The scrambling operators applied to multizeta algebra and singular perturbation analysis.

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Abstract. *The present paper addresses two seemingly unrelated topics – the analysis of singular-and-singularly-perturbed differential systems; and the arithmetics of multizetas – but with a strong unifying thread, provided by the three* scrambling operators.

The operators in question – scram, viscram, discram – *properly belong to the field of combinatorics and mould algebra. Their properties are many, but one stands out:* generating rich symmetries and sophisticated operations out of poorer or more elementary ones.

The formal solutions of singular differential systems, when expanded in inverse-power series of the 'critical variable' z, *tend to exhibit divergence, but of a regular and well-understood type: resummable and resurgent, with a resurgence regime completely governed by the now classical Bridge equation. When one introduces a singular perturbation parameter ϵ and expands the solution in powers of the same, divergence and resurgence still dominate, but the picture becomes incomparably more complex: the resurgence calls for two new Bridge equations, not one, and it takes the operator* scram *to fully unravel the mechanisms responsible for this new level of complexity.*

The closely related operators viscram *and* discram, *on their part, render distinguished services in multizeta algebra, especially for dissecting what is arguably the most pivotal case:* the bicoloured multizetas. *For one thing, they assist in proving the independence of the standard system of bicolour generators. But their real contribution lies elsewhere. The fact is that, due to the simultaneous play of* weigths $\mathrm{s_i} \in \mathbb{N}^*$ *and* colours $\epsilon_i \in \frac{1}{k}\mathbb{Z}/\mathbb{Z}$, *there exist for any given (large) total weight s, a huge number of* k*-coloured multizetas. Yet there is a saving grace: the double symmetry (known as* arithmetical dimorphy*) which constrains these multizetas induces so strong a rigidity that the* whole information *can be recovered from relatively sparse* boundary data *(somewhat like with harmonic or analytic functions). The phenomenon is*

particularly striking in the case of bicolours and their three satellites*: the 'lower satellite'* sa*, with all degrees set equal to* 0*; the 'first upper satellite'* sa**, with all colours (simultaneously) set equal to* 0 *or* $\frac{1}{2}$*; and the 'second upper satellite'* sa***, similar in shape to the first, but completely different in origin. We show, with ample assistance from* viscram *and* discram*, how each of these three satellite systems not only morphs into the other two, but also leads to the complete system of bicolours – each conversion finding its expression in remarkably explicit formulae.*

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1 Introduction. The three scrambling operators.

1.1 Roadmap and main results.

The present paper is about a family of operators – the *scrambling operators* – and their wide-ranging applications to Combinatorics, Algebra, and Analysis. In keeping with this prospectus, and although we shall present a fairly large number of new results along the way, our chief concern shall be one of bridge-building and unification, of bringing order and structure to a seemingly loosely-knit, in places even chaotic mathematical subject matter.

The scrambling operators.

They are three in number – *scram, discram, viscram* – and their proper setting is at the intersection of combinatorics and mould algebra. The secret of their usefulness lies in their two main properties. First, they turn the straightforward, uncomplicated, *uninflected* mould operations into the subtler, more complex, *inflected* operations which govern bimould algebra. Second, they transmute simple symmetries into double ones. Some of them, like *viscram*, also preserve double symmetries. This makes them ideally suited for tackling *arithmetical dimorphy.*

Singularly perturbed differential system and co-equational resurgence.

There is a distinct kinship, but also a sharp gap in complexity, between *equational resurgence* (i.e. the divergence-resurgence relative to the *critical variable* of a singular differential system) and *co-equational resurgence* (i.e. the divergence-resurgence relative to a *critical parameter* in such a system). The gap manifests at every level. At the global level: while equational resurgence is entirely described by *one* so-called Bridge equation (relating alien and ordinary differential operators), co-equational resurgence calls for *two* Bridge equations, each of a far more intricate structure. At the analytical level: while equational resurgence and equational Stokes analysis require only simple *resurgence monomials* (elementary resurgent functions) and *monics* (elementary transcendental numbers), co-equational resurgence calls for incomparably more complex monomials and monics. Lastly, at the methodological level: while the shape and nature of equational resurgence may be established almost calculation-free, by formal manipulations involving the alien derivations and supplemented by only a modicum of Analysis, co-equational resurgence allows no such short-cuts, not even for performing the very first step: locating the singularities on the various Riemann sheets of the 'Borel plane'.

As it happens, this gap in complexity faithfully reflects the divide between *uninflected mould algebra*, developed in the late seventies, largely as a handtool for equational resurgence, and *inflected bimould algebra*, developped from the mid-eighties for tackling co-equational resurgence. We survey the question in section §2.

An outstanding feature of co-equational resurgence is the centrality of *combinatorics* to the subject – a combinatorics moreover that is entirely dominated by the scramble transform, and even, in the case of ramified z-data, by a generalised version of it. One may balk at the complexity of certain developments, and resent the notational acrobatics they force on one, but one would do well to remember two things. First, the combinatorics in question has nothing artificial about it: it is entirely, rigidly, univocally imposed by the nature of this particular, very prevalent form of resurgence. Second, while the combinatorics is complex enough in its own terms, it neatly disentangles and tidies up mathematical situations that are incomparably *more* complex. Consider for instance this system, with generic, depth-4 hyperlogarithmic coefficients b_i:

$$(\partial_z + \omega_i\, x)\, Y_i(z) = Y_{i-1}(z)\, b_i(z) \qquad (1 \leqslant i \leqslant 4, Y_0 \equiv 1) \tag{1}$$

It is a honest-to-goodness differential system, linear to boot, and fairly simple. Yet its resurgence in x generates, in the corresponding Borel ξ-plane, close to 10^{10} distinct singularities, living on as many Riemann sheets. Situations like this may seem well-nigh intractable, yet the tool-kit presented here, in §2, leads to a complete, surveyable description of all their aspects. This should never be lost sight of when assessing the cost-effectiveness of the analytico-combinatorial apparatus introduced here.

Moreover, while combinatorics may dominate our *treatment* of coequational resurgence, when it comes to stating the results, it is two other objects that occupy center-stage. They are:
(i) the weighted multiplication or rather its Borel image, the *weighted convolution*, which leads to the specific 'resurgence monomials', which in turn manifest co-equational resurgence at the most basic level.[1]
(ii) the *tessellation coefficients*, indispensable but also sufficient for expressing the alien derivatives of these convolution products.

[1]More precisely, everything rests on two weighted multiplications, $wemu^\bullet$ and $welu^\bullet$, and the corresponding weighted convolution, $weco^\bullet$ and $welo^\bullet$. The *symmetral* operations $wemu^\bullet/weco^\bullet$ are essential for understanding the Second Bridge Equation; the *alternal* operations $welu^\bullet/welo^\bullet$ for understanding the Third Bridge Equation.

The passage from (i) to (ii) is precisely where combinatorics comes in: the integrals underlying *weighted convolution* are so intricate, so impossibly ramified, that the rules governing their alien differentiation cannot be established directly, but only over the detour through a special set of functions (- the hyperlogarithms -) sufficiently numerous to reflect the general picture, yet simple enough to allow a complete formalisation.

Multizeta algebra: monocolours and bicolours.

Soon after their introduction in Analysis, the scrambling operators and the flexion structure were found relevant to multizeta arithmetics, and began to be successfully applied there. This should not come as a surprise, since the multizetas are, among other things, one of the most basic systems of *monics* (they are the main transcendental ingredient in the Stokes constants of local resonant diffeomorphisms) and the most seminal instance of arithmetical *dimorphy.*

We have already devoted several investigations to the subject, and are planning many more, but in this paper (§3, §4, §5), we concentrate on just two classes of multizetas – the monocolours and bicolours – and keep the focus on one main issue: the search for a *suitable filtration*, as a way of overcoming the *curse of retro-action*. Let us explain.

Multizetas, whether taken in scalar form or collected inside the more convenient generating series $zag^\bullet/zig^\bullet$, admit three basic filtrations: by total weight s, by length r, and by degree[2].

The s-filtration is fine as far as it goes: the two basic 'symmetries' (i.e. the two, conjecturedly exhaustive, systems of 'quadratic relations') constraining the multizetas do indeed respect the filtration and even the gradation by weight, but as s increases, the multizetas of weight $\leqslant s$ get much too numerous for practical handling, especially in the case of bicolours.

The s-filtration, when refined by the s-filtration, looks more promising, but it remains blighted by the *curse of retro-action*. That curse, moreover, manifests in two sharply different, almost complementary ways for monocolours and bicolours, especially when one works in the relevant Lie algebra, namely $ARI^{al/il}_{ent}$. For *monocolours*, the two symmetries nicely allow the construction of a system of generators following the (s,r)-filtration, but do not fully determine the decomposition of the general element of $ARI^{al/il}_{ent}$ in terms of these generators: at each level (s,r) there is generally an indeterminacy which is removed only when we proceed to the level $(s,r+2)$. For *bicolours*,

[2]so-called, because in the approach based on the generating series $zag^{\boldsymbol{w}}$, d does indeed correspond to the global polynomial degree in the $\boldsymbol{u}$-variables.

the position is exactly the reverse: once we get hold of a system of generators, the decomposition of the general element of $ARI^{al/il}_{ent}$ is fully determined at each level (s, r), but the generators themselves resist construction according the (s, r)-filtration: at most levels (s, r) there appear parasitical degrees of freedom, which get removed only when we proceed to the higher levels $(s, r + 1)$, $(s, r + 2)$ etc.

That leaves the s-filtration refined by the d-filtration ($d = s–r$). It suffers from neither drawback (- no retro-action there, at least for bicolours -) but, starting as it does from low values of d and correspondingly high values of r, it saddles us with cumbersome polynomials of r variables.

These two distinct forms which retro-action can assume call for quite distinct remedies.

For *monocolours*, the best (though by no means the only) way out of trouble is to move from the *polynomial* to the *perinomal* setting. i.e. to work with plurivariate meromorphic functions with a very specific pole structure. We show in §5 how this simple and very natural trick enforces rigidity by removing all indeterminacy not only in the stepwise construction (along the r-filtration) of canonical generators of $ARI^{al/il}_{ent}$ but also in the stepwise decomposition (again along the r-filtration) of elements of $ARI^{al/il}_{ent}$ in terms of these generators.

For *bicolours*, the key notion is *satellisation*, i.e. the replacement of the huge quantity of multizetas (consequent on the introduction of colours) by sparse 'boundary data' or 'satellites', far smaller in size yet containing all the information, and that too in algorithmically retrievable form. There are three such 'boundary systems', each self-sufficient, but all three contributing in an essential way to the overall picture. The *lower* or *root* satellisation sa retains only the bicolours of zero degree.[3] The *first upper* satellisation sa^*, retains only the monochromous bicolours, either all-white (colour 0) are all-black (colour $\frac{1}{2}$). The *second upper* satellisation sa^{**} resembles the first in outward shape, but results from a completely different construction.[4]

Two remarkable phenomena are, in combination, responsible for the success of the satellisation scheme. *First*, the basic 'symmetries' that underpin *dimorphy*[5] impose on the bicolours a strong rigidity which makes it possible to recover the 'whole' from suitable 'parts', much as harmonicity or analyticity makes it possible to recover the whole of a function from its boundary data. *Second*, in the ARI algebra and the flexion structure in general, we ob-

[3]all their partial weights s_i are therefore equal to 1.

[4]It derives from the zero-degree multizetas by a procedure known as *amplification.*

[5]They are technically known as *symmetrality/symmetrelity* when we work with the scalar multizetas, and as *symmetality/symmtrility* (resp *alternality/alternility*) when we turn to the corresponding group (resp. algebra) of generating series.

serve a quite unexpected affinity of behaviour between $\boldsymbol{v}$-dependent, discrete bimoulds[6] and $\boldsymbol{u}$-dependent, polynomial-valued bimoulds.[7] As explained in §4, this *discrete* $\leftrightarrow$ *polynomial* duality governs the whole system of correspondences between the three satellites as also between each satellite and the 'global picture'.

Specific new results.

- We give a systematic account of the three scrambling operators and their main uses.
- We introduce the full analytical machinery necessary for tackling coequational resurgence — chiefly *weighted multiplication* and *weighted convolution*; the *resurgence monomials* $\mathcal{S}^\bullet$ and $\mathcal{T}^\bullet$; the universal *tesselation* and *texture* coefficients ($tes^\bullet$ and $tex^\bullet$) .
- We survey the Bridge equations II and III through the whole range of possible situations, from linear to non-linear, from meromorphic to hyperlogarithmic to general.
- We establish the independence of the basic bicolour bialternals. Though this was a conjecture of long standing, the proof (- based on a transparent formula -) turns out to be surprisingly, almost embarrassingly simple.
- We show that the first and second upper satellites, though a priori unrelated, in fact correspond under a remarkable involution $\mathfrak{K}$. That involution respects the d- rather than the r-filtration, but we revert to the more convenient r-filtration via an explicit $d \leftrightarrow r$ exchanging isomorphism.
- We give an elegant formula for deriving the *odd-degree* components of bimoulds in $ARI^{al/il}_{ent}$ from their *even-degree* components.
- We derive the 'Green-like' formulae, based on *viscram* and *discram*, that lead from the 'boundary data' (i.e. each of the three satellites) to the full system of bicolours.

[6]more precisely, bimoulds that depend only on the colours v_j (usually denoted ϵ_j) ranging through the discrete ring $\frac{1}{2}\mathbb{Z}/\mathbb{Z}$.

[7]that depend only on the complex variables u_i.

- Turning to monocolours, we give three pairs of formulae[8] that highlight the contrast between the rigidity of the *perinomal* and the looseness of the *polynomial* framework.

- The last section, alongside reminders (§6.1) and tables (§6.3, §6.7, §6.8), presents some scattered results (§6.2, §6.6) and conjectures (§6.4, §6.5) about multizetas and the flexion structure, including a rather mysterious *arithmetical interdependence* (modulo Bernoulli related numbers) for the length-4 bialternals.

1.2 Origin and properties of *scram*.

Origin:

The scramble operator is a bimould transform

$$\text{scram} \;:\; M^\bullet \mapsto SM^\bullet \quad \textit{with} \quad SM^{\boldsymbol{w}} = \sum_{\boldsymbol{w}'} \boldsymbol{\lambda}^{\boldsymbol{w}}_{\boldsymbol{w}'}\, M^{\boldsymbol{w}'} \tag{2}$$

$$\textit{and} \qquad \boldsymbol{w} = \binom{u_1\,,\ldots,\,u_r}{v_1\,,\ldots,\,v_r} \;,\; \boldsymbol{w}' = \binom{u'_1\,,\ldots,\,u'_r}{v'_1\,,\ldots,\,v'_r} \;,\; \boldsymbol{\lambda}^{\boldsymbol{w}}_{\boldsymbol{w}'} = \pm 1$$

that we first introduced in the late 1980's for calculating the *weighted convolution* products[9] $weco^{\binom{u_1\,,\ldots,\,u_r}{c_1\,,\ldots,\,c_r}}(\xi)$ of simple polar functions $c_i(\xi) := (\xi-\alpha_i)^{-1}$. It soon gave rise to the so-called *flexion structure*, with the algebra ARI and the group $GARI$ as its centre piece. These tools were later brought to bear on multizeta arithmetics.

Construction: In the expansion (2) of $SM^\bullet$ all new indices u'_i either reduce to some original u_j or to a gapless sum of such u_j's, while all new indices v'_i either reduce to some original v_j or to a pairwise difference of (not necessarily consecutive) v_j's. Moreover, the 'scalar product' is preserved: $\sum u_i\, v_i = \sum u'_i\, v'_i$. These, incidentally, are standard features of the flexion structure, as are the shorthand notations for partial sums and pairwise differences:

$$u_{i,\ldots,j} := u_1 + \cdots + u_j \qquad , \qquad v_{i:j} := v_i - v_j \tag{3}$$

To actually define the expansion (2) we proceed by induction on r and make use of the index removal operators $cutfi^{w_0}$ and $cutla^{w_0}$ (*fi* for *first*, *la* for *last*):

[8]See Propositions 5.1, 5.2, 5.3.

[9]They are central to co-equational resurgence. See §2.2 *infra*.

$$(\mathrm{cutfi}^{w_0}\, M)^{w_1,\ldots,w_r} \;=\; \begin{cases} M^{w_2,\ldots,w_r} & \textit{if} \quad w_0 = w_1 \\ 0 & \textit{otherwise} \end{cases} \tag{4}$$

$$(\mathrm{cutla}^{w_0}\, M)^{w_1,\ldots,w_r} \;=\; \begin{cases} M^{w_1,\ldots,w_{r-1}} & \textit{if} \quad w_0 = w_r \\ 0 & \textit{otherwise} \end{cases} \tag{5}$$

We have the choice between two very dissimilar, yet equivalent inductions:

Forward induction:

Let $SM^\bullet := \mathrm{scram} M^\bullet$ and $\boldsymbol{w} = \binom{u_1,\ldots,u_r}{v_1,\ldots,v_r}$. For $r = 1$, we start the induction by imposing $SM^{w_1} := M^{w_1}$, and for $r \geqslant 2$ by imposing $\mathrm{cutla}_M^{w_0}\, SM^{\boldsymbol{w}} \equiv 0$ except for w_0 of the form $\binom{u_r}{v_r}$, $\binom{u_i}{v_i - v_{i+1}}$,$\binom{u_i}{v_i - v_{i-1}}$, in which case we set:

$$\Big(\mathrm{cutla}_M^{\binom{u_r}{v_r}} SM\Big)^{\binom{u_1,\ldots,u_r}{v_1,\ldots,v_r}} \;=\; +SM^{\binom{u_1,\ldots,u_{r-1}}{v_1,\ldots,v_{r-1}}} \tag{6}$$

$$\Big(\mathrm{cutla}_M^{\binom{u_i}{v_i - v_{i+1}}} SM\Big)^{\binom{u_1,\ldots,u_r}{v_1,\ldots,v_r}} \;=\; +SM^{\binom{u_1\;,\ldots,\;u_i + u_{i+1}\;,\ldots,\;u_r}{v_1\;,\ldots,\;\;v_{i+1}\;\;,\ldots,\;v_r}} \quad (1 \leqslant i < r) \tag{7}$$

$$\Big(\mathrm{cutla}_M^{\binom{u_i}{v_i - v_{i-1}}} SM\Big)^{\binom{u_1,\ldots,u_r}{v_1,\ldots,v_r}} \;=\; -SM^{\binom{u_1\;,\ldots,\;u_{i-1} + u_i\;,\ldots,\;u_r}{v_1\;,\ldots,\;\;v_{i-1}\;\;,\ldots,\;v_r}} \quad (1 < i \leqslant r) \tag{8}$$

The lower index M in $\mathit{cutla}_M^{w_0}$ signals that this operator is made to act, not on $SM^\bullet$, but linearly on the various $M^\bullet$-summands of the expansion (2).

Backward induction:

Let again $SM^\bullet := \mathrm{scram} M^\bullet$ and $\boldsymbol{w} = \binom{u_1,\ldots,u_r}{v_1,\ldots,v_r}$. This time, we impose $\mathrm{cutfi}_M^{w_0}\, SM^{\boldsymbol{w}} \equiv 0$ except for w_0 of the form $\binom{u_1+\ldots+u_j}{v_i}$ with $i \leqslant j \leqslant r$, in which case we set:

$$\Big(\mathrm{cutfi}_M^{\binom{u_1+\ldots+u_j}{v_i}} SM\Big)^{\boldsymbol{w}} = \mathrm{concat}\Big(\mathrm{symlin}(SM_{v_i}^{\dot{\boldsymbol{w}}}, {}^*SM_{v_i}^{\ddot{\boldsymbol{w}}}), SM^{\vec{\boldsymbol{w}}}\Big) \tag{9}$$

with $\dot{\boldsymbol{w}} = \binom{u_1,\ldots,u_{i-1}}{v_1,\ldots,v_{i-1}}$, $\ddot{\boldsymbol{w}} = \binom{u_{i+1},\ldots,u_j}{v_{i+1},\ldots,v_j}$, $\vec{\boldsymbol{w}} = \binom{u_{j+1},\ldots,u_r}{v_{j+1},\ldots,v_r}$ and

$${}^*SM^{w_1,\ldots,w_r} := (-1)^r\, SM^{w_r,\ldots,w_1} \quad , \quad SM_{v_0}^{\binom{u_1\;,\ldots,\;u_r}{v_1\;,\ldots,\;v_r}} := SM^{\binom{u_1\;,\ldots,\;u_r}{v_1 - v_0\;,\ldots,\;v_r - v_0}}$$

and with bilinear operators *concat* ('concatenation') and *symlin* ('symmetral linearisation') so defined

$$\mathrm{concat}(SM^{\boldsymbol{w^1}}, SM^{\boldsymbol{w^2}}) \;:=\; SM^{\boldsymbol{w^1}\,\boldsymbol{w^2}} \tag{10}$$

$$\mathrm{symlin}(SM^{\boldsymbol{w^1}}, SM^{\boldsymbol{w^2}}) \;:=\; \sum_{\boldsymbol{w} \in \mathrm{sha}(\boldsymbol{w^1}, \boldsymbol{w^2})} SM^{\boldsymbol{w}} \tag{11}$$

Remark: As is well known, the relation $S^{\boldsymbol{\omega^1}} S^{\boldsymbol{\omega^2}} \equiv \sum_{\boldsymbol{\omega} \in \mathrm{sha}(\boldsymbol{\omega^1}, \boldsymbol{\omega^2})} S^{\boldsymbol{\omega}}$ characterises symmetral moulds. In the backward induction, however, the rule (11) *always* applies, whether $SM^\bullet$ is symmetral or not.[10]

Analytical expression:

The backward induction makes it clear that $scram A^{w_1,\dots,w_r}$ involves $r!! := 1.3.5\dots(2.r{-}1)$ summands. Of these, $(r!!{+}1)/2$ are preceded by a plus sign, and the remaining $(r!!{-}1)/2$ by a minus sign. Thus, for $r = 1,2,3$, we find:

$$
\begin{aligned}
(\mathrm{scram}\, M)^{\left(\begin{smallmatrix} u_1 \\ v_1 \end{smallmatrix}\right)} &= M^{\left(\begin{smallmatrix} u_1 \\ v_1 \end{smallmatrix}\right)} \\
(\mathrm{scram}\, M)^{\left(\begin{smallmatrix} u_1 , & u_2 \\ v_1 , & v_2 \end{smallmatrix}\right)} &= M^{\left(\begin{smallmatrix} u_1, & u_2 \\ v_1, & v_2 \end{smallmatrix}\right)} + M^{\left(\begin{smallmatrix} u_{1,2}, & u_1 \\ v_2 & , v_{1:2} \end{smallmatrix}\right)} - M^{\left(\begin{smallmatrix} u_{1,2}, & u_2 \\ v_1 & , v_{2:1} \end{smallmatrix}\right)} \\
(\mathrm{scram}\, M)^{\left(\begin{smallmatrix} u_1, & u_2, & u_3 \\ v_1, & v_2, & v_3 \end{smallmatrix}\right)} &= +M^{\left(\begin{smallmatrix} u_1, & u_2, & u_3 \\ v_1, & v_2, & v_3 \end{smallmatrix}\right)} + M^{\left(\begin{smallmatrix} u_1, & u_{2,3}, & u_2 \\ v_1, & v_3 & , v_{2:3} \end{smallmatrix}\right)} - M^{\left(\begin{smallmatrix} u_1, & u_{2,3}, & u_3 \\ v_1, & v_2 & , v_{3:2} \end{smallmatrix}\right)} \\
&\quad + M^{\left(\begin{smallmatrix} u_{1,2}, & u_1 & , u_3 \\ v_2 & , v_{1:2}, & v_3 \end{smallmatrix}\right)} - M^{\left(\begin{smallmatrix} u_{1,2}, & u_2 & , u_3 \\ v_1 & , v_{2:1}, & v_3 \end{smallmatrix}\right)} \\
&\quad + M^{\left(\begin{smallmatrix} u_{1,2}, & u_3, & u_1 \\ v_2 & , v_3, & v_{1:2} \end{smallmatrix}\right)} - M^{\left(\begin{smallmatrix} u_{1,2}, & u_3, & u_2 \\ v_1 & , v_3, & v_{2:1} \end{smallmatrix}\right)} \\
&\quad + M^{\left(\begin{smallmatrix} u_{1,2,3}, & u_{2,3}, & u_3 \\ v_1 & , v_{2:1}, & v_{3:2} \end{smallmatrix}\right)} - M^{\left(\begin{smallmatrix} u_{1,2,3}, & u_{2,3}, & u_2 \\ v_1 & , v_{3:1}, & v_{2:3} \end{smallmatrix}\right)} + M^{\left(\begin{smallmatrix} u_{1,2,3}, & u_3 , & u_2 \\ v_1 & , v_{3:1}, & v_{2:1} \end{smallmatrix}\right)} \\
&\quad - M^{\left(\begin{smallmatrix} u_{1,2,3}, & u_1 , & u_3 \\ v_2 & , v_{1:2}, & v_{3:2} \end{smallmatrix}\right)} - M^{\left(\begin{smallmatrix} u_{1,2,3}, & u_3 , & u_1 \\ v_2 & , v_{3:2}, & v_{1:2} \end{smallmatrix}\right)} \\
&\quad + M^{\left(\begin{smallmatrix} u_{1,2,3}, & u_1 , & u_2 \\ v_3 & , v_{1:3}, & v_{2:3} \end{smallmatrix}\right)} - M^{\left(\begin{smallmatrix} u_{1,2,3}, & u_{1,2}, & u_2 \\ v_3 & , v_{1:3}, & v_{2:1} \end{smallmatrix}\right)} + M^{\left(\begin{smallmatrix} u_{1,2,3}, & u_{1,2}, & u_1 \\ v_3 & , v_{2:3}, & v_{1:2} \end{smallmatrix}\right)}
\end{aligned}
$$

Main properties.

(i) **Turning uninflected into inflected operations:**
When acting on alternals, *scram* turns the ordinary *lu* bracket into *ari*, and when acting on symmetrals, it turns ordinary mould multiplication *mu* into the *gari* product:

$$
\begin{aligned}
\mathrm{scram}\,.\,\mathrm{lu}(A^\bullet, B^\bullet) &\equiv \mathit{ari}(\mathrm{scram}.A^\bullet, \mathrm{scram}.B^\bullet) && (12) \\
\mathrm{scram}\,.\,\mathrm{mu}(R^\bullet, S^\bullet) &\equiv \mathit{gari}(\mathrm{scram}.R^\bullet, \mathrm{scram}.S^\bullet) && (13)
\end{aligned}
$$

Actually, for (13) to hold, it is enough for the second factor $S^\bullet$ be symmetral. In (12), though, both factors have to be alternal.

(ii) **Respecting simple symmetries:**

$$
\begin{aligned}
\{A^\bullet \ \mathit{alternal}\} &\implies \{\mathrm{scram}.A^\bullet \ \mathit{alternal}\} && (14) \\
\{S^\bullet \ \mathit{symmetral}\} &\implies \{\mathrm{scram}.A^\bullet \ \mathit{symmetral}\} && (15)
\end{aligned}
$$

(iii) **Creating double symmetries:**
If $A^\bullet$ is alternal and *even* separately in each w_i, them $scram.A^\bullet$ is bialternal. Likewise, if $S^\bullet$ is symmetral and *even* separately in each w_i, them $scram.S^\bullet$ is bisymmetral.

[10] In actual fact, $SM^\bullet$ is symmetral if and only if $M^\bullet$ is.

1.3 Origin and properties of *discram*.

Origin:

The operator *discram* arose almost accidentally, while searching for a means of expressing all bicolored multizetas from a very small subset – the subset of 'all-blacks'.[11] Unlike *scram*, *discram* acts not on bimoulds, but on moulds $\mathcal{M}^\bullet$.[12] Like *scram*, *discram* produces bimoulds, but of a very special sort: their lower indices $v_i = \epsilon_i$ range through $\frac{1}{2}\mathbb{Z}/\mathbb{Z}$. They are 'colours', either 0 ('white') or $\frac{1}{2}$ ('black').

$$\text{discram} \ : \ \mathcal{M}^\bullet \mapsto S^\bullet_{\mathcal{M}} \quad \textit{with} \quad S^{\boldsymbol{w}}_{\mathcal{M}} = \sum_{\boldsymbol{u}'} \boldsymbol{\lambda}^{\boldsymbol{w}}_{\boldsymbol{u}'}\, \mathcal{M}^{\boldsymbol{u}'} \tag{16}$$

$$\textit{and} \quad \begin{cases} \boldsymbol{w} = \binom{u_1\ ,\dots,\ u_r}{\epsilon_1\ ,\dots,\ \epsilon_r} & , \ \ \boldsymbol{u}' = (u'_1, \dots, u'_r) \\ \epsilon_1, \dots, \epsilon_r \in \frac{1}{2}\mathbb{Z}/\mathbb{Z} & ; \ \ \boldsymbol{\lambda}^{\boldsymbol{w}}_{\boldsymbol{u}'} = \pm 1 \end{cases}$$

Construction:

(i) We start from the expansion (2) of $scram.M^\bullet$.
(ii) To each of the sequences $\boldsymbol{w}' = \binom{u'_1\ ,\dots,\ u'_r}{v'_1\ ,\dots,\ v'_r}$ occurring on the right-hand side, we attach two elementary sequences

$$\mu(\boldsymbol{w}') = (\epsilon'_1, \dots, \epsilon'_r) \quad , \quad \nu(\boldsymbol{w}') = (\sigma'_1, \dots, \sigma'_r)$$

defined in this way:

$$\epsilon'_i \ = \ \begin{cases} 0 & \textit{if at least one } v'_k \textit{ in } \boldsymbol{w}' \textit{ is of type } v_i - v_j \\ \frac{1}{2} & \textit{otherwise} \end{cases} \tag{17}$$

$$\sigma'_i \ = \ \begin{cases} -1 & \textit{if} \quad \epsilon'_i = 0 \\ +1 & \textit{if} \quad \epsilon'_i = \frac{1}{2} \end{cases} \tag{18}$$

(iii) For each sequence $(\epsilon_1, \dots, \epsilon_r)$ we set:

$$S^{\binom{u_1\ ,\dots,\ u_r}{\epsilon_1\ ,\dots,\ \epsilon_r}}_{\mathcal{M}} \ := \sum_{\mu(\boldsymbol{w}') = (\epsilon_1, \dots, \epsilon_r)} \boldsymbol{\lambda}^{\boldsymbol{w}}_{\boldsymbol{w}'}\ \mathcal{M}^{\sigma'_1 u'_1, \dots, \sigma'_r u'_r} \tag{19}$$

[11] i.e. the subset of multizetas carrying the sole colour $\frac{1}{2}$. See §4.2.
[12] In this paper, we shall have to handle moulds nearly as often as bimoulds. As far as feasible, we shall use calligraphic capitals $\mathcal{A}^\bullet, \mathcal{B}^\bullet$... for moulds and ordinary capitals $A^\bullet, B^\bullet$... for bimoulds.

The only trivial cases are

$$S_{\mathcal{M}}^{\binom{u_1\ ,\ldots,\ u_r}{\frac{1}{2}\ ,\ldots,\ \frac{1}{2}}} \quad = \quad \mathcal{M}^{u_1,\ldots,u_r} \qquad\qquad (\text{‘}\textit{all-blacks}\text{‘}) \tag{20}$$
$$S_{\mathcal{M}}^{\binom{u_1\ ,\ldots,\ u_r}{0\ ,\ldots,\ 0}} \quad = \quad 0 \qquad\qquad (\text{‘}\textit{all-whites}\text{‘}) \tag{21}$$

For most other sequences $(\epsilon_1, \ldots, \epsilon_r)$ the right-hand side of (19) inevitably carries a rather large number of summands, since according to (16) the $r!!$ terms in the expansion of $scram.M^{\boldsymbol{w}}$ get redistributed among only 2^r sequences $(\epsilon_1, \ldots, \epsilon_r)$.

