrm{leng}_r.\mathrm{senk}(\mathfrak{ess}^\bullet) \tag{5.22}$$

$$\mathrm{seng}_r(\mathfrak{ess}^\bullet) := \mathrm{adari}(\mathfrak{ess}^\bullet).\mathrm{senk}_r(\mathfrak{ess}^\bullet) \tag{5.23}$$

$$= \mathrm{adari}(\mathfrak{ess}^\bullet).\mathrm{leng}_r.\mathrm{adari}(\mathfrak{ess}^\bullet)^{-1}.\mathrm{seng}(\mathfrak{es}^\bullet) \tag{5.24}$$

Although the decomposition runs on different lines[82] in both cases, the resulting components share the same dimorphy-inducing properties:

$$\mathrm{senk}_r(\mathfrak{ess}^\bullet).S^\bullet \;\in\; \mathrm{ARI}^{\underline{al}/\underline{al}} \cap \mathrm{BIMU}_r \qquad \forall S^\bullet \in \mathrm{BIMU}_1 \tag{5.25}$$

$$\mathrm{seng}_r(\mathfrak{ess}^\bullet).S^\bullet \;\in\; \mathrm{ARI}^{\underline{al}/\underline{\mathfrak{ol}}} \cap \mathrm{BIMU}_{r\le} \qquad \forall S^\bullet \in \mathrm{BIMU}_1 \tag{5.26}$$

with

$$\mathrm{BIMU}_{r\le} := \oplus_{r\le r'} \; \mathrm{BIMU}_{r'} \tag{5.27}$$

But beware: the r-indexation is slightly confusing since, as an operator acting on $BIMU_1$, $\mathrm{senk}_r(\mathfrak{ess}^\bullet)$ is $(r-1)$-linear in $\mathfrak{E}$. Moreover, $\mathfrak{E}^{w_1}$ is odd in w_1. As a consequence, $\mathrm{senk}_r(\mathfrak{ess}^\bullet).S^\bullet$ and therefore $\mathrm{seng}_r(\mathfrak{ess}^\bullet).S^\bullet$ automatically vanish in exactly two cases: when S^{w_1} and r are both *even* or both *odd*.[83]

Dimorphic elements in the monogenous algebra $Flex(\mathfrak{E})$.
The above results also apply, of course, within $Flex(\mathfrak{E})$, but since the only

[82]The components $seng_r(\mathfrak{ess}^\bullet)$ fully depend on $\mathfrak{ess}^\bullet$ whereas the global operator $seng(\mathfrak{es}^\bullet)$ only depends on $\mathfrak{es}^\bullet = slash.\mathfrak{ess}^\bullet$.

[83]In the obvious sense: i.e. H^{w_1} as a function of w_1, and r as an integer.

singuland in $Flex_1(\mathfrak{E})$ is, up to scalar multiplication, the unit $\mathfrak{E}^\bullet$, which is *odd*, we only get bialternal singulates in $Flex_{2r}(\mathfrak{E})$. Moreover, the singulate in $Flex_2(\mathfrak{E})$ vanishes, because it essentially reduces to $oddari(\mathfrak{E}^\bullet, \mathfrak{E}^\bullet)$ (see 2.80). To sum up:

$$\mathrm{senk}_{2r-1}(\mathfrak{ess}^\bullet).\mathfrak{E}^\bullet \;=\; 0 \quad \forall r \qquad ; \qquad \mathrm{senk}_2(\mathfrak{ess}^\bullet).\mathfrak{E}^\bullet \;=\; 0 \tag{5.28}$$

$$\mathrm{senk}_{2r}(\mathfrak{ess}^\bullet).\mathfrak{E}^\bullet \;\in\; \mathrm{ARI}^{\underline{\mathrm{al}}/\underline{\mathrm{al}}} \cap \mathrm{Flex}_{2r}(\mathfrak{E}) \;\; \textit{and} \;\; \neq 0 \;\; \textit{if} \;\; r \geq 2 \tag{5.29}$$

5.4 Polar singulators: description and properties.

There is little point in considering the unit specialisation $\mathfrak{E} \mapsto Pi$, since it leads to the symmetry types $\underline{al}/\underline{ul}$ and $\underline{as}/\underline{us}$ which, as already pointed out, are not compatible with entireness. That leaves the specialisation $\mathfrak{E} \mapsto Pa$ and the symmetry types $\underline{al}/\underline{il}$ and $\underline{as}/\underline{is}$ that go with it. For the bisymmetral bimould, it induces the straightforward specialisation $\mathfrak{ess}^\bullet \mapsto par^\bullet$, but instead of $par^\bullet$ we may also consider $pal^\bullet$, which in fact turns out to be more convenient. This, however, has no impact on the specialisation *sang* of $seng(\mathfrak{es}^\bullet) = seng(slash.\mathfrak{ess}^\bullet)$ since $slash.par^\bullet = slash.pal^\bullet = paj^\bullet$. The definitions of §4.2 become:

$$\begin{aligned} 2\,\mathrm{sang}.S^\bullet \;&:=\; \big(\mathrm{id} + \mathrm{neg.adari}(\mathrm{paj}^\bullet)\big).\mathrm{mut}(\mathrm{paj}^\bullet).S^\bullet && (5.30)\\ &=\; \mathrm{mut}(\mathrm{paj}^\bullet).S^\bullet + \mathrm{garit}(\mathrm{paj}^\bullet).\mathrm{neg}.S^\bullet \\ &\quad\; -\mathrm{arit}\big(\mathrm{garit}(\mathrm{paj}^\bullet).\mathrm{neg}.S^\bullet\big).\mathrm{logari}(\mathrm{paj}^\bullet) && (5.31)\end{aligned}$$

and the equivalence between these two definitions is relatively easy to check, based on the fact that the bimoulds $vipaj^\bullet$ and $vimupaj^\bullet$ thus defined:

$$\mathrm{vipaj}^\bullet := \mathrm{adari}(\mathrm{paj}^\bullet).\mathrm{paj}^\bullet \quad , \quad \mathrm{vimupaj}^\bullet := \mathrm{adari}(\mathrm{paj}^\bullet).\mathrm{mupaj}^\bullet$$

admit the following expressions:

$$\begin{aligned} \mathrm{vipaj}^{w_1,\dots,w_r} \;&=\; (-1)^{r-1}\mathrm{mupaj}^{w_1,\dots,w_{r-1}}\, P(u_1 + \ldots + u_r)\\ \mathrm{vimupaj}^{w_1,\dots,w_r} \;&=\; (-1)^{r} \quad \mathrm{paj}^{w_2,\dots,w_r}\, P(u_1 + \ldots + u_r)\end{aligned}$$

This in turn enables us to recast definition (5.31) in more direct form:

$$\begin{aligned}
2\,(\mathrm{sang}.S)^{\boldsymbol{w}} \;=\; &+\sum_{\boldsymbol{a}\,w_i\boldsymbol{b}=\boldsymbol{w}} \mathrm{mupaj}^{\boldsymbol{a}}\, S^{w_i}\, \mathrm{paj}^{\boldsymbol{b}} \\
&+\sum_{\boldsymbol{a}\,w_i\boldsymbol{b}=\boldsymbol{w}} \mathrm{paj}^{\boldsymbol{a}\rfloor}\, (\mathrm{neg}.S)^{\lceil w_i\rceil}\, \mathrm{mupaj}^{\lfloor\boldsymbol{b}} \\
&+\sum_{\boldsymbol{a}\,w_i\boldsymbol{b}\,w_r=\boldsymbol{w}} \mathrm{paj}^{\boldsymbol{a}\rfloor}\, (\mathrm{neg}.S)^{\lceil w_i\rceil}\, \mathrm{mupaj}^{\lfloor\boldsymbol{b}}\, P(|\boldsymbol{u}|) \\
&-\sum_{w_1\boldsymbol{a}\,w_i\boldsymbol{b}=\boldsymbol{w}} \mathrm{paj}^{\boldsymbol{a}\rfloor}\, (\mathrm{neg}.S)^{\lceil w_i\rceil}\, \mathrm{mupaj}^{\lfloor\boldsymbol{b}}\, P(|\boldsymbol{u}|)
\end{aligned}$$

For the singulator *senk*($\mathfrak{ess}^\bullet$), however, we get two distinct specialisations *slank* and *srank*, based respectively on *pal*$^\bullet$ and *par*$^\bullet$:

$$\begin{aligned}
2\,\mathrm{slank}.S^\bullet &:= \mathrm{neginvar}.(\mathrm{adari}(\mathrm{pal}^\bullet))^{-1}.\mathrm{mut}(\mathrm{pal}^\bullet).S^\bullet && (5.32)\\
&= \mathrm{pushinvar}.\mathrm{mut}(\mathrm{neg}.\mathrm{pal}^\bullet).\mathrm{garit}(\mathrm{pal}^\bullet).S^\bullet && (5.33)
\end{aligned}$$

$$\begin{aligned}
2\,\mathrm{srank}.S^\bullet &:= \mathrm{neginvar}.(\mathrm{adari}(\mathrm{par}^\bullet))^{-1}.\mathrm{mut}(\mathrm{par}^\bullet).S^\bullet && (5.34)\\
&= \mathrm{pushinvar}.\mathrm{mut}(\mathrm{neg}.\mathrm{par}^\bullet).\mathrm{garit}(\mathrm{par}^\bullet).S^\bullet && (5.35)
\end{aligned}$$

Both *slank* and *srank* relate to *sang* under the predictable formulas:

$$\mathrm{sang} = \mathrm{adari}(\mathrm{pal}^\bullet).\mathrm{slank} = \mathrm{adari}(\mathrm{par}^\bullet).\mathrm{srank} \qquad (5.36)$$

and both *slank* and *srank* (resp. *sang*) turn arbitrary singulands $S^\bullet \in BIMU_1$ into dimorphic singulates of type $\underline{al}/\underline{al}$ (resp. $\underline{al}/\underline{il}$).

5.5 Simple polar singulators.

The polar singulators, like their universal models, have to be broken down into their constituent parts. For *slank* and *srank*, the formulas are straightforward:

$$\begin{aligned}
\mathrm{slank}_r &:= \mathrm{leng}_r.\mathrm{slank} && (5.37)\\
\mathrm{srank}_r &:= \mathrm{leng}_r.\mathrm{srank} && (5.38)
\end{aligned}$$

For *sang*, the decomposition is more roundabout, and depends on the choice of either *pal*$^\bullet$ or *par*$^\bullet$:

$$\begin{aligned}
\mathrm{slang}_r &:= \mathrm{adari}(\mathrm{pal}^\bullet).\mathrm{leng}_r.\big(\mathrm{adari}(\mathrm{pal}^\bullet)\big)^{-1}.\mathrm{sang} && (5.39)\\
&= \mathrm{adari}(\mathrm{pal}^\bullet).\mathrm{slank}_r \;\neq\; \mathrm{leng}_r.\mathrm{sang} && (5.40)\\
\mathrm{srang}_r &:= \mathrm{adari}(\mathrm{par}^\bullet).\mathrm{leng}_r.\big(\mathrm{adari}(\mathrm{par}^\bullet)\big)^{-1}.\mathrm{sang} && (5.41)\\
&= \mathrm{adari}(\mathrm{par}^\bullet).\mathrm{srank}_r \;\neq\; \mathrm{leng}_r.\mathrm{sang} && (5.42)
\end{aligned}$$

Thus, despite the similar-looking identities

$$\text{slank} = \sum_{r\geq 1} \text{slank}_r \quad , \quad \text{srank} = \sum_{r\geq 1} \text{srank}_r \quad , \quad \text{sang} = \sum_{r\geq 1} \text{slang}_r = \sum_{r\geq 1} \text{srang}_r$$

there is no way we can avoid *secondary* bimoulds (in this case, the bisymmetral $pal^\bullet$ or $par^\bullet$) even in the decomposition of the 'primary-looking' singulator *sang*.

5.6 Composite polar singulators.

To produce all possible dimorphic singularities, we require not just the singulator components, but also their Lie brackets. For reasons that shall be spelt out in §4.7, we settle for the choice $pal^\bullet$ and the corresponding singulators, and we set, for any arguments $S_1^\bullet, ..., S_l^\bullet$ in $BIMU_1$:

$$\text{slank}_{[r_1,\ldots,r_l]} \,.\, \text{mu}(S_1^\bullet,\ldots,S_l^\bullet) \;:=\; \text{ari}(\text{slank}_{r_1}.S_1^\bullet,\ldots,\text{slank}_{r_l}.S_l^\bullet) \;\in\; \text{ARI}_r^{\underline{\text{al}}/\underline{\text{al}}}$$
$$\text{slang}_{[r_1,\ldots,r_l]} \,.\, \text{mu}(S_1^\bullet,\ldots,S_l^\bullet) \;:=\; \text{ari}(\text{slang}_{r_1}.S_1^\bullet,\ldots,\text{slang}_{r_l}.S_l^\bullet) \;\in\; \text{ARI}_{r\leq}^{\underline{\text{al}}/\underline{\text{il}}}$$

with $r := r_1 + \ldots + r_l$ and of course:

$$\text{ARI}_r^{\underline{\text{al}}/\underline{\text{al}}} := \text{ARI}^{\underline{\text{al}}/\underline{\text{al}}} \cap \text{BIMU}_r \quad ; \quad \text{ARI}_{r\leq}^{\underline{\text{al}}/\underline{\text{il}}} := \text{ARI}^{\underline{\text{al}}/\underline{\text{il}}} \cap \big(\oplus_{r\leq r'} \text{BIMU}_{r'}\big)$$

and with the multiple *ari*-braket defined from left to right. By multilinearity, the above actions extend to mappings:

$$\text{slank}_{[r_1,\ldots,r_l]} \;:\; S^\bullet \mapsto \Sigma^\bullet \quad ; \quad \text{BIMU}_l \to \text{ARI}_r^{\underline{\text{al}}/\underline{\text{al}}} \tag{5.43}$$
$$\text{slang}_{[r_1,\ldots,r_l]} \;:\; S^\bullet \mapsto \Sigma^\bullet \quad ; \quad \text{BIMU}_l \to \text{ARI}_{r\leq}^{\underline{\text{al}}/\underline{\text{il}}} \tag{5.44}$$

It is sometimes convenient, nay indispensable,[84] to consider also the pre-Lie brackets of the singulator components. The formulas read:

$$\text{slank}_{r_1,\ldots,r_l} \,.\, \text{mu}(S_1^\bullet,\ldots,S_l^\bullet) \;:=\; \text{preari}(\text{slank}_{r_1}.S_1^\bullet,\ldots,\text{slank}_{r_l}.S_l^\bullet) \tag{5.45}$$
$$\text{slang}_{r_1,\ldots,r_l} \,.\, \text{mu}(S_1^\bullet,\ldots,S_l^\bullet) \;:=\; \text{preari}(\text{slang}_{r_1}.S_1^\bullet,\ldots,\text{slang}_{r_l}.S_l^\bullet) \tag{5.46}$$

with the multiple *pre-ari*-braket defined again from left to right, as in (2.49). By multilinearity, the above actions extend to mappings:

$$\text{slank}_{r_1,\ldots,r_l} \;:\; S^\bullet \mapsto \Sigma^\bullet \quad ; \quad \text{BIMU}_l \to \text{ARI}_r^{\text{al}} \tag{5.47}$$
$$\text{slang}_{r_1,\ldots,r_l} \;:\; S^\bullet \mapsto \Sigma^\bullet \quad ; \quad \text{BIMU}_l \to \text{ARI}_{r\leq}^{\text{al}} \tag{5.48}$$

[84] for example in *perinomal algebra*: see §6 and §8.

Here, the resulting singulates $\Sigma^\bullet$ are of course alternal, but their *swappees* exhibit no distinctive symmetry. In practical applications, however, these multiple singulators based on *preari* always occur in sums $\sum Q^\bullet slank_\bullet$ or $\sum Q^\bullet slang_\bullet$, with scalar moulds $Q^\bullet$ that are alternal (resp. symmetral), and these new composite operators *do* produce dimorphy: they turn arbitrary singulands $S^\bullet$ into singulates $\Sigma^\bullet$ of type $\underline{al}/\underline{al}$ or $\underline{al}/\underline{il}$ (resp. $\underline{as}/\underline{as}$ or $\underline{as}/\underline{is}$).

5.7 From $\underline{al}/\underline{al}$ to $\underline{al}/\underline{il}$. Nature of the singularities.

The reason for preferring the singulator *slank* (built from $pal^\bullet$) to the singulator *srank* (built from $par^\bullet$) is that it leads to simpler denominators. Indeed, for a singuland S^{w_1} regular at the origin and 'random', although the bialternal singulates $slank_r.S^{\boldsymbol{w}}$ and $srank_r.S^{\boldsymbol{w}}$, as functions of $\boldsymbol{w} = (w_1, ..., w_r)$, have both multipoles of order $r-1$ at the origin, the total number of factors differs sharply. After common denominator reduction, $slank_r.S^{\boldsymbol{w}}$ has only $r+1$ factors on its denominator, whereas $srank_r.S^{\boldsymbol{w}}$ has $r\,(r+1)/2$. More precisely:

$$\begin{aligned}
\mathrm{denom}(\mathrm{slank}_r.S^{\boldsymbol{w}}) &= u_0\, u_1 \dots u_{r-1}\, u_r \quad \textit{with} \;\; u_0 := -(u_1 + \cdots + u_r) \\
\mathrm{denom}(\mathrm{srank}_r.S^{\boldsymbol{w}}) &= \prod_{1\le i\le j\le r} \; \sum_{i\le k\le j} u_k
\end{aligned}$$

The results are slightly more complex for the singulates of type $\underline{al}/\underline{il}$, namely $slang_r.S^\bullet$ and $srang_r.S^\bullet$, since these, as a rule, possess non-vanishing components of any length $r' \ge r$, but here again the first choice leads to simpler denominators.

Another reason for preferring the $pal^\bullet$-based choice to the $par^\bullet$-based one is that $pal^\bullet$ possesses a trigonometric counterpart $tal^\bullet_c$ whereas $par^\bullet$ doesn't.

6 A natural basis for $ALIL \subset ARI^{\underline{al}/\underline{il}}$.

6.1 Singulation-desingulation: the general scheme.

This section is devoted to the construction of bimoulds $l\o ma^\bullet$ in *ALIL*. In other words:
– $l\o ma^{\boldsymbol{w}}$ should be $\boldsymbol{u}$-entire, i.e. in $\mathbb{C}[[u_1, \dots, u_r]]$.
– $l\o ma^{\boldsymbol{w}}$ should be $\boldsymbol{v}$-constant.
– $l\o ma^\bullet$ should be *alternal*.
– $l\o mi^\bullet := swap.l\o ma^\bullet$ should be *alternil* modulo *Center(ALIL)*
But we also add two key conditions:

(i) $l\o ma^{\boldsymbol{w}}$ should be in $\mathbb{Q}[[u_1,\dots,u_r]]$, i.e. carry rational Taylor coefficients.
(ii) the first component should be of the form:

$$l\o ma^{w_1} = u_1^2\,(1-u_1^2)^{-1} = u_1^2+u_1^4+u_1^6+u_1^8+\dots \tag{6.1}$$

Condition (ii) is there to ensure that in the iso-weight decomposition:

$$\mathrm{l\o ma}^\bullet = \mathrm{l\o ma}_3^\bullet + \mathrm{l\o ma}_5^\bullet + \mathrm{l\o ma}_7^\bullet + \mathrm{l\o ma}_9^\bullet + \dots \tag{6.2}$$

the part $l\o ma_s^\bullet$ of weight s be non-zero[85] and start with $l\o ma_s^{w_1} = u_1^{s-1}$, with the ultimate objective of getting a basis $\{l\o ma_s^\bullet\,;s\ odd \geq 3\}$ of $ALIL$.

The 'central correction' formula reads:

$$\mathrm{lomi}_s^\bullet = \mathrm{swap}(\mathrm{loma}_s^\bullet + \mathrm{Ca}_s^\bullet) \quad ; \quad \mathrm{Ca}_s^\bullet \in \mathrm{Center(ALIL)} \tag{6.3}$$

with a central bimould $Ca_s^\bullet$ which, due to condition (6.1), can be shown to be of the form:

$$\mathrm{Ca}_s^{w_1,\dots,w_s} = \frac{1}{s} \quad (\forall w_i) \quad ; \quad \mathrm{Ca}_s^{w_1,\dots,w_r} = 0 \quad if \quad r \neq s \quad (\forall w_i) \tag{6.4}$$

Expanding $l\o ma^\bullet$ into series of singulates.
Before decomposing $l\o ma_s^\bullet$ weight-by-weight, we must *construct* it as a series of singulates. There are actually two variants:

$$\mathrm{l\o ma}^\bullet = \overbrace{\Sigma^\bullet_{[1]}}^{r\leq 2} + \overbrace{\Sigma^\bullet_{[1,2]}}^{r\leq 4} + \overbrace{\Sigma^\bullet_{[1,4]} + \Sigma^\bullet_{[2,3]} + \Sigma^\bullet_{[1,1,3]} + \Sigma^\bullet_{[2,1,2]} + \Sigma^\bullet_{[1,1,1,2]}}^{r\leq 6} + \dots \tag{6.5}$$

$$\begin{aligned}\mathrm{l\o ma}^\bullet = {} & \overbrace{\Sigma^\bullet_{1}}^{r\leq 2} + \overbrace{\Sigma^\bullet_{1,2}+\Sigma^\bullet_{2,1}}^{r\leq 4} + \overbrace{\Sigma^\bullet_{1,4}+\Sigma^\bullet_{4,1}+\Sigma^\bullet_{2,3}+\Sigma^\bullet_{3,2}}^{r\leq 6} \\ & + \overbrace{\Sigma^\bullet_{1,1,3}+\Sigma^\bullet_{1,3,1}+\Sigma^\bullet_{3,1,1}+\Sigma^\bullet_{2,2,1}+\Sigma^\bullet_{2,1,2}+\Sigma^\bullet_{1,2,2}}^{r\leq 6} \\ & + \overbrace{\Sigma^\bullet_{1,1,1,2}+\Sigma^\bullet_{1,1,2,1}+\Sigma^\bullet_{1,2,1,1}+\Sigma^\bullet_{2,1,1,1}}^{r\leq 6} + \dots\end{aligned} \tag{6.6}$$

with

$$\Sigma^\bullet_{[r_1,\dots,r_l]} \ := \ \mathrm{slang}_{[r_1,\dots,r_l]}.S^\bullet_{[r_1,\dots,r_l]} \tag{6.7}$$
$$\Sigma^\bullet_{r_1,\dots,r_l} \ := \ \mathrm{slang}_{r_1,\dots,r_l}.S^\bullet_{r_1,\dots,r_l} \tag{6.8}$$

[85] s is odd ≥ 3. $l\o ma_s^\bullet$ (resp.$l\o mi_s^\bullet$) carries exactly $s-1$ (resp. s) nonzero components of length $r \in [1, s-1]$ (resp. $r \in [1,s]$) and degree $d = s-r$. Indeed, the last components are $l\o ma_s^{w_1,\dots,w_s} = 0$ and $l\o mi_s^{w_1,\dots,w_s} = 1/s$.

The *singulates* $\Sigma^{\bullet}_{[r_1,\ldots,r_l]}$ are going to be in $\mathrm{ARI}^{\underline{\mathrm{al}}/\underline{\mathrm{il}}}_{r\leq}$ but the *singulates* $\Sigma^{\bullet}_{r_1,\ldots,r_l}$ only in $\mathrm{ARI}^{\underline{\mathrm{al}}}_{r\leq}$. As for the *singulands* $S^{\bullet}_{[r_1,\ldots,r_l]}$ and $S^{\bullet}_{r_1,\ldots,r_l}$, they are merely in BIMU_l, but with a definite parity in each x_i, which is exactly opposite to the parity of r_i. Moreover, we can without loss of generality assume that they vanish as soon as one of the x_i's vanishes. Then again, they may be sought either in the form of power series or of meromorphic functions of a quite specific type:

$$S^{x_1,\ldots,x_l}_{[r_1,\ldots,r_l]} \in x_1^{\nu_1}\ldots x_l^{\nu_l}\,\mathbb{C}[[x_1^2,\ldots,x_l^2]] \qquad (\textit{power series}) \quad (6.9)$$
$$S^{x_1,\ldots,x_l}_{[r_1,\ldots,r_l]} = \sum_{n_i\in\mathbb{Z}^*} R^{n_1,\ldots,n_l}_{[r_1,\ldots,r_l]} P(n_1{+}x_1)\ldots P(n_l{+}x_l) \qquad (\textit{merom. funct.}) \quad (6.10)$$

with $\nu_i = 1$ (resp. 2) if r_i is even (resp. odd).

Both expansions (6.5) and (6.6) lead to the same results. The first expansion (6.5) relies on *ari*-brackets and has the advantage of involving fewer summands. The downside is that it forces us to choose a basis in the Lie algebra generated by the *simple* singulates $\Sigma^{\bullet}_{r_i}$ and that there exist no clear canonical choices for such bases. This arbitrariness, though, manifests only during the construction and doesn't show in the final result.

The second expansion (6.6) relies on *pre-ari*-brackets, and here the position is exactly the reverse: we have unicity and canonicity at every construction step, but more numerous summands.

Altogether, the *ari*-expansion is to be preferred in calculations, whereas the *pre-ari*-expansion is theoretically more appealing. In perinomal algebra, its use will even become mandatory (see §9). In any case, the conversion rules for changing from the one to the other are simple enough. Thus, up to length $r = 5$, we find:

$$S^{x_1,x_2}_{1,2} = +S^{x_1,x_2}_{[1,2]} \quad ; \quad S^{x_1,x_2}_{2,1} = -S^{x_2,x_1}_{[1,2]}$$
$$S^{x_1,x_2}_{1,4} = +S^{x_1,x_2}_{[1,4]} \quad ; \quad S^{x_1,x_2}_{4,1} = -S^{x_2,x_1}_{[1,4]}$$
$$S^{x_1,x_2}_{2,3} = +S^{x_1,x_2}_{[2,3]} \quad ; \quad S^{x_1,x_2}_{3,2} = -S^{x_2,x_1}_{[3,2]}$$

$$S^{x_1,x_2,x_3}_{1,1,3} = +S^{x_1,x_2,x_3}_{[1,1,3]} \quad ; \quad S^{x_1,x_2,x_3}_{1,3,1} = -S^{x_1,x_3,x_2}_{[1,1,3]} - S^{x_3,x_1,x_2}_{[1,1,3]} \quad ; \quad S^{x_1,x_2,x_3}_{3,1,1} = +S^{x_3,x_2,x_1}_{[1,1,3]}$$
$$S^{x_1,x_2,x_3}_{2,2,1} = -S^{x_1,x_3,x_2}_{[2,1,2]} \quad ; \quad S^{x_1,x_2,x_3}_{2,1,2} = +S^{x_1,x_2,x_3}_{[2,1,2]} + S^{x_3,x_2,x_1}_{[2,1,2]} \quad ; \quad S^{x_1,x_2,x_3}_{1,2,2} = -S^{x_3,x_1,x_2}_{[2,1,2]}$$

$$S^{x_1,x_2,x_3,x_4}_{1,1,1,2} = +S^{x_1,x_2,x_3,x_4}_{[1,1,1,2]}$$
$$S^{x_1,x_2,x_3,x_4}_{1,1,2,1} = -S^{x_1,x_2,x_4,x_3}_{[1,1,1,2]} - S^{x_1,x_4,x_2,x_3}_{[1,1,1,2]} - S^{x_4,x_1,x_2,x_3}_{[1,1,1,2]}$$
$$S^{x_1,x_2,x_3,x_4}_{1,2,1,1} = +S^{x_1,x_4,x_3,x_2}_{[1,1,1,2]} + S^{x_4,x_1,x_3,x_2}_{[1,1,1,2]} + S^{x_4,x_3,x_1,x_2}_{[1,1,1,2]}$$
$$S^{x_1,x_2,x_3,x_4}_{2,1,1,1} = -S^{x_4,x_3,x_2,x_1}_{[1,1,1,2]}$$

In the above table, as indeed throughout the sequel, we write down only the upper indices of the singulands (since, in the *colourless case* with which we are concerned here, the lower indices don't matter). Moreover, we write these upper indices of the singulands as "x_i" rather than "u_i", the better to bring out their independence from the u_i's that serve as upper indices for the singulates. Indeed, when expressing the *entireness condition* for the sums of singulates (see §6.3, §6.4 below), we may work either with $\Theta^\bullet_{r_*}$ itself or $swap.\Theta^\bullet_{r_*}$, and the distinct but equivalent constraints on the singulands which both approaches yield look much the same – all of which suggests that the singulands that go into the making of $l\o ma^\bullet$ stand, in a sense, halfway between that bimould and its swappee $l\o mi^\bullet$.

Singulation-desingulation.[86]
In keeping with the above remarks, we may (and shall), without loss of generality, limit ourselves to singulands $S^{x_1,\dots,x_l}_{[r_1,\dots,r_l]}$ and $S^{x_1,\dots,x_l}_{r_1,\dots,r_l}$ that are *even* (resp. *odd*) in each x_i if the corresponding index r_i is *odd* (resp. *even*). We may also (and shall), again without loss of generality, impose divisibility by $x_1 \dots x_l$.[87]

The construction of $l\o ma^{\boldsymbol{w}}$ is by induction, and goes like this.

Fix any odd integer r_* and assume we have already found singulates $\Sigma^\bullet_{[\boldsymbol{r}]}$ or $\Sigma^\bullet_{\boldsymbol{r}}$ of total index $|\boldsymbol{r}| := \sum r_i$ *odd* and $\leq r_*$, such that the truncated expansion:

$$\Theta^\bullet_{r_*} := \sum_{|\boldsymbol{r}|\leq r_*} \Sigma^\bullet_{[\boldsymbol{r}]} = \sum_{|\boldsymbol{r}|\leq r_*} \Sigma^\bullet_{\boldsymbol{r}} \tag{6.11}$$

has only *entire* components for all lengths $r \leq r_*$. One can then show the following:
(i) the component of $\Theta^\bullet_{r_*}$ of (even) length $1+r_*$ is automatically entire.
(ii) the component of $\Theta^\bullet_{r_*}$ of (odd) length $2+r_*$ is not entire, but possesses mulipoles of order r_* at the origin.
(iii) it is always possible to pick singulands $S^\bullet_{[\boldsymbol{r}]}$ or $S^\bullet_{\boldsymbol{r}}$ of total index $|\boldsymbol{r}| = 2+r_*$ and such that the corresponding singulates $\Sigma^\bullet_{[\boldsymbol{r}]}$ or $\Sigma^\bullet_{\boldsymbol{r}}$ exactly compensate the multipoles mentioned in (ii), so that the truncated sum $\Theta^\bullet_{2+\boldsymbol{r}}$ will coincide with $\Theta^\bullet_{\boldsymbol{r}}$ for all its components of length $r \leq 1+r_*$ but will have a singularity-free component of length $r = 2+r_*$.
(iv) the constraints on the newly added singulates are found by writing down, successively, the conditions for multipoles of order r_*, r_*-1, r_*-2 etc to be absent from the component $\Theta^{w_1,\dots,w_{2+r_*}}_{2+r_*}$.
(v) these constraints do not exactly determine the new singulates, but *very*

[86]We prefer this pair to the unwieldy *singularisation-desingularisation* not just for reasons of euphony, but also to keep close to the coinages *singulator, singuland, singulate.*

[87]The reason being that to a constant singuland $S^{w_1}_{r_1} \equiv 1$ there always answers a vanishing singulate $\Sigma^\bullet_{r_1} \equiv 0$.

nearly so[88], and in any case there exist two (closely related) privileged choices, leading to two closely related, canonical choices $lama^\bullet$, $loma^\bullet$ for $l\o ma^{\boldsymbol{w}}$.
(vi) there is also a third choice, $luma^\bullet$, whose components aren't sought in the ring of power series in $\boldsymbol{u}$ but rather in the space of meromorphic functions of $\boldsymbol{u}$, with multipoles located at the multiintegers $\boldsymbol{n}$, and with essentially bounded behaviour at infinity:[89]

$$S^{x_1,\dots,x_l}_{[r_1,\dots,r_l]} \overset{\text{ess}^{\text{lly}}}{=} \sum_{n_i\in\mathbb{Z}^*} R^{n_1,\dots,n_l}_{[r_1,\dots,r_l]}\, P(x_1{+}n_1)\dots P(x_l{+}n_l) \tag{6.12}$$

$$S^{x_1,\dots,x_l}_{r_1,\dots,r_l} \overset{\text{ess}^{\text{lly}}}{=} \sum_{n_i\in\mathbb{Z}^*} R^{n_1,\dots,n_l}_{r_1,\dots,r_l}\, P(x_1{+}n_1)\dots P(x_l{+}n_l) \tag{6.13}$$

Here, the solution $luma^\bullet$ turns out to be unique, its search essentially reducing to that of the multiresidues $R^{\boldsymbol{n}}_{[\boldsymbol{r}]}$ or $R^{\boldsymbol{n}}_{\boldsymbol{r}}$ carried by the multipoles of the singulands.[90] These multiresidues are uniquely determined rational numbers, and *perinomal functions*[91] of their argument $\boldsymbol{n}$. So the difficulty here is not the search for a canonical solution, but the elucidation of the arithmetical nature of the Taylor coefficients at the origin of the various components $luma^{\boldsymbol{w}}$, at least for lengths $r(\boldsymbol{w}) \geq 5$, since for lesser lengths the answer is elementary.

6.2 Singulation-desingulation up to length 2.

As usual, we set $1/t =: P(t) =: P^t$ throughout, and favour the third variant inside mould equations, for greater visual coherence. At lengths $r \leq 2$, one singuland only contributes to $l\o ma^\bullet$. At length 1, both singuland and singulate coincide. At length 2, the formula for the singulate involves poles of order 1, but these cancel out, duly yielding an entire $l\o ma^{w_1,w_2}$.

$$\text{løma}^{w_1} = \text{løma}^{w_1}_1 = \Sigma^{w_1}_{[1]} = S^{u_1}_{[1]} = u_1^2 + u_1^4 + u_1^6 + u_1^8 + \dots$$

$$\text{løma}^{w_1,w_2} = \text{løma}^{w_1,w_2}_1 = \Sigma^{w_1,w_2}_{[1]} =$$

$$\frac{1}{2}\,P^{u_1}\,(S^{u_{12}}_{[1]} - S^{u_2}_{[1]}) + \frac{1}{2}\,P^{u_2}\,(S^{u_1}_{[1]} - S^{u_{12}}_{[1]}) + \frac{1}{2}\,P^{u_{12}}\,(S^{u_2}_{[1]} - S^{u_1}_{[1]})$$

[88]In the sense that the *wandering bialternals*, which are ultimately responsible for this indeterminacy, are "few and far between". See §6.9 and the concluding comments in §9.1.

[89]Away from the multipoles, of course. Exactly what this means shall become clear in in the sequel: see §6.7 and §9. As for the warning *essentially* stacked over the = sign in the identities (6.12), (6.13), it means that we neglect simple corrective terms (with lower polar multiplicity) that ensure convergence on the right-hand side.

[90]These multiresidues $R^{n_1,\dots,n_l}_{[r_1,\dots,r_l]}$ have to be even (resp. odd) in n_i when r_i is even (resp. odd) to ensure that the singulate $S^{x_1,\dots,x_l}_{[r_1,\dots,r_l]}$ be odd (resp. even) when r_i is even (resp. odd).

[91]See §6.7 and §9.

6.3 Singulation-desingulation up to length 4.

The condition expressing that $l\o ma^{w_1,w_2,w_3}$ has no poles of order 1 at the origin involves only the singulands and singulates of indices [1] and $[1,2]$. For power series singulands, it reads:

$$\begin{aligned} 0 \;=\; &+\frac{1}{12}\left(P^{x_2}\,S^{x_{12}}_{[1]}-P^{x_{12}}\,S^{x_2}_{[1]}-P^{x_2}\,S^{x_1}_{[1]}+P^{x_{12}}\,S^{x_1}_{[1]}\right)\\ &+S^{x_1,x_2}_{[1,2]}+S^{x_2,x_{12}}_{[1,2]}-S^{x_1,x_{12}}_{[1,2]}-S^{x_{12},x_2}_{[1,2]} \end{aligned} \tag{6.14}$$

For meromorphic singulands (of type (6.12)), it translates into a condition on the multiresidues $R^{\bullet}_{[\bullet]}$, which reads:

$$0 = 1/12\left(\delta^{n_{12}}\,R^{n_1}_{[1]}-\delta^{n_2}\,R^{n_1}_{[1]}\right)+R^{n_1,n_2}_{[1,2]}-R^{n_1,n_{12}}_{[1,2]} \tag{6.15}$$

$$0 = 1/12\left(\delta^{n_2}\,R^{n_{12}}_{[1]}-\delta^{n_2}\,R^{n_1}_{[1]}-\delta^{n_{12}}\,R^{n_2}_{[1]}\right)+R^{n_1,n_2}_{[1,2]}+R^{n_2,n_{12}}_{[1,2]}-R^{n_{12},n_2}_{[1,2]} \tag{6.16}$$

When fulfilled, the above conditions ensure the entireness not just of $l\o ma^{w_1,\dots,w_3}$ but also of $l\o ma^{w_1,\dots,w_4}$.

6.4 Singulation-desingulation up to length 6.

At this stage of the construction, we are dealing with a component $l\o ma^{w_1,\dots,w_5}$ that may have multipoles of order 3, 2, 1 at the origin. Expressing that there are no such multipoles of order 3 leads to a single equation:

$$\mathcal{S}_{[1]}+\mathcal{S}_{[1,4]}+\mathcal{S}_{[1,4]}=0 \tag{6.17}$$

with contributions:

$$\begin{aligned} \mathcal{S}_{[1]} \;&:=\; +\frac{1}{120}\left(P^{x_2}\,S^{x_{12}}_{[1]}-P^{x_2}\,S^{x_1}_{[1]}\right)\\ \mathcal{S}_{[1,4]} \;&:=\; -S^{x_1,x_2}_{[1,4]}+S^{x_{12},x_2}_{[1,4]}\\ \mathcal{S}_{[2,3]} \;&:=\; +2\,S^{x_{12},x_2}_{[2,3]}+S^{x_1,x_2}_{[2,3]}-S^{x_1,x_{12}}_{[2,3]}-S^{x_2,x_{12}}_{[2,3]} \end{aligned}$$

We may note that the singulate $S_{[1,2]}$ remains, somewhat surprisingly, uninvolved at this stage.

Next, we must write down the condition for $l\o ma^{w_1,\dots,w_5}$ to have no multipoles of order 2 at the origin. This again leads to a single equation[92] that involves all singulands save the last one (i.e. $S_{[1,1,1,2]}$) :

$$\mathcal{S}^*_{[1]}+\mathcal{S}^*_{[1,2]}+\mathcal{S}^*_{[1,4]}+\mathcal{S}^*_{[2,3]}+\mathcal{S}^*_{[1,1,3]}+\mathcal{S}^*_{[2,1,2]}=0 \tag{6.18}$$

[92]This new condition, of course, makes sense, only *modulo* the earlier one, i.e. assuming the removal of order 3 multipoles.

with contributions:

$$
\begin{aligned}
720\,\mathcal{S}^*_{[1]} \;:=\; & -P^{x_2}\,P^{x_3}\,S^{x_{123}}_{[1]} - P^{x_1}\,P^{x_{23}}\,S^{x_2}_{[1]} + P^{x_1}\,P^{x_{23}}\,S^{x_3}_{[1]} \\
& +4\,P^{x_2}\,P^{x_{23}}\,S^{x_{123}}_{[1]} - 4\,P^{x_2}\,P^{x_{23}}\,S^{x_1}_{[1]} - 4\,P^{x_1}\,P^{x_{123}}\,S^{x_3}_{[1]} \\
& +11\,P^{x_{12}}\,P^{x_{123}}\,S^{x_2}_{[1]} - 11\,P^{x_{12}}\,P^{x_{123}}\,S^{x_1}_{[1]} - 11\,P^{x_1}\,P^{x_{123}}\,S^{x_2}_{[1]} \\
& +14\,P^{x_{12}}\,P^{x_3}\,S^{x_1}_{[1]} - 14\,P^{x_2}\,P^{x_3}\,S^{x_1}_{[1]} - 14\,P^{x_3}\,P^{x_{12}}\,S^{x_2}_{[1]} \\
& -15\,P^{x_1}\,P^{x_3}\,S^{x_{23}}_{[1]} - 15\,P^{x_2}\,P^{x_{123}}\,S^{x_{12}}_{[1]} + 15\,P^{x_2}\,P^{x_{123}}\,S^{x_3}_{[1]} \\
& +15\,P^{x_1}\,P^{x_3}\,S^{x_2}_{[1]} - 15\,P^{x_1}\,P^{x_3}\,S^{x_{12}}_{[1]} + 15\,P^{x_2}\,P^{x_{123}}\,S^{x_1}_{[1]} \\
& +15\,P^{x_1}\,P^{x_{123}}\,S^{x_{12}}_{[1]} + 15\,P^{x_1}\,P^{x_3}\,S^{x_{123}}_{[1]} + 15\,P^{x_2}\,P^{x_3}\,S^{x_{12}}_{[1]} \\
& -15\,P^{x_2}\,P^{x_{123}}\,S^{x_{23}}_{[1]} + 25\,P^{x_3}\,P^{x_{123}}\,S^{x_{23}}_{[1]} - 25\,P^{x_3}\,P^{x_{23}}\,S^{x_{123}}_{[1]} \\
& -25\,P^{x_3}\,P^{x_{123}}\,S^{x_1}_{[1]} + 25\,P^{x_3}\,P^{x_{23}}\,S^{x_1}_{[1]}
\end{aligned}
$$

$$
\begin{aligned}
12\,\mathcal{S}^*_{[1,2]} \;:=\; & +2\,P^{x_{123}}\,S^{x_3,x_{23}}_{[1,2]} - 2\,P^{x_{123}}\,S^{x_{23},x_3}_{[1,2]} + 2\,P^{x_{123}}\,S^{x_2,x_3}_{[1,2]} - 2\,P^{x_{123}}\,S^{x_2,x_{23}}_{[1,2]} \\
& -2\,P^{x_3}\,S^{x_{23},x_{123}}_{[1,2]} + 2\,P^{x_3}\,S^{x_{123},x_{23}}_{[1,2]} + 2\,P^{x_3}\,S^{x_1,x_{123}}_{[1,2]} - 2\,P^{x_3}\,S^{x_1,x_{23}}_{[1,2]} \\
& -3\,P^{x_1}\,S^{x_2,x_3}_{[1,2]} + 3\,P^{x_1}\,S^{x_{23},x_3}_{[1,2]} + 3\,P^{x_1}\,S^{x_2,x_{23}}_{[1,2]} - 3\,P^{x_1}\,S^{x_3,x_{23}}_{[1,2]} \\
& +3\,P^{x_1}\,S^{x_3,x_{123}}_{[1,2]} - 3\,P^{x_1}\,S^{x_{123},x_3}_{[1,2]} - 3\,P^{x_1}\,S^{x_{12},x_{123}}_{[1,2]} + 3\,P^{x_1}\,S^{x_{12},x_3}_{[1,2]} \\
& +3\,P^{x_2}\,S^{x_1,x_{23}}_{[1,2]} + 3\,P^{x_2}\,S^{x_{123},x_3}_{[1,2]} + 3\,P^{x_2}\,S^{x_{23},x_{123}}_{[1,2]} - 3\,P^{x_2}\,S^{x_1,x_{123}}_{[1,2]} \\
& -3\,P^{x_2}\,S^{x_{12},x_3}_{[1,2]} - 3\,P^{x_2}\,S^{x_3,x_{123}}_{[1,2]} + 3\,P^{x_2}\,S^{x_{12},x_{123}}_{[1,2]} - 3\,P^{x_2}\,S^{x_{123},x_{23}}_{[1,2]}
\end{aligned}
$$

$$
\begin{aligned}
12\,\mathcal{S}^*_{[1,4]} \;:=\; & -2\,P^{x_{123}}\,S^{x_3,x_{23}}_{[1,4]} + 2\,P^{x_{123}}\,S^{x_1,x_3}_{[1,4]} - 2\,P^{x_{123}}\,S^{x_2,x_3}_{[1,4]} + 2\,P^{x_{123}}\,S^{x_2,x_{23}}_{[1,4]} \\
& -2\,P^{x_{23}}\,S^{x_1,x_3}_{[1,4]} - 2\,P^{x_{23}}\,S^{x_2,x_{123}}_{[1,4]} + 2\,P^{x_{23}}\,S^{x_3,x_{123}}_{[1,4]} + 2\,P^{x_{23}}\,S^{x_{123},x3}_{[1,4]} \\
& +2\,P^{x_3}\,S^{x_1,x_{23}}_{[1,4]} - 2\,P^{x_3}\,S^{x_1,x_{123}}_{[1,4]} + 2\,P^{x_3}\,S^{x_2,x_{123}}_{[1,4]} - 2\,P^{x_3}\,S^{x_{123},x_{23}}_{[1,4]} \\
& +3\,P^{x_{12}}\,S^{x_1,x_3}_{[1,4]} - 3\,P^{x_{12}}\,S^{x_2,x_3}_{[1,4]} + 3\,P^{x_{12}}\,S^{x_1,x_{123}}_{[1,4]} - 3\,P^{x_{12}}\,S^{x_2,x_{123}}_{[1,4]} \\
& -3\,P^{x_1}\,S^{x_3,x_{23}}_{[1,4]} + 3\,P^{x_1}\,S^{x_2,x_{23}}_{[1,4]} - 3\,P^{x_1}\,S^{x_3,x_{123}}_{[1,4]} + 3\,P^{x_1}\,S^{x_2,x_{123}}_{[1,4]} \\
& -3\,P^{x_2}\,S^{x_1,x_3}_{[1,4]} - 3\,P^{x_2}\,S^{x_1,x_{23}}_{[1,4]} + 3\,P^{x_2}\,S^{x_{123},x_3}_{[1,4]} + 3\,P^{x_2}\,S^{x_{123},x_{23}}_{[1,4]}
\end{aligned}
$$

$$
\begin{aligned}
12\,\mathcal{S}^{*}_{[2,3]} \;:=\; & -P^{x_{123}}\,S^{x_3,x_1}_{[2,3]} + P^{x_{23}}\,S^{x_3,x_1}_{[2,3]} + P^{x_{23}}\,S^{x_{123},x_2}_{[2,3]} - P^{x_3}\,S^{x_{123},x_2}_{[2,3]}\\
& +2\,P^{x_{23}}\,S^{x_3,x_{123}}_{[2,3]} + 2\,P^{x_{23}}\,S^{x_{123},x_3}_{[2,3]} - 3\,P^{x_{123}}\,S^{x_1,x_3}_{[2,3]} + 3\,P^{x_{123}}\,S^{x_{23},x_3}_{[2,3]}\\
& +3\,P^{x_{12}}\,S^{x_1,x_3}_{[2,3]} - 3\,P^{x_{12}}\,S^{x_2,x_3}_{[2,3]} + 3\,P^{x_{12}}\,S^{x_2,x_{123}}_{[2,3]} - 3\,P^{x_{12}}\,S^{x_1,x_{123}}_{[2,3]}\\
& -3\,P^{x_1}\,S^{x_2,x_3}_{[2,3]} - 3\,P^{x_1}\,S^{x_{12},x_3}_{[2,3]} + 3\,P^{x_1}\,S^{x_{12},x_{123}}_{[2,3]} + 3\,P^{x_1}\,S^{x_2,x_{123}}_{[2,3]}\\
& +3\,P^{x_1}\,S^{x_{123},x_3}_{[2,3]} + 3\,P^{x_1}\,S^{x_{23},x_3}_{[2,3]} + 3\,P^{x_2}\,S^{x_1,x_3}_{[2,3]} - 3\,P^{x_2}\,S^{x_1,x_{123}}_{[2,3]}\\
& -3\,P^{x_2}\,S^{x_3,x_{123}}_{[2,3]} - 3\,P^{x_2}\,S^{x_{12},x_{123}}_{[2,3]} + 3\,P^{x_2}\,S^{x_{12},x_3}_{[2,3]} - 3\,P^{x_2}\,S^{x_{23},x_{123}}_{[2,3]}\\
& +3\,P^{x_3}\,S^{x_2,x_{123}}_{[2,3]} + 3\,P^{x_3}\,S^{x_{23},x_{123}}_{[2,3]} + 3\,P^{x_{23}}\,S^{x_2,x_{123}}_{[2,3]} - 3\,P^{x_{23}}\,S^{x_1,x_3}_{[2,3]}\\
& -5\,P^{x_{123}}\,S^{x_3,x_{23}}_{[2,3]} - 5\,P^{x_3}\,S^{x_{123},x_{23}}_{[2,3]} - 6\,P^{x_1}\,S^{x_3,x_{23}}_{[2,3]} - 6\,P^{x_1}\,S^{x_3,x_{123}}_{[2,3]}\\
& +6\,P^{x_2}\,S^{x_{123},x_3}_{[2,3]} + 6\,P^{x_2}\,S^{x_{123},x_{23}}_{[2,3]}
\end{aligned}
$$

$$
\begin{aligned}
2\,\mathcal{S}^{*}_{[1,1,3]} \;:=\; & +S^{x_1,x_2,x_3}_{[1,1,3]} - S^{x_1,x_2,x_{23}}_{[1,1,3]} - S^{x_1,x_{23},x_3}_{[1,1,3]} + S^{x_3,x_1,x_{23}}_{[1,1,3]} - S^{x_1,x_{12},x_3}_{[1,1,3]}\\
& +S^{x_2,x_{12},x_3}_{[1,1,3]} + S^{x_1,x_3,x_{23}}_{[1,1,3]} - S^{x_2,x_1,x_{23}}_{[1,1,3]} - S^{x_{123},x_2,x_3}_{[1,1,3]} + S^{x_1,x_{123},x_3}_{[1,1,3]}\\
& +S^{x_2,x_3,x_{123}}_{[1,1,3]} + S^{x_2,x_1,x_{123}}_{[1,1,3]} - S^{x_1,x_3,x_{123}}_{[1,1,3]} - S^{x_3,x_1,x_{123}}_{[1,1,3]} - S^{x_{123},x_3,x_{23}}_{[1,1,3]}\\
& -S^{x_3,x_{123},x_{23}}_{[1,1,3]} + S^{x_2,x_{123},x_{23}}_{[1,1,3]} + S^{x_{123},x_{23},x_3}_{[1,1,3]} + S^{x_{123},x_2,x_{23}}_{[1,1,3]} - S^{x_2,x_{23},x_{123}}_{[1,1,3]}\\
& -S^{x_2,x_{12},x_{123}}_{[1,1,3]} + S^{x_3,x_{23},x_{123}}_{[1,1,3]} + S^{x_1,x_{12},x_{123}}_{[1,1,3]} - S^{x_2,x_{123},x_3}_{[1,1,3]}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{S}^{*}_{[2,1,2]} \;:=\; & +S^{x_3,x_1,x_{23}}_{[2,1,2]} - S^{x_{123},x_2,x_3}_{[2,1,2]} - S^{x_3,x_1,x_{123}}_{[2,1,2]} + S^{x_{123},x_2,x_{23}}_{[2,1,2]} + S^{x_{123},x_{23},x_3}_{[2,1,2]}\\
& -S^{x_{123},x_3,x_{23}}_{[2,1,2]} - S^{x_3,x_{123},x_{23}}_{[2,1,2]} + S^{x_3,x_{23},x_{123}}_{[2,1,2]}
\end{aligned}
$$

Lastly, we must write down the condition for $l\o ma^{w_1,\dots,w_5}$ to have no poles of order 1 at the origin. This once again leads to a single equation, but one that now involves all seven relevant singulands:

$$
\mathcal{S}^{**}_{[1]} + \mathcal{S}^{**}_{[1,2]} + \mathcal{S}^{**}_{[1,4]} + \mathcal{S}^{**}_{[2,3]} + \mathcal{S}^{**}_{[1,1,3]} + \mathcal{S}^{**}_{[2,1,2]} + \mathcal{S}^{**}_{[1,1,1,2]} = 0 \tag{6.19}
$$

Though easy to compute, the various contributions $\mathcal{S}^{**}_{[\boldsymbol{r}]}$ are too unwieldy for us to write down. So we simply mention their number $\#(\mathcal{S}^{**}_{[\boldsymbol{r}]})$ of summands. Here is the list:

$$
\begin{aligned}
&\#(\mathcal{S}^{**}_{[1]}) = 126 \quad , \quad \#(\mathcal{S}^{**}_{[1,2]}) = 299 \quad , \quad \#(\mathcal{S}^{**}_{[1,4]}) = 176 \quad , \quad \#(\mathcal{S}^{**}_{[2,3]}) = 314\\
&\#(\mathcal{S}^{**}_{[1,1,3]}) = 288 \quad , \quad \#(\mathcal{S}^{**}_{[2,1,2]}) = 324 \quad , \quad \#(\mathcal{S}^{**}_{[1,1,1,2]}) = 192
\end{aligned}
$$

If we now look for *meromorphic* singulands of type (6.12), the absence of multipoles of order 3 at the origin is equivalent to a system of two independent identities of the form $\mathcal{R}_{[1]}+\mathcal{R}_{[1,4]}+\mathcal{R}_{[2,3]}=0$, namely:

$$0=-\frac{1}{120}\,\delta^{n_2}\,R^{n_1}_{[1]}-R^{n_1,n_2}_{[1,4]}+R^{n_1,n_2}_{[2,3]}-R^{n_1,n_{12}}_{[2,3]} \tag{6.20}$$

$$0=\frac{1}{120}\big(\delta^{n_2}\,R^{n_{12}}_{[1]}-\delta^{n_2}\,R^{n_1}_{[1]}\big)-R^{n_1,n_2}_{[1,4]}+R^{n_{12},n_2}_{[1,4]}+R^{n_1,n_2}_{[2,3]}-R^{n_2,n_{12}}_{[2,3]}+2\,R^{n_{12},n_2}_{[2,3]}$$

The absence of multipoles of order 2 at the origin is also equivalent to a system of two independent identities, with effective involvement of all singulands except the last one:

$$\mathcal{R}^*_{[1]}+\mathcal{R}^*_{[1,2]}+\mathcal{R}^*_{[1,4]}+\mathcal{R}^*_{[2,3]}+\mathcal{R}^*_{[1,1,3]}+\mathcal{R}^*_{[2,1,2]}=0 \tag{6.21}$$

$$\mathcal{R}^\dagger_{[1]}+\mathcal{R}^\dagger_{[1,2]}+\mathcal{R}^\dagger_{[1,4]}+\mathcal{R}^\dagger_{[2,3]}+\mathcal{R}^\dagger_{[1,1,3]}+\mathcal{R}^\dagger_{[2,1,2]}=0 \tag{6.22}$$

$$\begin{aligned}360\,\mathcal{R}^*_{[1]} \;=\;& -\delta^{n_1}\,\delta^{n_{23}}\,R^{n_2}_{[1]}-4\,\delta^{n_2}\,\delta^{n_{23}}\,R^{n_1}_{[1]}-11\,\delta^{n_1}\,\delta^{n_{123}}\,R^{n_2}_{[1]}-11\,\delta^{n_{12}}\,\delta^{n_{123}}\,R^{n_1}_{[1]}\\ &+14\,\delta^{n_{12}}\,\delta^{n_3}\,R^{n_1}_{[1]}-14\,\delta^{n_2}\,\delta^{n_3}\,R^{n_1}_{[1]}+15\,\delta^{n_1}\,\delta^{n_3}\,R^{n_2}_{[1]}+15\,\delta^{n_2}\,\delta^{n_{123}}\,R^{n_1}_{[1]}\\ &-15\,\delta^{n_1}\,\delta^{n_3}\,R^{n_{12}}_{[1]}+15\,\delta^{n_1}\,\delta^{n_{123}}\,R^{n_{12}}_{[1]}\end{aligned}$$

$$\begin{aligned}360\,\mathcal{R}^\dagger_{[1]} \;=\;& +\delta^{n_1}\,\delta^{n_{23}}\,R^{n_3}_{[1]}-\delta^{n_2}\,\delta^{n_3}\,R^{n_{123}}_{[1]}-14\,\delta^{n_2}\,\delta^{n_3}\,R^{n_1}_{[1]}-14\,\delta^{n_{12}}\,\delta^{n_3}\,R^{n_2}_{[1]}\\ &+15\,\delta^{n_1}\,\delta^{n_3}\,R^{n_2}_{[1]}+15\,\delta^{n_2}\,\delta^{n_3}\,R^{n_{12}}_{[1]}-15\,\delta^{n_1}\,\delta^{n_3}\,R^{n_{23}}_{[1]}+15\,\delta^{n_2}\,\delta^{n_{123}}\,R^{n_3}_{[1]}\\ &+25\,\delta^{n_{23}}\,\delta^{n_3}\,R^{n_1}_{[1]}+25\,\delta^{n_3}\,\delta^{n_{123}}\,R^{n_{23}}_{[1]}-25\,\delta^{n_{23}}\,\delta^{n_3}\,R^{n_{123}}_{[1]}\end{aligned}$$

$$\begin{aligned}2\,\mathcal{R}^*_{[1,2]} \;=\;& +\delta^{n_1}\,R^{n_2,n_3}_{[1,2]}-\delta^{n_1}\,R^{n_2,n_{23}}_{[1,2]}-\delta^{n_2}\,R^{n_1,n_{23}}_{[1,2]}+\delta^{n_2}\,R^{n_1,n_{123}}_{[1,2]}\\ &-\delta^{n_1}\,R^{n_{12},n_3}_{[1,2]}+\delta^{n_1}\,R^{n_{12},n_{123}}_{[1,2]}\end{aligned}$$

$$\begin{aligned}6\,\mathcal{R}^\dagger_{[1,2]} \;=\;& +2\,\delta^{n_3}\,R^{n_1,n_{23}}_{[1,2]}-2\,\delta^{n_{123}}\,R^{n_2,n_3}_{[1,2]}-2\,\delta^{n_{123}}\,R^{n_3,n_{23}}_{[1,2]}+2\,\delta^{n_{123}}\,R^{n_{23},n_3}_{[1,2]}\\ &-2\,\delta^{n_3}\,R^{n_{123},n_{23}}_{[1,2]}+2\,\delta^{n_3}\,R^{n_{23},n_{123}}_{[1,2]}+3\,\delta^{n_1}\,R^{n_2,n_3}_{[1,2]}+3\,\delta^{n_2}\,R^{n_{12},n_3}_{[1,2]}\\ &-3\,\delta^{n_1}\,R^{n_{23},n_3}_{[1,2]}+3\,\delta^{n_1}\,R^{n_3,n_{23}}_{[1,2]}+3\,\delta^{n_2}\,R^{n_3,n_{123}}_{[1,2]}-3\,\delta^{n_2}\,R^{n_{123},n_3}_{[1,2]}\end{aligned}$$

$$\begin{aligned}2\,\mathcal{R}^*_{[1,4]} \;=\;& \delta^{n_{12}}\,R^{n_1,n_3}_{[1,4]}-\delta^{n_2}\,R^{n_1,n_3}_{[1,4]}-\delta^{n_2}\,R^{n_1,n_{23}}_{[1,4]}+\delta^{n_1}\,R^{n_2,n_{23}}_{[1,4]}\\ &+\delta^{n_{12}}\,R^{n_1,n_{123}}_{[1,4]}+\delta^{n_1}\,R^{n_2,n_{123}}_{[1,4]}\end{aligned}$$

$$\begin{aligned}6\,\mathcal{R}^\dagger_{[1,4]} \;=\;& 2\,\delta^{n_3}\,R^{n_2,n_{123}}_{[1,4]}-2\,\delta^{n_{23}}\,R^{n_1,n_3}_{[1,4]}+2\,\delta^{n_3}\,R^{n_1,n_{23}}_{[1,4]}-2\,\delta^{n_{123}}\,R^{n_2,n_3}_{[1,4]}\\ &+2\,\delta^{n_{23}}\,R^{n_3,n_{123}}_{[1,4]}+2\,\delta^{n_{23}}\,R^{n_{123},n_3}_{[1,4]}-2\,\delta^{n_{123}}\,R^{n_3,n_{23}}_{[1,4]}-2\,\delta^{n_3}\,R^{n_{123},n_{23}}_{[1,4]}\\ &-3\,\delta^{n_2}\,R^{n_1,n_3}_{[1,4]}-3\,\delta^{n_{12}}\,R^{n_2,n_3}_{[1,4]}-3\,\delta^{n_1}\,R^{n_3,n_{23}}_{[1,4]}+3\,\delta^{n_2}\,R^{n_{123},n_3}_{[1,4]}\end{aligned}$$

$$
\begin{aligned}
2\,\mathcal{R}^{*}_{[2,3]} \;=\;& \delta^{n_2}\,R^{n_1,n_3}_{[2,3]} - \delta^{n_1}\,R^{n_2,n_3}_{[2,3]} + \delta^{n_{12}}\,R^{n_1,n_3}_{[2,3]} - \delta^{n_2}\,R^{n_1,n_{123}}_{[2,3]}\\
&+\delta^{n_1}\,R^{n_2,n_{123}}_{[2,3]} - \delta^{n_1}\,R^{n_{12},n_3}_{[2,3]} - \delta^{n_{12}}\,R^{n_1,n_{123}}_{[2,3]} + \delta^{n_1}\,R^{n_{12},n_{123}}_{[2,3]}\\
6\,\mathcal{R}^{\dagger}_{[2,3]} \;=\;& \delta^{n_{23}}\,R^{n_3,n_1}_{[2,3]} - \delta^{n_3}\,R^{n_{123},n_2}_{[2,3]} + 2\,\delta^{n_{23}}\,R^{n_{123},n_3}_{[2,3]} + 2\,\delta^{n_{23}}\,R^{n_3,n_{123}}_{[2,3]}\\
&+3\,\delta^{n_2}\,R^{n_1,n_3}_{[2,3]} - 3\,\delta^{n_{23}}\,R^{n_1,n_3}_{[2,3]} - 3\,\delta^{n_1}\,R^{n_2,n_3}_{[2,3]} - 3\,\delta^{n_{12}}\,R^{n_2,n_3}_{[2,3]}\\
&+3\,\delta^{n_2}\,R^{n_{12},n_3}_{[2,3]} + 3\,\delta^{n_1}\,R^{n_{23},n_3}_{[2,3]} + 3\,\delta^{n_3}\,R^{n_2,n_{123}}_{[2,3]} - 3\,\delta^{n_2}\,R^{n_3,n_{123}}_{[2,3]}\\
&+3\,\delta^{n_3}\,R^{n_{23},n_{123}}_{[2,3]} + 3\,\delta^{n_{123}}\,R^{n_{23},n_3}_{[2,3]} - 5\,\delta^{n_{123}}\,R^{n_3,n_{23}}_{[2,3]}\\
&-5\,\delta^{n_3}\,R^{n_{123},n_{23}}_{[2,3]} - 6\,\delta^{n_1}\,R^{n_3,n_{23}}_{[2,3]} + 6\,\delta^{n_2}\,R^{n_{123},n_3}_{[2,3]}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{R}^{*}_{[1,1,3]} \;=\;& R^{n_1,n_2,n_3}_{[1,1,3]} - R^{n_1,n_2,n_{23}}_{[1,1,3]} + R^{n_1,n_{12},n_{123}}_{[1,1,3]} - R^{n_1,n_{12},n3}_{[1,1,3]} + R^{n_2,n_1,n_{123}}_{[1,1,3]} - R^{n_2,n_1,n_{23}}_{[1,1,3]}\\
\mathcal{R}^{\dagger}_{[1,1,3]} \;=\;& R^{n_1,n_2,n_3}_{[1,1,3]} + R^{n_2,n_{12},n_3}_{[1,1,3]} - R^{n_1,n_{23},n_3}_{[1,1,3]} + R^{n_3,n_1,n_{23}}_{[1,1,3]} + R^{n_1,n_3,n_{23}}_{[1,1,3]}\\
&-R^{n_{123},n_2,n_3}_{[1,1,3]} - R^{n_2,n_{123},n_3}_{[1,1,3]} + R^{n_2,n_3,n_{123}}_{[1,1,3]} - R^{n_{123},n_3,n_{23}}_{[1,1,3]}\\
&-R^{n_3,n_{123},n_{23}}_{[1,1,3]} + R^{n_{123},n_{23},n_3}_{[1,1,3]} + R^{n_3,n_{23},n_{123}}_{[1,1,3]}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{R}^{*}_{[2,1,2]} \;=\;& 0\\
\mathcal{R}^{\dagger}_{[2,1,2]} \;=\;& 2\big(R^{n_3,n_1,n_{23}}_{[2,1,2]} - R^{n_{123},n_2,n_3}_{[2,1,2]} + R^{n_3,n_{23},n_{123}}_{[2,1,2]} - R^{n_{123},n_3,n_{23}}_{[2,1,2]} - R^{n_3,n_{123},n_{23}}_{[2,1,2]} + R^{n_{123},n_{23},n_3}_{[2,1,2]}\big)
\end{aligned}
$$

Lastly, the condition for $l\o ma^{w_1,\ldots,w_5}$ to have no poles of order 1 at the origin can be expressed by a single equation, that involves all seven relevant singulands :

$$
\mathcal{R}^{**}_{[1]} + \mathcal{R}^{**}_{[1,2]} + \mathcal{R}^{**}_{[1,4]} + \mathcal{R}^{**}_{[2,3]} + \mathcal{R}^{**}_{[1,1,3]} + \mathcal{R}^{**}_{[2,1,2]} + \mathcal{R}^{**}_{[1,1,1,2]} = 0 \qquad (6.23)
$$

Once again the $\mathcal{R}^{**}_{[\boldsymbol{r}]}$ are too unwieldy for us to write down, and we merely mention their number $\#(\mathcal{R}^{**}_{[\boldsymbol{r}]})$ of summands:

$$
\begin{aligned}
&\#(\mathcal{R}^{**}_{[1]}) = 34 \quad , \quad \#(\mathcal{R}^{**}_{[1,2]}) = 58 \quad , \quad \#(\mathcal{R}^{**}_{[1,4]}) = 40 \quad , \quad \#(\mathcal{R}^{**}_{[2,3]}) = 74\\
&\#(\mathcal{R}^{**}_{[1,1,3]}) = 48 \quad , \quad \#(\mathcal{R}^{**}_{[2,1,2]}) = 64 \quad , \quad \#(\mathcal{R}^{**}_{[1,1,1,2]}) = 24
\end{aligned}
$$

6.5 The basis $lama^{\bullet}/lami^{\bullet}$.

As already pointed out, the desingulation conditions listed above admit multiple solutions when the singulands are sought in the space of power series, even after imposing the proper parity in each variable. To ensure uniqueness, many additional constraints are theoretically possible, but two stand out as clearly privileged, in the sense that they, and they alone, guarantee coefficients with arithmetically simple denominators.

We mention here the first constraint, leading to the bimould *lama*$^\bullet$, for the first non-trivial singulands $S^\bullet_{[1,2]} = S^\bullet_{1,2}$. For the coefficients of weight s, the equation (6.14) admits exactly *one* solution of the form:

$$\mathrm{Sa}^{x_1,x_2}_{1,2} = \sum_{1\leq\delta\leq \mathrm{ent}(\frac{s-1}{2})-\mathrm{ent}(\frac{s+1}{6})} a_{2\,\delta}\ x_1^{2\,\delta}\ x_2^{s-2-2\,\delta} \tag{6.24}$$

This is, moreover, the choice for which the prime factors in the denominator admit the best universal bound $p \leq Cst\, s$. In fact, for this choice, the bound is $p \leq \frac{s}{3}$.

6.6 The basis *loma*$^\bullet$/*lomi*$^\bullet$.

Now, let us move on to the second type of constraints, leading to the bimould *loma*$^\bullet$, again for the first non-trivial singulands $S^\bullet_{[1,2]} = S^\bullet_{1,2}$. For the coefficients of weight s, the equation (6.14) admits exactly *one* solution of the form:

$$\mathrm{So}^{x_1,x_2}_{1,2} = x_1^2\, x_2 \sum_{0\leq\delta\leq \mathrm{ent}(\frac{s-3}{6})} a_{2\,\delta}\ \left(x_1^{2\,\delta}\ x_2^{s-5-2\,\delta} + x_1^{s-5-2\,\delta}\ x_2^{2\,\delta}\right) \tag{6.25}$$

which entails far fewer coefficients. This is basically the only other choice[93] for which the prime factors in the denominator admit a universal bound $p \leq Cst\, s$. In this case the bound is $p \leq \frac{2\,s-5}{3}$.

6.7 The basis *luma*$^\bullet$/*lumi*$^\bullet$.

Here, we may deal at once with all length-2 singulands:

$$S^{x_1,x_2}_{[r_1,r_2]} = S^{x_1,x_2}_{r_1,r_2} \overset{\mathrm{ess^{ly}}}{=} \sum_{n_i\in\mathbb{Z}^*} R^{n_1,n_2}_{r_1,r_2}\, P(x_1{+}n_1)\, P(x_2{+}n_2) \tag{6.26}$$

The multiresidues are simple enough:[94]

$$R^{n_1,n_2}_{[r_1,r_2]} = R^{n_1,n_2}_{r_1,r_2} = \gamma_{r_1,r_2}\ \mu(n_1,n_2)\ n_1^{n_2-1}\ n_2^{n_1-1} \tag{6.27}$$

[93]Leaving aside, of course, simple *averages* of the first and second choice.

[94]They cease to be simple for singulands of length $l \geq 3$. Here, we get full-blown 'perinomalness'. See §9.5.

with γ_{r_1,r_2} a simple rational constant, and with $\mu(n_1, n_2)$ being 1 (resp. 0) if n_1, n_2 are co-prime (resp. otherwise). The Taylor coefficients of the singulates, however, are less simple: they carry Bernoulli numbers in their denominators, and sometimes very large prime factors, that can exceed any given bound of the form *Cst s*:

$$\mathrm{Su}^{x_1,x_2}_{r_1,r_2}(s) \quad = \quad (-1)^{r_1} \frac{B_{r_1+r_2-1}}{r_1+r_2-1} \sum_{\substack{\delta_1 \geq r_1 \\ \delta_2 \geq r_2}}^{\delta_1+\delta_2=s+2} \frac{B^*_{\delta_1-r_1} B^*_{\delta_2-r_2}}{B^*_{\delta_1+\delta_2-r_1-r_2}} \, u_1^{\delta_2-2} u_2^{\delta_1-2} \tag{6.28}$$

with $B^*_n = \dfrac{B_n}{n!}$, $B_{2\,n}$:= *Bernoulli number* , $B_n := 0$ *for* n *odd or* < 0.

Pay attention to the exponents: it is $\delta_2 - 2$ on top of u_1 and $\delta_1 - 2$ on top of u_2. In fact, since both s and r_1+r_2 are always odd, the summation rule produces only *positive* powers of u_1, u_2 (one *even*, the other *odd*), except for the pairs $(r_1, r_2) = (1, 2)$ resp. $(2, 1)$ where constant monomials in u_1 resp. u_2 do appear – but these may be neglected, since they contribute nothing to the singulate. Of course, the usual identity $Su^{x_1,x_2}_{r_1,r_2} + Su^{x_2,x_1}_{r_2,r_1} = 0$ holds.

6.8 Arithmetical vs analytic smoothness.

To show how the three choices compare, arithmetically speaking, we list the weight-s component $S^{\bullet}_{1,2}(s)$ of the first non-trivial singuland in all three variants $Sa^{\bullet}_{1,2}(s)$, $So^{\bullet}_{1,2}(s)$, $Su^{\bullet}_{1,2}(s)$, up to the weight $s = 17$:

$$\mathrm{Sa}^{x_1,x_2}_{1,2}(5) = \mathrm{So}^{x_1,x_2}_{1,2}(5) = \mathrm{Su}^{x_1,x_2}_{1,2}(5) = -\frac{5}{12}\, x_1^2\, x_2$$

$$\mathrm{Sa}^{x_1,x_2}_{1,2}(7) = \mathrm{So}^{x_1,x_2}_{1,2}(7) = \mathrm{Su}^{x_1,x_2}_{1,2}(7) = -\frac{7}{24}\, x_1^2\, x_2^3 - \frac{7}{24}\, x_1^4\, x_2$$

$$\mathrm{Sa}^{x_1,x_2}_{1,2}(9) = \mathrm{So}^{x_1,x_2}_{1,2}(9) = \mathrm{Su}^{x_1,x_2}_{1,2}(9) = -\frac{5}{18}\, x_1^2\, x_2^5 - \frac{7}{36}\, x_1^4\, x_2^3 - \frac{5}{18}\, x_1^6\, x_2$$

$$\mathrm{Sa}^{x_1,x_2}_{1,2}(11) = -\frac{11}{8}\, x_1^2\, x_2^7 + \frac{55}{24}\, x_1^4\, x_2^5 - \frac{11}{6}\, x_1^6\, x_2^3$$

$$\mathrm{So}^{x_1,x_2}_{1,2}(11) = -\frac{11}{40}\, x_1^2\, x_2^7 - \frac{11}{60}\, x_1^4\, x_2^5 - \frac{11}{60}\, x_1^6\, x_2^3 - \frac{11}{40}\, x_1^8\, x_2$$

$$\mathrm{Su}^{x_1,x_2}_{1,2}(11) = \mathrm{So}^{x_1,x_2}_{1,2}(11)$$

$$\mathrm{Sa}^{x_1,x_2}_{1,2}(13) = -\frac{91}{48}\, x_1^4\, x_2^7 + \frac{65}{24}\, x_1^6\, x_2^5 - \frac{91}{48}\, x_1^8\, x_2^3,$$

$$\mathrm{So}^{x_1,x_2}_{1,2}(13) = -\frac{65}{252}\, x_1^2\, x_2^9 - \frac{143}{504}\, x_1^4\, x_2^7 - \frac{143}{504}\, x_1^8\, x_2^3 - \frac{65}{252}\, x_1^{10}\, x_2$$

$$\mathrm{Su}^{x_1,x_2}_{1,2}(13) = -\frac{2275}{8292}\, x_1^2\, x_2^9 - \frac{1001}{5528}\, x_1^4\, x_2^7 - \frac{715}{4146}\, x_1^6\, x_2^5 - \frac{1001}{5528}\, x_1^8\, x_2^3 - \frac{2275}{8292}\, x_1^{10}\, x_2$$

$$\mathrm{Sa}^{x_1,x_2}_{1,2}(15) = -\frac{691}{360}\,x_1^2\,x_2^{11} + \frac{665}{144}\,x_1^4\,x_2^9 - \frac{2233}{360}\,x_1^6\,x_2^7 + \frac{209}{48}\,x_1^8\,x_2^5 - \frac{21}{10}\,x_1^{10}\,x_2^3$$

$$\mathrm{So}^{x_1,x_2}_{1,2}(15) = -\frac{691}{2520}\,x_1^2\,x_2^{11} - \frac{13}{72}\,x_1^4\,x_2^9 - \frac{143}{840}\,x_1^6\,x_2^7 - \frac{143}{840}\,x_1^8\,x_2^5 - \frac{13}{72}\,x_1^{10}\,x_2^3 - \frac{691}{2520}\,x_1^{12}\,x_2$$

$$\mathrm{Su}^{x_1,x_2}_{1,2}(15) = \mathrm{So}^{x_1,x_2}_{1,2}(15)$$

$$\mathrm{Sa}^{x_1,x_2}_{1,2}(17) = -\frac{442}{15}\,x_1^2x_2^{13} + \frac{1105}{12}\,x_1^4x_2^{11} - \frac{1666}{15}\,x_1^6x_2^9 + \frac{187}{3}\,x_1^8x_2^7 - \frac{153}{10}\,x_1^{10}x_2^5$$

$$\mathrm{So}^{x_1,x_2}_{1,2}(17) = -\frac{17}{60}\,x_1^2\,x_2^{13} - \frac{17}{144}\,x_1^4\,x_2^{11} - \frac{221}{720}\,x_1^6\,x_2^9 - \frac{221}{720}\,x_1^{10}\,x_2^5 - \frac{17}{144}\,x_1^{12}\,x_2^3 - \frac{17}{60}\,x_1^{14}\,x_2$$

$$\mathrm{Su}^{x_1,x_2}_{1,2}(17) = -\frac{2975}{10851}\,x_1^2\,x_2^{13} - \frac{11747}{65106}\,x_1^4\,x_2^{11} - \frac{5525}{32553}\,x_1^6\,x_2^9 - \frac{2431}{14468}\,x_1^8\,x_2^7 - \frac{5525}{32553}\,x_1^{10}\,x_2^5 - \frac{11747}{65106}\,x_1^{12}\,x_2^3 - \frac{2975}{10851}\,x_1^{14}\,x_2$$

6.9 Singulator kernels and "wandering" bialternals.

Let $BIMU_l^s$ be the space of all bimoulds $M^\bullet$ whose only non-vanishing component $M^{w_1,\dots,w_l}$ is constant in the v_i-variables, and homogeneous polynomial of total degree $d = s - l$ in the u_i-variables.[95]

Likewise, let $BIMU^s_{r_1,\dots,r_l}$ be the subspace of $BIMU_l^s$ consisting of all bimoulds $M^\bullet$ whose only non-vanishing component $M^{w_1,\dots,w_l}$:
– is divisible by each u_i
– is *even* in u_i if r_i is *odd*, and *vice versa.*

For each pair r and s large enough ($s \geq s_r$), there always exist non-trivial collections of special *singulands* $S^\bullet_{\boldsymbol{r}}$:

$$\{S^\bullet_{r_1,\dots,r_l} \in \mathrm{BIMU}^s_{r_1,\dots,r_l} \;\;;\;\; 1 < l < r \;\;,\;\; r_1 + \cdots + r_l = r\} \tag{6.29}$$

such that the corresponding bialternal *singulates* $\Sigma^\bullet_{\boldsymbol{r}}$ combine to form a $\Theta^\bullet_r$ that is singularity-free, i.e. *polynomial*, with the predictable total degree $s{-}r$ and an unchanged 'weight' s:

$$\Theta^\bullet_r := \sum_{1<l<r} \;\sum_{r_1+\ldots+r_l=r} \mathrm{slank}_{r_1,\dots,r_l}.S^\bullet_{r_1,\dots,r_l} \in \mathrm{ALAL} \cap \mathrm{BIMU}^s_r \tag{6.30}$$

[95] so that s may be called the 'weight' of $M^\bullet$.

instead of presenting at the origin multipoles of order τ:

$$\tau := r - l_{\min} \quad \textit{with} \quad 2 \leq l_{\min} := \inf(l) \quad \textit{for} \quad S^{\bullet}_{r_1,\dots,r_l} \neq 0^{\bullet} \tag{6.31}$$

as would be the case for randomly chosen *singulands* $S^{\bullet}_{\boldsymbol{r}}$. The result holds even if we impose that there be a least one nonzero singuland $S^{\bullet}_{r_1,r_2}$ of minimal length $l = 2$.

These paradoxical *non-singular singulates* $\Theta^{\bullet}_r$ are known as *wandering bialternals*. They span a subspace of $BIMU$ which is in fact a (small) subalgebra $ALAL_{wander}$ of $ALAL \subset ARI^{\underline{al}/\underline{al}}$. On top of the natural gradation by r (the *length*), $ALAL_{wander}$ admits a natural filtration by τ (the *'avoided polar order'*).

The presence of these *wandering bialternals* is responsible for the *very slight* indeterminacy that exists in the construction by singulation-desingulation of a basis of $ALIL \subset ARI^{\underline{\mathrm{al}}/\underline{\mathrm{il}}}$. As we saw, to remove that indeterminacy, additional criteria (arithmetical or functional) are called for, leading to the three (distinct yet closely related) bases of §6.5, §6.6, §6.7.

7 A conjectural basis for $ALAL \subset ARI^{\underline{al}/\underline{al}}$. The three series of bialternals.

7.1 Basic bialternals: the enumeration problem.

We shall have to handle three series of bialternals, each with a single non-zero component, of length 1, 2, 4 respectively. Here they are, with their names and natural indexation:

$$\begin{aligned}
\mathrm{ekma}^{\bullet}_{d}/\mathrm{ekmi}^{\bullet}_{d} &\in \mathrm{BIMU}_1 \quad , \qquad d \;\; \textit{even} \;\; \geq 2\\
\mathrm{doma}^{\bullet}_{d,b}/\mathrm{domi}^{\bullet}_{d,b} &\in \mathrm{BIMU}_2 \quad , \qquad d \;\; \textit{even} \;\; \geq 6,\; 1 \leq b \leq \beta(d)\\
\mathrm{carma}^{\bullet}_{d,c}/\mathrm{carmi}^{\bullet}_{d,c} &\in \mathrm{BIMU}_4 \quad , \qquad d \;\; \textit{even} \;\; \geq 8,\; 1 \leq c \leq \gamma(d)
\end{aligned}$$

As usual, the vocalic alternation $a \leftrightarrow i$ is indicative of the basic involution *swap*. The integers $\alpha(d)$, $\beta(d)$, $\gamma(d)$ are given by the generating functions:

$$\sum \alpha(d)\, t^d := t^6\, (1-t^2)^{-1}\, (1-t^4)^{-1} = t^6 + t^8 + 2\,t^{10} + 2\,t^{12} + 3\,t^{14} \dots \tag{7.1}$$

$$\sum \beta(d)\, t^d := t^6\, (1-t^2)^{-1}\, (1-t^6)^{-1} = t^6 + t^8 + t^{10} + 2\,t^{12} + 2\,t^{14} \dots \tag{7.2}$$

$$\sum \gamma(d)\, t^d := t^8\, (1-t^4)^{-1}\, (1-t^6)^{-1} = t^8 + t^{12} + t^{14} + t^{16} + t^{18} + 2\,t^{20} \dots \tag{7.3}$$

and clearly verify $\alpha(d) \equiv \beta(d) + \gamma(d-2)$. Mark the absence of t^{10} in (7.3).

7.2 The regular bialternals: *ekma, doma*.

The *ekma* bialternals are utterly elementary

$$\mathrm{ekma}_d^{w_1} := u_1^d \; ; \; \mathrm{ekmi}_d^{w_1} := v_1^d \tag{7.4}$$

since, for length 1, bialternality reduces to *neg*-invariance. If the *ekmas* freely generated a subalgebra *EKMA* of *ALAL*, the dimension of $EKMA_{2,d}$ (length 2, degree d) would be exactly $\alpha(d)$. This, however, is not the case. Indeed, since the bialternality constraints for length 2 are *finitary*[96], Hilbert's invariant theory applies, and it is a simple matter to verify that $ALAL_2$
(i) is spanned by *ekma* brackets,
(ii) admits the following *domas* as a canonical (in the sense of 'simplest') basis:

$$\mathrm{doma}_{d,b}^{w_1,w_2} := \mathrm{fa}(u_1,u_2)\,(\mathrm{ga}(u_1,u_2))^{b-1}\,(\mathrm{ha}(u_1,u_2))^{d/2-3b} \tag{7.5}$$
$$\mathrm{domi}_{d,b}^{w_1,w_2} := \mathrm{fi}(v_1,v_2)\,(\mathrm{gi}(v_1,v_2))^{b-1}\,(\mathrm{hi}(v_1,v_2))^{d/2-3b} \tag{7.6}$$

with

$$\mathrm{fa}(u_1,u_2) := u_1u_2(u_1-u_2)(u_1+u_2)(2u_1+u_2)(2u_2+u_1) \tag{7.7}$$
$$\mathrm{ga}(u_1,u_2) := (u_1+u_2)^2u_1^2u_2^2 \quad ; \quad \mathrm{ha}(u_1,u_2) := u_1^2+u_1u_2+u_2^2 \tag{7.8}$$
$$\mathrm{fi}(v_1,v_2) := v_1v_2(v_1-v_2)(v_1+v_2)(2v_1-v_2)(2v_2-v_1) \tag{7.9}$$
$$\mathrm{gi}(v_1,v_2) := (v_1-v_2)^2v_1^2v_2^2 \quad ; \quad \mathrm{hi}(v_1,v_2) := v_1^2-v_1v_2+v_2^2 \tag{7.10}$$

Therefore $dim(EKMA_{2,d}) = dim(ALAL_{2,d}) = \beta(d) \leq \alpha(d)$ and, for each even degree $d{+}2$, the *ekma*-brackets verify exactly $\gamma(d)$ independent relations of the form:

$$\sum_{d_1+d_2=d+2} Q_c^{d_1,d_2}\mathrm{ari}(\mathrm{ekma}^\bullet_{d_1},\mathrm{ekma}^\bullet_{d_2}) = 0^\bullet \quad (1\leq c\leq \gamma(d),\; Q_c^{d_1,d_2}\in\mathbb{Q}) \tag{7.11}$$

easily derivable from the decompositions:

$$\mathrm{ari}(\mathrm{ekma}^\bullet_{d_1},\mathrm{ekma}^\bullet_{d_2}) = \sum_{1\leq b\leq\beta(d_1+d_2)} K^b_{d_1,d_2}\,\mathrm{doma}^\bullet_{d_1+d_2,b} \quad (K^b_{d_1,d_2}\in\mathbb{Q}) \tag{7.12}$$

7.3 The irregular bialternals: *carma*.

Not all bialternals of length $r = 4$ may be obtained as superpositions of *ekma* brackets. Thus, there exists (up to scalar multiplication) exactly *one*

[96] i.e. correspond to invariance under a finite subgroup of $Gl_2(\mathbb{C})$, which in the present instance is isomorphic to $\mathfrak{S}_3$. Finitariness ceases from length 3 onwards.

bialternal of length $r = 4$ and degree $d = 8$, which clearly cannot be generated by *ekmas*, since the first *ekma* has degree 2, and self-bracketting it four times yields nothing.

One of our conjectures (for which there is compelling theoretical and numerical evidence[97]) is that the number of these independent *exceptional* or *irregular* bialternals – we call them *carma* bialternals – is exactly $\gamma(d)$ as given by (7.3), and that these bialternals $carma_{d,c}$ $(1 \leq c \leq \gamma(d))$ are in one-to-one, constructive correspondence (see §7.7) with the elements (7.11) of length 2 and degree $d{+}2$ in the *ekma* ideal, under a transparent and quite universal *restoration mechanism* (see §7.9).

7.4 Main differences between regular and irregular bialternals.

For one thing, the algebra $EKMA \subset ALAL$ generated by the *ekmas* is intrinsical, while the algebra $CARMA \subset ALAL$ generated by the *carmas* depends, as we shall see in §8.5, on the choice of a basis for $ALIL$. (That said, there exist clearly canonical bases of $ALIL$, and therefore canonical choices for $CARMA$ as well.)

Then, the definition of the $ekma^\bullet_d$ is as elementary as the construction of the $carma^\bullet_{d,c}$ is complex. Unsurprisingly, this difference finds its reflection in the arithmetical properties (divisibility etc) and above all in the sheer size of their coefficients.[98] For instance, if we consider the first 'cells' $ALAL_{r,d}$ where elements of $EKMA$ and $DOMA$ coexist with *unique* elements of $CARMA$, namely the 'cells' $r = 4$ and $d \leq 18$, and then compare typical elements of $EKMA_{r,d}$ and $DOMA_{r,d}$ with those of $CARMA_{r,d}$, we find that the latter are strikingly more complex.

For illustration, here is, with self-explanatory labels, a list of representatives chosen in the three algebras, with *red* signalling that our polynomials are taken in their simplest form, i.e. with coprime coefficients:

$$\begin{aligned}
\mathrm{cara}_d &:= \mathrm{red}(\mathrm{carma}_{d,1}) && (d = 8, 12, 14, 16, 18)\\
\mathrm{eka}_d &:= \mathrm{red}(\mathrm{ari}(\mathrm{ekma}_{d-6}, \mathrm{ekma}_2, \mathrm{ekma}_2, \mathrm{ekma}_2)) && (d = 10, 12, 14, 16, 18)\\
\mathrm{doa}_{14} &:= \mathrm{red}(\mathrm{ari}(\mathrm{doma}_{6,1}, \mathrm{doma}_{8,1}))\\
\mathrm{doa}_{16} &:= \mathrm{red}(\mathrm{ari}(\mathrm{doma}_{6,1}, \mathrm{doma}_{10,1}))\\
\mathrm{doa}_{18} &:= \mathrm{red}(\mathrm{ari}(\mathrm{doma}_{6,1}, \mathrm{doma}_{12,2}))
\end{aligned}$$

The first table (below) mentions the exact number of monomials effectively present in each polynomial. That number is always larger in the $\boldsymbol{u}$-variables

[97] see §7.9, §8.5, §8.10.

[98] This applies equally to the $ekma^\bullet_d$, $carma^\bullet_{d,c}$ and their swappees $ekmi^\bullet_d$, $carmi^\bullet_{d,c}$.

(vowel *a*) than in the $\boldsymbol{v}$-variables (vowel *i*), and the figures in boldface represent the difference. For comparison, the first column *FULL* mentions the maximum number of monomials in general homogeneous polynomials of the corresponding degree.

d	FULL	CARMA	CARMI		EKMA	EKMI		DOMA	DOMI	
8	165	142	118	**24**						
10	286				254	254	**0**			
12	455	434	420	**14**	422	408	**14**			
14	680	658	640	**18**	650	586	**64**	498	420	**78**
16	969	946	924	**22**	940	752	**188**	778	616	**162**
18	1330	1306	1280	**26**	1300	922	**378**	930	798	**132**

The next table (below) mentions the approximate norms of our (reduced!) polynomials, i.e. the sum of the absolute values of all their co-prime coefficients. Here again, the norms are much larger for the $\boldsymbol{u}$- than for the $\boldsymbol{v}$-variables, and the numbers in boldface represent the approximate ratios of the two.

d	CARMA	CARMI		EKMA	EKMI		DOMA	DOMI	
8	$8.6\ 10^6$	$2.6\ 10^6$	**3**						
10				$1.9\ 10^4$	$1.4\ 10^4$	**1.3**			
12	$1.2\ 10^{11}$	$6.0\ 10^9$	**19**	$1.8\ 10^5$	$3.3\ 10^4$	**5**			
14	$6.8\ 10^{12}$	$7.9\ 10^{10}$	**87**	$2.0\ 10^6$	$8.5\ 10^4$	**23**	$3.6\ 10^5$	$2.5\ 10^4$	**14**
16	$7.6\ 10^{13}$	$3.8\ 10^{11}$	**200**	$9.5\ 10^7$	$1.0\ 10^6$	**95**	$5.2\ 10^6$	$8.8\ 10^4$	**59**
18	$1.3\ 10^{17}$	$1.6\ 10^{14}$	**845**	$1.5\ 10^9$	3.910^6	**379**	$4.9\ 10^6$	$2.3\ 10^5$	**21**

Thus, while the polynomials in *CARMA* are only marginally *fuller* (i.e. less lacunary) than those in *DOMA* and *EKMA*, the main difference lies in their dramatically *larger* coefficients. Arithmetically, too, their coefficients are more complex, as borne out by their various reductions *mod p*.

7.5 The *pre-doma* potentials.

Rectifying $\sigma_{1,1}$ to $\sigma^*_{1,1}$.

The mapping $(A^\bullet, B^\bullet) \in \mathrm{BIMU}_1 \times \mathrm{BIMU}_1 \mapsto C^\bullet := \mathrm{ari}(A^\bullet, B^\bullet) \in \mathrm{BIMU}_2$ induces by bilinearity a mapping $\sigma_{1,1} : S^\bullet \in \mathrm{BIMU}_2 \mapsto \Sigma^\bullet \in \mathrm{BIMU}_2$ with:

$$\begin{aligned}\Sigma^{\left(\begin{smallmatrix} u_1,\ u_2 \\ v_1,\ v_2 \end{smallmatrix}\right)} &= +S^{\left(\begin{smallmatrix} u_1,\ u_2 \\ v_1,\ v_2 \end{smallmatrix}\right)} + S^{\left(\begin{smallmatrix} u_2,\ u_{12} \\ v_{2:1},\ v_1 \end{smallmatrix}\right)} + S^{\left(\begin{smallmatrix} u_{12},\ u_1 \\ v_2,\ v_{1:2} \end{smallmatrix}\right)} \\ &\quad -S^{\left(\begin{smallmatrix} u_2,\ u_1 \\ v_2,\ v_1 \end{smallmatrix}\right)} - S^{\left(\begin{smallmatrix} u_{12},\ u_2 \\ v_1,\ v_{2:1} \end{smallmatrix}\right)} - S^{\left(\begin{smallmatrix} u_1,\ u_{12} \\ v_{1:2},\ v_2 \end{smallmatrix}\right)}\end{aligned}$$

For arguments S^{w_1,w_2} that are *even* in both w_1 and w_2, $\sigma_{1,1}$ coincides with the simpler mapping $\sigma^*_{1,1} : S^\bullet \in \mathrm{BIMU}_2 \mapsto \Sigma^\bullet_* \in \mathrm{BIMU}_2$ with:

$$\begin{aligned}
\Sigma_*^{\binom{u_1,\, u_2}{v_1,\, v_2}} &= +S^{\binom{u_1,\, u_2}{v_1,\, v_2}} + S^{\binom{u_2,\ -u_{12}}{v_{2:1},\ -v_1}} + S^{\binom{-u_{12},\ u_1}{-v_2,\ v_{1:2}}} \\
&\quad -S^{\binom{u_2,\, u_1}{v_2,\, v_1}} - S^{\binom{-u_{12},\ u_2}{-v_1,\ v_{2:1}}} - S^{\binom{u_1,\ -u_{12}}{v_{1:2},\ -v_2}} \\
&= +S^{\binom{u_1,\ \ u_2}{v_{1:0},\, v_{2:0}}} + S^{\binom{u_2,\ \ u_0}{v_{2:1},\, v_{0:1}}} + S^{\binom{u_0,\ \ u_1}{v_{0:2},\, v_{1:2}}} \\
&\quad -S^{\binom{u_2,\ \ u_1}{v_{2:0},\, v_{1:0}}} - S^{\binom{u_0,\ \ u_2}{v_{0:1},\, v_{2:1}}} - S^{\binom{u_1,\ \ u_0}{v_{1:2},\, v_{0:2}}}
\end{aligned}$$

which in the "long notation" (i.e. under adjunction of $u_0 := -u_{1,2}$ and $v_0 :=$ *free*) takes on the pleasant form:

$$\begin{aligned}
\Sigma_*^{[w_0],w_1,w_2} &= +S^{[w_0],w_1,w_2} + S^{[w_1],w_2,w_0} + S^{[w_2],w_0,w_1} \\
&\quad -S^{[w_0],w_2,w_1} - S^{[w_1],w_0,w_2} - S^{[w_2],w_1,w_0}
\end{aligned}$$

In this form, the 'finitariness' of $\sigma^*_{1,1}$ is conspicuous, since the right-hand side involves exactly all six permutations of the sequence (w_0, w_1, w_2). But $\sigma^*_{1,1}$ has another merit: it turns not just all *bi-even*, but also all *bi-odd* alternals S^{w_1,w_2} into bialternals $\Sigma_*^{w_1,w_2}$ (whereas $\sigma_{1,1}$ only turns *bi-even* alternals into bialternals). When acting on bi-even (resp. bi-odd) alternals, $\sigma^*_{1,1}$ bilinearly extends the action of *ari* (resp. that of *oddari*: see (2.80)). The mappings $\sigma_{1,1}$ and $\sigma^*_{1,1}$ are of course reminiscent of the mappings from *singulands* $S^\bullet$ to *singulates* $\Sigma^\bullet$ which we studied at length in §5, except that now neither $\Sigma^\bullet$ nor $\Sigma^\bullet_*$ carries poles.

The bi-even *pre-doma* potentials.
Before turning to our proper object – the kernel of $\sigma^*_{1,1}$ – let us look for *pre-doma*-potentials, i.e. for (alternal, bi-even) pre-images of the $doma^\bullet_{d,b}$ under $\sigma^*_{1,1}$. If we impose the a priori form:

$$\mathrm{predoma}^{x_1,x_2}_{d,b} = \sum_{1\leq\delta\,\mathrm{ent}(\frac{d}{6})} c_{d,b;\delta}\ (x_1^{2\,\delta}\, x_2^{d-2\,\delta} - x_2^{2\,\delta}\, x_1^{d-2\,\delta}) \quad \big(d\ even\ , 1 \leq b \leq \mathrm{ent}(\frac{d}{6})\big)$$

the solution is unique, and this is essentially the only choice that yields *arithmetical smoothness*, i.e. that ensures for the prime factors p in the denominators of the coefficients $c_{d,b;\delta}$ universal bounds of type $p \leq C\, d$. In fact, the bound here is $p \leq d - 3$.

The bi-odd *pre-doma* potentials.
Here again, there is only one (alternal, bi-odd) a priori form (analogous to the above) that ensures arithmetical smoothness.

Arithmetical smoothness.
So, even in the case of the atypical, *singularity-free singulator* $\sigma^*_{1,1}$ we encounter anew the phenomenon which, in the preceding section, led us to the privileged bases $lama^\bullet_s$ and $loma^\bullet_s$, namely the existence of very specific conditions on the singulates that ensure unicity and simple 'factorial' bounds for the coefficients' denominators.

7.6 The *pre-carma* potentials.

Natural basis for $\ker(\sigma^*_{1,1})$**.**
On the space of alternal bimoulds that are independent of (v_1, v_2) and polynomial in (u_1, u_2) of even (total) degree d, the dimension of $\ker(\sigma^*_{1,1})$ is $s_d := ent(\frac{d-1}{3})$. Let us look for a convenient basis. Reverting to the (x_1, x_2) variables favoured for singulands, we see that the alternal bimoulds

$$H_{d,s}^{\binom{x_1\,,\,x_2}{0\,,\,0}} := (x_1+x_2)^s\,(x_1^s\,x_2^{d-2\,s} - x_2^s\,x_1^{d-2\,s}) \tag{7.13}$$

clearly belong to $\ker(\sigma^*_{1,1})$. Consider now the sequences:

$$\mathcal{H}_{d;s_1,s_2} = \{H^\bullet_{d,s_1}, H^\bullet_{d,s_1+1}, \ldots, H^\bullet_{d,s_2-1}, H^\bullet_{d,s_2}\} \tag{7.14}$$

The main facts here are these:
(i) The elements of $\mathcal{H}_{d;1,s_d}$ constitute a basis of $\ker(\sigma^*_{1,1})$.
(ii) The same holds for the shifted sets $\mathcal{H}_{d;1+k,s_d+k}$.
(iii) But it is only the first basis $\mathcal{H}_{d;1,s_d}$ that leads to arithmetically smooth expansions.

Natural basis for the *pre-carma* **space.**
The *pre-carmas* (so-called because they are the raw material from which the *carmas* shall be built) are the elements of $\ker(\sigma^*_{1,1})$ which are bi-even (i.e. even separately in x_1 and x_2) and divisible by $x_1^2\,x_2^2$.[99] The main result here[100] is that there exists a complete system of arithmetically smooth *pre-carmas* of the form:

$$\mathrm{precarma}_{d,k}^{x_1,x_2} = Q_{\tau(d)}(x_1,x_2)\,R_8(x_1,x_2)^k\,S_4(x_1,x_2)^{\kappa(d)-k}\,T_{d,k}(x_1,x_2) \tag{7.15}$$

with $1 \le k \le \kappa(d) = \gamma(d-2)$ and γ as in (7.3) or, equivalently:

$$\begin{array}{lllll} \kappa(d) & = & \mathrm{ent}(\frac{d-2}{12}) & \textit{if} \quad d \neq 10 \mod 12 & (\mathrm{ent} = \textit{entire part}) \\ \kappa(d) & = & \mathrm{ent}(\frac{d-2}{12}) + 1 & \textit{if} \quad d = 10 \mod 12 & \end{array}$$

[99] We add this last condition for the reason that one-variable elements of $\ker(\sigma^*_{1,1})$ would contribute no *carmas*: see the construction in §7.7.

[100] arrived at by expanding the bi-even solutions of $\sigma^*_{1,1}.S^\bullet_{1,1} = 0$ in the 'good' basis $\mathcal{H}_{d;1,s_d}$.

The first factor depends on $\tau(d) := \gcd(d, 12)$. It is of degree $\tau^*(d) := \tau(d)$ except when $12|d$, in which case $\tau^*(d) := 8$. It is given for the four possible values of $\tau(d)$ by:

$$Q_2 := x_1^2 - x_2^2 \ , \ Q_4 := x_1^4 - x_2^4 \ , \ Q_6 := x_1^6 - x_2^6 \ , \ Q_{12} := \frac{Q_4\, Q_6}{Q_2}$$

The second factor, of degree 8, is given by:

$$R_8 := x_1^2\, x_2^2\, (x_1^2 - x_2^2)^2$$

and the reason for its spontaneous occurrence is that the six following polynomials are divisible by $x_1^2\, x_2^2\, (x_1{+}x_2)^2$:

$$R_8(x_i, x_j) \quad , \quad R_8(x_i, x_i{+}x_j) \quad , \quad R_8(x_i{+}x_j, x_j) \qquad \textit{with} \ \ i, j \subset \{1, 2\}$$

The third factor, of degree 4, can be chosen arbitrarily, provided it is symmetric in (x_1, x_2), *even* in each variable, and co-prime with R_8. The following choices:

$$S_4 := \frac{Q_4^2}{Q_2^2} = (x_1^2 + x_2^2)^2 \quad ; \quad S_4 := \frac{Q_6}{Q_2} = x_1^4 + x_1^2\, x_2^2 + x_2^4$$

are natural candidates to the extent that they introduce no *new* factors, but there seems to exist no really privileged choice, i.e. no choice that would render the last factor $T_{d,k}$ indisputably *simplest*.

That last factor, symmetric in x_1, x_2 and with the right degree $\delta(d,k)$,[101] is then fully determined by the condition $\sigma^*_{1,1}.precarma_{d,k} = 0$. It is thus simplest for k maximal, i.e. $k = \kappa$. The corresponding $precarma_{d,\kappa}$ is also the only fully canonical $precarma_{d,k}$, since it does not depend on the choice of S_4.

7.7 Construction of the *carma* bialternals.

The idea behind the construction.
Fix a polynomial basis $\{l\text{ø}ma^{\bullet}_s, s = 3, 5, 7...\}$ of $ALIL \subset ARI^{\underline{\mathrm{al}}/\underline{\mathrm{il}}}$ [102] and consider a pre-carma polynomial *precar* of total degree $d{+}2$ (recall that d has to be *even* and either $= 8$ or ≥ 12) with alternal coefficients $c_{2\,\delta_1, 2\,\delta_2}$:

$$\mathrm{precar}^{x_1,x_2} = \sum_{2\,(\delta_1+\delta_2)=d+2}^{\delta_i \geq 1} c_{2\,\delta_1, 2\,\delta_2}\ x_1^{2\,\delta_1}\, x_2^{2\,\delta_2} \tag{7.16}$$

[101] i.e. to ensure degree d for $precarma_{d,k}$. Thus $\delta(d,k) = d - \tau^*(d) - 4\,k - 4\,\kappa(d)$.

[102] As usual, $ALIL$ and $ALAL$ are short-hand for $ARI^{\underline{\mathrm{al}}/\mathrm{il}}_{ent/cst}$ and $ARI^{\underline{\mathrm{al}}/\mathrm{al}}_{ent/cst}$. Constructing a basis of $ALIL$ is of course no easy matter, as we saw in §6, but what we require here is only a basis up to length 3, which is quite simple to construct: see §6.3.

Next, form the bimould $c\o r^\bullet$ by bracketting the $l\o ma^\bullet_s$ with the coefficients $c_{2\,\delta_1,2\,\delta_2}$ as weights:

$$\mathrm{c\o r}^\bullet \;:=\; \sum_{2\,(\delta_1+\delta_2)=d+2}^{\delta_i\geq 1} c_{2\,\delta_1,2\,\delta_2}\;\; \mathrm{preari}(l\o ma^\bullet_{1+2\,\delta_1}, l\o ma^\bullet_{1+2\,\delta_2}) \;\in \mathrm{ALIL} \quad (7.17)$$

$$=\; \sum_{2\,(\delta_1+\delta_2)=d+2}^{\delta_i\geq 1} c_{2\,\delta_1,2\,\delta_2}\;\; \frac{1}{2}\,\mathrm{ari}(l\o ma^\bullet_{1+2\,\delta_1}, l\o ma^\bullet_{1+2\,\delta_2}) \;\in \mathrm{ALIL} \quad (7.18)$$

and consider the projection $c\o rma^\bullet$ of $c\o r^\bullet$ on $BIMU_4$. By construction, $c\o r^\bullet$ is of type $\underline{\mathrm{al}}/\underline{\mathrm{il}}$ and its *first* non-vanishing component is therefore, *on its own*, of type $\underline{\mathrm{al}}/\underline{\mathrm{al}}$, i.e. bialternal. That first component cannot have length $r = 2$, because *precar* is a pre-carma. It cannot have length $r = 3$ either, because the component of length 3 is a polynomial of *odd* degree $1+d$ and for that reason cannot possibly be bialternal. This implies, therefore, that $c\o rma^\bullet$, i.e. the component of length 4, is either $\equiv 0$ or a *non-trivial bialternal* of degree d. Based on extensive computational and theoretical evidence, we conjecture that *the latter is always the case*, and more precisely, that when *precar* runs through a basis of the precarma space, the corresponding $c\o rma^\bullet$ span a space $C\O RMA_4$ such that

$$\mathrm{C\O RMA}_4 \oplus \mathrm{EKMA}_4 = \mathrm{ALAL}_4 \subset \mathrm{ARI}_4^{\underline{\mathrm{al}}/\underline{\mathrm{al}}} \quad (7.19)$$

In simpler words: the $c\o rma^\bullet$ provide *all the missing bialternals* of length $r = 4$ and put them in one-to-one correspondance with the $precarma^\bullet$, i.e. with the "unproductive" brackets of $ekma^\bullet$.

The construction works for any basis $\{l\o ma^\bullet_s\}$ of *ALIL*. Specialising it to the three canonical bases $\{lama^\bullet_s\}$, $\{loma^\bullet_s\}$, $\{luma^\bullet_s\}$, we get three series of 'exceptional' bialternals $\{carma^\bullet_s\}$, $\{corma^\bullet_s\}$, $\{curma^\bullet_s\}$, spanning spaces $CARMA_4$, $CORMA_4$, $CURMA_4$ which, though distinct, each verify the complementarity relation (7.19).

7.8 Alternative approach.

In the expansion (6.5) for $\{l\o ma^\bullet_s\}$, let us retain only the first two singulates (those namely that contribute to the components of length $r \leq 4$) and then let us restrict everything to the homogeneous parts of weight s. We get:

$$l\o ma^\bullet_s = \Sigma^\bullet_{[1],s} + \Sigma^\bullet_{[1,2],s} \qquad (\text{ mod } \mathrm{BIMU}_{5\leq}) \quad (7.20)$$

If we now plug this into (7.17) for pairs $(s_1, s_2) = (1+2\,\delta_1, 1+2\,\delta_2)$, we get four contributions $\mathcal{P}_{[\boldsymbol{r^1}],[\boldsymbol{r^2}]}$, consisting of the terms linear in $\Sigma^\bullet_{[\boldsymbol{r^1}],1+2\,\delta_1}$ and

$\Sigma^{\bullet}_{[\boldsymbol{r^2}],1+2\,\delta_2}$. The contribution $\mathcal{P}_{[1,2],[1,2]}$ begins with a non-zero component of length 6 and therefore vanishes *modulo* $BIMU_{5\leq}$. The contribution $\mathcal{P}_{[1],[1]}$ vanishes *exactly*, for the reason that, $adari(pal^{\bullet})$ being an algebra isomorphism, $\mathcal{P}_{[1],[1]}$ is necessarily of the form $adari(pal^{\bullet}).leng_4.\mathcal{P}_{[1],[1]}$, with $leng_r$ denoting as usual the projector of $BIMU$ onto $BIMU_r$. But $leng_4.\mathcal{P}_{[1],[1]} \equiv 0$ since we have assumed $precar^{\bullet}$ of (7.16) to be a *pre-carma*. Thus $\mathcal{P}_{[1],[1]} \equiv 0$. That leaves only the two contributions $\mathcal{P}_{[1],[1,2]} + \mathcal{P}_{[1,2],[1]}$, whose component of length 4 is clearly a singulate of type $\Sigma\Sigma^{\bullet}_{[1,1,2]} := slank_{[1,1,2]}.S^{\bullet}_{[1,1,2]}$. It can in fact be shown to be of the form:

$$\begin{aligned}
\Sigma\Sigma^{w_1,w_2,w_3,w_4}_{[1,1,2]} &= (\mathrm{slank}_{[1,1,2]}.S_{[1,1,2]})^{w_1,w_2,w_3,w_4} \qquad (7.21)\\
&= +X^{u_1,u_2,u_3,u_4}\,P(u_0) + Y^{u_1,u_2,u_3,u_4}\,P(u_2{+}u_3)\\
&\quad +X^{u_2,u_3,u_4,u_0}\,P(u_1) + Y^{u_2,u_3,u_4,u_0}\,P(u_3{+}u_4)\\
&\quad +X^{u_3,u_4,u_0,u_1}\,P(u_2) + Y^{u_3,u_4,u_0,u_1}\,P(u_4{+}u_0)\\
&\quad +X^{u_4,u_0,u_1,u_2}\,P(u_3) + Y^{u_4,u_0,u_1,u_2}\,P(u_0{+}u_1)\\
&\quad +X^{u_0,u_1,u_2,u_3}\,P(u_4) + Y^{u_0,u_1,u_2,u_3}\,P(u_1{+}u_2)
\end{aligned}$$

with polynomials $X^{\bullet}$ and $Y^{\bullet}$ given by:

$$\begin{aligned}
2\,X^{u_1,u_2,u_3,u_4} = {} & S^{u_3,u_2,u_1}_{[1,1,2]} + S^{u_1,u_3,u_4}_{[1,1,2]} + S^{u_3,u_4,u_1}_{[1,1,2]} + S^{u_3,u_1,u_4}_{[1,1,2]} - S^{u_2,u_1,u_4}_{[1,1,2]} - S^{u_2,u_3,u_4}_{[1,1,2]}\\
& - S^{u_2,u_4,u_1}_{[1,1,2]} - S^{u_4,u_2,u_1}_{[1,1,2]} + S^{u_3,u_1,u_{12}}_{[1,1,2]} + S^{u_4,u_2,u_{12}}_{[1,1,2]} + S^{u_2,u_4,u_{12}}_{[1,1,2]} + S^{u_1,u_3,u_{12}}_{[1,1,2]} + S^{u_2,u_3,u_{34}}_{[1,1,2]}\\
& + S^{u_4,u_1,u_{34}}_{[1,1,2]} + S^{u_1,u_4,u_{34}}_{[1,1,2]} + S^{u_3,u_2,u_{34}}_{[1,1,2]} + S^{u_2,u_{12},u_4}_{[1,1,2]} + S^{u_2,u_{23},u_4}_{[1,1,2]} + S^{u_2,u_{23},u_1}_{[1,1,2]} + S^{u_4,u_{34},u_1}_{[1,1,2]}\\
& - S^{u_2,u_3,u_{12}}_{[1,1,2]} - S^{u_3,u_2,u_{12}}_{[1,1,2]} - S^{u_4,u_1,u_{12}}_{[1,1,2]} - S^{u_1,u_4,u_{12}}_{[1,1,2]} - S^{u_1,u_3,u_{34}}_{[1,1,2]} - S^{u_4,u_2,u_{34}}_{[1,1,2]} - S^{u_3,u_1,u_{34}}_{[1,1,2]}\\
& - S^{u_2,u_4,u_{34}}_{[1,1,2]} - S^{u_1,u_{12},u_4}_{[1,1,2]} - S^{u_3,u_{23},u_4}_{[1,1,2]} - S^{u_3,u_{23},u_1}_{[1,1,2]} - S^{u_3,u_{34},u_1}_{[1,1,2]} + S^{u_2,u_3,u_{123}}_{[1,1,2]} + S^{u_2,u_1,u_{123}}_{[1,1,2]}\\
& + S^{u_2,u_4,u_{234}}_{[1,1,2]} + S^{u_4,u_2,u_{234}}_{[1,1,2]} - S^{u_1,u_3,u_{123}}_{[1,1,2]} - S^{u_3,u_1,u_{123}}_{[1,1,2]} - S^{u_3,u_4,u_{234}}_{[1,1,2]} - S^{u_3,u_2,u_{234}}_{[1,1,2]}\\
& + S^{u_1,u_{12},u_{123}}_{[1,1,2]} + S^{u_3,u_{23},u_{123}}_{[1,1,2]} + S^{u_3,u_{23},u_{234}}_{[1,1,2]} + S^{u_3,u_{34},u_{234}}_{[1,1,2]} - S^{u_2,u_{23},u_{123}}_{[1,1,2]}\\
& - S^{u_2,u_{12},u_{123}}_{[1,1,2]} - S^{u_2,u_{23},u_{234}}_{[1,1,2]} - S^{u_4,u_{34},u_{234}}_{[1,1,2]}
\end{aligned}$$

$$\begin{aligned}
2\,Y^{u_1,u_2,u_3,u_4} = {} & S^{u_4,u_3,u_1}_{[1,1,2]} + S^{u_1,u_4,u_2}_{[1,1,2]} + S^{u_4,u_1,u_2}_{[1,1,2]} + S^{u_1,u_3,u_4}_{[1,1,2]} + S^{u_3,u_4,u_1}_{[1,1,2]} + S^{u_3,u_1,u_4}_{[1,1,2]}\\
& - S^{u_2,u_1,u_4}_{[1,1,2]} - S^{u_1,u_4,u_3}_{[1,1,2]} - S^{u_4,u_2,u_1}_{[1,1,2]} - S^{u_1,u_2,u_4}_{[1,1,2]} - S^{u_2,u_4,u_1}_{[1,1,2]} - S^{u_4,u_1,u_3}_{[1,1,2]} + S^{u_3,u_4,u_{123}}_{[1,1,2]}\\
& + S^{u_4,u_3,u_{123}}_{[1,1,2]} + S^{u_1,u_3,u_{234}}_{[1,1,2]} + S^{u_3,u_1,u_{234}}_{[1,1,2]} + S^{u_4,u_{123},u_3}_{[1,1,2]} + S^{u_1,u_{234},u_3}_{[1,1,2]} - S^{u_4,u_2,u_{123}}_{[1,1,2]}\\
& - S^{u_2,u_4,u_{123}}_{[1,1,2]} - S^{u_2,u_1,u_{234}}_{[1,1,2]} - S^{u_1,u_2,u_{234}}_{[1,1,2]} - S^{u_4,u_{123},u_2}_{[1,1,2]} - S^{u_1,u_{234},u_2}_{[1,1,2]} + S^{u_2,u_{1234},u_4}_{[1,1,2]}\\
& + S^{u_2,u_{1234},u_1}_{[1,1,2]} + S^{u_{1234},u_4,u_3}_{[1,1,2]} + S^{u_{1234},u_2,u_4}_{[1,1,2]} + S^{u_{1234},u_2,u_1}_{[1,1,2]} + S^{u_{1234},u_1,u_3}_{[1,1,2]} - S^{u_3,u_{1234},u_1}_{[1,1,2]}\\
& - S^{u_3,u_{1234},u_4}_{[1,1,2]} - S^{u_{1234},u_3,u_4}_{[1,1,2]} - S^{u_{1234},u_4,u_2}_{[1,1,2]} - S^{u_{1234},u_3,u_1}_{[1,1,2]} - S^{u_{1234},u_1,u_2}_{[1,1,2]} + S^{u_2,u_{1234},u_{123}}_{[1,1,2]}\\
& + S^{u_2,u_{1234},u_{234}}_{[1,1,2]} + S^{u_{1234},u_2,u_{123}}_{[1,1,2]} + S^{u_{1234},u_2,u_{234}}_{[1,1,2]} + S^{u_{1234},u_{123},u_2}_{[1,1,2]} + S^{u_{1234},u_{234},u_2}_{[1,1,2]}\\
& - S^{u_3,u_{1234},u_{123}}_{[1,1,2]} - S^{u_3,u_{1234},u_{234}}_{[1,1,2]} - S^{u_{1234},u_3,u_{123}}_{[1,1,2]} - S^{u_{1234},u_3,u_{234}}_{[1,1,2]} - S^{u_{1234},u_{123},u_3}_{[1,1,2]} - S^{u_{1234},u_{234},u_3}_{[1,1,2]}
\end{aligned}$$

and with a singuland $S^{\bullet}_{[1,1,2]}$ that has to be a homogeneous polynomial of total degree $1+d$ subject to three types of constraints.

First, it must be *even* in x_1, x_2, *odd* in x_3, and divisible by $x_1\,x_2\,x_3$.