Main properties:

(i) **Turning uninflected into inflected operations:**
When acting on alternals, *scram* turns the ordinary *lu* bracket into *ari*, and when acting on symmetrals, it turns ordinary mould multiplication *mu* into the *gari* product:

$$\text{discram}\,.\,\text{lu}(\mathcal{A}^\bullet, \mathcal{B}^\bullet) \equiv \text{ari}(\text{discram}.\mathcal{A}^\bullet, \text{discram}.\mathcal{B}^\bullet) \tag{22}$$
$$\text{discram}\,.\,\text{mu}(\mathcal{R}^\bullet, \mathcal{S}^\bullet) \equiv \text{gari}(\text{discram}.\mathcal{R}^\bullet, \text{discram}.\mathcal{S}^\bullet) \tag{23}$$

Once again, for (23) to hold, it is enough for the second factor $\mathcal{S}^\bullet$ to be symmetral.
(ii) **Respecting simple symmetries:**

$$\{\mathcal{A}^\bullet\ \textit{alternal}\} \implies \{\text{discram}.\mathcal{A}^\bullet\ \textit{alternal}\} \tag{24}$$
$$\{\mathcal{S}^\bullet\ \textit{symmetral}\} \implies \{\text{discram}.\mathcal{S}^\bullet\ \textit{symmetral}\} \tag{25}$$

(iii) **Creating double symmetries:** We know of no simple, non-tautological *necessary and sufficient* condition on $\mathcal{M}^\bullet$ for $S^\bullet_{\mathcal{M}}$ to be bialternal or bisymmetral, but there is an elementary sufficient (far from necessary) condition: if $\mathcal{M}^\bullet$ is *even* separately in each w_i and alternal (resp. symmetral), then $S^\bullet_{\mathcal{M}}$ is bialternal (resp. bisymmeral).
(iv) **"Recovering the whole from a part":**
If a bimould $M^\bullet$ with lower indices $\epsilon_i \in \frac{1}{2}\mathbb{Z}/\mathbb{Z}$ is bialternal and if we set $\mathcal{M}^{u_1,\ldots,u_r} := M^{\binom{u_1\ ,\ldots,\ u_r}{\frac{1}{2}\ ,\ldots,\ \frac{1}{2}}}$, then the reconstitution identity holds:

$$(\text{discram}.\mathcal{M})^{\binom{u_1\ ,\ldots,\ u_r}{\epsilon_1\ ,\ldots,\ \epsilon_r}} \equiv M^{\binom{u_1\ ,\ldots,\ u_r}{\epsilon_1\ ,\ldots,\ \epsilon_r}} \qquad \forall(\epsilon_1, \ldots, \epsilon_r) \neq (0, \ldots, 0) \tag{26}$$

1.4 Origin and properties of *viscram*.

Origin:

Here also, the first impulse came from multizeta algebra.[13]. But although *viscram* has a definition patterned on that of *discram*, in outward shape it more closely resembles *scram*. Like *scram*, it turns bimoulds into bimoulds:

$$\text{viscram} \;\; : \;\; M^\bullet \mapsto {}^{vi}\!SM^\bullet \quad \textit{with} \quad {}^{vi}\!SM^{\boldsymbol{w}} = \sum_{\boldsymbol{w}''} \epsilon^{\boldsymbol{w}}_{\boldsymbol{w}''}\, M^{\boldsymbol{w}''} \tag{27}$$

$$\textit{and} \qquad \boldsymbol{w} = (\begin{smallmatrix} u_1 \,,\,\ldots\,,\, u_r \\ v_1 \,,\,\ldots\,,\, v_r \end{smallmatrix}) \;\; , \;\; \boldsymbol{w}'' = (\begin{smallmatrix} u''_1 \,,\,\ldots\,,\, u''_r \\ v''_1 \,,\,\ldots\,,\, v''_r \end{smallmatrix}) \;\; , \;\; \epsilon^{\boldsymbol{w}}_{\boldsymbol{w}''} = \pm 1$$

However, compared with the sequences $\boldsymbol{w}'$ of (2), the new sequences $\boldsymbol{w}''$ exhibit slight *sign* changes, which look innocuous enough but greatly enhance the properties and usefulness of *viscram*.

Construction:

We start from (2) and define $\mu(\boldsymbol{w}'), \nu(\boldsymbol{w}')$ exactly as in §1.3. But this time we retain all lower indices v'_i and merely change the signs in front of some of them.

$$ {}^{vi}\!SM^{(\begin{smallmatrix} u_1 \,,\ldots,\, u_r \\ v_1 \,,\ldots,\, v_r \end{smallmatrix})} \;\; := \;\; \sum_{\boldsymbol{w}'} \;\boldsymbol{\lambda}^{\boldsymbol{w}}_{\boldsymbol{w}'} \;\; M^{(\begin{smallmatrix} \sigma'_1 u'_1 \,,\ldots,\, \sigma'_1 u'_r \\ \sigma'_1 v'_1 \,,\ldots,\, \sigma'_1 v'_r \end{smallmatrix})} \tag{28}$$

Since the upper and lower indices undergo exactly the same sign changes, we still have conservation of the scalar product $\sum u_i\, v_i = \sum u''_i\, v''_i$ in (27).

Main properties:

(i) **Turning uninflected into inflected operations:**
When acting on *neg*-invariant[14] alternals, *viscram* turns the ordinary *lu* bracket into *ari*, and when acting on *neg*-invariant symmetrals, it turns ordinary mould multiplication *mu* into the *gari* product:

$$\text{viscram}\,.\,\text{lu}(A^\bullet, B^\bullet) \;\; \equiv \;\; \textit{ari}(\text{viscram}.A^\bullet, \text{viscram}.B^\bullet) \tag{29}$$

$$\text{viscram}\,.\,\text{mu}(R^\bullet, S^\bullet) \;\; \equiv \;\; \textit{gari}(\text{viscram}.R^\bullet, \text{viscram}.S^\bullet) \tag{30}$$

As usual, for (30) to hold, it is enough for the second factor to be symmetral.

[13] See §4.6.
[14] We recall that $neg\, M^{w_1,\ldots,w_r} := M^{-w_1,\ldots,-w_r}$.

(ii) **Respecting simple symmetries or improving on them:**

$$\{A^{\bullet}\ \textit{alternal}\} \implies \{\text{viscram}.A^{\bullet}\ \textit{alternal}\} \tag{31}$$
$$\{S^{\bullet}\ \textit{symmetral}\} \implies \{\text{viscram}.A^{\bullet}\ \textit{symmetral}\} \tag{32}$$

If on top of the simple symmetry, we impose the mild requirement of *neg*-invariance on $A^{\bullet}$ and $S^{\bullet}$, then $\textit{viscram}.A^{\bullet}$ acquires *push*-invariance on top of its alternality: this amounts to "one symmetry and a half". Likewise, $\textit{viscram}.S^{\bullet}$ acquires *spush*-invariance[15] on top of its symmetrality.
(iii) **Creating double symmetries:**
If $A^{\bullet}$ is alternal and *even* separately in each w_i, them $\textit{viscram}.A^{\bullet}$ coincides with $\textit{scram}.A^{\bullet}$ and is therefore bialternal. Likewise, if $S^{\bullet}$ is symmetral and *even* separately in each w_i, them $\textit{viscram}.S^{\bullet}$ coincides with $\textit{scram}.S^{\bullet}$, which makes it bisymmetral.
(iv) **Respecting double symmetries:**

$$\{A^{\bullet}\ \textit{bialternal}\} \implies \{\text{viscram}.A^{\boldsymbol{w}} \equiv (2^{r(\boldsymbol{w})}-1).A^{\boldsymbol{w}}\} \tag{33}$$

Here, $r(\boldsymbol{w})$ denotes of course the length of $\boldsymbol{w}$. The above relation means that, up to a simple renormalisation, the *viscram* transform leaves all bialternals invariant. This is a huge improvement on *scram*. For the rest, property (i) for *scram* is slightly stronger than (i) for *viscram*, but property (ii) for *viscram* is much stronger than (ii) for *scram*. So – advantage *viscram*!

1.5 The scrambling operators: synopsis.

Origin and progeny:

operator	*origin*	*progeny*
scram	analysis, weighted convolution	co-equational resurgence
discram	multizeta algebra	flexion structure
viscram	multizeta algebra	flexion structure

Synoptic analytical expression:

$$\begin{array}{l|l|l l}
(\text{scram}\, M)^{\binom{u_1,u_2}{v_1,v_2}} & (\text{viscram}\, M)^{\binom{u_1,u_2}{v_1,v_2}} & (\text{discram}\, \mathcal{M})^{\binom{u_1,u_2}{\epsilon_1,\epsilon_2}} & (\epsilon_1,\epsilon_2) \\
+M^{\binom{u_1\ ,\ u_2}{v_1\ ,\ v_2}} & +M^{\binom{u_1\ ,\ u_2}{v_1\ ,\ v_2}} & +\mathcal{M}^{(u_1,u_2)} & (\frac{1}{2},\frac{1}{2}) \\
+M^{\binom{u_{1,2}\ ,\ u_1}{v_2\ ,\ v_{1:2}}} & +M^{\binom{u_{1,2}\ ,\ -u_1}{v_2\ ,\ v_{2:1}}} & \overline{+\mathcal{M}^{(u_{1,2},-u_1)}} & \overline{(0,\frac{1}{2})} \\
-M^{\binom{u_{1,2}\ ,\ u_2}{v_1\ ,\ v_{2:1}}} & -M^{\binom{u_{1,2}\ ,\ -u_2}{v_1\ ,\ v_{1:2}}} & \overline{-\mathcal{M}^{(u_{1,2},-u_2)}} & \overline{(\frac{1}{2},0)}
\end{array}$$

[15] *spush*-invariance is the natural equivalent in *GARI* of *push*-invariance in *ARI*. See...

$(\mathrm{scram}\, M)^{\binom{u_1,u_2,u_3}{v_1,v_2,v_3}}$	$(\mathrm{viscram}\, M)^{\binom{u_1,u_2,u_3}{v_1,v_2,v_3}}$	$(\mathrm{discram}\, \mathcal{M})^{\binom{u_1,u_2,u_3}{\epsilon_1,\epsilon_2,\epsilon_3}}$	$(\epsilon_1,\epsilon_2,\epsilon_3)$
$+M^{\binom{u_1\,,\,u_2\,,\,u_3}{v_1\,,\,v_2\,,\,v_3}}$	$+M^{\binom{u_1\,,\,u_2\,,\,u_3}{v_1\,,\,v_2\,,\,v_3}}$	$+\mathcal{M}^{(u_1,u_2,u_3)}$	$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$
$-M^{\binom{u_1\,,\,u_{2,3}\,,\,u_3}{v_1\,,\,v_2\,,\,v_{3:2}}}$	$-M^{\binom{u_1\,,\,u_{2,3}\,,\,-u_3}{v_1\,,\,v_2\,,\,v_{2:3}}}$	$-\mathcal{M}^{(u_1,u_{2,3},-u_3)}$	$(\frac{1}{2},\frac{1}{2},0)$
$-M^{\binom{u_{1,2,3}\,,\,u_{2,3}\,,\,u_2}{v_1\,,\,v_{3:1}\,,\,v_{2:3}}}$	$-M^{\binom{u_{1,2,3}\,,\,-u_{2,3}\,,\,u_2}{v_1\,,\,v_{1:3}\,,\,v_{2:3}}}$	$-\mathcal{M}^{(u_{1,2,3},-u_{2,3},u_2)}$	....
$+M^{\binom{u_1\,,\,u_{2,3}\,,\,u_2}{v_1\,,\,v_3\,,\,v_{2:3}}}$	$+M^{\binom{u_1\,,\,u_{2,3}\,,\,-u_2}{v_1\,,\,v_3\,,\,v_{3:2}}}$	$+\mathcal{M}^{(u_1,u_{2,3},-u_2)}$	$(\frac{1}{2},0,\frac{1}{2})$
$-M^{\binom{u_{1,2}\,,\,u_2\,,\,u_3}{v_1\,,\,v_{2:1}\,,\,v_3}}$	$-M^{\binom{u_{1,2}\,,\,-u_2\,,\,u_3}{v_1\,,\,v_{1:2}\,,\,v_3}}$	$-\mathcal{M}^{u(1,2,-u_2,u_3)}$	....
$-M^{\binom{u_{1,2}\,,\,u_3\,,\,u_2}{v_1\,,\,v_3\,,\,v_{2:1}}}$	$-M^{\binom{u_{1,2}\,,\,u_3\,,\,-u_2}{v_1\,,\,v_3\,,\,v_{1:2}}}$	$-\mathcal{M}^{(u_{1,2},u_3,-u_2)}$	....
$+M^{\binom{u_{1,2,3}\,,\,u_{2,3}\,,\,u_3}{v_1\,,\,v_{2:1}\,,\,v_{3:2}}}$	$+M^{\binom{u_{1,2,3}\,,\,-u_{2,3}\,,\,u_3}{v_1\,,\,v_{1:2}\,,\,v_{3:2}}}$	$+\mathcal{M}^{(u_{1,2,3},-u_{2,3},u_3)}$	....
$+M^{\binom{u_{1,2,3}\,,\,u_{1,2}\,,\,u_1}{v_3\,,\,v_{2:3}\,,\,v_{1:2}}}$	$+M^{\binom{u_{1,2,3}\,,\,-u_{1,2}\,,\,u_1}{v_3\,,\,v_{3:2}\,,\,v_{1:2}}}$	$+\mathcal{M}^{(u_{1,2,3},-u_{1,2},u_1)}$	....
$+M^{\binom{u_{1,2}\,,\,u_1\,,\,u_3}{v_2\,,\,v_{1:2}\,,\,v_3}}$	$+M^{\binom{u_{1,2}\,,\,-u_1\,,\,u_3}{v_2\,,\,v_{2:1}\,,\,v_3}}$	$+\mathcal{M}^{(u_{1,2},-u_1,u_3)}$	$(0,\frac{1}{2},\frac{1}{2})$
$+M^{\binom{u_{1,2}\,,\,u_3\,,\,u_1}{v_2\,,\,v_3\,,\,v_{1:2}}}$	$+M^{\binom{u_{1,2}\,,\,u_3\,,\,-u_1}{v_2\,,\,v_3\,,\,v_{2:1}}}$	$+\mathcal{M}^{(u_{1,2},u_3,-u_1)}$	....
$-M^{\binom{u_{1,2,3}\,,\,u_{1,2}\,,\,u_2}{v_3\,,\,v_{1:3}\,,\,v_{2:1}}}$	$-M^{\binom{u_{1,2,3}\,,\,-u_{1,2}\,,\,u_2}{v_3\,,\,v_{3:1}\,,\,v_{2:1}}}$	$-\mathcal{M}^{(u_{1,2,3},-u_{1,2},u_2)}$	....
$+M^{\binom{u_{1,2,3}\,,\,u_3\,,\,u_2}{v_1\,,\,v_{3:1}\,,\,v_{2:1}}}$	$+M^{\binom{u_{1,2,3}\,,\,-u_3\,,\,-u_2}{v_1\,,\,v_{1:3}\,,\,v_{1:2}}}$	$+\mathcal{M}^{(u_{1,2,3},-u_3,-u_2)}$	$(\frac{1}{2},0,0)$
$-M^{\binom{u_{1,2,3}\,,\,u_1\,,\,u_3}{v_2\,,\,v_{1:2}\,,\,v_{3:2}}}$	$-M^{\binom{u_{1,2,3}\,,\,-u_1\,,\,-u_3}{v_2\,,\,v_{2:1}\,,\,v_{2:3}}}$	$-\mathcal{M}^{(u_{1,2,3},-u_1,-u_3)}$	$(0,\frac{1}{2},0)$
$-M^{\binom{u_{1,2,3}\,,\,u_3\,,\,u_1}{v_2\,,\,v_{3:2}\,,\,v_{1:2}}}$	$-M^{\binom{u_{1,2,3}\,,\,-u_3\,,\,-u_1}{v_2\,,\,v_{2:3}\,,\,v_{2:1}}}$	$-\mathcal{M}^{(u_{1,2,3},-u_3,-u_1)}$	....
$+M^{\binom{u_{1,2,3}\,,\,u_1\,,\,u_2}{v_3\,,\,v_{1:3}\,,\,v_{2:3}}}$	$+M^{\binom{u_{1,2,3}\,,\,-u_1\,,\,-u_2}{v_3\,,\,v_{3:1}\,,\,v_{3:2}}}$	$+\mathcal{M}^{(u_{1,2,3},-u_1,-u_2)}$	$(0,0,\frac{1}{2})$

Synoptic properties:

- All three scrambling operators respect simple symmetries.
- When made to act on bimoulds separately *even* in *each* index, they even turn simple into double symmetries.
- When restricted to a proper setting, they have the remarkable property of turning the uninflected operations lu, mu into their inflected counterparts $ari, gari$.
- Only *viscram* has the distinction of leaving bialternals essentially invariant: it merely multiplies them by an elementary factor $(2^{r(\bullet)}-1)$.

The above list of properties is far from exhaustive. There is in fact every reason to believe that the scrambling operators are robust mathematical objects, destined to occur in more areas than the two (– singular perturbations and multizeta algebra –) examined in this paper, and that they possess more useful variants than the three just reviewed in this section. Consider for example the statements of §2.10 about the *local* constancy and *global* non-constancy of the bimould $scram.\underline{V}^\bullet$ derived from the hyperlogarithmic mould $V^\bullet$. These statements reflect a central fact about hyperlogarithms, rather recondite perhaps but ultimately not-to-be-missed. Which again means that, had *scram* not been already in existence, any thorough-going investigation of hyperlogarithms would have led to its discovery.

2 Singularly perturbed systems and co-equational resurgence.

2.1 Equational vs co-equational resurgence.

Model problem.

Consider the following paradigmatic instance of a *doubly singular* differential system — a system not only singular in itself (i.e. relative to the time variable t) but also singularly perturbed (by a small parameter ϵ):

$$\begin{aligned} 0 &= \epsilon\, t^2\, \partial_t\, y^i + \lambda_i\, y^i + b^i(t,\epsilon,y^1,\ldots,y^\nu) \qquad (1 \leqslant i \leqslant \nu) \\ t &\sim 0 \quad (\mathit{variable}) \\ \epsilon &\sim 0 \quad (\mathit{parameter}) \end{aligned} \tag{34}$$

It is advisable, both techically and theoretically, to change to the problem's 'critical variables' z and x, i.e. to set

$$z := 1/t \sim \infty \qquad , \qquad x := 1/\epsilon \sim \infty \tag{35}$$

so as to prepare for working in the conjugate Borel planes ζ and ξ. This leads to the system:

$$\begin{aligned} \partial_z Y &= x\,\Lambda\, Y + B(z,x,Y) \qquad \mathit{with} \\ & Y = \{Y^i\}\ ,\ B = \{B^i\}\ ,\ \Lambda = \mathit{diag.matr.}\{\lambda_i\} \\ & B^i \in \mathbb{C}\{z^{-1},x^{-1},Y^1,\ldots,Y^\nu\} \;\; \mathit{or} \;\; \in \mathbb{C}\{z^{-1},Y^1,\ldots,Y^\nu\} \end{aligned} \tag{36}$$

From the viewpoint of x-resurgence, choosing the series B^i independent of x, i.e. taking them in $\mathbb{C}\{z^{-1},Y\}$ rather than $\mathbb{C}\{z^{-1},x^{-1},Y\}$, makes little difference to the resurgence pattern in the ξ-plane, and none at all to the location of the singularities. So we shall henceforth stick with this simplifying

assumption.
To respect homogeneity, we may re-write our system thus:

$$\partial_z Y^i \quad = \quad x\,\lambda_i\,Y^i + \sum_{n_j \geqslant 0\; if\; j \neq i}^{1+n_i \geqslant 0} B^i_{n_1,\ldots,n_\nu}(z)\,Y^i \prod (Y^j)^{n_j} \quad (1 \leqslant i \leqslant \nu) \tag{37}$$

or in compact form:

$$\partial_z Y^i \quad = \quad Y^i \left(\lambda_i\,x + \sum_{n_j \geqslant 0\; if\; j \neq i}^{1+n_i \geqslant 0} B^i_{\boldsymbol{n}}(z)\,Y^{\boldsymbol{n}} \right) \qquad (1 \leqslant i \leqslant \nu) \tag{38}$$

with coefficients $B^i_{\boldsymbol{n}}(z) \in \mathbb{C}\{z^{-1}\}$ analytic at infinity and x-free.
Let us assume that the multipliers λ_i are neither resonant nor quasi-resonant.[16]
The general solution, with its full set $\{\tau_1, \ldots, \tau_\nu\}$ of integration parameters, may be formally[17] expanded in powers of either z^{-1} or x^{-1}:

$$\widetilde{Y} = \widetilde{Y}(z, x, \boldsymbol{\tau}) \in \mathbb{C}[[z^{-1}\; or\; x^{-1}]] \otimes \mathbb{C}\{\tau_1 z^{\rho_1} e^{\lambda_1 z x}, \ldots, \tau_\nu z^{\rho_\nu} e^{\lambda_\nu z x}\} \tag{39}$$

with $\rho_i \in \mathbb{C}$ denoting the coefficient of z^{-1} in $B^i_{\boldsymbol{0}}(z) = B^i_{0,\ldots,0}(z)$.

To get rid of the ramifications z^{ρ_i} (which complicate the formal expansions[18] without adding anything of substance to the Analysis) we shall set not only $\rho_i \equiv 0$ but also $B^i_{\boldsymbol{0}}(z) \equiv 0$.[19]

Double divergence, double resurgence.

Separating the exponentials from the power series, we get for (38) a formal solution of type:

$$\widetilde{Y}^i(z, x, \boldsymbol{\tau}) = \widetilde{Y}^i(z, x) + \sum_{n_j \geqslant 0\; if\; j \neq i}^{1+n_i \geqslant 0} \widetilde{Y}^i_{\boldsymbol{n}}(z, x)\; \tau_i \boldsymbol{\tau}^{\boldsymbol{n}}\; e^{(\lambda_i + <\boldsymbol{n}, \boldsymbol{\lambda}>)\, z x} \tag{40}$$

As just pointed out, our formal solution $\widetilde{Y}$, or rather its components $\widetilde{Y}^i_{\boldsymbol{n}}$, can be expanded in power series of z^{-1} or x^{-1}. Both types of expansions are generically divergent yet Borel-summable, but with distinctive *singular*

[16]meaning that the combinations $-\lambda_i + \sum_{n_j \geqslant 0} n_j \lambda)$ are all $\neq 0$ and do not approximate 0 abnormally fast (diophantine condition).

[17]The tildas, as usual in resurgence theory, signal formalness. They are often omitted, when the very context implies formalness.

[18]keeping the 'residues' ρ_i would merely force us to replace the exponential blocks $e^{(\lambda_i + <\boldsymbol{n}, \boldsymbol{\lambda}>)\, z x}$ in (40) by the mixed blocks $z^{\rho_i + <\boldsymbol{n}, \boldsymbol{\rho}>}\; e^{(\lambda_i + <\boldsymbol{n}, \boldsymbol{\lambda}>)\, z x}$.

[19]Once we have set $\rho_i \equiv 0$, a simple, *analytic* change of coordinates can also remove the whole of $B^i_{\boldsymbol{0}}(z)$.

points, *singularities* and *resurgence patterns.* Some form of the Bridge equation applies in both situations, but with distinct index reservoirs $\boldsymbol{\Omega_i}$ and above all with this crucial difference: whereas the ordinary, first-order differential operators $\mathbb{A}_\omega$ that govern the z-resurgence in $\mathbf{BE_1}$ do not depend on z, the differential operators $\mathbb{P}_\omega$ that govern the x-resurgence in $\mathbf{BE_2}$, have coefficients that are themselves divergent-resurgent in x and therefore require a third Bridge equation $\mathbf{BE_3}$ for their description:

Equational resurgence: $\widetilde{Y} = \widetilde{Y}(z, x, \boldsymbol{\tau})$ (expanded in z^{-1} with x fixed)

$$\mathbf{BE_1} \ : \qquad \Delta_{\omega_0} \widetilde{Y} = \mathbb{A}_{\omega_0} \widetilde{Y} \qquad \forall \ \omega_0 \in \Omega_1 \tag{41}$$

Co-equational resurgence: $\widetilde{Y} = \widetilde{Y}(z, x, \boldsymbol{\tau})$ (expanded in x^{-1} with z fixed)

$$\mathbf{BE_2} \ : \qquad \Delta_{\omega_0} \widetilde{Y} = \widetilde{\mathbb{P}}_{\omega_0} \widetilde{Y} \qquad \forall \ \omega_0 \in \Omega_2 \tag{42}$$

$$\mathbf{BE_3} \ : \qquad \Delta_{\omega_0} \widetilde{\mathbb{P}}_{\omega_1} = F_{\omega_0,\omega_1}(\{\widetilde{\mathbb{P}}_{\omega_j}\}) \qquad \forall \ \omega_0 \in \Omega_3 \tag{43}$$

Despite these far-going differences, there is bound to be a certain kinship between the two types of resurgence, since in the special case when $B^i_{\boldsymbol{n}}(z) = \beta^i_{\boldsymbol{n}}/z$ with $\beta^i_{\boldsymbol{n}}$ scalar, the variable z and the perturbation parameter x coalesce due to the underlying homogeneousness, so that the z- and x-expansions assume the same form:

$$\widetilde{Y}^i(z, x, \boldsymbol{\tau}) \ = \ \widetilde{Y}^i(z\,x) + \sum_{n_j \geqslant 0}^{j \neq i} \sum_{n_i \geqslant -1} \widetilde{Y}^i_{\boldsymbol{n}}(z\,x)\ \tau_i \boldsymbol{\tau}^{\boldsymbol{n}}\ e^{(\lambda_i + <\boldsymbol{n}, \boldsymbol{\lambda}>)\,zx} \tag{44}$$

with $\widetilde{Y}^i(zx)$ and $\widetilde{Y}^i_{\boldsymbol{n}}(zx) \in \mathbb{C}[[(zx)^{-1}]]$.

It is this loose kinship, or lax 'duality', that justifies the label *equational* for the z-resurgence (z being the variable with respect to which we differentiate in the system (38)) and *co-equational* for the x-resurgence. *Equational resurgence* is by far the simpler of the two, since the general shape of $\mathbf{BE_1}$ with its operators $\mathbb{A}_\omega$ and their indices ω, can be inferred from purely formal considerations, directly from the differential system (38). Equations $\mathbf{BE_2}$ and $\mathbf{BE_3}$ with their index reservoirs $\boldsymbol{\Omega_2}$, $\boldsymbol{\Omega_3}$, are harder to derive, yet here too we are fortunate in having a general machinery, with a strong algebraic-combinatorial flavour to it, that addresses the general case.

The normalisers $\Theta^{\pm 1}$.

Rather than handling the general solution $\widehat{Y}$ of our system, it is often advantageous to work with the information-equivalent but more flexible *normalis-*

ing operators $\Theta^{\pm 1}$:

$$\Theta \quad = \quad 1 + \sum_{i_k, \boldsymbol{n}_k}^{1 \leqslant r} \qquad e^{|\boldsymbol{u}|xz} \, \widetilde{\mathcal{W}}^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ B^{i_1}_{\boldsymbol{n}_1} & ,\dots, & B^{i_r}_{\boldsymbol{n}_r} \end{smallmatrix}\right)}(z,x) \;\; \mathbb{D}^{i_r}_{\boldsymbol{n}_r} \dots \mathbb{D}^{i_1}_{\boldsymbol{n}_1} \tag{45}$$

$$\Theta^{-1} \quad = \quad 1 + \sum_{i_k, \boldsymbol{n}_k}^{1 \leqslant r} (-1)^r e^{|\boldsymbol{u}|xz} \, \widetilde{\mathcal{W}}^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ B^{i_1}_{\boldsymbol{n}_1} & ,\dots, & B^{i_r}_{\boldsymbol{n}_r} \end{smallmatrix}\right)}(z,x) \;\; \mathbb{D}^{i_1}_{\boldsymbol{n}_1} \dots \mathbb{D}^{i_r}_{\boldsymbol{n}_r} \tag{46}$$

$$\text{with} \qquad \begin{cases} u_k := < \boldsymbol{n}_k, \boldsymbol{\lambda} > \quad , & \mathbb{D}^{i_k}_{\boldsymbol{n}_k} := \boldsymbol{\tau}^{\boldsymbol{n}_k} \, \tau^{i_k} \partial_{\tau_{i_k}} \\ 1 \leqslant i_k \leqslant \nu \quad , & \boldsymbol{\tau}^{\boldsymbol{n}}_k \, \tau_{i_k} \in \boldsymbol{\tau}^{\mathbb{N}} \end{cases} \tag{47}$$

and with a symmetral mould $\widehat{\mathcal{W}}^{\bullet}$ inductively defined by $\widehat{\mathcal{W}}^{\varnothing} = 1$ and

$$\partial_z \Big(e^{|\boldsymbol{u}|xz} \, \widetilde{\mathcal{W}}^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ B^{i_1}_{\boldsymbol{n}_1} & ,\dots, & B^{i_r}_{\boldsymbol{n}_r} \end{smallmatrix}\right)}(z,x) \Big) = -e^{|\boldsymbol{u}|xz} \, \widetilde{\mathcal{W}}^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_{r-1} \\ B^{i_1}_{\boldsymbol{n}_1} & ,\dots, & B^{i_{r-1}}_{\boldsymbol{n}_{r-1}} \end{smallmatrix}\right)}(z,x) \, B^{i_r}_{\boldsymbol{n}_r}(z) \tag{48}$$

Since $\widehat{\mathcal{W}}^{\bullet}$ is symmetral, the operators Θ and Θ^{-1} are (mutually inverse) formal automorphisms of $\mathbb{C}[[\boldsymbol{\tau}]] := \mathbb{C}[[\tau_1, \dots, \tau_\nu]]$:

$$\Theta^{\pm 1} \Big(\widetilde{\varphi}_1(\boldsymbol{\tau}).\widetilde{\varphi}_2(\boldsymbol{\tau}) \Big) \equiv \Big(\Theta^{\pm 1} \widetilde{\varphi}_1(\boldsymbol{\tau}) \Big) \Big(\Theta^{\pm 1} \widetilde{\varphi}_2(\boldsymbol{\tau}) \Big) \qquad (\widetilde{\varphi}_i \in \mathbb{C}[[\boldsymbol{\tau}]]) \tag{49}$$

Moreover, they exchange the general solution $\widehat{Y}$ of our system (38) and the elementary general solution Y_{nor} of the corresponding (linear) normal system:

$$\partial_z Y^i_{\text{nor}} = \lambda_i \, x \, Y^i_{nor} \quad ; \quad Y_{\text{nor}}(z,x,\boldsymbol{\tau}) = \tau_i \; e^{\lambda_i \, x \, z} \qquad (1 \leqslant i \leqslant \nu) \tag{50}$$

$$\Theta \, \widetilde{Y}^i(z,x,\boldsymbol{\tau}) \equiv Y^i_{\text{nor}}(z,x,\boldsymbol{\tau}) \quad ; \quad \Theta^{-1} \, Y^i_{\text{nor}}(z,x,\boldsymbol{\tau}) \equiv \widetilde{Y}^i(z,x,\boldsymbol{\tau}) \tag{51}$$

To check this, we first observe that the induction rule (48) translates into the following interaction between ∂_z and $\Theta^{\pm}$:

$$\partial_z \Theta \quad = \Theta \quad \partial_z \; - \Big(\sum_{i,\boldsymbol{n}} e^{u\,x\,z} B^i_{\boldsymbol{n}}(z) \mathbb{D}^i_{\boldsymbol{n}} \Big) \, \Theta \qquad \big(\textit{with } u := < \boldsymbol{n}, \boldsymbol{\lambda} > \big) \tag{52}$$

$$\partial_z \Theta^{-1} \; = \Theta^{-1} \partial_z + \Theta^{-1} \Big(\sum_{i,\boldsymbol{n}} e^{u\,x\,z} B^i_{\boldsymbol{n}}(z) \mathbb{D}^i_{\boldsymbol{n}} \Big) \qquad \big(\textit{with } u := < \boldsymbol{n}, \boldsymbol{\lambda} > \big) \tag{53}$$

Next, we define a 'tentative' solution $\widetilde{Y}_{\text{ten}}$ of our basic system (38) by setting $\widetilde{Y}_{\text{ten}} := \Theta^{-1} Y_{\text{nor}}$. Applying both sides of (53) to Y_{nor}, we find successively:[20]

$$\partial_z \Theta^{-1} Y_{\text{nor}} \quad = \quad \Theta^{-1} \partial_z Y_{\text{nor}} + \Theta^{-1} \Big(\sum_{j,\boldsymbol{n}} e^{u\,x\,z} B^j_{\boldsymbol{n}}(z) \mathbb{D}^j_{\boldsymbol{n}} \Big) \, Y^i_{\text{nor}} \tag{54}$$

$$\partial_z \widetilde{Y}_{\text{ten}} \quad = \quad \Theta^{-1} \lambda_i \, x \, Y_{\text{nor}} + \Theta^{-1} \Big(\sum_{\boldsymbol{n}} B^i_{\boldsymbol{n}}(z) \, Y^i_{\text{nor}} (\boldsymbol{Y}_{\text{nor}})^{\boldsymbol{n}} \Big) \tag{55}$$

$$\partial_z \widetilde{Y}_{\text{ten}} \quad = \quad \lambda_i \, x \, \widetilde{Y}_{\text{ten}} + \sum_{\boldsymbol{n}} B^i_{\boldsymbol{n}}(z) \, \widetilde{Y}^i_{\text{ten}} (\widetilde{\boldsymbol{Y}}_{\text{ten}})^{\boldsymbol{n}} \tag{56}$$

[20] We use the fact that Θ^{-1} is an automorphism to change $\Theta^{-1}(Y_{\text{nor}})^{\boldsymbol{n}}$ to $(\Theta^{-1} Y_{\text{nor}})^{\boldsymbol{n}}$.

Since the last equation (56) coincides with our initial system (38), it follows that $\widetilde{Y}_{\text{ten}} \equiv \widetilde{Y}$, which establishes (51).

2.2 The weighted convolution product *weco*.

Elementary multilinear inputs: biresurgent monomials.

In the above expansions of $\Theta^{\pm}$, the *sensitive* (i.e. generically divergent) ingredients are *symmetral* monomials $\mathcal{W}^{\bullet}(z,x)$ carrying a two-tier indexation $\left(\begin{smallmatrix} u_i \\ B^i_{\boldsymbol{n_i}} \end{smallmatrix}\right) = \left(\begin{smallmatrix} u_i \\ b_i \end{smallmatrix}\right)$ with scalar 'frequencies' $u_i \in \mathbb{C}$ and germs $b_i(z) \in \mathbb{C}\{z^{-1}\}$ analytic at $z=\infty$. Dispensing for simplicity with the tilda and removing the exponential factors, the induction rule (48) can be rewritten as

$$(\partial_z + |\boldsymbol{u}|\,x)\ \mathcal{W}^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ b_1 & ,\dots, & b_r \end{smallmatrix}\right)}(z,x) \quad = \quad -\mathcal{W}^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_{r-1} \\ b_1 & ,\dots, & b_{r-1} \end{smallmatrix}\right)}(z,x)\, b_r(z) \tag{57}$$

Equational resurgence: Under the z-Borel tranform

$$\mathcal{B}_z\ :\ z^{-n} \mapsto \frac{\zeta^{n-1}}{(n-1)!} \quad , \quad b(z) \mapsto \widehat{b}(\zeta) \quad , \quad \mathcal{W}^{\bullet}(z,x) \mapsto \widehat{\mathcal{W}}^{\bullet}(\zeta,x)$$

the induction rule (57) becomes

$$\widehat{\mathcal{W}}^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ b_1 & ,\dots, & b_r \end{smallmatrix}\right)}(\zeta,x) = \frac{1}{\zeta - |\boldsymbol{u}|\,x} \int_0^{\zeta} \widehat{\mathcal{W}}^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_{r-1} \\ b_1 & ,\dots, & \hat{b}_{r-1} \end{smallmatrix}\right)}(\zeta_1,x)\, b_r(\zeta-\zeta_1)\, dz_1 \tag{58}$$

and readily yields all the information we need: location of singularities, Stokes constants, pattern of z-resurgence, etc.

Coequational resurgence: Under the x-Borel tranform

$$\mathcal{B}_x\ :\ x^{-n} \mapsto \frac{\xi^{n-1}}{(n-1)!} \quad , \quad \mathcal{W}^{\bullet}(z,x) \mapsto \mathcal{B}_x\mathcal{W}^{\bullet}(z,\xi)$$

things are far more complex. The induction rule takes the form of a partial differential equation:

$$(\partial_z + |\boldsymbol{u}|\,\partial_\xi)\ \mathcal{B}_x\mathcal{W}^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ b_1 & ,\dots, & b_r \end{smallmatrix}\right)}(z,\xi) \quad = \quad -\,\mathcal{B}_x\mathcal{W}^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_{r-1} \\ b_1 & ,\dots, & b_{r-1} \end{smallmatrix}\right)}(z,\xi)\, b_r(z) \tag{59}$$

$$\text{with for } r \geqslant 2 \text{ the limit condition} \quad : \quad \mathcal{B}_x\mathcal{W}^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ b_1 & ,\dots, & b_r \end{smallmatrix}\right)}(z,0) = 0 \tag{60}$$

For $r=1$, solving (59) in decreasing powers of x and then applying the Borel transform $x \to \xi$, we find:

$$\mathcal{W}^{\left(\begin{smallmatrix} u_1 \\ b_1 \end{smallmatrix}\right)}(z,x) \quad = \quad -\sum_{n\geqslant 0} (u_1\,x)^{-1-n}\,(-\partial_z)^n\, b_1(z) \quad \Longrightarrow \tag{61}$$

$$\mathcal{B}_x\mathcal{W}^{\left(\begin{smallmatrix} u_1 \\ b_1 \end{smallmatrix}\right)}(z,\xi) \quad = \quad -\sum_{n\geqslant 0} \frac{1}{u_1}\,\frac{(-\xi/u_1)^n}{n!}\,\partial_z^n\, b_1(z) \quad = \quad -\frac{1}{u_1}\, b_1(z - \frac{\xi}{u_1}) \tag{62}$$

When $r \geqslant 2$, no such simplistic formula can be expected for $\mathcal{B}_x \mathcal{W}^{\binom{u_1\ ,\dots,\ u_r}{b_1\ ,\dots,\ b_r}}(z,\xi)$, and we must take recourse to the notion of *weighted convolution*.

Weighted convolution ***weco*.**

Proposition 2.1 *For $u_i \in \mathbb{C}$ and $\widehat{c}_i(\xi) \in \mathbb{C}\{x\}$, the following integrals*

$$\mathrm{weco}^{\binom{u_1}{\widehat{c}_1}}(\xi) = \frac{1}{u_1}\,\widehat{c}_1\Big(\frac{\xi}{u_1}\Big) \tag{63}$$

$$\mathrm{weco}^{\binom{u_1\ ,\ u_2}{\widehat{c}_1\ ,\ \widehat{c}_2}}(\xi) = \int_0^{\theta_*} \widehat{c}_2(\xi_2)\, d\xi_2\, \widehat{c}_1(\xi_1)\frac{1}{u_1} \tag{64}$$

$$\textit{with} \quad \begin{cases} u_1\,\xi_1 + u_2\,\xi_2 = \xi \\ \theta_* := \xi\,(u_1+u_2)^{-1} \end{cases}$$

....

$$\mathrm{weco}^{\binom{u_1\ ,\dots,\ u_r}{\widehat{c}_1\ ,\dots,\ \widehat{c}_r}}(\xi) = \begin{cases} \int_0^{\theta_*} \widehat{c}_r(\xi_r)\, d\xi_r \int_{\xi_r}^{\theta_r} \widehat{c}_{r-1}(\xi_{r-1})\, d\xi_{r-1} \dots \\ \dots \int_{\xi_4}^{\theta_4} \widehat{c}_3(\xi_3)\, d\xi_3 \int_{\xi_3}^{\theta_3} \widehat{c}_2(\xi_2)\, d\xi_2\, \widehat{c}_1(\xi_1)\frac{1}{u_1} \end{cases} \tag{65}$$

$$\textit{with} \quad \begin{cases} u_1\,\xi_1 + \dots + u_r\,\xi_r = \xi \\ \theta_i := (\xi - (u_i\,\xi_i + \dots + u_r\,\xi_r))(u_1 + \dots + u_{i-1})^{-1} \\ \theta_* := \xi\,(u_1 + \dots + u_r)^{-1} \end{cases}$$

unambiguously define germs $\mathrm{weco}^{\binom{u_1\ ,\dots,\ u_r}{c_1\ ,\dots,\ c_r}}(\xi) \in \mathbb{C}\{\xi\}$ *provided that* $u_1 + \dots + u_i \neq 0$. *The mould* $\mathrm{weco}^\bullet$ *is symmetral relative to the (ordinary) convolution product.*

A more symmetric definition reads

$$\mathrm{weco}^{\binom{u_1\ ,\dots,\ u_r}{c_1\ ,\dots,\ c_r}}(\xi) := \int_{W^{u_1,\dots,u_r}} c_1(\xi_1)\dots c_r(\xi_r)\, d\xi_1 \dots d\xi_r \tag{66}$$

with integration on a contorted multi-path:

$$\mathcal{P}^{u_1,\dots,u_r} = \begin{cases} 0 < \xi_r < \xi_{r-1} < \dots < \xi_2 < \xi_1 \\ (u_1 + \dots + u_i)\,\xi_i + (u_{i+1}\xi_{i+1} + \dots + u_r\xi_r) < \xi \qquad (2 \leqslant i \leqslant r) \\ u_1\xi_1 + \dots u_r\xi_r = \xi \end{cases}$$

While these integral representations have their use for majorising the weighted convolution products, for establishing the *symmetrality* of the mould $\mathit{weco}^\bullet(\xi)$, even for predicting where its singlarities will project on the ξ-plane, they are pretty hopeless for finding the precise addresses of these singularities on the wildly ramified ξ-surface or for deriving the corresponding resurgence equations. Fortunately, however, when the inputs $\widehat{c}_i$ or simple poles or even polylogarithms, there exist for *weco*$^\bullet$ transparent formulae that answer all these questions, as we shall see in the sequel.

Meanwhile, *weco*$^\bullet$ answers our immediate concern– expressing the biresurgent monomials $\mathcal{W}^\bullet$ in the Borel plane ξ.

Proposition 2.2 *The Borel transforms $x \to \xi$ of the biresurgent monomials $\mathcal{W}^\bullet$ can be expressed in terms of weighted convolution products*

$$\mathcal{B}_x \mathcal{W}^{\binom{u_1\ ,\dots,\ u_r}{b_1\ ,\dots,\ b_r}}(z,\xi) = \mathrm{weco}^{\binom{u_1\ ,\dots,\ u_r}{\widehat{c}_1\ ,\dots,\ \widehat{c}_r}}(\xi) \quad \textit{with} \quad \widehat{c}_i(\xi) := -b_i(z-\xi) \tag{67}$$

with z chosen close enough to ∞ for the inputs $\widehat{c}_i(\xi)$ to be regular at $\xi = 0$.

The proof, tedious but straightforward, lies in checking that the weighted convolution integrals (65) with the inputs $\widehat{c}_i$ as in (67) do indeed verify the partial differential relation (59) together with the limit condition (60).

We may note in passing a seeming incongruity: formula (67) defines $\widehat{c}_i$ (an analytic germ at 0 in the convolutive ξ-plane) directly as a translate of b_i (an analytic germ at ∞ in the multiplicative z-plane). This interference of the two structures is a standing feature of coequational resurgence.

Weighted multiplication ***wemu*.**

Proposition 2.3 *Just as ordinary convolution is the Borel image of ordinary multiplication, weighted convolution* weco *is the Borel image of a weighted multiplication* wemu*:*

$$c_1(x),\dots,c_r(x) \quad \overset{\text{Borel}}{\longrightarrow} \quad \widehat{c}_1(\xi),\dots,\widehat{c}_r(\xi) \tag{68}$$

$$\mathrm{wemu}^{\binom{u_1\ ,\dots,u_r}{c_1\ ,\dots,\ c_r}}(x) \quad \overset{\text{Borel}}{\longrightarrow} \quad \mathrm{weco}^{\binom{u_1\ ,\dots,u_r}{\widehat{c}_1\ ,\dots\ \widehat{c}_r}}(\xi) \tag{69}$$

For $u_i > 0$ and $\Re x$ positive and large, weighted multiplication is defined by the integrals:

$$\mathrm{wemu}^{\binom{u_1\ ,\dots,u_r}{c_1\ ,\dots,\ c_r}}(x) := \frac{1}{(2\pi i)^r} \int_{-i\infty}^{+i\infty} \frac{c_1(x_1)\dots c_r(x_r)\, dx_1 \dots dx_r}{\prod_{i=1}^{i=r} \big((u_1+\dots+u_i)\,x - (x_1+\dots+x_i)\big)} \tag{70}$$

Integration is along vertical axes $\Im x_j = \alpha_j < u_j \Re x$ but with α_j large enough for $c_j(x_j)$ to be holomorphic on $\alpha_j \leqslant \Re x_j$. The definition is then extended for general weights u_i by continuous contour deformation, which is always feasible provided the partial sums $u_1 + \dots + u_j$ remain $\neq 0$.

Proof: Obvious for $r=1$ since $wemu^{\binom{u_1}{c_1}}(x)=c_1(u_1\,x)$, $weco^{\binom{u_1}{\hat{c}_1}}(x)=\frac{1}{u_1}\widehat{c}_1(\frac{\xi}{u_1})$. But even for $r>1$ the argument is straightforward:

(i) assume $u_j>0$ and $\Re x >> 1$

(ii) write $c_j(x_j)=(2\pi i)^{-1}\int \widehat{c}_j(\xi_j)\exp(-x_j\xi_j)dx_j$

(iii) calculate $weco^{\binom{u_1\ ,\dots,\ u_r}{\hat{E}_{x_1}\ ,\dots,\ \hat{E}_{x_r}}}(\xi)$ for inputs $\widehat{E}_{x_j}(\xi_j):=\exp(-x_j\xi_j)$

(iv) subject $weco^{\binom{u_1\ ,\dots,\ u_r}{\hat{E}_{x_1}\ ,\dots,\ \hat{E}_{x_r}}}(\xi)$ to the Laplace transform.

Moreover, we clearly have weighted distributivity of the x-differentiation and x-shift:

$$\partial\,\mathrm{wemu}^{\binom{u_1\ ,\dots,\ u_r}{c_1\ ,\dots,\ c_r}}(x) \equiv \sum_{1\leqslant i\leqslant r} u_i\,\mathrm{wemu}^{\binom{u_1\ ,\dots,\ u_i\ ,\dots,\ u_r}{c_1\ ,\dots,\ \partial\, c_i\ ,\dots,\ c_r}}(x) \qquad \big(\partial := \partial_x\big) \tag{71}$$

$$\underline{\boldsymbol{\tau}}\,\mathrm{wemu}^{\binom{u_1\ ,\dots,\ u_r}{c_1\ ,\dots,\ c_r}}(x) \equiv \mathrm{wemu}^{\binom{u_1\ ,\dots,\ u_r}{\underline{\tau}_1\,c_1\ ,\dots,\ \underline{\tau}_r\,c_r}}(x) \qquad \big(\underline{\boldsymbol{\tau}} := e^{\tau\,\partial}, \underline{\boldsymbol{\tau}}_i := e^{u_i\,\tau\,\partial}\big) \tag{72}$$

Alternal marking.

One can easily check that the mould transforms *almark* and *almalk*:

$$\mathrm{almark}(M)^{t_1,\dots,t_i^\dagger,\dots,t_r} := \mathrm{concat}\Big(\mathrm{symlin}(M^{t_1,\dots,t_{i-1}}, {}^*M^{t_{i+1},\dots,t_r}), M^{t_i^\dagger}\Big) \tag{73}$$

$$\mathrm{almalk}(M)^{t_1,\dots,t_i^\dagger,\dots,t_r} := \mathrm{concat}\Big(M^{t_i^\dagger}, \mathrm{symlin}({}^*M^{t_1,\dots,t_{i-1}}, M^{t_{i+1},\dots,t_r})\Big) \tag{74}$$

$$\textit{with} \qquad \begin{cases} {}^*M^{t_1,\dots,t_r} := (-1)^r\, M^{t_r,\dots,t_1} \\ \mathrm{symlin}(M^{\boldsymbol{t}'}, M^{\boldsymbol{t}''}) := \sum_{\boldsymbol{t}\in sha(\boldsymbol{t}',\boldsymbol{t}'')} M^{\boldsymbol{t}} \\ \mathrm{concat}(M^{t_1,\dots,t_i}, M^{t_{i+1},\dots,t_r} := M^{t_1,\dots,t_r} \end{cases} \tag{75}$$

turn *any* mould $M^\bullet$ into *marked* moulds $\underline{M}^\bullet, \underline{\underline{M}}^\bullet$ of alternal type. Here, *'marked'* means that we distinguish one of the indices t_i by marking it with the 'dagger' sign †. If $M^\bullet$ itself is alternal, then $M^\bullet \equiv \underline{M}^\bullet \equiv \underline{\underline{M}}^\bullet$, but otherwise all three moulds tend to be quite distinct. If on the other hand $M^\bullet$ is symmetral, as will be the case in most of our applications, then the factor ${}^*M^\bullet$ occuring in the definitions (73)-(74) coincides with the multiplicative inverse $invmu M^\bullet$.

Of course, when the marked index is $t_i^\dagger$ happens to be the first or the last, the vanishing subsequence in the definition is simply neglected. Thus, if $\underline{M}^\bullet := almark M^\bullet$, we get:

$$\begin{aligned}
\underline{M}^{t_1^\dagger,t_2,t_3,t_4} &:= -M^{t_4,t_3,t_2,t_1} \\
\underline{M}^{t_1,t_2^\dagger,t_3,t_4} &:= +M^{t_1,t_4,t_3,t_2} + M^{t_4,t_1,t_3,t_2} + M^{t_4,t_3,t_1,t_2} \\
\underline{M}^{t_1,t_2,t_3^\dagger,t_4} &:= -M^{t_1,t_2,t_4,t_3} - M^{t_1,t_4,t_2,t_3} - M^{t_4,t_1,t_2,t_3} \\
\underline{M}^{t_1,t_2,t_3,t_4^\dagger} &:= +M^{t_1,t_2,t_3,t_4}
\end{aligned}$$

The alternal operations *welo* and *welu*.

In co-equational resurgence, one constantly requires the (alternal) weighted multiplication/convolution *welu/welo* derived from *wemu/weco* by right alternal marking:

$$\mathrm{welu}^\bullet := \mathrm{almark}.(\mathrm{wemu}^\bullet) \qquad ; \qquad \mathrm{welo}^\bullet := \mathrm{almark}.(\mathrm{weco}^\bullet) \tag{76}$$

Although this defines *welu/welo* as large sums of $\frac{(r-1)!}{(i-1)!(r-i)!}$ distinct terms of type *wemu/weco*, the form of the integrals does not become significantly more complex. Thus, the passage from *wemu* to *welu* reduces to changing a fully factorisable kernel $S^\bullet$ by an equally factorisable $\underline{S}^\bullet$:

$$\begin{cases} \mathrm{wemu}^{\binom{u_1 ,.., u_i ,.., u_r}{c_1 ,.., c_i ,.., c_r}}(x) = \frac{1}{(2\pi i)^r} \int S^{\binom{u_1 ,.., u_i ,.., u_r}{x_1 ,.., x_i ,.., x_r}}(x) \prod c_i(x_i) dx_i \\ \mathrm{welu}^{\binom{u_1 ,.., u_i^\dagger ,.., u_r}{c_1 ,.., c_i{}^\dagger ,.., c_r}}(x) = \frac{1}{(2\pi i)^r} \int \underline{S}^{\binom{u_1 ,.., u_i{}^\dagger ,.., u_r}{x_1 ,.., x_i{}^\dagger ,.., x_r}}(x) \prod c_i(x_i) dx_i \end{cases} \tag{77}$$

$$\begin{cases} S^{\binom{u_1 ,.., u_i ,.., u_r}{x_1 ,.., x_i ,.., x_r}}(x) = \prod_{i=1}^{i=r} \Big((u_1 + .. + u_i)\, x - (x_1 + .. + x_i) \Big)^{-1} \\ \underline{S}^{\binom{u_1 ,.., u_i{}^\dagger ,.., u_r}{x_1 ,.., x_i{}^\dagger ,.., x_r}}(x) = \begin{cases} (-1)^{r-j} S^{\binom{u_1 ,.., u_{j-1}}{x_1 ,.., x_{j-1}}}(x)\ S^{\binom{u_r ,.., u_{j+1}}{x_r ,.., x_{j+1}}}(x) \times \\ \Big((u_1 + .. + u_r)\, x - (x_1 + .. + x_r) \Big)^{-1} \end{cases} \end{cases} \tag{78}$$

This would not be the case at all, had we defined *welu*$^\bullet$ and $\underline{S}^\bullet$ based on the left alternal marking *almalk*. Thus, of the two alternal markings, the one we require also happens to be the simpler (in this instance). Similar sweeping simplifications occur in the definition of the integration multi-path behind the alternal convolution *welo*. The reader is invited to work out the form of that multi-path for himself.

Remark 2: Simple vs weighted convolution.

The basic weighted convolution *weco* is symmetral, but otherwise devoid of any associativity-like properties. The following pair of formulae bring out the difference with ordinary convolution:

$$(\widehat{e}_{s_1} * \cdots * \widehat{e}_{s_r})(\xi) \ \equiv\ \widehat{e}_{s_1+\cdots+s_r}(\xi) \qquad \textit{with} \quad \widehat{e}_s(\xi) := \frac{\xi^{s-1}}{(s-1)!} \tag{79}$$

$$\mathrm{weco}^{\binom{u_1 ,..., u_r}{\widehat{e}_{s_1} ,..., \widehat{e}_{s_r}}}(\xi) \ \equiv\ \widehat{e}_{s_1+\cdots+s_r}(\xi)\, H^{\binom{u_1 ,..., u_r}{s_1 ,..., s_r}} \tag{80}$$

The symmetral mould $H^\bullet$ doesn't depend on ξ. For any fixed positive integers s_1, the coefficient $H^{\binom{\boldsymbol{u}}{\boldsymbol{s}}}$ is a rational funcion in the weights u_i, of the form:

$$H^{\binom{u_1 ,..., u_r}{s_1 ,..., s_r}} \ =\ P^{\binom{u_1 ,..., u_r}{s_1 ,..., s_r}} \prod_{1 \leqslant j \leqslant r} (u_1 + \cdots + u_j)^{j-1-(s_1+\cdots+s_j)} \tag{81}$$

The numerator $P^{\binom{\boldsymbol{u}}{\boldsymbol{s}}}$ is a homogeneous polynomial, with non-negative integer coefficients and with total degree in $\boldsymbol{u}$:

$$\deg(P^{\binom{u_1\ ,\dots,\ u_r}{s_1\ ,\dots,\ s_r}}) \quad = \quad \sum_{1\leqslant j\leqslant r-1} (r-j)\, s_j - \frac{1}{2}\, r\,(r-1) \quad \textit{if} \quad s_i \in \mathbb{N} \tag{82}$$

This makes $H^{\bullet}$ homogeneous in $\boldsymbol{u}$ of total degree $d = -\sum s_i$.
(i) For identical powers $s_i \equiv s > 0$ and a fixed set of weights $\{u_1, \dots, u_r\}$, the coefficients $H^{\binom{\boldsymbol{u}}{\boldsymbol{s}}}$ are always largest (resp. smallest) when the weights u_i are arranged in increasing (resp. decreasing) order.
(ii) Conversely, for identical weights $u_i \equiv u > 0$ and a fixed set of positive powers $\{s_1, \dots, s_r\}$, the coefficients $H^{\binom{\boldsymbol{u}}{\boldsymbol{s}}}$ are always largest (resp. smallest) when the weights u_i are arranged in decreasing (resp. increasing) order.
(iii) Since the weighted convolution product remains defined for all complex valued weights s_i (see below), the coefficients $H^{\binom{\boldsymbol{u}}{\boldsymbol{s}}}$ possess an analytic extension to the whole of $\mathbb{C}^{2\,r}$, single-valued in $\boldsymbol{s}$ but multivalued in $\boldsymbol{u}$, with singularity locus $\cup_i\{u_1 + ... + u_i = 0\}$.
(iv) For real positive powers s_i, the influence of the weights is strongest (resp. weakest) when the powers increase to $+\infty$ (resp. decrease to 0). In particular, $\lim_{s_i \downarrow 0} H^{\binom{u_1\ ,\dots,\ u_r}{s_1\ ,\dots,\ s_r}} = \frac{1}{r!}$ irrespective of the weights u_i.
(v) Apart from symmetrality, $\boldsymbol{u}$-homogeneousness, and the $\boldsymbol{s}$-shift relations

$$H^{\binom{u_1\ ,\dots,\ u_r}{s_1\ ,\dots,\ s_r}} = \sum_{1\leqslant i\leqslant r} \frac{u_i\, s_i}{s_1 + \cdots + s_r} H^{\binom{u_1\ ,\dots,\ u_1\ \ ,\dots,\ u_r}{s_1\ ,\dots,\ 1+s_i\ ,\dots,\ s_r}} \tag{83}$$

which simply reflect (71), the coefficients $H^{\binom{\boldsymbol{u}}{\boldsymbol{s}}}$ do not appear to be subject to other algebraic constraints.
(vi) Whereas r-multiple convolution products tend to decrease like $\mathit{Const}/r!$, r-multiple *weighted* convolution products tend to decrease like $\mathit{Const}/(r!)^2$. This is particularly obvious in the case of positive weights u_i, which precludes sign compentations in the following sum

$$\sum_{\sigma\mathfrak{S}_r} \mathrm{weco}^{\binom{u_{\sigma(1)}\ ,\dots,\ u_{\sigma(r)}}{\hat{c}_{\sigma(1)}\ ,\dots,\ \hat{c}_{\sigma(r)}}}(\xi) \equiv \left(\mathrm{weco}^{\binom{u_1}{\hat{c}_1}} * \cdots * \mathrm{weco}^{\binom{u_r}{\hat{c}_r}}\right)(\xi) \tag{84}$$

and makes each of its summands, on average, equal to $1/r!$ times the right-hand side of (84), which is itself small of order $1/r!$. This, however, appears to lead to an anomaly: the very same biresurgent monomials $\mathcal{W}^{\binom{u_1\ ,\dots,\ u_r}{b_1\ ,\dots,\ b_r}}(z,x)$ give rise, in the ζ-plane, essentially[21] to ordinary convolution products that

[21]Indeed, if we neglect the factors $(\zeta - |\boldsymbol{u}|x)^{-1}$ which have almost no impact on the rate of decrease at a given ζ, the induction (58) amounts to an ordinary convolution product with r factors.

decrease roughly like $C_1/r!$, and in the ξ-plane to weighted convolution products that decrease roughly like $C_2/(r!)^2$. The answer lies simply with the *convolands*, which differ in both cases: in the ζ-plane, we have the rather small $\widehat{b}_i(\zeta)$, and in the ξ-plane the much larger[22] $\widehat{c}_i(\xi) := -b_i(z-\xi)$. So on the whole things balance out just fine.

Remark 3: The case of non-integrable minors $\widehat{c}_i$.

Like with ordinary convolution, when dealing with convolands $\widehat{c}_j$ that are non-integrable at $\xi = 0$, we must resort to so-called majors $\check{c}_j$[23] and replace the *path integrals* (65) by suitable *loop integrals* that avoid the origin. Fortunately, in §2.11 we shall come across a formula which gives us the exact form of these loop integrals.[24]

In particular, when all convolands $\widehat{c}_j$ are equal to the convolution unit δ (dirac distribution at the origin), we find that the weighted convolution ceases to depend on the weights:

$$\mathrm{weco}^{\binom{u_1}{\delta}, \dots, \binom{u_r}{\delta}}(\xi) \equiv \frac{1}{r!}\,\delta \qquad \forall u_1, \dots, u_r \tag{85}$$

Remark 5: Weighted convolutuion and the diracs.

This last remark takes us to the case when one or several convolands $\widehat{c}_i$ are equal to δ. When only one is, and the others are regular, we find:

$$\mathrm{weco}^{\binom{u_1}{\hat{c}_1}, \dots, \binom{u_r}{\hat{c}_r}}(\xi) := \begin{cases} \mathrm{weco}^{\binom{u_1}{\hat{c}_1}, \dots, \binom{u_{r-1}}{\hat{c}_{r-1}}}(\xi) & \textit{if} \quad \widehat{c}_r = \delta \\ 0 \quad \textit{otherwise} \end{cases} \tag{86}$$

When exactly k convolands $\widehat{c}_i$ are equal to δ and the others are regular, we find:

$$\mathrm{weco}^{\binom{u_1}{\hat{c}_1}, \dots, \binom{u_r}{\hat{c}_r}}(\xi) := \begin{cases} \frac{1}{k!}\,\mathrm{weco}^{\binom{u_1}{\hat{c}_1}, \dots, \binom{u_{r-k}}{\hat{c}_{r-k}}}(\xi) & \textit{if} \quad \widehat{c}_{r-k+1} = \widehat{c}_{r-k+2} = \dots \widehat{c}_r = \delta \\ 0 \quad \textit{otherwise} \end{cases}$$

Remark 6: The case of vanishing sums $u_1 + \cdots + u_i$.

When some of the partial sums $u_1 + \cdots + u_i$ vanish, the integration multipath in (65) ceases to be finite. This either renders the integral meaningless

[22] Compare for instance $\widehat{b}_i(\zeta) := \zeta^{n_i - 1}/(n_i - 1)!$ and $b_i(z) := z^{-n_i}$.

[23] verify $\widehat{c}_j(\xi) = -\frac{1}{2\pi i}\big(\check{c}_j(\xi e^{\pi i \xi} - \check{c}_j(\xi e^{-\pi i \xi}\big)$.

[24] Unlike with ordinary convolution, the multiplicative plane is of no direct help here, since non-integrability at $\xi = 0$ in the integrals (65) automatically translates in non-integrability at $z = \pm i\infty$ in the integrals (70).

(when the germs $\widehat{c}_i$ cannot be continued to infinity) or again (when they can, but display singularities) opens the way to massive indeterminacy. In our problem, however, two fortunate circumstances save the day:
(i) in the *Second Bridge Equation*, the $\widehat{c}_i$ that occur are all of the form $\widehat{c}_i(\xi) = -b_i(z-\xi)$, with z large and b_i analytic and small at ∞. So here we have in the ξ-plane a privileged path to infinity[25], *which we choose.* We shall see in §2.8 how this translates in analytical terms: we must replace the resurgence monomials $S^{\boldsymbol{w}}(x)$ by the *amended* monomials $S^{\boldsymbol{w}}_{am}(x)$.
(ii) in the *Third Bridge Equation*, the convolands $\widehat{c}_i$ carry no z-shift, but here all terms with vanishing sums $u_1 + \cdots + u_i$ cancel out!

Remark 5: The need for a detour through combinatorics.

After the *weighted convolution* products, the other tool required for mastering coequational resurgence is a recipe for *alien-differentiating them*, more precisely, for expressing $\Delta_\omega weco^{\binom{u_1\ ,\dots,\ u_r}{\widehat{c}_1\ ,\dots,\ \widehat{c}_r}}$ and $\Delta_\omega welo^{\binom{u_1\ ,\dots,\ u_r}{\widehat{c}_1\ ,\dots,\ \widehat{c}_r}}$ in terms of weighted convolution products of the alien derivatives $\Delta_{\omega_i}\widehat{c}_i$ of the individual convolands. However, the integrals (65) that define weighted convolution, and especially their *analytic continuation in the large*[26] are so impossibly long, intricate and contorted that they defy visualisation. So an analytical-combinatorial approach is required instead, which consists in focusing on well-chosen convolands $\widehat{c}_j$, with *well-chosen* meaning two things:
(i) the c_i should be sufficiently simple to yield explicit weighted convolution products
(ii) they should be numerous enough to approximate all ramified functions.

Fortunately, there is a set of functions that meets both conditions and that will eventually lead us to the rule for alien-differentiating the convolution products: they are the hyperlogarithms. The next sections (§2.3 through §2.8) will be devoted to them or to constructions based on them.

2.3 The elementary monomials $\mathcal{V}^\bullet(z)$ and monics $V^\bullet$.

The z-resurgence ('equational'), which manifests in the dual ζ-plane, turns out to be totally independent of what singularities the coefficients $B^i_{\boldsymbol{n}}(z)$ of our model system (38) may or may not possess: they depend only on its 'multipliers' λ_i. The x-resurgence ('co-equational'), however, which manifests in the dual ξ-plane, depends on both the multipliers λ_i *and* the singularities of the $B^i_{\boldsymbol{n}}(z)$, which live directly in the z-plane, *at* or *over* some points α_j.

[25]namely $\arg(z-\xi) = \arg(z)$.

[26]technically: the *weightedly self-symmetrical* and *self-symmetrically shrinkable* multipaths that we would have to consider for a direct 'geometric' treatment.

The same holds for our resurgence-carrying monomials $\mathcal{W}^\bullet$: the singularities of $\mathcal{B}_z\,\mathcal{W}$ in the ζ-plane depend only on the weights u_i, while those of $\mathcal{B}_x\,\mathcal{W}$ in the ξ-plane depend on the u_i's *and* on the singularities α_i of the coefficients $b_i(z)$ in the z-plane. More concretely, the former singularities lie *over* points of the form $x\,(u_1+\cdots+u_i)$ and the latter *over* subtle bilinear combinations of the u_i's and the differences $z-\alpha_i$.

So we find ourselves once again facing a highly unusual but inescapable interference of two structures:
(i) the *multiplicative* structure, which leaves the singularities in place,
(ii) the *convolutive* structure, which *adds* singularities, in the sense that:
(singularity over ω_1)$*$(singularity over ω_2)$\Rightarrow$ (singularities over $\omega_1+\omega_2$).