Second, it must verify the identity:

$$\begin{aligned}
0 = &-S^{x_1,x_2,x_3}_{[1,1,2]} + S^{x_1,x_2,x_{23}}_{[1,1,2]} + S^{x_2,x_1,x_{23}}_{[1,1,2]} + S^{x_1,x_{23},x_3}_{[1,1,2]} + S^{x_1,x_{12},x_3}_{[1,1,2]} - S^{x_1,x_3,x_{23}}_{[1,1,2]}\\
&-S^{x_3,x_1,x_{23}}_{[1,1,2]} - S^{x_2,x_{12},x_3}_{[1,1,2]} + S^{x_1,x_3,x_{123}}_{[1,1,2]} + S^{x_3,x_1,x_{123}}_{[1,1,2]} + S^{x_2,x_{123},x_3}_{[1,1,2]} + S^{x_{123},x_2,x_3}_{[1,1,2]}\\
&-S^{x_2,x_1,x_{123}}_{[1,1,2]} - S^{x_2,x_3,x_{123}}_{[1,1,2]} - S^{x_1,x_{123},x_3}_{[1,1,2]} + S^{x_2,x_{23},x_{123}}_{[1,1,2]} + S^{x_2,x_{12},x_{123}}_{[1,1,2]} + S^{x_3,x_{123},x_{23}}_{[1,1,2]}\\
&+S^{x_{123},x_3,x_{23}}_{[1,1,2]} - S^{x_1,x_{12},x_{123}}_{[1,1,2]} - S^{x_3,x_{23},x_{123}}_{[1,1,2]} - S^{x_2,x_{123},x_{23}}_{[1,1,2]} - S^{x_{123},x_2,x_{23}}_{[1,1,2]} - S^{x_{123},x_{23},x_3}_{[1,1,2]}
\end{aligned}$$

which ensures the absence of poles at the origin and therefore, in the terminology of §5.9, makes $\Sigma\Sigma^{\bullet}_{[1,1,2]}$ into a *wandering bialternal*.

Lastly, it must verify a third, similar-looking identity, which reflects the fact that *precar*$^{\bullet}$ is a *pre-carma* and, by so doing, guarantees that the bialternal $\Sigma\Sigma^{\bullet}_{[1,1,2]}$ won't be in *EKMA*.

Caveat: for each d, there is exactly one *carma* bialternal that is not captured by the above formula (7.21) but by a slight modification of the same.[103] This, however, is a minor technicality.

7.9 The global bialternal ideal and the universal 'restoration' mechanism.

Suppose that, contrary to all evidence (see §8.5) the ideal *IDEKMA* is not generated by *IDEKMA*$_2$, i.e. by the sole *pre-carmas*. There would then exist at least one $r>2$ and one identity of the form:

$$\sum_{2\,(\delta_1+...+\delta_r)=2\,\delta}^{\delta_i\geq 1} c_{2\,\delta_1,..,2\,\delta_r}\;\; x_1^{2\,\delta_1}...x_r^{2\,\delta_r}\,\mathrm{ari}(\mathrm{ekma}^{\bullet}_{2\,\delta_1},...,\mathrm{ekma}^{\bullet}_{2\,\delta_r})\equiv 0 \qquad (7.22)$$

corresponding to a *'prime'* (i.e. non-derivative) element of *IDEKMA*$_r$. We might then form the polynomial *prehar*:

$$\mathrm{prehar}^{x_1,...,x_r} = \sum_{2\,(\delta_1+...+\delta_r)=2\,\delta}^{\delta_i\geq 1} c_{2\,\delta_1,..,2\,\delta_r}\;\; x_1^{2\,\delta_1}...x_r^{2\,\delta_r} \qquad (7.23)$$

[103]Due to the presence of the corrective term $Ca^{\bullet}_3/Ci^{\bullet}_3$ in the formula linking the components of length 3 and weight 3 of *løma*$^{\bullet}$/*lømi*$^{\bullet}$. See (6.3), (6.4).

as an analogue of *precar* (see §7.7) and then use the alternal coefficients of *prehar* to construct a bimould $h\o r^\bullet$:

$$\mathrm{h\o r}^\bullet := \sum_{2\,(\delta_1+\ldots+\delta_r)=2\,\delta}^{\delta_i\geq 1} c_{2\,\delta_1,\ldots,2\,\delta_r}\ \mathrm{preari}(l\o ma^\bullet_{1+2\,\delta_1},\ldots,l\o ma^\bullet_{1+2\,\delta_r}) \in ALIL \quad (7.24)$$

$$= \sum_{2\,(\delta_1+\ldots+\delta_r)=2\,\delta}^{\delta_i\geq 1} c_{2\,\delta_1,\ldots,2\,\delta_r}\ \frac{1}{r}\ \mathrm{ari}(l\o ma^\bullet_{1+2\,\delta_1},\ldots,l\o ma^\bullet_{1+2\,\delta_r}) \in ALIL \quad (7.25)$$

exactly analogous to $c\o r^\bullet$. By arguing on the same lines as in §7.7, we would see that the first non-vanishing component $h\o rma^\bullet$ of $h\o r^\bullet$, necessarily of *even* degree $2\,\delta-2\,k$ and therefore of length $r+2\,k$ with $k\geq 1$, would automatically provide an 'exceptional' bialternal that would 'make up' for the missing element of *EKMA* corresponding to (7.22). Although, in keeping with our general conjectures, the existence of *prime* relations (7.22) is most unlikely, it is reasonable to speculate that, *if perchance they exist*, the corresponding $h\o rma^\bullet$ must then have length $r+2$ and degree $2\,\delta-2$, although they might conceivably have length $r+2\,k$ and degree $2\,\delta-2\,k$ for some $k\geq 2$. In any case, we have here a transparent *stop-gap mechanism* which automatically associates one *exceptional* bialternal to any 'missing' *regular* bialternal.

8 The enumeration of bialternals. Conjectures and computational evidence.

8.1 Primary, sesquary, secondary algebras.

Before addressing the enumeration of bialternals, let us return to the main subalgebras $\mathcal{A}$ of *ARI* listed in §2.5, but in the special case of bimoulds that are polynomial in $\boldsymbol{u}$ and constant in $\boldsymbol{v}$. For each such subalgebra $\mathcal{A}$, we tabulate the dimension $dim(\mathcal{A}_{r,d})$ of the cells of length $r\geq 3$ and of total $\boldsymbol{u}$-degree d. The reason for neglecting the length $r=1$ resp. 2 is that the results there are trivial resp. elementary.[104] As in §2.5, we reserve bold-face for the *secondary* subalgebras.

[104] since for $r=2$ the constraints that define $\mathcal{A}$ are always *finitary*.

$r=3$ \\ $d=$	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*
···········		·	·	··	··	··	··	··	··	··	··	··	··	··
$\mathbf{ARI}^{\mathrm{al}/*}$	**1**	**2**	**3**	**5**	**7**	**9**	**12**	**15**	**18**	**22**	**26**	**30**	**35**	**40**
$\mathrm{ARI}^{\mathrm{mantar}/*}$	2	4	6	9	12	16	20	25	30	36	42	49	56	64
$\mathrm{ARI}^{\mathrm{pusnu}/*}$	2	4	6	10	14	18	24	30	36	44	52	60	70	80
$\mathrm{ARI}^{\mathrm{pusnu}/*}_{\mathrm{mantar}/*}$	1	2	3	5	7	9	12	15	18	22	26	30	35	40
$\mathbf{ARI}^{\underline{\mathrm{al}}/\underline{\mathrm{al}}}$	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**0**	**2**	**0**	**2**	**0**	**4**
$\mathrm{ARI}^{\mathrm{al}/\mathrm{push}}$	0	1	0	2	1	3	2	5	3	7	5	9	7	12
$\mathrm{ARI}^{\mathrm{push}}$	0	2	2	5	4	8	8	13	12	18	18	25	24	32
$\mathrm{ARI}^{\overline{\mathrm{pusnu}}/\overline{\mathrm{pusnu}}}_{\mathrm{mantar}/.}$	1	0	1	0	2	0	3	1	4	2	6	2	8	4
$\mathrm{ARI}^{\overline{\mathrm{pusnu}}/\overline{\mathrm{pusnu}}}$	1	2	2	5	7	8	12	15	17	22	26	29	35	40

$r=4$ \\ $d=$	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*
···········		·	··	··	··	··	··	···	···	···	···	···	···	···
$\mathbf{ARI}^{\mathrm{al}/*}$	**1**	**2**	**5**	**8**	**14**	**20**	**30**	**40**	**55**	**70**	**91**	**112**	**140**	**168**
$\mathrm{ARI}^{\mathrm{mantar}/*}$	2	5	10	16	28	40	60	80	110	140	182	224	280	336
$\mathrm{ARI}^{\mathrm{pusnu}/*}$	3	7	15	25	42	62	90	122	165	213	273	339	420	508
$\mathrm{ARI}^{\mathrm{pusnu}/*}_{\mathrm{mantar}/*}$	2	4	9	14	24	34	50	66	90	114	147	180	224	268
$\mathbf{ARI}^{\underline{\mathrm{al}}/\underline{\mathrm{al}}}$	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**0**	**1**	**0**	**3**	**0**	**5**
$\mathrm{ARI}^{\mathrm{al}/\mathrm{push}}$	0	0	1	1	3	3	6	7	11	13	18	21	28	32
$\mathrm{ARI}^{\mathrm{push}}$	1	4	5	7	12	16	24	33	44	58	72	91	112	136
$\mathrm{ARI}^{\overline{\mathrm{pusnu}}/\overline{\mathrm{pusnu}}}_{\mathrm{mantar}/*}$	1	2	4	6	10	15	20	28	35	48	56	74	84	109
$\mathrm{ARI}^{\overline{\mathrm{pusnu}}/\overline{\mathrm{pusnu}}}$	2	4	10	15	28	40	60	79	110	140	182	223	280	336

$r=5 \quad \backslash\!\backslash \quad d=$	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*
$\mathbf{ARI}^{al/*}$	**1**	**3**	**7**	**14**	**25**	**42**	**66**	**99**	**143**	**200**	**273**	**364**	**476**	**612**
$\mathrm{ARI}^{mantar/*}$	3	9	19	38	66	110	170	255	365	511	693	924	1204	1548
$\mathrm{ARI}^{pusnu/*}$	4	12	28	56	100	168	264	396	572	800	1092	1456	1904	2448
$\mathrm{ARI}^{pusnu/*}_{mantar/*}$	2	6	14	28	50	84	132	198	286	400	546	728	952	1224
$\mathbf{ARI}^{\underline{al}/\underline{al}}$	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**0**	**2**	**0**	**5**
$\mathrm{ARI}^{al/push}$	0	1	1	3	3	9	9	19	22	36	42	66	74	108
ARI^{push}	1	5	6	12	20	38	52	85	118	169	224	310		
$\mathrm{ARI}^{\overline{pusnu}/\overline{pusnu}}_{mantar/*}$	2	3	8	14	26	42	69	99		200		364		612
$\mathrm{ARI}^{\overline{pusnu}/\overline{pusnu}}$	2	6	14	28	50	84	132	198	286	400	546	728	952	1224

Let us now tabulate the corresponding generating functions. These are always *rational*. For brevity, we set $X_m^n := (1-x^m)^{-n}$.

$r=3$	*generating function* $\sum \dim(d)\, x^d$
$\mathbf{ARI}^{al/*}$	$x\, X_1^2\, X_3^1$
$\mathrm{ARI}^{mantar/*}$	$x\,(2-2\,x^2+x^3)\, X_1^2\, X_2^1$
$\mathrm{ARI}^{pusnu/*}$	$2\,x\, X_1^2\, X_3^1$
$\mathrm{ARI}^{pusnu/*}_{mantar/.}$	$x\, X_1^2\, X_3^1$
$\mathbf{ARI}^{\underline{al}/\underline{al}}$	$x^8\,(1+x^2-x^4)\, X_2^1\, X_4^1\, X_6^1$
$\mathrm{ARI}^{al/push}$	$x^2\, X_2^2\, X_3^1$
ARI^{push}	$x^2\,(2+x^2-x^3-x^4+x^5)\, X_1^1\, X_2^1\, X_4^1$
$\mathrm{ARI}^{\overline{pusnu}/\overline{pusnu}}_{mantar/*}$	$x\,(1+x^7+x^9+x^{10}-x^{11})\, X_2^1\, X_4^1\, X_6^1$
$\mathrm{ARI}^{\overline{pusnu}/\overline{pusnu}}$	$x\,(1-x+x^2)\,(1+x-x^2)\, X_1^2\, X_3^1$

$r=4$	*generating function* $\sum \dim(d)\, x^d$
$\mathbf{ARI}^{\mathrm{al}/*}$	$x\, X_1^2\, X_2^2$
$\mathrm{ARI}^{\mathrm{mantar}/*}$	$2\, x\, X_1^2\, X_2^2$
$\mathrm{ARI}^{\mathrm{pusnu}/*}$	$x\,(3+x+x^2+x^3)\, X_1^2\, X_2^1\, X_4^1$
$\mathrm{ARI}^{\mathrm{pusnu}/*}_{\mathrm{mantar}/*}$	$x\,(2+x^2)\, X_1^2\, X_2^1\, X_4^1$
$\mathbf{ARI}^{\underline{\mathrm{al}}/\underline{\mathrm{al}}}$	$x^8\,(1+2\,x^4+x^6+x^8+2\,x^{10}+x^{14}-x^{16})\, X_2^1\, X_6^1\, X_8^1\, X_{12}^1$
$\mathrm{ARI}^{\mathrm{al}/\mathrm{push}}$	$x^3\, X_1^1\, X_2^2\, X_5^1$
$\mathrm{ARI}^{\mathrm{push}}$	$x\,(1+x-4x^2+3x^3+2x^4-5x^5+4x^6+x^7-2x^8-x^9+x^{10})\, X_1^3\, X_5^1$
$\mathrm{ARI}^{\overline{\mathrm{pusnu}}/\overline{\mathrm{pusnu}}}_{\mathrm{mantar}/*}$	$x\,(1+x+x^5-x^6)\, X_1^1\, X_2^2\, X_4^1$
$\mathrm{ARI}^{\overline{\mathrm{pusnu}}/\overline{\mathrm{pusnu}}}$	$x\,(2+2\,x^2-x^3+2\,x^4-2\,x^6+x^7)\, X_1^2\, X_2^1\, X_4^1$

$r=5$	*generating function* $\sum \dim(d)\, x^d$
$\mathbf{ARI}^{\mathrm{al}/*}$	$x\,(1+x^3)\, X_1^3\, X_2^1\, X_5^1$
$\mathrm{ARI}^{\mathrm{mantar}/*}$	$x\,(3-5\,x^2+5\,x^3+x^4-3\,x^5+x^6)\, X_1^3\, X_2^2$
$\mathrm{ARI}^{\mathrm{pusnu}/*}$	$4\,x\,(1+x^3)\, X_1^3\, X_2^1\, X_5^1$
$\mathrm{ARI}^{\mathrm{pusnu}/*}_{\mathrm{mantar}/*}$	$2\,x\,(1+x^3)\, X_1^3\, X_2^1\, X_5^1$
$\mathbf{ARI}^{\underline{\mathrm{al}}/\underline{\mathrm{al}}}$	$x^{10}\,(1+2\,x^2+3\,x^4+3\,x^6+2\,x^8)\, X_4^2\, X_6^2\, X_{10}^1$
$\mathrm{ARI}^{\mathrm{al}/\mathrm{push}}$	$x^2\,(1+x+x^2+3x^4+2x^5+x^6+x^7+2x^8)\, X_2^2\, X_3^1\, X_5^1\, X_6^1$
$\mathrm{ARI}^{\mathrm{push}}$	???
$\mathrm{ARI}^{\overline{\mathrm{pusnu}}/\overline{\mathrm{pusnu}}}_{\mathrm{mantar}/*}$	???
$\mathrm{ARI}^{\overline{\mathrm{pusnu}}/\overline{\mathrm{pusnu}}}$	$2\,x\,(1+x^3)\, X_1^3\, X_2^1\, X_5^1$

8.2 The 'factor' algebra *EKMA* and its subalgebra *DOMA*.

Of these two subalgebras of *ALAL*, generated respectively by the *ekmas* and *domas*, the first is obviously far from free (though all relations between the *ekmas* are conjectured to be generated by the sole *bilinear* relations) but the second is conjectured to be free, with the $doma^{\bullet}_{d,b}$, of length 2, as canonical generators.

The main unresolved point, even at the conjectural level, is this: how much of *EKMA* must one 'add' to *DOMA* to recover (ideally, with unique decomposition) the whole of *EKMA*? While the inclusion

$$\mathrm{EKMA}_1 \oplus \mathrm{DOMA} \oplus \mathrm{ari}(\mathrm{DOMA}, \mathrm{EKMA}_1) \subset \mathrm{EKMA}$$

is strict, the (rather small) gap between the two spaces would seem to be bridgeable, but exactly how is unclear at the moment.

8.3 The 'factor' algebra *CARMA*.

Like *DOMA*, *CARMA* is conjectured to be free (the theoretical case as well as the computational evidence here are even more overwhelming) but, unlike *DOMA*, it is not intrinsically defined: it exists in various isomorphic realisations (some canonical), all of which are conjectured to verify:

$$\mathrm{EKMA} \,\widehat{\otimes}\, \mathrm{CARMA} = \mathrm{ALAL}$$

with the notation $E \,\widehat{\otimes}\, C = A$ (not a tensor product!) signalling that A is freely generated by E and C, i.e. without constraints other than those internal to E and C: see §8.5, $\boldsymbol{C_1}$ *infra*.

8.4 The total algebra of bialternals *ALAL* and the original BK-conjecture.

How many multizeta irreducibles of weight s and length r must one retain to freely generate the $\mathbb{Q}$-ring $\mathbb{Z}eta$ of formal (uncoloured) multizetas? How many independent bialternals of weight s and length r are there in *ALAL*? It is easy to show that the answer to both questions is the same number $\mathcal{D}_{s,r}$, but harder to find these numbers. Based on their numerical investigation of *genuine* rather that *formal* multizetas, and on the assumptions that both rings are actually "the same", Broadhurst and Kreimer conjectured in [B] that the $\mathcal{D}_{s,r}$ are deducible, after Möbius inversion, from the formula:

$$\prod_{2\leq d, 1\leq r} \big(1 - z^s\, y^r\big)^{\mathcal{D}_{s,r}} = 1 - \frac{z^3\, y}{1-z^2} + \frac{z^{12} y^2 (1-y^2)}{(1-z^4)(1-z^6)} \tag{8.1}$$

8.5 The factor algebras and our sharper conjectures.

$\boldsymbol{C_1}$ **:** Under the *ari*-bracket, the factor algebras *EKMA* and *CARMA* *freely* generate the total algebra *ALAL* of all polynomial bialternals. *Freely* means: without other relations than those internal to each factor algebra.

$\boldsymbol{C_2}$ **:** Only the factor *EKMA* has internal relations, and all of these are generated by the bilinear relations between the $\{ekma^{\bullet}_d\,;\, d = 2, 4, 6\ldots\}$. We recall[105] that for each even degree d there are exactly $[[\frac{d-2}{4}]] - [[\frac{d}{6}]]$ such re-

[105] See §7.2.

lations.[106]

$\boldsymbol{C_3}$ **:** The $\{doma^{\bullet}_{d,\delta}\,;\ d{=}6,8\ldots,\delta\le[[\frac{d}{6}]]\}$ freely generate *DOMA*.

$\boldsymbol{C_4}$ **:** The $\{carma^{\bullet}_{d,\delta};\ d=8,12,14\ldots,\delta\le[[\frac{d}{4}]]-[[\frac{d+2}{6}]]\}$ freely generate *CARMA*.

If we now denote $D_{d,r}, D^{ek}_{d,r}, D^{do}_{d,r}, D^{car}_{d,r}$ the dimensions of the cells of *ALAL, EKMA, DOMA, CARMA* of degree d and length r, the above conjectures translate into the following formulas:

$$\boldsymbol{C_1^*}:\ \prod_{2\le d,1\le r}\left(1-x^d\,y^r\right)^{D_{d,r}}=1-\frac{x^2\,y}{1-x^2}+\frac{x^8y^2(x^2-y^2)}{(1-x^4)(1-x^6)}\tag{8.2}$$

$$\boldsymbol{C_2^*}:\ \prod_{2\le d,1\le r}\left(1-x^d\,y^r\right)^{D^{ek}_{d,r}}=1-\frac{x^2\,y}{1-x^2}+\frac{x^{10}y^2}{(1-x^4)(1-x^6)}\tag{8.3}$$

$$\boldsymbol{C_3^*}:\ \prod_{6\le d,1\le r}\left(1-x^d\,y^r\right)^{D^{do}_{d,r}}=1-\frac{x^6\,y^2}{(1-x^2)\,(1-x^6)}\tag{8.4}$$

$$\boldsymbol{C_4^*}:\ \prod_{8\le d,1\le r}\left(1-x^d\,y^r\right)^{D^{car}_{d,r}}=1-\frac{x^8\,y^4}{(1-x^4)(1-x^6)}\tag{8.5}$$

Formula $\boldsymbol{C_1^*}$ merely restates the classical BK-conjecture in the (d,r)-parameters, but $\boldsymbol{C_2^*}$, $\boldsymbol{C_3^*}$, $\boldsymbol{C_4^*}$ are sharp improvements. Above all, these formulas, together with the compellingly natural *restoration mechanism*[107] that underpins them, provide a convincing explanation for the complicated y^4-term in $\boldsymbol{C_1^*}$ and completely divest it of its mysterious character.

For explicitness, we shall now list the partial generating functions $D^*_r(x)=\sum D^*_{d,r}x^d$ for each algebra and the first lengths r.

[106] with $[[x]]:=$ *entire part of* x.
[107] see §7.7 and §7.9.

8.6 Cell dimensions for $ALAL$.

$$D_1 = \frac{x^2}{(1-x^2)}$$

$$D_2 = \frac{x^6}{(1-x^2)\,(1-x^6)}$$

$$D_3 = \frac{x^8\,(1+x^2-x^4)}{(1-x^2)\,(1-x^4)\,(1-x^6)}$$

$$D_4 = \frac{x^8\,(1+2\,x^4+x^6+x^8+2\,x^{10}+x^{14}-x^{16})}{(1-x^2)\,(1-x^6)\,(1-x^8)\,(1-x^{12})}$$

$$D_5 = \frac{x^{10}\,(1+2\,x^2+3\,x^4+3\,x^6+2\,x^8)}{(1-x^4)^2\,(1-x^6)^2\,(1-x^{10})}$$

$$\begin{aligned} D_6 = \; & x^{12}\,(1+x^8-2\,x^{10}+x^{14}-4\,x^{16}+4\,x^{18}-2\,x^{20}-x^{22}+2\,x^{24}-2\,x^{26} \\ & +2\,x^{28}-x^{32}+3\,x^{34}-3\,x^{36}+x^{38})(1-x^2)^{-3}(1-x^4)^{-1}\times \\ & (1-x^6)^{-1}(1-x^8)^{-1}(1-x^{12})^{-1}(1-x^{18})^{-1} \end{aligned}$$

$$\begin{aligned} D_7 = \; & x^{14}\,(1+4\,x^2+8\,x^4+8\,x^6+6\,x^8+4\,x^{10}+5\,x^{12}+6\,x^{14}+3\,x^{16}-2\,x^{18} \\ & -3\,x^{20}-x^{22}+x^{24}+x^{26})(1-x^4)^{-3}(1-x^6)^{-3}(1-x^{14})^{-1} \end{aligned}$$

$$\begin{aligned} D_8 = \; & x^{16}\,(1+3\,x^2+7\,x^4+8\,x^6+13\,x^8+14\,x^{10}+15\,x^{12}+16\,x^{14}+8\,x^{16} \\ & +10\,x^{18}+4\,x^{22}-3\,x^{24}+x^{26}-2\,x^{28}+x^{30}+x^{34})\times \\ & (1-x^2)^{-2}(1-x^6)^{-2}(1-x^8)^{-2}(1-x^{12})^{-2} \end{aligned}$$

8.7 Cell dimensions for *EKMA*.

$$D_1^{ek} = \frac{x^2}{(1-x^2)}$$

$$D_2^{ek} = \frac{x^6}{(1-x^2)\,(1-x^6)}$$

$$D_3^{ek} = \frac{x^8\,(1+x^2-x^4)}{(1-x^2)\,(1-x^4)\,(1-x^6)}$$

$$D_4^{ek} = \frac{x^{10}\,(1+x^2+2\,x^4+x^6+2\,x^8+x^{10}-x^{16})}{(1-x^2)\,(1-x^6)\,(1-x^8)\,(1-x^{12})}$$

$$D_5^{ek} = \frac{x^{12}\,(1+3\,x^2+4\,x^4+3\,x^6+x^8+x^{10}-x^{14}-x^{16})}{(1-x^4)^2\,(1-x^6)^2\,(1-x^{10})}$$

$$\begin{aligned} D_6^{ek} = {} & x^{14}\,(1+x^4-x^6+x^8-2\,x^{10}-x^{14}+x^{16}-x^{18}-x^{20}+x^{22} \\ & -x^{26}+2\,x^{28}+x^{34}-x^{36})(1-x^2)^{-3}(1-x^4)^{-1}(1-x^8)^{-1}\times \\ & (1-x^6)^{-1}(1-x^{12})^{-1}(1-x^{18})^{-1} \end{aligned}$$

$$\begin{aligned} D_7^{ek} = {} & x^{16}\,(1+4\,x^2+8\,x^4+10\,x^6+8\,x^8+6\,x^{10}+6\,x^{12}+6\,x^{14}+2\,x^{16} \\ & -3\,x^{18}-5\,x^{20}-3\,x^{22}+x^{26})(1-x^4)^{-3}(1-x^6)^{-3}(1-x^{14})^{-1} \end{aligned}$$

$$\begin{aligned} D_8^{ek} = {} & x^{18}\,(1+2\,x^2+7\,x^4+8\,x^6+17\,x^8+14\,x^{10}+23\,x^{12}+13\,x^{14}+17\,x^{16} \\ & +6\,x^{18}+3\,x^{20}-x^{22}-5\,x^{24}-2\,x^{26}-5\,x^{28}-x^{30}-x^{32}+x^{36})\times \\ & (1-x^2)^{-2}(1-x^6)^{-2}(1-x^8)^{-2}(1-x^{12})^{-2} \end{aligned}$$

8.8 Cell dimensions for *DOMA*.

$$D_1^{do} = D_3^{do} = D_5^{do} \ldots = 0$$

$$D_2^{do} = \frac{x^6}{(1-x^2)\,(1-x^6)}$$

$$D_4^{do} = \frac{x^{14}\,(1+x^4)}{(1-x^2)\,(1-x^4)\,(1-x^6)\,(1-x^{12})}$$

$$D_6^{do} = \frac{x^{20}\,(1+x^{10})}{(1-x^2)^2\,(1-x^4)\,(1-x^6)^2\,(1-x^{18})}$$

$$D_8^{do} = \frac{x^{26}\,(1+x^2)\,(1+x^4)\,(1+x^8)}{(1-x^2)\,(1-x^4)^3\,(1-x^6)^2\,(1-x^{12})^2}$$

$$D_{10}^{do} = \frac{x^{32}(1+x^4+2x^8+2x^{10}+x^{12}+x^{14}+4x^{18}+x^{22}+x^{24}+2x^{26}+2x^{28}+x^{32}+x^{36})}{(1-x^2)^3\,(1-x^4)\,(1-x^6)^3\,(1-x^{10})\,(1-x^{12})\,(1-x^{30})}$$

8.9 Cell dimensions for *CARMA*.

$$D_1^{car} = D_2^{car} = D_3^{car} = 0$$

$$D_4^{car} = \frac{x^8}{(1-x^4)\,(1-x^6)}$$

$$D_5^{car} = D_6^{car} = D_7^{car} = 0$$

$$D_8^{car} = \frac{x^{20}}{(1-x^2)\,(1-x^6)\,(1-x^8)\,(1-x^{12})}$$

$$D_9^{car} = D_{10}^{car} = D_{11}^{car} = 0$$

$$D_{12}^{car} = \frac{x^{28}\,(1+x^{12})}{(1-x^2)\,(1-x^4)\,(1-x^6)\,(1-x^8)\,(1-x^{12})\,(1-x^{18})}$$

Predictably, for the two *free* subalgebras of *ALAL*, i.e. *DOMA* and *CARMA*, the generating functions verify self-symmetry relations:

$$(x)^{-2\,n}\,D_{2\,n}^{do}(x) = (\frac{1}{x})^{-2\,n}\,D_{2\,n}^{do}(\frac{1}{x}) \tag{8.6}$$

$$(x)^{-3\,n}\,D_{4\,n}^{car}(x) = (\frac{1}{x})^{-3\,n}\,D_{4\,n}^{car}(\frac{1}{x}) \tag{8.7}$$

8.10 Computational checks (Sarah Carr).

We checked conjecture $\boldsymbol{C_3^*}$ (which of course is not independent of $\boldsymbol{C_2^*}$) for $r \leq 8$ and $d \leq 100$, by using the following, highly efficient method:
(i) form the *domi*-generating functions (see notations *fi*, *hi*, *gi* in §7.2):

$$\mathrm{gedomi}_{t;a,b}^{w_1,w_2} := \frac{t^6\, fi(v_1,v_2)}{(1-t^2\, a\, hi(v_1,v_2))(1-t^6\, b\, gi(v_1,v_2))} \tag{8.8}$$

(ii) form the *ari*-brackets of several copies of $gedomi_{t;a_i,b_i}^{\boldsymbol{w}}$; keep the variables v_1, v_2 and parameters a_i, b_i provisionally unassigned; and studiously refrain from simplifying the rational functions obtained in the process;
(iii) assign random entire values to the v_1, v_2 and a_i, b_i and reduce everything modulo some moderately large prime number p (8 or 9 digits);
(iv) expand everything into power series of t and, for each d, study the dimensions of the spaces generated by the coefficient in front of t^d.

We then requested Sarah Carr, during her 2010 stay at Orsay, to computationally check the other conjectures $\boldsymbol{C_1^*}, \boldsymbol{C_2^*}, \boldsymbol{C_4^*}$ for lengths r up to 8 and degrees d up to 100. To that end, we supplied her with a complete system of independent *carma/carmi*-polynomials[108] of degree $d \leq 40$ (there are exactly 44 such polynomials). Here is her own account of the method she used and the scope of her verifications.

Checking the conjectures $\boldsymbol{C_2^*}$ about *EKMA*.

Checking $\boldsymbol{C_2^*}$ is equivalent to checking conjecture $\boldsymbol{C_2^{**}}$, according to which all ari*-relations between the* $\mathrm{ekma}_d^\bullet$ *are generated by the sole bilinear relations (whose exact number is known from the theory). To test $\boldsymbol{C_2^{**}}$, I created the generators in the lengths and degrees given in Table A* infra. *To slightly reduce the complexity of the calculations, I opted for working with the* $\mathrm{ekmi}_d^\bullet$ *rather than the* $\mathrm{ekma}_d^\bullet$*, so as to deal with pair-wise differences of* $\mathrm{v_i}$*'s rather than multiple sums of* $\mathrm{u_i}$*'s.*

For each length r *and degree* d*, I calculated and stored all elements of the form* $\mathrm{ari}(\mathrm{f_{d',r'}}, \mathrm{f_{d-d',r-r'}})$ *where* $1 \leq \mathrm{r'} \leq [\mathrm{r}/2]$ *and* $2\,r'+2 \leq d' \leq d$*, and where* $\mathrm{f_{d',r'}}$ *(resp.* $\mathrm{f_{d-d',r-r'}}$*) is a basis element of the length* $\mathrm{r'}$ *(resp.* $\mathrm{r-r'}$*) degree* d' *(resp.* $\mathrm{d-d'}$*) graded part of the Lie algebra. Let the number of such generators be denoted by* $\mathrm{G_{d,r}^{ek}}$ *and let the elements in the set of generators be denoted by* $(\mathrm{g^{ek}})_{\mathrm{d,r}}^{\mathrm{i}}; 1 \leq \mathrm{i} \leq \mathrm{G_{d,r}^{ek}}$.

[108] they are those constructed from the *lama/lami*-basis of *ALIL* (see §6 and §7).

Since we know that the integers $\mathrm{D}^{\mathrm{ek}}_{\mathrm{d,r}}$ *are upper bounds for the dimensions, we need to verify that we have at least* $\mathrm{D}^{\mathrm{ek}}_{\mathrm{d,r}}$ *linearly independent elements. To check this, I created the generating series* $\sum_{1\leq i\leq G^{\mathrm{ek}}_{d,r}} \alpha_i (g^{ek})^i_{d,r}$. *The polynomials have many terms with large coefficients. I first zeroed out some of the terms in this series by setting a number of variables (between none and 5, depending on the length and degree) equal to zero. Then I defined* $\mathrm{G}^{\mathrm{ek}}_{\mathrm{d,r}}$ *randomly generated vectors from the series, by substituting a randomly chosen number (using the* Linear Algebra [Random Vector] Maple function*) between 1 and 20 for each of the variables, and repeating the process* $\mathrm{G}^{\mathrm{ek}}_{\mathrm{d,r}}$ *times. Lastly, I reduced these vectors modulo either 101 or 100003. Now, given the linear system defined by these matrices, there are a number of options for solving it. Since we expect this system to have some relations coming from the universal Jacobi identity and from the bilinear relations special to our problem, I tested the efficiency of the Maple commands* linalg[rank],linalg[ker] *and* solve. *The* solve *command proved to be the most efficient. I then used the solution of the linear system to find a basis for the length* r, *degree* d *part.*

The tests confirmed the conjecture $\boldsymbol{C_2^{**}}$ *to lengths and degrees given in Table A. More precisely, the dimensions of all degrees between* $2+2\times$ length *and the highest degree entered in the table were verified.*

Table A.

Length	Highest degree generators	Dimension highest degree
1	100	100
2	100	100
3	100	100
4	100	58
5	50	40
6	38	32
7	32	28
8	26	24

Checking the conjectures $\boldsymbol{C_4^{*}}$ about *CARMA*.

The calculations were done with the same method as for EKMA. *The scope of the verification is indicated in the following Table.*

Table B.

Length	Highest degree generators	Range of degrees verifying $\boldsymbol{C_4^*}$
4	46	8 −− 46
8	54	20 −− 54
12	58	28 −− 58

Checking the conjectures $\boldsymbol{C_1^*}$ about ***EKMA*** $\widehat{\otimes}$ ***CARMA*****.**[109]

Here again I used the same method as for the two previous conjectures. The results are consigned in the following Table.

Table C.

Length	Highest degree generators	Range of degrees verifying $\boldsymbol{C_1^*}$
4	46	8 −− 46
8	54	20 −− 54
12	58	28 −− 58

Acknowledgments. *My computations were done on the calculation servers at the Max Planck Institut für Math. in Bonn, the Medicis servers at the Ecole Polytechnique and the calculation servers at the Math. Dept. of Orsay University. I would like to thank these institutions for their permission and trust, and warmly thank the system administrators for their indispensable and patient guidance. (Sarah Carr).*

9 Canonical irreducibles and perinomal algebra.

9.1 The general scheme.

The trifactorisation of $\boldsymbol{Zag^\bullet}$**.**
Let $Zag^\bullet$ denote the generating functions of the (uncoloured) multizetas, defined as in (1.9), but with all $\epsilon_i = 0$ and all $e_i = 1$. This generating function $Zag^\bullet$ admits a remarkable trifactorisation in *GARI*, with a first factor $\mathrm{Zag}_{\mathrm{I}}^\bullet$ which in turn splits into three subfactors:

$$\mathrm{Zag}^\bullet \;:=\; \mathrm{gari}\big(\mathrm{Zag}_{\mathrm{I}}^\bullet, \mathrm{Zag}_{\mathrm{II}}^\bullet, \mathrm{Zag}_{\mathrm{III}}^\bullet\big) \tag{9.1}$$

$$\mathrm{Zag}_{\mathrm{I}}^\bullet \;:=\; \mathrm{gari}\big(\mathrm{tal}^\bullet\,,\, \mathrm{invgari.pal}^\bullet\,,\, \mathrm{Røma}^\bullet\big) \tag{9.2}$$

$$\mathrm{Zag}_{\mathrm{I}}^\bullet \;:=\; \mathrm{gari}\big(\mathrm{tal}^\bullet\,,\, \mathrm{invgari.pal}^\bullet\,,\, \mathrm{expari.røma}^\bullet\big) \tag{9.3}$$

[109] For the meaning of $\widehat{\otimes}$, see §8.3.

Here is where the three factors or sub-factors belong:

$$\begin{aligned}
\mathrm{tal}^\bullet\ ,\ \mathrm{pal}^\bullet\ &\in\ \mathrm{GARI}^{\mathrm{as/as}} && (9.4)\\
\mathrm{invgari.pal}^\bullet\ ,\ \mathrm{Zag}^\bullet\ ,\ \mathrm{Zag}^\bullet_{\mathrm{I}}\ &\in\ \mathrm{GARI}^{\mathrm{as/is}} && (9.5)\\
\mathrm{R\o ma}^\bullet\ ,\ \mathrm{Zag}^\bullet_{\mathrm{II}}\ ,\ \mathrm{Zag}^\bullet_{\mathrm{III}}\ &\in\ \mathrm{GARI}^{\underline{\mathrm{as}}/\underline{\mathrm{is}}} && (9.6)\\
\mathrm{r\o ma}^\bullet\ ,\ \mathrm{logari.Zag}^\bullet_{\mathrm{II}}\ ,\ \mathrm{logari.Zag}^\bullet_{\mathrm{III}}\ &\in\ \mathrm{ARI}^{\underline{\mathrm{al}}/\underline{\mathrm{il}}} && (9.7)
\end{aligned}$$

and here is their real meaning in terms of multizeta irreducibles:
(*i*) The factor $\mathrm{Zag}^\bullet_{\mathrm{I}}$ carries only powers of the special irreducibe $\zeta(2) = \pi^2/6$, of weight 2.
(*ii*) The factor $\mathrm{Zag}^\bullet_{\mathrm{II}}$ carries only irreducibles of even weight $s \geq 4$ and their products.
(*iii*) The factor $\mathrm{Zag}^\bullet_{\mathrm{III}}$ carries only irreducibles of odd weight $s \geq 3$ and their products.

Now, since *weight, length, and degree* are related by $s = r+d$, it is obvious that under the involution *neg.pari*:
(*j*) elements of *ARI* or *GARI* that carry only *even* weights remain unchanged
(*jj*) elements of *ARI* that carry only *odd* weights change sign, and their exponentials in *GARI* change into their *gari*-inverses.

With respect to our three factors, this yields:

$$\begin{aligned}
\mathrm{neg.pari.Zag}^\bullet_{\mathrm{I}}\ &=\ \mathrm{Zag}^\bullet_{\mathrm{I}} && (9.8)\\
\mathrm{neg.pari.Zag}^\bullet_{\mathrm{II}}\ &=\ \mathrm{Zag}^\bullet_{\mathrm{II}} && (9.9)\\
\mathrm{neg.pari.Zag}^\bullet_{\mathrm{III}}\ &=\ \mathrm{invgari.Zag}^\bullet_{\mathrm{III}} && (9.10)\\
\mathrm{gari}(\mathrm{Zag}^\bullet_{\mathrm{III}},\mathrm{Zag}^\bullet_{\mathrm{III}})\ &=\ \mathrm{gari}(\mathrm{neg.pari.invgari.Zag}^\bullet,\mathrm{Zag}^\bullet) && (9.11)
\end{aligned}$$

Since all elements of *GARI* have one well-defined square-root,[110] the last identity (9.11) readily yields $Zag^\bullet_{\mathrm{III}}$. Separating the last factor from the first two is thus an easy matter (assuming the flexion machinery). But separating $Zag^\bullet_{\mathrm{I}}$ from $Zag^\bullet_{\mathrm{II}}$ is much trickier, and requires the construction of a bimould $r\o ma^\bullet$ rather analogous to $l\o ma^\bullet$ but not quite. More precisely, the sought-after $r\o ma^\bullet$
– must (like $l\o ma^\bullet$) be of type $\underline{al}/\underline{il}$
– must (unlike $l\o ma^\bullet$) carry multipoles at the origin that are so chosen as to cancel those of $tal^\bullet$ and $pal^\bullet$ in the trifactorisation (9.3).

The auxiliary bimoulds $l\o ma^\bullet, r\o ma^\bullet$.
The building blocks are the elementary singulands $sa^\bullet_{s_1} \in BIMU_1$ and the corresponding elementary singulates $sa^\bullet_{\binom{s_1}{r_1}} \in ARI^{\underline{al}/\underline{il}}$:

$$\mathrm{sa}^{w_1}_{s_1} := u_1^{s_1-1} \qquad ; \qquad \mathrm{sa}^\bullet_{\binom{s_1}{r_1}} := \mathrm{slang}_{r_1}.\mathrm{sa}^\bullet_{s_1} \tag{9.12}$$

[110] Apply *expari*.$\frac{1}{2}$.*logari*.

The singulates $sa^{\bullet}_{\binom{s_1}{r_1}}$ are $\neq 0$ *iff* $s_1{+}r_1$ is even and $s_1 \geq 2$.

We then define $l\o ma^{\bullet}$ and $r\o ma^{\bullet}$ as sums of their homogeneous components of weight s:

$$l\o ma^{\bullet} := \sum_{s\,\mathrm{odd}\geq 3} l\o ma^{\bullet}_{s} \quad ; \quad r\o ma^{\bullet} := \sum_{s\,\mathrm{odd}\geq 2} r\o ma^{\bullet}_{s} \tag{9.13}$$

and proceed to construct these homogeneous components by bracketting the singulates, in *PREARI* rather than *ARI* (- because that is by far the theoretically cleaner way -), with the multibrackets always defined from *left to right*, as in (2.49).

$$\mathrm{l\o ma}^{\bullet}_{s} = \sum_{1\leq l} \sum_{\substack{\{ {s_i+\dots+s_l=s \atop r_1+\dots+r_l\ odd} \} \\ \{ {1\leq s_i, 1\leq r_i \atop s_i+r_i\ even} \}}} \mathrm{l\o m}^{\binom{s_1\ ,\dots,\ s_l}{r_1\ ,\dots,\ r_l}}\ \mathrm{preari}(\mathrm{sa}^{\bullet}_{\binom{s_1}{r_1}},\dots,\mathrm{sa}^{\bullet}_{\binom{s_l}{r_l}})\ (\forall s\ odd) \tag{9.14}$$

$$\mathrm{r\o ma}^{\bullet}_{s} = \sum_{1\leq l} \sum_{\substack{\{ {s_i+\dots+s_l=s \atop r_1+\dots+r_l\ even} \} \\ \{ {1\leq s_i, 1\leq r_i \atop s_i+r_i\ even} \}}} \mathrm{r\o m}^{\binom{s_1\ ,\dots,\ s_l}{r_1\ ,\dots,\ r_l}}\ \mathrm{preari}(\mathrm{sa}^{\bullet}_{\binom{s_1}{r_1}},\dots,\mathrm{sa}^{\bullet}_{\binom{s_l}{r_l}})\ (\forall s\ even) \tag{9.15}$$

As for $R\o ma^{\bullet}$, it may be sought either in the form $expari.r\o ma^{\bullet}$ or, equivalently but more directly, in the form:

$$\mathrm{R\o ma}^{\bullet} = 1^{\bullet} + \sum_{1\leq l} \sum_{\substack{\{ {s_i+\dots+s_l\ even \atop r_1+\dots+r_l\ even} \} \\ \{ {1\leq s_i, 1\leq r_i \atop s_i+r_i\ even} \}}} \mathrm{R\o m}^{\binom{s_1\ ,\dots,\ s_l}{r_1\ ,\dots,\ r_l}}\ \mathrm{preari}(\mathrm{sa}^{\bullet}_{\binom{s_1}{r_1}},\dots,\mathrm{sa}^{\bullet}_{\binom{s_l}{r_l}})\quad (\forall s\ even)$$

Of course, in the above expansions, all summands must be true singulates,[111] with a least a pole of order 1 at the origin, so that at least one of their indices r_i must be ≥ 2.

Due to the condition $\sum s_i = s$, the right-hand sides of (9.14) and (9.15) carry only finitely many summands. Each summand that goes into the making of $l\o ma^{\bullet}_{s}$ or $r\o ma^{\bullet}_{s}$ is of type $\underline{al}/\underline{il}$ and its *shortest component* is of *even* degree $d = \sum(s_i - r_i)$, which is compatible with its being of type $\underline{al}/\underline{al}$.

The moulds $l\o m^{\bullet}$ or $r\o m^{\bullet}$ (resp. $R\o m^{\bullet}$) must be alternal (resp. symmetral) and one goes from $r\o m^{\bullet}$ to $R\o m^{\bullet} = expmu(r\o m^{\bullet})$ by the straightforward mould exponential.

At this stage (i.e. provisionally setting aside all considerations of canonicity) the only additional constraints on the alternal moulds $l\o m^{\bullet}$, $r\o m^{\bullet}$, and

[111] with the sole exception of the first summand in the expansion (9.14) for $l\o ma^{\bullet}_{s}$, which is of the form $l\o m^{\binom{s}{1}}\, sa^{\bullet}_{\binom{s}{1}}$ with $l\o m^{\binom{s}{1}} = 1$.

the symmetral mould $R\o m^\bullet$ are these:
(k) $l\o m^\bullet$ must make $l\o ma_s^\bullet$ singularity-free;
(kk) $r\o m^\bullet$ (or $R\o m^\bullet$) must, within the *gari*-product:

$$\begin{aligned} \mathrm{Zag}_{\mathrm{I}}^\bullet &:= \mathrm{gari}\big(\mathrm{tal}^\bullet\,,\,\mathrm{invgari.pal}^\bullet\,,\,\mathrm{Roma}^\bullet\big) && (9.16)\\ &:= \mathrm{gari}\big(\mathrm{tal}^\bullet\,,\,\mathrm{invgari.pal}^\bullet\,,\,\mathrm{expari.}\sum_s \mathrm{roma}_{\mathrm{s}}^\bullet\big) && (9.17)\end{aligned}$$

eliminate all the singularities present in $gari(tal^\bullet, invgari.pal^\bullet)$;
(kkk) the moulds $l\o m^\bullet$ or $r\o m^\bullet$ must be rational-valued.

Explicit decomposition of multizetas into irreducibles.
Anticipating on the construction of $l\o ma^\bullet$ and its iso-weight parts $l\o ma_s^\bullet$, the *preari*-product gives us an extremely elegant and explicit representation of the multizetas in terms of irreducibles:

$$\mathrm{Zag}_{\mathrm{II}}^\bullet := 1^\bullet + \sum_{\substack{1\le l\\ l\ even}} \sum_{\substack{3\le s_i\\ s_i\ odd}} \mathrm{Irr\o}_{\mathrm{II}}^{s_1,\dots,s_l}\mathrm{preari}(\mathrm{l\o ma}_{s_1}^\bullet,\dots,\mathrm{l\o ma}_{s_l}^\bullet) \quad (9.18)$$

$$\mathrm{Zag}_{\mathrm{III}}^\bullet = 1^\bullet + \sum_{\substack{1\le l\\ l\ free}} \sum_{\substack{3\le s_i\\ s_i\ odd}} \mathrm{Irr\o}_{\mathrm{III}}^{s_1,\dots,s_l}\mathrm{preari}(\mathrm{l\o ma}_{s_1}^\bullet,\dots,\mathrm{l\o ma}_{s_l}^\bullet) \quad (9.19)$$

$$\mathrm{logari.Zag}_{\mathrm{II}}^\bullet = \sum_{\substack{1\le l\\ l\ even}} \sum_{\substack{3\le s_i\\ s_i\ odd}} \mathrm{irr\o}_{\mathrm{II}}^{s_1,\dots,s_l}\mathrm{preari}(\mathrm{l\o ma}_{s_1}^\bullet,\dots,\mathrm{l\o ma}_{s_l}^\bullet) \quad (9.20)$$

$$\mathrm{logari.Zag}_{\mathrm{III}}^\bullet := \sum_{\substack{1\le l\\ l\ odd}} \sum_{\substack{3\le s_i\\ s_i\ odd}} \mathrm{irr\o}_{\mathrm{III}}^{s_1,\dots,s_l}\mathrm{preari}(\mathrm{l\o ma}_{s_1}^\bullet,\dots,\mathrm{l\o ma}_{s_l}^\bullet) \quad (9.21)$$

The irreducible carriers $Irr\o_{\mathrm{III}}^\bullet$, $Irr\o_{\mathrm{III}}^\bullet$ (resp. $irr\o_{\mathrm{II}}^\bullet$, $irr\o_{\mathrm{II}}^\bullet$) are scalar moulds of symmetral (resp. alternal) type. They are related under ordinary mould exponentiation:

$$\begin{aligned} \mathrm{Irr\o}_{\mathrm{II}}^\bullet &= \mathrm{expmu.irr\o}_{\mathrm{II}}^\bullet && (9.22)\\ \mathrm{Irr\o}_{\mathrm{III}}^\bullet &= \mathrm{expmu.irr\o}_{\mathrm{III}}^\bullet && (9.23)\end{aligned}$$

The pair $irr\o_{\mathrm{II}}^\bullet$, $Irr\o_{\mathrm{II}}^\bullet$ has only (non-vanishing) components of even length. In the pair $irr\o_{\mathrm{III}}^\bullet$, $Irr\o_{\mathrm{III}}^\bullet$, however, $irr\o_{\mathrm{III}}^\bullet$ has only (non-vanishing) components of odd length, but $Irr\o_{\mathrm{III}}^\bullet$ has of course components of any length, even or odd.

There are two ways of looking at the expansions (9.18)-(9.21).

If we are dealing with *formal multizetas*, then our four moulds (9.22)-(9.23) are subject to no other constraints than the above, i.e. symmetrality or alternality, and a definite length parity. They subsume all multizeta irreducibles other than π^2 in the theoretically most satisfactory manner, i.e. without introducing any artificial dissymmetry.[112]

In practice, to decompose *formal* multizetas into irreducibles, one may:
– calculate $Zag_{_I}^\bullet$ according to (9.2) or (9.3);
– calculate $Zag_{_{II}}^\bullet$ and $Zag_{_{III}}^\bullet$ according to (9.18) and (9.19);
– calculate $Zag^\bullet$ according to the trifactorisation (9.1);
– calculate the *swappee* $Zig^\bullet$ of $Zag^\bullet$;
– harvest the Taylor coefficients of $Zig^\bullet$.

Since any given multizeta appears once and only once as Taylor coefficient of $Zig^\bullet$, it can thus be expressed in purely algorithmic manner, via the flexion machinery, in terms of $irr\text{ø}_{_{II}}^\bullet$ and $irr\text{ø}_{_{III}}^\bullet$, or $Irr\text{ø}_{_{II}}^\bullet$ and $Irr\text{ø}_{_{III}}^\bullet$.

When dealing with the *genuine* multizetas, on the other hand, the irreducibles are well-defined *numbers* and the five-step procedure works in both directions: it also enables one to express $irr\text{ø}_{_{II}}^\bullet$, $irr\text{ø}_{_{III}}^\bullet$ and $Irr\text{ø}_{_{II}}^\bullet$, $Irr\text{ø}_{_{III}}^\bullet$ in terms of the multizetas. This 'reverse expression', however, is not unique. To get a unique, privileged expression of the irreducibles – not in terms of multizetas, but of *perinomal* numbers – there is no (known) alternative to the approach sketched in §9.4 *infra*.

Explicit decomposition of multizetas into canonical irreducibles. To qualify as *canonical*, the irreducible carriers $irr\text{ø}_{II}{}^\bullet$, $irr\text{ø}_{III}{}^\bullet$ or $Irr\text{ø}_{II}{}^\bullet$, $Irr\text{ø}_{III}{}^\bullet$ just defined must correspond to a *compellingly natural* solution ($l\text{ø}ma_s^\bullet, r\text{ø}ma_s^\bullet$). The constraints (*k*),(*kk*),(*kkk*), however, do not quite suffice to uniquely determine the solution – due to the existence of *wandering bialternals*, which was pointed out in §6.9.

One cannot stress enough that this *residual indeterminacy*, compared with the huge *a priori indeterminacy* inherent in all other approaches, is quite negligible, and that too in a precise and measurable sense. Indeed, let $\mathcal{I}rr(r,s)$ be the space of prime irreducibles of length r and total weight s. Next, let $Wander(r,s)$ be the indeterminacy (i.e. number of free parameters) in the definition of the irreducibles in $\mathcal{I}rr(r,s)$ that comes from the existence of *wandering bialternals.* Lastly, let $Naive(r,s)$ be the indeterminacy that we would be stuck with in the *naive approach*, i.e. if we had no criteria for

[112] If one wishes for a basis of scalar irreducibles totally free of constraints, one can readily produce one by picking any minimal system of components of, say, $irr\text{ø}_{_{II}}^\bullet$ and $irr\text{ø}_{_{III}}^\bullet$, that is large enough to determine all other components by alternality. That essentially amounts to selecting a basis in the Lie algebra freely generated by the symbols $\epsilon_3, \epsilon_5, \epsilon_7$.... Many such bases exist (Lyndon's etc) but none is truly canonical. Thus, while in calculations it may often be convenient to opt for *free* i.e. *unconstrained* systems of irreducibles, from a theoretical viewpoint it is far preferable to stick with the *constrained* systems implicit in $irr\text{ø}_{_{II}}^\bullet$ and $irr\text{ø}_{_{III}}^\bullet$ or their symmetral counterparts $Irr\text{ø}_{_{II}}^\bullet$ and $Irr\text{ø}_{_{III}}^\bullet$.

privileging any given irreducible $\rho_{r,s}$ in $\mathcal{I}rr(r,s)$ over all its variants of the form:

$$\rho_{r,s} + \sum_{s_1+\ldots+s_l=s}^{l\geq 2} c^{r_1,\ldots,r_l}_{s_1,\ldots,s_l} \prod_{1\leq i\leq l} \rho_{r_i,s_i} \quad with \quad c^{r_1,\ldots,r_l}_{s_1,\ldots,s_l} \in \mathbb{Q} \ ; \ \rho_{r_i,s_i} \in \mathcal{I}rr(r_i,s_i) \tag{9.24}$$

One shows that, for each r fixed and $s \to \infty$, we have:

$$Wander(r,s)/Naive(r,s) = \mathcal{O}(s^{-1}) \tag{9.25}$$

So this small residual indeterminacy due to the wandering bialternals is something we could live with. We can remove it, however, and ensure both unicity and canonicity, by imposing additional conditions – of arithmetical or function-theoretical nature. As we shall see, there are three basic choices (two arithmetical options and a function-theoretical one) but we go with relative ease from the one to the others, so that we are still justified in speaking, in the singular, of *the* canonical choice.

9.2 Arithmetical criteria.

One way of lifting the residual indeterminacy in the construction of the pair $(l\o ma^\bullet_s, r\o ma^\bullet_s)$ is to impose additional linear constraints on the Taylor coefficients of the singulates $S^\bullet_{\boldsymbol{r}}$ being used in the successive[113] inductive steps. As it happens, there are two natural systems of linear constraints that do the trick. We mentioned them in §6.5 and §6.6 in the case of $l\o ma^\bullet_s$ and only at the first occurence (i.e. for $r = 3$) but they extend to all lengths, and have their exact counterparts for $r\o ma^\bullet_s$. They lead to two distinct pairs $(lama^\bullet_s, rama^\bullet_s)$ and $(loma^\bullet_s, roma^\bullet_s)$, which stand out on account of their arithmetical properties. Very roughly speaking: with the first pair, both singulators and singulates possess "more" independent Taylor coefficients but these have "smaller" denominators, whereas with the second pair the position is exactly reversed. In both cases, however, the denominators of the Taylor coefficients are always divisors of simple *factorials* that depend only on *length* and *degree*. That changes completely with the third pair $(luma^\bullet_s, ruma^\bullet_s)$, which we shall examine next and which is characterised by its functional properties.

9.3 Functional criteria.

To transport entire multipoles, we require dilation operators δ^n:
(i) that define a group action: $\delta^{n_1}\,\delta^{n_2} \equiv \delta^{n_1 n_2}\,, \forall n_i \in \mathbb{Q}^+$;

[113] For *odd* lengths r in the case of $l\o ma^\bullet_s$ and *even* lengths in the case of $r\o ma^\bullet_s$.

(ii) that act as flexion automorphisms;
(iii) that commute with the singulators (simple or composite);
(iv) that conserve multiresidues.

This imposes the definition:

$$(\delta^n.A)^{\binom{u_1\ ,\dots,\ u_r}{v_1\ ,\dots,\ v_r}} \ := \ n^{-r}\, A^{\binom{u_1/n\ ,\dots,\ u_r/n}{v_1.n\ ,\dots,\ v_r.n}} \qquad (\forall n \in \mathbb{Z}) \tag{9.26}$$

which ensures the required properties:

$$\delta^n \quad : \quad ARI^{\underline{\mathrm{al}}/\underline{\mathrm{al}}} \overset{\text{isom.}}{\to} ARI^{\underline{\mathrm{al}}/\underline{\mathrm{al}}} \ , \ ARI^{\underline{\mathrm{al}}/\underline{\mathrm{il}}} \overset{\text{isom.}}{\to} ARI^{\underline{\mathrm{al}}/\underline{\mathrm{il}}} \tag{9.27}$$

$$\delta^n\, \mathrm{slank}_{r_1,\dots,r_l}\ S^\bullet \ \equiv \ \mathrm{slank}_{r_1,\dots,r_l}\ \delta^n\ S^\bullet \tag{9.28}$$
$$\delta^n\, \mathrm{slang}_{r_1,\dots,r_l}\ S^\bullet \ \equiv \ \mathrm{slang}_{r_1,\dots,r_l}\ \delta^n\ S^\bullet \tag{9.29}$$

Next, to reflect the change from *power series* to *meromorphic functions*, we must replace
– the monomial singulands $sa^\bullet_{s_1} \in BIMU_1$ of singulates $sa^\bullet_{\binom{s_1}{r_1}} \in ARI^{\underline{\mathrm{al}}/\underline{\mathrm{il}}}_{r_1\leq}$
– by monopolar singulands $ta^\bullet_{n_1} \in BIMU_1$ of singulates $ta^\bullet_{\binom{n_1}{r_1}} \in ARI^{\underline{\mathrm{al}}/\underline{\mathrm{il}}}_{r_1\leq}$.