Then, messing up things still further, we must contend with the *weighted* convolution *weco*, which also *adds* singularities, but via weighted rather than straightforward sums. This forces us to juggle two systems of notation:

- *incremental*, with sequences $(\omega_1,\ldots,\omega_r)$ $\qquad(\omega_i=\alpha_i-\alpha_{i-1})$
- *positional*, with sequences $[\alpha_1,\ldots,\alpha_r]$ $\qquad(\alpha_i=\omega_1+\ldots+\omega_i)$

The ideal tool for understanding this hybrid structure is the *hyperlogarithms*, with their *two encodings*[27], their stability under *two products*[28] and *two sets of exotic derivations* [29] and, not least, their *density* property: any given resurgent function in the Borel plane is the limit, uniformly on any compact set of its Riemann surface, of a suitable series of hyperlogarithms. Here are the main definitions and properties:

Hyperlogarithms in the $\boldsymbol{\alpha}$ and $\boldsymbol{\omega}$-encodings:

$$\overset{\frown}{\mathcal{V}}^{[\alpha_1,\ldots,\alpha_r]}(\tau) \;:=\; \int_0^\tau \frac{d\tau_r}{\tau_r-\alpha_r}\cdots\int_0^{\tau_3}\frac{d\tau_2}{\tau_2-\alpha_2}\int_0^{\tau_2}\frac{d\tau_1}{\tau_1-\alpha_1} \tag{87}$$

$$\overset{\frown}{\mathcal{V}}^{\omega_1,\ldots,\omega_r}(\tau) \;\equiv\; \overset{\frown}{\mathcal{V}}^{[\alpha_1,\ldots,\alpha_r]}(\tau)\quad \textit{with}\quad \alpha_i\equiv\omega_1+\ldots+\omega_i\quad(\forall i) \tag{88}$$

$$\widehat{\mathcal{V}}^{[\alpha_1,\ldots,\alpha_r]}(\tau) \;:=\; \partial_\tau\,\overset{\frown}{\mathcal{V}}^{[\alpha_1,\ldots,\alpha_r]}(\tau) \tag{89}$$

$$\widehat{\mathcal{V}}^{\omega_1,\ldots,\omega_r}(\tau) \;:=\; \partial_\tau\,\overset{\frown}{\mathcal{V}}^{\omega_1,\ldots,\omega_r}(\tau) \tag{90}$$

[27] i.e. *incremental* and *positional*.
[28] i.e. ordinary *pointwise multiplication* and *convolution*.
[29] i.e. the *alien derivations* Δ_{ω_0} and the less important *foreign derivations* ∇_{ω_0} (which shall play no part in this paper).

Functional dimorphy:

$$\big(\widetilde{\mathcal{V}}^{[\boldsymbol{\alpha}']} \,.\, \widetilde{\mathcal{V}}^{[\boldsymbol{\alpha}'']}\big)(\tau) \;\equiv\; \sum_{\boldsymbol{\alpha}\in \mathrm{sha}(\boldsymbol{\alpha}',\boldsymbol{\alpha}'')} \widetilde{\mathcal{V}}^{[\boldsymbol{\alpha}]}(\tau) \tag{91}$$

$$\big(\widetilde{\mathcal{V}}^{\boldsymbol{\omega}'} \,\widetilde{*}\, \widetilde{\mathcal{V}}^{\boldsymbol{\omega}''}\big)(\tau) \;\equiv\; \sum_{\boldsymbol{\omega}\in \mathrm{sha}(\boldsymbol{\omega}',\boldsymbol{\omega}'')} \widetilde{\mathcal{V}}^{\boldsymbol{\omega}}(\tau) \tag{92}$$

$$\big(\widehat{\mathcal{V}}^{\boldsymbol{\omega}'} \,\widehat{*}\, \widehat{\mathcal{V}}^{\boldsymbol{\omega}''}\big)(\tau) \;\equiv\; \sum_{\boldsymbol{\omega}\in \mathrm{sha}(\boldsymbol{\omega}',\boldsymbol{\omega}'')} \widehat{\mathcal{V}}^{\boldsymbol{\omega}}(\tau) \tag{93}$$

(91) says that $\widetilde{\mathcal{V}}^{[\bullet]}$ is symmetral relative to pointwise multiplication. (92) and (93) say that $\widetilde{\mathcal{V}}^{\bullet}$ and $\widehat{\mathcal{V}}^{\bullet}$ are symmetral relative to the convolutions $\widetilde{*}$ and $\widehat{*}$ respectively.

Remark 1: Here $\widetilde{*}$ stands for the convolution

$$(\widetilde{\varphi}_1 \,\widetilde{*}\, \widetilde{\varphi}_2)(\tau) := \int_0^\tau \widetilde{\varphi}_1(\tau-\tau_2)\; d\,\widetilde{\varphi}_2(\tau_2) \tag{94}$$

whose unit (namely $\widetilde{\varphi}\,(\tau) \equiv 1$) coincides with the unit of point-wise multiplication – a definite advantage in this context. To fall back on the more familiar convolution $\hat{*}$ or simply $*$ (whose unit is the dirac at 0):

$$(\widehat{\varphi}_1 \,\widehat{*}\, \widehat{\varphi}_2)(\tau) := \int_0^\tau \widehat{\varphi}_1(\tau-\tau_2)\; \widehat{\varphi}_2(\tau_2)\, d\tau_2 \tag{95}$$

it is enough to change $\widetilde{\varphi}_i\,(\tau)$ to $\widehat{\varphi}_i(\tau) := \partial_\tau\, \widetilde{\varphi}_i\,(\tau)$.

Remark 2: When some α_i's coincide or, equivalently, when some ω_i-sums vanish, the definition (87) remains in force, but the conversion rule (88) has to be slightly modified.[30] Indeed, in the extreme case when all α_i's and therefore all ω_i's vanish, to ensure the double *symmetrality*, the definitions have to be:

$$\widetilde{\mathcal{V}}^{\overbrace{[\,0,\ldots,0\,]}^{r \text{ times}}}(\tau) \quad = \frac{(\log\tau)^r}{r!} \qquad\qquad (\boldsymbol{\alpha}\text{-}encoding) \tag{96}$$

$$\widetilde{\mathcal{V}}^{\overbrace{0,\ldots,0}^{r \text{ times}}}(\tau) \;= \frac{(\log\tau)^r}{r!} + \cdots = \Big[(-\partial_\sigma)^r \big(\tfrac{\tau^\sigma}{\Gamma(1+\sigma)}\big)\Big]_{\sigma=0} \quad (\boldsymbol{\omega}\text{-}encoding) \tag{97}$$

with the dots in (97) standing for a polynomial in $\log\tau$ of degree $< r$.

[30]The modification is imposed by the need to adopt two different *re-normalisations* in presence of divergence. It has an exact analogue for multizetas, namely the factor $mono^\bullet$ which tweaks the conversion rule for $zag^\bullet$ and $zig^\bullet$. See §3.2.

Hyperlogarithmic monics.

In the *incremental* encoding, the hyperlogarithmic monics $V^\bullet$ are defined inductively by:

$$\Delta_{\omega_1+\cdots+\omega_r}\mathcal{V}^{\omega_1,\ldots,\omega_r}(z) = V^{\omega_1,\ldots,\omega_r} + \sum_{\omega_{i+1}+\ldots+\omega_r=0} V^{\omega_1,\ldots,\omega_i}\,\mathcal{V}^{\omega_{i+i},\ldots,\omega_r}(z) \tag{98}$$

and in the *positional* encoding by the usual re-indexation:

$$V^{[\alpha_1,\ldots,\alpha_r]} \equiv V^{\alpha_1,\alpha_2-\alpha_1,\ldots,\alpha_r-\alpha_{r-1}} \tag{99}$$

The hyperlogarithmic monics are central to *equational resurgence*, where they serve as elementary building blocks in the calculation of the Stokes constants, and to *co-equational resurgence*, where they enter the definition of the important *tesselation* and *texture* coefficients.

2.4 Index dependence of $\mathcal{V}^\bullet(z)$ and $V^\bullet$.

In the sequel, we shall have repeatedly to differentiate the hyperlogarithmic monomials and monics with respect to their variable *and* their indices, and that too in both models (multiplicative and convolutive) and in both encodings (incremental and positional). So we collect in this section the main formulae:

Monomials in incremental indexation.

$$\begin{aligned}
\omega_1\partial_{\omega_1}\mathcal{V}^{\omega_1}(z) = z\partial_z\mathcal{V}^{\omega_1}(z) &= -1-\omega_1\,z\,\mathcal{V}^{\omega_1}(z)\\
\omega_1(\partial_{\omega_1}+z)\,\mathcal{V}^{\omega_1,\ldots,\omega_r}(z) &= -\mathcal{V}^{\omega_1+\omega_2,\ldots,\omega_r}(z)\\
\omega_j(\partial_{\omega_j}+z)\,\mathcal{V}^{\omega_1,\ldots,\omega_r}(z) &= +\mathcal{V}^{\omega_1,\ldots,\omega_{j-1}+\omega_j,\ldots,\omega_r}(z)-\mathcal{V}^{\omega_1,\ldots,\omega_j+\omega_{j+1},\ldots,\omega_r}(z)\\
\omega_r(\partial_{\omega_r}+z)\,\mathcal{V}^{\omega_1,\ldots,\omega_r}(z) &= +\mathcal{V}^{\omega_1,\ldots,\omega_{r-1}+\omega_r}(z)-\mathcal{V}^{\omega_1,\ldots,\omega_{r-1}}(z)\\
z(\partial_z+|\boldsymbol{\omega}|)\,\mathcal{V}^{\omega_1,\ldots,\omega_r}(z) &= -\mathcal{V}^{\omega_1,\ldots,\omega_{r-1}}(z)
\end{aligned}$$

$$\begin{aligned}
\omega_1\partial_{\omega_1}\widehat{\mathcal{V}}^{\omega_1}(\zeta) = -\zeta\partial_\zeta\widehat{\mathcal{V}}^{\omega_1}(\zeta) &= -\,\zeta\,(\zeta-\omega_1)^{-1}\\
\omega_1(\partial_{\omega_1}+\partial_\zeta)\,\widehat{\mathcal{V}}^{\omega_1,\ldots,\omega_r}(\zeta) &= -\,\widehat{\mathcal{V}}^{\omega_1+\omega_2,\ldots,\omega_r}(\zeta)\\
\omega_j(\partial_{\omega_j}+\partial_\zeta)\,\widehat{\mathcal{V}}^{\omega_1,\ldots,\omega_r}(\zeta) &= +\,\widehat{\mathcal{V}}^{\omega_1,\ldots,\omega_{j-1}+\omega_j,\ldots,\omega_r}(\zeta)-\widehat{\mathcal{V}}^{\omega_1,\ldots,\omega_j+\omega_{j+1},\ldots,\omega_r}(\zeta)\\
\omega_r(\partial_{\omega_r}+\partial_\zeta)\,\widehat{\mathcal{V}}^{\omega_1,\ldots,\omega_r}(\zeta) &= +\,\widehat{\mathcal{V}}^{\omega_1,\ldots,\omega_{r-1}+\omega_r}(\zeta)-\widehat{\mathcal{V}}^{\omega_1,\ldots,\omega_{r-1}}(\zeta)\\
(\zeta-|\boldsymbol{\omega}|)\,\partial_\zeta\,\widehat{\mathcal{V}}^{\omega_1,\ldots,\omega_r}(\zeta) &= -\,\widehat{\mathcal{V}}^{\omega_1,\ldots,\omega_{r-1}}(\zeta)
\end{aligned}$$

Monics in incremental indexation.

$$\begin{aligned}
\omega_1\partial_{\omega_1}V^{\omega_1} &= 0,\\
\omega_1\partial_{\omega_1}V^{\omega_1,\omega_2} &= -V^{\omega_1+\omega_2} = -1\\
\omega_2\partial_{\omega_2}V^{\omega_1,\omega_2} &= +V^{\omega_1+\omega_2} = +1\\
\omega_1\partial_{\omega_1}V^{\omega_1,\ldots,\omega_r} &= -V^{\omega_1+\omega_2,\ldots,\omega_r}\\
\omega_j\partial_{\omega_j}V^{\omega_1,\ldots,\omega_r} &= +V^{\omega_1,\ldots,\omega_{j-1}+\omega_j,\ldots,\omega_r} - V^{\omega_1,\ldots,\omega_j+\omega_{j+1},\ldots,\omega_r}\\
\omega_r\partial_{\omega_r}V^{\omega_1,\ldots,\omega_r} &= +V^{\omega_1,\ldots,\omega_{r-1}+\omega_r}
\end{aligned}$$

Monomials in positional indexation.

$$\begin{aligned}
\partial_{\alpha_1}\,\overset{\frown}{\mathcal{V}}^{[\alpha_1]}(\zeta) &= (\alpha_1-\zeta)^{-1} - (\alpha_1)^{-1}\\
\partial_{\zeta}\,\overset{\frown}{\mathcal{V}}^{[\alpha_1]}(\zeta) &= (\zeta-\alpha_1)^{-1}\\
\partial_{\alpha_1}\,\overset{\frown}{\mathcal{V}}^{[\alpha_1,\ldots,\alpha_r]}(\zeta) &= \begin{cases} -\,\overset{\frown}{\mathcal{V}}^{[\widehat{\alpha}_1,\alpha_2,\ldots,\alpha_r]}(\zeta)\,\big(\alpha_1^{-1}+(\alpha_2-\alpha_1)^{-1}\big)\\ +\,\overset{\frown}{\mathcal{V}}^{[\alpha_1,\widehat{\alpha}_2,\ldots,\alpha_r]}(\zeta)\,(\alpha_2-\alpha_1)^{-1}\end{cases}\\
\partial_{\alpha_j}\,\overset{\frown}{\mathcal{V}}^{[\alpha_1,\ldots,\alpha_r]}(\zeta) &= \begin{cases} +\,\overset{\frown}{\mathcal{V}}^{[\ldots,\widehat{\alpha}_{j-1},\alpha_j,\alpha_{j+1},\ldots]}(\zeta)\,(\alpha_j-\alpha_{j-1})^{-1}\\ -\,\overset{\frown}{\mathcal{V}}^{[\ldots,\alpha_{j-1},\widehat{\alpha}_j,\alpha_{j+1},\ldots]}(\zeta)\,\big((\alpha_j-\alpha_{j-1})^{-1}+(\alpha_{j+1}-\alpha_j)^{-1}\big)\\ +\,\overset{\frown}{\mathcal{V}}^{[\ldots,\alpha_{j-1},\alpha_j,\widehat{\alpha}_{j+1},\ldots]}(\zeta)\,(\alpha_{j+1}-\alpha_j)^{-1}\end{cases}\\
\partial_{\alpha_r}\,\overset{\frown}{\mathcal{V}}^{[\alpha_1,\ldots,\alpha_r]}(\zeta) &= \begin{cases} +\,\overset{\frown}{\mathcal{V}}^{[\alpha_1,\ldots,\widehat{\alpha}_{r-1},\alpha_r]}(\zeta)\,(\alpha_r-\alpha_{r-1})^{-1}\\ -\,\overset{\frown}{\mathcal{V}}^{[\alpha_1,\ldots,\alpha_{r-1},\widehat{\alpha}_r]}(\zeta)\,\big((\alpha_r-\alpha_{r-1})^{-1}+(\zeta-\alpha_r)^{-1}\big)\end{cases}\\
\partial_{\zeta}\,\overset{\frown}{\mathcal{V}}^{[\alpha_1,\ldots,\alpha_r]}(\zeta) &= +\,\overset{\frown}{\mathcal{V}}^{[\alpha_1,\ldots,\alpha_{r-1},\widehat{\alpha}_r]}(\zeta)\,(\zeta-\alpha_r)^{-1}
\end{aligned}$$

The hat above $\widehat{v}_j$ always signals the omission of v_j.

Monics in positional indexation.

$$\begin{aligned}
\partial_{\alpha_1} V^{[\alpha_1,\alpha_2]} &= -V^{[\widehat{\alpha}_1,\alpha_2]}\big((\alpha_1)^{-1}+(\alpha_2-\alpha_1)^{-1}\big) = -(\alpha_1)^{-1}-(\alpha_2-\alpha_1)^{-1}\\
\partial_{\alpha_2} V^{[\alpha_1,\alpha_2]} &= +V^{[\widehat{\alpha}_1,\alpha_2]}(\alpha_2-\alpha_1)^{-1} = (\alpha_2-\alpha_1)^{-1}\\
\partial_{\alpha_1} V^{[\alpha_1,\dots,\alpha_r]} &= \begin{cases} -V^{[\widehat{\alpha}_1,\alpha_2,\dots,\alpha_r]}\,\big(\alpha_1^{-1}+(\alpha_2-\alpha_1)^{-1}\big)\\ +V^{[\alpha_1,\widehat{\alpha}_2,\dots,\alpha_r]}\,(\alpha_2-\alpha_1)^{-1}\end{cases}\\
\partial_{\alpha_j} V^{[\alpha_1,\dots,\alpha_r]} &= \begin{cases} +V^{[\dots,\widehat{\alpha}_{j-1},\alpha_j,\alpha_{j+1},\dots]}\,(\alpha_j-\alpha_{j-1})^{-1}\\ -V^{[\dots,\alpha_{j-1},\widehat{\alpha}_j,\alpha_{j+1},\dots]}\big((\alpha_j-\alpha_{j-1})^{-1}+(\alpha_{j+1}-\alpha_j)^{-1}\big)\\ +V^{[\dots,\alpha_{j-1},\alpha_j,\widehat{\alpha}_{j+1},\dots]}\,(\alpha_{j+1}-\alpha_j)^{-1}\end{cases}\\
\partial_{\alpha_{r-1}} V^{[\alpha_1,\dots,\alpha_r]} &= \begin{cases} +V^{[\dots,\widehat{\alpha}_{r-2},\alpha_{r-1},\alpha_r]}\,(\alpha_{r-1}-\alpha_{r-2})^{-1}\\ -V^{[\dots,\alpha_{r-2},\widehat{\alpha}_{r-1},\alpha_r]}\big((\alpha_{r-1}-\alpha_{r-2})^{-1}+(\alpha_r-\alpha_{r-1})^{-1}\big)\end{cases}\\
\partial_{\alpha_r} V^{[\alpha_1,\dots,\alpha_r]} &= +V^{[\alpha_1,\dots,\widehat{\alpha}_{r-1},\alpha_r]}\,(\alpha_r-\alpha_{r-1})^{-1}
\end{aligned}$$

Transition equations for the monics.

Outside a finite number of singular *points*, the resurgence monomials $\mathcal{V}^\bullet$ are ramified, holomorphic functions of their indices ω_i or α_i and of their variable z (in the multiplicative plane) or ζ (in the Borel plane). Not so the corresponding monics $V^\bullet$: these are uniform, non-ramified analytic functions of their indices on a number of domains of $\mathbb{C}^r$, but undergo discontinuous changes of determination from domain to domain,[31] according to the formula:

$$D_{\frac{\omega_1+\dots+\omega_i}{\omega_{i+1}+\dots+\omega_r}} V^{\omega_1,\dots,\omega_r} \equiv 2\pi i\; V^{\omega_1,\dots,\omega_i}\, V^{\omega_{i+1},\dots,\omega_r} \tag{100}$$

$$D_{\frac{\alpha_i}{\alpha_r}} V^{[\alpha_1,\dots,\alpha_r]} = D_{\frac{\alpha_i}{\alpha_r-\alpha_i}} V^{[\alpha_1,\dots,\alpha_r]} \equiv 2\pi i\; V^{[\alpha_1,\dots,\alpha_i]}\, V^{[\alpha_{i+1}-\alpha_i,\dots,\alpha_r-\alpha_i]} \tag{101}$$

with jump operators

$$D_x\, F(x) := \lim_{\epsilon\to 0}(F(x+i\,\epsilon)-F(x-i\,\epsilon)) \qquad (t,\epsilon\in\mathbb{R}^+) \tag{102}$$

2.5 The special monomials $\mathcal{S}^\bullet(x)$.

To construct the monomials $\mathcal{S}^\bullet(x)$ and the associated tesselation coefficients $tes^\bullet$, we first turn the moulds $\mathcal{V}^\bullet(x), V^\bullet$ into bimoulds $\mathcal{V}^\bullet(x), \underline{V}^\bullet$ and then

[31] This reason lies their definition (**??**): it involves the operators Δ_{ω_0}, which are themselves uniformly defined for all $\omega_0 \in \mathbb{C}_\bullet := \widetilde{\mathbb{C}-\{0\}}$, but whose action on a given resurgent function is of course discontinuous in ω_0

subject them to the scramble transform:

$$\mathcal{S}^\bullet(x) := \mathrm{scram}\,.\,\underline{\mathcal{V}}^\bullet(x) \quad \textit{with} \quad \underline{\mathcal{V}}^{\binom{u_1\ ,\ldots,\ u_r}{v_1\ ,\ldots,\ v_r}}(x) := \mathcal{V}^{u_1v_1,\ldots,u_rv_r}(x) \tag{103}$$

$$\mathrm{tes}^\bullet = \mathrm{S}^\bullet := \mathrm{scram}\,.\,\underline{\mathrm{V}}^\bullet \quad \textit{with} \quad \underline{\mathrm{V}}^{\binom{u_1\ ,\ldots,\ u_r}{v_1\ ,\ldots,\ v_r}} := \mathrm{V}^{u_1v_1,\ldots,u_rv_r} \tag{104}$$

Thus, we the usual shorthand $u_{1,2} := u_1 + u_2, v_{1:2} := v_1 - v_2$, we get:

$$\mathcal{S}^{\binom{u_1}{v_1}}(x) \ := \ \mathcal{V}^{u_1v_1}(x)$$

$$\mathcal{S}^{\binom{u_1\ ,\ u_2}{v_1\ ,\ v_2}}(x) \ := \ \mathcal{V}^{u_1\,v_1\,,\,u_2\,v_2}(x) - \mathcal{V}^{u_{1,2}\,v_1\,,\,u_2\,v_{2:1}}(x) + \mathcal{V}^{u_{1,2}\,v_2\,,\,u_1\,v_{1:2}}(x)$$

Proposition 2.4 (Weighted convolution for polar inputs) .
We assume here that all partial sums $u_1 + \cdots + u_i$ are $\neq 0$, so that all integration bounds θ_i in (65) are finite. Then the weighted convolution of simple polar functions $\pi_i(\xi) = (\xi - \alpha_i)^{-1}$ coincides with the x-Borel transform $\widehat{\mathcal{S}}^\bullet(\xi)$ of the bimould $\mathcal{S}^\bullet(x)$ for indices $w_i = \binom{u_i}{\alpha_i}$. Similarly, the bi-resurgent monomials[32] *$\mathcal{W}^\bullet(z,x)$ of (57) with polar inputs $b_i(z) := (z-\alpha_i)^{-1}$, coincide with the bimouls $\mathcal{S}^\bullet(x)$ for indices $w_i = \binom{u_i}{z-\alpha_i}$. In other words:*

$$\mathrm{weco}^{\binom{u_1\ ,\ldots,\ u_r}{\pi_1\ ,\ldots,\ \pi_r}}(\xi) = \widehat{\mathcal{S}}^{\binom{u_1\ ,\ldots,\ u_r}{\alpha_1\ ,\ldots,\ \alpha_r}}(\xi) \qquad \textit{with} \quad \pi_i(\xi) = \frac{1}{\xi - \alpha_i} \tag{105}$$

$$\mathcal{W}^{\binom{u_1\ ,\ldots,\ u_r}{b_1\ ,\ldots,\ b_r}}(z,x) = \mathcal{S}^{\binom{u_1\ ,\ldots,\ u_r}{z-\alpha_1\ ,\ldots,\ z-\alpha_r}}(x) \quad \textit{with} \quad b_i(z) = \frac{1}{z-\alpha_i} \tag{106}$$

Sketch of proof: Based on the rules of §2.4 for the ω_i-differentiation of the hyperlogarithmic monomials $\mathcal{V}$, we find that the $\mathcal{S}^\bullet(x)$, defined as superpositions of $\mathcal{V}(x)$-monomials, verify

$$(\partial_z + |\boldsymbol{u}(\bullet)|\,x)\,\mathcal{S}^\bullet(x) \ = \ -\mathcal{S}^\bullet(x) \times \mathcal{J}^\bullet \tag{107}$$

$$\mathcal{J}^{w_1} := \frac{1}{v_1} \qquad , \qquad \mathcal{J}^{w_1,\ldots,w_r} = 0 \quad \textit{if} \quad r \neq 1 \tag{108}$$

2.6 The generalised scramble.

Approximating ramified z-functions by hyperlogarithms.

Our singular, singularly perturbed model system (38) may, instead of meromorphic coefficients $B^i_{n_i}(z)$, possess coefficients which, though analytic at ∞ on the Riemann sphere, are ramified away from ∞. The proper framework for approximating such creatures is the space of hyperlogarithmic functions:

$$\overset{\frown}{\mathfrak{V}}{}^{[\alpha_1,\ldots,\alpha_r]}(\zeta) \ := \ \overset{\frown}{\mathcal{V}}{}^{[\alpha_1^{-1},\ldots,\alpha_r^{-1}]}(\zeta^{-1}) \tag{109}$$

$$\widehat{\mathfrak{V}}^{[\alpha_1,\ldots,\alpha_r]}(\zeta) \ := \ \widehat{\mathcal{V}}^{[\alpha_1^{-1},\ldots,\alpha_r^{-1}]}(\zeta^{-1}) \tag{110}$$

[32] viewed as resurgent functions of their second variable x, in any of the multiplicative models – formal or geometric.

considered in the Borel plane, but with ζ replaced by z ! This overlap of *convolutive* and *multiplicative* structure, of ζ- and z-planes, should not surprise us in view of the remarks at the beginning of §2.3.The theoretical superiority $\widehat{\mathfrak{V}}^{\bullet}$ and $\widehat{\mathfrak{V}}^{\bullet}$ over $\widehat{\mathcal{V}}^{\bullet}$ and $\widehat{\mathcal{V}}^{\bullet}$ is also clear: while the two sets of hyperlogarithms have exactly the same singularities[33] and are easily convertible into one another, *the former are regular at infinity, the latter ramified.*[34]

In this brief paper, however, to avoid the headache of yet another system of monomials, we shall stick with the familiar hyperlogarithms $\widehat{\mathcal{V}}^{\bullet}$ and $\widehat{\mathcal{V}}^{\bullet}$. In a sense, this shall enable us to tackle the more general situation of coefficients $B^i_{n_i}(z)$ in $\mathbb{C}\{z^{-1}\}\otimes\mathbb{C}[[\log z]]$ rather than in $\mathbb{C}\{z^{-1}\}$. But there is a downside: our hyperlogarithms having no natural, privileged determination at ∞ in their Borel plane, we shall have to specify one.[35]

From indices $w_i = \binom{u_i}{v_i}$ to indices $\underline{w}_i = \binom{u_i}{\underline{v}_i}$.

In §2.5 we succeeded in expressing the complicated weighted convolution of elementary polar functions $\pi_i(\xi) := (\xi-\alpha_i)^{-1}$ with the help of the scramble transform $\mathcal{S}^{\binom{u_1\ ,\dots,\ u_r}{v_1\ ,\dots,\ v_r}}(x)$ of the hyperlogarithmic bimould $\underline{\mathcal{V}}^{\binom{u_1\ ,\dots,\ u_r}{v_1\ ,\dots,\ v_r}}(x)$, with simple lower indices $v_i = \alpha_i$ standing for poles. In the light of the preceding considerations, the challenge is now to repeat the trick for general hyperlogarithmic inputs $\pi_i(\xi) := \widehat{\mathcal{V}}^{[\alpha_{i,1},\dots,\alpha_{i,m_i}]}(\xi)$ taken, for technical convenience, in the *positional* encoding.[36] This will force us to construct a generalised scramble transform

$$\text{scram} : M^{\boldsymbol{w}} \mapsto SM^{\underline{\boldsymbol{w}}} \qquad \textit{with} \qquad \begin{cases} M^{\boldsymbol{w}} = M^{\binom{u_1\ ,\dots,\ u_r}{v_1\ ,\dots,\ v_r}} \\ SM^{\underline{\boldsymbol{w}}} = M^{\binom{u_1\ ,\dots,\ u_r}{\underline{v}_1\ ,\dots,\ \underline{v}_r}} \end{cases} \tag{111}$$

leading to a bimould $SM^{\underline{\boldsymbol{w}}}$ whose lower indices

$$\underline{v}_i = (v_1, v'_i, \dots, v^{\ddagger}_i, v^{\dagger}_i) = (\alpha_{i,1}, \dots, \alpha_{i,m_i}) \tag{112}$$

are no longer simple scalars, but finite sequences of arbitrary length m_i.

The present section is devoted to that construction. As in the case of the ordinary scramble, it relies on two equivalent inductions – *forward* and *backward* – both of which are indispensible for a rounded picture. The next section shall validate the construction after the event and dispel its seeming artificiality by providing the link with the weighted convolution product.

[33]away from 0 and ∞.

[34]but with simple logarithmic singularities.

[35]usually by analytic continuation from 0 to ∞ along a singularity-free axis.

[36]For $m_i = 1$ we recover the polar inputs since $\widehat{\mathcal{V}}^{[\alpha_i]}(\xi) = (\xi-\alpha_i)^{-1}$.

Forward induction for the generalised scramble $M^{\boldsymbol{w}} \mapsto SM^{\underline{\boldsymbol{w}}}$**:**

For $r{=}1$ and $\underline{w}_1 = \binom{u_1}{\underline{v}_1} = \binom{u_1}{v_1, v_1', v_i'' \ldots, v_i^{\ddagger}, v_i^{\dagger}}$ we start the induction by setting:

$$SM^{\binom{u_1}{\underline{v}_1}} \;:=\; M^{\binom{u_1}{v_1},\binom{u_1}{v_1'-v_1},\binom{u_1}{v_1''-v_1'},\binom{u_1}{v_1'''-v_1''},\ldots,\binom{u_1}{v_i^{\dagger}-v_i^{\ddagger}}}$$

To continue the induction, we must distinguish four types of sequences $\boldsymbol{w}$, depending on the nature of the last index w_m of the sequences $\boldsymbol{w}$ in the various summands $M^{\boldsymbol{w}}$ occuring in the expansion of $SM^{\underline{\boldsymbol{w}}}$:

$$w_0 \;=\; \begin{pmatrix} u_r \\ v_r \end{pmatrix} \qquad \textit{with} \quad \#(\underline{v}_r) = 1 \;\; \textit{and} \;\; r = \#(\underline{\boldsymbol{w}}) \tag{113}$$

$$w_0 \;=\; \begin{pmatrix} u_i \\ \underline{v}_i^{\dagger} - \underline{v}_i^{\ddagger} \end{pmatrix} \qquad \textit{with} \quad \#(\underline{v}_i) \geqslant 2 \tag{114}$$

$$w_0 \;=\; \begin{pmatrix} u_i \\ \underline{v}_i^{\dagger} - \underline{v}_{i+1}^{\ddagger} \end{pmatrix} \qquad \textit{with} \quad i < r = \#(\underline{\boldsymbol{w}}) \tag{115}$$

$$w_0 \;=\; \begin{pmatrix} u_i \\ \underline{v}_i^{\dagger} - \underline{v}_{i-1}^{\ddagger} \end{pmatrix} \qquad \textit{with} \quad 1 < i \tag{116}$$

The linear operators $\mathrm{cutla}_M^{w_0}$ are defined as in §1.2. They act by removing the last index of $M^{\boldsymbol{w}}$ (not of $SM^{\underline{\boldsymbol{w}}}$!) if that last index happens to be w_0, and by annihilating $M^{\boldsymbol{w}}$ otherwise.

$$\mathrm{cutla}_M^{w_0}\, SM^{\underline{w}_1,\ldots,\underline{w}_r} \;=\; 0 \;\; \textit{if} \;\; w_0 \;\; \textit{not of type} \; (113)\text{-}(116)$$

$$\mathrm{cutla}_M^{\binom{u_r}{\underline{v}_r^{\dagger}}}\, SM^{\underline{w}_1,\ldots,\underline{w}_r} \;=\; +SM^{\underline{w}_1,\ldots,\underline{w}_{r-1}} \tag{117}$$

$$\mathrm{cutla}_M^{\binom{u_i}{\underline{v}_i^{\dagger}-\underline{v}_i^{\ddagger}}}\, SM^{\underline{w}_1,\ldots,\underline{w}_r} \;=\; +SM^{\underline{w}_1,\ldots,\underline{w}_i^{\dagger},\ldots,\underline{w}_r} \qquad \textit{with} \;\; \underline{w}_i^{\dagger} = \begin{pmatrix} u_i \\ \underline{v}_i^{\dagger} \end{pmatrix} \tag{118}$$

$$\mathrm{cutla}_{\omega_m}^{\binom{u_i}{\underline{v}_i^{\dagger}-\underline{v}_{i+1}^{\ddagger}}}\, SM^{\underline{w}_1,\ldots,\underline{w}_r} \;=\; +\sum_{\underline{w}_{i,i+1}^{+}\in W_{i,i+1}^{+}} SM^{\underline{w}_1,\ldots,\underline{w}_{i,i+1}^{+},\ldots,\underline{w}_r} \tag{119}$$

$$\mathrm{cutla}_M^{\binom{u_i}{\underline{v}_i^{\dagger}-\underline{v}_{i-1}^{\ddagger}}}\, SM^{\underline{w}_1,\ldots,\underline{w}_r} \;=\; -\sum_{\underline{w}_{i-1,i}^{-}\in W_{i-1,i}^{-}} SM^{\underline{w}_1,\ldots,\underline{w}_{i-1,i}^{-},\ldots,\underline{w}_r} \tag{120}$$

with indices $\underline{w}_{i,i+1}^{+}$ and $\underline{w}_{i-1,i}^{-}$ running through the sets

$$W_{i,i+1}^{+} \;:=\; \bigcup_{\underline{v}_{i,i+1}^{*}\in \mathrm{sha}(\underline{v}_i^{*},\underline{v}_{i+1}^{*})} \left\{ \begin{pmatrix} u_i + u_{i+1} \\ \underline{v}_{i,i+1}^{*},\, v_{i+1}^{\dagger} \end{pmatrix} \right\} \tag{121}$$

$$W_{i-1,i}^{-} \;:=\; \bigcup_{\underline{v}_{i-1,i}^{*}\in \mathrm{sha}(\underline{v}_{i-1}^{*},\underline{v}_{i}^{*})} \left\{ \begin{pmatrix} u_{i-1} + u_i \\ \underline{v}_{i-1,i}^{*},\, v_{i-1}^{\dagger} \end{pmatrix} \right\} \tag{122}$$

When each $\underline{v}_i$ reduces to a single element v_i, the case (118) is automatically ruled out, and the rules (117),(119),(120) simplify to the earlier rules (6),(7),(8) governing the ordinary scramble.

Interpretation: To construct the set $W^+_{i,i+1}$ of indices $\underline{w}^+_{i,i+1}$ we always take $u_i + u_{i+1}$ as upper index. To define the lower indices, we start from the sequences $\underline{v}^*_i, \underline{v}^*_{i+1}$ obtained by depriving $\underline{v}_i, \underline{v}_{i+1}$ of their last element $v^\dagger_i, v^\dagger_{i+1}$. Next, we consider all sequences $\underline{v}^*_{i,i+1}$ obtainable by shuffling the sequences $\underline{v}^*_i, \underline{v}^*_{i+1}$. Lastly, to each of these $\underline{v}^*_{i,i+1}$ we attach, as last element, the last element $v^\dagger_{i+1}$ of $\underline{v}_{i+1}$. Since $\#(\underline{v}^*_{i,i+1}, v^\dagger_{i+1}) = \#(\underline{v}^*_i) + \#(\underline{v}^*_{i+1}) - 1$, the rule (119) amounts to a proper induction step.

Of course, when either $\underline{v}_i$ or $\underline{v}_{i+1}$ reduce to a single element, the set $W^+_{i,i+1}$ also reduces to a single element. And when *both* $\underline{v}_i$ or $\underline{v}_{i+1}$ reduce to a single element, the set $W^+_{i,i+1}$'s single element is $\binom{u_i+u_{i+1}}{v_{i+1}}$, so that we fall back on the induction rule (7) for the ordinary scramble.

The same remarks apply for the set $W^-_{i-1,i}$. We may note in passing that the induction steps (118),(119),(120) essentially respect the left-right symmetry.[37] So we might expect the generalised scramble to obey a *backward* induction very similar to the *forward* one. As we shall see in a moment, this is not at all the case. The reason lies in the innocuous-looking rule (117), which on its own completely upsets the left-right symmetry.

Backward induction for the generalised scramble $M^{\boldsymbol{w}} \mapsto SM^{\underline{\boldsymbol{w}}}$:

The linear operators $\mathrm{cutfi}_M^{w_0}$ are defined as in §1.2. They act by removing the first index of $M^{\boldsymbol{w}}$ (not of $SM^{\underline{\boldsymbol{w}}}$!) if that first index happens to be w_0, and by annihilating $M^{\boldsymbol{w}}$ otherwise.

The backward induction says that the only operators $\mathrm{cutfi}_M^{w_0}$ acting non-trivially (i.e. without yielding 0) on the $SM^{\underline{\boldsymbol{w}}}$ (viewed as a sum of $M^{\boldsymbol{w}}$ summands) are those with indices w_0 of the form $\binom{u_1+\ldots+u_j}{v_i}$, where v_i is the first element of some sequence $\underline{v}_i$ with $1 \leqslant i \leqslant j$. And for those particular w_0, the backward induction rule reads:

$$\mathrm{cutfi}_M^{\binom{u_1+\ldots+u_j}{v_i}} SM^{\underline{\boldsymbol{w}}} = \mathrm{concat}\Big(\mathrm{concat}\big(\mathrm{symlin}(SM^{\dot{\underline{\boldsymbol{w}}}}_{v_i}, {}^*SM^{\ddot{\underline{\boldsymbol{w}}}}_{v_i}), {}^\sharp SM^{\underline{w}_i}_{v_i}\big), SM^{\vec{\underline{\boldsymbol{w}}}}\Big)$$
$$\textit{with} \quad \begin{cases} \underline{\boldsymbol{w}} := (\underline{w}_1, \ldots, \underline{w}_r) & , \ \dot{\underline{\boldsymbol{w}}} := (\underline{w}_1, \ldots, \underline{w}_{i-1}) \\ \ddot{\underline{\boldsymbol{w}}} := (\underline{w}_{i+1}, \ldots, \underline{w}_j) & , \ \vec{\underline{\boldsymbol{w}}} := (\underline{w}_{j+1}, \ldots, \underline{w}_r) \end{cases} \tag{123}$$

Some of the three factor sequences $\dot{\underline{\boldsymbol{w}}}$, $\ddot{\underline{\boldsymbol{w}}}$, $\vec{\underline{\boldsymbol{w}}}$, may be empty. The operators *concat* and *symlin* are defined as in §1.2. They act directly on the $SM^\bullet$ terms,

[37] Apart from the opposite signs in front of the right-hand sides of (119) and (120).

not on their $M^\bullet$ summands. Regarding the four $SM^\bullet$-terms occurring on the right-hand side of (123), the notations are as follows:

$$SM_{v_0}^{\binom{u_1\ ,\dots\dots,\ u_r}{\underline{v}_1\ ,\dots\dots,\ \underline{v}_r}} \;:=\; SM^{\binom{u_1\ ,\dots\dots,\ u_r}{\underline{v}_1-v_0\ ,\dots\dots,\ \underline{v}_r-v_0}} \tag{124}$$

$$^*SM_{v_0}^{\binom{u_1\ ,\dots\dots,\ u_r}{\underline{v}_1\ ,\dots\dots,\ \underline{v}_r}} \;:=\; (-1)^r SM^{\binom{u_r\ ,\dots\dots,\ u_1}{\underline{v}_r-v_i\ ,\dots\dots,\ \underline{v}_1-v_0}} \tag{125}$$

$$^\sharp SM_{v_0}^{\binom{u_i}{v_i,v_i',v_i'',v_i'''\dots}} \;:=\; {}^\sharp SM^{\binom{u_i}{v_i'-v_0,v_i''-v_0,v_i'''-v_0\dots}} \quad (v_i \;\; \textit{gets removed}) \tag{126}$$

Here and henceforth, we use the self-explanatory shorthand:

$$\underline{v}_i - v_0 := (v_i - v_0, v_i' - v_0, v_i'' - v_0 \dots) \qquad \textit{if} \qquad \underline{v}_i := (v_i, v_i', v_i'' \dots) \tag{127}$$

Proposition 2.5 *The forward-going formulae (113)-(116), which tell how to add an index in final position, and the backward-going formulae (123), which tell how to add an index in initial position, are equivalent. They define the general scramble transform* scram*, which turns symmetral (resp. alternal) w_i-indexed bimoulds into symmetral (resp. alternal) $\underline{w}_i$-indexed bimoulds:*

$$\mathrm{scram} \;:\; M^\bullet \mapsto SM^\bullet \quad \textit{with} \quad SM^{\underline{\boldsymbol{w}}} = \sum_{\boldsymbol{w}'} \epsilon^{\underline{\boldsymbol{w}}}_{\boldsymbol{w}'} \, M^{\boldsymbol{w}'} \tag{128}$$

$$\textit{and} \qquad \underline{\boldsymbol{w}} = \binom{u_1\ , \dots, \ u_r}{\underline{v}_1\ , \dots, \ \underline{v}_r} \;\;,\;\; \boldsymbol{w}' = \binom{u_1'\ , \dots, \ u_{r'}'}{v_1'\ , \dots, \ v_{r'}'} \;\;,\;\; \epsilon^{\underline{\boldsymbol{w}}}_{\boldsymbol{w}'} = \pm 1$$

The $\boldsymbol{w}'$-sequences on the right-hand side of (128) tend to be much longer than the $\underline{\boldsymbol{w}}$-sequence on the left-hand side, since their common length r' is $\sum \#(\underline{v}_i)$. Their most important feature[38]*, however, has to do with their contracted initial sums $\sum u_i' \, v_i'$, which are all of the form:*

$$u_1' \, v_1' + \dots + u_s' \, v_s' = |\boldsymbol{u}^{\boldsymbol{1}}| \, v_{1*} + \dots + |\boldsymbol{u}^{\boldsymbol{s}}| \, v_{s*} \tag{129}$$

relative to some factorisation $\underline{\boldsymbol{w}} = \underline{\boldsymbol{w}}^{\boldsymbol{1}} \dots \underline{\boldsymbol{w}}^{\boldsymbol{s}} \vec{\underline{\boldsymbol{w}}}$ and to a selection of indices v_{i*}, each of which belongs to the lower sequence $\underline{v}_{i*}$ of some simple index $\underline{w}_{i*} = \binom{u_{i*}}{\underline{v}_{i*}}$ inside $\underline{\boldsymbol{w}}^{\boldsymbol{i}}$.

Sketch of proof: One way of verifying the equivalence of the two induction rules – forward and backward – is to iterate each one and check that "they meet in the middle". In this regard, we may mention that, in §2.9, in order to find the form of the alien derivatives of the monomials $\mathcal{S}^{\underline{\boldsymbol{w}}}$, we shall perform an operation which, in fact, is tantamount to iterating the backward induction rule for the generalised scramble.

[38]It shall determine the form of the alien derivations Δ_{ω_0} that act effectively on the monomials $\mathcal{S}^\bullet(x)$. See §.. below.

Remark 1: huge number of $M^\bullet$-summands in $SM^\bullet$.
The number $\mu(\underline{\boldsymbol{w}}) = \mu(m_1,\dots,m_r)$ of $M^\bullet$-summands in the standard expansion (128) of $SM^\bullet$ depends only on the lengths $m_i := \#(\underline{v}_i)$ of the partial sequences $\underline{v}_i$. It tends to be huge. Thus:

$$
\begin{array}{lcrcrl}
\mu(\overbrace{1,\dots,1}^{r \text{ times}}) & = & 1.3.5\dots(2\,r-1) & = & r!! & \\
\mu(5,5,5) & = & 29\,135\,106 & \sim & 29 & 10^6 \\
\mu(4,5,6) & = & 22\,855\,560 & \sim & 23 & 10^6 \\
\mu(6,5,4) & = & 23\,963\,940 & \sim & 24 & 10^6 \\
\mu(4,4,4,4) & = & 10\,050\,665\,625 & \sim & 10 & 10^9 \\
\mu(1,3,5,7) & = & 349\,098\,750 & \sim & 0.4 & 10^9 \\
\mu(7,5,3,1) & = & 539\,188\,650 & \sim & 0.5 & 10^9 \\
\mu(3,3,3,3,3) & = & 60\,575\,515\,000 & \sim & 60 & 10^9 \\
\mu(1,2,3,4,5) & = & 6\,067\,061\,000 & \sim & 6 & 10^9 \\
\mu(5,4,3,2,1) & = & 9\,641\,071\,440 & \sim & 10 & 10^9
\end{array}
$$

Remark 2: multiplication and symmetral linearisation.
When applied to a symmetral $M^\bullet$, the generalised scramble transform produces a symmetral $SM^\bullet$ defined as a sum of symmetral $M^\bullet$-summands. This opens two paths for the calculation of products $SM^{\underline{\boldsymbol{w}}'}.\,SM^{\underline{\boldsymbol{w}}''}$:

$$
\begin{array}{ccc}
SM^{\underline{\boldsymbol{w}}'}\ .\ SM^{\underline{\boldsymbol{w}}''} & \overset{\text{symmetral linearisatiom}}{\longrightarrow} & \sum SM^{\underline{\boldsymbol{w}}} \\
\text{M}^\bullet\text{-expansion} \downarrow \qquad \downarrow \text{M}^\bullet\text{-expansion} & & \downarrow \text{M}^\bullet\text{-expansion} \\
\left(\sum \epsilon_{\boldsymbol{w}'} M^{\boldsymbol{w}'}\right)\ .\ \left(\sum \epsilon_{\boldsymbol{w}''} M^{\boldsymbol{w}''}\right) & \overset{\text{symmetral linearisation}}{\longrightarrow} & \sum \epsilon_{\boldsymbol{w}} M^{\boldsymbol{w}}
\end{array}
$$

The path *expansion followed by linearisation* always leads to a number of $M^\bullet$-summands considerably less than the path *linearisation followed by expansion*, but the latter gives rise to massive (pair-wise) cancellations, ensuring the same end result.

2.7 The general monomials $\mathcal{S}^\bullet(x)$ and $\mathcal{S}^\bullet_{cor}(x)$.

Definition 2.1 (The general monomials $\mathcal{S}^\bullet(x)$) .
The general monomials $\mathcal{S}^\bullet(x)$ are simply the general scramble transform of the familiar hyperlogarithmic bimould $\underline{\mathcal{V}}^{\binom{u_1\ ,\dots,\ u_r}{v_1\ ,\dots,\ v_r}}(x) := \mathcal{V}^{u_1 v_1,\dots,u_r v_r}(x)$

Since $\mathcal{V}^\bullet(x)$ and $\underline{\mathcal{V}}^\bullet(x)$ are both symmetral, $\mathcal{S}^\bullet(x)$ is symmetral as well.
For r=1 and $\underline{w}_1 = \binom{u_1}{\underline{v}_1} = \binom{u_1}{v_1, v_1', \underline{v}_i''\dots}$, the definition yields

$$
\begin{aligned}
\mathcal{S}^{\underline{w}_1}(x) &:= \mathcal{V}^{[u_1 v_{1,1}\,,\,u_1 v_1\,,\dots,\,u_1 v_1'',v_1'''\dots]}(x) \\
&:= \mathcal{V}^{u_1 v_1\,,\,u_1(v_1'-v_1)\,,\,u_1(v_1''-v_1')\,,\,u_1(v_1'''-v_1'')\dots}(x)
\end{aligned}
$$

In this case, the *positional* notation is obviously more advantageous, but it ceases to be so when r grows. For instance, for $r{=}2$ and $\underline{w}_1 = \binom{u_1}{v_1, v_1'}$, $\underline{w}_2 = \binom{u_2}{v_2, v_2'}$, we find in the *incremental* notation the following expansion, which would look much worse in the *positional* notation:[39]

$$\begin{aligned}
\mathcal{S}^{\binom{u_1\ ,\ u_2}{\underline{v}_1\ ,\ \underline{v}_2}}(x) = \; & +\mathcal{V}^{u_1v_1,u_2v_2,u_2v_{2':2},u_1v_{1':1}}(x) && +\mathcal{V}^{u_{12}v_2,u_{12}v_{1:2},u_{12}v_{2':1},u_1v_{1':2'}}(x)\\
& +\mathcal{V}^{u_1v_1,u_2v_2,u_1v_{1':1},u_2v_{2':2}}(x) && -\mathcal{V}^{u_{12}v_1,u_2v_{2:1},u_1v_{1':1},u_2v_{2':2}}(x)\\
& +\mathcal{V}^{u_1v_1,u_1v_{1':1},u_2v_2,u_2v_{2':2}}(x) && -\mathcal{V}^{u_{12}v_1,u_{12}v_{1':1},u_2v_{2:1'},u_2v_{2':2}}(x)\\
& +\mathcal{V}^{u_{12}v_1,u_{12}v_{2:1},u_1v_{1':2},u_2v_{2':2}}(x) && -\mathcal{V}^{u_{12}v_2,u_{12}v_{1:2},u_{12}v_{1':1},u_2v_{2':1'}}(x)\\
& +\mathcal{V}^{u_{12}v_1,u_{12}v_{2:1},u_{12}v_{2':2},u_1v_{1':2'}}(x) && -\mathcal{V}^{u_{12}v_1,u_{12}v_{2:1},u_{12}v_{1':2},u_2v_{2':1'}}(x)\\
& +\mathcal{V}^{u_{12}v_2,u_{12}v_{2':2},u_1v_{1:2'},u_1v_{1':1}}(x) && -\mathcal{V}^{u_{12}v_2,u_{12}v_{1:2},u_2v_{2':1},u_1v_{1':1}}(x)\\
& +\mathcal{V}^{u_{12}v_2,u_1v_{1:2},u_2v_{2':2},u_1v_{1':1}}(x) && -\mathcal{V}^{u_{12}v_1,u_2v_{2:1},u_2v_{2':2},u_1v_{1':1}}(x)\\
& +\mathcal{V}^{u_{12}v_2,u_1v_{1:2},u_1v_{1':1},u_2v_{2':2}}(x)
\end{aligned}$$

According to (67), the bi-resurgent monomials $\mathcal{W}^{\bullet}(z,x)$ with inputs $b_i(z)$ reduce, in the ξ-plane, to weighted convolution products with inputs $c_i(\xi) := b_i(z-\xi)$. Thus, to get rid of the variable z in $b_i(z-\xi)$ for hyperlogarithmic data b_i, we require an addition identity for hyperlogarithms:

Proposition 2.6 (The addition law for hyperlogarithms) .
For suitable determinations of our multivalued functions[40]*, we have:*

$$\overset{\frown}{\mathcal{V}}^{[\bullet]}(t_1+t_2) = \overset{\frown}{\mathcal{V}}^{[\bullet]}(t_1) \times \overset{\frown}{\mathcal{V}}^{[\bullet - t_1]}(t_2) \tag{130}$$

Or again, more explicitely

$$\overset{\frown}{\mathcal{V}}^{[\alpha_1,\ldots,\alpha_r]}(t_1+t_2) = \overset{\frown}{\mathcal{V}}^{[\alpha_1,\ldots,\alpha_r]}(t_1) + \sum_{1\leqslant j\leqslant r} \overset{\frown}{\mathcal{V}}^{[\alpha_1,\ldots,\alpha_{j-1}]}(t_1)\, \overset{\frown}{\mathcal{V}}^{[\alpha_j - t_1,\ldots,\alpha_r - t_1]}(t_2) \tag{131}$$

Sketch of proof: It is again a question of checking that the above addition formula is stable under $\partial_{\alpha_i}, \partial_{t_1}, \partial_{t_2}$, with the proper limit conditions. Thus, using the rules of §2.4 and applying ∂_{t_2} to the identity (131) with $r = r_0$, we find the same identity with $r = r_0 - 1$.

What we shall require is actually the following variant of this addition law. Setting $t_1 := z$, $t_2 := -\xi$ in (131), then using the homogeneousness $\overset{\frown}{\mathcal{V}}^{[\bullet - z]}(-\xi) \equiv \overset{\frown}{\mathcal{V}}^{[z-\bullet]}(\xi)$, and lastly applying ∂_ξ, we find:

$$\widehat{\mathcal{V}}^{[\alpha_1,\ldots,\alpha_r]}(z-\xi) = -\sum_{1\leqslant j\leqslant r} \overset{\frown}{\mathcal{V}}^{[\alpha_1,\ldots,\alpha_{j-1}]}(z)\, \widehat{\mathcal{V}}^{[z-\alpha_j,\ldots,z-\alpha_r]}(\xi) \tag{132}$$

Note here the unusual juxtaposition of monomials $\widehat{\mathcal{V}}$ and $\overset{\frown}{\mathcal{V}}$.

[39]Beside the usual abbreviations $u_{1,2} := u_1 + u_2$, $v_{1:2} := v_1 - v_2$ we write $v_{1':1} := v_1' - v_1$.
[40]See the important remark below

Definition 2.2 (The general monomials $\mathcal{S}^{\bullet}_{cor}(x)$).
The monomials $\mathcal{S}^{\bullet}_{cor}(x)$ carry lower indices of the form

$$\underline{v}_i = z - \underline{\alpha}_i = (z-\alpha_i, z-\alpha'_i, z-\alpha''_i, \dots) \tag{133}$$

and are derived from the monomials $\mathcal{S}^{\bullet}(x)$ under the adjunction of corrective, x-constant, z-dependent terms of type $\big(-\overset{\frown}{\mathcal{V}}^{[\boldsymbol{\alpha}]}(z)\big)$, which should be taken as $\equiv 1$ when $\boldsymbol{\alpha}$ reduces to the empty sequence:

$$\mathcal{S}_{cor}^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ z-\underline{\alpha}_1 & ,\dots, & z-\underline{\alpha}_r\end{smallmatrix}\right)}(x) := \sum_{\underline{\alpha}_i^* \underline{\alpha}_i^{**}=\underline{\alpha}_i}^{\underline{\alpha}_i^{**}\neq\varnothing} \big(-\overset{\frown}{\mathcal{V}}^{[\underline{\alpha}_1^*]}(z)\big)\dots\big(-\overset{\frown}{\mathcal{V}}^{[\underline{\alpha}_r^*]}(z)\big)\, \mathcal{S}^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ z-\underline{\alpha}_1^{**} & ,\dots, & z-\underline{\alpha}_r^{**}\end{smallmatrix}\right)}(x) \tag{134}$$

$$\text{with}\quad \begin{cases} \underline{\alpha}_i = (\alpha_i,\alpha'_i,\dots,\alpha_i^{(n_i-1)}) & , \quad z-\underline{\alpha}_i = (z-\alpha_i,\dots,z-\alpha_i^{(n_i-1)}) \\ \underline{\alpha}_i^* = (\alpha_i,\alpha'_i,\dots,\alpha_i^{(m_i-1)}) & \quad (0\leqslant m_i < n_i) \\ \underline{\alpha}_i^{**} = (\alpha_i^{(m_i)},\dots,\alpha_i^{(n_i-1)}) & , \quad z-\underline{\alpha}_i^{**} = (z-\alpha_i^{(m_i)},\dots,z-\alpha_i^{(n_i-1)}) \end{cases}$$

Note that in (131) the sequences $\boldsymbol{\alpha_i^*}$ are always $\neq\varnothing$, unlike the sequences $\boldsymbol{\alpha_i^{**}}$, which turn empty when $m_i=0$, in which case one should of course set $\big(-\overset{\frown}{\mathcal{V}}^{\varnothing}(z)\big) := 1$. Therefore:

$$\mathcal{S}_{cor}^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ z-\underline{\alpha}_1 & ,\dots, & z-\underline{\alpha}_r\end{smallmatrix}\right)}(x) = \mathcal{S}^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ z-\underline{\alpha}_1 & ,\dots, & z-\underline{\alpha}_r\end{smallmatrix}\right)}(x) + \textit{shorter monomials}$$

Proposition 2.7 (Weighted convolution with hyperlog inputs) .
We still assume here that all partial sums $u_1+\dots+u_i$ are $\neq 0$. Then the weighted convolution of hyperlogarithmic functions $\pi_i(\xi) = \widehat{\mathcal{V}}^{[\alpha_i,\alpha'_i\dots]}(\xi)$ coincides with the x-Borel transform $\widehat{\mathcal{S}}^{\bullet}(\xi)$ of the bimould $\mathcal{S}^{\bullet}(x)$ for indices $\underline{w}_i = \binom{u_i}{\underline{\alpha}_i} = \binom{u_i}{\alpha_i,\alpha'_i,\dots}$. Similarly, the bi-resurgent monomials $\mathcal{W}^{\bullet}(z,x)$ of (??) with hyperlogarithmic inputs $b_i(z) = \hat{\mathcal{V}}^{[\alpha_i,\alpha'_i,\dots]}(z)$, when viewed as resurgent functions of their second variable x, coincide with the corrected bimould $\mathcal{S}^{\bullet}_{cor}(x)$ for indices $\underline{w}_i = \binom{u_i}{z-\underline{\alpha}_i} = \binom{u_i}{z-\alpha_i, z-\alpha'_i,\dots}$.

$$\mathrm{weco}^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ \pi_1 & ,\dots, & \pi_r\end{smallmatrix}\right)}(\xi) = \widehat{\mathcal{S}}^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ \underline{\alpha}_1 & ,\dots, & \underline{\alpha}_r\end{smallmatrix}\right)}(\xi) \qquad \textit{with} \quad \pi_i(\xi) = \widehat{\mathcal{V}}^{[\alpha_i,\alpha'_i,\dots]}(\xi) \tag{135}$$

$$\mathcal{W}^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ b_1 & ,\dots, & b_r\end{smallmatrix}\right)}(z,x) = \mathcal{S}_{cor}^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ z-\underline{\alpha}_1 & ,\dots, & z-\underline{\alpha}_r\end{smallmatrix}\right)}(x) \quad \textit{with} \quad b_i(z) = \widehat{\mathcal{V}}^{[\alpha_i,\alpha'_i,\dots]}(z) \tag{136}$$

Sketch of proof: As in the case of the simple $\mathcal{S}^{\boldsymbol{w}}(x)$, it is a matter of pure combinatorial drudgery. Here again, we make massive use of the differentiation rules of §2.4 to check that

$$(\partial_z + (u_1+\dots+u_r)\,x)\,\mathcal{S}^{\underline{w}_1,\dots,\underline{w}_r}(x) = -\mathcal{S}^{\underline{w}_1,\dots,\underline{w}_{r-1}}(x)\times \hat{\mathcal{V}}^{[\underline{v}_r]}(z) \tag{137}$$

Mark the alternation of variables: x inside $\mathcal{S}^{\boldsymbol{w}}(x)$ but z inside $\widehat{\mathcal{V}}^{[\underline{v}_r]}(z)$.

Remark 1: Both $\mathcal{S}^{\bullet}_{cor}(x)$ and $\mathcal{V}^{\bullet}(x)$ behave as *symmetral* moulds under ordinary multiplication (as power series of x^{-1}). The existence of a unique expansion of $\mathcal{S}^{\underline{\boldsymbol{w}}}_{cor}(x)$ into a finite sum of $\mathcal{V}^{\boldsymbol{\omega}}(x)$-terms leads therefore to a commutative diagram:

$$\begin{array}{ccc} \mathcal{S}^{\underline{\boldsymbol{w}}'}_{cor} * \mathcal{S}^{\underline{\boldsymbol{w}}''}_{cor} & \xrightarrow{\text{symmetral linearisatiom}} & \sum \mathcal{S}^{\underline{\boldsymbol{w}}}_{cor} \\ {\scriptstyle\text{hyperlogarithmic expansion}}\downarrow \quad \downarrow{\scriptstyle\text{hyperlogarithmic expansion}} & & \downarrow{\scriptstyle\text{hyperlogarithmic expansion}} \\ (\sum \epsilon_{\boldsymbol{\omega}'} \mathcal{V}^{\boldsymbol{\omega}'}) * (\sum \epsilon_{\boldsymbol{\omega}''} \mathcal{V}^{\boldsymbol{\omega}''}) & \xrightarrow{\text{symmetral linearisation}} & \sum \epsilon_{\boldsymbol{\omega}} \mathcal{V}^{\boldsymbol{\omega}} \end{array}$$

The same already held true, of course, for the mould $\mathcal{S}^{\underline{\boldsymbol{w}}}(x)$ but this immediately followed from the definition 2.1 combined with the earlier commutative diagram involving $SM^{\underline{\boldsymbol{w}}}$ and $M^{\boldsymbol{w}}$. The point here is the preservation of the diagram's commutativity *after* the change (134) from $\mathcal{S}^{\underline{\boldsymbol{w}}}(x)$ to $\mathcal{S}^{\underline{\boldsymbol{w}}}_{cor}(x)$.

Remark 2: Bounds for $\widehat{\mathcal{S}}^{\underline{\boldsymbol{w}}}(\xi)$ to $\widehat{\mathcal{S}}^{\underline{\boldsymbol{w}}}_{cor}(\xi)$. The huge number of hyperlogarithmic summands $\mathcal{V}^{\bullet}(x)$ present in the expansion of $\mathcal{S}^{\underline{\boldsymbol{w}}}(x)$ and $\mathcal{S}^{\underline{\boldsymbol{w}}}_{cor}(x)$ (see Remark 1 towards the end of §2.6.) doesn't prevent our monomials from admitting excellent bounds on the compact sets of the ramified Borel ξ-'plane'. The hyperlogarithmic expansions are useful, indispensable even, for understanding the resurgence pattern. But for the purpose of majorisation one should turn to the weighted convolution product *weco*$^{\bullet}$. The corresponding integral may look messy, but it leads to even better bounds than the ordinary convolution integral: for r convolands, a second factor $\frac{1}{r!}$ comes into play instead of just one!

2.8 Vanishing u_i-sums and amended monomials $\mathcal{S}^{\bullet}_{am}(x)$.

When some of the partial sums $(u_1+\cdots+u_i)$ vanish, some of the end points θ_i in the multiple integral (65) become infinite. Since we consider integrands of the form $c_i(\xi_i) := b_i(z-\xi_i)$ for z large and for inputs $b_i(z)$ which, even when ramified away from ∞, are assumed to be analytic in some neighbourhood of ∞, this is no obstacle to the continued existence of the weighted convolution: we can always arrange for all integration variables ξ_i to move within the safe neighbourhood of ∞. However, the analytic expression of $\mathcal{W}^{\bullet}(z,x)$ in terms of $\mathcal{S}^{\bullet}(x)$ (polar case) or $\mathcal{S}^{\bullet}_{cor}(x)$ (ramified case) ceases to be valid, forcing us to resort to 'amended' monomials $\mathcal{S}^{\bullet}_{am}(x)$ or $\mathcal{S}^{\bullet}_{coram}(x)$. Let us begin with the polar case:

Proposition 2.8 (z-derivative of $\mathcal{S}^\bullet(x)$) .
In presence of vanishing u_i-sums, the x-derivative of $\mathcal{S}^{\binom{u_1\ ,\dots,\ u_r}{z-\alpha_1\ ,\dots,\ z-\alpha_r}}(x)$ *no longer verifies the relation (476), but a modified form of it:*

$$(\partial_z + |\boldsymbol{u}(\bullet)|\, x)\, \mathcal{S}^\bullet(x) = -\mathcal{S}^\bullet(x) \times \mathcal{J}^\bullet + \mathcal{H}^\bullet(x) \times \mathcal{S}^\bullet(x) \tag{138}$$

The definition of the elementary alternal bimoulds $\mathcal{J}^\bullet$ remains unchanged. That of the corrective alternal bimould $\mathcal{H}^\bullet$ is as follows:

$$\mathcal{J}^{w_1} \;:=\; \frac{1}{v_1} \quad , \quad \mathcal{J}^{w_1,\dots,w_r} = 0 \quad \text{if} \quad r \neq 1 \tag{139}$$

$$\mathcal{H}^{\boldsymbol{w}}(x) \;:=\; \begin{cases} \sum_{\boldsymbol{w}' w_j \boldsymbol{w}''=\boldsymbol{w}} \mathcal{S}_{v_j}^{\boldsymbol{w}'}(x)\, \mathcal{J}^{w_j}\, inv\mathcal{S}_{v_j}^{\boldsymbol{w}''}(x) & \text{if} \ \ |\boldsymbol{u}| = 0 \\ 0 & \textit{otherwise} \end{cases} \tag{140}$$

$$\mathcal{S}_{v_j}^{\binom{u_1\ ,\dots,\ u_r}{v_1\ ,\dots,\ v_r}}(x) \;:=\; \mathcal{S}^{\binom{u_1\ ,\dots,\ u_r}{v_{1:j}\ ,\dots,\ v_{r:j}}}(x) \quad \text{with} \quad v_{i:j} := v_i - v_j \tag{141}$$

Sketch of proof: The repetition of consecutive v_i's modifies the behaviour of $\mathcal{S}^{\boldsymbol{w}}$ under ∂_{u_i}, while the vanishing of partial u_i-sums modifies the behaviour of $\mathcal{S}^{\boldsymbol{w}}$ under ∂_{v_i} (mark the criss-cross). The exact rules are these:

$$\partial_{u_j}\mathcal{S}^{\boldsymbol{w}} \;=\; P(u_j)\left(\delta(v_{j-1}-v_j)\,\mathcal{S}^{\boldsymbol{w}^{\widehat{j-1,j}}} + \delta(v_j - v_{j+1})\,\mathcal{S}^{\boldsymbol{w}^{\widehat{j,j+1}}}\right) \tag{142}$$

$$\partial_{v_j}\mathcal{S}^{\boldsymbol{w}} \;=\; P(v_j) \sum_{\boldsymbol{w^1} w_j \boldsymbol{w^2}\boldsymbol{w^3}=\boldsymbol{w}} \delta(|\boldsymbol{u^1} u_j \boldsymbol{u^2}|)\ \mathcal{S}^{\boldsymbol{w^1}}\, inv\mathcal{S}^{\boldsymbol{w^2}}\mathcal{S}^{\boldsymbol{w^3}} \tag{143}$$

with δ standing here for the discrete dirac.[41] From (142) we then derive the modified formula (138) with its corrective term $\mathcal{H}^\bullet(x) \times \mathcal{S}^\bullet(x)$.