Concretely, we set:

$$\mathrm{ta}^{w_1} \ := \ (1-u_1)^{-1} \ \ , \ \ \mathrm{ta}^{w_1,\dots,w_r} := 0 \quad \textit{if} \quad r \neq 1 \tag{9.30}$$
$$\mathrm{ta}^\bullet_{n_1} \ := \ \delta^{n_1}.\mathrm{ta}^\bullet \ \ , \ \ ta^\bullet_{\binom{n_1}{r_1}} := \mathrm{slang}_{r_1}.\delta^{n_1}.\mathrm{ta}^\bullet = \delta^{n_1}.\mathrm{slang}_{r_1}.\mathrm{ta}^\bullet \tag{9.31}$$

We may now look for bimoulds *luma*$^\bullet$ and *ruma*$^\bullet$ given by expansions of the form:

$$\mathrm{luma}^\bullet = \sum_{1\leq l}\ \sum_{1\leq n_i, 1\leq r_i}^{\left\{ {n_i\ coprime \atop r_1+\dots+r_l\ odd} \right\}} \mathrm{lum}^{\binom{n_1\ ,\dots,\ n_l}{r_1\ ,\dots,\ r_l}}\ \mathrm{preari}(\mathrm{ta}^\bullet_{\binom{n_1}{r_1}},\dots,\mathrm{ta}^\bullet_{\binom{n_l}{r_l}}) \tag{9.32}$$

$$\mathrm{ruma}^\bullet = \sum_{1\leq l}\ \sum_{1\leq n_i, 1\leq r_i}^{\left\{ {n_i\ anything \atop r_1+\dots+r_l\ even} \right\}} \mathrm{rum}^{\binom{n_1\ ,\dots,\ n_l}{r_1\ ,\dots,\ r_l}}\ \mathrm{preari}(\mathrm{ta}^\bullet_{\binom{n_1}{r_1}},\dots,\mathrm{ta}^\bullet_{\binom{n_l}{r_l}}) \tag{9.33}$$

that run exactly parallel to (9.14) and (9.15), and may also be rewritten as:

$$\mathrm{luma}^\bullet = \sum_{1\leq l}\ \sum_{1\leq n_i\, ,\, 1\leq r_i}^{\left\{ {n_i\ coprime \atop r_1+\dots+r_l\ odd} \right\}} \mathrm{lum}^{\binom{n_1\ ,\dots,\ n_l}{r_1\ ,\dots,\ r_l}}\ \mathrm{slang}_{r_1,\dots,r_l}.\mathrm{mu}(\delta^{n_1}\mathrm{ta}^\bullet,\dots,\delta^{n_l}\mathrm{ta}^\bullet) \tag{9.34}$$

$$\mathrm{ruma}^\bullet = \sum_{1\leq l}\ \sum_{1\leq n_i\, ,\, 1\leq r_i}^{\left\{ {n_i\ anything \atop r_1+\dots+r_l\ even} \right\}} \mathrm{rum}^{\binom{n_1\ ,\dots,\ n_l}{r_1\ ,\dots,\ r_l}}\ \mathrm{slang}_{r_1,\dots,r_l}.\mathrm{mu}(\delta^{n_1}\mathrm{ta}^\bullet,\dots,\delta^{n_l}\mathrm{ta}^\bullet) \tag{9.35}$$

The remarkable fact is that if we impose:[114]

$$\mathrm{lum}^{\binom{1}{1}} = 1 \ \ , \ \ \mathrm{lum}^{\binom{n_1}{1}} = 0 \qquad \forall n_1 \geq 2$$

and

$$\mathrm{lum}^{\binom{n_1\ ,\ldots,\ n_l}{1\ ,\ldots,\ 1}} \ = \ 0 \qquad \forall l \geq 2\ ,\ \forall n_i \tag{9.36}$$

$$\mathrm{rum}^{\binom{n_1\ ,\ldots,\ n_l}{1\ ,\ldots,\ 1}} \ = \ 0 \qquad \forall l \geq 2\ ,\ \forall n_i \tag{9.37}$$

then there is only one mould *lum*$^\bullet$ (resp. *rum*$^\bullet$) such that *luma*$^\bullet$ be *free* of singularities at the origin (resp that *ruma*$^\bullet$ carry exactly the *right* singularities[115] there). So the problem now is no longer that of determining a canonical solution, but of ascertaining the arithmetical nature of the Taylor coefficients at the origin of the unique *luma*$^\bullet$ and the unique *ruma*$^\bullet$. With *luma*$^\bullet$ the problem arises only for lengths $r \geq 5$, and with *ruma*$^\bullet$ only for lengths $r \geq 4$. This, however, is not a matter for this Survey.

But even without addressing this question, we may note that the pair *luma*$^\bullet$, *ruma*$^\bullet$ leads to a trifactorisation (9.1) of *Zag*$^\bullet$ exactly as the pair *løma*$^\bullet$, *røma*$^\bullet$ did at the end of §9.1. Explicitly:

$$\mathrm{Zag}^\bullet_{\mathrm{I}} \ := \ \mathrm{gari}\Big(\mathrm{tal}^\bullet\ ,\ \mathrm{invgari}(\mathrm{pal}^\bullet)\ ,\ \mathrm{expari}(\sum_{1\leq n} \delta^n \mathrm{ruma}^\bullet)\Big) \tag{9.38}$$

$$\mathrm{Zag}^\bullet_{\mathrm{II}} \ := \ 1^\bullet + \sum_{1\leq r}\sum_{1\leq n_i} \mathrm{Urr}_{\mathrm{II}}^{n_1,\ldots,n_l}\mathrm{preari}(\delta^{n_1}\mathrm{luma}^\bullet,\ldots,\delta^{n_l}\mathrm{luma}^\bullet) \tag{9.39}$$

$$\mathrm{Zag}^\bullet_{\mathrm{III}} \ = \ 1^\bullet + \sum_{1\leq r}\sum_{1\leq n_i} \mathrm{Urr}_{\mathrm{III}}^{n_1,\ldots,n_l}\mathrm{preari}(\delta^{n_1}\mathrm{luma}^\bullet,\ldots,\delta^{n_l}\mathrm{luma}^\bullet) \tag{9.40}$$

$$\mathrm{logari.Zag}^\bullet_{\mathrm{II}} = \sum_{1\leq r}\sum_{1\leq n_i} \mathrm{urr}_{\mathrm{II}}^{n_1,\ldots,n_l}\mathrm{preari}(\delta^{n_1}\mathrm{luma}^\bullet,\ldots,\delta^{n_l}\mathrm{luma}^\bullet) \tag{9.41}$$

$$\mathrm{logari.Zag}^\bullet_{\mathrm{III}} := \sum_{1\leq r}\sum_{1\leq n_i} \mathrm{urr}_{\mathrm{III}}^{n_1,\ldots,n_l}\mathrm{preari}(\delta^{n_1}\mathrm{luma}^\bullet,\ldots,\delta^{n_l}\mathrm{luma}^\bullet) \tag{9.42}$$

Instead of the symmetral pair of irreducible carriers $Irrø^\bullet_{II}$, $Irrø^\bullet_{III}$ and the alternal pair $irrø^\bullet_{II}$, $irrø^\bullet_{III}$, we now have the symmetral pair $Irru^\bullet_{II}$, $Irru^\bullet_{III}$ and the alternal pair $irru^\bullet_{II}$, $irru^\bullet_{III}$, with indices no longer running through $\{3, 5, 7...\}$ but through $\mathbb{N}^*$. Moreover, when dealing with the genuine (rather than formal) multizetas, these four new moulds are well-determined, *rational-valued*, and, for any given length r, *perinomal* functions of their indices n_i. So it is about time to explain what perinomal functions are, and what they can accomplish.

[114]No such condition is requires for *rum*$^\bullet$ since it automatically vanishes when the sum r_1+...+r_l is odd, and in particular when it reduces to $r_1 = 1$.

[115]i.e. singularities capable of compensating those of *tal*$^\bullet$ and *pal*$^\bullet$ and of ensuring the regularity of $Zag^\bullet_I$ at the origin.

9.4 Notions of perinomal algebra.

A function $\rho \in \mathcal{C}(\mathbb{Z}^r, \mathbb{C})$ is said to be *perinomal* (of arity r and rank r^*) iff:
(i) there exist $S_1, \ldots, S_{r^*} \in Sl_r(\mathbb{Z})$ such that the functions $\rho \circ S_1, \ldots, \rho \circ S_{r^*}$ be linearly independent
(ii) for any $r^{**} > r^*$ and any $S_1, \ldots, S_{r^{**}} \in Sl_r(\mathbb{Z})$, the $\rho \circ S_1, \ldots, \rho \circ S_{r^{**}}$ are linearly dependent.

We set $S\rho := \rho \circ S$, which defines an anti-action of $Sl_r(\mathbb{Z})$ on $\mathbb{C}(\mathbb{Z}^r, \mathbb{C})$.

If $T \in Sl_r(\mathbb{Z})$, $\boldsymbol{S} := [S_1, \ldots, S_{r^*}] \in (Sl_r(\mathbb{Z}))^{r^*}$ and $\boldsymbol{n} \in \mathbb{Z}^r$, we also set:

$$\begin{array}{rclcl} \boldsymbol{S}\rho & := & [S_1\rho, \ldots, S_{r^*}\rho] & = & [\rho \circ S_1, \ldots, \rho \circ S_{r^*}] \\ T\boldsymbol{S}\rho & := & [TS_1\rho, \ldots, TS_d\rho] & = & [\rho \circ S_1 \circ T, \ldots, \rho \circ S_{r^*} \circ T] \\ T\boldsymbol{S}\rho(\boldsymbol{n}) & := & [TS_1\rho(\boldsymbol{n}), \ldots, TS_{r^*}\rho(\boldsymbol{n})] & = & [(\rho \circ S_1 \circ T)(\boldsymbol{n}), \ldots, (\rho \circ S_{r^*} \circ T)(\boldsymbol{n})] \end{array}$$

If the S_i are now chosen so as to make $S_1\rho, \ldots, S_{r^*}\rho$ linearly independent, for each T there must exist scalars $M_i^j(\boldsymbol{S}\rho\,;T)$ such that

$$\begin{array}{rcl} TS_i\rho(\boldsymbol{n}) & \equiv & \displaystyle\sum_{1 \leq j \leq d} S_j\rho(\boldsymbol{n})\, M_i^j(\boldsymbol{S}\rho\,;T) \quad (\forall\, i, \forall \boldsymbol{n}) \quad \textit{i.e. in matrix notation}: \\ T\boldsymbol{S}\rho(\boldsymbol{n}) & \equiv & (\boldsymbol{S}\rho(\boldsymbol{n}))\,.\,M(\boldsymbol{S}.\rho\,;T) \end{array} \tag{9.43}$$

But changing $\boldsymbol{S}$ into another choice $\boldsymbol{S'}$ would simply subject M to some T*-independent* matrix conjugation $M \to M'$:

$$M'(\boldsymbol{S'}\rho\,;T) = C(\boldsymbol{S'}\rho\,;\boldsymbol{S}\rho)\, M(\boldsymbol{S}\rho\,;T)\, C(\boldsymbol{S}\rho\,;\boldsymbol{S'}\rho) \tag{9.44}$$

Moreover, we clearly have:

$$M(\boldsymbol{S}\rho\,;T_1T_2) \equiv M(\boldsymbol{S}\rho\,;T_1)M(\boldsymbol{S}\rho\,;T_2) \tag{9.45}$$

The upshot is that the identity (9.43) defines a linear representation of $Sl_r(\mathbb{Z})$ into $Gl_{r^*}(\mathbb{Z})$ or rather $Sl_{r^*}(\mathbb{Z})$:

$$\mathrm{Sl}_r(\mathbb{Z}) \;\to\; \mathrm{Sl}_{r^*}(\mathbb{Z}) \tag{9.46}$$

$$T \;\mapsto\; M(\boldsymbol{S}\rho\,;T) \;\sim\; M_\rho(T) \tag{9.47}$$

This representation M_ρ in turn splits into irreducible factor representations M_{ρ, r_i^*}:

$$M_\rho = M_{\rho, r_1^*} \otimes \cdots \otimes M_{\rho, r_s^*} \qquad \textit{with} \qquad r_1^* + \ldots r_s^* = r^* \tag{9.48}$$

Analogy with polynomials and action of $sl_r(\mathbb{Z})$.

Let ρ be perinomal of type (r, r^*). For $T \in Sl_r(\mathbb{Z})$ of the form $id + nilpotent$ and with logarithm $t = \log(T) \in sl_r(\mathbb{Z})$, the image $M_\rho(T)$ of T in $TSl_{r^*}(\mathbb{Z})$ is also of the form $id + nilpotent$. For any $\boldsymbol{n}$ fixed in $\mathbb{Z}^r$ the sequence $\{T^k\rho(\boldsymbol{n}), k = \mathbb{Z}\}$ is therefore polynomial in k and it makes sense to set:

$$t\rho(\boldsymbol{n}) := \big[\partial_k \, T^k \, \rho(\boldsymbol{n})\big]_{k=0} \qquad (\forall \boldsymbol{n}\, , \, T{=}\exp(t)) \tag{9.49}$$

as if k were a continuous variable. This defines a *coherent* anti-action on $Peri_r$ (the ring of perinomal functions of arity r), first of the nilpotent part of $sl_r(\mathbb{Z})$, and then, by composition, of $sl_r(\mathbb{Z})$ in its entirety. This applies in particular for the elementary operators:

$$\begin{array}{lll} e_{i,j} \in \mathrm{sl}_r(\mathbb{Z}) & \text{“=”} & n_j \partial_{n_i} \\ E_{i,j} \in \mathrm{Sl}_r(\mathbb{Z}) & E_{i,j} & : \ \boldsymbol{n} \mapsto \boldsymbol{n'} \ \ \textit{with} \ \ n'_i := n_i + n_j \ \ \textit{and} \ \ n'_k = n_k \ \ \textit{if} \ \ k \neq i \end{array}$$

But despite this analogy with polynomial functions, perinomal functions as a rule do not admit sensible extensions beyond $\mathbb{Z}^r$: they are essentially discrete creatures.

Perinomal continuation.
Even for functions ρ defined only on a “full-measure” cone of $\mathbb{Z}^r$, e.g. on $\mathbb{N}^r$, the above definitions of perinomalness still applies, but under restriction to the sub-semigroup of $\Gamma \subset Sl_r(\mathbb{Z})$ that sends that cone into itself. When these conditions of “partial perinomalness” are fulfilled, on can then pick in Γ elements of the form $id + nilpotent$ and take advantage of the polynomial dependence of $T^k\rho(\boldsymbol{n})$ in k for $k \in \mathbb{N}$ to extend, in unique and coherent manner, the function ρ to the whole of $\mathbb{Z}^r$, and then define, on this extended function, the anti-action not just of Γ but of the whole of $Sl_r(\mathbb{Z}) \supset \Gamma$.

Stability properties of perinomal functions.
Perinomal functions are stable under most common operations, such as:
(i) ordinary addition and multiplication (assuming a common arity r);
(ii) concatenation or, what amounts to the same, mould mutiplication;
(iii) the whole range of flexion operations, and notably *ari/gari*.

The latter means that bimoulds $A^{\boldsymbol{w}}$ whose indices $w_i = \binom{u_i}{v_i}$ assume only entire values and whose dependence on the sequences $\boldsymbol{u}$ and/or $\boldsymbol{v}$ is *perinomal*, are stable under *ari, gari* etc.

Basic transforms $\rho \leftrightarrow \rho^* \leftrightarrow \rho^{\#}$.
The definitions read:

$$\rho^*(s_1, \ldots, s_r) \quad := \quad \sum_{n_i \in \mathbb{N}^*} \rho(n_1, \ldots, n_r)\, n_1^{-s_1} \ldots n_r^{-s_r} \quad (s_i \in \mathbb{C} \ or\ \mathbb{N}) \tag{9.50}$$

$$\rho^{\#}(x_1, \ldots, x_r) \quad \overset{\text{dom.}}{:=} \quad \sum_{n_i \in \mathbb{N}^*} \frac{\rho(n_1, \ldots, n_r)}{(n_1 - x_1) \ldots (n_r - x_r)} \qquad (x_i \in \mathbb{C}) \tag{9.51}$$

For $s_i \in \mathbb{C}$ and $\Re(s_i) > C_i$ with C_i large enough, the sum (9.50) converges to an analytic function ρ^* which may or may not possess a meromorphic continuation to the whole of $\mathbb{C}^r$. But one usually considers entire arguments s_i. The corresponding *perinomal numbers* $\rho^*(\boldsymbol{s})$ constitute a remarkable $\mathbb{Q}$-ring that not only extends the $\mathbb{Q}$-ring of multizetas, but is also the proper framework for studying the *"impartial"* multizeta irreducibles.

As for the sum (9.51), it usually converges only if we subtract from the generic summand suitable corrective monomials (of bounded degrees) in the x_i. Hence the caveat *"dom."* i.e. *"dominant"* over the sign = . The resulting meromorphic function $\rho^{\#}(\boldsymbol{x})$ is known as a *perinomal carrier.* Its Taylor coefficients are clearly related to the *perinomal numbers* $\rho^*(\boldsymbol{s})$ and its multiresidues $\rho(\boldsymbol{n})$ are *perinomal functions* of $\boldsymbol{n}$.

9.5 The all-encoding perinomal mould *peri*$^\bullet$.

Definition of *peri*$^\bullet$.
For any $l \geq 1$ and any integers n_i , $r_i \geq 1$ we set:

$$\begin{aligned}
\mathrm{peri}^{\binom{n_1\ ,\dots,\ n_l}{r_1\ ,\dots,\ r_l}} &:= \mathrm{urr}_{\mathrm{III}}^{n_1,\dots,n_l} && \textit{if}\ \ r_i \equiv 1\ \forall i && \textit{and}\ \ \textstyle\sum r_i = l\ \ \textit{is odd}\\
&:= \mathrm{urr}_{\mathrm{II}}^{n_1,\dots,n_l} && \textit{if}\ \ r_i \equiv 1\ \forall i && \textit{and}\ \ \textstyle\sum r_i = l\ \ \textit{is even}\\
&:= \mathrm{lum}^{\binom{n_1\ ,\dots,\ n_l}{r_1\ ,\dots,\ r_l}} && \textit{if}\ \ \max_i(r_i) > 1 && \textit{and}\ \ \textstyle\sum r_i\ \ \textit{is odd}\\
&:= \mathrm{rum}^{\binom{n_1\ ,\dots,\ n_l}{r_1\ ,\dots,\ r_l}} && \textit{if}\ \ \max_i(r_i) > 1 && \textit{and}\ \ \textstyle\sum r_i\ \ \textit{is even}
\end{aligned}$$

The following table recalls the origin and role of the four *parts* of *peri*$^\bullet$, depending on l and $\boldsymbol{r}$:

peri$^\bullet$	$\sum r_i$ *odd*	$\sum r_i$ *even*
$r_i = 1 \quad \forall i$	*constructs* $\mathrm{Zag}^\bullet_{\mathrm{III}}$ *from* $\mathrm{luma}^\bullet$	*constructs* $\mathrm{Zag}^\bullet_{\mathrm{II}}$ *from* $\mathrm{luma}^\bullet$
$max(r_i) > 1$	*constructs* $\mathrm{luma}^\bullet$ *from* $\mathrm{ta}^\bullet$	*constructs* $\mathrm{ruma}^\bullet$ *from* $\mathrm{ta}^\bullet$

In view of its definition, this holds-all mould *peri*$^\bullet$ may seem a hopelessly heterogeneous and ramshackle construct. However, upon closer examination, its four *parts* turn out to be so closely interrelated that they cannot be described or understood in isolation. This amply justifies our welding them together into a unique mould *peri*$^\bullet$ which, far from being composite, is almost "seamless" .

Properties of *peri*$^\bullet$.
(i) As a mould[116] with indices $\binom{n_i}{r_i}$, *peri*$^\bullet$ is *alternal.*[117]

[116]Despite having two-layered indices $\binom{n_i}{r_i}$, *peri*$^\bullet$ should be viewed as a mould rather than a bimould, since it would be meaningless to subject the r_i-part (as opposed to the n_i-part) to the flexion operations.

[117]As a consequence, is is enough to know a rather small subset of all numbers

(ii) For any fixed sequence $(r_1,\ldots,r_l)$, $\mathrm{peri}^{\binom{n_1\ ,\ldots,\ n_l}{r_1\ ,\ldots,\ r_l}}$ is a *perinomal* function of $(n_1,\ldots,n_l)$.
(iii) Although the above formulas define $peri^{\binom{\boldsymbol{n}}{\boldsymbol{r}}}$ only for an upper sequence $\boldsymbol{n}$ in $\mathbb{N}^l$, *perinomal continuation* ensures a unique extension to $\mathbb{Z}^l$.
(iv) There is another natural way of extending $peri^\bullet$ for $\boldsymbol{n}\in\mathbb{Z}^l$, namely by parity continuation, according to the formula:

$$\mathrm{peri}^{\binom{n_1\ ,\ldots,\ n_l}{r_1\ ,\ldots,\ r_l}} := (\mathrm{sign}(n_1))^{r_1}\ldots(\mathrm{sign}(n_l))^{r_l}\ \ \mathrm{peri}^{\binom{|n_1|\ ,\ldots,\ |n_l|}{r_1\ ,\ldots,\ r_l}} \tag{9.52}$$

(v) Whether the *perinomal* and *parity* continuations coincide – wholly, partially, or not at all – depends on the sequence $\boldsymbol{r}$ via simple criteria.
(vi) The *perinomal numbers* associated with $urr^\bullet_{II}$ and $urr^\bullet_{III}$ generate a $\mathbb{Q}$-ring that contains the $\mathbb{Q}$-ring of multizetas.
(vii) The *perinomal numbers* associated with $lum^\bullet$ resp. $rum^\bullet$ "tend" to be in $\mathbb{Q}$ resp. $\mathbb{Q}[\pi^2]$ (they are definitely there for very small sequence lengths l) but it is still a moot point whether this holds true for all l.

9.6 A glimpse of perinomal splendour.

As an illustration, we shall mention the remarkable perinomal equations involving the elementary transformations $E_{i,j}$ and $e_{i,j}$ relative to neighbouring indices i,j. So let us set:

$$\begin{aligned} E_i^+ &:= E_{i,i+1}\in \mathrm{Sl_l}(\mathbb{Z}) \quad ; \quad e_i^+ \text{ ``:=''}\ \ n_{i+1}\partial_{n_i}\in \mathrm{sl_l}(\mathbb{Z}) \\ E_i^- &:= E_{i,i-1}\in \mathrm{Sl_l}(\mathbb{Z}) \quad ; \quad e_i^- \text{ ``:=''}\ \ n_{i-1}\partial_{n_i}\in \mathrm{sl_l}(\mathbb{Z}) \end{aligned}$$

E_i^+ and E_i^- clearly commute, and so do e_i^+ and e_i^-.
Given a sequence $\boldsymbol{r}=(r_1,\ldots,r_l)$ and $1\le i\le l$, we set

$$r_i^+ := \sum_{i<j\le l} r_j \quad ; \quad r_i^- := \sum_{1\le j<i} r_j \tag{9.53}$$

$$(E_i^+ - id)^{1+r_i^+}\ (E_i^- - id)^{r_i^-}\ \mathrm{peri}^{\binom{n_1\ ,\ldots,\ n_l}{r_1\ ,\ldots,\ r_l}} \ \equiv\ 0 \quad (\forall \boldsymbol{r}\ ,\ \forall i) \tag{9.54}$$

$$(E_i^+ - id)^{r_i^+}\ (E_i^- - id)^{1+r_i^-}\ \mathrm{peri}^{\binom{n_1\ ,\ldots,\ n_l}{r_1\ ,\ldots,\ r_l}} \ \equiv\ 0 \quad (\forall \boldsymbol{r}\ ,\ \forall i) \tag{9.55}$$

In particular, for extreme values of i:

$$(E_1^+ - id)^{1+r_2+\ldots+r_l}\ \ \mathrm{peri}^{\binom{n_1\ ,\ldots,\ n_l}{r_1\ ,\ldots,\ r_l}} \ \equiv\ 0 \quad (\forall \boldsymbol{r}) \tag{9.56}$$

$$(E_l^- - id)^{1+r_1+\ldots+r_{l-1}}\ \mathrm{peri}^{\binom{n_1\ ,\ldots,\ n_l}{r_1\ ,\ldots,\ r_l}} \ \equiv\ 0 \quad (\forall \boldsymbol{r}) \tag{9.57}$$

$peri^{\binom{n_1\ ,\ldots,\ n_l}{r_1\ ,\ldots,\ r_l}}$, e.g. those with $r_1=\min(r_i)$, to know them all.

In the above identities, the discrete difference operators $E_i^{\pm}-id$ may of course be replaced by the derivations $e_i^{\pm}$. But the most interesting identities are these:

$$(E_1^+ - id)^{r_2+..+r_l}\ \mathrm{peri}^{\binom{n_1\ ,\dots,\ n_l}{r_1\ ,\dots,\ r_l}} \equiv \mathrm{peri}_L^{\binom{n_2\ ,\dots,\ n_l}{r_2\ ,\dots,\ r_l}} \quad (\forall \boldsymbol{r}) \tag{9.58}$$

$$(E_l^- - id)^{r_1+..+r_{l-1}}\ \mathrm{peri}^{\binom{n_1\ ,\dots,\ n_l}{r_1\ ,\dots,\ r_l}} \equiv \mathrm{peri}_R^{\binom{n_1\ ,\dots,\ n_{l-1}}{r_1\ ,\dots,\ r_{l-1}}} \quad (\forall \boldsymbol{r}) \tag{9.59}$$

$$(e_1^+)^{r_2+..+r_l}\ \mathrm{peri}^{\binom{n_1\ ,\dots,\ n_l}{r_1\ ,\dots,\ r_l}} \equiv \mathrm{peri}_{L*}^{\binom{n_2\ ,\dots,\ n_l}{r_2\ ,\dots,\ r_l}} \quad (\forall \boldsymbol{r}) \tag{9.60}$$

$$(e_l^-)^{r_1+..+r_{l-1}}\ \mathrm{peri}^{\binom{n_1\ ,\dots,\ n_l}{r_1\ ,\dots,\ r_l}} \equiv \mathrm{peri}_{R*}^{\binom{n_1\ ,\dots,\ n_{l-1}}{r_1\ ,\dots,\ r_{l-1}}} \quad (\forall \boldsymbol{r}) \tag{9.61}$$

because they yield new, simpler perinomal fonctions $peri_L^{\bullet}$, $peri_R^{\bullet}$ (or their infinitesimal variants $peri_{L*}^{\bullet}$, $peri_{R*}^{\bullet}$) that are themselves closely related to the *jump functions* that measure the differences between the 2^l perinomal continuations $C^{\epsilon_1,\dots,\epsilon_l} peri^{\binom{n_1\ ,\dots,\ n_l}{r_1\ ,\dots,\ r_l}}$ of $peri^{\binom{n_1\ ,\dots,\ n_l}{r_1\ ,\dots,\ r_l}}$ starting from the 'multioctant':

$$\mathcal{O}^{\epsilon_1,\dots,\epsilon_l} := \{(n_1,\dots,n_l) \in \mathbb{Z}^l \quad with \quad \epsilon_i n_i \in \mathbb{N}^* \ \ , \ \ \epsilon_i \in \{+,-\}\} \tag{9.62}$$

They are also related to the shorter components of $peri^{\bullet}$.

It is probably no exaggeration to say that this wondrous, double-layered mould $peri^{\bullet}$ is some sort of *algebraic Mandelbrot set* – its equal in terms of complexity and richness of sub-structure at all scales, but much tidier, because here the structure is algebraic in nature, consisting as it does of:
– the infinite series of perinomal fonctions encoded in $peri^{\bullet}$;
– their seemingly inexhaustible properties and relations;
– the degrees of the induced representations of $Sl_l(\mathbb{Z})$ for all l;
– the irreducible factor representations of these induced representations;
– the arithmetic properties of the corresponding perinomal numbers;
etc etc...

10 Provisional conclusion.

10.1 Arithmetical and functional dimorphy.

The word 'dimorphy' points to the parallel existence of two distinct multiplication rules, but the interpretation differs for *functions* and for *numbers.* For functions, the two multiplication rules define *distinct and independent products.* For numbers, they are merely *distinct and independent expressions* of one and the same product.

• **Dimorphy for functions rings.**

A function space $\mathbb{F}$ is said to be dimorphic if it is endowed with, and stable under, two distinct (bilinear) products – usually, *pointwise multiplication* and some form or other of *convolution.* One often adds the requirement that both products should have the same unit – usually, the constant function 1. Moreover, dimorphic function rings often possess two sets of *exotic derivations*, i.e. linear operators irreducible to ordinary differentiation but acting as abstract derivations respective to the first or second product. (It would be tempting to attach to these dimorphic function rings the label "bialgebra", had it not long ago acquired a different connotation – namely, stability under a product and a coproduct.)

• Dimorphy for numbers rings.
A countable $\mathbb{Q}$-ring $\mathbb{D} \subset \mathbb{C}$ is dimorphic if it has two countable prebases[118] $\{\alpha_m\}$ and $\{\beta_n\}$, with a simple conversion rule linking the two, and a multiplication rule[119] attached to each prebasis:

$$\alpha_m = \sum\nolimits^* H_m^n \ \beta_n \quad , \quad \beta_n = \sum\nolimits^* K_n^m \ \alpha_m \qquad (H_m^n \, , \, K_n^m \in \mathbb{Q})$$
$$\alpha_{m_1} \alpha_{m_2} = \sum\nolimits^* A_{m_1,m_2}^{m_3} \ \alpha_{m_3} \quad , \quad \beta_{n_1} \beta_{n_2} = \sum\nolimits^* B_{n_1,n_2}^{n_3} \ \beta_{n_3} \quad (A_{n_1,n_2}^{n_3} \, , \, B_{n_1,n_2}^{n_3} \in \mathbb{Q})$$

All sums Σ^* have to be finite. Moreover, the two multiplication rules must be "independent", in the precise sense that neither should follow *algebraically* from the other under the conversion rule. This in turn implies that neither $\{\alpha_m\}$ nor $\{\beta_n\}$ can be a $\mathbb{Q}$-basis of $\mathbb{D}$: there have to be non-trivial, linear $\mathbb{Q}$-relations between the α_m, and others between the β_n. The main challenges, when studying a dimorphic $\mathbb{Q}$-ring $\mathbb{D} \subset \mathbb{C}$, are therefore:
(i) ascertaining whether $\mathbb{D}$ is a *polynomial algebra* (generated by a countable set of *irreducibles*) or the quotient of a polynomial algebra by some ideal;
(ii) pruning each prebasis $\{\alpha_m\}$ and $\{\beta_n\}$ of redundant elements, so as to turn them into true bases;
(iii) whenever possible, constructing an *impartial* or *'non-aligned'* basis $\{\gamma_p\}$, positioned 'halfway' between $\{\alpha_m\}$ and $\{\beta_n\}$.
(iv) whenever possible, finding for the impartial γ_p's a *direct* expression that is itself *impartial* and leans neither towards the α_m's nor the β_n's.

• Kinship and difference between the two types of dimorphy: functional and numerical.
The two notions have much in common: indeed, most dimorphic number rings are derived from dimorphic function rings either via *function evaluation* at some special points, or via *function integration*, or again via the application of *exotic derivations* to the functions and the harvesting of the

[118] see definition at the beginning of §1.1.
[119] compatible with $\mathbb{D}$'s natural product, which is induced by that of $\mathbb{C}$.

constants produced in the process. And yet there is this striking difference: whereas the notion of dimorphic ring is entirely objective (- the two products are just *there* -), that of numerical dimorphy is embarrassingly subjective: on any countable $\mathbb{Q}$-ring $\mathbb{D} \subset \mathbb{C}$, one may always construct two prebases $\{\alpha_m\}$ and $\{\beta_n\}$ with the required properties. So what makes a $\mathbb{Q}$-ring $\mathbb{D}$ truly dimorphic is the existence of genuinely *natural* prebases, and the – often considerable difficulty – of solving the four problems (i), (ii), (iii), (iv) listed above. The irony, withal, is that the notion of numerical dimorphy, despite its conceptual shakiness, is much more interesting and basic than that of functional dimorphy, and throws up much harder problems.

• Hyperlogarithmic functions: the dimorphic ring $\mathcal{H}$.
An interesting dimorphic space is the space $\mathcal{H}$ of hyperlogarithmic functions, which is spanned by the $\overline{\mathcal{H}}^{\boldsymbol{\alpha}}$ thus defined:[120]

$$\overline{\mathcal{H}}^{\alpha_1,\dots,\alpha_r}(\zeta) = \int_0^\zeta \overline{\mathcal{H}}^{\alpha_1,\dots,\alpha_{r-1}}(\zeta_r)\,\frac{d\zeta_r}{\zeta_r - \alpha_r} \qquad \textit{with} \quad \mathcal{H}^{\emptyset}(\zeta) \equiv 1 \tag{10.1}$$

$\mathcal{H}$ is stable under pointwise multiplication and under the unit-preserving convolution $\star$:

$$(\underline{\mathcal{H}}_1 \star \underline{\mathcal{H}}_2)(\zeta) = \int_0^\zeta d\underline{\mathcal{H}}_1(\zeta_1)\,\underline{\mathcal{H}}_2(\zeta-\zeta_1) = \int_0^\zeta \underline{\mathcal{H}}_1(\zeta-\zeta_2)\,d\underline{\mathcal{H}}_2(\zeta_2) \tag{10.2}$$

Side by side with the $\boldsymbol{\alpha}$-encoding, it is convenient to consider an $\boldsymbol{\omega}$-encoding via the correspondence:

$$\underline{\mathcal{H}}^{\omega_1,\dots,\omega_r} := \overline{\mathcal{H}}^{\omega_1,\,\omega_1+\omega_2,\dots,\,\omega_1+\dots+\omega_r} \tag{10.3}$$

if all $\alpha_i := \omega_1 + \dots + \omega_i$ are $\neq 0$, and by a slightly modified formula otherwise.

For this function ring $\mathcal{H}$, the basic dimorphic stability follows from the fact that the moulds $\overline{\mathcal{H}}^{\bullet}$ and $\underline{\mathcal{H}}^{\bullet}$ are both *symmetral*, the former under pointwise multiplication, the latter under convolution. Moreover, there exist on $\mathcal{H}$ two rich arrays of exotic derivations: the *foreign derivations* ∇_{α_0} and the *alien derivations* Δ_{ω_0}. These are linear operators that basically 'analyse' the singularities 'over'[121] the points α_0 or ω_0, but in such a way as to make the ∇_{α_0} and Δ_{ω_0} act as *derivations* on $\mathcal{H}$ relative to, respectively, multiplication and convolution.

• Hyperlogarithmic numbers: the dimorphic ring $\mathbb{H}$.
If we now restrict ourselves to rational-complex sequences $\boldsymbol{\alpha}$ or $\boldsymbol{\omega}$ (i.e. with

[120] first for ζ small, and then in the large by analytic continuation.
[121] Since we are dealing here with highly ramified functions, we have to consider various leaves *over* any given point.

all indices α_i or ω_i in $\mathbb{Q}+i\,\mathbb{Q}$) and evaluate the corresponding $\overline{\mathcal{H}}^{\boldsymbol{\alpha}}$ or $\underline{\mathcal{H}}^{\boldsymbol{\omega}}$ *at* or *over* rational-complex points ζ, the space $\mathbb{Q}$-spanned by these numbers is actually a $\mathbb{Q}$-ring: the $\mathbb{Q}$-ring $\mathbb{H}$ of so-called *hyperlogarithmic numbers*, which is in fact dimorphic, since it possesses two natural prebases $\{\overline{H}^{\boldsymbol{\alpha}}\}$ and $\{\underline{H}^{\boldsymbol{\omega}}\}$, each with its own, independent multiplication rule.[122]

Clearly, $\mathcal{H}$ contains the space of polylogarithms with singularities over the unit roots. Likewise, $\mathbb{H}$ contains the dimorphic $\mathbb{Q}$-ring of all (colourless or coloured) multizetas, but it also contains much more: in fact, the structure of $\mathbb{H}$ is still farther from a complete elucidation than that of the ring of multizetas.

10.2 Moulds and bimoulds. The flexion structure.

• Moulds have their origin in alien calculus.
Alien calculus deals with the totally non-commutative derivations Δ_ω and with the Hopf algebra $\boldsymbol{\Delta}$ freely generated by them. Let $\mathbb{A}$ be any commutative algebra. Multiplying several elements $B_i \in \mathbb{A}\otimes\boldsymbol{\Delta}$:

$$B_i = \sum_{\bullet} A_i^{\bullet}\,\Delta_{\bullet} = \sum_{r\geq 0}\sum_{\omega_i} A_i^{\omega_1,\dots,\omega_r}\;\Delta_{\omega_1}\dots\Delta_{\omega_r} \tag{10.4}$$

reduces to multiplying the corresponding moulds $A_i^{\bullet}$, which in many contexts (e.g. in formal computation) is much more convenient.

Alien calculus led straightaway to the four hyperlogarithmic moulds:

$$\mathcal{U}^{\bullet}(z)\;,\;\mathcal{V}^{\bullet}(z)\;\;(\textit{resurgent-valued})\quad;\quad U^{\bullet}\;,\;V^{\bullet}\;\;(\textit{scalar-valued}) \tag{10.5}$$

with their many properties and symmetries, and this is what really got the whole subject of mould calculus started.

One way of looking at moulds is to think of them as permitting the handling of non-commutative objects by means of commutative operations.

Another is to view them as permitting the explicit calculation of objects (like the Taylor coefficients of the power series expansions of the solutions of very complex, non-linear equations) that would otherwise resist explicitation.

• Moulds have found their second largest application in local differential geometry.
Expansions of type (10.4) but with scalar- or function-valued moulds $A_i^{\bullet}$

[122]Their origin, roughly, is as follows: when we subject some 'monomial' $\overline{\mathcal{H}}^{\boldsymbol{\alpha}}$ (resp. $\underline{\mathcal{H}}^{\boldsymbol{\omega}}$) to an exotic derivation ∇_{α_0} (resp. Δ_{ω_0}), what we get is a linear combination of simpler monomials $\overline{\mathcal{H}}^{\boldsymbol{\alpha}'}$ (resp. $\underline{\mathcal{H}}^{\boldsymbol{\omega}'}$) with constant coefficients $\overline{H}^{\boldsymbol{\alpha}''}$ (resp. $\underline{H}^{\boldsymbol{\omega}''}$), which are precisely the elements of our two prebases.

and with homogeneous (ordinary) differential operators in place of the Δ_ω, are very useful in local differential geometry (especially when all data are analytic) for expressing and investigating normal forms, normalising transformations, fractional iterates etc. Here again, moulds make it possible to render explicit the seemingly inexplicitable – with all the benefits that accrue from transparency.

• **Mould operations and mould symmetries.**
Moulds of natural origin usually come with a definite symmetry type – *symmetral* or *symmetrel*, *alternal* or *alternel* – and most mould operations either preserve these symmetries or transmute them in a predictable manner.

• **Moulds and arborification.**
When natural mould-comould expansions such as (10.4) display normal divergence and yet "ought to converge" (because they stand for really existing function germs or 'local' geometric objects), a general and very effective remedy is at hand: the transform known as *arborification-coarborification* nearly always suffices to restore normal convergence. Roughly speaking, the transform in question replaces, dually in $A^\bullet$ and $\Delta_\bullet$, the totally ordered sequences $\boldsymbol{\omega}$ by sequences carrying a weaker, arborescent order, and it does so in such a way as to leave the global series formally unchanged, while effecting the proper internal reordering that restores convergence.

• **Bimoulds.**
There is much more to being a bimould than just carrying double-layered indices $w_i := (\begin{smallmatrix} u_i \\ v_i \end{smallmatrix})$. On top of being subject to the usual mould operations, like *mu* and *lu*, and being eligible for the four basic mould symmetries (see above), bimoulds can also display new symmetries *sui generis*, and can be subjected to numerous (unary or binary) operations without 'classical' equivalents. These are the so-called *flexion operations*, under which the u_i get added bunch-wise, and the v_i subtracted pair-wise, in such a way as to preserve $\sum u_i v_i$ and $\sum du_i \wedge dv_i$.

• **The flexion structure.**
A non-pedantic, if slightly cavalier, way of defining the *flexion structure* is to characterise it as the collection of all interesting objects (unary or binary operations, symmetry types, algebras, groups etc) that may be constructed on bimoulds from the sole *flexions*. It turns out that, up to isomorphy, the flexion structure consists of exactly:
(i) seven algebras, notably ARI and ALI.
(ii) seven groups, notably $GARI$ and $GALI$.
(iii) five super-algebras, notably $SUARI$ and $SUALI$.

• **Recovering most classical moulds from bimoulds.**

Many classical moulds (especially when, as is often the case, their analytical expression involves partial sums or pairwise differences of their indices ω_i) can be recovered, and their properties better understood, when viewed as special bimoulds with one vanishing row of indices (either $\boldsymbol{v} = \mathbf{0}$ or $\boldsymbol{u} = \mathbf{0}$)

• **Monogenous substructures.**
These are the spaces $Flex(\mathfrak{E}) = \oplus_{0 \leq r} Flex_r(\mathfrak{E})$ generated by a single length-one bimould $\mathfrak{E}^\bullet$ under *all* flexion operations. The most natural monogenous structures correspond to the case when $\mathfrak{E}^{w_1}$ is totally 'random' (i.e. when there are no unexpected relations in its flexion offspring) or possesses a given parity in u_1 and v_1 (four possibilities).

• **Flexion units and their offspring.**
In terms of applications the most important monogenous structures $Flex(\mathfrak{E})$ correspond to special generators $\mathfrak{E}^\bullet$ that verify the so-called *tripartite identity* (3.9). These $\mathfrak{E}^\bullet$ are known as *flexion units* and admit various realisations as concrete functions of w_1: polar, trigonometric, elliptic, 'flat' etc.

• **Algebraisation of the substructures.**
Each type of abstract generator $\mathfrak{E}^\bullet$ subject to a given set of constraints[123] may admit several realisations (as a function or distribution etc), or just one, or none at all. But in all cases the flexion structure $Flex(\mathfrak{E}) = \oplus_{0 \leq r} Flex_r(\mathfrak{E})$ generated by $\mathfrak{E}^\bullet$ is a well-defined algebraic object, with an integer sequence $d_r = \dim(Flex_r(\mathfrak{E}))$ that reflects the strength of the constraints on $\mathfrak{E}^\bullet$. Moreover, in most cases, the length-r component $Flex_r(\mathfrak{E})$ of $Flex(\mathfrak{E})$ possesses one (or several) *natural* bases $\{\mathfrak{e}^\bullet_{\boldsymbol{t}}\} = \{\mathfrak{e}^{w_1,\dots,w_r}_{\boldsymbol{t}}\}$, with basis elements *naturally* indexed by r-node *trees* $\boldsymbol{t}$ of a well-defined sort – like for instance *binary trees* if $\mathfrak{E}^\bullet$ is a *flexion unit* or *ternary trees* if $\mathfrak{E}^\bullet$ is 'random'.[124] This automatically endows the abstract space spanned by those trees with the full flexion structure and all its wealth of operations, opening the way for fascinating (and as yet largely unexplored) developments in combinatorics.[125]

• **Origins of the flexion structure.**
The flexion structure arose in the early 1990s in an analysis context, as a tool for describing a very specific type of resurgence, variously known as *quantum*

[123]like (3.9) or (3.28) or (3.29) etc.

[124]but with a given parity in u_1 and v_1.

[125]There exists of course an abundant botanical literature on trees of various descriptions, their enumeration, generation, classification etc. But so far these trees have not been studied, generated, classified etc from the angle of the flexion operations, for the obvious reason that these operations are new.

resurgence[126] or *parametric resurgence*[127] or *co-equational resurgence*.[128]

• Present and future of the flexion structure.
In the early 2000s, the flexion structure began to be used, to great effect, in the investigation of multizeta arithmetics and numerical dimorphy, and this is likely to remain the theory's main area of application for quite some time to come. However, the *algebraisation* of monogenous (resp. polygenous) structures like $Flex(\mathfrak{E})$ (resp. $Flex(\mathfrak{E}_1, ..., \mathfrak{E}_n)$) also suggests promising applications in algebra and combinatorics. We can even discern the outlines of a future 'flexion Galois theory' that would concern itself with the way in which a given type of constraints on $\mathfrak{E}^\bullet$ or on the $\mathfrak{E}_i^\bullet$ impacts the structure, dimensions, etc, of such objects as $Flex_r(\mathfrak{E})$ or $Flex_r(\mathfrak{E}_1, ..., \mathfrak{E}_n)$.

10.3 *ARI/GARI* and the handling of double symmetries.

• Simple symmetries or subsymmetries at home in *LU/MU*.
The uninflected mould bracket *lu* preserves *alternality* and its two subsymmetries: *mantar*-invariance and *pus*-neutrality.[129] Similarly, the uninflected mould product *mu* preserves *symmetrality* and its two subsymmetries: *gantar*-invariance and *gus*-neutrality.[130] And that's about all. Even when *lu* or *mu* are made to act on bimoulds, they preserve none of the double symmetries[131] and none of their induced subsymmetries[132] – not even the so crucial *push*- or *gush*-invariance.

• Double symmetries or subsymmetries at home in *ARI/GARI*.
Things change when we go over to the inflected operations, or rather to the right ones, since of all seven pairs consisting of a flexion Lie algebra and its group, only *ARI//GARI* and *ALI//GALI* are capable of preserving double symmetries and subsymmetries. In the case of *ARI* (resp. *GARI*) the full picture has been summarised on the table of §2.5 (resp. §2.6). Things differ slightly with *ALI* (resp. *GALI*), but we need not bother with these differ-

[126]because often encountered in the 'semi-classical' mechanics – i.e. when expanding formal solutions of the Schrödinger equation in power series of the Planck constant $\hbar$. See §11.1, §11.2, §11.3 *infra*.
[127]since it is typically encountered in power series of a (singular perturbation) parameter.
[128]because it is loosely dual to 'equational resurgence', that is to say, to the type of resurgence encountered in power series of the equation's proper variable.
[129]As defined in §2.4.
[130]As defined in §2.4.
[131]i.e. symmetries affecting simultaneously a bimould $M^\bullet$ and its swappee $swap.M^\bullet$.
[132]meaning of course the *strictly double* subsymmetries – i.e. those that don't follow from a *single* symmetry.

ences since, when restricted to bimoulds of type $\underline{al}/\underline{al}$ (resp. $\underline{as}/\underline{as}$), the Lie brackets *ari* and *ali* (resp. the group laws *gari* and *gali*) exactly coincide.

All the above, it should be noted, applies to *straight* (i.e. uninflected) double symmetries, but similar results hold for the *twisted*[133] double symmetries that really matter, beginning with $\underline{al}/\underline{il}$ and $\underline{as}/\underline{is}$.

• Ubiquity of poles at the origin: associator.
In the canonical trifactorisation of $Zag^\bullet$, the leftmost factor $Zag^\bullet_I$ which, we recall, encodes all the information about the canonical-rational associator, admits in its turn a trifactorisation of the form

$$\mathrm{Zag}^\bullet_{\mathrm{I}} = \mathrm{gari}(\mathrm{tal}^\bullet, \mathrm{invgari.pal}^\bullet, \mathrm{Roma}^\bullet) \tag{10.6}$$

and the strange thing is that, although $Zag^\bullet_I$, as a function of the u_i variables, is of course free of poles at the origin, all three factors are replete with them.
(i) The (polar) mid-factor $pal^\bullet$ contains nothing but multipoles at the origin, and so does its *gari*-inverse.
(ii) The (trigonometric) first factor $tal^\bullet$, which is a *periodised* variant of $pal^\bullet$, carries multipoles *at* and *off* the origin, and those *at* the origin are roughly the same as those of $pal^\bullet$.
(iii) Since the multipoles of $pal^\bullet$ and $tal^\bullet$ very nearly, but not exactly, cancel out at the origin, a (highly transcendental) third factor $Roma^\bullet$ is called for to remove the remaining singularities, and the construction of that third factor involves at every step special operators, the so-called *singulators*, whose function it is to introduce, in a systematic and controlled manner, all the required corrective singularities at the origin.

• Ubiquity of poles at the origin: singulators and generation of $ALIL \subset ARI^{\underline{al}/\underline{il}}$.
To construct any of the three alternative bases $\{luma^\bullet_s\}$, $\{loma^\bullet_s\}$, $\{lama^\bullet_s\}$ of $ALIL$, we start from the arch-elementary bimoulds *ekmas*, purely of length-1 and trivially of type $\underline{al}/\underline{al}$, and then apply $adari(pal^\bullet)$ to produce new bimoulds, this time of the right type $\underline{al}/\underline{il}$ but ridden with unwanted singularities at the origin. To remove these without losing the property $\underline{al}/\underline{il}$, we must then engage in a double process of *singularity destruction* and *singularity re-introduction* (at higher lengths), which is painstakingly described in §6. The operators behind the construction, the so-called *singulators*, are themselves built from the *purely singular*, polar bimould $pal^\bullet$. Poles, therefore, completely dominate the process – first as obstacles, then as remedies.

[133]or, should we say, *half-twisted*, since it is not the bimould $M^\bullet$ itself, but only its swappee $swap.M^\bullet$, that may display a twisted symmetry. No other combination would be stable under the flexion operations.

• **Ubiquity of poles at the origin: singulators and generation of** $ALAL \subset ARI^{\underline{al}/\underline{al}}$**. The exceptional bialternals.**
That the construction of pole-free bases for *ALIL* should involve poles at all intermediary steps, is surprising enough, but still halfway understandable, since the very definition of *alternility* involves (mutually cancelling) polar terms. But the really weird thing is that poles should also be required to construct bases of *ALAL*, since the double symmetry here is completely straight. Nevertheless, such is the case: to the elementary *ekma* bialternals, one must adjoin the exceptional and very complex *carma* bialternals, whose construction cannot bypass the introduction of poles, since it requires the prior knowledge of an *ALIL*-basis up to length $r = 3$ (but, thankfully, no farther), as shown in §7.

• **Ubiquity of poles away from the origin: perinomal analysis.**
Perinomal analysis deals with meromorphic functions that possess multipoles all over the place: their location admits a natural indexation over $\mathbb{Z}^r$, their multiresidues are also defined on $\mathbb{Z}^r$ and are of *perinomal* nature. So, here again, multipoles have a way of inviting themselves into all calculations.

• *ARI* **and the Ihara algebra.**
The fact that the Ihara algebra is isomorphic to a *twee tiny little* subalgebra of *ARI*[134] – namely, the subalgebra of bimoulds of type $\underline{al}/\underline{il}$, polynomial in $\boldsymbol{u}$ and constant in $\boldsymbol{v}$ – is no reason for 'equating' the two structures, or even their Lie bracket. But since there still reigns much confusion around this fraught issue, a short clarification is in order.
(i) To begin with, none of the dozens of pole-carrying bimoulds such as $pal^\bullet$ or $tal^\bullet$ or $r\o ma^\bullet$, which are key to the understanding of $Zag^\bullet$, possess any counterpart in the Ihara algebra. As a consequence, neither can the *carma* bialternals be constructed in that framework, nor can the reason behind their presence be understood, nor can anything even remotely resembling $l\o ma^\bullet$ be constructed.
(ii) Second, unlike the Ihara algebra, the *ARI* approach puts both symmetries – alternal and alternil – on exactly the same footing and does full justice to the duality that underpins multizeta (and general arithmetical) dimorphy. Indeed, with its involution *swap*, its built-in duality between upper and lower indices, and all the main bimoulds like $pal^\bullet/pil^\bullet$, $tal^\bullet/til^\bullet$ etc that always occur in pairs, *ARI* is itself 'dimorphic' to the marrow.
(iii) Third, the whole subject of perinomal algebra and of canonical irreducibles is beyond not just the computational reach of the Ihara algebra, but

[134]Though it houses the multizetas themselves (in a formalised version), the subalgebra in question is too cramped a framework for their complete elucidation, since most auxiliary constructions required in the process lie *outside*.

even its means of conception.
(iv) Fourth, unlike the Ihara algebra, *ARI*, with its double row of indices, lends itself effortlessly to the passage from uncoloured to coloured multizetas.
(v) Lastly, *ARI* arose independently of the Ihara algebra, in direct answer to a problem of analysis and resurgence. In fact, unlike the Ihara algebra, *ARI* is serviceable in analysis no less than in algebra.

10.4 What has already been achieved.

Finding the proper setting was the first and arguably main step. The rest followed rather naturally.

• Correction formula.
Moving from the scalar multizetas $Wa^\bullet/Ze^\bullet$ to the generating functions $Zag^\bullet/Zig^\bullet$ makes it much easier to understand the reason for the corrective terms $Mana^\bullet/Mini^\bullet$ in (1.27), (1.28). As meromorphic functions, $Zag^\bullet$ and $Zig^\bullet$ are both given by *semi-convergent* series of multipoles. Formally, the involution *swap* exchanges both series *exactly*, but alters their *summation order*, leading to simple corrective terms constructed from monozetas.

• Meromorphic continuation of multizetas and arithmetical nature at negative points.
When taken in the *Ze* encoding, the scalar multizetas $Ze^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{s_1\ ,\dots,\ s_r}}$ possess a meromorphic extension to the whole of $\mathbb{C}^r$, with all their multipoles on $\mathbb{Z}^r$.
(i) the density of multipoles decreases with the 'coloration' of the multizetas, i.e. with the number of non-vanishing ϵ_i's.
(ii) the values (resp. residues) found at the regular (resp. irregular) places $\boldsymbol{s} \in \mathbb{Z}^r \dot{-} \mathbb{N}^r$ are themselves rational combinations of *simpler* multizetas.[135]
(iii) the symmetrelity relations verified by $Ze^\bullet$, which hold for positive s_i's, extend by meromorphic continuation to the whole of $\mathbb{C}^r$, including to the points of $\mathbb{Z}^r$ where, in view of (ii), they *might* – but in fact *do not* – generate new multizeta relations.[136]

• Unit-cleansing.
Any 'uncoloured' multizeta $Ze^{\binom{0\ ,\dots,\ 0}{s_1\ ,\dots,\ s_r}}$ with $s_i \in (\mathbb{N}^*)^r$ can in fact be expressed (in non-unique manner) as a rational-linear combinations of analogous but *unit-free* multizetas (i.e. with $s_i \geq 2$). The proof rests on a reformulation of the problem in terms of bialternals, and then on the so-called *redistribution* identities (of rich combinatorial content) which make it

[135] i.e. of multizetas of length $r' < r$. The more 'negative' s_i's there are, the smaller the number r' becomes.
[136] for details, see [E2].

possible to recover any bialternal polynomial $Mi^{\binom{0,\ldots,0}{v_1,\ldots,v_r}}$ from its *essential part*, i.e. from the collection of its constituant monomials that are divisible by $v_1 \ldots v_r$.

• Parity reduction.

Any 'uncoloured' multizeta $Ze^{\binom{0,\ldots,0}{s_1,\ldots,s_r}}$ with $s_i \in (\mathbb{N}^*)^r$ can in fact be expressed as a rational-linear combinations of analogous multizetas of *even* degree.[137] While this follows from the general result on the decomposition of multizetas into *irreducibles* (these correspond here to uncoloured bialternal polynomials, which are necessarily of *even* degree d), there exists a more elementary derivation, based on the properties of the symmetrel bimould $Tig^\bullet(z)$, or "multitangent bimould", thus defined:[138]

$$Tig^{\binom{\epsilon_1,\ldots,\epsilon_r}{v_1,\ldots,v_r}}(z) \quad := \quad \sum_{s_i\geq 1} Te^{\binom{\epsilon_1,\ldots,\epsilon_r}{s_1,\ldots,s_r}}(z)\, v_1^{s_1-1}\ldots v_r^{s_r-1} \tag{10.7}$$

$$Te^{\binom{\epsilon_1,\ldots,\epsilon_r}{s_1,\ldots,s_r}}(z) \quad := \sum_{+\infty>n_1>\ldots>n_r>-\infty} e_1^{-n_1}\ldots e_r^{-n_r}\,(n_1+z)^{-s_1}\ldots(n_r+z)^{-s_r} \tag{10.8}$$

and on the *two different ways* of expressing each uncoloured multitangent $Tig^{\binom{\mathbf{0}}{\mathbf{s}}}(z)$ as sums of uncoloured monotangents $Tig^{\binom{0}{s_1}}(z)$ with uncoloured multizeta coefficients. See §11.7.

• The senary relation and palindromy formula.

The *senary relations* on bimoulds of type $\underline{al}/\underline{il}$ are the only double subsymmetries of finite arity – they involve exactly six terms. In polar (resp. universal) mode, they assume the form (3.64) (resp. (3.58)). They result from the double symmetry $\underline{al}/\underline{il}$ of a bimould $M^\bullet$, more precisely from the *mantar*-invariance of $M^\bullet$ (consequence of its alternality) and the *mantir*-invariance of $swap.M^\bullet$ (consequence of its alternility).

The *palindromy relations*, on the other hand, apply to homogeneous elements $C \in IHARA \subset \mathbb{Q}[x_0,x_1]$ of the Ihara algebra (x_0 and x_1 don't commute), or more precisely to their left or right decompositions:

$$C = A_0\,x_0 + A_1\,x_1 = x_0\,B_0 + x_1\,B_1 \quad (A_i, B_i \in \mathbb{Q}[x_0,x_1]) \tag{10.9}$$

and state that the sums $A_0 + A_1$ and $B_0 + B_1$ are invariant under the palindromic involution:

$$x_{\epsilon_1}\,x_{\epsilon_2}\ldots x_{\epsilon_s} \mapsto (-1)^s\,x_{\epsilon_s}\ldots x_{\epsilon_2}\,x_{\epsilon_1} \tag{10.10}$$

[137] recall that $d := s - r$: the degree d of a scalar multizeta (in the *Ze* encoding) is equal to its total weight s minus its length r.

[138] In the second sum, $e_j := \exp(2\pi i\epsilon_j)$ as usual, and we apply standard symmetrel renormalisation to get a finite result when either s_1 or s_r is $= 1$.

The palindromy relations[139], which according to the above involve *four clusters of terms*, can easily be shown to be equivalent to a special case of the senary relations[140], which involve *six*.

• **Coloured multizetas. Bicolours and tricolours.**
The statement about the eliminability of unit weights[141] in uncoloured multizetas still applies in the coloured case, but here another, almost opposite results holds: every bicoloured or tricoloured multizeta[142] with arbitrary weights can be (in non-unique manner) expressed as a rational-linear combination of multizetas with unit-weights only.

• **Canonical-rational associator and explicit decomposition into canonical irreducibles.**
We would rate this as the second-most encouraging result obtained so far with the flexion apparatus. The existence of a truly canonical decomposition[143] was by no means a foregone conclusion – in fact, it had gone completely unsuspected. Moreover, since everything rests on the construction of an explicit basis of $ALIL \subset ARI^{\underline{al}/\underline{il}}$, which in turn requires the repeated introduction and elimination of *singularities at the origin*,[144] the construction cannot be duplicated in any other framework than the flexion structure.

• **The impartial expression of irreducibles as perinomal numbers.**
We would, in all humility, regard this as the crowning achievement of the flexion method so far. The two circumstances which made it possible are: the exact adequation of *ARI//GARI* to dimorphy; and the 'vastness' of the structure, which accommodates not just polynomials in the $\boldsymbol{u}$ or $\boldsymbol{v}$ variables, but also meromorphic functions (and much else).

• **The first forays into perinomal territory.**
Though we only stand at the beginning of what looks like an open-ended exploration, we can already rely on two firm facts to guide the search: one is the *perinomal nature* of the multiresidues 'hidden' in the constituent parts of $Zag^\bullet/Zig^\bullet$; the other is the existence, attached to each integer sequence $\boldsymbol{r} := (r_1, \ldots, r_l)$, of a *specific linear representation* of $Sl_l(\mathbb{Z})$.

[139]They were empirically observed by followers of the Ihara approach, and pointed out to me, as conjectures, by L. Schneps in March 2010.

[140]Namely for $\boldsymbol{u}$-polynomial and $\boldsymbol{v}$-constant bimoulds. The senary relations first appeared, among many similar consequences of *double symmetries*, in a 2002 paper by us and were mentioned, the next year, during a series of Orsay lectures.

[141]i.e. of all indices s_i that are equal to 1.

[142]i.e. with $\epsilon_i \in \frac{1}{2}\mathbb{Z}/\mathbb{Z}$ or $\epsilon_i \in \frac{1}{3}\mathbb{Z}/\mathbb{Z}$

[143]The existence of three closely related variants (see §9.1 and §9.3) in no way detracts from the canonicity.

[144]The infinite process is described in §6.

10.5 Looking ahead : what is within reach and what beckons from afar.

• **Arithmetical and analytic properties of** $lama^\bullet/lami^\bullet$**.**
Of all three 'co-canonical' pairs, this is the simplest, arithmetically speaking. As power series of $\boldsymbol{u}$ or $\boldsymbol{v}$, these bimoulds carry Taylor coefficients that have, globally, the smallest possible denominators. But the series themselves are divergent-resurgent – with a resurgence pattern that is still poorly understood.[145]

• **Arithmetical and analytic properties of** $loma^\bullet/lomi^\bullet$**.**
Arithmetically, this second pair is less simple (the Taylor coefficients have slightly larger denominators) but the associated power series are convergent, with a finite multiradius of convergence. At the moment, however, it is unclear whether the corresponding functions admit endless analytic continuation and, if so, what the exact nature of their isolated singularities might be.

• **Arithmetical and analytic properties of** $luma^\bullet/lumi^\bullet$**.**
This last pair, being defined by semi-convergent series of multipoles, has a completely transparent meromorphic structure. The difficulty, here, is with the arithmetics of the Taylor coefficients: up to length $r = 4$, they are all rational, but (for $3 \leq r \leq 4$) with very irregular denominators.[146] Beyond that (for $5 \geq r$), it is not even known whether the coefficients are rational.[147]

Needless to say, analogous questions arise for the three parallel pairs $rama^\bullet/rami^\bullet$, $roma^\bullet/romi^\bullet$, and $ruma^\bullet/rumi^\bullet$.

• **Perinomal algebra. Ranks of** $Sl_r(\mathbb{Z})$ **representations.**
As repeatedly noted, to each integer sequence $\boldsymbol{r} := (r_1, \ldots, r_l)$, our approach to multizeta algebra attaches a perinomal function $\boldsymbol{n} \mapsto peri(^{\boldsymbol{n}}_{\boldsymbol{r}})$, which in turn induces a linear representation $\mathcal{R}_{\boldsymbol{r}}$ of $Sl_l(\mathbb{Z})$. The (clearly fast increasing) ranks of these $\mathcal{R}_{\boldsymbol{r}}$ are unknown except in a few special cases, and their structure (e.g. their decomposition into irreducible representations) is equally unknown.

• **Links between the four series of perinomal functions.**
To each perinomal function carried by $peri^\bullet$, identities such as (9.58) or (9.59) attach simpler but related perinomal functions, but a clear overall picture is probably still a long way off. For aught we know, the two-layered mould

[145] For any given length r, the resulting resurgence algebra is probably finite dimensional, which would be an additional incentive for unravelling its structure.
[146] See §6.7.
[147] See §9.3.

peri$^\bullet$ may turn out to be as complex (though more tidy) than the Mandelbrot set, with algebraic (rather than fractal-geometric) detail "as far as the sight reaches".

• **Arithmetical nature of all perinomal numbers.**
The $\mathbb{Q}$-ring *PERI* of all perinomal numbers (see §8.4) exceeds the $\mathbb{Q}$-ring $\mathbb{Z}eta$ of multizetas (even if we allow colour) but the range and structure of the *difference* remains unexplored.

• **The quest for numerical derivations.**
Does there exist on *PERI* an algebra *DERI* of *direct* numerical derivations, that is to say, of linear operators D verifying:

$$D(x.y) \equiv Dx.y + x.Dy \qquad (\forall x, y \in \text{PERI}, \ \forall D \in \text{DERI}) \quad (10.11)$$
$$D.\mathbb{Q} = \{0\} \ , \ \{0\} \neq D.\text{PERI} \subset \text{PERI} \qquad (\forall D \in \text{DERI}) \quad (10.12)$$

The emphasis here is on *direct*, meaning that the action of D on any $x \in$ *PERI* ought to be defined in universal terms, i.e. based on a universal expansion (decimal, continued fraction, etc) of x, and not on its *mode of construction*. This at the moment is little more than a dream, but if it came true, it would give us a key – possibly, the only workable key – to unlock the *exact*, as opposed to *formal*, arithmetics[148] of *PERI* and its subring $\mathbb{Z}eta$. But this is purest *terra incognita* and, as it said on ancient maps where unchartered territory began, *ibi sunt leones...*

11 Complements.

11.1 Origin of the flexion structure.

The flexion structure has its origin (ca 1990) in the investigation of *parametric resurgence* – typically, the sort of resurgence associated with formal expansions in series of a *singular perturbation parameter* ϵ. [149] Set $x = \epsilon^{-1}$ (x large, ϵ small) and consider the standard system:

$$((u_1 + \ldots + u_r)\, x + \partial_z)\, \mathcal{W}^{\binom{u_1\ ,\ldots,\ u_r}{v_1\ ,\ldots,\ v_r}}(z, x) = \mathcal{W}^{\binom{u_1\ ,\ldots,\ u_{r-1}}{v_1\ ,\ldots,\ v_{r-1}}}(z, x)\, \frac{1}{z - v_r} \qquad (11.1)$$

with $\mathcal{W}^{\binom{\emptyset}{\emptyset}}(z, x) := 1$ to start the induction.

[148] No one would seriously expects the two arithmetics – exact and formal – to differ, but proving their identity is another matter.

[149] Think for definiteness of a differential equation with a small ϵ sitting in front of the highest order derivative.

We may fix x and expand the solutions as formal power series of z^{-1}. These turn out to be divergent, Borel-summable, and resurgent, with a simple resurgence locus[150] consisting of the sums of u_i indices.

We may also fix z and expand the solutions as formal power series of x^{-1}. These are again divergent, Borel-summable, and resurgent, but with a much more intricate resurgence locus generated (bi-linearly) by the two sets of indices, the u_i and v_i, under 'flexion operations'.

As functions of z, the $\mathcal{W}^{\binom{\boldsymbol{u}}{\boldsymbol{v}}}(z,x)$ do not differ significantly[151] from the standard resurgence monomials $\mathcal{V}^{\boldsymbol{\omega}}(z) := \mathcal{W}^{\binom{\boldsymbol{\omega}}{\boldsymbol{0}}}(z,1)$ defined by the induction:

$$(\omega_1+\ldots+\omega_r+\partial_z)\,\mathcal{V}^{\omega_1,..,\omega_r}(z) = \mathcal{V}^{\omega_1,..,\omega_{r-1}}(z)\,\frac{1}{z} \quad \textit{with} \quad \mathcal{V}^{\emptyset}(z) := 1 \qquad (11.2)$$

As functions of x, on the other hand, the $\mathcal{W}^{\binom{\boldsymbol{u}}{\boldsymbol{v}}}(z,x)$ can be expressed as linear combinations[152] of standard resurgence monomials $\mathcal{V}^{\boldsymbol{\omega}}(x) = \mathcal{V}^{\omega_1,\ldots,\omega_r}(x)$, with indices ω_j that depend bilinearly on the indices u_i and v_i (to which one must add z itself). Formally, the u_j's and v_j's contribute in much the same fashion to the ω_j's, although the natural way of expressing the ω_j's is via *sums* of (several consecutive) u_j's and *differences* of (two non-necessarily consecutive) v_j's or of v_j's and z.

As to their origin, however, the u_j's and v_j's could not differ more. In all natural problems, the u_j's depend only on the *principal part* of the differential equation or system and tend to be generated by a finite number of scalars (such as the system's *multipliers*, i.e. the eigenvalues of its linear part). There is thus considerable *rigidity* about the u_j's. With the v_j's, on the other hand, we have complete *flexibility*: they reflect pre-existing singularities in the (multiplicative) z-plane and can be *anything.*

11.2 From simple to double symmetries. The *scramble* transform.

Originally, the *scramble* transform arose during the search for a systematic expression of the complex $\mathcal{W}^\bullet$ of (11.1) in terms of the simpler $\mathcal{V}^\bullet$ of (11.2). Our reason for mentioning it here is because the transform in question led:
(i) to the first systematic use of flexions;
(ii) to the first systematic production of double symmetries.

[150]The *resurgence locus* of a resurgent function f is the set $\Omega \subset \mathbb{C}_\bullet := \widetilde{(\mathbb{C}\dot{-}0)}$ of all ω_0 that give rise to non-vanishing alien derivatives $\Delta_{\omega_0} f$ or $\Delta_{\omega_0}\Delta_{\omega_1}\ldots\Delta_{\omega_r} f$.

[151]in terms of their resurgence properties.

[152]The number of summands is exactly $r!! := 1.3.5\ldots(2\,r-1)$ and all coefficients are of the form ± 1.

The *scramble* is a linear transform on *BIMU*:

$$M^\bullet \to S^\bullet = \text{scram}.M^\bullet \qquad \textit{with} \qquad S^{\boldsymbol{w}} := \sum_{\boldsymbol{w}^*} \epsilon(\boldsymbol{w},\boldsymbol{w}^*)\, M^{\boldsymbol{w}^*} \tag{11.3}$$

which not only preserves simple symmetries (*alternal* or *symmetral*) but, in the case of *all-even* bimoulds[153] $M^\bullet$, turns simple into double symmetries (*alternal* into *bialternal* and *symmetral* into *bisymmetral*).

$$\begin{array}{lllllll}
\text{scramble}: & M^\bullet \mapsto S^\bullet & & & & & \\
\text{scramble}: & \text{LU}^{\text{al}} & \to & \text{ARI}^{\text{al}} & \| \; \text{LU}^{\text{al}}_{\text{all-even}} & \to & \text{ARI}^{\underline{\text{al}}/\underline{\text{al}}} \\
\text{scramble}: & \text{MU}^{\text{as}} & \to & \text{GARI}^{\text{as}} & \| \; \text{MU}^{\text{as}}_{\text{all-even}} & \to & \text{GARI}^{\underline{\text{as}}/\underline{\text{as}}}
\end{array}$$

To define the sums $S^{\boldsymbol{w}}$ in (11.3) we have the choice between a *forward* and *backward* induction, quite dissimilar in outward form but equivalent nonetheless. They involve respectively the 'mutilation' operators *cut* and *drop*:

$$\begin{array}{llll}
(\text{cut}_{w_0} M)^{w_1,\dots,w_r} & := & M^{w_2,\dots,w_r} & \textit{if} \quad w_0 = w_1 \\
 & := & 0 & \textit{if} \quad w_0 \neq w_1 \\
(\text{drop}_{w_0} M)^{w_1,\dots,w_r} & := & M^{w_1,\dots,w_{r-1}} & \textit{if} \quad w_0 = w_r \\
 & := & 0 & \textit{if} \quad w_0 \neq w_r
\end{array}$$

We get each induction started by setting $S^{w_1} := M^{w_1}$ and then apply the following rules.

Forward induction rule:
We set $(\text{cut}_{w_0}.S)^{\mathbf{w}} := 0$ unless w_0 be of the form $\lceil w_i \rceil$ with respect to some sequence factorisation $\mathbf{w} = \mathbf{a} w_i \mathbf{b}\mathbf{c}$, in which case we set :

$$(\text{cut}_{\lceil w_i \rceil} S)^{\mathbf{w}} := (-1)^{r(\mathbf{b})} \sum_{\mathbf{w}' \in \text{sha}\left(\mathbf{a}\rfloor\,,\, \lfloor\tilde{\mathbf{b}}\,,\, \mathbf{c}\right)} S^{\mathbf{w}'} \qquad (\textit{if} \;\; \mathbf{w} = \mathbf{a} w_i \mathbf{b}\mathbf{c}) \tag{11.4}$$

with $\tilde{\boldsymbol{b}}$ denoting the sequence $\boldsymbol{b}$ in reverse order. If $M^\bullet$ is symmetral, so is $S^\bullet$ (see below). In that important case the forward induction rules assumes the much simpler form :

$$(\text{cut}_{\lceil w_i \rceil} S)^{\mathbf{w}} := S^{\mathbf{a}}\, (\text{invmu}.S)^{\mathbf{b}}\, S^{\mathbf{c}} \qquad (\textit{if} \;\; \mathbf{w} = \mathbf{a} w_i \mathbf{b}\mathbf{c}) \tag{11.5}$$

Backward induction rule:

[153] i.e. in the case of bimoulds $M^{\boldsymbol{w}}$ that are *even* separately in each double index w_i.

We set $(\mathrm{cut}_{w_0}.S)^{\mathbf{w}} := 0$ unless w_0 be of the form $\lfloor w_i$ or $w_i \rfloor$ with respect to some sequence factorisation $\mathbf{w} = \mathbf{a} w_i \mathbf{b}$, in which case we set :

$$(\mathrm{drop}_{\lfloor w_i} S)^{\mathbf{w}} \ := \ -S^{\mathbf{a}\rceil\,\mathbf{b}} \qquad (\textit{if} \ \ \mathbf{w} = \mathbf{a} w_i \mathbf{b}) \tag{11.6}$$

$$(\mathrm{drop}_{w_i \rfloor} S)^{\mathbf{w}} \ := \ +S^{\mathbf{a}\,\lceil\mathbf{b}} \qquad (\textit{if} \ \ \mathbf{w} = \mathbf{a} w_i \mathbf{b}) \tag{11.7}$$

Remark 1: *mu* is bilinear whereas *gari* is heavily non-linear in its second argument. So how can the scramble inject $\mathrm{MU}^{\mathrm{as}}$ into $\mathrm{GARI}^{\mathrm{as}}$? The answer is that under the above algebra morphism, the non-linearity of *gari* gets “absorbed” by the bimoulds’ symmetrality. This is easy to check up to length 3, on the formulas:

$$\begin{array}{lcl}
S^{\left(\begin{smallmatrix} u1 \\ v1 \end{smallmatrix}\right)} & = & +M^{\left(\begin{smallmatrix} u1 \\ v1 \end{smallmatrix}\right)} \\
S^{\left(\begin{smallmatrix} u_1 , & u_2 \\ v_1 , & v_2 \end{smallmatrix}\right)} & = & +M^{\left(\begin{smallmatrix} u_1 , & u_2 \\ v_1 , & v_2 \end{smallmatrix}\right)} + M^{\left(\begin{smallmatrix} u_{12} , & u_1 \\ v_2 , & v_{1:2} \end{smallmatrix}\right)} - M^{\left(\begin{smallmatrix} u_{12} , & u_2 \\ v_1 , & v_{2:1} \end{smallmatrix}\right)} \\
S^{\left(\begin{smallmatrix} u_1 , & u_2 , & u_3 \\ v_1 , & v_2 , & v_3 \end{smallmatrix}\right)} & = & +M^{\left(\begin{smallmatrix} u_1 , & u_2 , & u_3 \\ v_1 , & v_2 , & v_3 \end{smallmatrix}\right)} + M^{\left(\begin{smallmatrix} u_1 , & u_{23} , & u_2 \\ v_1 , & v_3 , & v_{2:3} \end{smallmatrix}\right)} - M^{\left(\begin{smallmatrix} u_1 , & u_{23} , & u_3 \\ v_1 , & v_2 , & v_{3:2} \end{smallmatrix}\right)} \\
 & & +M^{\left(\begin{smallmatrix} u_{12} , & u_1 , & u_3 \\ v_2 , & v_{1:2} , & v_3 \end{smallmatrix}\right)} - M^{\left(\begin{smallmatrix} u_{12} , & u_2 , & u_3 \\ v_1 , & v_{2:1} , & v_3 \end{smallmatrix}\right)} \\
 & & +M^{\left(\begin{smallmatrix} u_{12} , & u_3 , & u_1 \\ v_2 , & v_3 , & v_{1:2} \end{smallmatrix}\right)} - M^{\left(\begin{smallmatrix} u_{12} , & u_3 , & u_2 \\ v_1 , & v_3 , & v_{2:1} \end{smallmatrix}\right)} \\
 & & +M^{\left(\begin{smallmatrix} u_{123} , & u_{23} , & u_3 \\ v_1 , & v_{2:1} , & v_{3:2} \end{smallmatrix}\right)} - M^{\left(\begin{smallmatrix} u_{123} , & u_{23} , & u_2 \\ v_1 , & v_{3:1} , & v_{2:3} \end{smallmatrix}\right)} + M^{\left(\begin{smallmatrix} u_{123} , & u_3 , & u_2 \\ v_1 , & v_{3:1} , & v_{2:1} \end{smallmatrix}\right)} \\
 & & -M^{\left(\begin{smallmatrix} u_{123} , & u_1 , & u_3 \\ v_2 , & v_{1:2} , & v_{3:2} \end{smallmatrix}\right)} - M^{\left(\begin{smallmatrix} u_{123} , & u_3 , & u_1 \\ v_2 , & v_{3:2} , & v_{1:2} \end{smallmatrix}\right)} \\
 & & +M^{\left(\begin{smallmatrix} u_{123} , & u_1 , & u_2 \\ v_3 , & v_{1:3} , & v_{2:3} \end{smallmatrix}\right)} - M^{\left(\begin{smallmatrix} u_{123} , & u_{12} , & u_2 \\ v_3 , & v_{1:3} , & v_{2:3} \end{smallmatrix}\right)} + M^{\left(\begin{smallmatrix} u_{123} , & u_{12} , & u_1 \\ v_3 , & v_{2:3} , & v_{1:2} \end{smallmatrix}\right)}
\end{array}$$

The number of summands $M^{\boldsymbol{w}^*}$ in the expression of $S^{w_1,\dots,w_r}$ is exactly $r!! := 1.3.5\dots(2\,r-1)$.

Remark 2: Extending the *scramble* to ordinary moulds.
We must often let the *scramble* act on moulds $M^\bullet$ by first ‘lifting’ these into bimoulds $\underline{M}^\bullet$ according to the rule: $\underline{M}^{\left(\begin{smallmatrix} u_1 , & \dots, & u_r \\ v_1 , & \dots, & v_r \end{smallmatrix}\right)} = M^{u_1 v_1 + \dots + u_r v_r}$. Of course, the *scramble* of a mould is a bimould – not a mould. Thus, the bimould $\mathcal{W}^\bullet$ of (11.1) is essentially the *scramble* of the mould $\mathcal{V}^\bullet$ of (11.2).

11.3 The bialternal tesselation bimould.

Let $V^\bullet$ be the classical scalar mould produced under alien derivation from the equally classical resurgent mould $\mathcal{V}^\bullet(z)$:

$$\Delta_{\omega_0} \mathcal{V}^{\boldsymbol{\omega}}(z) = \sum_{\boldsymbol{\omega}=\boldsymbol{\omega}'\boldsymbol{\omega}''}^{\|\boldsymbol{\omega}'\|=\omega_0} V^{\boldsymbol{\omega}'}\, \mathcal{V}^{\boldsymbol{\omega}''}(z) \tag{11.8}$$

$\mathcal{V}^\bullet(z)$ is symmetral; $V^\bullet$ is alternal.

If we now apply the *scramble* transform to the alternal mould $V^\bullet$ (see Remark 2 *supra* about the lift $V^\bullet \mapsto \underline{V}^\bullet$), we get a bialternal bimould $tes^\bullet$:[154]

$$\mathrm{tes}^\bullet = \mathrm{scram}.V^\bullet \qquad \textit{with} \qquad \mathrm{tes}^{\boldsymbol{w}} := \sum_{\boldsymbol{w}^*} \epsilon(\boldsymbol{w}, \boldsymbol{w}^*)\, \underline{V}^{\boldsymbol{w}^*} \tag{11.9}$$

which (surprisingly) turns out to be piecewise constant in each u_i and v_i, despite being a sum of hyperlogarithmic summands $\underline{V}^{\boldsymbol{w}^*}$. This begs for an alternative, simpler expression of $tes^\bullet$. The following induction formula provides such an elementary alternative:

$$\mathrm{tes}^{\mathbf{w}} = \sum_{0 \le n \le r(\mathbf{w})} \mathrm{push}^n \sum_{\mathbf{w}'\mathbf{w}''=\mathbf{w}} \mathrm{sig}^{\mathbf{w}',\mathbf{w}''}\, \mathrm{tes}^{\mathbf{w}^*}\, \mathrm{tes}^{\mathbf{w}^{**}} \tag{11.10}$$

The notations are as follows.
We fix $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ and set $\Re_\theta : z \in \mathbb{C} \mapsto \Re(e^{i\theta} z) \in \mathbb{R}$. Then we define:

$$f^{\mathbf{w}'}_{\mathbf{w}} := <\mathbf{u}', \mathbf{v}'><\mathbf{u}, \mathbf{v}>^{-1} \quad , \quad g^{\mathbf{w}'}_{\mathbf{w}} := <\mathbf{u}', \Re_\theta \mathbf{v}'><\mathbf{u}, \Re_\theta \mathbf{v}>^{-1} \tag{11.11}$$

$$f^{\mathbf{w}''}_{\mathbf{w}} := <\mathbf{u}'', \mathbf{v}''><\mathbf{u}, \mathbf{v}>^{-1} \quad , \quad g^{\mathbf{w}''}_{\mathbf{w}} := <\mathbf{u}'', \Re_\theta \mathbf{v}''><\mathbf{u}, \Re_\theta \mathbf{v}>^{-1} \tag{11.12}$$

From these scalars we construct the crucial sign factor *sig* which takes its values in $\{-1, 0, 1\}$. Here, the abbreviation *si*(.) stands for *sign*($\Im$(.)).

$$\begin{aligned} \mathrm{sig}^{\mathbf{w}',\mathbf{w}''} = \mathrm{sig}^{\mathbf{w}',\mathbf{w}''}_\theta := \frac{1}{8} & \Big(\mathrm{si}(f^{\mathbf{w}'}_{\mathbf{w}} - f^{\mathbf{w}''}_{\mathbf{w}}) - \mathrm{si}(g^{\mathbf{w}'}_{\mathbf{w}} - g^{\mathbf{w}''}_{\mathbf{w}})\Big) \times \\ & \Big(1 + \mathrm{si}(f^{\mathbf{w}'}_{\mathbf{w}}/g^{\mathbf{w}'}_{\mathbf{w}})\ \mathrm{si}(f^{\mathbf{w}'}_{\mathbf{w}} - g^{\mathbf{w}'}_{\mathbf{w}})\Big) \times \\ & \Big(1 + \mathrm{si}(f^{\mathbf{w}''}_{\mathbf{w}}/g^{\mathbf{w}''}_{\mathbf{w}})\ \mathrm{si}(f^{\mathbf{w}''}_{\mathbf{w}} - g^{\mathbf{w}''}_{\mathbf{w}})\Big) \end{aligned} \tag{11.13}$$

Lastly, the pair $(\mathbf{w}^*, \mathbf{w}^{**})$ is constructed from the pair $(\mathbf{w}', \mathbf{w}'')$ according to:

$$\mathbf{u}^* := \mathbf{u}' \quad , \quad \mathbf{v}^* := \mathbf{v}'\ <\mathbf{u}, \mathbf{v}>^{-1}\ \Im g^{\mathbf{w}'}_{\mathbf{w}} - \Re_\theta \mathbf{v}'\ <\mathbf{u}, \Re_\theta \mathbf{v}>^{-1}\ \Im g^{\mathbf{w}'}_{\mathbf{w}} \tag{11.14}$$

$$\mathbf{u}^{**} := \mathbf{u}'' \quad , \quad \mathbf{v}^{**} := \mathbf{v}''\ <\mathbf{u}, \mathbf{v}>^{-1}\ \Im g^{\mathbf{w}''}_{\mathbf{w}} - \Re_\theta \mathbf{v}''\ <\mathbf{u}, \Re_\theta \mathbf{v}>^{-1}\ \Im g^{\mathbf{w}''}_{\mathbf{w}} \tag{11.15}$$

Remark 1: The above induction for $tes^\bullet$ is elementary in the sense of being non-transcendental: it depends only on the *sign function*. But on the face of it, it looks non-intrinsical. Indeed, the partial sum:

$$\mathrm{urtes}^{\mathbf{w}}_\theta := \sum_{\mathbf{w}'\mathbf{w}''=\mathbf{w}} \mathrm{sig}^{\mathbf{w}',\mathbf{w}''} \mathrm{tes}^{\mathbf{w}^*} \mathrm{tes}^{\mathbf{w}^{**}} = \sum_{\mathbf{w}'\mathbf{w}''=\mathbf{w}} \mathrm{sig}^{\mathbf{w}',\mathbf{w}''}_\theta \mathrm{tes}^{\mathbf{w}^*_\theta} \mathrm{tes}^{\mathbf{w}^{**}_\theta} \tag{11.16}$$

[154] Its proper place is in resurgence theory – in the description of the “geometry” of *co-equational resurgence.*

is *polarised*, i.e. θ-dependent. However, its *push*-invariant offshoot:

$$\mathrm{tes}^\bullet := \sum_{0\leq n\leq r(\mathbf{w})} \mathrm{push}^n\, \mathrm{urtes}^\bullet_\theta \tag{11.17}$$

is duly *unpolarised.* We might of course remove the polarisation in $urtes^\bullet_\theta$ itself by replacing it by this isotropic variant:

$$\mathrm{urtes}^\bullet_{\mathrm{iso}} := \frac{1}{2\,\pi} \int_0^{2\pi} \mathrm{urtes}^\bullet_\theta \;\; d\theta \tag{11.18}$$

but at the cost of rendering it less elementary, since $urtes^\bullet_{iso}$ would assume its value in $\mathbb{R}$ rather than $\{-1,0,1\}$. It would also depend hyperlogarithmically on its indices, and thus take us back to something rather like formula (11.9), which we wanted to get away from. So the alternative for $tes^\bullet$ is: *either* an intrinsical but heavily transcendental expression *or* an elementary but heavily polarised one!

Remark 2: In the induction (11.10) we might exchange everywhere the role of $\boldsymbol{u}$ and $\boldsymbol{v}$ and still get the correct answer $tes^\bullet$, but via a different polarised intermediary $urtes^\bullet_\theta$. The natural setting for studying $tes^\bullet$ is the *biprojective* space $\mathbb{P}^{r,r}$ equal to $\mathbb{C}^{2r}$ quotiented by the relation $\{\boldsymbol{w}^1 \sim \boldsymbol{w}^2\} \Leftrightarrow \{\boldsymbol{u}^1 = \lambda \boldsymbol{u}^2\,,\, \boldsymbol{v}^1 = \mu\,\boldsymbol{v}^2\ (\lambda,\mu\in\mathbb{C}^*)\}$. But rather than using biprojectivity to get rid of two coordinates (u_i,v_i), it is often useful, on the contrary, to resort to the *augmented* or *long* notation, by *adding* two redundant coordinates (u_0,v_0). The *long* coordinates (u_i^*,v_i^*) relate to the short ones (u_i,v_i) under the rules:

$$u_i = u_i^* \quad , \quad v_i = v_i^* - v_0^* \qquad (1\leq i\leq r) \tag{11.19}$$

The *long* u_i^* are constrained by $u_0^*+\cdots+u_r^* = 0$ while the *long* v_i^* are, dually, regarded as defined up to a common additive constant. Thus we have $<u^*,v^*>=<u,v>$.The indices i of the *long* coordinates are viewed as elements of $\mathbb{Z}_{r+1} = \mathbb{Z}/(r{+}1)\mathbb{Z}$ with the natural circular ordering on triplets $circ(i_1 < i_2 < i_3)$ that goes with it. Lastly, we require $r^2{-}1$ basic “homographies” $H_{i,j}$ on $\mathbb{P}^{r,r}$, defined by:

$$H_{i,j}(\mathbf{w}) \;:=\; Q_{i,j}(\mathbf{w})/Q^*_{i,j}(\mathbf{w}) \qquad (i-j\neq 0; i,j\in\mathbb{Z}_{r+1}) \tag{11.20}$$

$$Q_{i,j}(\mathbf{w}) \;:=\; \sum_{\mathrm{circ}(j\leq q<i)} u_q^*\,(v_q^*-v_j^*) \tag{11.21}$$

$$Q^*_{i,j}(\mathbf{w}) \;:=\; \sum_{\mathrm{circ}(i\leq q<j)} u_q^*\,(v_q^*-v_j^*) \;\neq\; Q_{j,i}(\mathbf{w}) \tag{11.22}$$

Main properties of $\boldsymbol{tes^\bullet}$.

$\boldsymbol{P_1}$: the bimould $tes^\bullet$ is bialternal, i.e. alternal and of alternal *swappee.*

$\boldsymbol{P_2}$: in fact *swap* $tes^\bullet = tes^\bullet$.
$\boldsymbol{P_3}$: $tes^\bullet$ is *push*-invariant.
$\boldsymbol{P_4}$: $tes^\bullet$ is *pus*-variant, i.e. of zero *pus*-average.
$\boldsymbol{P_5}$: $tes^\bullet$ assumes the sole values -1,0,1.
$\boldsymbol{P_6}$: for r fixed but large, the sets $S_\pm \subset \mathbb{P}^{r,r}$ where $tes^{\mathbf{w}}$ is ± 1, have positive but incredibly small Lebesgue measure.
$\boldsymbol{P_7}$: for r fixed, all three sets S_-, S_0, S_+ are path-connected.
$\boldsymbol{P_8}$: for r fixed, the hypersurfaces $\Im(H_{i,j}(\mathbf{w})) = 0$ *limit*[155] but do not *separate*[156] the sets S_-, S_0, S_+.
$\boldsymbol{P_9}$: $tes^{\mathbf{w}} = 0$ whenever $\mathbf{w}$ is semi-real, i.e. whenever one of its two components $\mathbf{u}$ or $\mathbf{v}$ is real. [157]

11.4 Polar, trigonometric, bitrigonometric symmetries.

The trigonometric symmetries *iil* and *uul* coincide *modulo c* with the polar symmetries *il* and *ul*, but their exact expression is much more complex. So let us first restate the polar symmetries in terms that lend themselves to the extension to the trigonometric case.

Polar symmetries: symmetril/alternil.

A bimould $M^\bullet$ is symmetril (resp. alternil) iff for all pairs $\mathbf{w}', \mathbf{w}'' \neq \emptyset$ the identity holds:

$$\sum_{\mathbf{w} \in \text{shi}(\mathbf{w}', \mathbf{w}'')} M^{\mathbf{w}} \prod_{1 \le k \le r(\mathbf{w})} \text{li}^{w_k} \equiv M^{\mathbf{w}'} M^{\mathbf{w}''} \quad (\textit{resp.} \equiv 0) \tag{11.23}$$

with a sum ranging over all sequences $\mathbf{w}$ that are order-compatible with $(\mathbf{w}', \mathbf{w}'')$ and whose indices w_k are of the form:
(i) either w'_i or w''_j, in which case $\text{li}^{w_k} := 1$
(ii) or $\binom{u'_i + u''_j}{v'_i}$, in which case $\text{li}^{w_k} := -P(v''_j - v'_i)$
(iii) or $\binom{u'_i + u''_j}{v''_j}$, in which case $\text{li}^{w_k} := -P(v'_i - v''_j)$

Polar symmetries: symmetrul/alternul.

A bimould $M^\bullet$ is symmetrul (resp. alternul) iff for all pairs $\mathbf{w}', \mathbf{w}'' \neq \emptyset$ the

[155] that is to say, the boundaries of these sets lie on the hypersurfaces.
[156] that is to say, none of the three sets can be defined in terms of the sole signs $si(H_{i,j}(\mathbf{w})) := sign(\Im(H_{i,j}(\mathbf{w})))$, at least for $r \ge 3$. For $r = 1$, $tes^\bullet \equiv 1$ and for $r = 2$, $tes^\bullet = \pm 1$ iff $si(H_{0,1}(\mathbf{w})) = si(H_{1,2}(\mathbf{w})) = si(H_{2,0}(\mathbf{w})) = \pm$ and 0 otherwise.
[157] or purely imaginary, since under biprojectivity this amounts to the same. Of course, $tes^{\mathbf{w}}$ vanishes in many more cases. In fact it vanishes most of the time: see $\boldsymbol{P_6}$ above.

identity holds:

$$\sum_{\mathbf{w}\in\mathrm{shu}(\mathbf{w}',\mathbf{w}'')} M^{\mathbf{w}} \prod_{1\le k\le r(\mathbf{w})} \mathrm{lu}^{w_k} \equiv M^{\mathbf{w}'} M^{\mathbf{w}''} \quad (\mathit{resp.} \equiv 0) \tag{11.24}$$

with a sum ranging over all sequences $\mathbf{w}$ that are order-compatible with $(\mathbf{w}', \mathbf{w}'')$ and whose indices w_k are of the form
(i) either w'_i or w''_j, in which case $\mathrm{lu}^{w_k} := 1$
(ii) or $\binom{u'_i+u''_j}{v'_i}$, in which case $\mathrm{lu}^{w_k} := -P(u''_j)$
(iii) or $\binom{u'_i+u''_j}{v''_j}$, in which case $\mathrm{lu}^{w_k} := -P(u'_i)$

Trigonometric symmetries: auxiliary functions.
To handle the trigonometric case, we require four series of rational coefficients:
(*) $xii_{p,q}$, $zii_{p,q}$, $xuu_{p,q}$, $zuu_{p,q}$
which are best defined as Taylor coefficients of the following functions:
(**) $Xii(x,y)$, $Zii(x,y)$, $Xuu(x,y)$, $Zuu(x,y)$.
Here are the definitions:

$$Q(t) := \frac{1}{\tan(t)} \quad \| \quad R(t) := \frac{1}{\arctan(t)} \tag{11.25}$$

$$\mathrm{Xii}(x,y) := \frac{x^{-1}+y^{-1}}{Q(x)+Q(y)} \quad \| \quad \mathrm{Xuu}(x,y) := \frac{x^{-1}+y^{-1}}{R(x)+R(y)} \tag{11.26}$$

$$\mathrm{Zii}(x,y) := \frac{x^{-1}Q(x)-y^{-1}Q(y)}{Q(x)+Q(y)} \quad \| \quad \mathrm{Zuu}(x,y) := \frac{x^{-1}R(x)-y^{-1}R(y)}{R(x)+R(y)} \tag{11.27}$$

Thus:

$$\begin{aligned}
\mathrm{Xii}(x,y) = {} & 1+\frac{1}{3}xy+\frac{1}{45}y^3+\frac{4}{45}x^2y^2+\frac{1}{45}x^3y \\
& +\frac{2}{945}xy^5+\frac{4}{315}x^2y^4+\frac{23}{945}x^3y^3+\frac{4}{315}x^4y^2+\frac{2}{945}x^5y+\dots \\
\mathrm{Xuu}(x,y) = {} & 1-\frac{1}{3}xy+\frac{4}{45}xy^3+\frac{1}{45}x^2y^2+\frac{4}{45}x^3y \\
& -\frac{44}{945}xy^5-\frac{4}{315}x^2y^4-\frac{23}{945}x^3y^3-\frac{4}{315}x^4y^2-\frac{44}{945}x^5y+\dots
\end{aligned}$$

$$\mathrm{Zii}(x,y) = x^{-1} - y^{-1} - \frac{1}{3}x + \frac{1}{3}y - \frac{1}{45}x^3 - \frac{4}{45}x^2y + \frac{4}{45}xy^2 + \frac{1}{45}y^3$$
$$-\frac{2}{945}x^5 - \frac{4}{315}x^4y - \frac{16}{945}x^3y^2 + \frac{16}{945}x^2y^3 + \frac{4}{315}xy^4 + \frac{2}{945}y^5 + \ldots$$
$$\mathrm{Zuu}(x,y) = x^{-1} - y^{-1} + \frac{1}{3}x - \frac{1}{3}y - \frac{4}{45}x^3 - \frac{1}{45}x^2y + \frac{1}{45}xy^2 + \frac{4}{45}y^3$$
$$+\frac{44}{945}x^5 + \frac{4}{315}x^4y - \frac{1}{189}x^3y^2 + \frac{1}{189}x^2y^3 - \frac{4}{315}xy^4 - \frac{44}{945}y^5 + \ldots$$

Trigonometric symmetries: symmetriil/alterniil.

A bimould $M^\bullet$ is symmetriil (resp. alterniil) iff for all pairs $\mathbf{w}', \mathbf{w}'' \neq \emptyset$ the identity holds:

$$\sum_{\mathbf{w}\in \mathrm{shii}(\mathbf{w}',\mathbf{w}'')} M^{\mathbf{w}} \prod_{1\le k\le r(\mathbf{w})} \mathrm{lii}^{w_k} \equiv M^{\mathbf{w}'} M^{\mathbf{w}''} \quad (resp. \equiv 0) \tag{11.28}$$

with a sum ranging over all sequences $\mathbf{w}$ that are order-compatible with $(\mathbf{w}', \mathbf{w}'')$ and whose indices w_k are of the form

(i) either w'_i or w''_j, in which case $\mathrm{lii}^{w_k} := 1$

(ii) or $\binom{u'_i+\ldots u'_{i+p}+u''_j+\ldots u''_{j+q}}{v'_i}$ with $p, q \geq 0$, in which case

$$\mathrm{lii}^{w_k} := -c^{p+q}\,\mathrm{xii}_{p,q}\, Q_c(v''_j - v'_i) - c^{p+q+1}\,\mathrm{zii}_{p,q} \tag{11.29}$$

(iii) or $\binom{u'_i+\ldots u'_{i+p}+u''_j+\ldots u''_{j+q}}{v''_j}$ with $p, q \geq 0$, in which case

$$\mathrm{lii}^{w_k} := +c^{p+q}\,\mathrm{xii}_{p,q}\, Q_c(v'_i - v''_j) + c^{p+q+1}\,\mathrm{zii}_{p,q} \tag{11.30}$$

Trigonometric symmetries: symmetruul/alternuul.

A bimould $M^\bullet$ is symmetruul (resp. alternuul) iff for all pairs $\mathbf{w}', \mathbf{w}'' \neq \emptyset$ the identity holds:

$$\sum_{\mathbf{w}\in \mathrm{shuu}(\mathbf{w}',\mathbf{w}'')} M^{\mathbf{w}} \prod_{1\le k\le r(\mathbf{w})} \mathrm{luu}^{w_k} \equiv M^{\mathbf{w}'} M^{\mathbf{w}''} \quad (resp. \equiv 0) \tag{11.31}$$

with a sum ranging over all sequences $\mathbf{w}$ that are order-compatible with $(\mathbf{w}', \mathbf{w}'')$ and whose indices w_k are of the form

(i) either w'_i or w''_j, in which case $\mathrm{luu}^{w_k} := 1$

(ii) or $\binom{u'_i+u''_j}{v'_i}$, in which case $\mathrm{luu}^{w_k} := -Q_c(u''_j)$

(iii) or $\binom{u'_i+u''_j}{v''_j}$, in which case $\mathrm{luu}^{w_k} := -Q_c(u'_i)$

(iv) or $\binom{u'_i+\ldots u'_{i+p}+u''_j+\ldots u''_{j+q}}{v'_i}$ with $p,q\geq 0$ and $p+q\geq 1$, in which case

$$\mathrm{luu}^{w_k} := -\sum_{\substack{0\leq p_1\leq p\\ 0\leq q_1\leq q}} c^{p+q+1}\,\mathrm{zuu}_{p_1,q_1}\,\mathrm{Sym}_{p-p_1}\Big(\bigcup_{i<s<i+p} Q_c(u'_s)\Big)\,\mathrm{Sym}_{q-q_1}\Big(\bigcup_{j<s<j+q} Q_c(u''_s)\Big)$$

(v) or $\binom{u'_i+\ldots u'_{i+p}+u''_j+\ldots u''_{j+q}}{v''_j}$ with $p,q\geq 0$ and $p+q\geq 1$, in which case

$$\mathrm{luu}^{w_k} := +\sum_{\substack{0\leq p_1\leq p\\ 0\leq q_1\leq q}} c^{p+q+1}\,\mathrm{zuu}_{p_1,q_1}\,\mathrm{Sym}_{p-p_1}\Big(\bigcup_{i<s<i+p} Q_c(u'_s)\Big)\,\mathrm{Sym}_{q-q_1}\Big(\bigcup_{j<s<j+q} Q_c(u''_s)\Big)$$

with $Sym_s(x_1,\ldots,x_r)$ standing for the s-th symmetric function of the x_i :

$$\mathrm{Sym}_s(x_1,\ldots,x_r) := \sum_{1\leq i_1<\cdots<i_s\leq r} x_{i_1}\ldots x_{i_s} \tag{11.32}$$

However, to get the formula for luu^{w_k} right, we must observe the following convention:

$$\begin{array}{lcll}\mathrm{Sym}_0(x_1,\ldots,x_r) & := & 1 & (\textit{even if } r=0)\\ \mathrm{Sym}_s(x_1,\ldots,x_r) & := & 0 \ \textit{ if } 1\leq r<s & (\textit{but } \mathrm{Sym}_0(\emptyset):=1)\end{array}$$

We may also note the complete absence, from the expression of luu^{w_k}, of the four extreme terms $Q_c(u'_i), Q_c(u'_{i+p}), Q_c(u''_j), Q_c(u''_{j+q})$.

Dimorphic transport.
As in the polar case, the adjoint action of the bisymmetrals $tal_c^\bullet$ and $til_c^\bullet$ exchanges double symmetries, but without respecting entireness.

$$\begin{array}{ccc}\mathrm{GARI}^{\underline{\mathrm{as}}/\underline{\mathrm{as}}} & \xrightarrow{\mathrm{adgari}(\mathrm{tal}_c^\bullet)} & \mathrm{GARI}^{\underline{\mathrm{as}}/\underline{\mathrm{iis}}}\\ \mathrm{logari}\downarrow\uparrow\mathrm{expari} & & \mathrm{logari}\downarrow\uparrow\mathrm{expari}\\ \mathrm{ARI}^{\underline{\mathrm{al}}/\underline{\mathrm{al}}} & \xrightarrow{\mathrm{adari}(\mathrm{tal}_c^\bullet)} & \mathrm{ARI}^{\underline{\mathrm{al}}/\underline{\mathrm{iil}}}\end{array}$$

$$\begin{array}{ccc}\mathrm{GARI}^{\underline{\mathrm{as}}/\underline{\mathrm{as}}} & \xrightarrow{\mathrm{adgari}(\mathrm{til}_c^\bullet)} & \mathrm{GARI}^{\underline{\mathrm{as}}/\underline{\mathrm{uus}}}\\ \mathrm{logari}\downarrow\uparrow\mathrm{expari} & & \mathrm{logari}\downarrow\uparrow\mathrm{expari}\\ \mathrm{ARI}^{\underline{\mathrm{al}}/\underline{\mathrm{al}}} & \xrightarrow{\mathrm{adari}(\mathrm{til}_c^\bullet)} & \mathrm{ARI}^{\underline{\mathrm{al}}/\underline{\mathrm{uul}}}\end{array}$$

Bitrigonometric symmetries.

As usual, the trigonometric case fully determines the bitrigonmetric extension.

Symmetries associated with other approximate units $\mathfrak{E}^\bullet$.

$\mathfrak{E}^\bullet$ of course replaces Q_c in the expressions of lii^{w_k} and luu^{w_k} but the structure coefficients $xii_{p,q}, zii_{p,q}, xuu_{p,q}, zuu_{p,q}$ do not change, and must still be calculated from Q and the related R (see (11.25)) even in the case of the *flat* approximate units $Sa^\bullet$ or $Si^\bullet$ of (3.25).

Remark 1. While bimoulds polynomial or entire in the u_i and v_i variables may be alternil or symmetril or alterniil or symmetriil, they *can never* be alternul nor symmetrul nor alternuul nor symmetruul.

Remark 2. Of course, just as with the *straight* symmetries (see §2.4), when expressing the new, *twisted* symmetries, one should take care to allow only sequences $\mathbf{w}$ that are order-compatible with $\mathbf{w}'$ and $\mathbf{w}''$, i.e. that never carry pairs u'_i, v'_i or u''_j, v''_j (whether in isolation or within sums or differences) in an order that clashes with their relative position within the parent sequences $\mathbf{w}'$ or $\mathbf{w}''$.

11.5 The separative algebras $Inter(Qi_c)$ and $Exter(Qi_c)$.

Introduction.
The subalgebra $Exter(Qi_c)$ of $Flex(Qi_c)$ is the trigonometric equivalent of the polar subalgebra $ARI_{<pi>}$ of $Flex(Pi)$ which itself is but the specialisation, for $\mathfrak{E} = Pi$, of the subalgebra $ARI_{<\mathfrak{re}>}$ of $Flex(\mathfrak{E})$ which was investigated in §3.6. Both $Exter(Qi_c)$ and $ARI_{<pi>}$ consist of $\boldsymbol{u}$-constant, $\boldsymbol{v}$-dependent, alternal bimoulds, and both are indispensable to an in-depth understanding of the fundamental bialternals $pil^\bullet$ and $til_c^\bullet$ since they house their *ari*-logarithms $logari.pil^\bullet$ and $logari.til_c^\bullet$ as well as the corresponding infinitesimal dilators.

However, due to $Pi^\bullet$ being an *exact* flexion unit, the algebra $ARI_{<pi>}$ has a very simple structure: it is spanned by the bimoulds $pi_r^\bullet$ $(1 \leq r)$, which self-reproduce under the *ari*-bracket: $ari(pi_{r_1}^\bullet, pi_{r_2}^\bullet) \equiv (r_1 - r_2) pi_{r_1+r_2}^\bullet$.

Its trigonometric counterpart $Exter(Qi_c)$, on the other hand, is vaster and much more complex: it does indeed contain a series of bimoulds $qi_r^\bullet$ defined in the same way as the $pi_r^\bullet$ or the $\mathfrak{re}_r^\bullet$ of (4.5), but these $qi_r^\bullet$ no longer self-reproduce under the *ari*-bracket: they do so only modulo c^2.

Nonetheless, the structure of $Exter(Qi_c)$ is highly interesting, and can be exhaustively described by decomposing $Exter(Qi_c)$ into a direct sum of subspaces $\mathfrak{g}^n.Inter(Qi_c)$ $(0 \leq n)$ which are all derived from a subalgebra $Inter(Qi_c) \subset Exter(Qi_c)$ consisting of all alternals in $Flex(Qi_c)$ that depend only on the differences $v_i - v_j$.[158] The algebra $Inter(Qi_c)$ and its elements shall be called *internal*, whereas elements of $Exter(Qi_c) \dot{-} Inter(Qi_c)$ shall be called *external.* The internal algebra is quite elementary: on it, most flexion

[158] *in the short notation*, of course. In the long notation (with the additional variable v_0), this is automatic and implies no constraint at all.

operations reduce to non-inflected operations. Thus, the *ari*-bracket of two internals coincides (up to a sign change) with their *lu*-bracket.

The external and internal algebras are also called *separative*, since under the action of the operator *separ*, which is to *ARI* what the operator *gepar* of §4.1. was to *GARI*:

$$\text{separ}.M^\bullet := \text{anti.swap}.M^\bullet + \text{swap}.M^\bullet \quad (11.33)$$

$$\text{gepar}.M^\bullet := \text{mu}(\text{anti.swap}.M^\bullet, \text{swap}.M^\bullet) \quad (11.34)$$

their bimoulds experience a *separation* of their variables[159] and assume the elementary form:

$$\{M^\bullet \in \text{Exter}(\text{Qi}_c)\} \Rightarrow \{(\text{separ}.M)^{w_1,\ldots,w_r} \in \mathbb{C}[c^2, Q_c(u_1),\ldots,Q_c(u_r)]\}$$

Remark: strictly speaking, elements of $Flex(Qi_c)$ can involve only *even* powers of c, but it is convenient to enlarge $Exter(Qi_c)$ and $Inter(Qi_c)$ with *odd* powers of c, so as to make room for the bimoulds $qin_r^\bullet$ and the operators $\mathfrak{h}_n$ (defined *infra*). Ultimately, however, we shall end up with structure formulas where these $qin_r^\bullet$ and $\mathfrak{h}_n$ appear only in pairs, thus ensuring that there is no violation of c-parity.

The external $qi_r^\bullet$ and the internal $qin_r^\bullet$.
They are the first ingredients of the 'separative' structure. These alternal bimoulds of $BIMU_r$ are defined by the induction:

$$\text{qi}_1^{w_1} := \text{Qi}_c^{w_1} = Q_c(v_1) = \tfrac{c}{\tan{(c\,v1)}} \quad \| \quad \text{qi}_r^\bullet := \text{arit}(\text{qi}_{r-1}^\bullet).\text{qi}_1^\bullet \quad \forall (r \geq 2)$$

$$\text{qin}_1^{w_1} := c \quad \| \quad \text{qin}_r^\bullet := \text{ari}(\text{qin}_1^\bullet, \text{qi}_{r-1}^\bullet) \quad \forall (r \geq 2)$$

The auxiliary mould $har^\bullet$.
Our second ingredient is a scalar mould whose only non-vanishing components have *odd* length. Here again, the definition is by induction:

$$\text{har}^{n_1,\ldots,n_r} := 0 \quad \forall r \;\; even \geq 0 \quad (11.35)$$

$$\text{har}^{n_1} := \frac{1}{n_1} \quad (11.36)$$

$$\text{har}^{n_1,\ldots,n_r} := \frac{1}{n_1+\ldots+n_r} \sum_{1<i<r} \text{har}^{n_1,\ldots,n_{i-1}} \text{har}^{n_{i+1},\ldots,n_r} \quad \forall r \;\; odd \geq 3 \quad (11.37)$$

Thus:

$$\text{har}^{n_1,n_2,n_3} = \frac{1}{n_1 n_3 n_{123}} \quad (11.38)$$

$$\text{har}^{n_1,n_2,n_3,n_4,n_5} = \frac{1}{n_1 n_3 n_5 n_{12345}} \left(\frac{1}{n_{123}} + \frac{1}{n_{345}}\right) \quad (11.39)$$

[159] due to the *swap* which is implicit in the definition of *separ* and *gepar*, the new variables are no longer v_i's but u_i's.

The operators $\mathfrak{g}^n$, $\mathfrak{h}_n$.
These linear operators of $BIMU_r$ into $BIMU_{r+n}$ are our third ingredient. The first are mere powers of a single operator $\mathfrak{g}$ defined by:

$$\mathfrak{g} : \mathcal{A}^\bullet \ \rightarrow \ \mathcal{B}^\bullet \ := \ \mathrm{arit}(\mathcal{A}^\bullet)\ \mathrm{qi}_1^\bullet = \mathrm{arit}(\mathcal{A}^\bullet)\ \mathrm{Qi}_\mathrm{c}^\bullet \tag{11.40}$$

which, since $Qi_c^\bullet \in BIMU_1$, may be rewritten as:

$$\mathcal{B}^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ v_1 & ,\dots, & v_r \end{smallmatrix}\right)} \ = \ \mathcal{A}^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_{r-1} \\ v_{1:r} & ,\dots, & v_{r-1:r} \end{smallmatrix}\right)} Q_c(v_r) - \mathcal{A}^{\left(\begin{smallmatrix} u_2 & ,\dots, & u_r \\ v_{2:1} & ,\dots, & v_{r:1} \end{smallmatrix}\right)} Q_c(v_1) \tag{11.41}$$

The operators $\mathfrak{h}_n$, on the other hand, must be defined singly:

$$\mathfrak{h}_n\, A^\bullet \ := \ \sum_{1\leq s} \ \sum_{\substack{1\leq n_i \\ n_1+\dots n_s=n}} \mathrm{har}^{n_1,\dots,n_r}\, [\mathrm{qin}^\bullet_{n_1}[\mathrm{qin}^\bullet_{n_2}\dots[\mathrm{qin}^\bullet_{n_s}, A^\bullet]..]]_{\mathrm{lu}} \tag{11.42}$$

Due to the imparity of $har^\bullet$, the $\mathfrak{h}_n$ too are strictly odd in c.

The operators of $\mathfrak{G}$ and $\mathfrak{H}$.
If we set:

$$\mathfrak{G} := \mathrm{id} + \sum_{1\leq n} \mathfrak{g}^n \quad ; \quad \mathfrak{H} := + \sum_{1\leq n} \mathfrak{h}_n \tag{11.43}$$

the operators of $\mathfrak{G}$ and $\mathfrak{H}$ so defined verify the identities:

$$\mathfrak{G}\,\mathrm{mu}(A^\bullet, B^\bullet) \ \equiv \ \mathrm{mu}(\mathfrak{G}\,A^\bullet, \mathfrak{G}\,B^\bullet) + \mathfrak{G}\,\mathrm{mu}(\mathfrak{H}\,A^\bullet, \mathfrak{H}\,B^\bullet) \tag{11.44}$$

$$\mathfrak{H}\,\mathrm{mu}(A^\bullet, B^\bullet) \ \equiv \ \mathrm{mu}(\mathfrak{H}\,A^\bullet, B^\bullet) + \mathrm{mu}(A^\bullet, \mathfrak{H}\,B^\bullet) + \mathfrak{H}\,\mathrm{mu}(\mathfrak{H}\,A^\bullet, \mathfrak{H}\,B^\bullet) \tag{11.45}$$

and of course analogous identities with lu in place of mu. The only restriction is that in (11.44) the inputs $A^\bullet, B^\bullet$ must be *internal*.

If we now 'iterate' these identites so as to rid their right-hand sides of all terms $\mathfrak{G}.mu(...,...)$ and $\mathfrak{H}.mu(...,...)$, we find that $\mathfrak{G}$ and $\mathfrak{H}$ verify the co-products:

$$\mathfrak{G} \ \rightarrow \ \mathfrak{G}\otimes\mathfrak{G} + \sum_{1\leq s} \mathfrak{G}\mathfrak{H}^s \otimes \mathfrak{G}\mathfrak{H}^s \tag{11.46}$$

$$\mathfrak{H} \ \rightarrow \ \mathfrak{H}\otimes 1 + 1\otimes\mathfrak{H} + \sum_{1\leq s} (\mathfrak{H}^{s+1}\otimes\mathfrak{H}^s + \mathfrak{H}^s\otimes\mathfrak{H}^{s+1}) \tag{11.47}$$

Again, the coproducts (11.45),(11.47) for $\mathfrak{H}$ hold on the full algebra of bimoulds, whereas the coproducts (11.44),(11.46) for $\mathfrak{G}$ hold only on the algebra of *internals*.

The rectified operators $\mathfrak{G}_*$ and $\mathfrak{H}_*$.
As the above coproducts show, $\mathfrak{H}$ is an 'approximate' derivation and $\mathfrak{G}$ an 'approximate' automorphism. However, if we set:

$$\begin{aligned} \mathfrak{H}_* &:= \arctan(\mathfrak{H}) \;=\; \mathfrak{H} - \frac{1}{3}\mathfrak{H}^3 + \frac{1}{5}\mathfrak{H}^5 \dots && (11.48)\\ \mathfrak{G}_* &:= \mathfrak{G}(\mathrm{id} + \mathfrak{H})^{-\frac{1}{2}} \;=\; \mathfrak{G} - \frac{1}{2}\mathfrak{G}\mathfrak{H}^2 + \frac{3}{8}\mathfrak{G}\mathfrak{H}^4 \dots && (11.49)\\ &= \mathfrak{G}\cos(\mathfrak{H}_*) \;=\; \mathfrak{G} - \frac{1}{2}\mathfrak{G}\mathfrak{H}_*^2 + \frac{1}{24}\mathfrak{G}\mathfrak{H}_*^4 \dots && (11.50) \end{aligned}$$

we get an operator $\mathfrak{H}_*$ that is an exact derivation and an operator $\mathfrak{G}_*$ that is an exact automorphism.

The chain $Inter(Qi_c) \subset Exter(Qi_c) \subset Flex(Qi_c)$**.**
The space $Inter(Qi_c)$ is separative, and so is the space $Exter(Qi_c)$ defined as the (direct) sum of all the $\mathfrak{g}$-translates of $Inter(Qi_c)$.

$$\mathrm{Exter}(\mathrm{Qi}_c^\bullet) := \bigoplus_{0\le n} \mathfrak{g}^n.\mathrm{Inter}(\mathrm{Qi}_c^\bullet) \qquad (11.51)$$

In fact, both spaces are stable under the *ari*-bracket, and we shall now give a complete description of their structure with the help of our two series of operators $\mathfrak{g}^n$ and $\mathfrak{h}_n$.