Let us now decompose $\mathcal{H}^{\boldsymbol{w}}$ into a finite sum of terms $\mathcal{H}_{v_j}^{\boldsymbol{w}} := \mathcal{S}_{v_j}^{\boldsymbol{w}'}\,\mathcal{J}^{w_j}\, inv\mathcal{S}_{v_j}^{\boldsymbol{w}''}$ and then set

$$\mathcal{K}^{\boldsymbol{w}}(x) := \sum_{\boldsymbol{w^1}\dots\boldsymbol{w^s}\,=\,\boldsymbol{w}}^{|\boldsymbol{w^1}|=0,\dots,|\boldsymbol{w^s}|=0} \ \sum_{v_{j_1},\dots,v_{j_s}} \mathcal{H}_{v_{j_1}}^{\boldsymbol{w^1}}(x)\dots\mathcal{H}_{v_{j_s}}^{\boldsymbol{w^s}}(x)\ \mathcal{X}^{v_{j_1},\dots,v_{j_s}} \tag{144}$$

with an elementary symmetral mould unambiguously defined by the conditions

$$\partial_z\, \mathcal{X}^{v_1,\dots,v_s} \;=\; -\mathcal{X}^{v_1,\dots,v_{s-1}}\frac{1}{v_s} \qquad (\textit{recall that}\ v_s := \frac{1}{z-\alpha_s}) \tag{145}$$

$$\mathcal{X}^{v_1,\dots,v_s} \;\sim\; \frac{(-1)^s}{s!}(\log z)^s \qquad \textit{for}\ z \sim \infty \quad \textit{on main sheet} \tag{146}$$

We are then in a position to construct the *amended* mould $\mathcal{S}_{am}^\bullet$:

$$\mathcal{S}_{am}^\bullet(x) \;:=\; \mathcal{K}^\bullet(x) \times \mathcal{S}^\bullet(x) \tag{147}$$

[41] $\delta(0) = 1$, $\delta(t) = 0$ if $t \neq 0$.

Proposition 2.9 (The amended monomials $\mathcal{S}^{\bullet}_{am}(x)$) .
As the product of two symmetral factors, the bimould $\mathcal{S}^{\bullet}_{am}(x)$ is symmetral and clearly verifies

$$(\partial_z+|u_1+...u_r|.x)\mathcal{S}^{w_1,...,w_r}_{am}(x) := -\mathcal{S}^{w_1,...,w_{r-1}}_{am}(x)\,\frac{1}{v_r} \quad \Big(w_i := (\begin{smallmatrix} u_i \\ z-\alpha_i \end{smallmatrix})\Big) \tag{148}$$

Changing $\mathcal{S}^{\bullet}(x)$ to $\mathcal{S}^{\bullet}_{am}(x)$, we can extend the earlier identities (105)-(106) to identities valid in all cases:

$$\mathrm{weco}^{(\begin{smallmatrix} u_1 & ,..., & u_r \\ \pi_1 & ,..., & \pi_r \end{smallmatrix})}(\xi) = \widehat{\mathcal{S}}^{(\begin{smallmatrix} u_1 & ,..., & u_r \\ \alpha_1 & ,..., & \alpha_r \end{smallmatrix})}_{am}(\xi) \qquad \textit{for} \quad \pi_i(\xi) = \frac{1}{\xi - \alpha_i} \tag{149}$$

$$\mathcal{W}^{(\begin{smallmatrix} u_1 & ,..., & u_r \\ b_1 & ,..., & b_r \end{smallmatrix})}(z,x) = \mathcal{S}^{(\begin{smallmatrix} u_1 & ,..., & u_r \\ z-\alpha_1 & ,..., & z-\alpha_r \end{smallmatrix})}_{am}(x) \quad \textit{for} \quad b_i(z) = \frac{1}{z-\alpha_i} \tag{150}$$

2.9 Alien derivatives of the monomials $\mathcal{S}^{\bullet}(x), \mathcal{S}^{\bullet}_{cor}(x)$.

In a sense, we already 'know' the answer: having expanded $\mathcal{S}^{\bullet}(x)$ and $\mathcal{S}^{\bullet}_{am}(x)$ into finite sums of hyperlogarithms $\mathcal{V}^{\bullet}(x)$ and possessing with formula (98) a prescription for alien-differentiating each $\mathcal{V}^{\bullet}(x)$, we have all it takes to calculate $\Delta_{\omega_0}\mathcal{S}^{\bullet}(x)$ and $\Delta_{\omega_0}\mathcal{S}^{\bullet}_{cor}(x)$. In practice, however, we require explicit and compact formulae covering each of the many possible situations. This is the object of the present section.

The special monomials $\mathcal{S}^{\boldsymbol{w}}(x)$.

Proposition 2.10 (Alien derivatives of $\mathcal{S}^{\boldsymbol{w}}(x)$) .
The only alien derivations Δ_{ω_0} acting effectively on a given monomial $\mathcal{S}^{\boldsymbol{w}}(x) = \mathcal{S}^{(\begin{smallmatrix} u_1 & ,..., & u_* & ,..., & u_r \\ v_1 & ,..., & v_* & ,..., & v_r \end{smallmatrix})}$ correspond either to simple indices ω_0 of the form

$$\omega_0 = |\boldsymbol{u}|\, v_* \quad \textit{with} \quad \begin{cases} \boldsymbol{w} = \dot{\boldsymbol{w}}.w_*.\ddot{\boldsymbol{w}}.\vec{\boldsymbol{w}} \\ |\boldsymbol{u}| = |\dot{\boldsymbol{u}}| + u_* + |\ddot{\boldsymbol{u}}| \end{cases}$$

or to composite ones of the form

$$\omega_0 = |\boldsymbol{u}^{\boldsymbol{1}}|\, v_{1,*} + ... + |\boldsymbol{u}^{\boldsymbol{s}}|\, v_{s,*} \quad \textit{with} \quad \begin{cases} \boldsymbol{w} = \dot{\boldsymbol{w}}^{\boldsymbol{1}}.w_{1*}.\ddot{\boldsymbol{w}}^{\boldsymbol{1}} \dots \dot{\boldsymbol{w}}^{\boldsymbol{s}}.w_{s*}.\ddot{\boldsymbol{w}}^{\boldsymbol{s}}.\vec{\boldsymbol{w}} \\ |\boldsymbol{u}^{\boldsymbol{i}}| = |\dot{\boldsymbol{u}}^{\boldsymbol{i}}| + u_{i*} + |\ddot{\boldsymbol{u}}^{\boldsymbol{i}}| \end{cases}$$

For a simple index ω_0, the operator Δ_{ω_0} acts as follows:

$$\Delta_{\omega_0}\,\mathcal{S}^{\boldsymbol{w}}(x) \;=\; \mathcal{T}_{v_*}^{\dot{\boldsymbol{w}};\ddot{\boldsymbol{w}}}(x)\,\mathcal{S}^{\vec{\boldsymbol{w}}}(x) \tag{151}$$

$$\textit{with}\qquad \begin{cases} \mathcal{T}^{\dot{\boldsymbol{w}};\ddot{\boldsymbol{w}}} := \mathcal{S}^{\dot{\boldsymbol{w}}}(\mathrm{invmu}.\mathcal{S})^{\ddot{\boldsymbol{w}}} \\ \mathcal{T}_{v_*}^{\dot{\boldsymbol{w}};\ddot{\boldsymbol{w}}} := \mathcal{S}_{v_*}^{\dot{\boldsymbol{w}}}(\mathrm{invmu}.\mathcal{S}_{v_*})^{\ddot{\boldsymbol{w}}} \\ (\mathrm{invmu}.\mathcal{S})^{w_1,\dots,w_r} = (-1)^r\,\mathcal{S}^{w_r,\dots,w_1} \\ \mathcal{S}_{v_*}^{\binom{u_1\ ,\dots,\ u_r}{v_1\ ,\dots,\ v_r}} := \mathcal{S}^{\binom{u_1\ \ ,\dots,\ \ u_r}{v_1-v_*\ ,\dots,\ v_r-v_*}} \end{cases} \tag{152}$$

For a composite index ω_0, the action involves a new ingredient: the locally constant bimould $\mathrm{tes}^\bullet$*, or* tessellation bimould*, defined as the scramble transform of the hyperlogarithmic mould $V^\bullet$.*

$$\Delta_{\omega_0}\,\mathcal{S}^{\boldsymbol{w}}(x) \;=\; \mathrm{tes}^{\binom{|\boldsymbol{u^1}|\ ,\dots,\ |\boldsymbol{u^s}|}{v_1\ \ ,\dots,\ \ v_s}}\,\mathcal{T}_{v_{1*}}^{\dot{\boldsymbol{w}}^{\boldsymbol{1}};\ddot{\boldsymbol{w}}^{\boldsymbol{1}}}(x)\dots\mathcal{T}_{v_{s*}}^{\dot{\boldsymbol{w}}^{\boldsymbol{s}};\ddot{\boldsymbol{w}}^{\boldsymbol{s}}}(x)\,\mathcal{S}^{\vec{\boldsymbol{w}}}(x) \tag{153}$$

$$\textit{with}\qquad \mathrm{tes}^\bullet := \mathrm{scram}.\underline{V}^\bullet \quad \textit{and}\quad \underline{V}^{\binom{u_1\ ,\dots,\ u_r}{v_1\ ,\dots,\ v_r}} := V^{u_1\,v_1,\dots,u_r\,v_r} \tag{154}$$

The general monomials $\mathcal{S}^{\underline{\boldsymbol{w}}}(x)$.

To enunciate suitably compact statements, we need the following:

Definition 2.3 (Notion of v_*-splitting) .
Let v_* be some element (-first, middle, last-) of some lower $\underline{v}_*$ in a sequence $\underline{\boldsymbol{w}} = \binom{u_1\ ,\dots,\ u_*\ ,\dots,\ u_r}{\underline{v}_1\ ,\dots,\ \underline{v}_*\ ,\dots,\ \underline{v}_r}$. A $\underline{v}_*$-splitting of $\underline{\boldsymbol{w}}$ is a joint factorisation of all $\underline{v}_i$ such that

$$\begin{aligned} \underline{v}_i &= (\underline{v}'_i,\underline{v}''_i) \qquad \textit{if}\ \ \underline{v}_i \neq \underline{v}_* \qquad &&(\textit{only } \underline{v}''_i \textit{ may be } \varnothing) \\ \underline{v}_* &= (\underline{v}'_*, v_*, \underline{v}''_*) &&(\textit{both } \underline{v}'_* \textit{ and } \underline{v}''_* \textit{ may be } \varnothing) \end{aligned}$$

To each $\underline{v}_*$-splitting we associate

- *a non-ordered sequence $\{\underline{\boldsymbol{v}}'\}$ consisting of ordered sequences $\underline{v}'_i$*
- *two ordered sequences $\underline{\dot{\boldsymbol{w}}}''$ and $\underline{\ddot{\boldsymbol{w}}}''$*
- *a lone index $\underline{w}''_*$ (that may be empty)*

defined in this way:

$$\begin{aligned} \{\underline{\boldsymbol{v}}'\} &:= \{\underline{v}'_1;\underline{v}'_2;\dots;\underline{v}'_*;\dots;\underline{v}'_{r-1};\underline{v}'_r\} \\ \underline{\dot{\boldsymbol{w}}}'' &:= (\underline{w}''_1,\dots,\underline{w}''_i,\dots) = \binom{u_1\ ,\dots,\ u_i\ ,\dots}{\underline{v}''_1\ ,\dots,\ \underline{v}''_i\ ,\dots} \quad \textit{with}\ \ \underline{w}_i\ \ \textit{earlier than}\ \underline{w}_* \\ \underline{\ddot{\boldsymbol{w}}}'' &:= (\dots,\underline{w}''_i,\dots,\underline{w}_r) = \binom{\dots,\ u_i\ ,\dots,\ u_r}{\dots,\ \underline{v}''_i\ ,\dots,\ \underline{v}''_r} \quad \textit{with}\ \ \underline{w}_i\ \ \textit{later than}\ \ \underline{w}_* \\ \underline{w}''_* &:= \binom{u_*}{\underline{v}''_*} \qquad \big(\underline{w}''_* := \varnothing\ \ \textit{if}\ \ \underline{v}''_* := \varnothing\big) \end{aligned}$$

Proposition 2.11 (Alien derivatives of $\mathcal{S}^{\underline{\boldsymbol{w}}}(x)$) .
As was the case with simple monomials $\mathcal{S}^{\boldsymbol{w}}(x)$, the only alien derivations Δ_{ω_0} acting effectively on a general monomial $\mathcal{S}^{\underline{\boldsymbol{w}}}(x) = \mathcal{S}^{\binom{u_1\ ,\dots,\ u_*\ ,\dots,\ u_r}{\underline{v}_1\ ,\dots,\ \underline{v}_*\ ,\dots,\ \underline{v}_r}}$ correspond to simple (155) or composite (155) index ω_0:

$$\omega_0 = |\boldsymbol{u}|\, v_* \qquad \textit{with} \quad \begin{cases} \boldsymbol{w} = \dot{\underline{\boldsymbol{w}}}.\underline{w}_*.\ddot{\underline{\boldsymbol{w}}}.\vec{\underline{\boldsymbol{w}}} \\ |\boldsymbol{u}| = |\dot{\boldsymbol{u}}| + u_* + |\ddot{\boldsymbol{u}}| \end{cases} \tag{155}$$

$$\omega_0 = \sum_{1\leqslant i\leqslant s} |\boldsymbol{u^i}|\, v_{i*} \quad \textit{with} \quad \begin{cases} \underline{\boldsymbol{w}} = \dot{\boldsymbol{w}}^{\boldsymbol{1}}.\underline{w}_{1*}.\ddot{\underline{\boldsymbol{w}}}^{\boldsymbol{1}} \dots \dot{\underline{\boldsymbol{w}}}^{\boldsymbol{s}}.\underline{w}_{s*}.\ddot{\underline{\boldsymbol{w}}}^{\boldsymbol{s}}.\vec{\underline{\boldsymbol{w}}} \\ |\boldsymbol{u^i}| = |\dot{\boldsymbol{u}}^{\boldsymbol{i}}| + u_{i*} + |\ddot{\boldsymbol{u}}^{\boldsymbol{i}}| \end{cases} \tag{156}$$

but with this important difference that v_* (resp. v_{i*}) now denotes some element[42] *of the sequence $\underline{v}_*$ (resp. $\underline{v}_{i*}$).*
For a simple index ω_0, the action of Δ_{ω_0} involves the so-called texture mould tex$^\bullet$ *which, unlike the tessellation bimould, doesn't depend on the weights u_i:*

$$\Delta_{\omega_0}\, \mathcal{S}^{\underline{\boldsymbol{w}}}(x) \quad = \sum_{v_*\text{-}split} \mathrm{tex}_{v_*}^{\{\underline{v}'\}}\ \mathcal{T}_{v_*}^{\dot{\underline{\boldsymbol{w}}}'',\underline{w}_*''^{\dagger},\ddot{\underline{\boldsymbol{w}}}''}(x)\, \mathcal{S}^{\vec{\underline{\boldsymbol{w}}}}(x) \tag{157}$$

$$\textit{with} \qquad \begin{cases} \mathcal{T}^{\dot{\underline{\boldsymbol{w}}},\underline{w}_*^{\dagger},\ddot{\underline{\boldsymbol{w}}}} := \mathrm{concat}\Big(\mathrm{symlin}\big(\mathcal{S}^{\dot{\underline{\boldsymbol{w}}}}, (\mathrm{invmu}.\mathcal{S})^{\ddot{\underline{\boldsymbol{w}}}}\big), \mathcal{S}^{\underline{w}_*}\Big) \\ \mathcal{T}_{v_*}^{\dot{\underline{\boldsymbol{w}}},\underline{w}_*^{\dagger},\ddot{\underline{\boldsymbol{w}}}} := \mathrm{concat}\Big(\mathrm{symlin}\big(\mathcal{S}_{v_*}^{\dot{\underline{\boldsymbol{w}}}}, (\mathrm{invmu}.\mathcal{S}_{v_*})^{\ddot{\underline{\boldsymbol{w}}}}\big), \mathcal{S}_{v_*}^{\underline{w}_*}\Big) \\ \mathrm{tex}_{v_*}^{\{\underline{v}_1;\dots;\underline{v}_s\}} := \sum_{\underline{v}^\sharp\in\mathrm{sha}(\underline{v}_1;\dots;\underline{v}_s)} V^{[\underline{v}^\sharp, v_*]} \end{cases}$$

When $\underline{w}_* = \varnothing$ the definition of $\mathcal{T}^{\dot{\underline{\boldsymbol{w}}},\underline{w}_*^{\dagger},\ddot{\underline{\boldsymbol{w}}}}$ reduces to

$$\mathcal{T}^{\dot{\underline{\boldsymbol{w}}},\underline{w}_*^{\dagger},\ddot{\underline{\boldsymbol{w}}}} := \mathrm{symlin}\big(\mathcal{S}^{\dot{\underline{\boldsymbol{w}}}}, (\mathrm{invmu}.\mathcal{S})^{\ddot{\underline{\boldsymbol{w}}}}\big) \ = \ \mathcal{S}^{\dot{\underline{\boldsymbol{w}}}}\,.\,(\mathrm{invmu}.\mathcal{S})^{\ddot{\underline{\boldsymbol{w}}}}$$

For a composite index ω_0, the action involves both tes$^\bullet$ and tex$^\bullet$:

$$\Delta_{\omega_0}\, \mathcal{S}^{\underline{\boldsymbol{w}}}(x) = \sum_{v_*\text{-}splits} \mathrm{tes}^{\binom{|\boldsymbol{u^1}|\ ,\dots,\ |\boldsymbol{u^s}|}{\underline{v}_1'\ ,\dots,\ \underline{v}_s'}} \Big(\prod_{j=1}^{j=s} \mathcal{T}_{\underline{v}_j'}^{\dot{\underline{\boldsymbol{w}}}^{j\prime\prime},\underline{w}_{j*}^{\dagger},\ddot{\underline{\boldsymbol{w}}}^{j\prime\prime}}(x)\Big)\, \mathcal{S}^{\vec{\underline{\boldsymbol{w}}}}(x) \tag{158}$$

The sum (157) extends to all v_*-splittings of $(\dot{\underline{\boldsymbol{w}}}, \underline{w}_*, \ddot{\underline{\boldsymbol{w}}})$, and the sum (158) to all v_*-splittings of $(\dot{\underline{\boldsymbol{w}}}^{\boldsymbol{i}}, \underline{w}_{i*}, \ddot{\underline{\boldsymbol{w}}}^{\boldsymbol{i}})$ For simple sequences $\underline{\boldsymbol{w}}$, all texture coefficients become $tex_{v_*}^{\{\varnothing\}} \equiv 1$, so that (157) reduces to (151) and (158) to (153).

Short proof: The index postponement identity.

$$(\mathrm{post}_i A)^{\dot{\boldsymbol{\omega}},\omega_i,\ddot{\boldsymbol{\omega}}} \equiv (-1)^{r(\ddot{\boldsymbol{\omega}})} \sum_{\ddot{\boldsymbol{\omega}}\in\mathrm{sha}(\dot{\boldsymbol{\omega}},\tilde{\ddot{\boldsymbol{\omega}}})} A^{\ddot{\boldsymbol{\omega}},\omega_i} \qquad \forall A^\bullet \in \textit{alternal} \tag{159}$$

[42]not necessarily the first or last, but *any* element.

applies only for alternal moulds $A^\bullet$, but since the expansion on the right-hand side of (159) is fully determined, it follows that the postponement operators always verify

$$\mathrm{post}_j \ \mathrm{post}_i \equiv \mathrm{post}_j \qquad\qquad (\forall i, j) \tag{160}$$

whether the moulds on which they act are alternal or not. If we now write the backward induction rule in the case $\vec{\underline{\boldsymbol{w}}} = \emptyset$, we get

$$\mathrm{cutfi}_M^{\binom{|\boldsymbol{u}|}{v_i}} SM^{\underline{\boldsymbol{w}}} = \mathrm{concat}\big(\mathrm{symlin}(SM^{\underline{\dot{\boldsymbol{w}}}}_{v_i}, {}^*SM^{\underline{\ddot{\boldsymbol{w}}}}_{v_i}), {}^\sharp SM^{\underline{w}_i}_{v_i}\big)$$

Formally, this is nothing but a postponement identity for the index $\underline{w}_i$, followed by the removal of the first element v_i of $\underline{v}_i$ and by the subtraction of that same v_i from *all* elements of *all* lower sequences $\underline{v}_j$. We can easily iterate the process. For a v_*-splitting of $\underline{\boldsymbol{w}}$ and $\underline{v}^\diamond \in sha(\{\underline{\boldsymbol{v}}'\})$

$$\underline{v}^\diamond := (\underline{v}_1^\diamond, \ldots, \underline{v}_n^\diamond) \in \mathrm{sha}(\{\underline{\boldsymbol{v}}'\}) = \mathrm{sha}(\underline{v}_1'; \ldots; \underline{v}_r')$$

let us calculate

$$\mathrm{cutfi}_M^{\binom{|\boldsymbol{u}|}{v_* - v_n^\diamond}} \mathrm{cutfi}_M^{\binom{|\boldsymbol{u}|}{v_n^\diamond - v_{n-1}^\diamond}} \ldots \mathrm{cutfi}_M^{\binom{|\boldsymbol{u}|}{v_2^\diamond - v_1^\diamond}} \mathrm{cutfi}_M^{\binom{|\boldsymbol{u}|}{v_1^\diamond}} SM^{\underline{\boldsymbol{w}}}$$

Using the crucial identity (160) , we arrive at a result

$$\mathrm{concat}\big(\mathrm{symlin}(SM^{\underline{\dot{\boldsymbol{w}}}''}_{v_*}, {}^*SM^{\underline{\ddot{\boldsymbol{w}}}''}_{v_*}), SM^{\underline{w}_*''}_{v_*}\big)$$

that does not depend on the choice of $\underline{v}^\diamond$ in $sha(\{\underline{\boldsymbol{v}}'\})$.
As a consequence, if we now calculate

$$\Delta_{|\boldsymbol{u}|v_*} \mathcal{S}^{\underline{\boldsymbol{w}}}(x) = \Delta_{(|\boldsymbol{u}|(v_* - v_n^\diamond) + |\boldsymbol{u}|(v_n^\diamond - v_{n-1}^\diamond) + \ldots |\boldsymbol{u}|(v_2^\diamond - v_1^\diamond) + |\boldsymbol{u}|(v_1^\diamond)} \mathcal{S}^{\underline{\boldsymbol{w}}}(x)$$

and apply the backward induction rule (123) and the prescription (98) for alien-differentiation , we find

$$\Delta_{\omega_0} \mathcal{S}^{\underline{\boldsymbol{w}}}(x) \;\; = \sum_{v_*\text{-}split} \big(\sum V^{\,[\underline{\boldsymbol{v}}^\diamond, v_*]}\big)\, \mathcal{T}^{\underline{\dot{\boldsymbol{w}}}'', \underline{w}_*''^\dagger, \underline{\ddot{\boldsymbol{w}}}''}_{v_*}(x)$$

which, in view of the definition of $tex^\bullet$ (see after (157)), is exactly the identity (157) in the case $\vec{\underline{\boldsymbol{w}}} = \emptyset$.

The argument for proving (157) when $\vec{\underline{\boldsymbol{w}}} \neq \emptyset$ is no different. Lastly, to establish (158) for composite indices ω_0 of type (155), the only additional result required is the factorisation lemma for *gentes*$^\bullet$ in Proposition 2.11.

2.10 The tessellation and texture coefficients $tes^\bullet/tex^\bullet$.

Since both the tesselation coefficients $tes^{\boldsymbol{w}} := (scram.\underline{V})^{\boldsymbol{w}}$ and their generalised variant $gentes^{\underline{\boldsymbol{w}}} := (gen.scram.\underline{V})^{\underline{\boldsymbol{w}}}$, despite being defined in terms of the transcendental hyperlogarithms $V^{\boldsymbol{\omega}}$, turn out to possess remarkable properties of local-constancy in their upper and lower indices, and since both encapsulate some sort of 'universal geometry' that governs co-equational resurgence, we must take a closer look at them.

The tesselation bimould $tes^\bullet$.

We recall its definition, which is based on the scramble transform of the monics $V^\bullet$ taken in incremental notation:

$$\begin{array}{rcll} \mathrm{tes}^\bullet & := & \mathrm{scram}\,.\,\underline{\mathrm{V}}^\bullet & \textit{with} \quad \underline{\mathrm{V}}^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ v_1 & ,\dots, & v_r \end{smallmatrix}\right)} := \mathrm{V}^{\,u_1 v_1,\dots,u_r v_r} \\ \Longrightarrow \quad \mathrm{tes}^{\boldsymbol{w}} & := & \sum_{\boldsymbol{w}'} \epsilon^{\boldsymbol{w}}_{\boldsymbol{w}'}\, \underline{V}^{\boldsymbol{w}'} & \textit{with} \qquad \epsilon^{\boldsymbol{w}}_{\boldsymbol{w}'} \in \{\pm 1\} \ \ , \ \ \sum |\epsilon^{\boldsymbol{w}}_{\boldsymbol{w}'}| = r!! \end{array}$$

The natural setting for studying $tes^\bullet$ is the *biprojective* space $\mathbb{P}^{r,r}$ equal to $\mathbb{C}^{2r}$ quotiented by the relation $\{\boldsymbol{w}^1 \sim \boldsymbol{w}^2\} \Leftrightarrow \{\boldsymbol{u}^1 = \lambda \boldsymbol{u}^2\,,\, \boldsymbol{v}^1 = \mu\, \boldsymbol{v}^2\ (\lambda, \mu \in \mathbb{C}^*)\}$. But rather than using biprojectivity to get rid of two coordinates (u_i, v_i), it is often useful, on the contrary, to resort to the *augmented* or *long* notation, by *adding* two redundant coordinates (u_0, v_0). The *long* coordinates $(u_i^\sharp, v_i^\sharp)$ relate to the short ones (u_i, v_i) under the rules:

$$u_i = u_i^\sharp \quad , \quad v_i = v_i^\sharp - v_0^\sharp \qquad (1 \leqslant i \leqslant r) \tag{161}$$

The *long* $u_i^\sharp$ are constrained by $u_0^\sharp + \cdots + u_r^\sharp = 0$ while the *long* $v_i^\sharp$ are, dually, regarded as defined up to a common additive constant. Thus we have $<u^\sharp, v^\sharp>=<u,v>$.The indices i of the *long* coordinates are viewed as elements of $\mathbb{Z}_{r+1} = \mathbb{Z}/(r+1)\mathbb{Z}$ with the natural circular ordering on number triplets $circ(i_1 < i_2 < i_3)$ that goes with it. Lastly, we require r^2-1 basic "homographies" $H_{i,j}$ on $\mathbb{P}^{r,r}$, defined by:

$$H_{i,j}(\boldsymbol{w}) \ := \ Q^*_{i,j}(\boldsymbol{w})/Q^{**}_{i,j}(\boldsymbol{w}) \qquad (i-j \neq 0; i,j \in \mathbb{Z}_{r+1}) \tag{162}$$

$$Q^*_{i,j}(\boldsymbol{w}) \ := \sum_{\mathrm{circ}(i<q\leqslant j)} u_q^\sharp\,(v_q^\sharp - v_i^\sharp) \tag{163}$$

$$Q^{**}_{i,j}(\boldsymbol{w}) \ := \sum_{\mathrm{circ}(j<q\leqslant i)} u_q^\sharp\,(v_q^\sharp - v_i^\sharp) \ = \ <\boldsymbol{u},\boldsymbol{v}> - Q^*_{i,j}(\boldsymbol{w}) \tag{164}$$

Proposition 2.12 (Local constancy of $tes^{\boldsymbol{w}}$) .
Outside a finite number of hypersurfaces $\Im(H_{i,j}(\boldsymbol{w})) = 0$ of $\mathbb{C}^{2r}$ (see (162) below), the tessellation coefficients $\mathrm{tes}^{\boldsymbol{w}}$ are constant in each upper index u_i and each lower index v_i.

Sketch of proof: By repeated application of the formulae in §2.4 for the partial differentiation of the hyperlogarithmic monics.

On the other hand the tessellation coefficients are not *globally* constant as soon as $r > 1$ (for $r = 1$, $tes^{w_1} \equiv 1$). Indeed:

Proposition 2.13 (The jump rule for $tes^{\boldsymbol{w}}$) .
It is only when $\boldsymbol{w}$ crosses a hypersurface $\mathcal{H}^+_{i,j} = \{\boldsymbol{w} \in \mathbb{C}^{2r}\,;\, H_{i,j}(\boldsymbol{w}) \in \mathbb{R}^+\}$ that $\mathrm{tes}^{\boldsymbol{w}}$ can change its value. More precisely, let $\boldsymbol{w}$ be any point on $\mathcal{H}^+_{i,j}$ and let $\boldsymbol{w}^+, \boldsymbol{w}^-$ be two points close by, with $\Im \boldsymbol{w}^+ > 0$, $\Im \boldsymbol{w}^- < 0$. Then

$$\boldsymbol{tes}^{\boldsymbol{w}^+} - \boldsymbol{tes}^{\boldsymbol{w}^-} = \boldsymbol{tes}^{\boldsymbol{w}^*}\, \boldsymbol{tes}^{\boldsymbol{w}^{**}} \tag{165}$$

$$\textit{with} \quad \begin{cases} \boldsymbol{w}^* := \left(\begin{smallmatrix} u_{i+1} & ,\ldots, & u_p & ,\ldots, & u_j \\ v_{i+1}-v_i & ,\ldots, & v_p-v_i & ,\ldots, & v_j-v_i \end{smallmatrix}\right) & \qquad \left(\mathrm{circ}(i < p \leqslant j) \ \in \mathbb{Z}_{r+1}\right) \\ \boldsymbol{w}^{**} := \left(\begin{smallmatrix} u_{j+1} & ,\ldots, & u_q & ,\ldots, & u_{i-1} \\ v_{j+1}-v_i & ,\ldots, & v_q-v_i & ,\ldots, & v_{i-1}-v_i \end{smallmatrix}\right) & \qquad \left(\mathrm{circ}(j < q < i) \ \in \mathbb{Z}_{r+1}\right) \end{cases}$$

Proof: Start from the hyperlogarithmic expansion of $tes^{\boldsymbol{w}}$ and apply the jump formula (100)-(101) to each individual hyperlogarithmic summand.

This begs for an alternative, simpler expression of $tes^{\boldsymbol{w}}$, or rather, to get rid of the $2\pi i$ factors, of its normalized variant $tes^{\boldsymbol{w}}_{nor}$:

$$\mathrm{tes}^{w_1,\ldots,w_r}_{\mathrm{nor}} \;:=\; (2\pi i)^{r-1}\, \mathrm{tes}^{w_1,\ldots,w_r} \tag{166}$$

The following induction rule, based on the jump formula (100)-(101) applied to each individual hyperlogarithmic summand, provides such an elementary alternative:

Proposition 2.14 (Calculation of $tes^{\boldsymbol{w}}$) .
We fix some $c \in \mathbb{C}^*$ and set $\Re_c : z \in \mathbb{C} \mapsto \Re(c\, z) \in \mathbb{R}$. Then we define:

$$f^{\boldsymbol{w}'}_{\boldsymbol{w}} := <\boldsymbol{u}', \boldsymbol{v}'><\boldsymbol{u}, \boldsymbol{v}>^{-1} \quad , \quad g^{\boldsymbol{w}'}_{\boldsymbol{w}} := <\boldsymbol{u}', \Re_\theta \boldsymbol{v}'><\boldsymbol{u}, \Re_\theta \boldsymbol{v}>^{-1} \tag{167}$$
$$f^{\boldsymbol{w}''}_{\boldsymbol{w}} := <\boldsymbol{u}'', \boldsymbol{v}''><\boldsymbol{u}, \boldsymbol{v}>^{-1} \quad , \quad g^{\boldsymbol{w}''}_{\boldsymbol{w}} := <\boldsymbol{u}'', \Re_\theta \boldsymbol{v}''><\boldsymbol{u}, \Re_\theta \boldsymbol{v}>^{-1} \tag{168}$$

From these scalars we construct the crucial sign factor sig which takes its values in $\{-1, 0, 1\}$. Here, the abbreviation $si(.)$ stands for $sign(\Im(.))$.

$$\mathrm{sig}^{\boldsymbol{w}',\boldsymbol{w}''} = \mathrm{sig}^{\boldsymbol{w}',\boldsymbol{w}''}_c := \frac{1}{8} \begin{cases} \left(\mathrm{si}(f^{\boldsymbol{w}'}_{\boldsymbol{w}} - f^{\boldsymbol{w}''}_{\boldsymbol{w}}) - \mathrm{si}(g^{\boldsymbol{w}'}_{\boldsymbol{w}} - g^{\mathbf{w}''}_{\boldsymbol{w}})\right) \times \\ \left(1 + \mathrm{si}(f^{\boldsymbol{w}'}_{\boldsymbol{w}}/g^{\mathbf{w}'}_{\boldsymbol{w}})\ \ \mathrm{si}(f^{\boldsymbol{w}'}_{\boldsymbol{w}} - g^{\boldsymbol{w}'}_{\boldsymbol{w}})\right) \times \\ \left(1 + \mathrm{si}(f^{\boldsymbol{w}''}_{\boldsymbol{w}}/g^{\boldsymbol{w}''}_{\boldsymbol{w}})\ \mathrm{si}(f^{\boldsymbol{w}''}_{\boldsymbol{w}} - g^{\boldsymbol{w}''}_{\boldsymbol{w}})\right) \end{cases} \tag{169}$$

Next, from the pair $(\boldsymbol{w}', \boldsymbol{w}'')$ we derive a pair $(\boldsymbol{w}^*, \boldsymbol{w}^{**})$ by setting:

$$\boldsymbol{u}^* := \boldsymbol{u}' \ , \ \boldsymbol{v}^* := \boldsymbol{v}' \ <\boldsymbol{u}, \boldsymbol{v}>^{-1} \Im g^{\boldsymbol{w}'}_{\boldsymbol{w}} \ - \ \Re_c \boldsymbol{v}' \ <\boldsymbol{u}, \Re_c \boldsymbol{v}>^{-1} \Im f^{\boldsymbol{w}'}_{\boldsymbol{w}} \tag{170}$$
$$\boldsymbol{u}^{**} := \boldsymbol{u}'' \ , \ \boldsymbol{v}^{**} := \boldsymbol{v}'' \ <\boldsymbol{u}, \boldsymbol{v}>^{-1} \Im g^{\boldsymbol{w}''}_{\boldsymbol{w}} \ - \ \Re_c \boldsymbol{v}'' \ <\boldsymbol{u}, \Re_c \boldsymbol{v}>^{-1} \Im f^{\boldsymbol{w}''}_{\boldsymbol{w}} \tag{171}$$

or more symmetrically:

$$\boldsymbol{v}^* := \det\begin{pmatrix} \frac{\boldsymbol{v}'}{<\boldsymbol{u},\boldsymbol{v}>} & \frac{\Re_c \boldsymbol{v}'}{<\boldsymbol{u},\Re_c\boldsymbol{v}>} \\ \Im\frac{<\boldsymbol{u}',\boldsymbol{v}'>}{<\boldsymbol{u},\boldsymbol{v}>} & \Im\frac{<\boldsymbol{u}',\Re_c\boldsymbol{v}'>}{<\boldsymbol{u},\Re_c\boldsymbol{v}>}\end{pmatrix} , \ \boldsymbol{v}^{**} := \det\begin{pmatrix} \frac{\boldsymbol{v}''}{<\boldsymbol{u},\boldsymbol{v}>} & \frac{\Re_c \boldsymbol{v}''}{<\boldsymbol{u},\Re_c\boldsymbol{v}>} \\ \Im\frac{<\boldsymbol{u}'',\boldsymbol{v}''>}{<\boldsymbol{u},\boldsymbol{v}>} & \Im\frac{<\boldsymbol{u}'',\Re_c\boldsymbol{v}''>}{<\boldsymbol{u},\Re_c\boldsymbol{v}>}\end{pmatrix}$$

Lastly, from all these ingredients, we construct an auxilliary bimould $\mathrm{urtes}^{\bullet}_{\mathrm{nor}}$ *by setting:*

$$\mathrm{urtes}^{\boldsymbol{w}}_{\mathrm{nor}} \;=\; \sum_{\boldsymbol{w}'\boldsymbol{w}''=\boldsymbol{w}} \mathrm{sig}^{\boldsymbol{w}'\boldsymbol{w}''}\ \mathrm{tes}^{\boldsymbol{w}^*}_{\mathrm{nor}}\ \mathrm{tes}^{\boldsymbol{w}^{**}}_{\mathrm{nor}} \qquad \Big((\boldsymbol{w}',\boldsymbol{w}'') \mp (\boldsymbol{w}^*,\boldsymbol{w}^{**})\Big) \tag{172}$$

Then the tessellation bimould can be inductively calculated from:

$$\mathrm{tes}^{\bullet}_{\mathrm{nor}} \;=\; \sum_{0\leqslant n\leqslant r(\bullet)} \mathrm{push}^n\ \mathrm{urtes}^{\bullet}_{\mathrm{nor}} \qquad (\forall c\in\mathbb{C}^*) \tag{173}$$

Proof: The jump formulae (100)-(101) make it clear that the locally conctant $tes^{\boldsymbol{w}}$ can change values only when $\boldsymbol{w}$ crosses one of the r^2-1 hypersurfaces $\Im(H_{i,j}(\boldsymbol{w}))=0$, which themselves can be derived from the $r-1$ hypersurfaces $\Im\frac{<\boldsymbol{u}',\boldsymbol{v}'>}{<\boldsymbol{u}'',\boldsymbol{v}''>}=0$ under repeated application of the *push*-transform. We also note that $tes^{\boldsymbol{w}}$ takes the same value at the points $\boldsymbol{w}=\binom{\boldsymbol{u}}{\boldsymbol{v}}$ and $\overline{\boldsymbol{w}}=\binom{\boldsymbol{u}}{\overline{\boldsymbol{v}}}$ with $\overline{\boldsymbol{v}}:=\boldsymbol{v}<\boldsymbol{u},\boldsymbol{v}>^{-1}$, and further that $tes^{\underline{\boldsymbol{w}}}=0$ at the semi-real point $\underline{\boldsymbol{w}}=\binom{\boldsymbol{u}}{\underline{\boldsymbol{v}}}$ with $\underline{\boldsymbol{v}}:=\Re_c\boldsymbol{v}<\boldsymbol{u},\Re_c\boldsymbol{v}>^{-1}$. So it all becomes a question of comparing $tes^{\overline{\boldsymbol{w}}}$ and $tes^{\underline{\boldsymbol{w}}}$. To that end, we set $\boldsymbol{w}(t):=\binom{\boldsymbol{u}}{\boldsymbol{v}(t)}$ with $\boldsymbol{v}(t):=\underline{\boldsymbol{v}}+t.(\overline{\boldsymbol{v}}-\underline{\boldsymbol{v}})$. The line $\{\boldsymbol{w}(t);t\in\mathbb{R}\}$ joins the points $\underline{\boldsymbol{w}}$ (for $t=0$) and $\overline{\boldsymbol{w}}$ (for $t=1$) and crosses the hypersurface $\Im\frac{<\boldsymbol{u}',\boldsymbol{v}'>}{<\boldsymbol{u}'',\boldsymbol{v}''>}=0$, for some critical $t=t_0$, at the point $\boldsymbol{w}^*\boldsymbol{w}^{**}=\binom{\boldsymbol{u}^*\boldsymbol{u}^{**}}{\boldsymbol{v}^*\boldsymbol{v}^{**}}$, with $\boldsymbol{u}^*,\boldsymbol{v}^*$ and $\boldsymbol{u}^{**},\boldsymbol{v}^{**}$ as above. Lastly, regarding the three factors in the expression (169) of $sig^{\boldsymbol{w}';\boldsymbol{w}''}$, their interpretation is as follows:
(i) the first factor is ± 2 (resp.0) if $\overline{\boldsymbol{w}}$ and $\underline{\boldsymbol{w}}$ lie on distinct sides of the hypersurface $\Im\frac{<\boldsymbol{u}',\boldsymbol{v}'>}{<\boldsymbol{u}'',\boldsymbol{v}''>}=0$ (resp. on the same side).
(ii) the second factor is 2 (resp.0) if the critical value t_0 is >0 (resp.<0).
(iii) the third factor is 2 (resp.0) if the critical value t_0 is <1 (resp.>1).
Thus, formulae (172)-(173) exactly reflect the changes which $tes^{\boldsymbol{w}}$ undergoes when $\boldsymbol{w}$ moves from the semi-real $\underline{\boldsymbol{w}}$ to $\overline{\boldsymbol{w}}\sim\boldsymbol{w}$ after crossing some of the r^2-1 hypersurfaces $\Im(H_{i,j}(\boldsymbol{w}))=0$. $\square$

Remark 1: In the induction (174) we might exchange everywhere the role of $\boldsymbol{u}$ and $\boldsymbol{v}$ and still get the correct answer $tes^{\bullet}_{nor}$, but via a different auxilliary bimould $urtes^{\bullet}_{nor}$.

Remark 2: The above induction for $tes^{\bullet}$ is elementary[43] in the sense of

[43] and easily programmable.

being non-transcendental: it depends only on the *sign function.* But on the face of it, it looks non-intrinsical. Indeed, the partial sum relative to the choice $c = e^{i\theta}$:

$$\mathrm{urtes}_{\theta}^{\boldsymbol{w}} := \sum_{\boldsymbol{w}'\boldsymbol{w}''=\boldsymbol{w}} \mathrm{sig}^{\boldsymbol{w}',\boldsymbol{w}''}\mathrm{tes}_{\mathrm{nor}}^{\boldsymbol{w}^*}\mathrm{tes}_{\mathrm{nor}}^{\boldsymbol{w}^{**}} = \sum_{\boldsymbol{w}'\boldsymbol{w}''=\boldsymbol{w}} \mathrm{sig}_{(\theta)}^{\boldsymbol{w}',\boldsymbol{w}''}\mathrm{tes}_{\mathrm{nor}}^{\boldsymbol{w}_{\theta}^*}\mathrm{tes}_{\mathrm{nor}}^{\boldsymbol{w}_{\theta}^{**}} \tag{174}$$

is *polarised*, i.e. θ-dependent. However, its *push*-invariant offshoot :

$$\mathrm{tes}_{\mathrm{nor}}^{\bullet} := \sum_{0\leqslant n\leqslant r(\boldsymbol{w})} \mathrm{push}^{n}\ \mathrm{urtes}_{\theta}^{\bullet} \tag{175}$$

is duly *unpolarised.* We might of course remove the polarisation in $urtes_{\theta}^{\bullet}$ itself by replacing it by this isotropic variant:

$$\mathrm{urtes}_{\mathrm{iso}}^{\bullet} := \frac{1}{2\,\pi}\int_{0}^{2\pi} \mathrm{urtes}_{\theta}^{\bullet}\ \ d\theta \tag{176}$$

but at the cost of rendering it less elementary, since $urtes_{iso}^{\bullet}$ would assume its value in $\mathbb{R}$ rather than $\{-1,0,1\}$. It would also depend hyperlogarithmically on its indices, and thus take us back to something rather like formula (104), which we wanted to get away from. Thus, the alternative so far for our bimould $\mathrm{tes}^{\bullet}$ is: *either an intrinsical but heavily transcendental expression, or an elementary but heavily polarised one.*

Remark 3: Let $h_{i,j} := \mathrm{sign}\big(\Im H_{i,j}(\boldsymbol{w})\big)$.
(i)For $r = 1$, we have trivially $tes^{w_1} \equiv 1$.
(ii) For $r = 2$, we find:

$$H_{0,1}(\boldsymbol{w}) = \frac{u_1 v_1}{u_2 v_2}\ ,\ H_{1,2}(\boldsymbol{w}) = \frac{u_2(v_2-v_1)}{(u_1+u_2)v_1}\ ,\ H_{2,0}(\boldsymbol{w}) = \frac{(u_1+u_2)v_2}{u_1(v_1-v_2)}$$

and the corresponding signs $h_{i,j}$ determine $tes^{\boldsymbol{w}}$:

$$\mathrm{tes}^{w_1,w_2} = \begin{cases} \pm 2\pi i \ \ \textit{iff} \ \ h_{0,1}(\boldsymbol{w}) = h_{1,2}(\boldsymbol{w}) = h_{2,0}(\boldsymbol{w}) = \pm 1 \\ 0 \quad \textit{otherwise} \end{cases} \tag{177}$$

(iii) For $r \geqslant 3$, the r^2-1 independent signs $\{h_{i,j}\,;\, i,j \in \mathbb{Z}_{r+1}, j-i \neq r\}$ do not suffice to determine $tes^{\boldsymbol{w}}$, except in some very special cases, like:

$$\{h_{i,j}(\boldsymbol{w}) \equiv +1 \ \ \forall i,j\} \Longrightarrow \{tes^{w_1,..,w_r} = (+2\pi i)^{r-1}\} \tag{178}$$
$$\{h_{i,j}(\boldsymbol{w}) \equiv -1 \ \ \forall i,j\} \Longrightarrow \{tes^{w_1,..,w_r} = (-2\pi i)^{r-1}\} \tag{179}$$

Remark 4: To be able to determine the tessellation coefficients purely in terms of 'signs', we must revert to their expression as sums of $r!!$ hyperlogarithms $tes^{\boldsymbol{w}} := \sum \epsilon_i V^{\boldsymbol{\omega^i}} = \sum \epsilon_i V^{\omega_1^i,\dots,\omega_r^i}$ and set:

$$h^i_{j_1,j_2,j_3}(\boldsymbol{w}) := sign \Im\Big(\frac{\sum_{j_1<j\leqslant j_2}\omega_j^i}{\sum_{j_2<j\leqslant j_3}\omega_j^i}\Big) \qquad \begin{cases}\forall j_1,j_2,j_3\\ 0\leqslant j_1<j_2<j_3\leqslant r\end{cases} \tag{180}$$

Unfortunately, these $h^i_{j_1,j_2,j_3}(\boldsymbol{w})$ are far too numerous (even taking into account their dependence relations) to be of practical assistance, and we know of no simple rule for inferring $tes^{\boldsymbol{w}}$ from them. So, at the moment, the induction formula (173) remains the simplest way of calculating $tes^{\boldsymbol{w}}$.

Proposition 2.15 (Main properties of $\mathrm{tes}^\bullet$) .

***P*₁**: *$tes^\bullet$ is invariant under the involution swap and the iden-potent push:*

$$\mathrm{swap}.A^{\binom{u_1\ ,\dots,\ u_r}{v_1\ ,\dots,\ v_r}} = A^{\binom{v_r\ ,\dots,\ v_3-v_4\ ,\ v_2-v_3\ ,\ v_1-v_2}{u_1+\dots+u_r\ ,\dots,\ u_1+u_2+u_3\ ,\ u_1+u_2\ ,\ u_1}} \qquad (\mathrm{swap}^2=\mathrm{iden})$$

$$\mathrm{push}.A^{\binom{u_1\ ,\dots,\ u_r}{v_1\ ,\dots,\ v_r}} = A^{\binom{-u_1\dots-u_r\ ,\ u_1\ ,\ u_2\ ,\dots,\ u_{r-1}}{-v_r\ ,\ v_1-v_r\ ,\ v_2-v_r\ ,\dots,\ v_{r-1}-v_r}} \qquad (\mathrm{push}^{r+1}=\mathrm{iden})$$

***P*₂**: *the bimould $tes^\bullet$ is bialternal, i.e. alternal and of alternal swappee.*

***P*₃**: *$tes^\bullet_{nor}$ assumes all its sole values in $\mathbb{Z}$ and $|tes^{w_1,\dots,w_r}|<(r-1)!(r+1)!$ (far from sharp)*

***P*₄**: *As r increases, the set where $\mathrm{tes}^\bullet \neq 0$ has surprisingly small Lebesgue measure.*

$$\begin{array}{ll}
\mathrm{tes}^{w_1}\equiv 1 & \\
\mathrm{tes}^{w_1,w_2}\in\{0,\pm1\} & \mathcal{P}(|\mathrm{tes}^{w_1,w_2}|=1)\sim 0.21\\
\mathrm{tes}^{w_1,w_2,w_3}\in\{0,\pm1\} & \mathcal{P}(|\mathrm{tes}^{w_1,w_2,w_3}|=1)\sim 0.026\\
\mathrm{tes}^{w_1,\dots,w_4}\in\{0,\pm1,\pm2\} & \begin{cases}\mathcal{P}(|\mathrm{tes}^{w_1,\dots,w_4}|=1)\sim 0.0037\\ \mathcal{P}(|\mathrm{tes}^{w_1,\dots,w_4}|=2)\sim 0.0000037\end{cases}
\end{array}$$

***P*₅**: *in presence of vanishing u_i-sums, we no longer have local constancy in the v_js.*

***P*₆**: *conversely, in presence of v_i-repetitions, we no longer have local constancy in the u_js.*

***P*₇**: *in the semi-real case, i.e. when either all u_is or all v_is are aligned with the origin, the tessellation coefficients altogether exit the picture, since in that case $test^{w_1,\dots,w_r}\equiv 0$ as soon as $2\leqslant r$.*

***P*₈**: *for r fixed, the hypersurfaces $\Im(H_{i,j}(\boldsymbol{w}))=0$ limit*[44] *but do not separate*[45] *the sets $\mathcal{T}_k := \{\boldsymbol{w}\ ,\ tes^{\boldsymbol{w}}=k\}$.*

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[44]that is to say, the boundaries of these sets lie on the hypersurfaces.

[45]that is to say, none of the three sets can be defined in terms of the sole signs $h_{i,j}(\boldsymbol{w}) := sign(\Im(H_{i,j}(\boldsymbol{w})))$, at least for $r\geqslant 3$. See Remark 3 and 4 *supra*.

The *swap*-invariance of $tes^\bullet$ is quite unexpected, since the involution *swap* exchanges the upper and lower indices which, in this context, have completely different origins.

The texture mould $tex^\bullet$.

We recall its definition, which is based on the monics $V^{[\bullet]}$ taken in positional notation:

$$\mathrm{tex}_{v_*}^{\{\varnothing\}} := 1 \quad , \quad \mathrm{tex}_{v_*}^{\{\underline{v}_1;\ldots;\underline{v}_s\}} := \sum_{\underline{v}^\sharp \in \mathrm{sha}(\underline{v}_1;\ldots;\underline{v}_s)} V^{[\underline{v}^\sharp, v_*]} \tag{181}$$

The system of texture coefficients is stable under differentiation:

$$\partial_{v_{i,1}} \mathrm{tex}_{v_*}^{\{\underline{v}_1;\ldots;\underline{v}_r\}} = \begin{cases} -\mathrm{tex}_{v_*}^{\{\underline{v}_1;\ldots;\underline{v}_{i||\hat{1}};\ldots;\underline{v}_r\}} \big((v_{i,1})^{-1} + (v_{i,2}-v_{i,1})^{-1}\big) \\ +\mathrm{tex}_{v_*}^{\{\underline{v}_1;\ldots;\underline{v}_{i||\hat{2}};\ldots;\underline{v}_r\}} (v_{i,2}-v_{i,1})^{-1} \end{cases}$$

$$\partial_{v_{i,k}} \mathrm{tex}_{v_*}^{\{\underline{v}_1;\ldots;\underline{v}_r\}} = \begin{cases} +\mathrm{tex}_{v_*}^{\{\underline{v}_1;\ldots;\underline{v}_{i||\widehat{k-1}};\ldots;\underline{v}_r\}} (v_{i,k}-v_{i,k-1})^{-1} \\ -\mathrm{tex}_{v_*}^{\{\underline{v}_1;\ldots;\underline{v}_{i||\hat{k}};\ldots;\underline{v}_r\}} \big((v_{i,k}-v_{i,k-1})^{-1} + (v_{i,k+1}-v_{i,k})^{-1}\big) \\ +\mathrm{tex}_{v_*}^{\{\underline{v}_1;\ldots;\underline{v}_{i||\widehat{k+1}};\ldots;\underline{v}_r\}} (v_{i,k+1}-v_{i,k})^{-1} \end{cases}$$

$$\partial_{v_*} \mathrm{tex}_{v_*}^{\{\underline{v}_1;\ldots;\underline{v}_r\}} = + \sum_{1\leqslant i\leqslant r} \mathrm{tex}_{\underline{v}_*}^{\{\underline{v}_1;\ldots;\underline{v}_i^*;\ldots;v_r\}} (v_* - v_i^\dagger)^{-1}$$

Here, $\underline{v}_{i,\hat{k}}$ and $\underline{v}_{i,\widehat{k\pm1}}$ denote the sequence $\underline{v}_i$ minus its element $v_{i,k}$ or $v_{i,k\pm1}$, and $\underline{v}_i^*$ is simply $\underline{v}_i$ minus its last element $v_i^\dagger$. If $\underline{v}_{i,k}$ happens to be the last element of $\underline{v}_i$, the corresponding identity should be changed to:

$$\partial_{v_{i,k}} \mathrm{tex}_{v_*}^{\{\underline{v}_1;\ldots;\underline{v}_r\}} = \begin{cases} +\mathrm{tex}_{v_*}^{\{\underline{v}_1;\ldots;\underline{v}_{i||\widehat{k-1}};\ldots;\underline{v}_r\}} (v_{i,k}-v_{i,k-1})^{-1} \\ -\mathrm{tex}_{v_*}^{\{\underline{v}_1;\ldots;\underline{v}_{i||\hat{k}};\ldots;\underline{v}_r\}} \big((v_{i,k}-v_{i,k-1})^{-1} + (v_*-v_{i,k})^{-1}\big) \end{cases}$$

These identities are clearly compatible with the 0-homogeneity of the texture coefficients:

$$\big(v_*\partial_* + \sum_i \sum_k v_{i,k}\partial_{v_{i,k}}\big) \mathrm{tex}_{v_*}^{\{\underline{v}_1;\ldots;\underline{v}_r\}} \equiv 0$$

For single-element sequences $\underline{v}_i = \{v_i\}$, the whole system reduces to:

$$\partial_{\alpha_i} \mathrm{tex}_{v_*}^{\{v_1;\ldots;v_r\}} = -\mathrm{tex}_{v_*}^{\{v_1;\ldots;\widehat{v}_i;\ldots;v_r\}} \big((v_i)^{-1} + (v_*-v_i)^{-1}\big) \tag{182}$$

$$\partial_{v_*} \mathrm{tex}_{v_*}^{\{v_1;\ldots;v_r\}} = + \sum_{1\leqslant i\leqslant r} \mathrm{tex}_{v_*}^{\{v_1;\ldots;\widehat{v}_i;\ldots;v_r\}} (v_*-v_i)^{-1} \tag{183}$$

where $\widehat{v}_i$ signals the omission of the term v_i.

The generalised tesselation bimould $gentes^\bullet$**.**

Proposition 2.16 (Local semi-constancy of $gentes^{\underline{w}}$**)** .
The coefficients $\mathrm{gentes}^{\underline{w}}$ *are locally constant in each weight* u_i *but not in the indices* $v_i, v_i', v_i'' \ldots$ *that constitute the lower sequences* $\underline{v}_i$. *However, they admit a decomposition of the form:*

$$\mathrm{gentes}^{\binom{u_1\ ,\ldots,\ u_r}{\underline{v}_1\ ,\ldots,\ \underline{v}_r}} \quad \equiv \quad \mathrm{tes}^{\binom{u_1\ ,\ldots,\ u_r}{v_1^\dagger\ ,\ldots,\ v_r^\dagger}} V^{[\underline{v}_1]} \ldots V^{[\underline{v}_r]} \tag{184}$$

with $v_i^\dagger$ *denoting the last element of* $\underline{v}_i$ *and with the hyperlogarithms* $V^{[\underline{v}_i]}$ *absorbing all the non-trivial part of the* $\underline{v}_i$*-dependence.*

Sketch of proof: The above decomposition follows from the following three facts, which are straightforward in a sense but extremely tedious to check:[46]
(i) the expression of all partial derivatives of $\mathrm{gentes}^{\underline{w}}$, whether in the upper or lower indices, are compatible with (174)
(ii) all jump formulae (derivable from (100)) that describe the transition from one domain of holomorphy of $\mathrm{tes}^{\underline{w}}$ to the next, are compatible with (174).
(iii) for axially aligned upper indices u_i, i.e when all ratios u_i/u_j are > 0, both sides of (174) simultaneously vanish.

2.11 The three Bridge equations at the molecular level.

Equational resurgence. First Bridge equation.

At monomial level, the alien derivatives in z are exceedingly simple, and totally insensitive to the ramifications that the lower indices $b_i(z)$ (they are regular germs at ∞) may or may not possess away from ∞:

$$\Delta_\omega \mathcal{W}^{\binom{\boldsymbol{u}}{\boldsymbol{b}}}(z,x) = \sum_{\omega = x\,|\boldsymbol{u^1}|}^{\binom{\boldsymbol{u^1}}{\boldsymbol{b^1}}\binom{\boldsymbol{u^2}}{\boldsymbol{b^2}}=\binom{\boldsymbol{u}}{\boldsymbol{b}}} W^{\binom{\boldsymbol{u^1}}{\boldsymbol{b^1}}}(x)\, \mathcal{W}^{\binom{\boldsymbol{u^2}}{\boldsymbol{b^2}}}(z,x) \tag{185}$$

The new ingredients – the alternal monics $W^\bullet(x)$ – do not depend on z. They are well-defined entire functions of x – Stokes constants, basically. The above equation can therefore be indefinitely iterated and contains all the information about the z-resurgence of $\mathcal{W}^\bullet(z,x)$.

[46] For an alternative approach, based on an a priori formula for the alien differentiation of weighted convolution products, see §2.11.

Coequational resurgence. From the atomic to the molecular level.

The position is altogether different, and far more complex, with the x-resurgence. Our monomials $\mathcal{W}^{(\boldsymbol{u} \atop \boldsymbol{b})}(z,x)$ must now be viewed as weighted products $wemu^{(\boldsymbol{u} \atop \boldsymbol{c})}(x)$, and their Borel transforms as weighted convolutions $weco^{(\boldsymbol{u} \atop \hat{\boldsymbol{c}})}(\xi)$. The z-dependence migrates to the lower indices $\widehat{c}_i$, which are themselves defined in terms of the b_i via $\widehat{c}_i(\xi) := -b_i(z-\xi)$. So, while the z-resurgence demands only the local analyticity of the *germs* $b_i(z)$ at ∞, to get full x-resurgence[47] we must assume the endless analytical continuability of these same $b_i(z)$.

The alien derivatives in x still consist of two factors. One of these (the equivalent of the monics $W^\bullet$ in the z-resurgence) sheds its z-dependence, but both retain their dependence on, and resurgence in, x. This complicates the calculation of higher-order alien derivatives. It also forces us to negotiate two quite distinct levels of complexity: even when the data $\widehat{c}_i$ (the *'atoms'*) are simple (poles or hyperlogarithms), their weighted convolutions (the *'molecules'*) tend to be superpositions of huge numbers of such atoms. This accounts for the emergence[48] of completely new properties and operations (the *flexion structure*).

Ridding the general tessellator of the $\boldsymbol{v}$-dependence.

The aim is to move from the general tesselation coefficients $tes^{\underline{\boldsymbol{w}}}$ which are locally $\boldsymbol{u}$-constant (like the special $tes^{\boldsymbol{w}}$) but not locally $\boldsymbol{v}$-constant (*unlike* the $tes^{\boldsymbol{w}}$), to coefficients $Tes^{\underline{\boldsymbol{w}}}$ that are locally $\boldsymbol{u}$- and $\boldsymbol{v}$-constant and (barring the case of alignments) assume integer values. The reason for the absence of local $\boldsymbol{v}$-constancy in the general $tes^{\underline{\boldsymbol{w}}}$ is that the formula we gave in §2.9 for $\Delta_\omega S^{\underline{\boldsymbol{w}}}(x)$ involves shifts that apply to the sequences $\underline{v}_i := [v_i, v_i', v_i''...]$ defining the hyperlogarithm associated with a given u_i, and *not* shifts bearing on the variable of that hyperlogarithm (in the ξ-plane). It is precisely the $\boldsymbol{v}$-dependent part of $tes^{\underline{\boldsymbol{w}}}$ (essentially, the 'texture' part) that, in accordance with the addition formula (130) combines with the shift on $\underline{v}_i = [v_i, v_i', v_i''...]$ to produce what is ultimately needed – a shift purely on the variable ξ. In concrete terms, it takes us from formula (158) (recalled here as (186)) to

[47]Actually, even when the $b_i(z)$ are not endlessly continuable, something of the x-resurgence survives – all the relations namely which do not take us outside the maximum domain of definition of these $b_i(z)$.

[48]somewhat like in organic chemistry, one might be tempted to say.

formula (187):

$$\Delta_{\omega_0}\, \mathcal{S}^{\underline{\boldsymbol{w}}}(x) = \sum_{v_*\text{-}splits} \mathrm{tes}^{\left({|\boldsymbol{u^1}| \atop \underline{v}'_1} {,\dots, \atop ,\dots,} {|\boldsymbol{u^s}| \atop \underline{v}'_s} \right)} \Big(\prod_{j=1}^{j=s} \mathcal{T}_{\underline{v}'_j}^{\underline{\dot{\boldsymbol{w}}}^{j''}, \underline{w}^{\dagger}_{j*}, \underline{\ddot{\boldsymbol{w}}}^{j''}}(x) \Big)\, \mathcal{S}^{\underline{\vec{\boldsymbol{w}}}}(x) \tag{186}$$

$$\Delta_{\omega_0}\, \mathcal{S}^{\underline{\boldsymbol{w}}}(x) = \sum_{\check{v}_*\text{-}shifts} \mathrm{Tes}^{\left({|\boldsymbol{u^1}| \atop \check{v}_1} {,\dots, \atop ,\dots,} {|\boldsymbol{u^s}| \atop \check{v}_s} \right)} \Big(\prod_{j=1}^{j=s} \mathcal{T}_{\check{v}_j}^{\underline{\dot{\boldsymbol{w}}}^{j}, \underline{w}^{\dagger}_{j*}, \underline{\ddot{\boldsymbol{w}}}^{j}}(x) \Big)\, \mathcal{S}^{\underline{\vec{\boldsymbol{w}}}}(x) \tag{187}$$

However, the hyperlogarithms being ramified, a shift operator on them cannot be defined by a single complex scalar v, but
(i) either by taut broken[49] lines $\check{v} = [v_1, v_2, \dots, v_k]$ starting at the origin and ending at v
(ii) or (preferably) by concatenations $\Delta_{v_j} \dots \Delta_{v_1}$ followed by a straight[50] shift $v_{j+1} + ..v_k$.

The two operations are clearly not equivalent, but their linear combinations are. Both lead to different definitions of the general tesselation mould $Tes^\bullet$, but in both cases
(i) the double local constancy (in the upper and lower indices) of the tesselation coefficients $Tes^\bullet$ is restored, barring the usual exceptions[51]
(ii) the change from (186) to (187) leads to expressions of the (locally constant) coefficients $Tes^\bullet$ as superpositions of hyperlogarithms.

From the hyperlogarithmic $S^{\underline{\boldsymbol{w}}}$ to the general $weco^{\binom{\boldsymbol{u}}{c}}$.

Let RES_{reg} be the algebra of *regular* resurgent functions, i.e. of all $\widetilde{\varphi}(x)$ such that $\widehat{\varphi}(\xi)$ and all its (simple and multiple) alien derivatives are regular (non-ramified) germs at the origin $\xi = 0$. Since the hyperlogarithms (as functions of ξ) span a dense subset of $R\acute{E}S_{reg}$ (for that space's natural topology), the information we have collected on the behaviour of hyperlogarithms under weighted convolutions is sufficient to determine the properties of that operation on $R\acute{E}S_{reg}$. Actually, if we were to allow vanishing indices ω_i (in the incremental notation) or identical consecutive indices α_i (in the positional natation), the enlarged class of hyperlogarithms so defined would become dense in the whole $R\acute{E}S$, and their behaviour under weighted convolution (readily given by an easy extension of the formulae of §2.7) would completely clarify the situation in $R\acute{E}S$ itself. But for the moment let us stick with $R\acute{E}S_{reg}$.

[49] with summits at the singular points of the test function.

[50] or, in the case of intervening singularities, by an unambiguous prescription for bypassing them, e.g. by systematic right or left circumvention.

[51] i.e. vanishing partial sums of u_i's or partial coinciding of $\underline{v}_i$'s.

Alien derivatives of weighted products.

Although the system of all symmetral weighted convolutions *weco* is closed under alien differentiation, in order to get compact expressions (and for other reasons as well) we must supplement it with the alternal weighted convolutions *welo*, whose definition we recall:[52]

$$\begin{cases} \mathrm{welo}^{\binom{u_1}{c_1},\dots,\binom{u_j}{c_j}^{\dagger},\dots,\binom{u_r}{c_r}} = \\ \mathrm{concat}\Big(\mathrm{symlin}\big(\mathrm{weco}^{\binom{u_1}{c_1}\,\dots\,\binom{u_{j-1}}{c_{j-1}}},{}^{*}\mathrm{weco}^{\binom{u_{j+1}}{c_{j+1}}\,\dots\,\binom{u_r}{c_r}}\big)\mathrm{weco}^{\binom{u_j}{c_j}}\Big) \end{cases} \tag{188}$$

When $c_j \equiv 1$, i.e. when $\widehat{c}_j$ is the convolution unit δ, the definition reduces to

$$\mathrm{welo}^{\binom{u_1}{c_1},\dots,\binom{u_j}{\delta}^{\dagger},\dots,\binom{u_r}{c_r}} = \mathrm{weco}^{\binom{u_1}{c_1}\,\dots\,\binom{u_{j-1}}{c_{j-1}}} * \big({}^{*}\mathrm{weco}^{\binom{u_{j+1}}{c_{j+1}}\,\dots\,\binom{u_r}{c_r}}\big) \tag{189}$$

$$= \mathrm{weco}^{\binom{u_1}{c_1}\,\dots\,\binom{u_{j-1}}{c_{j-1}}} * \mathrm{weco}^{\binom{u_r}{c_r}\,\dots\,\binom{u_{j+1}}{c_{j+1}}}\,(-1)^{r-j} \tag{190}$$

This is a case of frequent occurence, because in the applications the marked index is usually of the form $\binom{u_i}{\hat{\Delta}_\omega \widehat{c}_i}$, which $\hat{\Delta}_\omega \widehat{c}_i$ often equal to δ.