Full structure of the ***ari*****-algebra** $Inter(Qi_c)$**.**
The space $Inter(Qi_c^\bullet)$ is obviously stable under the *lu*-bracket, and also under the *ari*-bracket, due to the elementary identities:

$$\begin{aligned} \mathrm{ari}(A^\bullet, B^\bullet) &= -\mathrm{lu}(A^\bullet, B^\bullet) \qquad \forall A^\bullet, B^\bullet \in \mathrm{Inter}(\mathrm{Qi_c}) && (11.52)\\ \mathrm{arit}(A^\bullet).B^\bullet &= +\mathrm{lu}(A^\bullet, B^\bullet) \qquad \forall A^\bullet, B^\bullet \in \mathrm{Inter}(\mathrm{Qi_c}) && (11.53) \end{aligned}$$

Full structure of the ***ari*****-algebra** $Exter(Qi_c)$**.**
The space $Exter(Qi_c)$, though not closed under the *lu*-bracket, is stable under the *ari*-bracket and the *arit*-operation. Its full structure is given by the three

following identities, where $A^\bullet, B^\bullet$ stand for arbitrary elements of *Inter*(Qi_c):

$$\begin{aligned}\mathrm{ari}(\mathfrak{g}^p A^\bullet, \mathfrak{g}^q B^\bullet) \;\equiv\; & -\mathfrak{g}^q\,\mathrm{arit}(\mathfrak{g}^p A^\bullet)B^\bullet + \mathfrak{g}^p\,\mathrm{arit}(\mathfrak{g}^q B^\bullet)A^\bullet \\ & +\;\mathfrak{g}^{p+q}\,\mathrm{lu}(A^\bullet, B^\bullet) \\ & -\sum_{\substack{1\le p_1\le p\\ 1\le q_1\le q}} \mathfrak{g}^{p+q-p_1-q_1}\,\mathrm{lu}(\mathfrak{h}_{p_1}A^\bullet, \mathfrak{h}_{q_1}B^\bullet) \end{aligned} \tag{11.54}$$

$$\begin{aligned}\mathrm{arit}(\mathfrak{g}^p A^\bullet)\,\mathfrak{g}^q B^\bullet \;\equiv\; & +\mathfrak{g}^q\,\mathrm{arit}(\mathfrak{g}^p A^\bullet)B^\bullet \\ & -\sum_{0\le q_1\le q-1} \mathrm{lu}(\mathfrak{g}^{p+q-q_1}A^\bullet, \mathfrak{g}^{q_1}B^\bullet) \\ & -\sum_{\substack{1\le q_1\le q-1\\ p+1\le p_1\le p+q-q_1}} \mathfrak{g}^{p+q-p_1-q_1}\,\mathrm{lu}(\mathfrak{h}_{p_1}A^\bullet, \mathfrak{h}_{q_1}B^\bullet) \end{aligned} \tag{11.55}$$

$$\mathrm{arit}(\mathfrak{g}^p A^\bullet)\,\mathrm{tin}_q^\bullet \;\equiv\; \mathfrak{h}_{p,q}A^\bullet \tag{11.56}$$

Since the above identities are linear in each *internal* argument $A^\bullet$ or $B^\bullet$ and since any *external* bimould $M^\bullet$ uniquely decomposes into a sum $\sum \mathfrak{g}^n.M^\bullet_{(n)}$ of $\mathfrak{g}$-tanslates of internal $M^\bullet_{(n)}$, one readily sees that the above identities do indeed encapsulate the whole structure of *Exter*(Qi_c), provided one adds to *Inter*(Qi_c) a symbolic bimould $\square^\bullet \in BIMU_0$ subject to the following rules:[160]

$$\mathrm{qi}_n^\bullet \;:=\; +\mathfrak{g}^n\,\square^\bullet \tag{11.57}$$
$$\mathrm{qin}_n^\bullet \;:=\; -\mathfrak{h}_n\,\square^\bullet \tag{11.58}$$

$$\mathrm{lu}(A^\bullet, \square^\bullet) \;:=\; -r(\bullet)\,A^\bullet \tag{11.59}$$
$$\mathrm{ari}(A^\bullet, \square^\bullet) \;:=\; +r(\bullet)\,A^\bullet \tag{11.60}$$
$$\mathrm{arit}(A^\bullet)\,\square^\bullet \;:=\; -r(\bullet)\,A^\bullet \tag{11.61}$$
$$\mathrm{arit}(\square^\bullet)\,A^\bullet \;:=\; +r(\bullet)\,A^\bullet \tag{11.62}$$

and of course

$$\mathrm{lu}(\square^\bullet, \square^\bullet) = \mathrm{ari}(\square^\bullet, \square^\bullet) = \mathrm{arit}(\square^\bullet)\,\square^\bullet = 0^\bullet \tag{11.63}$$

11.6 Multizeta cleansing: elimination of unit weights.

Main statement.
The present section is devoted to proving the following:

[160] One should refrain from applying to $\square^\bullet$ any other rules than these, and never forget than $\square^\bullet$ is just a convenient symbol rather than a true bimould. Indeed, the only *bona fide* bimould of $BIMU_0$ is (up to a scalar factor) the multiplication unit $1^\bullet$ with $1^\emptyset := 1$.

***P_0* :** *(Unit cleansing)*
Every uncoloured multizeta $\zeta(s_1,\ldots,s_r)$ can be expressed as a finite sum, with rational coefficients, of unit-free multizetas.[161] *The result extends to all coloured multizetas, but it is less relevant there.*[162]

We shall provide an effective algorithm for achieving the *unit-cleansing.* Along the way, we shall also come across some really fine combinatorics about bialternals, and construct a new *infinitary* subalgebra of ARI larger than $ALAL$.

Some heuristics.
As is natural with heuristics, we proceed backwards:

Step 4: restriction of the problem to bialternals.
Since scalar irreducibles accompany homogeneous bialternals, it will be both necessary and sufficient to express the latter without recourse to unit weights.

Step 3: the need for "reconstitution identities".
Since in the v_i-encoding, unit weights correspond to monomials not divisible by $v_1...v_r$, the challenge it to reconstitute any homogeneous bialternal from its "essential part", i.e. the part that is divisible by $v_1...v_r$.

Step 2: the need for "redistribution identities".
To do this, it is more or less clear beforehand that we shall have to find a means of expressing any homogeneous bialternal $M^{\boldsymbol{w}}$ with one or several vanishing v_i's as a superpositions of $M^{\boldsymbol{w}^*}$, with new v_i^*'s formed from the sole non-vanishing v_i's.

Step 1: the need for "pairing identities".
To be able to extend the procedure to *coloured* multizetas (and also to respect the spirit of dimorphy), we must find a way of restating the redistribution identities for arbitrary bialternals that effectively depend on the u_i's as well as on the v_i's.

Step 1: The pairing identities.
An *endoflexion* (of length r) is any self-mapping of $BIMU_r$ of the form

$$\mathrm{flex}.M^{w_1,\ldots,w_r} = M^{w_1^*,\ldots,w_r^*} \qquad \Big(w_i = \binom{u_i}{v_i}\ ,\ w_i^* = \binom{u_i^*}{v_i^*}\Big) \qquad (11.64)$$

[161]i.e. of multizetas $\zeta(s'_1,\ldots,s'_{r_*})$ with partial weights $s'_i \geq 2$.

[162]for two reasons: first, because the removal of the unit-weights necessitates a remixing of the colours; and second, because one may on the contrary play on the colours to express everything in terms of multizetas with *nothing but unit-weights!*

with

$$
\begin{aligned}
u_i^* &:= \overbrace{u_{m_i}+...+u_{n_i}}^{\text{circular}} = \sum^{(m_i\le k\le n_i)_{\mathbb{Z}_{r+1}}} u_k \\
v_i^* &:= v_{p_i}-v_{q_i} \qquad \mathit{and} \qquad p_i\in\mathcal{P}^+, q_i\in\mathcal{P}^- \\
\sum_{1\le i\le r} {u_i}^* v_i^* &\equiv \sum_{1\le i\le r} u_i v_i
\end{aligned}
$$

Here, all indices m_i, n_i, p_i, q_i are in the set $\{0,1,...,r\} \sim \mathbb{Z}_{r+1}$ and $\mathcal{P} = (\mathcal{P}^+, \mathcal{P}^-)$ is any given (strict) partition of $\{0,1,...,r\}$. We say that *flex* is $\mathcal{P}$-compatible. Whereas *flex* determines $\mathcal{P}$ if we impose (as we shall do) that 0 be in $\mathcal{P}^-$, there are usually many endoflexions *flex* compatible with a given partition $\mathcal{P}$.

$\boldsymbol{P_1}$ **:** *(Existence and unicity of the pairing identities.)*
For any strict partition $\mathcal{P}$ of $\{0,1,...,r\}$ into $\mathcal{P}^+$ ("white indices") and $\mathcal{P}^-$ ("black indices") there exists a self-mapping $\mathrm{flex}_{\mathcal{P}}$ *of* $\mathrm{BIMU_r}$ *of the form:*

$$\mathrm{flex}_{\mathcal{P}} = \sum_{\mathrm{flex_n}\ \mathcal{P}\text{-compatible}} \epsilon_n\, \mathrm{flex_n} \qquad (\epsilon_n \in \{0,1,-1\}) \tag{11.65}$$

whose restriction to the bialternals is the identity:

$$\mathrm{flex}_{\mathcal{P}}.M^\bullet \equiv M^\bullet \qquad \forall M^\bullet \in \mathrm{ARI}_r^{\underline{\mathrm{al}}/\underline{\mathrm{al}}} \tag{11.66}$$

Furthermore, $\mathrm{flex}_{\mathcal{P}}$ *is unique modulo the alternality (not bialternality!) relations on* $\mathrm{ARI}_r^{\underline{\mathrm{al}}/\underline{\mathrm{al}}}$*.*

Let us now return to the graph pairs $\boldsymbol{g} = (\boldsymbol{ga}, \boldsymbol{gi})$ defined in §3.1 (see also the examples and pictures *infra*). We say that such a pair $\boldsymbol{g}$ is $\mathcal{P}$-compatible if all edges of $\boldsymbol{gi}$ connect a "white" vertex S_{p_i} with a "black" vertex S_{q_i}. Now, both $\boldsymbol{gi}$ and $\boldsymbol{ga}$ have r edges each, and every edge of $\boldsymbol{gi}$ intersects exactly *one* edge of $\boldsymbol{ga}$ at exactly *one* point x, and there clearly exists a (topologically) unique graph $\boldsymbol{gai}$ with those r intersection points x as vertices, and with edges that intersect neither the unit circle nor the edges of $\boldsymbol{ga}$ nor those of $\boldsymbol{gi}$. To each vertex x_* of $\boldsymbol{gai}$ there corresponds one unique coherent orientation $\mathcal{O}_{\boldsymbol{g},x_*}$ of the edges of $\boldsymbol{gai}$ or, what amounts to the same, one coherent arborescent order, also noted $\mathcal{O}_{\boldsymbol{g},x_*}$, on the vertices x of $\boldsymbol{gai}$, with x_* as the lowest vertex.

Next, for any $\boldsymbol{g}$ that is $\mathcal{P}$-compatible and for any vertex x_* of the corresponding $\boldsymbol{gai}$, let $\boldsymbol{\gamma}$ be a total order on the vertices of $\boldsymbol{gai}$ that is compatible with the arborescent order $\mathcal{O}_{\boldsymbol{g},x_*}$. We write $\boldsymbol{\gamma} \in \mathcal{O}_{\boldsymbol{g},x_*}$ to denote this compatibility and associate with $\boldsymbol{\gamma}$ the following endoflexion:

$$\mathrm{flex}_{\boldsymbol{\gamma}}.M^{w_1,...,w_r} = M^{w_1^*(\boldsymbol{\gamma}),...,w_r^*(\boldsymbol{\gamma})} \quad \Big(w_i = \big({}^{u_i}_{v_i}\big)\ ,\ w_i^*(\boldsymbol{\gamma}) = \big({}^{u_i^*(\boldsymbol{\gamma})}_{v_i^*(\boldsymbol{\gamma})}\big)\Big) \tag{11.67}$$

with

$$
\begin{aligned}
u_i^*(\boldsymbol{\gamma}) &:= \overbrace{u_{m_i}+...+u_{n_i}}^{\text{circular}} = \sum^{(m_i\leq k\leq n_i)_{\mathbb{Z}_{r+1}}} u_k \\
v_i^*(\boldsymbol{\gamma}) &:= v_{p_i}-v_{q_i} \qquad \textit{and} \qquad p_i\in\mathcal{P}^+, q_i\in\mathcal{P}^- \\
\sum_{1\leq i\leq r} {u_i}^*(\boldsymbol{\gamma})\, v_i^*(\boldsymbol{\gamma}) &\equiv \sum_{1\leq i\leq r} u_i\, v_i
\end{aligned}
$$

and with the following notations:
– $x_i(\boldsymbol{\gamma})$ is the i-th vertex of $\boldsymbol{gai}$ in the total order $\boldsymbol{\gamma}$;
– $ga_i(\boldsymbol{\gamma})$ is the unique edge of $\boldsymbol{ga}$ passing through $x_i(\boldsymbol{\gamma})$;
– $gi_i(\boldsymbol{\gamma})$ is the unique edge of $\boldsymbol{gi}$ passing through $x_i(\boldsymbol{\gamma})$;
– $u_i^*(\boldsymbol{\gamma})$ is the sum of all u_k with Si_k on the 'correct' side of $ga_i(\boldsymbol{\gamma})$, i.e. on the side that contains the "white" vertex Si_{p_i} of $gi_i(\boldsymbol{\gamma})$;
– $v_i^*(\boldsymbol{\gamma})$ is the difference $v_{p_i}-v_{q_i}$ with Si_{p_i} and Si_{q_i} being the "white" and "black" vertices joined by $gi_i(\boldsymbol{\gamma})$.

Next, we set:

$$
\begin{aligned}
\text{flex}_{\boldsymbol{g},x_*} &:= \sum_{\boldsymbol{\gamma}\in\mathcal{O}_{\boldsymbol{g},x_*}} \epsilon_{\boldsymbol{\gamma}}\,\text{flex}_{\boldsymbol{\gamma}} \qquad \textit{with} && (11.68)\\
\epsilon_{\boldsymbol{\gamma}} &:= \prod_{gai_k\in\text{Edge}(\boldsymbol{gai})} \epsilon(\text{gai}_k)\in\{1,-1\} && (11.69)
\end{aligned}
$$

with a product in (11.69) extending to all $r-1$ edges gai_k of $\boldsymbol{gai}$, and with factor signs $\epsilon(gai_k)$ defined as follows. Each edge gai_k of $\boldsymbol{gai}$ touches two edges $gi_{k'}$ and $gi_{k''}$ of $\boldsymbol{gi}$, which in turn meet at a vertex Si_{k^*} of $\boldsymbol{gi}$. What counts is the colour of that vertex Si_{k^*}, and the position of the triangle $\{gai_k, gi_{k'}, gi_{k''}\}$ respective to the oriented vertex $\vec{gai}_k$. Concretely, we set:
(i) $\epsilon(gai_k) := +1$ if $\vec{gai}_k$ sees a white Si_{k^*} to its right or a black Si_{k^*} to its left.
(ii) $\epsilon(gai_k) := -1$ if $\vec{gai}_k$ sees a white Si_{k^*} to its left or a black Si_{k^*} to its right.

It is readily seen that the operator $\text{flex}_{\boldsymbol{g},x_*}$, *when applied to alternal bimoulds*, is independent of the choice of the base vertex: indeed, replacing x_* by a neighbouring vertex x_{**} simultaneously changes the signs of $\epsilon(\boldsymbol{\gamma})$ and $\mathit{flex}_{\boldsymbol{\gamma}}.M^\bullet$, for any alternal $M^\bullet$. We shall therefore drop x_* and write simply $\text{flex}_{\boldsymbol{g}}$ whenever the operator $\text{flex}_{\boldsymbol{g},x_*}$ is made to act on alternals (or *a fortiori* on bialternals).

Remark: each one of the graphs $\boldsymbol{ga}$ or $\boldsymbol{gi}$ completely determines the other as well as $\boldsymbol{gai}$. It also determines the only partition $\mathcal{P}$ of $\{0,1,...,r\}$ with which it is compatible, since 0 is automatically *black*, and so Si_0 is declared *black* too, and the coloring then extends to all Si_k by following $\boldsymbol{gi}$. On the other hand, the number of graph pairs $\boldsymbol{g} = \{\boldsymbol{ga}, \boldsymbol{gi}\}$ compatible with a given partition $\mathcal{P}$ is *on average* equal to $\frac{(3\,r)!}{(2\,r+1)!r!2^r}$ and therefore tends to be very large.

***P*₂ :** *(Explicit formula for the pairing identities.)*
For each partition $\mathcal{P}$ of $\{0,1,...,r\}$, the pairing operator $\mathrm{flex}_{\mathcal{P}}$ *of (11.65) is explicitely given by:*

$$\mathrm{flex}_{\mathcal{P}} := \sum_{\boldsymbol{g}\ \mathcal{P}\text{-compatible}} \mathrm{flex}_{\boldsymbol{g}} \qquad \textit{with}\ \ \epsilon_{\boldsymbol{g}} \in \{1,-1\} \tag{11.70}$$

with a sum extending to all graph pairs $\boldsymbol{g} = (\boldsymbol{ga}, \boldsymbol{gi})$ compatible with the white-black partition $\mathcal{P}$.

P_3 : *(Unitary criterion for bialternality.)*
A bimould $M^\bullet \in \mathrm{BIMU}_r$ is bialternal if and only if it verifies all pairing identities $\mathrm{flex}_{\mathcal{P}}.M^\bullet \equiv M^\bullet$*, for all partitions $\mathcal{P} = \mathcal{P}^+ \sqcup \mathcal{P}^-$ of $\{0,1,...,r\}$.*

This is the only known characterisation of bialternality that is *unitary* – by which we mean that, unlike all the others, it does not split into two distinct sets of conditions, one bearing on $M^\bullet$ and another on $swap.M^\bullet$.

Step 2: The redistribution identities.
P_4 : *(Redistribution identity on* $\mathrm{ARI}^{\underline{\mathrm{al}}/\underline{\mathrm{al}}}$ *and* swap.ALAL.*)*
If we take a bialternal $M^\bullet \in \mathrm{ARI}_r^{\underline{\mathrm{al}}/\underline{\mathrm{al}}}$ and a partition $\mathcal{P} = \mathcal{P}^+ \sqcup \mathcal{P}^-$ of $\{0,1,...,r\}$ and then turn all *u_i's into 0 and also turn all black v_i's (i.e. all v_i's with black indices) into 0 but leave all white v_i's unchanged, the* pairing identity *of Proposition P_2 becomes a* redistribution identity*:*

$$\{\mathrm{flex}_P.M^\bullet = M^\bullet\} \Longrightarrow \{\mathrm{redis}_P.M^\bullet = M^\bullet\} \tag{11.71}$$

so-called because it has the effect of 'spreading' or 'redistributing' the total multiplicity μ_0 of the vanishing black v_i's[163] *among the multiplicities μ_i of the remaining white v_i's, with $\mu_0 - 1 = \sum(\mu_i - 1)$. The redistribution identities apply in particular to all bimoulds of* swap.ALAL*, since they are bialternal,* $\boldsymbol{u}$*-constant and polynomial in* $\boldsymbol{v}$*.*

P_5 : *(The infinitary redistribution algebra.)*
The set of all "redistributive" bimoulds, i.e, of all bimoulds that are

[163]in the augmented notation, i.e. considering $\{v_0, v_1, ..., v_r\}$, with v_0 automatically regarded as black. When v_0 is the only black variable, i.e. when $\mathcal{P}^- = \{0\}$ and $\mathcal{P}^+ = \{1,...,r\}$, then $\mu_0 = 1$ and both flex_P and redis_P reduce to the identity, so that in this case the pairing and redistribution identities become trivial.

– $\boldsymbol{u}$*-constant*
– *alternal*
– *and verify all redistribution identities*
constitutes a subalgebra of ARI that is
– *much larger than that of the* $\boldsymbol{u}$*-constant bialternals*
– *not subject to* neg*-invariance (unlike the bialternals)*
– *and yet defined by an infinitary group of constraints (like the bialternals).*

Although the *redistribution identities* have a more elementary appearance than the *pairing identities*, they are in fact
– theoretically derivative,
– distinctly weaker (since they do not imply bialternality),
– and less transparent (since the terms on the right-hand side are *composite*[164] and preceded by general integers rather than by ± signs.)

Step 3: The reconstitution identities.
For any bimould $M^\bullet$, we denote by *essen.*$M^\bullet$ the "essential part" of $M^\bullet$, i.e. the "part" of $M^\bullet$ that is "divisible" by each v_i. In precise terms:

$$(\text{essen}.M)^{\binom{u_1\ ,\ \dots\ ,\ u_r}{v_1\ ,\ \dots\ ,\ v_r}} = \sum_{\epsilon_i\in\{0,1\}} \Big(\prod_{1\leq i\leq r}(-1)^{1+\epsilon_i}\Big)\, M^{\binom{u_1\ ,\ \dots\ ,\ u_r}{\epsilon_1 v_1\ ,\ \dots\ ,\ \epsilon_r v_r}} \tag{11.72}$$

Likewise, to each partition $\mathcal{P}$ that makes 0 *black*, we associate the "slice" of $M^\bullet$ that is "divisible" by all *white* v_i's and constant in all *black* v_i's:[165]

$$(\text{slice}_{\mathcal{P}}.M)^{\binom{u_1\ ,\ \dots\ ,\ u_r}{v_1\ ,\ \dots\ ,\ v_r}} = \sum_{\left\{\substack{\epsilon_i\in\{0,1\}\ if\ i\in\mathcal{P}^+\\ \epsilon_i=0\ if\ i\in\mathcal{P}^-}\right\}}^{0\in\mathcal{P}^-} \Big(\prod_{i\in\mathcal{P}^+}(-1)^{1+\epsilon_i}\Big)\, M^{\binom{u_1\ ,\ \dots\ ,\ u_r}{\epsilon_1 v_1\ ,\ \dots\ ,\ \epsilon_r v_r}} \tag{11.73}$$

$M^\bullet$ is clearly the sum of all its slices:

$$M^\bullet = \sum_{\mathcal{P}\ with\ 0\in\mathcal{P}^-} \text{slice}_{\mathcal{P}}.M^\bullet \tag{11.74}$$

and if $M^\bullet$ happens to be bialternal, each slice may be separately recovered from *essen.*$M^\bullet$ by means of the redistribution identites, since:

$$\text{slice}_{\mathcal{P}}.M^\bullet \equiv \text{redis}_{\mathcal{P}}\,.\,\text{essen}.M^\bullet \tag{11.75}$$

as we can see by applying the $\mathcal{P}$-related redistribution identity separately to each summand on the right-hand side of (11.72). Therefore:

[164]in the sense that they often conflate several contributions, which were clearly distinct in the pairing identities.
[165]If $\mathcal{P}^+ = \{1,\dots,r\}$ and $\mathcal{P}^- := \{0\}$, the slice *slice*$_{\mathcal{P}}.M^\bullet$ coincides with *essen.*$M^\bullet$.

$\boldsymbol{P_6}$ **:** *(Reconstitution identity on* $ARI^{\underline{\mathrm{al}}/\underline{\mathrm{al}}}$*.)*
For each bialternal bimould $M^\bullet$ *(purely of length* r*), the identity holds:*

$$M^\bullet \equiv \mathrm{induc.essen}.M^\bullet \tag{11.76}$$

with the linear operator

$$\mathrm{induc} := \sum_{\mathcal{P}\ \mathit{with}\ 0\in\mathcal{P}^-} \mathrm{redis}_{\mathcal{P}} \tag{11.77}$$

This applies in particular to all elements of *swap.ALAL*, i.e. to all $\boldsymbol{u}$-constant, $\boldsymbol{v}$-polynomial, and bialternal bimoulds. For such bialternals, the possiblity of recovering $M^\bullet$ from $essen.M^\bullet$ was by no means a foregone conclusion, since for a not too large ratio $d/r := degree/length$[166] the essential part $essen.M^\bullet$ carries but a minute fraction of the total data of $M^\bullet$.

$\boldsymbol{P_7}$ **:** *(Involutive nature of* induc*.)*
While essen *is (trivially) a projector,* induc *becomes (non-trivially) an involution when restricted to the space of* $\boldsymbol{u}$*-constant bialternals.*

Step 4: The unit-cleansing algorithm.
The algorithm applies to all multizetas, coloured or uncoloured, but let us focus on the uncoloured case for simplicity.

Fix any basis $\{l\o ma^\bullet_s\,;\ s = 3, 5, 7...\}$ of *ALIL*. That automatically fixes a system of irreducibles $\{irr\o^\bullet_{II}, irr\o^\bullet_{III}\}$ and provides a way of expressing all multizetas in terms of these.

Now, reason inductively. Assume that all irreducibles of length $r < r_0$ have already been expressed in terms of unit-free multizetas $\zeta(s_1, ..., s_r)$. The machinery of §6 makes it possible to exactly determine the contribution that these "earlier" irreducibles (including π^2) are going to make to $Zig^\bullet := swap.Zag^\bullet$, at all higher lengths, including at length r_0. Next, subtract from $leng_{r_0}.Zig^\bullet$ (i.e. from the length-r_0 component of $Zig^\bullet$) all these contributions from the "earlier" irreducibles. What is left is a superposition $M^\bullet$ of independent bialternals $M^\bullet_j$ of length r_0:

$$M^\bullet = \sum \mathrm{irr}\o_j\, M^\bullet_j \quad \mathit{with} \quad M^\bullet_j \in \mathrm{swap.ALAL}_r \ \ \mathit{and}\ \mathrm{irr}\o_j \in \mathbb{C} \tag{11.78}$$

with scalar coefficients $irr\o_j$ that are irreducibles of length r_0. But, as we just saw, $M^\bullet$, and therefore all $M^\bullet_j$ and all $irr\o_j$, can be recovered from $essen.M^\bullet$, and as a consequence expressed in terms of *unit-free* multizetas $\zeta(s_1, ..., s_r)$. By induction, this applies to *all* irreducibles subsumed in the moulds $irr\o^\bullet_{II}$, $irr\o^\bullet_{III}$ and of course also to the exceptional irreducible $\pi^2 = 6\,\zeta(2)$.

[166] say, for $2 < d/r < 3$. (Recall that d/r can in no case be ≤ 2).

But since every multizeta $\zeta(s_1, ..., s_r)$ can be (algorithmically) expressed in terms of irreducibles, this means that every multizeta can be expressed as a polynomial of *unit-free* multizetas $\zeta(s_1, ..., s_r)$, with rational coefficients. After *symmetrel linearisation*, this polynomial becomes a linear combination of multizetas, still unit-free and still with rational coefficients. $\square$.

Example of pairing identities.
For r=5, $\mathcal{P}^+ = \{1, 2, 4\}$, $\mathcal{P}^- = \{0, 3, 5\}$, the pairing identity $M^\bullet \equiv \mathit{flex}_{\mathcal{P}}.M^\bullet$ takes the form:

$$
\begin{array}{lll}
M^{\binom{u_1,\, u_2,\, u_3,\, u_4,\, u_5,}{v_1,\, v_2,\, v_3,\, v_4,\, v_5,}} \equiv & & \qquad (***) \\
-M^{\binom{u_{5*0},\, u_{4*5},\, u_{1*4},\, u_{4*3},\, u_{2*1}}{v_{1:0},\, v_{1:5},\, v_{1:3},\, v_{4:3},\, v_{2:3}}} & -M^{\binom{u_{5*0},\, u_{4*5},\, u_{1*4},\, u_{2*1},\, u_{4*3}}{v_{1:0},\, v_{1:5},\, v_{1:3},\, v_{2:3},\, v_{4:3}}} & -M^{\binom{u_{1*0},\, u_{5*1},\, u_{4*5},\, u_{2*4},\, u_{4*3}}{v_{1:0},\, v_{2:0},\, v_{2:5},\, v_{2:3},\, v_{4:3}}} \\
-M^{\binom{u_{5*0},\, u_{1*5},\, u_{4*1},\, u_{2*4},\, u_{4*3}}{v_{1:0},\, v_{1:5},\, v_{2:5},\, v_{2:3},\, v_{4:3}}} & -M^{\binom{u_{4*5},\, u_{5*3},\, u_{3*1},\, u_{1*0},\, u_{2*3}}{v_{4:5},\, v_{4:0},\, v_{2:0},\, v_{1:0},\, v_{2:3}}} & -M^{\binom{u_{4*5},\, u_{5*3},\, u_{3*1},\, u_{2*3},\, u_{1*0}}{v_{4:5},\, v_{4:0},\, v_{2:0},\, v_{2:3},\, v_{1:0}}} \\
+M^{\binom{u_{2*1},\, u_{1*3},\, u_{3*0},\, u_{5*3},\, u_{4*5}}{v_{2:3},\, v_{1:3},\, v_{4:0},\, v_{1:0},\, v_{4:5}}} & -M^{\binom{u_{1*0},\, u_{2*1},\, u_{5*2},\, u_{4*5},\, u_{2*3}}{v_{1:0},\, v_{2:0},\, v_{4:0},\, v_{4:5},\, v_{4:3}}} & -M^{\binom{u_{1*0},\, u_{2*1},\, u_{5*2},\, u_{2*3},\, u_{4*5}}{v_{1:0},\, v_{2:0},\, v_{4:0},\, v_{4:3},\, v_{4:5}}} \\
+M^{\binom{u_{2*1},\, u_{1*3},\, u_{5*1},\, u_{1*0},\, u_{4*5}}{v_{2:3},\, v_{4:3},\, v_{4:0},\, v_{1:0},\, v_{4:5}}} & +M^{\binom{u_{2*1},\, u_{1*3},\, u_{5*1},\, u_{4*5},\, u_{1*0}}{v_{2:3},\, v_{4:3},\, v_{4:0},\, v_{4:5},\, v_{1:0}}} & +M^{\binom{u_{2*1},\, u_{1*0},\, u_{0*3},\, u_{5*0},\, u_{4*5}}{v_{2:3},\, v_{1:3},\, v_{4:3},\, v_{4:0},\, v_{4:5}}} \\
-M^{\binom{u_{1*0},\, u_{5*1},\, u_{2*5},\, u_{5*3},\, u_{4*5}}{v_{1:0},\, v_{2:0},\, v_{2:3},\, v_{4:3},\, v_{4:5}}} & +M^{\binom{u_{4*5},\, u_{5*3},\, u_{1*5},\, u_{5*0},\, u_{2*1}}{v_{4:5},\, v_{4:3},\, v_{1:3},\, v_{1:0},\, v_{2:3}}} & +M^{\binom{u_{4*5},\, u_{5*3},\, u_{1*5},\, u_{2*1},\, u_{5*0}}{v_{4:5},\, v_{4:3},\, v_{1:3},\, v_{2:3},\, v_{1:0}}} \\
-M^{\binom{u_{5*0},\, u_{1*5},\, u_{4*1},\, u_{1*3},\, u_{2*1}}{v_{1:0},\, v_{1:5},\, v_{4:5},\, v_{4:3},\, v_{2:3}}} & -M^{\binom{u_{1*0},\, u_{5*1},\, u_{2*5},\, u_{4*2},\, u_{2*3}}{v_{1:0},\, v_{2:0},\, v_{2:5},\, v_{4:5},\, v_{4:3}}} & -M^{\binom{u_{2*1},\, u_{1*3},\, u_{3*5},\, u_{5*0},\, u_{4*3}}{v_{2:3},\, v_{1:3},\, v_{1:5},\, v_{1:0},\, v_{4:5}}} \\
-M^{\binom{u_{2*1},\, u_{1*3},\, u_{3*5},\, u_{4*3},\, u_{5*0}}{v_{2:3},\, v_{1:3},\, v_{1:5},\, v_{4:5},\, v_{1:0}}} & -M^{\binom{u_{5*0},\, u_{1*5},\, u_{2*1},\, u_{4*2},\, u_{2*3}}{v_{1:0},\, v_{1:5},\, v_{2:5},\, v_{4:5},\, v_{4:3}}} & +M^{\binom{u_{1*0},\, u_{5*1},\, u_{3*5},\, u_{4*3},\, u_{2*3}}{v_{1:0},\, v_{2:0},\, v_{2:5},\, v_{4:5},\, v_{2:3}}} \\
+M^{\binom{u_{1*0},\, u_{5*1},\, u_{3*5},\, u_{2*3},\, u_{4*3}}{v_{1:0},\, v_{2:0},\, v_{2:5},\, v_{2:3},\, v_{4:5}}} & +M^{\binom{u_{5*0},\, u_{1*5},\, u_{3*1},\, u_{4*3},\, u_{2*3}}{v_{1:0},\, v_{1:5},\, v_{2;5},\, v_{4:5},\, v_{2:3}}} & +M^{\binom{u_{5*0},\, u_{1*5},\, u_{3*1},\, u_{2*3},\, u_{4*3}}{v_{1:0},\, v_{1:5},\, v_{2:5},\, v_{2:3},\, v_{4:5}}}
\end{array}
$$

with the usual convention $u_0 := -(u_1 + \cdots + u_r)$, $v_0 := 0$ and the convenient abbreviations:

$$
\begin{aligned}
u_{i*j} &:= su_i - su_j \quad \mathit{with} \quad su_k := u_0 + u_1 + ... + u_k = -u_{k+1} - u_{k+2} ... - u_r \\
v_{i:j} &:= v_i - v_j
\end{aligned}
$$

To arrive at the pairing identity $(***)$, we form all graph triples $\boldsymbol{g} = \{\boldsymbol{ga}, \boldsymbol{gi}, \boldsymbol{gai}\}$ compatible with the partition $\mathcal{P}$. There exist exactly 16 such triples. They are pictured on Figure 3, with split lines for the edges of $\boldsymbol{ga}$, plain lines for those of $\boldsymbol{gi}$, and large plain lines for those of $\boldsymbol{gai}$. Next, on each $\boldsymbol{gai}$, we pick a vertex x_* so chosen as to minimise the number $\nu(\boldsymbol{gai}, x_*)$ of total orders $\boldsymbol{\gamma}$ on $\boldsymbol{gai}$ compatible with the partial order induced by x_*. In each case, x_* has to be at the extremity of the longest branch of $\boldsymbol{gai}$. For eight graphs $\boldsymbol{gai}$, this minimal number $\nu_{min}(\boldsymbol{gai})$ is 1; for the remaining eight graphs, $\nu_{min}(\boldsymbol{gai})$ is 2. Altogether, this yields the 24 elementary flexions $\mathit{flex}_{\boldsymbol{\gamma}}$ that contribute to the pairing identity $(***)$.

Lastly, to show how to calculate each $\mathit{flex}_{\boldsymbol{g}}$, we focus on the first graph triple (the one in top-left position on Figure 3) and reproduce it, enlarged, in Figure 4. Applying the rules just after (11.67), we see that the flexion

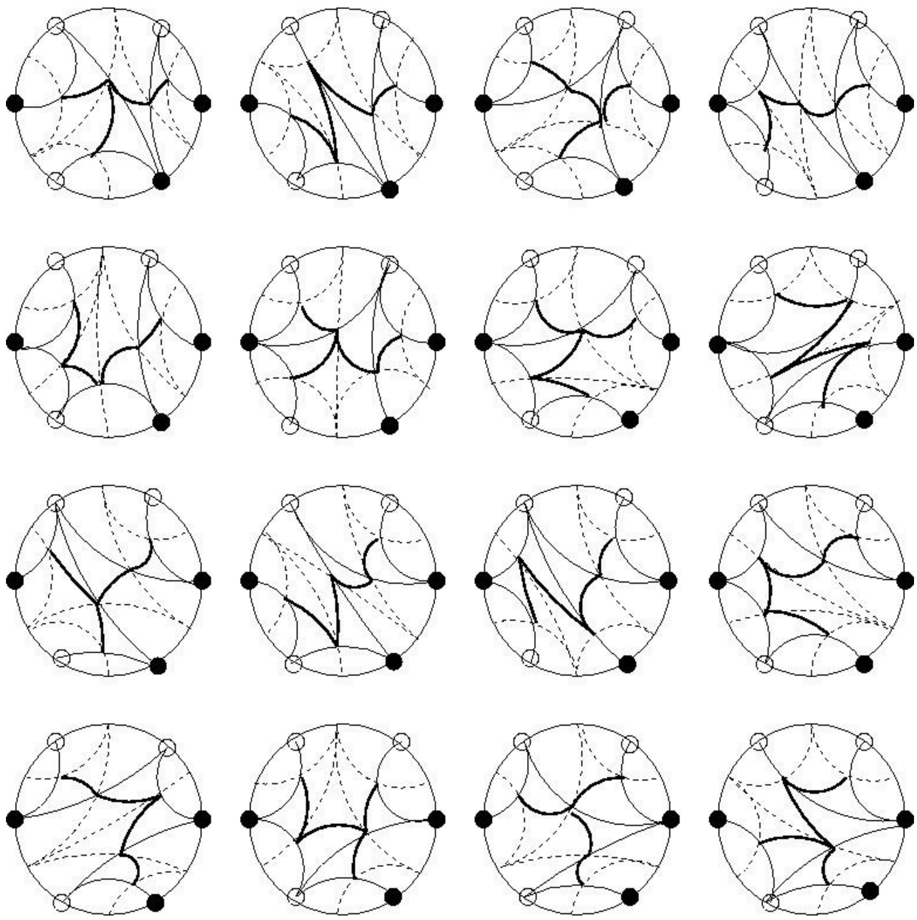

Figure 1: The 16 graph triads $\boldsymbol{g} = \{\boldsymbol{ga}, \boldsymbol{gi}, \boldsymbol{gai}\}$ compatible with the partition $\mathcal{P}$ of $\{0,1,2,3,4,5\}$ defined by $\mathcal{P}^+ = \{1,2,4\}$, $\mathcal{P}^- = \{0,3,5\}$.

indices $w_i^* = \binom{u_i^*}{v_i^*}$ corresponding to the five vertices of $\boldsymbol{gai}$ are given by:

$$
\begin{array}{lllll}
u_1^* = u_{1,2,3,4,5} & & \| & v_1^* = v_1 - v_0 = v_1 \\
u_2^* = u_{0,1} & = -u_{2,3,4,5} & \| & v_2^* = v_1 - v_5 \\
u_3^* = u_{2,3} & & \| & v_3^* = v_2 - v_5 \\
u_4^* = u_{4,5,0,1,2} & = -u_3 & \| & v_4^* = v_2 - v_3 \\
u_5^* = u_4 & & \| & v_5^* = v_4 - v_5
\end{array}
$$

with the expected identity $\sum_{1 \le i \le 5} u_i^*\, v_i^* \equiv \sum_{1 \le i \le 5} u_i\, v_i$. There are three possible roots, w_1^*, w_4^*, w_5^*, with three corresponding flexions:

$$(\mathrm{flex}_{\boldsymbol{g},w_1^*}.M)^{w_1,w_2,w_3,w_4,w_5} = +M^{w_1^*,w_2^*,w_3^*,w_4^*,w_5^*} + M^{w_1^*,w_2^*,w_3^*,w_5^*,w_4^*}$$

$$(\mathrm{flex}_{\boldsymbol{g},w_4^*}.M)^{w_1,w_2,w_3,w_4,w_5} = -M^{w_4^*,w_3^*,w_2^*,w_1^*,w_5^*} - M^{w_4^*,w_3^*,w_2^*,w_5^*,w_1^*} - M^{w_4^*,w_3^*,w_5^*,w_2^*,w_1^*}$$

$$(\mathrm{flex}_{\boldsymbol{g},w_5^*}.M)^{w_1,w_2,w_3,w_4,w_5} = -M^{w_5^*,w_3^*,w_2^*,w_1^*,w_4^*} - M^{w_5^*,w_3^*,w_2^*,w_4^*,w_1^*} - M^{w_5^*,w_3^*,w_4^*,w_2^*,w_1^*}$$

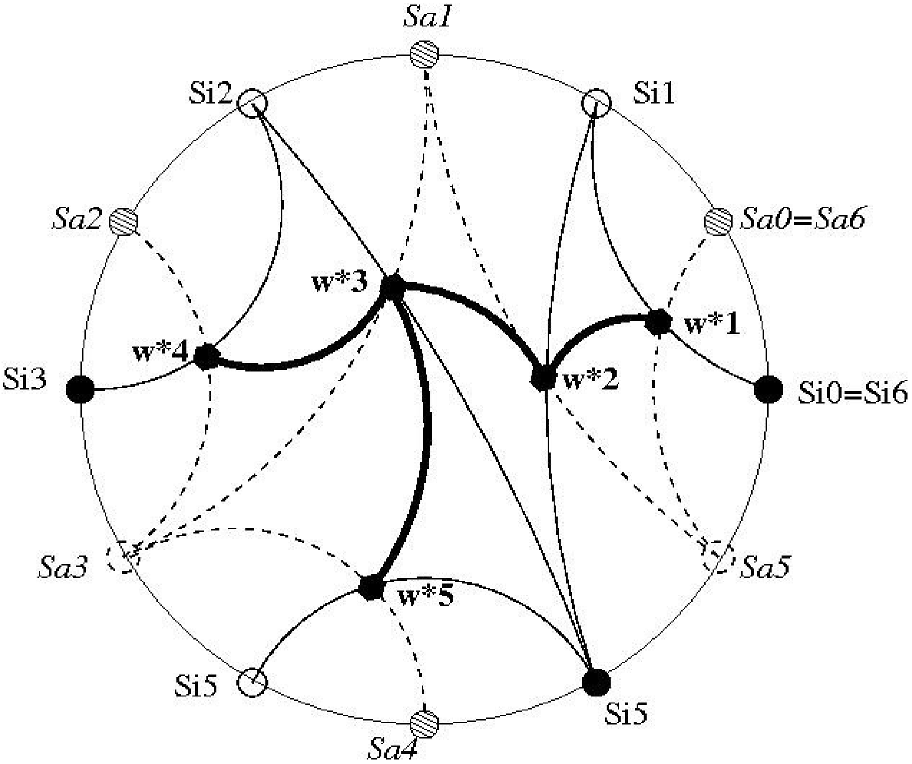

Figure 2: Flexion $\mathit{flex}_{\boldsymbol{g}}$ associated with a graph triad $\boldsymbol{g} = \{\boldsymbol{ga}, \boldsymbol{gi}, \boldsymbol{gai}\}$.

which coincide modulo the alternality relations:

$$\mathrm{flex}_{\boldsymbol{g},w_1^*}.M^\bullet \equiv \mathrm{flex}_{\boldsymbol{g},w_4^*}.M^\bullet \equiv \mathrm{flex}_{\boldsymbol{g},w_5^*}.M^\bullet \qquad \forall M^\bullet \ \mathit{alternal}$$

One might also take $\mathit{flex}_{\boldsymbol{g},w_2^*}.M^\bullet$ and $\mathit{flex}_{\boldsymbol{g},w_3^*}.M^\bullet$, but here the number of summands would be much larger: 8 and 12 respectively.

Example of redistribution identity.
For r=5, $\mathcal{P}^+ = \{1,2,4\}$ and $\mathcal{P}^- = \{0,3,5\}$, we have a black multiplicity $\mu_0 = 3$, and the *redistribution identity* $M^\bullet \equiv \mathit{redis}_{\mathcal{P}}.M^\bullet$ follows from the preceding *pairing identity* $M^\bullet \equiv \mathit{flex}_{\mathcal{P}}.M^\bullet$ by setting all black v_i's equal to zero in $(***)$. For simplicity, we write the redistribution identity only for $\boldsymbol{u}$-constant bilaternals, and since for them the u_i's don't matter, we don't mention them.

$$\begin{aligned} M^{v_1,v_2,0,v_4,0} \equiv & -M^{v_1,v_1,v_1,v_4,v_2} && -M^{v_1,v_1,v_1,v_2,v_4} && -M^{v_4,v_4,v_2,v_1,v_2} && -M^{v_4,v_4,v_2,v_2,v_1} \\ & +M^{v_2,v_1,v_1,v_4,v_4} && -2\,M^{v_1,v_2,v_4,v_4,v_4} && +M^{v_2,v_4,v_4,v_1,v_4} && +M^{v_2,v_4,v_4,v_4,v_1} \\ & +M^{v_2,v_1,v_4,v_4,v_4} && -2\,M^{v_1,v_2,v_2,v_4,v_4} && +M^{v_4,v_4,v_1,v_1,v_2} && +M^{v_4,v_4,v_1,v_2,v_1} \\ & -M^{v_1,v_1,v_4,v_4,v_2} && -M^{v_2,v_1,v_1,v_1,v_4} && -M^{v_2,v_1,v_1,v_4,v_1} && -M^{v_1,v_1,v_2,v_4,v_4} \\ & +M^{v_1,v_2,v_2,v_4,v_2} && +M^{v_1,v_1,v_2,v_4,v_2} \end{aligned}$$

Examples of reconstitution identities.
Up to length 2, the operator *induc* is trivial, but the number N_r of terms involved increases sharply thereafter. Thus:[167]

$$N_1 = 1, N_2 = 2, N_3 \sim 7, N_4 \sim 38, N_5 \sim 273, N_6 \sim 1837, N_7 \sim 15199, etc\ldots$$

Here are the formulas up to length 4, for the case of $\boldsymbol{u}$-constant bimoulds (and after removal of the u_i's):

$$(\text{induc}.M)^{v_1} := M^{v_1} \quad ; \quad (\text{induc}.M)^{v_1,v_2} := M^{v_1,v_2}$$

$$\begin{aligned}(\text{induc}.M)^{v_1,v_2,v_3} := &+M^{v_1,v_2,v_3} + M^{v_1,v_1,v_2} + M^{v_1,v_2,v_2} + M^{v_1,v_3,v_1}\\ &+M^{v_3,v_1,v_3} + M^{v_2,v_2,v_3} + M^{v_2,v_3,v_3}\end{aligned}$$

$$\begin{aligned}(\text{induc}.M)^{v_1,v_2,v_3,v_4} := &+M^{v_1,v_2,v_3,v_4} + M^{v_1,v_1,v_2,v_3} + M^{v_1,v_2,v_2,v_3} + M^{v_1,v_2,v_3,v_3}\\ &+M^{v_1,v_4,v_1,v_2} + M^{v_1,v_2,v_4,v_2} + M^{v_1,v_4,v_2,v_4} + M^{v_4,v_1,v_4,v_2}\\ &+M^{v_4,v_1,v_2,v_4} + M^{v_3,v_4,v_1,v_4} + M^{v_3,v_1,v_3,v_4} + M^{v_1,v_3,v_1,v_4}\\ &+M^{v_3,v_1,v_4,v_1} + M^{v_1,v_3,v_4,v_1} + M^{v_2,v_3,v_4,v_4} + M^{v_2,v_3,v_3,v_4}\\ &+M^{v_2,v_2,v_3,v_4} + M^{v_1,v_1,v_1,v_2} + M^{v_1,v_1,v_3,v_1} + M^{v_1,v_4,v_1,v_1}\\ &+M^{v_1,v_2,v_2,v_2} + M^{v_2,v_2,v_2,v_3} + M^{v_2,v_2,v_4,v_2} + M^{v_3,v_1,v_3,v_3}\\ &+M^{v_2,v_3,v_3,v_3} + M^{v_3,v_3,v_3,v_4} + M^{v_4,v_4,v_1,v_4} + M^{v_4,v_2,v_4,v_4}\\ &+M^{v_3,v_4,v_4,v_4} + M^{v_1,v_1,v_2,v_2} + M^{v_3,v_3,v_1,v_1} + M^{v_1,v_3,v_1,v_3}\\ &+M^{v_1,v_1,v_4,v_4} + M^{v_4,v_1,v_4,v_1} + M^{v_2,v_2,v_3,v_3} + M^{v_4,v_4,v_2,v_2}\\ &+M^{v_2,v_4,v_2,v_4} + M^{v_3,v_3,v_4,v_4}\end{aligned}$$

11.7 Multizeta cleansing: elimination of odd degrees.

We shall now construct a simple algorithm for *expressing every multizeta of odd degree as a finite sum, with rational coefficients, of multizetas of even degree.*[168]

[167] Recall that the expression of *induc* is unique only modulo the alternality relations. Hence the sign $\sim$ to caution that there is at least *one* expression of *induc* with the number N_r of summands mentioned. In any case, the minimal number N_r^{min} cannot be significantly less.
[168] Recall that the degree $d := s - r$ of a multizeta is defined as its total weight s minus its length (or depth) r.

We take as our starting point the symmetrel multitangent mould $Te^\bullet(z)$ and its generating function, the symmetril mould $Tig^\bullet(z)$, with definitions transparently patterned on those of $Ze^\bullet$ and $Zig^\bullet$:

$$\mathrm{Te}^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{s_1\ ,\dots,\ s_r}}(z) \ := \sum_{+\infty>n_1>\dots>n_r>-\infty} \prod_{i=1}^{i=r} \Big(e_i^{-n_i}\,(n_i+z)^{-s_1} \Big) \tag{11.79}$$

$$\mathrm{Tig}^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{v_1\ ,\dots,\ v_r}}(z) \ := \sum_{s_i\geq 1} Te^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{s_1\ ,\dots,\ s_r}}(z)\, v_1^{s_1-1}\dots v_r^{s_r-1} \tag{11.80}$$

The next step is to express the multitangents in terms of multizetas. Here, we have the choice between an *uninflected* formula which leaves z spread over all terms, and an *inflected* formula which concentrates z in a few elementary central terms:

$$\mathrm{Tig}^{\boldsymbol{w}}(z) = \sum_{\boldsymbol{w}=\boldsymbol{w}^+\boldsymbol{w}^-} \mathrm{Zig}^{\boldsymbol{w}^+}(z)\,\mathrm{viZig}^{\boldsymbol{w}^-}(z) - \sum_{\boldsymbol{w}=\boldsymbol{w}^+w_0\boldsymbol{w}^-} \mathrm{Zig}^{\boldsymbol{w}^+}(z)\,\mathrm{Pi}^{w_0}(z)\,\mathrm{viZig}^{\boldsymbol{w}^-}(z)$$

$$\mathrm{Tig}^{\boldsymbol{w}}(z) = \ \mathrm{Rig}^{\boldsymbol{w}} - \sum_{\boldsymbol{w}=\boldsymbol{w}^+w_0\boldsymbol{w}^-} \mathrm{Zig}^{\boldsymbol{w}^+\rfloor}\ \mathrm{Qii}^{\lceil w_0\rceil}(z)\ \mathrm{viZig}^{\lfloor\boldsymbol{w}^-}$$

The ingredient $Rig^\bullet$ in the above formulas is defined as follows:

$$\mathrm{Rig}^{w_1,\dots,w_r} \ := \ 0 \quad \mathit{for}\ \ r=0\ \ \mathit{or}\ \ r\ \ \mathit{odd}$$

$$\mathrm{Rig}^{w_1,\dots,w_r} \ := \ \frac{(\pi i)^r}{r!}\,\delta(u_1)\dots\delta(u_r) \qquad \mathit{for}\ \ r\ \ \mathit{even}\ >0$$

with δ denoting as usual the discrete dirac.[169] The length-1 bimoulds $Pi^\bullet$ and $Qii^\bullet := Qii^\bullet_\pi$ denote the *polar* and *bitrigonometric* flexion units of §3.2, and $viZig^\bullet := neg.pari.anti.Zig^\bullet$. Lastly, the bimoulds $Pi^\bullet(z)$, $Qii^\bullet(z)$, $Zig^\bullet(z)$, $viZig^\bullet(z)$ are deduced from $Pi^\bullet$, $Qii^\bullet$, $Zig^\bullet$, $viZig^\bullet$ under the change $v_i \to v_i - z$ $(\forall i)$.

By equating our *uninflected* and *inflected* expressions of $Tig^\bullet(z)$ and then setting $z=0$, we get the remarkable identity:

$$\sum_{\boldsymbol{w}=\boldsymbol{w}^+\boldsymbol{w}^-} \mathrm{Zig}^{\boldsymbol{w}^+}\,\mathrm{viZig}^{\boldsymbol{w}^-} - \sum_{\boldsymbol{w}=\boldsymbol{w}^+w_0\boldsymbol{w}^-} \mathrm{Zig}^{\boldsymbol{w}^+}\,\mathrm{Pi}^{w_0}\,\mathrm{viZig}^{\boldsymbol{w}^-} \ =$$

$$\mathrm{Rig}^{\boldsymbol{w}} - \sum_{\boldsymbol{w}=\boldsymbol{w}^+w_0\boldsymbol{w}^-} \mathrm{Zig}^{\boldsymbol{w}^+\rfloor}\ \mathrm{Qii}^{\lceil w_0\rceil}\ \mathrm{viZig}^{\lfloor\boldsymbol{w}^-} \qquad (\forall \boldsymbol{w}) \tag{11.81}$$

where the factor sequences $\boldsymbol{w}^\pm$ can be $\boldsymbol{\emptyset}$. As a consequence, (11.81) is of the form:

$$\mathrm{Zig}^{w_1,\dots,w_r} + (-1)^r\,\mathrm{Zig}^{-w_r,\dots,-w_1} \ = \ \text{“\textit{shorter terms}”} \tag{11.82}$$

[169] $\delta(0) := 1$ and $\delta(t) := 0$ *for* $t \neq 0$.

But $Zig^\bullet$ is symmetril and therefore *mantir*-invariant (see §3.4), which again yields an identity of the form:

$$\mathrm{Zig}^{-w_1,\dots,-w_r} + (-1)^r\,\mathrm{Zig}^{-w_r,\dots,-w_1} \;=\; \text{“\textit{shorter terms}”} \qquad (11.83)$$

If we now take ‘colourless’ indices w_i, i.e. indices $w_i := (\,{}^{0}_{v_i}\,)$, then subtract (11.84) from (11.82), and calculate therein the coefficient of $\prod v_i^{s_i-1}$, we find:

$$(1-(-1)^d)\;\mathrm{Ze}^{\left(\begin{smallmatrix}0&,\dots,&0\\ s_1&,\dots,&s_r\end{smallmatrix}\right)} \;=\; \text{“\textit{shorter terms}”} \qquad (d := \sum s_i - r) \qquad (11.84)$$

with quite explicit ‘shorter terms’.

We have here a very effective algorithm for the ‘elimination’ of all *uncoloured* multizetas $\zeta(s_1,\dots,s_r)$ of *odd* degree d. The argument extends to the case of *bicoloured* multizetas $Ze^{\left(\begin{smallmatrix}\epsilon_1&,\dots,&\epsilon_r\\ s_1&,\dots,&s_r\end{smallmatrix}\right)}$ with $\epsilon_i \in \frac{1}{2}\mathbb{Z}/\mathbb{Z}$, since we then have $\epsilon_i \equiv -\epsilon_i$. In the case of more than two colours, however, equation (11.84) becomes a *singular* linear system, which allows the elimination of *most*, but not *all*, multizetas of odd degree.

Remark 1: elimination of irreducibles other than π^2.
A simple argument shows that identity (11.81) still holds if we neglect all irreducibles other than π^2, i.e. if we retain only the first factor $Zig_I{}^\bullet$ in the trifactorisation (9.1) of $Zig^\bullet$. But since $Zig_I^\bullet$ is invariant under *pari.neg*, we clearly have $viZig_I = anti.Zig_I{}^\bullet$, so that (11.81) becomes:

$$\mathrm{mu}(\mathrm{Zig_I}^\bullet, 1^\bullet - \mathrm{Pi}^\bullet, \mathrm{anti.Zig_I}^\bullet) = \mathrm{Rig}^\bullet - \mathrm{giwat}(\mathrm{Zig_I}^\bullet).\mathrm{Qi}^\bullet \qquad (11.85)$$

Remark 2: separation of π^2 from the rationals.
Actually, we may retain in (11.85) only the first two factors of $Zig_I{}^\bullet$ (see (9.2)) namely $gira(til^\bullet, sripil^\bullet)$ with $sripil^\bullet := invgira(pil^\bullet)$. Furthermore, since in (11.85) the ‘trigometric’ part (which carries π^2) and the ‘polar’ part (which carries only rationals) do not mix, (11.85) leads to two distinct identities, to wit:

$$\mathrm{mu}(\mathrm{sripil}^\bullet, \mathrm{anti.sripil}^\bullet) \;=\; \mathrm{mu}(\mathrm{sripil}^\bullet, \mathrm{Pi}^\bullet, \mathrm{anti.sripil}^\bullet) \qquad (11.86)$$

$$\mathrm{mu}(\mathrm{til}^\bullet, \mathrm{anti.til}^\bullet) \;=\; \mathrm{Rig}^\bullet - \mathrm{giwat}(\mathrm{til}^\bullet).\mathrm{Qi}^\bullet \qquad (11.87)$$

Remark 3: universalisation.
Identity (11.86) admits an automatic extension to all exact units, namely:

$$\mathrm{mu}(\mathfrak{esz}^\bullet, \mathrm{anti}.\mathfrak{esz}^\bullet) = \mathrm{mu}(\mathfrak{esz}^\bullet, \mathfrak{E}^\bullet, \mathrm{anti}.\mathfrak{esz}^\bullet) \qquad (11.88)$$

Identity (11.87), which involves the approximate unit $Qi^\bullet$, *does not* admit *extensions* to all approximate units,[170] but it does possess a *restriction* to the polar unit $Pi^\bullet$ [171] and hence an *extension* to all exact units:

$$\mathrm{mu}(\mathfrak{ess}^\bullet\,,\,\mathrm{anti}.\mathfrak{ess}^\bullet) = -\mathrm{giwat}(\mathfrak{ess}^\bullet).\mathfrak{E}^\bullet \qquad (11.89)$$

11.8 $GARI_{\mathfrak{se}}$ and the two separation lemmas.

Let $\mathfrak{E}$ be an exact flexion unit and $\mathfrak{O}$ its conjugate unit. Reverting to the notations of §4.1, with any $f(x) := x + \sum_{1\le r} x^{r+1}$ in the group *GIFF*, we associate its image $\mathfrak{Se}^\bullet_f$ in the group $GARI_{<\mathfrak{se}>} \subset GARI^{as}$. Being the exponential of an alternal bimould of *ARI*, $\mathfrak{Se}^\bullet_f$ is automatically symmetral but its swappee $\mathfrak{S\ddot{o}}^\bullet_f := swap.\mathfrak{Se}^\bullet_f$ is only exceptionnally so. It does possess, however, two remarkable *separation properties*, which may be viewed as weakened forms of symmetrality. Indeed, if we set

$$\mathrm{gepar}.\mathfrak{Se}^\bullet_f \;:=\; \mathrm{mu}(\mathrm{anti.swap}.\mathfrak{Se}^\bullet_f, \mathrm{swap}.\mathfrak{Se}^\bullet_f) \qquad (11.90)$$

$$\mathrm{hepar}.\mathfrak{Se}^\bullet_f \;:=\; \sum_{1\le r\le r(\bullet)} \mathrm{pus}^k.\mathrm{logmu.swap}.\mathfrak{Se}^\bullet_f \qquad (11.91)$$

then both $\mathrm{gepar}.\mathfrak{Se}^\bullet_f$ and $\mathrm{hepar}.\mathfrak{Se}^\bullet_f$ turn out to be expressible as simple, uninflected products of the conjugate unit $\mathfrak{O}$. More precisely:

$$\mathrm{gepar}.\mathfrak{Se}_f^{w_1,\dots,w_r} \;:=\; a_r^*\,\mathfrak{O}^{w_1}\dots\mathfrak{O}^{w_r}\;\; with \quad a_r^* := (r{+}1)\,a_r \qquad (11.92)$$

$$\mathrm{hepar}.\mathfrak{Se}_f^{w_1,\dots,w_r} \;:=\; a_r^{**}\mathfrak{O}^{w_1}\dots\mathfrak{O}^{w_r}\;\; with \;\sum_{1\le r} a_r^{**}x^r := \frac{x}{2}\frac{f''(x)}{f'(x)} \qquad (11.93)$$

Remark 1: The definition of *hepar* involves *logmu*, which is of course the logarithm relative to the *mu*-product. It should be noted, however, that after simplification all rational coefficients disappear from the right-hand side of (11.91) and the only coefficients left are ± 1. In fact, the right-hand side of (11.91) is none other than the left-hand side of (2.75).

Remark 2: If $\mathfrak{S\ddot{o}}_f$ were exactly symmetral, it would verify the two subsymmetries implied by symmetrality, namely *gantar*-invariance (see (2.74)) and *gus*-neutrality (see (2.75)) and we would have $mu(pari.anti.\mathfrak{S\ddot{o}}^\bullet_f, \mathfrak{S\ddot{o}}^\bullet_f) \equiv 1^\bullet$ and $\sum_{1\le r\le r(\bullet)} pus^k.logmu.\mathfrak{S\ddot{o}}^\bullet_f \equiv 0^\bullet \; mod\; BIMU_1$. As it is, we merely have the separation properties (11.92) and (11.93), with the addded twist that

[170] it has no simple counterpart with $(Qa^\bullet\,,\, tal^\bullet)$ in place of $(Qi^\bullet\,,\, til^\bullet)$.

[171] after automatic elimination of the $Rig^\bullet$ part.

separ involves $mu(anti.\mathfrak{So}^\bullet_f, \mathfrak{So}^\bullet_f)$ rather than $mu(pari.anti.\mathfrak{So}^\bullet_f, \mathfrak{So}^\bullet_f)$.

Remark 3: The simplest way to prove the separation identities is to consider the infinitesimal dilator $f_\#(x) = \sum_{1\leq r} \eta_r\, x^{r+1}$ of f and to form its image $\mathfrak{Te}^\bullet_f = \sum_{1\leq r} \eta_r \mathfrak{te}^\bullet_r$ in *ARI*. One of the defining identities for $\mathfrak{Se}^\bullet_f$ then reads:

$$r(\bullet)\,\mathfrak{Se}^\bullet_f = \text{preari}(\mathfrak{Se}^\bullet_f, \mathfrak{Te}^\bullet_f) = \text{preawi}(\mathfrak{Se}^\bullet_f, \mathfrak{Te}^\bullet_f) \tag{11.94}$$

Under the *swap* transform this becomes:[172]

$$r(\bullet)\,\mathfrak{S\ddot{o}}^\bullet_f = \text{preira}(\mathfrak{S\ddot{o}}^\bullet_f, \mathfrak{T\ddot{o}}^\bullet_f) = \text{preiwa}(\mathfrak{S\ddot{o}}^\bullet_f, \mathfrak{T\ddot{o}}^\bullet_f) \tag{11.95}$$

If we then set:

$$\ddot{\mathfrak{O}}^{w_1,\dots,w_r}_* := a^*_r\, \mathfrak{O}^{w_1} \dots \mathfrak{O}^{w_r} \quad ; \quad \ddot{\mathfrak{O}}^{w_1,\dots,w_r}_{**} := a^{**}_r\, \mathfrak{O}^{w_1} \dots \mathfrak{O}^{w_r} \tag{11.96}$$

we readily sees that (11.92) is equivalent to the rather elementary identity:

$$r(\bullet)\,\ddot{\mathfrak{O}}^\bullet_* = \text{iwat}(\mathfrak{T\ddot{o}}^\bullet_f).\ddot{\mathfrak{O}}^\bullet_* + \text{mu}(\ddot{\mathfrak{O}}^\bullet_*\,,\, \mathfrak{T\ddot{o}}^\bullet_f) + \text{mu}(\text{anti}.\mathfrak{T\ddot{o}}^\bullet_f\,,\, \ddot{\mathfrak{O}}^\bullet_*) \tag{11.97}$$

The proof of the (11.93) follows the same pattern, with $\ddot{\mathfrak{O}}_*$ replaced by $\ddot{\mathfrak{O}}_{**}$, but is less direct.

Remark 4: In view of these two separation identities (11.92),(11.93), which involve respectively the coefficients a^*_r and a^{**}_r of f' and f''/f', i.e. of the differential operators of first and second order that give rise to simple *composition laws*, one may speculate about the existence of a third separation identity that would involve the coefficients a^{***}_r of the Schwarzian derivative of f. At the moment no such identity is known, but it may be pointed out that the formulas in Table 3 below also fall into the broad category of separation identities: see Remark 1 in §12.3.

11.9 Bisymmetrality of $\mathfrak{ess}^\bullet$: conceptual proof.

The bimould $\mathfrak{ess}^\bullet$ of §4.2 is a special element $\mathfrak{Se}^\bullet_f$ of $GARI_{<\mathfrak{se}>}$ whose preimage f and dilator $f_\#$ are given by:

$$f(x) := 1 - e^{-x} \quad , \quad f_\#(x) := 1 + x - e^x \quad , \quad \frac{x}{2}\frac{f''(x)}{f'(x)} := -\frac{x}{2} \tag{11.98}$$

[172]The reasons why in this particular instance one may replace the pair *ari/ira* by the more convenient pair *awi/iwa* were explained in §4.1.

As a consequence, the two *separation lemmas* of §11.8 yield:

$$\mathrm{mu}(\mathrm{anti}.\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet\,,\,\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet) = \mathrm{expmu}(-\mathfrak{O}^\bullet) \tag{11.99}$$

$$\sum_{1\leq k\leq r} \mathrm{pus}^k.\mathrm{logmu}.\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet = -\frac{1}{2}\,\mathfrak{O}^\bullet \tag{11.100}$$

Both relations exhibit the only possibly form compatible with $\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet$ being symmetral, but we aren't quite there yet. To collect more information, let us harken back to the relation that defines $\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet$ in terms of its dilator $\ddot{\mathfrak{o}}\mathfrak{tt}^\bullet$. It reads:

$$r(\bullet)\,\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet = \mathrm{preiwa}(\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet, \ddot{\mathfrak{o}}\mathfrak{tt}^\bullet) \tag{11.101}$$

with

$$\ddot{\mathfrak{o}}\mathfrak{tt}^\bullet := -\sum_{1\leq r} \frac{1}{(2\,r+1)!}\,\mathfrak{r}\ddot{\mathfrak{o}}^\bullet_{2r} = -\sum_{1\leq r} \frac{1}{(2\,r+1)!}\,\mathrm{swap}.\mathfrak{re}^\bullet_{2r} \tag{11.102}$$

Let us further *mu*-factorise $\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet$ as in (4.46), with the same elementary right factor $\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_\star$ but with a left factor $\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_{\star\star}$ whose properties are a priori unknown:

$$\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_\star = \mathrm{mu}(\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_{\star\star}, \ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_\star) \quad \textit{with} \quad \ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_\star := \mathrm{expmu}(-\frac{1}{2}\mathfrak{O}^\bullet) \tag{11.103}$$

Elementary calculations show that (11.101) transforms into:

$$r(\bullet)\,\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_{\star\star} = \mathrm{preiwa}(\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_{\star\star}, \ddot{\mathfrak{o}}\mathfrak{tt}^\bullet_{\star\star}) + \frac{1}{2}\,\mathrm{mu}(\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_{\star\star}, \ddot{\mathfrak{o}}\mathfrak{tt}^\bullet_\star) \tag{11.104}$$

with

$$\ddot{\mathfrak{o}}\mathfrak{tt}^\bullet_\star \;:=\; +\sum_{1\leq r} \frac{1}{(2\,r)!}\,\mathrm{mu}(\overbrace{\mathfrak{O}^\bullet,...,\mathfrak{O}^\bullet}^{2\,\mathrm{r\ times}}) \;=\; \mathrm{coshmu}(\mathfrak{O}^\bullet) \tag{11.105}$$

$$\ddot{\mathfrak{o}}\mathfrak{tt}^\bullet_{\star\star} \;:=\; -\sum_{1\leq r} \frac{1}{(2\,r+1)!}\,\mathfrak{r}\ddot{\mathfrak{o}}^\bullet_{2r} \tag{11.106}$$

But since $\ddot{\mathfrak{o}}\mathfrak{tt}^\bullet_\star$ and $\ddot{\mathfrak{o}}\mathfrak{tt}^\bullet_{\star\star}$ have only non-vanishing components of *even* length, (11.104) shows that the same must hold for $\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet_{\star\star}$. Reverting to the factorisation (11.101) and the separation identity (11.99) and using the invariance of $\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet$, we deduce from all this:

$$\mathrm{mu}(\mathrm{pari}.\mathrm{anti}.\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet\,,\,\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet) \;=\; 1^\bullet \tag{11.107}$$

(11.107) expresses the *gantar*-invariance of $\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet$ and (11.100) expresses its *gus*-neutrality. In other words, $\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet$ possesses *the two fundamental subsymmetries implied by symmetrality.* Yet this still doesn't imply full symmetrality. Fortunately, two crucial facts save the situation:

(i) since $\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet$ has a swappee $\mathfrak{ess}^\bullet$ that is obviously symmetral, and therefore *gantar*-invariant, the *gantar*-invariance of $\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet$, in view of the factorisation (4.46), also implies its invariance under *neg.gush* or, what here amounts to the same, *pari.gush*.
(ii) between themselves, the *neg.gush*-invariance and *gus*-neutrality of $\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet$ ensure its symmetrality.[173]

This fact is akin to the analogous implication valid in the algebras:

{ *pus*-neutrality + *push*- or *neg-push*-invariance} ⇒ {alternality}

Ultimately, it rests on the fact that *pus* and *push*, interpreted in the *short* and *long* notations,[174] amount to circular permutations of order r and $r+1$ respectively, which together generate the full symmetric group $\mathfrak{S}_{r+1}$. More precisely, each $\sigma \in \mathfrak{S}_{r+1}$ can be written as a product $\alpha^{m_1}\beta^{n_1}\ldots\alpha^{m_{r-1}}\beta^{n_{r-1}}$ with $\alpha = pus$ and $\beta = push$.

11.10 Bisymmetrality of $\mathfrak{ess}^\bullet$: combinatorial proof.

This alternative proof uses the inductive expression of $\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet$ in terms of its dilators $\ddot{\mathfrak{o}}\mathfrak{tt}^\bullet$ (direct) and $\ddot{\mathfrak{o}}\mathfrak{dd}^\bullet$ (inverse). Explicitely:

$$r(\bullet)\,\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet = +\mathrm{preiwa}(\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet, \ddot{\mathfrak{o}}\mathfrak{tt}^\bullet) \tag{11.108}$$
$$r(\bullet)\,\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet = -\mathrm{giwa}(\ddot{\mathfrak{o}}\mathfrak{dd}^\bullet, \ddot{\mathfrak{o}}\mathfrak{ss}^\bullet) \tag{11.109}$$

with

$$\ddot{\mathfrak{o}}\mathfrak{tt}^\bullet := \mathrm{swap}.\mathfrak{ett}^\bullet \quad \textit{and} \quad \mathfrak{ett}^\bullet := -\sum_{1\le r}\frac{1}{(r+1)!}\,\mathfrak{re}^\bullet_r \tag{11.110}$$
$$\ddot{\mathfrak{o}}\mathfrak{dd}^\bullet := \mathrm{swap}.\mathfrak{edd}^\bullet \quad \textit{and} \quad \mathfrak{edd}^\bullet := +\sum_{1\le r}\frac{1}{r\,(r+1)}\,\mathfrak{re}^\bullet_r \tag{11.111}$$

These identities flow from the fact that the preimage of $\mathfrak{ess}^\bullet$ in *GIFF* is the diffeo $f(x) := 1 - e^{-x}$ with a reciprocal diffeo $f^{-1}(x) = -\log(1-x)$. The corresponding dilators therefore admit the expansions

$$f_\#(x) = 1 + x - e^x = -\sum_{1\le r}\frac{1}{(r+1)!}\,x^{r+1} \tag{11.112}$$
$$(f^{-1})_\#(x) = x + (1-x)\,\log(1-x) = +\sum_{1\le r}\frac{1}{r\,(r+1)}\,x^{r+1} \tag{11.113}$$

[173] which *gantar*-invariance + *gus*-neutrality do not!
[174] See at the beginning of §5.1, right before (5.2).

which provide us with the defining coefficients of $\mathfrak{ett}^\bullet$ and $\mathfrak{edd}^\bullet$.

On the face of it, relation (11.108), being linear in $\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet$, would seem a more promising starting point than relation (11.109), whose right-hand side is heavily non-linear in $\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet$. This appearance is deceptive, though, because the bimould $\mathfrak{ett}^\bullet$ possesses only a simple symmetry (alternal), unlike the bimould $\mathfrak{edd}^\bullet$, which possesses a double one: it is alternal, with an $\mathfrak{O}$-alternal swappee, as already observed in §4.1. Indeed, $\mathfrak{edd}^\bullet$ coincides with the bimould $\mathfrak{sre}^\bullet$ of (4.6). We shall therefore take our stand on (11.109) rather than (11.108). But first we require a general bimould identity.

For any two bimoulds $S^\bullet, D^\bullet$ in $BIMU^* \times BIMU_*$, i.e. such that $S^\emptyset = 1$ and $D^\emptyset = 0$, we introduce the following abbreviations

$$\boldsymbol{S}^{\{\{\boldsymbol{w^1};\boldsymbol{w^2}\}\}} \;:=\; -S^{\boldsymbol{w^1}}\,S^{\boldsymbol{w^2}} + \sum_{\boldsymbol{w}\in \mathrm{sha}(\boldsymbol{w^1};\boldsymbol{w^2})} S^{\boldsymbol{w}} \tag{11.114}$$

$$\boldsymbol{D}^{[[\boldsymbol{w^1};\boldsymbol{w^2}]]} \;:=\; \Bigg[\sum_{\boldsymbol{w}\in \mathrm{sh}\mathfrak{o}(\boldsymbol{w^1};\boldsymbol{w^2})} D^{\boldsymbol{w}}\Bigg]_{\mathfrak{O}^\bullet=-2\,S_1^\bullet} \tag{11.115}$$

$$\boldsymbol{S}^{\{\boldsymbol{w}\}} \;:=\; \Big[\mathrm{mu}(S^\bullet,\mathrm{anti}.S^\bullet) + \mathrm{giwat}(S^\bullet).\mathfrak{O}^\bullet\Big]_{\mathfrak{O}^\bullet=-2\,S_1^\bullet} \tag{11.116}$$

In all the above, $S_1^\bullet$ denotes the projection of $S^\bullet$ onto $BIMU_1$, and the interpretation of the three symbols is as follows:

(i) $\boldsymbol{S}^{\{\{\bullet;\bullet\}\}}$ measures the failure of $S^\bullet$ to be symmetral

(ii) $\boldsymbol{D}^{[[\bullet;\bullet]]}$ measures the failure of $D^\bullet$ to be $\mathfrak{O}$-alternal, with $\mathfrak{O}$-alternality defined as in §3.4, but *after replacement of the flexion unit* $\mathfrak{O}^\bullet$ *by* $-2\,S_1^\bullet$*, which is not required to be a flexion unit!*

(iii) $\boldsymbol{S}^{\{\bullet\}}$ measure the failure of $S^\bullet$ to verify a property closely related to *gantar*-invariance, which is a subsymmetry of symmetrality.

Thus, for $r(\boldsymbol{w^1}) = r(\boldsymbol{w^2}) = 1$ and for any $\boldsymbol{w}$, we get (mark the signs and the position of *anti*):

$$\boldsymbol{S}^{\{\{(w_1);(w_2)\}\}} = -S^{w_1}\,S^{w_2} + S^{w_1,w_2} + S^{w_2,w_1}$$

$$\boldsymbol{D}^{[[(w_1);(w_2)]]} = D^{w_1,w_2} + D^{w_2,w_1} + 2\,D^{w_1\rceil}\,S^{\lfloor w_2} + 2\,D^{w_2\rceil}\,S^{\lfloor w_1}$$

$$\boldsymbol{S}^{\{\boldsymbol{w}\}} = \sum_{\boldsymbol{w^1}.\boldsymbol{w^2}=\boldsymbol{w}} S^{\boldsymbol{w^1}}(\mathrm{anti}.S)^{\boldsymbol{w^2}} - 2\sum_{\boldsymbol{w^1}.w_0.\boldsymbol{w^2}=\boldsymbol{w}} S^{\boldsymbol{w^1}\rfloor}S^{\lceil w_0\rceil}(\mathrm{anti}.S)^{\lfloor \boldsymbol{w^2}}$$

We now require the following lemma:

If the bimoulds $S^\bullet, D^\bullet$ are related under the identity:[175]

$$-r(\bullet)\, S^\bullet = \mathrm{giwa}(D^\bullet, S^\bullet) \tag{11.117}$$

then for any two $\boldsymbol{w^1}, \boldsymbol{w^2}$ the identity holds:

$$0 = (r_1 + r_2)\, \boldsymbol{S}^{\{\{\boldsymbol{w^1}\,;\,\boldsymbol{w^2}\}\}} + \boldsymbol{D}^{[[\boldsymbol{w^1}\,;\,\boldsymbol{w^2}]]} + \boldsymbol{\Sigma_1} + \boldsymbol{\Sigma_2} + \boldsymbol{\Sigma_3} \tag{11.118}$$

(i) with a sum $\boldsymbol{\Sigma_1}$ linear in earlier terms $\boldsymbol{D}^{[[\boldsymbol{w'}\,;\,\boldsymbol{w''}]]}$ and multilinear in earlier terms $S^{\boldsymbol{w^*}}$, "earlier" meaning that $r'+r''$ and r^* are always $< r_1+r_2$.
(ii) with a sum $\boldsymbol{\Sigma_2}$ bilinear in earlier terms $\boldsymbol{S}^{\{\{\boldsymbol{w'}\,;\,\boldsymbol{w''}\}\}}, D^{\boldsymbol{w'''}}$ and multilinear in earlier terms $S^{\boldsymbol{w^*}}$.
(iii) with a sum $\boldsymbol{\Sigma_3}$ bilinear in earlier terms $\boldsymbol{S}^{\{\boldsymbol{w'}\}}, D^{\boldsymbol{w''}}$ and multilinear in earlier terms $S^{\boldsymbol{w^*}}$.
Moreover, in all three sums, the coefficients in front of the monomials made up of 'earlier' terms are always equal to +1.

The way to prove (11.118) is:
(i) to start from the identity

$$(r_1{+}r_2)\, \boldsymbol{S}^{\{\{\boldsymbol{w^1}\,;\,\boldsymbol{w^2}\}\}} = -(r_1\, S^{\boldsymbol{w^1}})\, S^{\boldsymbol{w^2}} - S^{\boldsymbol{w^1}}\, (r_2\, S^{\boldsymbol{w^2}}) + \sum_{\boldsymbol{w^1}.\boldsymbol{w^2} \,=\, \boldsymbol{w}} (r_1{+}r_2)\, S^{\boldsymbol{w}}$$

(ii) to replace therein all terms of the form $r(\bullet)\, S^\bullet$ by $-giwa(D^\bullet, S^\bullet)$
(iii) to replace (- this clearly is the crucial step -) the usual definition of *giwa* for totally ordered sequences $\boldsymbol{w}$ by an analogous expression valid for sequences $\boldsymbol{w}$ carrying a weaker, arborescent order[176] – in the present instance, for sequences $\boldsymbol{w}$ consisting of two totally ordered, but mutually non comparable branches $\boldsymbol{w^1}, \boldsymbol{w^2}$.

Thus, in the (very elementary) case $r_1 = 1, r_2 = 2$, we find

$$0 = (1+2)\, \boldsymbol{S}^{\{\{(\begin{smallmatrix} u_1 \\ v_1 \end{smallmatrix});(\begin{smallmatrix} u_2 \\ v_2 \end{smallmatrix}, \begin{smallmatrix} u_3 \\ v_3 \end{smallmatrix})\}\}} + \boldsymbol{D}^{[[(\begin{smallmatrix} u_1 \\ v_1 \end{smallmatrix});(\begin{smallmatrix} u_2 \\ v_2 \end{smallmatrix}, \begin{smallmatrix} u_3 \\ v_3 \end{smallmatrix})]]} + \boldsymbol{\Sigma_1} + \boldsymbol{\Sigma_2} + \boldsymbol{\Sigma_3}$$

[175] We recall that *GIWA* is the unary subgroup of *GAXI* relative to the involution $\mathcal{M}_R = anti.\mathcal{M}_L$. Under normal circumstances, $giwa(A^\bullet, B^\bullet)$ has its two arguments $A^\bullet, B^\bullet$ in $BIMU^*$. Here, however, we have to consider $\mathrm{giwa}(D^\bullet, S^\bullet)$, with a first argument in $BIMU_*$, but we can take recourse to the usual definition $giwa(D^\bullet, S^\bullet) := mu(giwat(S^\bullet).D^\bullet, S^\bullet)$, which still makes perfect sense.

[176] This, of course, does not apply for *giwa* alone: *all* flexion operations without exception extend to the case of arborescent sequences $\boldsymbol{w}$, provided we suitably redefine the product *mu* and the four flexions $\rfloor, \lceil, \rceil, \lfloor$ in accordance with the new order.

with

$$
\begin{aligned}
\boldsymbol{\Sigma_1} &= \boldsymbol{D}^{[[(\begin{smallmatrix}u_1\\ v_1\end{smallmatrix});(\begin{smallmatrix}u_2\\ v_2\end{smallmatrix})]]}\, S^{(\begin{smallmatrix}u_3\\ v_3\end{smallmatrix})} + \boldsymbol{D}^{[[(\begin{smallmatrix}u_1\\ v_1\end{smallmatrix});(\begin{smallmatrix}u_{23}\\ v_3\end{smallmatrix})]]}\, S^{(\begin{smallmatrix}u_2\\ v_{2:3}\end{smallmatrix})} + \boldsymbol{D}^{[[(\begin{smallmatrix}u_1\\ v_1\end{smallmatrix});(\begin{smallmatrix}u_{23}\\ v_2\end{smallmatrix})]]}\, S^{(\begin{smallmatrix}u_3\\ v_{3:2}\end{smallmatrix})}\\
\boldsymbol{\Sigma_2} &= \boldsymbol{S}^{\{\{(\begin{smallmatrix}u_1\\ v_1\end{smallmatrix});(\begin{smallmatrix}u_3\\ v_3\end{smallmatrix})\}\}}\, S^{(\begin{smallmatrix}u_2\\ v_2\end{smallmatrix})} + \boldsymbol{S}^{\{\{(\begin{smallmatrix}u_1\\ v_{1:3}\end{smallmatrix});(\begin{smallmatrix}u_2\\ v_{2:3}\end{smallmatrix})\}\}}\, S^{(\begin{smallmatrix}u_{123}\\ v_3\end{smallmatrix})} + \boldsymbol{S}^{\{\{(\begin{smallmatrix}u_1\\ v_{1:2}\end{smallmatrix});(\begin{smallmatrix}u_3\\ v_{3:2}\end{smallmatrix})\}\}}\, S^{(\begin{smallmatrix}u_{123}\\ v_2\end{smallmatrix})}\\
\boldsymbol{\Sigma_3} &= \boldsymbol{S}^{\{\begin{smallmatrix}u_2\\ v_{2:1}\end{smallmatrix}\, ,\, \begin{smallmatrix}u_3\\ v_{3:1}\end{smallmatrix}\}}\, T^{(\begin{smallmatrix}u_{123}\\ v_1\end{smallmatrix})}
\end{aligned}
$$

At this point, all we have to do is:
(i) replace $S^\bullet$ by $\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet$ and $D^\bullet$ by $\ddot{\mathfrak{o}}\mathfrak{dd}^\bullet$ in (11.118)
(ii) observe that since in this case $\mathfrak{D}^\bullet = -2\,\ddot{\mathfrak{o}}\mathfrak{dd}^\bullet_1$, all terms $\boldsymbol{D}^{[[\bullet,\bullet]]}$ automatically vanish, since $D^\bullet \equiv \ddot{\mathfrak{o}}\mathfrak{dd}^\bullet$ is indeed $\mathfrak{D}$-alternal
(iii) observe that the identities $\boldsymbol{S}^{\{\bullet\}} = 0$ (up to length $r-1$) are an easy consequence of the symmetrality of $S^\bullet \equiv \ddot{\mathfrak{o}}\mathfrak{ss}^\bullet$ (up to length $r-1$) and of the factorisation (4.46). Besides, these identities $\boldsymbol{S}^{\{\bullet\}} = 0$ are also capable of an elementary, direct derivation, as we saw towards the end of §11.7: see (11.89).

Altogether, the identity (11.118) shows that if $\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet$ is symmetral up to length $r-1$, it is automatically symmetral up to length r. $\square$

There exist several other strategies for establishing the symmetrality of $\ddot{\mathfrak{o}}\mathfrak{ss}^\bullet$, all of more or less equal length,[177] but apparently no completely elementary proof.

12 Tables, index, references.

12.1 Table 1: basis for $Flex(\mathfrak{E})$.

Here are the bases of the first cells of the free *monogenous* structure $\oplus Flex_r(\mathfrak{E})$ generated by a general $\mathfrak{E}$ subject only to one of the four possible parity constraints (3.1): it doesn't matter which. By retaining only the first $\frac{(2\,r)!}{(r+1)!\,r!}$ elements, one also obtains bases for the *eumonogenous* structure $\oplus Flex_r(\mathfrak{E})$

[177] Thus, there exists a heavily calculational proof based on formula (12.6) of §12.3.

generated by an exact flexion unit $\mathfrak{E}$.

$$
\begin{array}{lcl|lcl}
\mathfrak{e}_{1,1}^{w_1} & := & \mathfrak{E}^{\binom{u_1}{v_1}} & & & \\
\mathfrak{e}_{2,1}^{w_1,w_2} & := & \mathfrak{E}^{\binom{u_{12}}{v_2}}\mathfrak{E}^{\binom{u_1}{v_{1:2}}} & \mathfrak{e}_{2,3}^{w_1,w_2} & := & \mathfrak{E}^{\binom{u_1}{v_1}}\mathfrak{E}^{\binom{u_2}{v_2}} \\
\mathfrak{e}_{2,2}^{w_1,w_2} & := & \mathfrak{E}^{\binom{u_{12}}{v_1}}\mathfrak{E}^{\binom{u_2}{v_{2:1}}} & & & \\
\mathfrak{e}_{3,1}^{w_1,w_2,w_3} & := & \mathfrak{E}^{\binom{u_{123}}{v_3}}\,\mathfrak{E}^{\binom{u_{12}}{v_{2:3}}}\,\mathfrak{E}^{\binom{u_1}{v_{1:2}}} & \mathfrak{e}_{3,6}^{w_1,w_2,w_3} & := & \mathfrak{E}^{\binom{u_{123}}{v_1}}\mathfrak{E}^{\binom{u_2}{v_{2:1}}}\mathfrak{E}^{\binom{u_3}{v_{3:1}}} \\
\mathfrak{e}_{3,2}^{w_1,w_2,w_3} & := & \mathfrak{E}^{\binom{u_{123}}{v_3}}\,\mathfrak{E}^{\binom{u_{12}}{v_{1:3}}}\,\mathfrak{E}^{\binom{u_2}{v_{2:1}}} & \mathfrak{e}_{3,7}^{w_1,w_2,w_3} & := & \mathfrak{E}^{\binom{u_{123}}{v_3}}\mathfrak{E}^{\binom{u_1}{v_{1:3}}}\mathfrak{E}^{\binom{u_2}{v_{2:3}}} \\
\mathfrak{e}_{3,3}^{w_1,w_2,w_3} & := & \mathfrak{E}^{\binom{u_{123}}{v_2}}\,\mathfrak{E}^{\binom{u_1}{v_{1:2}}}\,\mathfrak{E}^{\binom{u_3}{v_{3:2}}} & \mathfrak{e}_{3,8}^{w_1,w_2,w_3} & := & \mathfrak{E}^{\binom{u_{23}}{v_2}}\,\mathfrak{E}^{\binom{u_1}{v_1}}\,\mathfrak{E}^{\binom{u_3}{v_{3:2}}} \\
\mathfrak{e}_{3,4}^{w_1,w_2,w_3} & := & \mathfrak{E}^{\binom{u_{123}}{v_1}}\,\mathfrak{E}^{\binom{u_{23}}{v_{3:1}}}\,\mathfrak{E}^{\binom{u_2}{v_{2:3}}} & \mathfrak{e}_{3,9}^{w_1,w_2,w_3} & := & \mathfrak{E}^{\binom{u_{23}}{v_3}}\,\mathfrak{E}^{\binom{u_1}{v_1}}\,\mathfrak{E}^{\binom{u_2}{v_{2:3}}} \\
\mathfrak{e}_{3,5}^{w_1,w_2,w_3} & := & \mathfrak{E}^{\binom{u_{123}}{v_1}}\,\mathfrak{E}^{\binom{u_{23}}{v_{2:1}}}\,\mathfrak{E}^{\binom{u_3}{v_{3:2}}} & \mathfrak{e}_{3,10}^{w_1,w_2,w_3} & := & \mathfrak{E}^{\binom{u_{12}}{v_2}}\,\mathfrak{E}^{\binom{u_1}{v_{1:2}}}\,\mathfrak{E}^{\binom{u_3}{v_3}} \\
 & & & \mathfrak{e}_{3,11}^{w_1,w_2,w_3} & := & \mathfrak{E}^{\binom{u_{12}}{v_1}}\,\mathfrak{E}^{\binom{u_2}{v_{2:1}}}\,\mathfrak{E}^{\binom{u_3}{v_3}} \\
 & & & \mathfrak{e}_{3,12}^{w_1,w_2,w_3} & := & \mathfrak{E}^{\binom{u_1}{v_1}}\,\mathfrak{E}^{\binom{u_2}{v_2}}\,\mathfrak{E}^{\binom{u_3}{v_3}}
\end{array}
$$

Here follows the graphic interpretation of the bases, with full lines for the graphs ***gi*** and broken lines for the graphs ***ga*** (see §3.3).

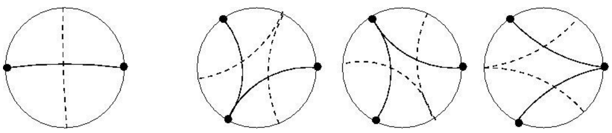

Figure 3: Length $r = 1, 2$. Basis vectors $\{\mathfrak{e}^{\bullet}_{1,1}\}$ and $\{\mathfrak{e}^{\bullet}_{2,1}, \mathfrak{e}^{\bullet}_{2,2}\} \cup \{\mathfrak{e}^{\bullet}_{2,3}\}$.

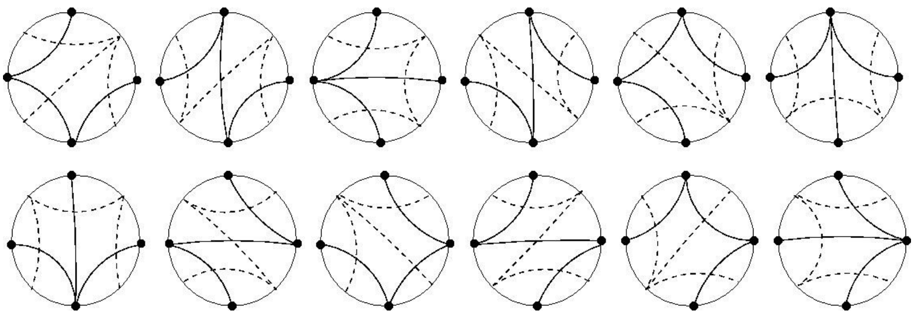

Figure 4: Length $r = 3$. Basis vectors $\{\mathfrak{e}^{\bullet}_{3,1}, \dots, \mathfrak{e}^{\bullet}_{3,5}\} \cup \{\mathfrak{e}^{\bullet}_{3,6}, \dots, \mathfrak{e}^{\bullet}_{3,12}\}$.

$$\mathfrak{e}_{4,1}^{w_1..w_4} = \mathfrak{E}^{\binom{u_1}{v_{1:2}}}\ \mathfrak{E}^{\binom{u_{12}}{v_{2:3}}}\ \mathfrak{E}^{\binom{u_{123}}{v_{3:4}}}\ \mathfrak{E}^{\binom{u_{1234}}{v_4}}$$
$$\mathfrak{e}_{4,2}^{w_1..w_4} = \mathfrak{E}^{\binom{u_{12}}{v_{1:3}}}\ \mathfrak{E}^{\binom{u_2}{v_{2:1}}}\ \mathfrak{E}^{\binom{u_{123}}{v_{3:4}}}\ \mathfrak{E}^{\binom{u_{1234}}{v_4}}$$
$$\mathfrak{e}_{4,3}^{w_1..w_4} = \mathfrak{E}^{\binom{u_1}{v_{1:2}}}\ \mathfrak{E}^{\binom{u_{123}}{v_{2:4}}}\ \mathfrak{E}^{\binom{u_3}{v_{3:2}}}\ \mathfrak{E}^{\binom{u_{1234}}{v_4}}$$
$$\mathfrak{e}_{4,4}^{w_1..w_4} = \mathfrak{E}^{\binom{u_{123}}{v_{1:4}}}\ \mathfrak{E}^{\binom{u_2}{v_{2:3}}}\ \mathfrak{E}^{\binom{u_{23}}{v_{3:1}}}\ \mathfrak{E}^{\binom{u_{1234}}{v_4}}$$
$$\mathfrak{e}_{4,5}^{w_1..w_4} = \mathfrak{E}^{\binom{u_{123}}{v_{1:4}}}\ \mathfrak{E}^{\binom{u_{23}}{v_{2:1}}}\ \mathfrak{E}^{\binom{u_3}{v_{3:2}}}\ \mathfrak{E}^{\binom{u_{1234}}{v_4}}$$
$$\mathfrak{e}_{4,6}^{w_1..w_4} = \mathfrak{E}^{\binom{u_1}{v_{1:2}}}\ \mathfrak{E}^{\binom{u_{12}}{v_{2:3}}}\ \mathfrak{E}^{\binom{u_{1234}}{v_3}}\ \mathfrak{E}^{\binom{u_4}{v_{4:3}}}$$
$$\mathfrak{e}_{4,7}^{w_1..w_4} = \mathfrak{E}^{\binom{u_{12}}{v_{1:3}}}\ \mathfrak{E}^{\binom{u_2}{v_{2:1}}}\ \mathfrak{E}^{\binom{u_{1234}}{v_3}}\ \mathfrak{E}^{\binom{u_4}{v_{4:3}}}$$
$$\mathfrak{e}_{4,8}^{w_1..w_4} = \mathfrak{E}^{\binom{u_1}{v_{1:2}}}\ \mathfrak{E}^{\binom{u_{1234}}{v_2}}\ \mathfrak{E}^{\binom{u_3}{v_{3:4}}}\ \mathfrak{E}^{\binom{u_{34}}{v_{4:2}}}$$
$$\mathfrak{e}_{4,9}^{w_1..w_4} = \mathfrak{E}^{\binom{u_1}{v_{1:2}}}\ \mathfrak{E}^{\binom{u_{1234}}{v_2}}\ \mathfrak{E}^{\binom{u_{34}}{v_{3:2}}}\ \mathfrak{E}^{\binom{u_4}{v_{4:3}}}$$
$$\mathfrak{e}_{4,10}^{w_1..w_4} = \mathfrak{E}^{\binom{u_{1234}}{v_1}}\ \mathfrak{E}^{\binom{u_2}{v_{2:3}}}\ \mathfrak{E}^{\binom{u_{23}}{v_{3:4}}}\ \mathfrak{E}^{\binom{u_{234}}{v_{4:1}}}$$
$$\mathfrak{e}_{4,11}^{w_1..w_4} = \mathfrak{E}^{\binom{u_{1234}}{v_1}}\ \mathfrak{E}^{\binom{u_{23}}{v_{2:4}}}\ \mathfrak{E}^{\binom{u_3}{v_{3:2}}}\ \mathfrak{E}^{\binom{u_{234}}{v_{4:1}}}$$
$$\mathfrak{e}_{4,12}^{w_1..w_4} = \mathfrak{E}^{\binom{u_{1234}}{v_1}}\ \mathfrak{E}^{\binom{u_2}{v_{2:3}}}\ \mathfrak{E}^{\binom{u_{234}}{v_{3:1}}}\ \mathfrak{E}^{\binom{u_4}{v_{4:3}}}$$
$$\mathfrak{e}_{4,13}^{w_1..w_4} = \mathfrak{E}^{\binom{u_{1234}}{v_1}}\ \mathfrak{E}^{\binom{u_{234}}{v_{2:1}}}\ \mathfrak{E}^{\binom{u_3}{v_{3:4}}}\ \mathfrak{E}^{\binom{u_{34}}{v_{4:2}}}$$
$$\mathfrak{e}_{4,14}^{w_1..w_4} = \mathfrak{E}^{\binom{u_{1234}}{v_1}}\ \mathfrak{E}^{\binom{u_{234}}{v_{2:1}}}\ \mathfrak{E}^{\binom{u_{34}}{v_{3:2}}}\ \mathfrak{E}^{\binom{u_4}{v_{4:3}}}$$

$$\mathfrak{e}_{4,15}^{w_1..w_4} = \mathfrak{E}^{\binom{u_1}{v_{1:2}}}\ \mathfrak{E}^{\binom{u_{12}}{v_2}}\ \mathfrak{E}^{\binom{u_3}{v_3}}\ \mathfrak{E}^{\binom{u_4}{v_4}}$$
$$\mathfrak{e}_{4,16}^{w_1..w_4} = \mathfrak{E}^{\binom{u_{12}}{v_1}}\ \mathfrak{E}^{\binom{u_2}{v_{2:1}}}\ \mathfrak{E}^{\binom{u_{34}}{v_3}}\ \mathfrak{E}^{\binom{u_4}{v_{4:3}}}$$
$$\mathfrak{e}_{4,17}^{w_1..w_4} = \mathfrak{E}^{\binom{u_{12}}{v_1}}\ \mathfrak{E}^{\binom{u_2}{v_{2:1}}}\ \mathfrak{E}^{\binom{u_3}{v_3}}\ \mathfrak{E}^{\binom{u_4}{v_4}}$$
$$\mathfrak{e}_{4,18}^{w_1..w_4} = \mathfrak{E}^{\binom{u_1}{v_{1:2}}}\ \mathfrak{E}^{\binom{u_{12}}{v_2}}\ \mathfrak{E}^{\binom{u_{34}}{v_3}}\ \mathfrak{E}^{\binom{u_4}{v_{4:3}}}$$
$$\mathfrak{e}_{4,19}^{w_1..w_4} = \mathfrak{E}^{\binom{u_1}{v_{1:2}}}\ \mathfrak{E}^{\binom{u_{12}}{v_2}}\ \mathfrak{E}^{\binom{u_3}{v_{3:4}}}\ \mathfrak{E}^{\binom{u_{34}}{v_4}}$$
$$\mathfrak{e}_{4,20}^{w_1..w_4} = \mathfrak{E}^{\binom{u_1}{v_1}}\ \mathfrak{E}^{\binom{u_{234}}{v_2}}\ \mathfrak{E}^{\binom{u_3}{v_{3:4}}}\ \mathfrak{E}^{\binom{u_{34}}{v_{4:2}}}$$
$$\mathfrak{e}_{4,21}^{w_1..w_4} = \mathfrak{E}^{\binom{u_{123}}{v_1}}\ \mathfrak{E}^{\binom{u_{23}}{v_{2:1}}}\ \mathfrak{E}^{\binom{u_3}{v_{3:2}}}\ \mathfrak{E}^{\binom{u_4}{v_4}}$$
$$\mathfrak{e}_{4,22}^{w_1..w_4} = \mathfrak{E}^{\binom{u_{12}}{v_1}}\ \mathfrak{E}^{\binom{u_2}{v_{2:1}}}\ \mathfrak{E}^{\binom{u_3}{v_{3:4}}}\ \mathfrak{E}^{\binom{u_{34}}{v_4}}$$
$$\mathfrak{e}_{4,23}^{w_1..w_4} = \mathfrak{E}^{\binom{u_1}{v_1}}\ \mathfrak{E}^{\binom{u_2}{v_2}}\ \mathfrak{E}^{\binom{u_3}{v_{3:4}}}\ \mathfrak{E}^{\binom{u_{34}}{v_4}}$$
$$\mathfrak{e}_{4,24}^{w_1..w_4} = \mathfrak{E}^{\binom{u_1}{v_1}}\ \mathfrak{E}^{\binom{u_2}{v_2}}\ \mathfrak{E}^{\binom{u_3}{v_3}}\ \mathfrak{E}^{\binom{u_4}{v_4}}$$
$$\mathfrak{e}_{4,25}^{w_1..w_4} = \mathfrak{E}^{\binom{u_1}{v_1}}\ \mathfrak{E}^{\binom{u_2}{v_2}}\ \mathfrak{E}^{\binom{u_{34}}{v_3}}\ \mathfrak{E}^{\binom{u_4}{v_{4:3}}}$$
$$\mathfrak{e}_{4,26}^{w_1..w_4} = \mathfrak{E}^{\binom{u_1}{v_1}}\ \mathfrak{E}^{\binom{u_2}{v_{2:4}}}\ \mathfrak{E}^{\binom{u_3}{v_{3:4}}}\ \mathfrak{E}^{\binom{u_{234}}{v_4}}$$
$$\mathfrak{e}_{4,27}^{w_1..w_4} = \mathfrak{E}^{\binom{u_1}{v_1}}\ \mathfrak{E}^{\binom{u_2}{v_{2:3}}}\ \mathfrak{E}^{\binom{u_{23}}{v_3}}\ \mathfrak{E}^{\binom{u_4}{v_4}}$$
$$\mathfrak{e}_{4,28}^{w_1..w_4} = \mathfrak{E}^{\binom{u_1}{v_1}}\ \mathfrak{E}^{\binom{u_{23}}{v_2}}\ \mathfrak{E}^{\binom{u_3}{v_{3:2}}}\ \mathfrak{E}^{\binom{u_4}{v_4}}$$
$$\mathfrak{e}_{4,29}^{w_1..w_4} = \mathfrak{E}^{\binom{u_1}{v_1}}\ \mathfrak{E}^{\binom{u_2}{v_{2:3}}}\ \mathfrak{E}^{\binom{u_{23}}{v_{3:4}}}\ \mathfrak{E}^{\binom{u_{234}}{v_4}}$$
$$\mathfrak{e}_{4,30}^{w_1..w_4} = \mathfrak{E}^{\binom{u_1}{v_1}}\ \mathfrak{E}^{\binom{u_{23}}{v_{2:4}}}\ \mathfrak{E}^{\binom{u_3}{v_{3:2}}}\ \mathfrak{E}^{\binom{u_{234}}{v_4}}$$
$$\mathfrak{e}_{4,31}^{w_1..w_4} = \mathfrak{E}^{\binom{u_1}{v_1}}\ \mathfrak{E}^{\binom{u_2}{v_{2:3}}}\ \mathfrak{E}^{\binom{u_{234}}{v_3}}\ \mathfrak{E}^{\binom{u_4}{v_{4:3}}}$$
$$\mathfrak{e}_{4,32}^{w_1..w_4} = \mathfrak{E}^{\binom{u_1}{v_1}}\ \mathfrak{E}^{\binom{u_{234}}{v_2}}\ \mathfrak{E}^{\binom{u_3}{v_{3:2}}}\ \mathfrak{E}^{\binom{u_4}{v_{4:2}}}$$
$$\mathfrak{e}_{4,33}^{w_1..w_4} = \mathfrak{E}^{\binom{u_1}{v_1}}\ \mathfrak{E}^{\binom{u_{234}}{v_2}}\ \mathfrak{E}^{\binom{u_{34}}{v_{3:2}}}\ \mathfrak{E}^{\binom{u_4}{v_{4:3}}}$$
$$\mathfrak{e}_{4,34}^{w_1..w_4} = \mathfrak{E}^{\binom{u_1}{v_{1:3}}}\ \mathfrak{E}^{\binom{u_2}{v_{2:3}}}\ \mathfrak{E}^{\binom{u_{123}}{v_3}}\ \mathfrak{E}^{\binom{u_4}{v_4}}$$
$$\mathfrak{e}_{4,35}^{w_1..w_4} = \mathfrak{E}^{\binom{u_1}{v_{1:2}}}\ \mathfrak{E}^{\binom{u_{12}}{v_{2:3}}}\ \mathfrak{E}^{\binom{u_{123}}{v_3}}\ \mathfrak{E}^{\binom{u_4}{v_4}}$$
$$\mathfrak{e}_{4,36}^{w_1..w_4} = \mathfrak{E}^{\binom{u_{12}}{v_{1:3}}}\ \mathfrak{E}^{\binom{u_2}{v_{2:1}}}\ \mathfrak{E}^{\binom{u_{123}}{v_3}}\ \mathfrak{E}^{\binom{u_4}{v_4}}$$
$$\mathfrak{e}_{4,37}^{w_1..w_4} = \mathfrak{E}^{\binom{u_1}{v_{1:2}}}\ \mathfrak{E}^{\binom{u_{123}}{v_2}}\ \mathfrak{E}^{\binom{u_3}{v_{3:2}}}\ \mathfrak{E}^{\binom{u_4}{v_4}}$$
$$\mathfrak{e}_{4,38}^{w_1..w_4} = \mathfrak{E}^{\binom{u_{123}}{v_1}}\ \mathfrak{E}^{\binom{u_2}{v_{2:1}}}\ \mathfrak{E}^{\binom{u_3}{v_{3:1}}}\ \mathfrak{E}^{\binom{u_4}{v_4}}$$
$$\mathfrak{e}_{4,39}^{w_1..w_4} = \mathfrak{E}^{\binom{u_{123}}{v_1}}\ \mathfrak{E}^{\binom{u_2}{v_{2:3}}}\ \mathfrak{E}^{\binom{u_{23}}{v_{3:1}}}\ \mathfrak{E}^{\binom{u_4}{v_4}}$$
$$\mathfrak{e}_{4,40}^{w_1..w_4} = \mathfrak{E}^{\binom{u_{1234}}{v_1}}\ \mathfrak{E}^{\binom{u_2}{v_{2:1}}}\ \mathfrak{E}^{\binom{u_3}{v_{3:1}}}\ \mathfrak{E}^{\binom{u_4}{v_{4:1}}}$$
$$\mathfrak{e}_{4,41}^{w_1..w_4} = \mathfrak{E}^{\binom{u_1}{v_{1:2}}}\ \mathfrak{E}^{\binom{u_{1234}}{v_2}}\ \mathfrak{E}^{\binom{u_3}{v_{3:2}}}\ \mathfrak{E}^{\binom{u_4}{v_{4:2}}}$$
$$\mathfrak{e}_{4,42}^{w_1..w_4} = \mathfrak{E}^{\binom{u_1}{v_{1:3}}}\ \mathfrak{E}^{\binom{u_2}{v_{2:3}}}\ \mathfrak{E}^{\binom{u_{1234}}{v_3}}\ \mathfrak{E}^{\binom{u_4}{v_{4:3}}}$$
$$\mathfrak{e}_{4,43}^{w_1..w_4} = \mathfrak{E}^{\binom{u_{1234}}{v_1}}\ \mathfrak{E}^{\binom{u_{234}}{v_{2:1}}}\ \mathfrak{E}^{\binom{u_3}{v_{3:2}}}\ \mathfrak{E}^{\binom{u_4}{v_{4:2}}}$$
$$\mathfrak{e}_{4,44}^{w_1..w_4} = \mathfrak{E}^{\binom{u_{1234}}{v_1}}\ \mathfrak{E}^{\binom{u_{23}}{v_{2:1}}}\ \mathfrak{E}^{\binom{u_3}{v_{3:2}}}\ \mathfrak{E}^{\binom{u_4}{v_{4:1}}}$$
$$\mathfrak{e}_{4,45}^{w_1..w_4} = \mathfrak{E}^{\binom{u_{1234}}{v_1}}\ \mathfrak{E}^{\binom{u_2}{v_{2:3}}}\ \mathfrak{E}^{\binom{u_{23}}{v_{3:1}}}\ \mathfrak{E}^{\binom{u_4}{v_{4:1}}}$$
$$\mathfrak{e}_{4,46}^{w_1..w_4} = \mathfrak{E}^{\binom{u_{1234}}{v_1}}\ \mathfrak{E}^{\binom{u_2}{v_{2:4}}}\ \mathfrak{E}^{\binom{u_3}{v_{3:4}}}\ \mathfrak{E}^{\binom{u_{234}}{v_{4:1}}}$$
$$\mathfrak{e}_{4,47}^{w_1..w_4} = \mathfrak{E}^{\binom{u_{1234}}{v_1}}\ \mathfrak{E}^{\binom{u_2}{v_{2:1}}}\ \mathfrak{E}^{\binom{u_{34}}{v_{3:1}}}\ \mathfrak{E}^{\binom{u_4}{v_{4:3}}}$$
$$\mathfrak{e}_{4,48}^{w_1..w_4} = \mathfrak{E}^{\binom{u_{1234}}{v_1}}\ \mathfrak{E}^{\binom{u_2}{v_{2:1}}}\ \mathfrak{E}^{\binom{u_3}{v_{3:4}}}\ \mathfrak{E}^{\binom{u_{34}}{v_{4:1}}}$$
$$\mathfrak{e}_{4,49}^{w_1..w_4} = \mathfrak{E}^{\binom{u_1}{v_{1:4}}}\ \mathfrak{E}^{\binom{u_2}{v_{2:4}}}\ \mathfrak{E}^{\binom{u_3}{v_{3:4}}}\ \mathfrak{E}^{\binom{u_{1234}}{v_4}}$$
$$\mathfrak{e}_{4,50}^{w_1..w_4} = \mathfrak{E}^{\binom{u_{123}}{v_{1:4}}}\ \mathfrak{E}^{\binom{u_2}{v_{2:1}}}\ \mathfrak{E}^{\binom{u_3}{v_{3:1}}}\ \mathfrak{E}^{\binom{u_{1234}}{v_4}}$$
$$\mathfrak{e}_{4,51}^{w_1..w_4} = \mathfrak{E}^{\binom{u_1}{v_{1:3}}}\ \mathfrak{E}^{\binom{u_2}{v_{2:3}}}\ \mathfrak{E}^{\binom{u_{123}}{v_{3:4}}}\ \mathfrak{E}^{\binom{u_{1234}}{v_4}}$$
$$\mathfrak{e}_{4,52}^{w_1..w_4} = \mathfrak{E}^{\binom{u_1}{v_{1:4}}}\ \mathfrak{E}^{\binom{u_{23}}{v_{2:4}}}\ \mathfrak{E}^{\binom{u_3}{v_{3:2}}}\ \mathfrak{E}^{\binom{u_{1234}}{v_4}}$$
$$\mathfrak{e}_{4,53}^{w_1..w_4} = \mathfrak{E}^{\binom{u_1}{v_{1:4}}}\ \mathfrak{E}^{\binom{u_2}{v_{2:3}}}\ \mathfrak{E}^{\binom{u_{23}}{v_{3:4}}}\ \mathfrak{E}^{\binom{u_{1234}}{v_4}}$$
$$\mathfrak{e}_{4,54}^{w_1..w_4} = \mathfrak{E}^{\binom{u_{12}}{v_{1:4}}}\ \mathfrak{E}^{\binom{u_2}{v_{2:1}}}\ \mathfrak{E}^{\binom{u_3}{v_{3:4}}}\ \mathfrak{E}^{\binom{u_{1234}}{v_4}}$$
$$\mathfrak{e}_{4,55}^{w_1..w_4} = \mathfrak{E}^{\binom{u_1}{v_{1:2}}}\ \mathfrak{E}^{\binom{u_{12}}{v_{2:4}}}\ \mathfrak{E}^{\binom{u_3}{v_{3:4}}}\ \mathfrak{E}^{\binom{u_{1234}}{v_4}}$$

We end with bases for the first cells of the structures $\oplus Flex_r(\mathfrak{E})$ and $\oplus Flex_r(\mathfrak{O})$ for an approximate flexion unit $\mathfrak{E}$ verifying the same tripartite equation (3.30) as Qaa_c and an approximate flexion unit $\mathfrak{O}$ verifying the same tripartite equation (3.31) as Qii_c. Here, $\mathfrak{d}$ denotes the discrete dirac multiplied by c. In other words: $\mathfrak{d}^t := c\,\delta(t)$.

$$
\begin{array}{lllllll|lllllll}
\mathfrak{ed}_1^{w_1} & = & \mathfrak{E}^{\binom{u_1}{v_1}} & & & & & \mathfrak{od}_1^{w_1} & = & \mathfrak{O}^{\binom{u_1}{v_1}} & & & & \\
\\
\mathfrak{ed}_1^{w_1,w_2} & = & \mathfrak{E}^{\binom{u_1}{v_{1:2}}} & \mathfrak{E}^{\binom{u_{12}}{v_2}} & & & & \mathfrak{od}_1^{w_1,w_2} & = & \mathfrak{O}^{\binom{u_1}{v_{1:2}}} & \mathfrak{O}^{\binom{u_{12}}{v_2}} & & & \\
\mathfrak{ed}_2^{w_1,w_2} & = & \mathfrak{E}^{\binom{u_{12}}{v_1}} & \mathfrak{E}^{\binom{u_2}{v_{2:1}}} & & & & \mathfrak{od}_2^{w_1,w_2} & = & \mathfrak{O}^{\binom{u_{12}}{v_1}} & \mathfrak{O}^{\binom{u_2}{v_{2:1}}} & & & \\
\mathfrak{ed}_3^{w_1,w2} & = & \mathfrak{d}^{v_1} & \mathfrak{d}^{v_2} & & & & \mathfrak{od}_3^{w_1,w2} & = & \mathfrak{d}^{u_1} & \mathfrak{d}^{u_2} & & & \\
\\
\mathfrak{ed}_1^{w_1..w_3} & = & \mathfrak{E}^{\binom{u_1}{v_{1:2}}} & \mathfrak{E}^{\binom{u_{12}}{v_{2:3}}} & \mathfrak{E}^{\binom{u_{123}}{v_3}} & & & \mathfrak{od}_1^{w_1..w_3} & = & \mathfrak{O}^{\binom{u_1}{v_{1:2}}} & \mathfrak{O}^{\binom{u_{12}}{v_{2:3}}} & \mathfrak{O}^{\binom{u_{123}}{v_3}} & & \\
\ldots\ldots & = & \ldots\ldots & \ldots\ldots & \ldots\ldots & & & \ldots\ldots & = & \ldots\ldots & \ldots\ldots & \ldots\ldots & & \\
\mathfrak{ed}_5^{w_1..w_3} & = & \mathfrak{E}^{\binom{u_{123}}{v_1}} & \mathfrak{E}^{\binom{u_{23}}{v_{2:1}}} & \mathfrak{E}^{\binom{u_3}{v_{3:2}}} & & & \mathfrak{od}_5^{w_1..w_3} & = & \mathfrak{O}^{\binom{u_{123}}{v_1}} & \mathfrak{O}^{\binom{u_{23}}{v_{2:1}}} & \mathfrak{O}^{\binom{u_3}{v_{3:2}}} & & \\
\mathfrak{ed}_6^{w_1..w_3} & = & \mathfrak{E}^{\binom{u_1}{v_1}} & \mathfrak{d}^{v_2} & \mathfrak{d}^{v_3} & & & \mathfrak{od}_6^{w_1..w_3} & = & \mathfrak{O}^{\binom{u_1}{v_1}} & \mathfrak{d}^{u_2} & \mathfrak{d}^{u_3} & & \\
\mathfrak{ed}_7^{w_1..w_3} & = & \mathfrak{E}^{\binom{u_2}{v_2}} & \mathfrak{d}^{v_1} & \mathfrak{d}^{v_3} & & & \mathfrak{od}_7^{w_1..w_3} & = & \mathfrak{O}^{\binom{u_1}{v_{1:2}}} & \mathfrak{d}^{u_{12}} & \mathfrak{d}^{u_3} & & \\
\mathfrak{ed}_8^{w_1..w_3} & = & \mathfrak{E}^{\binom{u_3}{v_3}} & \mathfrak{d}^{v_1} & \mathfrak{d}^{v_2} & & & \mathfrak{od}_8^{w_1..w_3} & = & \mathfrak{O}^{\binom{u_3}{v_{3:2}}} & \mathfrak{d}^{u_1} & \mathfrak{d}^{u_{23}} & & \\
\mathfrak{ed}_9^{w_1..w_3} & = & \mathfrak{E}^{\binom{u_{123}}{v_1}} & \mathfrak{d}^{v_{2:1}} & \mathfrak{d}^{v_{3:1}} & & & \mathfrak{od}_9^{w_1..w_3} & = & \mathfrak{O}^{\binom{u_3}{v_3}} & \mathfrak{d}^{u_1} & \mathfrak{d}^{u_2} & & \\
\\
\mathfrak{ed}_1^{w_1..w_4} & = & \mathfrak{E}^{\binom{u_1}{v_{1:2}}} & \mathfrak{E}^{\binom{u_{12}}{v_{2:3}}} & \mathfrak{E}^{\binom{u_{123}}{v_{3:4}}} & \mathfrak{E}^{\binom{u_{1234}}{v_4}} & & \mathfrak{od}_1^{w_1..w_4} & = & \mathfrak{O}^{\binom{u_1}{v_{1:2}}} & \mathfrak{O}^{\binom{u_{12}}{v_{2:3}}} & \mathfrak{O}^{\binom{u_{123}}{v_{3:4}}} & \mathfrak{O}^{\binom{u_{1234}}{v_4}} & \\
\ldots\ldots & = & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots & & \ldots\ldots & = & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \\
\mathfrak{ed}_{14}^{w_1..w_4} & = & \mathfrak{E}^{\binom{u_{1234}}{v_1}} & \mathfrak{E}^{\binom{u_{234}}{v_{2:1}}} & \mathfrak{E}^{\binom{u_{34}}{v_{3:2}}} & \mathfrak{E}^{\binom{u_4}{v_{4:3}}} & & \mathfrak{od}_{14}^{w_1..w_4} & = & \mathfrak{O}^{\binom{u_{1234}}{v_1}} & \mathfrak{O}^{\binom{u_{234}}{v_{2:1}}} & \mathfrak{O}^{\binom{u_{34}}{v_{3:2}}} & \mathfrak{O}^{\binom{u_4}{v_{4:3}}} & \\
\mathfrak{ed}_{15}^{w_1..w_4} & = & \mathfrak{E}^{\binom{u_1}{v_1}} & \mathfrak{E}^{\binom{u_3}{v_3}} & \mathfrak{d}^{v_2} & \mathfrak{d}^{v_4} & & \mathfrak{od}_{15}^{w_1..w_4} & = & \mathfrak{O}^{\binom{u_1}{v_1}} & \mathfrak{O}^{\binom{u_3}{v_{3:2}}} & \mathfrak{d}^{u_{23}} & \mathfrak{d}^{u_4} & \\
\mathfrak{ed}_{16}^{w_1..w_4} & = & \mathfrak{E}^{\binom{u_1}{v_1}} & \mathfrak{E}^{\binom{u_4}{v_4}} & \mathfrak{d}^{v_2} & \mathfrak{d}^{v_3} & & \mathfrak{od}_{16}^{w_1..w_4} & = & \mathfrak{O}^{\binom{u_1}{v_1}} & \mathfrak{O}^{\binom{u_4}{v_{4:3}}} & \mathfrak{d}^{u_{34}} & \mathfrak{d}^{u_2} & \\
\mathfrak{ed}_{17}^{w_1..w_4} & = & \mathfrak{E}^{\binom{u_2}{v_2}} & \mathfrak{E}^{\binom{u_4}{v_4}} & \mathfrak{d}^{v_1} & \mathfrak{d}^{v_3} & & \mathfrak{od}_{17}^{w_1..w_4} & = & \mathfrak{O}^{\binom{u_2}{v_{2:3}}} & \mathfrak{O}^{\binom{u_4}{v_4}} & \mathfrak{d}^{u_{23}} & \mathfrak{d}^{u_1} & \\
\mathfrak{ed}_{18}^{w_1..w_4} & = & \mathfrak{E}^{\binom{u_{12}}{v_2}} & \mathfrak{E}^{\binom{u_1}{v_{1:2}}} & \mathfrak{d}^{v_3} & \mathfrak{d}^{v_4} & & \mathfrak{od}_{18}^{w_1..w_4} & = & \mathfrak{O}^{\binom{u_1}{v_{1:2}}} & \mathfrak{O}^{\binom{u_4}{v_4}} & \mathfrak{d}^{u_{12}} & \mathfrak{d}^{u_3} & \\
\mathfrak{ed}_{19}^{w_1..w_4} & = & \mathfrak{E}^{\binom{u_{12}}{v_1}} & \mathfrak{E}^{\binom{u_2}{v_{2:1}}} & \mathfrak{d}^{v_3} & \mathfrak{d}^{v_4} & & \mathfrak{od}_{19}^{w_1..w_4} & = & \mathfrak{O}^{\binom{u_2}{v_{2:1}}} & \mathfrak{O}^{\binom{u_{12}}{v_1}} & \mathfrak{d}^{u_3} & \mathfrak{d}^{u_4} & \\
\mathfrak{ed}_{20}^{w_1..w_4} & = & \mathfrak{E}^{\binom{u_{23}}{v_3}} & \mathfrak{E}^{\binom{u_2}{v_{2:3}}} & \mathfrak{d}^{v_1} & \mathfrak{d}^{v_4} & & \mathfrak{od}_{20}^{w_1..w_4} & = & \mathfrak{O}^{\binom{u_1}{v_{1:2}}} & \mathfrak{O}^{\binom{u_{12}}{v_2}} & \mathfrak{d}^{u_3} & \mathfrak{d}^{u_4} & \\
\mathfrak{ed}_{21}^{w_1..w_4} & = & \mathfrak{E}^{\binom{u_{23}}{v_2}} & \mathfrak{E}^{\binom{u_3}{v_{3:2}}} & \mathfrak{d}^{v_1} & \mathfrak{d}^{v_4} & & \mathfrak{od}_{21}^{w_1..w_4} & = & \mathfrak{O}^{\binom{u_4}{v_{4:3}}} & \mathfrak{O}^{\binom{u_{34}}{v_3}} & \mathfrak{d}^{u_1} & \mathfrak{d}^{u_2} & \\
\mathfrak{ed}_{22}^{w_1..w_4} & = & \mathfrak{E}^{\binom{u_{34}}{v_4}} & \mathfrak{E}^{\binom{u_3}{v_{3:4}}} & \mathfrak{d}^{v_1} & \mathfrak{d}^{v_2} & & \mathfrak{od}_{22}^{w_1..w_4} & = & \mathfrak{O}^{\binom{u_3}{v_{3:4}}} & \mathfrak{O}^{\binom{u_{34}}{v_4}} & \mathfrak{d}^{u_1} & \mathfrak{d}^{u_2} & \\
\mathfrak{ed}_{23}^{w_1..w_4} & = & \mathfrak{E}^{\binom{u_{34}}{v_3}} & \mathfrak{E}^{\binom{u_4}{v_{4:3}}} & \mathfrak{d}^{v_1} & \mathfrak{d}^{v_2} & & \mathfrak{od}_{23}^{w_1..w_4} & = & \mathfrak{O}^{\binom{u_1}{v_{1:2}}} & \mathfrak{O}^{\binom{u_3}{v_{3:4}}} & \mathfrak{d}^{u_{12}} & \mathfrak{d}^{u_{34}} & \\
\mathfrak{ed}_{24}^{w_1..w_4} & = & \mathfrak{E}^{\binom{u_{1234}}{v_4}} & \mathfrak{E}^{\binom{u_1}{v_{1:4}}} & \mathfrak{d}^{v_{2:4}} & \mathfrak{d}^{v_{3:4}} & & \mathfrak{od}_{24}^{w_1..w_4} & = & \mathfrak{O}^{\binom{u_1}{v_{1:4}}} & \mathfrak{O}^{\binom{u_{1234}}{v_4}} & \mathfrak{d}^{u_2} & \mathfrak{d}^{u_3} & \\
\mathfrak{ed}_{25}^{w_1..w_4} & = & \mathfrak{E}^{\binom{u_{1234}}{v_3}} & \mathfrak{E}^{\binom{u_2}{v_{2:3}}} & \mathfrak{d}^{v_{1:3}} & \mathfrak{d}^{v_{4:3}} & & \mathfrak{od}_{25}^{w_1..w_4} & = & \mathfrak{O}^{\binom{u_4}{v_{4:1}}} & \mathfrak{O}^{\binom{u_{1234}}{v_1}} & \mathfrak{d}^{u_2} & \mathfrak{d}^{u_3} & \\
\mathfrak{ed}_{26}^{w_1..w_4} & = & \mathfrak{E}^{\binom{u_{1234}}{v_2}} & \mathfrak{E}^{\binom{u_3}{v_{3:2}}} & \mathfrak{d}^{v_{4:2}} & \mathfrak{d}^{v_{1:2}} & & \mathfrak{od}_{26}^{w_1..w_4} & = & \mathfrak{O}^{\binom{u_2}{v_{2:1}}} & \mathfrak{O}^{\binom{u_{12}}{v_{1:3}}} & \mathfrak{d}^{u_{123}} & \mathfrak{d}^{u_4} & \\
\mathfrak{ed}_{27}^{w_1..w_4} & = & \mathfrak{E}^{\binom{u_{1234}}{v_1}} & \mathfrak{E}^{\binom{u_4}{v_{4:1}}} & \mathfrak{d}^{v_{3:1}} & \mathfrak{d}^{v_{2:1}} & & \mathfrak{od}_{27}^{w_1..w_4} & = & \mathfrak{O}^{\binom{u_1}{v_{1:2}}} & \mathfrak{O}^{\binom{u_{12}}{v_{2:3}}} & \mathfrak{d}^{u_{123}} & \mathfrak{d}^{u_4} & \\
\mathfrak{ed}_{28}^{w_1..w_4} & = & \mathfrak{E}^{\binom{u_{1234}}{v_4}} & \mathfrak{E}^{\binom{u_{123}}{v_{1:4}}} & \mathfrak{d}^{v_{2:1}} & \mathfrak{d}^{v_{3:1}} & & \mathfrak{od}_{28}^{w_1..w_4} & = & \mathfrak{O}^{\binom{u_{34}}{v_{4:2}}} & \mathfrak{O}^{\binom{u_3}{v_{3:4}}} & \mathfrak{d}^{u_{234}} & \mathfrak{d}^{u_1} & \\
\mathfrak{ed}_{29}^{w_1..w_4} & = & \mathfrak{E}^{\binom{u_{1234}}{v_1}} & \mathfrak{E}^{\binom{u_{234}}{v_{4:1}}} & \mathfrak{d}^{v_{3:4}} & \mathfrak{d}^{v_{2:4}} & & \mathfrak{od}_{29}^{w_1..w_4} & = & \mathfrak{O}^{\binom{u_{34}}{v_{3:2}}} & \mathfrak{O}^{\binom{u_4}{v_{4:3}}} & \mathfrak{d}^{u_{234}} & \mathfrak{d}^{u_1} & \\
\mathfrak{od}_{30}^{w_1..w_4} & = & \mathfrak{d}^{v_1} & \mathfrak{d}^{v_2} & \mathfrak{d}^{v_3} & \mathfrak{d}^{v_4} & & \mathfrak{od}_{30}^{w_1..w_4} & = & \mathfrak{d}^{u_1} & \mathfrak{d}^{u_2} & \mathfrak{d}^{u_3} & \mathfrak{d}^{u_4} &
\end{array}
$$

12.2 Table 2: basis for *Flexin*($\mathfrak{E}$).

In §4.1 we introduced three series of bimoulds $\{\mathfrak{me}_r^\bullet\}$, $\{\mathfrak{ne}_r^\bullet\}$, $\{\mathfrak{re}_r^\bullet\}$, each of which, under *mu*-multiplication, produces a linear basis for *Flexin*($\mathfrak{E}$). For the first two series, the inductive definitions $\mathfrak{me}_r^\bullet := amit(\mathfrak{me}_{r-1}^\bullet).\mathfrak{E}^\bullet$ and $\mathfrak{ne}_r^\bullet := anit(\mathfrak{ne}_{r-1}^\bullet).\mathfrak{E}^\bullet$ straightaway generate atoms

$$\mathfrak{me}_r^{\binom{u_1\ ,\dots,\ u_r}{v_1\ ,\dots,\ v_r}} = \mathfrak{E}^{\binom{u_1}{v_{1:2}}}\ \mathfrak{E}^{\binom{u_{1,2}}{v_{2:3}}}\ \dots\ \mathfrak{E}^{\binom{u_{1,\dots,r}}{v_r}} \tag{12.1}$$

$$\mathfrak{ne}_r^{\binom{u_1\ ,\dots,\ u_r}{v_1\ ,\dots,\ v_r}} = \mathfrak{E}^{\binom{u_{1,\dots,r}}{v_1}}\mathfrak{E}^{\binom{u_{2,\dots,r}}{v_{2:1}}}\dots\mathfrak{E}^{\binom{u_r}{v_{r:r-1}}} \tag{12.2}$$

in *all cases*, i.e. whether $\mathfrak{E}$ is a flexion unit or not. Not so with the more important – because *alternal* – third series. Here, the inductive rule $\mathfrak{re}_r^\bullet := arit(\mathfrak{re}_{r-1}^\bullet).\mathfrak{E}^\bullet$ produces 2^{r-1} summands. If $\mathfrak{E}$ is a flexion unit, this far exceeds the minimal number of atoms required, which is always r. Moreover, in the polar realisation $\mathfrak{E} = Pi$, the mechanical application of the induction rule produces illusory poles. To remedy these drawbacks, we may use any one of these three alternative expressions:

$$\mathfrak{re}_r^\bullet = \sum_{r_1+r_2=r}^{0\le r_1} (-1)^{r_1}\, r_2\, \mathrm{mu}(\mathfrak{ne}_{r_1}^\bullet, \mathfrak{me}_{r_2}^\bullet) \tag{12.3}$$

$$\mathfrak{re}_r^\bullet = \sum_{r_1+r_2=r}^{0\le r_2} -(-1)^{r_1}\, r_1\, \mathrm{mu}(\mathfrak{ne}_{r_1}^\bullet, \mathfrak{me}_{r_2}^\bullet) \tag{12.4}$$

$$\mathfrak{re}_r^\bullet = \sum_{r_1+r_2=r}^{0\le r_1,r_2} (-1)^{r_1}\, \frac{r_2-r_1}{2}\, \mathrm{mu}(\mathfrak{ne}_{r_1}^\bullet, \mathfrak{me}_{r_2}^\bullet) \tag{12.5}$$

with the convention $\mathfrak{me}_0^\bullet = \mathfrak{ne}_0^\bullet := 1$. These sums produce indeed the minimal number of atoms[178] and do away with illusory poles, but they are of course valid only if $\mathfrak{E}$ is a flexion unit. Only the last expression is left-right symmetric, and renders the alternality of the $\mathfrak{re}_r^\bullet$ 'visually' obvious.

12.3 Table 3: basis for *Flexinn*($\mathfrak{E}$).

To produce an explicit basis, we must first express the iterated *preari* products $\mathfrak{Re}_{\boldsymbol{r}}^\bullet$ of the basic alternal bimoulds $\mathfrak{re}_r^\bullet$, calculated as usual from left to right:

$$\mathfrak{Re}_{r_1}^\bullet := \mathfrak{re}_{r_1}^\bullet \quad \textit{and} \quad \mathfrak{Re}_{r_1,\dots,r_s}^\bullet := \mathrm{preari}(\mathfrak{Re}_{r_1,\dots,r_{s-1}}^\bullet, \mathfrak{re}_{r_s}^\bullet)$$

[178] Strictly speaking, this applies only to the first two sums. For r odd, the last sum involves a supererogatory atom.

Technically, however, it is more convenient to consider the swappees $\mathfrak{Ro}^{\bullet}_{\boldsymbol{r}} := swap.\mathfrak{Re}^{\bullet}_{\boldsymbol{r}}$. The formula for expressing them as *minimal* sums of *inflected atoms* may seem forbiddingly complex, but it is still very useful, and in some contexts even indispensable. It reads:

$$\mathfrak{R\ddot{o}}^{\bullet}_{r_1,...,r_s} := \sum \mathfrak{Po}^{\bullet}_{\binom{n_1}{\boldsymbol{r^1}} \, \substack{,...,\\ ,...,} \, \binom{n_t}{\boldsymbol{r^t}}} \; H^{\boldsymbol{r^1}}_{\overline{n}_1,\underline{n}_1} \dots H^{\boldsymbol{r^t}}_{\overline{n}_t,\underline{n}_t} \qquad (12.6)$$

(i) with a sum extending to all partitions of $\boldsymbol{r} = (r_1, ..., r_s)$ into any number of partial sequences $\boldsymbol{r^i}$, and to all choices of integers n_i, subject only to the following constraints: the internal order of each $\boldsymbol{r^i}$ must be compatible with that of $\boldsymbol{r}$, whereas the various $\boldsymbol{r^i}$ may be positioned in *any* order; and the integers n_i need only verify $r^*_{i-1} < n_i \leq r^*_i$ with $r^*_i := ||\boldsymbol{r^1}|| + ... + ||\boldsymbol{r^i}||$.
(ii) with half-integers $\underline{n}_i$ and integers $\overline{n}_i$ defined by

$$\begin{aligned} \underline{n}_i &:= n_i - r^*_i - \tfrac{1}{2} && \textit{with} \;\; r^*_i := ||\boldsymbol{r^1}|| + ... + ||\boldsymbol{r^i}|| \\ \overline{n}_i &:= 1 + r^* - n_i && \textit{with} \;\; r^* := ||\boldsymbol{r^1}|| + ... + ||\boldsymbol{r^t}|| = r_1 + ... + r_s \end{aligned}$$

(iii) with inflected atoms of type:[179]

$$\mathfrak{Po}^{\bullet}_{\binom{n_1}{\boldsymbol{r^1}} \, \substack{,...,\\ ,...,} \, \binom{n_t}{\boldsymbol{r^t}}} := \prod_{1 \leq i \leq t} \left(\mathfrak{O}^{\binom{u_1+...+u_{r^*_i}}{v_{n_i} - v_{n_{i+1}}}} \prod_{\binom{n^*_{i-1} < n \leq n^*_i}{n \neq n_i}} \mathfrak{O}^{\binom{u_n}{v_n - v_{n_i}}} \right) \qquad (12.7)$$

(iv) with coefficients $H^{\boldsymbol{r}}_{\overline{n},\underline{n}}$ given by the sums

$$H^{\boldsymbol{r}}_{\overline{n},\underline{n}} := \sum_{\boldsymbol{r}^+ \cup \, \boldsymbol{r}^- = \boldsymbol{r}} sign(||\boldsymbol{r}^+|| - \underline{n}) \; F_{\boldsymbol{r}^+}(\overline{n}) \; F_{\boldsymbol{r}^-}(\overline{n}) \qquad (12.8)$$

ranging over all partitions $\boldsymbol{r}^+ \cup \boldsymbol{r}^-$ of $\boldsymbol{r}$.

If $\boldsymbol{r}^+ = (r^+_1, ..., r^+_p)$ and $\boldsymbol{r}^- = (r^-_1, ..., r^-_q)$, the two factors $F_{\boldsymbol{r}^\pm}$ are defined as follows:

$$\begin{aligned} F_{\boldsymbol{r}^+}(\overline{n}) &:= (\overline{n})\,(r^+_2 + r^+_3 + .. + r^+_p - \overline{n})(r^+_3 + .. + r^+_p - \overline{n}) \dots (r^+_p - \overline{n}) \\ F_{\boldsymbol{r}^-}(\overline{n}) &:= (\overline{n})\,(r^-_2 + r^-_3 + .. + r^-_q + \overline{n})(r^-_3 + .. + r^-_q + \overline{n}) \dots (r^-_p + \overline{n}) \end{aligned}$$

If p (resp. q) is 1, then $F_{\boldsymbol{r}^+}(\overline{n})$ (resp $F_{\boldsymbol{r}^-}(\overline{n})$) reduces to $\overline{n}$.
Lastly, for the extreme partitions $(\boldsymbol{r}^+, \boldsymbol{r}^-) = (\boldsymbol{r}, \boldsymbol{\emptyset})$ or $(\boldsymbol{\emptyset}, \boldsymbol{r})$, we must replace the product $F_{\boldsymbol{r}^+}(\overline{n})\, F_{\boldsymbol{r}^-}(\overline{n})$ respectively by

$$\begin{aligned} F_{\boldsymbol{r},\boldsymbol{\emptyset}}(\overline{n}) &:= +\overline{n}\,(r_2 + r_3 + .. + r_p - \overline{n})(r_3 + .. + r_p - \overline{n}) \dots (r_p - \overline{n}) \\ F_{\boldsymbol{\emptyset},\boldsymbol{r}}(\overline{n}) &:= -\overline{n}\,(r_2 + r_3 + .. + r_q + \overline{n})(r_3 + .. + r_q + \overline{n}) \dots (r_q + \overline{n}) \end{aligned}$$

[179] For extreme values of the index i, we must of course set $n^*_0 := 0$ and $v_{n_{t+1}} := 0$.

Remark 1: massive pole cancellations.
From formula (12.6) and the shape (12.7) of the atoms involved, we immediately infer a huge difference between the specialisations $(\mathfrak{E},\mathfrak{O}) = (Pa, Pi)$ and $(\mathfrak{E},\mathfrak{O}) = (Pi, Pa)$. In the first case, the $\mathfrak{Re}^{\bullet}_{\boldsymbol{r}}$ and $\mathfrak{R\ddot{o}}^{\bullet}_{\boldsymbol{r}}$ are saddled with a maximal number of poles, namely $r\,(r+1)/2$. In the second case, they possess far fewer – as little as $2\,r-1$. This results from massive and rather extraordinary compensations that occur during the iteration of the *preari* product *when applied to the* $\mathfrak{re}_{r_i}$. Were we, however, to subject the *separate components* of the $\mathfrak{re}_{r_i}$ (as given for instance by (12.5)) to *preari*-iteration, no such compensations would take place.

Remark 2: bases of $Flexinn(\mathfrak{E})$**.**
On their own, the $\mathfrak{Re}^{\bullet}_{\boldsymbol{r}}$ span, not $Flexinn(\mathfrak{E})$, but the larger $Flexin(\mathfrak{E})$. If however we restrict ourselves to combinations of the form[180]

$$ {}^{\Gamma}\mathfrak{Re}^{\bullet}_{\{r_1,\ldots,r_s\}} := \sum_{\{\boldsymbol{r}'\}=\{\boldsymbol{r}\}} \Gamma^{r'_1,\ldots,r'_s}\ \ \mathfrak{Re}^{\bullet}_{r'_1,\ldots,r'_s} \qquad (\Gamma\ \textit{symmetral}) \tag{12.9} $$

then the new ${}^{\Gamma}\mathfrak{Re}^{\bullet}_{\{\boldsymbol{r}\}}$ do constitute a basis of $Flexinn(\mathfrak{E})$, and that too *irrespective of the choice of the symmetral mould* Γ, provided Γ^{r_1} be $\neq 0$ for all indices r_1. Three choices stand out:

$$ \Gamma_1^{r_1,\ldots,r_s} \;:=\; 1/s! \tag{12.10} $$

$$ \Gamma_2^{r_1,\ldots,r_s} \;:=\; \prod_{1\le i\le s} \frac{1}{r_1+\ldots+r_i} \tag{12.11} $$

$$ \Gamma_3^{r_1,\ldots,r_s} \;:=\; (-1)^s \prod_{1\le i\le s} \frac{1}{r_i+\ldots+r_s} \tag{12.12} $$

(i) The basis ${}^{\Gamma_1}\mathfrak{Re}^{\bullet}_{\{\boldsymbol{r}\}}$ permits the expression of the elements $\mathfrak{Se}^{\bullet}_{f}$ of $GARI_{<\mathfrak{se}>}$ in terms of the coefficients ϵ_r of the infinitesimal generator f_* of f.

(ii) The basis ${}^{\Gamma_2}\mathfrak{Re}^{\bullet}_{\{\boldsymbol{r}\}}$ permits the expression of the elements $\mathfrak{Se}^{\bullet}_{f}$ of $GARI_{<\mathfrak{se}>}$ in terms of the coefficients γ_r of the (direct) infinitesimal dilator $f_\#$ of f.

(iii) The basis ${}^{\Gamma_3}\mathfrak{Re}^{\bullet}_{\{\boldsymbol{r}\}}$ permits the expression of the elements $\mathfrak{Se}^{\bullet}_{f}$ of $GARI_{<\mathfrak{se}>}$ in terms of the coefficients δ_r of the inverse infinitesimal dilator $(f^{-1})_\#$.

[180]with a sum ranging over all permutations $\boldsymbol{r}'$ of the sequence $\boldsymbol{r}$.)

12.4 Table 4: the universal bimould $\mathfrak{ess}^\bullet$.

$$
\begin{aligned}
\mathfrak{ess}^{w_1} &= -\tfrac{1}{2}\ \mathfrak{E}^{\left(\begin{smallmatrix} u_1 \\ v_1 \end{smallmatrix}\right)} \\[6pt]
\mathfrak{ess}^{w_1,w_2} &= +\tfrac{1}{12}\ \mathfrak{E}^{\left(\begin{smallmatrix} u_1 \\ v_{1:2} \end{smallmatrix}\right)} \mathfrak{E}^{\left(\begin{smallmatrix} u_{12} \\ v_2 \end{smallmatrix}\right)} \\
&\quad +\tfrac{1}{12}\ \mathfrak{E}^{\left(\begin{smallmatrix} u_1 \\ v_1 \end{smallmatrix}\right)} \mathfrak{E}^{\left(\begin{smallmatrix} u_2 \\ v_2 \end{smallmatrix}\right)} \\[6pt]
\mathfrak{ess}^{w_1,w_2,w_3} &= -\tfrac{1}{24}\ \mathfrak{E}^{\left(\begin{smallmatrix} u_1 \\ v_{1:2} \end{smallmatrix}\right)} \mathfrak{E}^{\left(\begin{smallmatrix} u_{12} \\ v_2 \end{smallmatrix}\right)} \mathfrak{E}^{\left(\begin{smallmatrix} u_3 \\ v_3 \end{smallmatrix}\right)} \\[6pt]
\mathfrak{ess}^{w_1,w_2,w_3,w_4} &= -\tfrac{1}{720}\ \mathfrak{E}^{\left(\begin{smallmatrix} u_1 \\ v_{1:2} \end{smallmatrix}\right)} \mathfrak{E}^{\left(\begin{smallmatrix} u_{12} \\ v_{2:3} \end{smallmatrix}\right)} \mathfrak{E}^{\left(\begin{smallmatrix} u_{123} \\ v_{3:4} \end{smallmatrix}\right)} \mathfrak{E}^{\left(\begin{smallmatrix} u_{1234} \\ v_4 \end{smallmatrix}\right)} \\
&\quad -\tfrac{1}{240}\ \mathfrak{E}^{\left(\begin{smallmatrix} u_1 \\ v_1 \end{smallmatrix}\right)} \mathfrak{E}^{\left(\begin{smallmatrix} u_2 \\ v_{2:3} \end{smallmatrix}\right)} \mathfrak{E}^{\left(\begin{smallmatrix} u_{23} \\ v_{3:4} \end{smallmatrix}\right)} \mathfrak{E}^{\left(\begin{smallmatrix} u_{234} \\ v_4 \end{smallmatrix}\right)} \\
&\quad -\tfrac{1}{240}\ \mathfrak{E}^{\left(\begin{smallmatrix} u_{12} \\ v_1 \end{smallmatrix}\right)} \mathfrak{E}^{\left(\begin{smallmatrix} u_2 \\ v_{2:1} \end{smallmatrix}\right)} \mathfrak{E}^{\left(\begin{smallmatrix} u_3 \\ v_{3:4} \end{smallmatrix}\right)} \mathfrak{E}^{\left(\begin{smallmatrix} u_{34} \\ v_4 \end{smallmatrix}\right)} \\
&\quad +\tfrac{1}{180}\ \mathfrak{E}^{\left(\begin{smallmatrix} u_1 \\ v_{1:2} \end{smallmatrix}\right)} \mathfrak{E}^{\left(\begin{smallmatrix} u_{12} \\ v_{2:3} \end{smallmatrix}\right)} \mathfrak{E}^{\left(\begin{smallmatrix} u_{123} \\ v_3 \end{smallmatrix}\right)} \mathfrak{E}^{\left(\begin{smallmatrix} u_4 \\ v_4 \end{smallmatrix}\right)} \\
&\quad +\tfrac{1}{120}\ \mathfrak{E}^{\left(\begin{smallmatrix} u_1 \\ v_1 \end{smallmatrix}\right)} \mathfrak{E}^{\left(\begin{smallmatrix} u_2 \\ v_{2:3} \end{smallmatrix}\right)} \mathfrak{E}^{\left(\begin{smallmatrix} u_{23} \\ v_3 \end{smallmatrix}\right)} \mathfrak{E}^{\left(\begin{smallmatrix} u_4 \\ v_4 \end{smallmatrix}\right)} \\
&\quad -\tfrac{1}{720}\ \mathfrak{E}^{\left(\begin{smallmatrix} u_{12} \\ v_1 \end{smallmatrix}\right)} \mathfrak{E}^{\left(\begin{smallmatrix} u_2 \\ v_{2:1} \end{smallmatrix}\right)} \mathfrak{E}^{\left(\begin{smallmatrix} u_3 \\ v_3 \end{smallmatrix}\right)} \mathfrak{E}^{\left(\begin{smallmatrix} u_4 \\ v_4 \end{smallmatrix}\right)} \\[6pt]
\mathfrak{ess}^{w_1,w_2,w_3,w_4,w_5} &= -\tfrac{1}{240}\ \mathfrak{E}^{\left(\begin{smallmatrix} u_1 \\ v_{1:2} \end{smallmatrix}\right)} \mathfrak{E}^{\left(\begin{smallmatrix} u_{12} \\ v_2 \end{smallmatrix}\right)} \mathfrak{E}^{\left(\begin{smallmatrix} u_3 \\ v_{3:4} \end{smallmatrix}\right)} \mathfrak{E}^{\left(\begin{smallmatrix} u_{34} \\ v_4 \end{smallmatrix}\right)} \mathfrak{E}^{\left(\begin{smallmatrix} u_5 \\ v_5 \end{smallmatrix}\right)} \\
&\quad +\tfrac{1}{480}\ \mathfrak{E}^{\left(\begin{smallmatrix} u_1 \\ v_{1:2} \end{smallmatrix}\right)} \mathfrak{E}^{\left(\begin{smallmatrix} u_{12} \\ v_2 \end{smallmatrix}\right)} \mathfrak{E}^{\left(\begin{smallmatrix} u_3 \\ v_{3:4} \end{smallmatrix}\right)} \mathfrak{E}^{\left(\begin{smallmatrix} u_{34} \\ v_{4:5} \end{smallmatrix}\right)} \mathfrak{E}^{\left(\begin{smallmatrix} u_{345} \\ v_5 \end{smallmatrix}\right)} \\
&\quad +\tfrac{1}{480}\ \mathfrak{E}^{\left(\begin{smallmatrix} u_1 \\ v_{1:2} \end{smallmatrix}\right)} \mathfrak{E}^{\left(\begin{smallmatrix} u_{123} \\ v_2 \end{smallmatrix}\right)} \mathfrak{E}^{\left(\begin{smallmatrix} u_3 \\ v_{3:2} \end{smallmatrix}\right)} \mathfrak{E}^{\left(\begin{smallmatrix} u_4 \\ v_{4:5} \end{smallmatrix}\right)} \mathfrak{E}^{\left(\begin{smallmatrix} u_{45} \\ v_5 \end{smallmatrix}\right)} \\
&\quad +\tfrac{1}{1440}\ \mathfrak{E}^{\left(\begin{smallmatrix} u_1 \\ v_{1:2} \end{smallmatrix}\right)} \mathfrak{E}^{\left(\begin{smallmatrix} u_{12} \\ v_{2:3} \end{smallmatrix}\right)} \mathfrak{E}^{\left(\begin{smallmatrix} u_{123} \\ v_{3:4} \end{smallmatrix}\right)} \mathfrak{E}^{\left(\begin{smallmatrix} u_{1234} \\ v_4 \end{smallmatrix}\right)} \mathfrak{E}^{\left(\begin{smallmatrix} u_5 \\ v_5 \end{smallmatrix}\right)} \\
&\quad +\tfrac{1}{1440}\ \mathfrak{E}^{\left(\begin{smallmatrix} u_1 \\ v_{1:2} \end{smallmatrix}\right)} \mathfrak{E}^{\left(\begin{smallmatrix} u_{123} \\ v_2 \end{smallmatrix}\right)} \mathfrak{E}^{\left(\begin{smallmatrix} u_3 \\ v_{3:2} \end{smallmatrix}\right)} \mathfrak{E}^{\left(\begin{smallmatrix} u_4 \\ v_4 \end{smallmatrix}\right)} \mathfrak{E}^{\left(\begin{smallmatrix} u_5 \\ v_5 \end{smallmatrix}\right)}
\end{aligned}
$$

For $r = 6$ or larger, the number of summands increases dramatically. However, one gets markedly simpler expressions when expanding $\mathfrak{ess}^\bullet$ in the bases $\{\mathfrak{me}^\bullet_{r_1,\ldots,r_s}\}, \{\mathfrak{ne}^\bullet_{r_1,\ldots,r_s}\}, \{\mathfrak{re}^\bullet_{r_1,\ldots,r_s}\}$ of $\mathit{Flexin}(\mathfrak{E}) \subset \mathit{Flex}(\mathfrak{E})$: see §4.1.

12.5 Table 5: the universal bimould $\mathfrak{esz}_\sigma^{\bullet}$.

$$\mathfrak{esz}_\sigma^{w_1} =$$

$$+\sigma \quad \times\ \mathfrak{E}^{\binom{u_1}{v_1}}$$

$$\mathfrak{esz}_\sigma^{w_1,w_2} =$$

$$\begin{aligned}
&+\tfrac{1}{3}\,\sigma\,(1+2\,\sigma) &&\times\ \mathfrak{E}^{\binom{u_{12}}{v_2}}\ \mathfrak{E}^{\binom{u_1}{v_{1:2}}}\\
&-\tfrac{1}{3}\,\sigma\,(1-\sigma) &&\times\ \mathfrak{E}^{\binom{u_{12}}{v_1}}\ \mathfrak{E}^{\binom{u_2}{v_{2:1}}}
\end{aligned}$$

$$\mathfrak{esz}_\sigma^{w_1,w_2,w_3} =$$

$$\begin{aligned}
&+\tfrac{1}{6}\,\sigma\,(1+2\,\sigma)\,(1+\sigma) &&\times\ \mathfrak{E}^{\binom{u_{123}}{v_3}}\ \mathfrak{E}^{\binom{u_{12}}{v_{2:3}}}\ \mathfrak{E}^{\binom{u_1}{v_{1:2}}}\\
&-\tfrac{1}{6}\,\sigma\,(1-\sigma) &&\times\ \mathfrak{E}^{\binom{u_{123}}{v_3}}\ \mathfrak{E}^{\binom{u_{12}}{v_{1:3}}}\ \mathfrak{E}^{\binom{u_2}{v_{2:1}}}\\
&-\tfrac{1}{3}\,(1-\sigma)\,\sigma^2 &&\times\ \mathfrak{E}^{\binom{u_{123}}{v_2}}\ \mathfrak{E}^{\binom{u_1}{v_{1:2}}}\ \mathfrak{E}^{\binom{u_3}{v_{3:2}}}\\
&-\tfrac{1}{6}\,\sigma\,(1-\sigma) &&\times\ \mathfrak{E}^{\binom{u_{123}}{v_1}}\ \mathfrak{E}^{\binom{u_{23}}{v_{3:1}}}\ \mathfrak{E}^{\binom{u_2}{v_{2:3}}}\\
&+\tfrac{1}{6}\,\sigma\,(1-\sigma) &&\times\ \mathfrak{E}^{\binom{u_{123}}{v_1}}\ \mathfrak{E}^{\binom{u_{23}}{v_{2:1}}}\ \mathfrak{E}^{\binom{u_3}{v_{3:2}}}
\end{aligned}$$

$$\mathfrak{esz}_\sigma^{w_1,w_2,w_3,w_4} =$$

$$\begin{aligned}
&+\tfrac{1}{30}\,\sigma\,(1+2\,\sigma)\,(1+\sigma)\,(3+2\,\sigma) &&\times\ \mathfrak{E}^{\binom{u_{1234}}{v_4}}\ \mathfrak{E}^{\binom{u_{123}}{v_{3:4}}}\ \mathfrak{E}^{\binom{u_{12}}{v_{2:3}}}\ \mathfrak{E}^{\binom{u_1}{v_{1:2}}}\\
&-\tfrac{1}{90}\,\sigma\,(1-\sigma)\,(9+2\,\sigma-2\,\sigma^2) &&\times\ \mathfrak{E}^{\binom{u_{1234}}{v_4}}\ \mathfrak{E}^{\binom{u_{123}}{v_{3:4}}}\ \mathfrak{E}^{\binom{u_{12}}{v_{1:3}}}\ \mathfrak{E}^{\binom{u_2}{v_{2:1}}}\\
&-\tfrac{1}{6}\,(1-\sigma)\,\sigma^2 &&\times\ \mathfrak{E}^{\binom{u_{1234}}{v_4}}\ \mathfrak{E}^{\binom{u_{123}}{v_{2:4}}}\ \mathfrak{E}^{\binom{u_1}{v_{1:2}}}\ \mathfrak{E}^{\binom{u_3}{v_{3:2}}}\\
&-\tfrac{1}{90}\,\sigma\,(1-\sigma)\,(9+2\,\sigma-2\,\sigma^2) &&\times\ \mathfrak{E}^{\binom{u_{1234}}{v_4}}\ \mathfrak{E}^{\binom{u_{123}}{v_{1:4}}}\ \mathfrak{E}^{\binom{u_{23}}{v_{3:1}}}\ \mathfrak{E}^{\binom{u_2}{v_{2:3}}}\\
&+\tfrac{1}{90}\,\sigma\,(1-\sigma)\,(9-8\,\sigma+8\,\sigma^2) &&\times\ \mathfrak{E}^{\binom{u_{1234}}{v_4}}\ \mathfrak{E}^{\binom{u_{123}}{v_{1:4}}}\ \mathfrak{E}^{\binom{u_{23}}{v_{2:1}}}\ \mathfrak{E}^{\binom{u_3}{v_{3:2}}}\\
&-\tfrac{1}{9}\,(1-\sigma)\,(1+2\,\sigma)\,\sigma^2 &&\times\ \mathfrak{E}^{\binom{u_{1234}}{v_3}}\ \mathfrak{E}^{\binom{u_{12}}{v_{2:3}}}\ \mathfrak{E}^{\binom{u_1}{v_{1:2}}}\ \mathfrak{E}^{\binom{u_4}{v_{4:3}}}\\
&+\tfrac{1}{9}\,\sigma^2\,(1-\sigma)^2 &&\times\ \mathfrak{E}^{\binom{u_{1234}}{v_3}}\ \mathfrak{E}^{\binom{u_{12}}{v_{1:3}}}\ \mathfrak{E}^{\binom{u_2}{v_{2:1}}}\ \mathfrak{E}^{\binom{u_4}{v_{4:3}}}\\
&-\tfrac{1}{6}\,(1-\sigma)\,\sigma^2 &&\times\ \mathfrak{E}^{\binom{u_{1234}}{v_2}}\ \mathfrak{E}^{\binom{u_1}{v_{1:2}}}\ \mathfrak{E}^{\binom{u_{34}}{v_{4:2}}}\ \mathfrak{E}^{\binom{u_3}{v_{3:4}}}\\
&+\tfrac{1}{6}\,(1-\sigma)\,\sigma^2 &&\times\ \mathfrak{E}^{\binom{u_{1234}}{v_2}}\ \mathfrak{E}^{\binom{u_1}{v_{1:2}}}\ \mathfrak{E}^{\binom{u_{34}}{v_{3:2}}}\ \mathfrak{E}^{\binom{u_4}{v_{4:3}}}\\
&-\tfrac{1}{90}\,\sigma\,(1-\sigma)\,(9+2\,\sigma-2\,\sigma^2) &&\times\ \mathfrak{E}^{\binom{u_{1234}}{v_1}}\ \mathfrak{E}^{\binom{u_{234}}{v_{4:1}}}\ \mathfrak{E}^{\binom{u_{23}}{v_{3:4}}}\ \mathfrak{E}^{\binom{u_2}{v_{2:3}}}\\
&+\tfrac{1}{90}\,\sigma\,(1-\sigma)\,(9-8\,\sigma+8\,\sigma^2) &&\times\ \mathfrak{E}^{\binom{u_{1234}}{v_1}}\ \mathfrak{E}^{\binom{u_{234}}{v_{4:1}}}\ \mathfrak{E}^{\binom{u_{23}}{v_{2:4}}}\ \mathfrak{E}^{\binom{u_3}{v_{3:2}}}\\
&+\tfrac{1}{9}\,\sigma^2\,(1-\sigma)^2 &&\times\ \mathfrak{E}^{\binom{u_{1234}}{v_1}}\ \mathfrak{E}^{\binom{u_{234}}{v_{3:1}}}\ \mathfrak{E}^{\binom{u_2}{v_{2:3}}}\ \mathfrak{E}^{\binom{u_4}{v_{4:3}}}\\
&+\tfrac{1}{90}\,\sigma\,(1-\sigma)\,(9-8\,\sigma+8\,\sigma^2) &&\times\ \mathfrak{E}^{\binom{u_{1234}}{v_1}}\ \mathfrak{E}^{\binom{u_{234}}{v_{2:1}}}\ \mathfrak{E}^{\binom{u_{34}}{v_{4:2}}}\ \mathfrak{E}^{\binom{u_3}{v_{3:4}}}\\
&-\tfrac{1}{90}\,\sigma\,(1-\sigma)\,(9+2\,\sigma-2\,\sigma^2) &&\times\ \mathfrak{E}^{\binom{u_{1234}}{v_1}}\ \mathfrak{E}^{\binom{u_{234}}{v_{2:1}}}\ \mathfrak{E}^{\binom{u_{34}}{v_{3:2}}}\ \mathfrak{E}^{\binom{u_4}{v_{4:3}}}
\end{aligned}$$

12.6 Table 6: the bitrigonometric bimould $taal^{\bullet}/tiil^{\bullet}$.

For simplicity, we drop the c in Qaa_c and Qii_c.

$$\mathrm{taal}^{w_1} = -\tfrac{1}{2}\,\mathrm{Qaa}^{\binom{u_1}{v_1}} \qquad \Vert\ \mathrm{tiil}^{w_1} = -\tfrac{1}{2}\,\mathrm{Qii}^{\binom{u_1}{v_1}}$$

$$\mathrm{taal}^{w_1,w_2} = \qquad \Vert\ \mathrm{tiil}^{w_1,w_2} =$$
$$+\tfrac{1}{12}\,\mathrm{Qaa}^{\binom{u_{12}}{v_1}}\,\mathrm{Qaa}^{\binom{u_2}{v_{2:1}}} \qquad \Vert\ +\tfrac{1}{6}\,\mathrm{Qii}^{\binom{u_{12}}{v_2}}\,\mathrm{Qii}^{\binom{u_1}{v_{1:2}}}$$
$$+\tfrac{1}{6}\,\mathrm{Qaa}^{\binom{u_{12}}{v_2}}\,\mathrm{Qaa}^{\binom{u_1}{v_{1:2}}} \qquad \Vert\ +\tfrac{1}{12}\,\mathrm{Qii}^{\binom{u_{12}}{v_1}}\,\mathrm{Qii}^{\binom{u_2}{v_{2:1}}}$$
$$+\tfrac{1}{8}\,c^2\,\delta^{v_1}\,\delta^{v_2} \qquad \Vert\ -\tfrac{1}{8}\,c^2\,\delta^{u_1}\,\delta^{u_2}$$

$$\mathrm{taal}^{w_1,w_2,w_3} = \qquad \Vert\ \mathrm{tiil}^{w_1,w_2,w_3} =$$
$$-\tfrac{1}{24}\,\mathrm{Qaa}^{\binom{u_1}{v_{1:2}}}\,\mathrm{Qaa}^{\binom{u_{12}}{v_2}}\,\mathrm{Qaa}^{\binom{u_3}{v_3}} \qquad \Vert\ -\tfrac{1}{24}\,\mathrm{Qii}^{\binom{u_1}{v_{1:2}}}\,\mathrm{Qii}^{\binom{u_{12}}{v_2}}\,\mathrm{Qii}^{\binom{u_3}{v_3}}$$
$$-\tfrac{1}{48}\,c^2\,\mathrm{Qaa}^{\binom{u_1}{v_1}}\,\delta^{v_2}\delta^{v_3} \qquad \Vert\ -\tfrac{1}{24}\,c^2\,\mathrm{Qii}^{\binom{u_1}{v_{1:2}}}\,\delta^{u_{12}}\,\delta^{u_3}$$
$$+\tfrac{1}{24}\,c^2\,\mathrm{Qaa}^{\binom{u_2}{v_2}}\,\delta^{v_1}\,\delta^{v_3} \qquad \Vert\ +\tfrac{1}{24}\,c^2\,\mathrm{Qii}^{\binom{u_2}{v_{2:3}}}\,\delta^{u_1}\delta^{u_{23}}$$
$$-\tfrac{1}{24}\,c^2\,\mathrm{Qaa}^{\binom{u_3}{v_3}}\,\delta^{v_1}\,\delta^{v_2} \qquad \Vert\ +\tfrac{1}{48}\,c^2\,\mathrm{Qii}^{\binom{u_3}{v_3}}\,\delta^{u_1}\,\delta^{u_2}$$

$$\mathrm{taal}^{w_1,w_2,w_3,w_4} = \qquad \Vert\ \mathrm{tiil}^{w_1,w_2,w_3,w_4} =$$
$$-\tfrac{1}{720}\,\mathrm{Qaa}^{\binom{u_1}{v_1}}\mathrm{Qaa}^{\binom{u_2}{v_2}}\mathrm{Qaa}^{\binom{u_3}{v_3}}\mathrm{Qaa}^{\binom{u_4}{v_4}} \qquad \Vert\ -\tfrac{1}{720}\,\mathrm{Qii}^{\binom{u_1}{v_{1:2}}}\mathrm{Qii}^{\binom{u_{12}}{v_{2:3}}}\mathrm{Qii}^{\binom{u_{123}}{v_{3:4}}}\mathrm{Qii}^{\binom{u_{1234}}{v_4}}$$
$$-\tfrac{1}{240}\,\mathrm{Qaa}^{\binom{u_1}{v_{1:4}}}\mathrm{Qaa}^{\binom{u_2}{v_{2:4}}}\mathrm{Qaa}^{\binom{u_3}{v_{3:4}}}\mathrm{Qaa}^{\binom{u_{1234}}{v_4}} \qquad \Vert\ -\tfrac{1}{240}\,\mathrm{Qii}^{\binom{u_1}{v_1}}\mathrm{Qii}^{\binom{u_2}{v_{2:3}}}\mathrm{Qii}^{\binom{u_{23}}{v_{3:4}}}\mathrm{Qii}^{\binom{u_{234}}{v_4}}$$
$$+\tfrac{1}{240}\,\mathrm{Qaa}^{\binom{u_1}{v_{1:3}}}\mathrm{Qaa}^{\binom{u_2}{v_{2:3}}}\mathrm{Qaa}^{\binom{u_4}{v_{4:3}}}\mathrm{Qaa}^{\binom{u_{1234}}{v_3}} \qquad \Vert\ -\tfrac{1}{240}\,\mathrm{Qii}^{\binom{u_{12}}{v_1}}\mathrm{Qii}^{\binom{u_2}{v_{2:1}}}\mathrm{Qii}^{\binom{u_3}{v_{3:4}}}\mathrm{Qii}^{\binom{u_{34}}{v_4}}$$
$$+\tfrac{1}{180}\,\mathrm{Qaa}^{\binom{u_1}{v_{1:2}}}\mathrm{Qaa}^{\binom{u_{12}}{v_2}}\mathrm{Qaa}^{\binom{u_3}{v_3}}\mathrm{Qaa}^{\binom{u_4}{v_4}} \qquad \Vert\ +\tfrac{1}{180}\,\mathrm{Qii}^{\binom{u_1}{v_{1:2}}}\mathrm{Qii}^{\binom{u_{12}}{v_{2:3}}}\mathrm{Qii}^{\binom{u_{123}}{v_3}}\mathrm{Qii}^{\binom{u_4}{v_4}}$$
$$+\tfrac{1}{120}\,\mathrm{Qaa}^{\binom{u_1}{v_{1:2}}}\mathrm{Qaa}^{\binom{u_{12}}{v_{2:4}}}\mathrm{Qaa}^{\binom{u_3}{v_{3:4}}}\mathrm{Qaa}^{\binom{u_{1234}}{v_4}} \qquad \Vert\ +\tfrac{1}{120}\,\mathrm{Qii}^{\binom{u_1}{v_1}}\mathrm{Qii}^{\binom{u_2}{v_{2:3}}}\mathrm{Qii}^{\binom{u_{23}}{v_3}}\mathrm{Qii}^{\binom{u_4}{v_4}}$$
$$+\tfrac{1}{720}\,\mathrm{Qaa}^{\binom{u_1}{v_{1:2}}}\mathrm{Qaa}^{\binom{u_{12}}{v_{2:3}}}\mathrm{Qaa}^{\binom{u_4}{v_{4:3}}}\mathrm{Qaa}^{\binom{u_{1234}}{v_3}} \qquad \Vert\ -\tfrac{1}{720}\,\mathrm{Qii}^{\binom{u_2}{v_{2:1}}}\mathrm{Qii}^{\binom{u_{12}}{v_1}}\mathrm{Qii}^{\binom{u_3}{v_3}}\mathrm{Qii}^{\binom{u_4}{v_4}}$$
$$+\tfrac{7}{720}\,c^2\,\mathrm{Qaa}^{\binom{u_1}{v_1}}\mathrm{Qaa}^{\binom{u_2}{v_2}}\delta^{v_3}\delta^{v_4} \qquad \Vert\ -\tfrac{1}{480}\,\mathrm{Qii}^{\binom{u_1}{v_1}}\,\mathrm{Qii}^{\binom{u_4}{v_4}}\,\delta^{u_2}\,\delta^{u_3}$$
$$+\tfrac{7}{1440}\,c^2\,\mathrm{Qaa}^{\binom{u_1}{v_{1:2}}}\,\mathrm{Qaa}^{\binom{u_{12}}{v_2}}\,\delta^{v_3}\,\delta^{v_4} \qquad \Vert\ -\tfrac{1}{480}\,\mathrm{Qii}^{\binom{u_3}{v_3}}\,\mathrm{Qii}^{\binom{u_4}{v_4}}\,\delta^{u_1}\,\delta^{u_2}$$
$$-\tfrac{5}{288}\,c^2\,\mathrm{Qaa}^{\binom{u_1}{v_1}}\,\mathrm{Qaa}^{\binom{u_3}{v_3}}\,\delta^{v_2}\,\delta^{v_4} \qquad \Vert\ -\tfrac{5}{288}\,c^2\,\mathrm{Qii}^{\binom{u_2}{v_{2:3}}}\,\mathrm{Qii}^{\binom{u_4}{v_4}}\,\delta^{u_1}\,\delta^{u_{23}}$$
$$+\tfrac{19}{1440}\,c^2\,\mathrm{Qaa}^{\binom{u_1}{v_1}}\,\mathrm{Qaa}^{\binom{u_4}{v_4}}\,\delta^{v_2}\,\delta^{v_3} \qquad \Vert\ +\tfrac{1}{360}\,c^2\,\mathrm{Qii}^{\binom{u_3}{v_{3:4}}}\,\mathrm{Qii}^{\binom{u_{34}}{v_4}}\,\delta^{u_1}\,\delta^{u_2}$$
$$-\tfrac{1}{480}\,c^2\,\mathrm{Qaa}^{\binom{u_2}{v_2}}\,\mathrm{Qaa}^{\binom{u_3}{v_3}}\,\delta^{v_1}\,\delta^{v_4} \qquad \Vert\ +\tfrac{19}{1440}\,c^2\,\mathrm{Qii}^{\binom{u_1}{v_{1:2}}}\,\mathrm{Qii}^{\binom{u_4}{v_4}}\,\delta^{u_{12}}\,\delta^{u_3}$$
$$+\tfrac{1}{1440}\,c^2\,\mathrm{Qaa}^{\binom{u_2}{v_2}}\,\mathrm{Qaa}^{\binom{u_4}{v_4}}\,\delta^{v_1}\,\delta^{v_3} \qquad \Vert\ -\tfrac{1}{288}\,c^2\,\mathrm{Qii}^{\binom{u_1}{v_1}}\,\mathrm{Qii}^{\binom{u_3}{v_{3:4}}}\,\delta^{u_2}\,\delta^{u_{34}}$$
$$+\tfrac{1}{288}\,c^2\,\mathrm{Qaa}^{\binom{u_3}{v_3}}\,\mathrm{Qaa}^{\binom{u_4}{v_4}}\,\delta^{v_1}\,\delta^{v_2} \qquad \Vert\ -\tfrac{11}{1440}\,c^2\,\mathrm{Qii}^{\binom{u_1}{v_1}}\,\mathrm{Qii}^{\binom{u_3}{v_{3:2}}}\,\delta^{u_{23}}\,\delta^{u_4}$$
$$-\tfrac{1}{480}\,c^2\,\mathrm{Qaa}^{\binom{u_{1234}}{v_2}}\,\mathrm{Qaa}^{\binom{u_1}{v_{1:2}}}\,\delta^{v_{2:4}}\,\delta^{v_{2:3}} \qquad \Vert\ +\tfrac{1}{480}\,c^2\,\mathrm{Qii}^{\binom{u_{12}}{v_1}}\,\mathrm{Qii}^{\binom{u_2}{v_{2:1}}}\,\delta^{u_3}\,\delta^{u_4}$$
$$-\tfrac{1}{288}\,c^2\,\mathrm{Qaa}^{\binom{u_{1234}}{v_1}}\,\mathrm{Qaa}^{\binom{u_2}{v_{2:1}}}\,\delta^{v_{1:4}}\,\delta^{v_{1:3}} \qquad \Vert\ +\tfrac{1}{1440}\,c^2\,\mathrm{Qii}^{\binom{u_1}{v_{1:2}}}\,\mathrm{Qii}^{\binom{u_3}{v_{3:4}}}\,\delta^{u_{12}}\,\delta^{u_{34}}$$
$$+\tfrac{11}{1440}\,c^2\,\mathrm{Qaa}^{\binom{u_{1234}}{v_1}}\,\mathrm{Qaa}^{\binom{u_3}{v_{3:1}}}\,\delta^{v_{1:4}}\,\delta^{v_{1:2}} \qquad \Vert\ +\tfrac{1}{288}\,c^2\,\mathrm{Qii}^{\binom{u_1}{v_{1:2}}}\,\mathrm{Qii}^{\binom{u_{12}}{v_{2:3}}}\,\delta^{u_{123}}\,\delta^{u_4}$$
$$-\tfrac{1}{480}\,c^2\,\mathrm{Qaa}^{\binom{u_{1234}}{v_1}}\,\mathrm{Qaa}^{\binom{u_4}{v_{4:1}}}\,\delta^{v_{1:3}}\,\delta^{v_{1:2}} \qquad \Vert\ -\tfrac{1}{480}\,c^2\,\mathrm{Qii}^{\binom{u_2}{v_{2:3}}}\,\mathrm{Qii}^{\binom{u_{23}}{v_{3:4}}}\,\delta^{u_1}\,\delta^{u_{234}}$$
$$+\tfrac{7}{5760}\,c^4\,\delta^{v_1}\,\delta^{v_2}\,\delta^{v_3}\,\delta^{v_4} \qquad \Vert\ -\tfrac{1}{640}\,c^4\,\delta^{u_1}\,\delta^{u_2}\,\delta^{u_3}\,\delta^{u_4}$$

12.7 Index of terms and notations.

Slight liberties have been taken with the alphabetical order, so as to regroup similar objects or notions.

ALAL: §2.4, §5.7, §7, §8.4.
ASAS: §2.8.
al/al, $\underline{al}/\underline{al}$: §2.7.
as/as, $\underline{as}/\underline{as}$: §2.8.
ALIL: §4.7 §5.7.
ASIS: §4.7.
ALIIL, ASIIS: §4.7
al/il, $\underline{al}/\underline{il}$: §5.7.
as/is, $\underline{as}/\underline{is}$: §4.7.
alternal: §2.4, (2.72).
alternil: §3.4.
anti: §2.1, (2.6).
ami, amit, ani, anit, ari, arit: §2.2.
axi, axit: §2.1.
approximate flexion unit: §3.2 (towards the end).
bialternal: §2.7, §7, §8.
bisymmetral: §2.8, §9.1.
carma$^\bullet$/carmi$^\bullet$, corma$^\bullet$/cormi$^\bullet$, curma$^\bullet$/curmi$^\bullet$: §7.3, §7.7.
conjugate flexion units: §3.2.
dilator (infinitesimal): §4.1, §11.8, §11.10, §12.3.
dimorphy, *dimorphic*: §1.1, §2, §10.1.
doma$^\bullet$/domi$^\bullet$: §7.2.
ekma$^\bullet$/ekmi$^\bullet$: §7.3.
$\mathfrak{E}^\bullet$: §3.1, §3.2.
$\mathfrak{E}^\bullet$-alternal: §3.4.
$\mathfrak{E}^\bullet$-symmetral: §3.4.
$\mathfrak{E}^\bullet$-mantar: §3.4, (3.46).
$\mathfrak{E}^\bullet$-gantar: §3.4, (3.49).
$\mathfrak{E}^\bullet$-push: §3.4, (3.53), (3.54).
$\mathfrak{E}^\bullet$-gush: §3.4, (3.60).
$\mathfrak{E}^\bullet$-neg: §3.4, (3.52).
$\mathfrak{E}^\bullet$-geg: §3.4, (3.59).
$\mathfrak{es}^\bullet, \mathfrak{ez}^\bullet$: §4.3, (4.70), (4.71).
$\mathfrak{ess}^\bullet, \mathfrak{esz}^\bullet$: §4.2, (4.35), (4.36), §11.9, §11.10, §12.4, §12.5.
expari: §2.2, (2.50).
Exter(Qi_c): §11.5.
flexion: §2.1.

flexion unit: §3.2.
flexion structure: §2.
gami, gamit, gani, ganit, gari, garit: §2.2.
gantar, gantir: §2.3, (2.74), (2.75), §3.4.
gepar: §4.1, (4.10), §11.8.
hepar: §11.8, (4.10).
gegu, gegi: §3.5, (3.65).
gus: §2.4, (2.74), (2.75).
gusi, gusu: §3.4.
gush: §2.4, (2.76).
gushi, gushu: §3.4.
invmu: §2.1, (2.2).
invgami, invgani, invgari: §2.2, (2.58).
$\mathrm{Inter}(Qi_c)$: §11.5.
$\mathrm{lama}^\bullet/\mathrm{lami}^\bullet$: §6.5.
$\mathrm{loma}^\bullet/\mathrm{lomi}^\bullet$: §6.6.
$\mathrm{luma}^\bullet/\mathrm{lumi}^\bullet$: §6.7.
$\mathfrak{O}^\bullet$: §3.2.
$\mathfrak{me}_r^\bullet$: §4.1, §12.2.
$\mathfrak{ne}_r^\bullet$: §4.1, §12.2.
mantar, mantir: §2.1, (2.7), §3.4.
minu: §2.1, (2.4).
neg: §2.1, (2.8).
negi, negu: §3.4, (3.61).
pari: §2.1, (2.5).
P: $P(t) := 1/t$.
$\mathrm{pac}^\bullet/\mathrm{pic}^\bullet, \mathrm{paj}^\bullet/\mathrm{pij}^\bullet$: §4.3.
$\mathrm{pal}^\bullet/\mathrm{pil}^\bullet, \mathrm{par}^\bullet/\mathrm{pir}^\bullet$: §4.2 (last but one para).
perinomal: §9.4, §9.5, §9.6.
preami, preani, preari: §2.2.
predoma: §7.5.
precarma: §7.6.
pus: §2.1, (2.10).
pusi, pusu: §3.4.
push: §2.1, (2.11), (2.12).
pushi, pushu: §3.4, (3.62), (3.63).
Q, Q_c: $Q(t) := 1/\tan(t), Q_c(t) := c/\tan(c\,t)$.
$\mathfrak{re}_r^\bullet$: §4.1, §12.2.
$\mathfrak{Re}_f^\bullet, \mathfrak{R\ddot{o}}_f^\bullet$: §12.3.
sap, swap, syap: §2.2, (2.9), §3.3, (4.37), (4.38), (4.70), (4.71).
separ: §10.9.

$\mathfrak{se}^\bullet_r$: §4.1.
$\mathfrak{sse}^\bullet_{12}$: §4.2.
$\mathfrak{Se}^\bullet_f, \mathfrak{S\ddot{o}}^\bullet_f$: §4.1, §11.8.
slank, srank, sang: §5.4, §5.5.
sen: §5.1.
senk, seng: §5.3.
singulator, singuland, singulate etc: §5.
symmetral: §2.4 (2.72).
symmetril: §3.5.
symmetry types (straight): §2.4.
symmetry types (twisted): §3.5.
subsymmetries (simple or double, straight): §2.4.
subsymmetries (simple or double, twisted): §3.5.
$\text{tac}^\bullet/\text{tic}^\bullet, \text{taj}^\bullet/\text{tij}^\bullet$: §4.2.
$\text{tal}^\bullet/\text{til}^\bullet, \text{taal}^\bullet/\text{tiil}^\bullet$: §12.6.
tripartite relation: §3.2, (3.9).
wandering bialternals: §6.9, §9.1.
$\text{Wa}^\bullet$: §1.1.
$\text{Za}^\bullet$: §1.2 (after (2.13)).
$\text{Ze}^\bullet$: §1.1.
$\text{Zag}^\bullet/\text{Zig}^\bullet$: §1.2, §9.

12.8 References.

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N.B. There exists of course a vast literature on multizetas and related lore: polylogarithms, associators, knots, Feynman diagrams, etc. Ample references are readily available at the end of papers dealing with any of these topics. The present article, however, is not primarily about multizetas, but about the *flexion structure*, which happens to be a new subject. Hence the paucity of our bibliographical references.