Second Bridge equation.

Purely for notational convenience, we shall state the results in the x-plane, i.e. in terms of the multiplicative counterparts *wemu* and *welu* of *weco* and *welo*.

Proposition 2.17 (Alien derivatives of *wemu*, hence *weco*) .
The only alien derivatives Δ_{ω_0} acting effectively on $\mathrm{wemu}^{\binom{u_1}{c_1}\,\dots\,\binom{u_r}{c_r}}(x)$ *correspond either to simple ($s = 1$) or composite ($s > 1$) indices ω_0 of the form*

$$\omega_0 = |\boldsymbol{u^1}|\, v^1_{i_1} + \dots + |\boldsymbol{u^s}|\, v^s_{i_s} \quad \textit{with} \begin{cases} \boldsymbol{u^1}\,\boldsymbol{u^2}\dots\boldsymbol{u^{s-1}}\,\boldsymbol{u^s}\,\boldsymbol{u^*} = \boldsymbol{u} \\ \Delta_{v^k_{i_k}}\, c^k_{i_k} \neq 0 \textit{ and } \binom{u^k_{i_k}}{c^k_{i_k}} \in \binom{\boldsymbol{u^k}}{\boldsymbol{c^k}} \end{cases} \tag{191}$$

with each factor sequence $\binom{\boldsymbol{u^k}}{\boldsymbol{c^k}}$ re-indexed for convenience as $\binom{u^k_1}{c^k_1},\dots,\binom{u^k_{r_k}}{c^k_{r_k}}$. The corresponding alien derivative is given by:

$$\Delta_{\omega_0}\mathrm{wemu}^{\binom{u_1}{c_1}\,\dots\,\binom{u_r}{c_r}}(x) = \begin{cases} \sum_{\check{v}^k_j \text{ over } v^k_{i_k}} \mathrm{Tes}^{\binom{|\boldsymbol{u^1}|}{\check{v}^1_1,\dots,\check{v}^1_{r_1}};\dots;\binom{|\boldsymbol{u^s}|}{\check{v}^s_1,\dots,\check{v}^s_{r_s}}} \times \\ \prod_{1\leqslant k\leqslant s} \mathrm{welu}^{\binom{u^k_1}{\check{v}^k_1\, c^k_1},\dots,\binom{u^k_{i_k}}{\Delta_{\check{v}^k_{i_k}}\, c^k_{i_k}}^{\dagger},\dots,\binom{u^k_{r_k}}{\check{v}^k_{r_k}\, c^k_{r_k}}}(x) \times \\ \mathrm{wemu}^{\binom{u^*_1}{c^*_1}\,\dots\,\binom{u^*_{r_*}}{c^*_{r_*}}}(x) \end{cases} \tag{192}$$

[52] for details, see §2.2.

Third Bridge equation.

Let us now move on to the *welu* products. Since they resolve themselves into sums (189) of *wemu*'s and we have just seen how to alien-differentiate these, the lazy option would be to say that we know how to alien-differentiate the *welu*'s, and leave it at that. But that would yield unwieldy expressions; worse, it would mean missing important cancellations and encumber us with parasitical terms.

Consider for instance a length-9 term like $welu^{\binom{u_1}{c_1}\,\ldots,\,\binom{u_5}{c_5}^{\dagger}\,\ldots,\,\binom{u_9}{c_9}}(x)$ with the marker $\dagger$ on the 5-th index. Formula (189) produces 70 summands, all of the form $wemu^{\left(\begin{smallmatrix} u_{\sigma(1)} & \ldots, & u_{\sigma(8)}, & u_5 \\ c_{\sigma(1)} & \ldots, & c_{\sigma(8)}, & c_5 \end{smallmatrix}\right)}(x)$. Taken singly, some respond non-trivially to alien derivations Δ_ω with indices such as

$$\omega = u_1 v_1 \quad , \quad \omega = u_1\, v_1 + u_{8,9}\, v_8 \quad , \quad \omega = u_1\, v_1 + u_2\, v_2 + u_{7,8,9}\, v_9 \quad , \quad etc$$

and yield non-zero terms, which however vanish from the final result, due to cancellations resulting from the alternality of $welu^\bullet$ or that of $Tes^\bullet$ or both. For other indices, like

$$\omega = u_{1,2}\, v_1 + u_{2,3}\, v_4 \quad , \quad \omega = u_{7,8,9}\, v_8 \quad , \quad \omega = u_{1,2,3}\, v_3 + u_{4,5,6,7,8,9}\, v_7 \quad , \quad etc$$

the non-zero terms do not vanish, but eventually re-group into single terms. Once these cancellations and these clusters are taken into account, we get a result both simpler, more elegant, and relying on *welu* alone.

Proposition 2.18 (Alien derivatives of *welu*, hence *welo*) .
The only alien derivatives Δ_{ω_0} *acting effectively on* $\mathrm{welu}^{\binom{u_1}{c_1}\,\ldots,\,\binom{u_j}{c_j}^{\dagger}\,\ldots,\,\binom{u_r}{c_r}}(x)$ *correspond either to simple* ($s = 1$) *or composite* ($s > 1$) *indices* ω_0 *of three possible types – initial, final, global. Respectively:*

$$\omega_0^{ini} = |\boldsymbol{u^1}|\, v^1_{i_1} + \cdots + |\boldsymbol{u^s}|\, v^s_{i_s} \;\; with \;\; \begin{cases} \boldsymbol{u^1} \,\ldots\, \boldsymbol{u^s}\, \boldsymbol{u^*} = \boldsymbol{u} \;\; ; \;\; \binom{u_j}{c_j}^{\dagger} \in \binom{\boldsymbol{u^*}}{\boldsymbol{c^*}} \\ \Delta_{v^k_{i_k}}\, c^k_{i_k} \neq 0 \;\; and \;\; \binom{u^k_{i_k}}{c^k_{i_k}} \in \binom{\boldsymbol{u^k}}{\boldsymbol{c^k}} \end{cases} \tag{193}$$

$$\omega_0^{fin} = |\boldsymbol{u^1}|\, v^1_{i_1} + \cdots + |\boldsymbol{u^s}|\, v^s_{i_s} \;\; with \;\; \begin{cases} {}^*\boldsymbol{u}\, \boldsymbol{u^1} \,\ldots\, \boldsymbol{u^s} = \boldsymbol{u} \;\; ; \;\; \binom{u_j}{c_j}^{\dagger} \in \binom{{}^*\boldsymbol{u}}{{}^*\boldsymbol{c}} \\ \Delta_{v^k_{i_k}}\, c^k_{i_k} \neq 0 \;\; and \;\; \binom{u^k_{i_k}}{c^k_{i_k}} \in \binom{\boldsymbol{u^k}}{\boldsymbol{c^k}} \end{cases} \tag{194}$$

$$\omega_0^{glo} = |\boldsymbol{u^1}|\, v^1_{i_1} + \cdots + |\boldsymbol{u^s}|\, v^s_{i_s} \;\; with \;\; \begin{cases} \boldsymbol{u^1} \,\ldots\, \boldsymbol{u^s} = \boldsymbol{u} \\ \Delta_{v^k_{i_k}}\, c^k_{i_k} \neq 0 \;\; and \;\; \binom{u^k_{i_k}}{c^k_{i_k}} \in \binom{\boldsymbol{u^k}}{\boldsymbol{c^k}} \end{cases} \tag{195}$$

with each factor sequence $\binom{\boldsymbol{u^k}}{\boldsymbol{c^k}}$ *re-indexed for convenience as* $\binom{u^k_1\,,\ldots,\,u^k_{r_k}}{c^k_1\,,\ldots,\,c^k_{r_k}}$. *The* *corresponding alien derivatives are given by:*

$$\Delta_{\omega_0^{ini}} \mathrm{welu}^{\left(\begin{smallmatrix} u_1 & \dots, & (u_j)^\dagger & \dots, & u_r \\ c_1 & ,\dots, & (c_j) & ,\dots, & c_r \end{smallmatrix}\right)}(x) = \begin{cases} + \sum_{\check{v}_j^k \text{ over } v_{i_k}^k} \mathrm{Tes}^{\left(\begin{smallmatrix} |\boldsymbol{u^1}| & ;\dots; & |\boldsymbol{u^s}| \\ \check{v}_1^1,\dots,\check{v}_{r_1}^1 & ;\dots; & \check{v}_1^s,\dots,\check{v}_{r_s}^s \end{smallmatrix}\right)} \times \\ \prod_{1\leqslant k\leqslant s} \mathrm{welu}^{\left(\begin{smallmatrix} u_1^k & ,\dots, & \left(\begin{smallmatrix} u_{i_k}^k \\ \Delta_{\check{v}_{i_k}^k} c_{i_k}^k \end{smallmatrix}\right)^\dagger & ,\dots, & u_{r_k}^k \\ \check{v}_1^k\, c_1^k & ,\dots, & & ,\dots, & \check{v}_{r_k}^k\, c_{r_k}^k \end{smallmatrix}\right)}(x) \times \\ \mathrm{welu}^{\left(\begin{smallmatrix} u_1^* & \dots, & \left(\begin{smallmatrix} u_j \\ c_j \end{smallmatrix}\right)^\dagger & \dots, & u_{r^*}^* \\ c_1^* & ,\dots, & & ,\dots, & c_{r^*}^* \end{smallmatrix}\right)}(x) \end{cases}$$

$$\Delta_{\omega_0^{fin}} \mathrm{welu}^{\left(\begin{smallmatrix} u_1 & \dots, & (u_j)^\dagger & \dots, & u_r \\ c_1 & ,\dots, & (c_j) & ,\dots, & c_r \end{smallmatrix}\right)}(x) = \begin{cases} - \sum_{\check{v}_j^k \text{ over } v_{i_k}^k} \mathrm{Tes}^{\left(\begin{smallmatrix} |\boldsymbol{u^1}| & ;\dots; & |\boldsymbol{u^s}| \\ \check{v}_1^1,\dots,\check{v}_{r_1}^1 & ;\dots; & \check{v}_1^s,\dots,\check{v}_{r_s}^s \end{smallmatrix}\right)} \times \\ \prod_{1\leqslant k\leqslant s} \mathrm{welu}^{\left(\begin{smallmatrix} u_1^k & ,\dots, & \left(\begin{smallmatrix} u_{i_k}^k \\ \Delta_{\check{v}_{i_k}^k} c_{i_k}^k \end{smallmatrix}\right)^\dagger & ,\dots, & u_{r_k}^k \\ \check{v}_1^k\, c_1^k & ,\dots, & & ,\dots, & \check{v}_{r_k}^k\, c_{r_k}^k \end{smallmatrix}\right)}(x) \times \\ \mathrm{welu}^{\left(\begin{smallmatrix} {}^*u_1 & \dots, & \left(\begin{smallmatrix} u_j \\ c_j \end{smallmatrix}\right)^\dagger & \dots, & {}^*u_{r^*} \\ {}^*c_1 & ,\dots, & & ,\dots, & {}^*c_{r^*} \end{smallmatrix}\right)}(x) \end{cases}$$

$$\Delta_{\omega_0^{glo}} \mathrm{welu}^{\left(\begin{smallmatrix} u_1 & \dots, & (u_j)^\dagger & \dots, & u_r \\ c_1 & ,\dots, & (c_j) & ,\dots, & c_r \end{smallmatrix}\right)}(x) = \begin{cases} + \sum_{\check{v}_j^k \text{ over } v_{i_k}^k} \mathrm{Tes}^{\left(\begin{smallmatrix} |\boldsymbol{u^1}| & ;\dots; & |\boldsymbol{u^s}| \\ \check{v}_1^1,\dots,\check{v}_{r_1}^1 & ;\dots; & \check{v}_1^s,\dots,\check{v}_{r_s}^s \end{smallmatrix}\right)} \times \\ \prod_{1\leqslant k\leqslant s} \mathrm{welu}^{\left(\begin{smallmatrix} u_1^k & ,\dots, & \left(\begin{smallmatrix} u_{i_k}^k \\ \Delta_{\check{v}_{i_k}^k} c_{i_k}^k \end{smallmatrix}\right)^\dagger & ,\dots, & u_{r_k}^k \\ \check{v}_1^k\, c_1^k & ,\dots, & & ,\dots, & \check{v}_{r_k}^k\, c_{r_k}^k \end{smallmatrix}\right)}(x) \end{cases}$$

Remark 1: In the last equation the marking (of the j-th index, on the left-hand side) disappears and is replaced by the marking of the i_k-index of *the* factor sequence $\left(\begin{smallmatrix} \boldsymbol{u^k} \\ \boldsymbol{c^k} \end{smallmatrix}\right)$ that contains $\left(\begin{smallmatrix} u_j \\ c_j \end{smallmatrix}\right)^\dagger$. This general rule – *when occuring inside the same sequence, the second marking abolishes the first* – results from a simple, but not entirely trivial combinatorial fact : let $\underline{M}^\bullet$ be the alternal marking of some mould $M^\bullet$ (with † as marker), and let $\underline{\underline{M}}^\bullet$ be the alternal marking of $\underline{M}^\bullet$ (with ‡ as new marker). Then ‡ replaces (and removes) †. Thus:

$$\underline{\underline{M}}^{t_1,\dots,t_i^\dagger,\dots,t_j^\ddagger,\dots,t_r} = \underline{M}^{t_1,\dots,t_i,\dots,t_j^\ddagger,\dots,t_r}$$

If the initial mould $M^\bullet$ is already alternal, this is obvious, since in that case *almark* amounts to the postponement identity of a marked index for alternal moulds. But the statement holds for any $M^\bullet$.

Remark 2: $\Delta_{\omega_0^{glo}} \mathrm{welu}^{\left(\begin{smallmatrix} u_1 & \dots, & (u_j)^\dagger & \dots, & u_r \\ c_1 & ,\dots, & (c_j) & ,\dots, & c_r \end{smallmatrix}\right)}(x) \equiv 0$ whenever the †-marked index c_j is $\equiv 1$ (i.e. when $\widehat{c}_j \equiv \delta$). Since this marked index in practice is itself an alien derivative, this is often the case – and always so for meromorphic convolands $\widehat{c}_i$.

Discrete coequational resurgence. Some examples.

Example 1: the case $u_i, v_i \in \mathbb{N}$.

Let $Ram(\mathbb{N})$ be the space spanned by the hyperlogarithmic monomials taken in incremental notation $\widehat{\mathcal{V}}^{\omega_1,\ldots,\omega_s}(\xi)$ ($\omega_i \in \mathbb{N}^*$). Let $\xi^{\boldsymbol{\epsilon}} = \xi^{\epsilon_1,\ldots,\epsilon_{n-1},\bullet}$ with $\epsilon_i \in \{\pm\}$ be *the* point of $\widetilde{\mathbb{C}-\mathbb{N}^*}$ of address[53] $\boldsymbol{\epsilon}$, and let $\pi^{\boldsymbol{\epsilon}}(\xi)$ be *the* element of $Ram(\mathbb{N})$ with a simple pole (of residue 1) at $\xi^{\boldsymbol{\epsilon}}$ and at no other point. Since $\{\pi^{\boldsymbol{\epsilon}}\}$ is clearly an alternative basis of $Ram(\mathbb{N})$, which is itself stable under convolution and weighted convolution (for weights u_i in $\mathbb{N}^*$), both products can be expressed in that basis, leading for these two structures to a discretisation of sorts:

$$(\pi^{\boldsymbol{\epsilon_1}} * \pi^{\boldsymbol{\epsilon_2}})(\xi) = \sum_{\boldsymbol{\epsilon}} H_{\boldsymbol{\epsilon}}^{\boldsymbol{\epsilon_1},\boldsymbol{\epsilon_r}}\, \pi^{\epsilon}(\xi) \qquad \begin{cases} \boldsymbol{\epsilon} := (\epsilon_1,\ldots,\epsilon_{n-1},\bullet) \\ \boldsymbol{\epsilon_i} := (\epsilon_{i,1},\ldots,\epsilon_{i,n_i-1},\bullet) \\ n = n_1+n_2 \end{cases} \tag{196}$$

$$\mathrm{weco}^{\left(\begin{smallmatrix} u_1 & ,\ldots, & u_r \\ \pi^{\boldsymbol{\epsilon_1}} & ,\ldots, & \pi^{\boldsymbol{\epsilon_r}}\end{smallmatrix}\right)}(\xi) = \sum_{\boldsymbol{\epsilon}} K_{\boldsymbol{\epsilon}}^{\left(\begin{smallmatrix} u_1 & ,\ldots, & u_r \\ \boldsymbol{\epsilon_1} & ,\ldots, & \boldsymbol{\epsilon_r}\end{smallmatrix}\right)}\, \pi^{\epsilon}(\xi) \quad \begin{cases} \boldsymbol{\epsilon} := (\epsilon_1,\ldots,\epsilon_{n-1},\bullet) \\ \boldsymbol{\epsilon_i} := (\epsilon_{i,1},\ldots,\epsilon_{i,n_i-1},\bullet) \\ n = u_1\, n_1 + \cdots + u_r\, n_r \end{cases} \tag{197}$$

In the case of convolution, we arrive at a structure already known from another context: the Solomon algebra, with structure coefficients $H^\bullet \in \mathbb{Z}$. In the case of weighted convolution, the structure coefficients $K^\bullet$ are in $\mathbb{Q}$. The theory provides for these $K^\bullet$ a rather weird expression, polynomial in the hyperlogarithmic monics $v^\bullet$. However, based on the *jump rules* for these monics, this expression translates into a more convenient induction rule, *which translates back into algebraic relations for the transcendental monics.*

Example 2: the case $u_i, v_i \in \mathbb{Z}$ or $u_i, v_i \in \mathbb{Z} + i\mathbb{Z}$.

The construction can be repeated for u_i, v_i ranging through various discrete rings such as $\mathbb{Z}$ or $\mathbb{Z}+i\mathbb{Z}$ or complex quadratic rings. Here, the *self-symmetrically shrinkable* integration multi-paths for convolution, simple or weighted, soon become so unimaginably complex that the hyperlogarithmic expression for the structure constants $K^\bullet$ looks, by comparison, simple.

[53] $\xi^{\boldsymbol{\epsilon}}$ is defined as accessible from 0 by moving forward under right (resp. left) circumvention of j if $\epsilon_j = +$ (resp. $-$)

2.12 The three Bridge equations at the global level.

Equational resurgence. First Bridge equation.

It is the classical identity:

$$\textbf{BE1} \qquad\qquad [\mathbb{\Delta}_\omega, \Theta^{-1}] = \mathbb{A}_\omega\, \Theta^{-1} \tag{198}$$

with $\mathbb{\Delta}_\omega := e^{-\omega z}\, \Delta_\omega$ (z-resurgence) and

$$\begin{aligned}
\mathbb{A}_\omega &= -\sum (-1)^r \sum W^{\binom{u_1\ ,\ldots,\ u_r}{B^{i_1}_{\boldsymbol{n}_1}\ ,\ldots,\ B^{i_r}_{\boldsymbol{n}_r}}}(x)\ \ \mathbb{D}^{i_1}_{\boldsymbol{n}_1} \mathbb{D}^{i_2}_{\boldsymbol{n}_2} \ldots \mathbb{D}^{i_r}_{\boldsymbol{n}_r} \\
&= -\sum \frac{(-1)^r}{r} \sum W^{\binom{u_1\ ,\ldots,\ u_r}{B^{i_1}_{\boldsymbol{n}_1}\ ,\ldots,\ B^{i_r}_{\boldsymbol{n}_r}}}(x)\ \ [..[\mathbb{D}^{i_1}_{\boldsymbol{n}_1}, \mathbb{D}^{i_1}_{\boldsymbol{n}_2}] \ldots \mathbb{D}^{i_r}_{\boldsymbol{n}_r}]
\end{aligned}$$

Since any two $\mathbb{D}_{\omega_1}$ and $\mathbb{A}_{\omega_2}$ commute, formula (198) lends itself to indefinite iteration (but mark the order on both sides):

$$[\mathbb{\Delta}_{\omega_r} \ldots [\mathbb{\Delta}_{\omega_2}, [\mathbb{\Delta}_{\omega_1}, \Theta^{-1}]..] = \mathbb{A}_{\omega_1}\, \mathbb{A}_{\omega_2} \ldots \mathbb{A}_{\omega_r}\, \Theta^{-1} \tag{199}$$

To prepare for the comparison with coequational resurgence, let us also mention the case of a singular, singularly perturbed Riccati equation:

$$\partial_z Y = x\, Y + b_-(z) + b_+(z)\, Y^2 \qquad\qquad (b_\pm(z) \in z^{-1}\, \mathbb{C}\{z^{-1}\}) \tag{200}$$

Its general solution may be written in the form:

$$Y(z, x; \tau) = \frac{\tau\, e^z\, T_1(z, x) + T_2(z, x)}{\tau\, e^z\, T_3(z, x) + T_4(z, x)} \quad \textit{with} \quad \det\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \equiv 1 \tag{201}$$

with

$$\begin{aligned}
T_1(z, x) &= 1 + \sum \mathcal{W}^{u_+,\ldots,u_-}(z, x) \quad , \quad & T_2(z, x) &= \sum \mathcal{W}^{u_-,\ldots,u_-}(z, x) \\
T_3(z, x) &= \sum \mathcal{W}^{u_+,\ldots,u_+}(z, x) \quad , \quad & T_4(z, x) &= 1 + \sum \mathcal{W}^{u_-,\ldots,u_+}(z, x)
\end{aligned}$$

$\widehat{T}_1(\zeta, x)$ and $\widehat{T}_3(\zeta, x)$ have all their singularities over $\{0, x\, u_+\}$.
$\widehat{T}_2(\zeta, x)$ and $\widehat{T}_4(\zeta, x)$ have all their singularities over $\{0, x\, u_-\}$.
The (very elementary) resurgence equations read in this case:

$$\begin{array}{llll}
\Delta_{xu_+} T_1 = \alpha_+\, T_2 & \Delta_{xu_+} T_2 = 0 & \Delta_{xu_+} T_3 = \alpha_+\, T_4 & \Delta_{xu_+} T_4 = 0 \\
\Delta_{xu_-} T_2 = \alpha_-\, T_1 & \Delta_{xu_-} T_1 = 0 & \Delta_{xu_-} T_4 = \alpha_-\, T_3 & \Delta_{xu_-} T_3 = 0
\end{array}$$

Coequational resurgence. From the molecular to the higher levels.

Coequational resurgence already forced us to distinguish two levels of complexity – *'atomic'* and *'molecular'*. It will shortly impose two more:
(i) a *'microscopic'* level. The objects here are derivation operators $\mathbb{Q}_\omega$ obtained by contracting alternal products *welu* with ordinary differential operators. The resulting sums being usually infinite, the gap from *molecular* to *microscopic* is large.[54]
(ii) a *'macroscopic'* level. The objects here are new derivation operators $\mathbb{P}_\omega$ obtained by contracting the tessellation mould with the previous $\mathbb{Q}_\omega$. These new sums, too, tend to be infinite, making the gap from *microscopic* to *macroscopic* as large as the earlier ones, although in some relatively rare but important instances the relation between the $\mathbb{Q}_\omega$'s and the $\mathbb{P}_\omega$'s simplifies.

Some heuristics.

1) Recall first that alternate moulds $A^\bullet$, when contacted with ordinary derivations, always produce formal derivations:

$$\begin{aligned}\sum A^{\omega_1,\dots,\omega_r} D_{\omega_1}\dots D_{\omega_r} &\equiv \sum \frac{1}{r}\, A^{\omega_1,\dots,\omega_r}[...[D_{\omega_1},D_{\omega_2}]\dots D_{\omega_r}]\\ &\equiv \sum \frac{1}{r}\, A^{\omega_1,\dots,\omega_r}[D_{\omega_1}\dots[D_{\omega_{r-1}},D_{\omega_r}]..]\end{aligned}$$

2) The distance between the $\mathbb{P}_\omega$'s and the $\mathbb{Q}_\omega$'s will be least when the tessellation coefficients $Tes^\bullet$ connecting the two will be simplest. In the case of elementary indices $w_i = \binom{u_i}{v_i}$, $Tes^\bullet$ coincides with $tes^\bullet$ and each of the four following conditions, when met, tends to simplify the coefficients:
(i) no vanishing u_i-sums.
(ii) no identical consecutive v_i's.
(iii) all u_i are aligned with the origin
(iv) all v_i are aligned with the origin

Imposing (i) in our model equation amounts to imposing that the critical coefficients $B^i_{\boldsymbol{n}}$ in our model problem (i.e. the ν coefficients without Y factors in front of them) vanish.[55] This renders the problem uninteresting, as its reduces each component $Y^i_{\boldsymbol{n}}$ of the general solution to a finite sum of monomials $\mathcal{W}^\bullet(z,x)$.

[54] even if the convergence of these infinite sums in the space of resurgent functions is not really an issue

[55] This is the so-called *unilateral* case, where all weights have the form $u := \sum_{n_i\geqslant 0}\lambda_i$, as opposed to the general or *sesquilateral* case, where $u := -\lambda_j + \sum_{n_i\geqslant 0}\lambda_i$.

Imposing (ii) means restricting oneself to the linear case, which leads to interesting results provided we are dealing not with a single equation, but with a true system, i.e. when with $\nu \geqslant 2$.

The conditions (iii) or (iv), are perfectly reasonable. They lead to massive simplifications by ensuring that $\text{test}^{\boldsymbol{w}} = 0$ for all $\boldsymbol{w}$ of length $r(\boldsymbol{w}) > 1$ that meet the conditions (i) and (ii). For $\boldsymbol{w}$ of length 1 we have of course $tes^{w_1} \equiv 1$.
3) We should expect, and do in fact get, particularly simple results when the convolands $\widehat{c}_i$ are meromorphic, or hyperlogarithmic, or again, like in the case (211) *infra*, when they enjoy special closure properties under ω-shifts and Δ_ω-derivations, globally for the same ω's. In any case, since $\widehat{c}_i(\xi) = -b_i(z-\xi)$, it stands to reason that to get full x-resurgence we must assume each $b_i(z)$ to possess endless analytic continuation (on the Riemann sphere, starting from ∞), whereas for z-resurgence it was enough for the $b_i(z)$ to be locally analytic at ∞ (with suitable uniformity conditions in i, of course).

Some examples.

Let us give some illustrations, mostly in the meromorphic context. To lighten notations, we write the results when our model system (37) reduces to a single (non-linear) equation, i.e. when $\nu = 1$, because in that case the operators $\mathbb{D}^i_{\boldsymbol{n}} = \tau_i \boldsymbol{\tau}^{\boldsymbol{n}} \partial_{\tau_i}$ correspond one-to-one with the weights u and can be re-indexed as $\mathbb{D}_{\| u} = \tau^{n+1}\partial_\tau$. The transposition to the case $\nu > 1$ offers only notational complications but deserves special consideration because it allows non-aligned weights $u =< \boldsymbol{\lambda}, \boldsymbol{n} >$.

Second Bridge equation.

$$(\mathbf{BE2}) \qquad [\mathbb{\Delta}_\omega, \Theta^{-1}] \;=\; \mathbb{P}_\omega \, \Theta^{-1} \tag{202}$$

with $\mathbb{\Delta}_\omega := e^{-\omega x}\Delta_\omega$ (x-resurgence) and:

$$\mathbb{P}_\omega \;:=\; \sum_{\sum u_i(z-\alpha_i)=\omega} \mathrm{Tes}^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ z-\alpha_1 & ,\dots, & z-\alpha_r\end{smallmatrix}\right)} \mathbb{Q}_{\left[\begin{smallmatrix} u_1 \\ \alpha_1\end{smallmatrix}\right]} \dots \mathbb{Q}_{\left[\begin{smallmatrix} u_r \\ \alpha_r\end{smallmatrix}\right]} \tag{203}$$

$$\mathbb{Q}_{\left[\begin{smallmatrix} u_0 \\ \alpha_0\end{smallmatrix}\right]} := \; e^{u_0 \alpha_0 x} \sum_{\sum u_i = u_0} \mathrm{welu}^{\left(\begin{smallmatrix} u_1 & ,\dots, & \left(\begin{smallmatrix} u_i \\ \Delta_{\alpha_0} c_i\end{smallmatrix}\right)^\dagger & ,\dots, & u_r \\ \bar{\alpha}_0.c_1 & ,\dots, & & ,\dots, & \bar{\alpha}_0.c_r\end{smallmatrix}\right)} \mathbb{D}_{\| u_1} \dots \mathbb{D}_{\| u_r} \tag{204}$$

Here $Tes^{\bullet}$ coincides with the elementary $tes^{\bullet}$.

Third Bridge equation.

$$(\mathbf{BE3}) \qquad \mathbb{\Delta}_\omega \mathbb{Q}_{\left[\begin{smallmatrix} u_0 \\ \alpha_0\end{smallmatrix}\right]} \;=\; \begin{cases} + \sum_{u_1+u_2=u_0} \mathbb{P}_{\omega, \left[\begin{smallmatrix} u_1 \\ \alpha_0\end{smallmatrix}\right]} \mathbb{Q}_{\left[\begin{smallmatrix} u_2 \\ \alpha_0\end{smallmatrix}\right]} \\ - \sum_{u_1+u_2=u_0} \mathbb{Q}_{\left[\begin{smallmatrix} u_1 \\ \alpha_0\end{smallmatrix}\right]} \mathbb{P}_{\omega, \left[\begin{smallmatrix} u_2 \\ \alpha_0\end{smallmatrix}\right]} \end{cases} \tag{205}$$

with

$$\mathbb{P}_{\omega,[{u_0 \atop \alpha_0}]} := \sum_{\sum u_i(\alpha_0-\alpha_i)=\omega}^{\sum u_i=u_0} \mathrm{Tes}^{({u_1 \atop \alpha_0-\alpha_1} {,\ldots, \atop ,\ldots,} {u_r \atop \alpha_0-\alpha_r})} \mathbb{Q}_{[{u_1 \atop \alpha_1}]} \ldots \mathbb{Q}_{[{u_r \atop \alpha_r}]} \tag{206}$$

Remark 1: With the notations of (206), the operator $\mathbb{P}_\omega$ of **BE2** may be rewritten as $\mathbb{P}_\omega = \sum_u \mathbb{P}_{\omega,[{u \atop z}]}$. It should be noted that $\mathbb{P}_\omega$ in **BE2** is locally (though not globally) constant in z, just as the operators $\mathbb{P}_{\omega,[{u \atop \alpha_0}]}$ in **BE3** are locally (though not globally) constant in α_0.

Remark 2: In the important instances when the tessellation coefficients $Tes^{w_1,\ldots,w_r}$ turn trivial (i.e. $\equiv 1$ for $r=1$ and $\equiv 0$ for $r \neq 1$), the Third Bridge equation simplifies:

$$(\mathbf{BE3}) \qquad \mathbb{\Delta}_\omega \mathbb{Q}_{[{u_0 \atop \alpha_0}]} \;=\; \sum_{u_1+u_2=u_0}^{u_1(\alpha_0-\alpha_1)=\omega} \left[\mathbb{Q}_{[{u_1 \atop \alpha_1}]}, \mathbb{Q}_{[{u_2 \atop \alpha_0}]}\right] \tag{207}$$

and one can checks the equality of the exponential factors on both sides:

(i) $\mathbb{\Delta}_\omega$ carries a factor $e^{-\omega\, x} = e^{-u_1(\alpha_0-\alpha_1)\, x}$

(ii) $\mathbb{Q}_{[{u_0 \atop \alpha_0}]}$ carries a factor $e^{u_0\alpha_0\, x} = e^{(u_1+u_2)\alpha_0\, x}$

(iii) $\mathbb{Q}_{[{u_1 \atop \alpha_1}]}$ carries a factor $e^{u_1\alpha_1\, x}$

(iv) $\mathbb{Q}_{[{u_2 \atop \alpha_0}]}$ carries a factor $e^{u_2\alpha_0\, x}$

Remark 3. In the opposite directions, the results also extend to the case of hyperlogarithmic (instead of meromorpic) inputs $\mathrm{b}_i(z)$ (and thus $\widehat{c}_i(\xi)$), except that we must switch to a multiple indexation $\alpha_i \to \check{\alpha}_i$ and that the third Bridge equation inherits a third term, corresponding to the case $\Delta^{glo}_\omega welu^\bullet$ of Proposition 2.16. We get:

$$(\mathbf{BE3}) \qquad \Delta_\omega \mathbb{Q}_{[{u_0 \atop \check{\alpha}_0}]} \;=\; \begin{cases} +\sum_{u_1+u_2=u_0} \mathbb{P}_{\omega,[{u_1 \atop \check{\alpha}_0}]}\, \mathbb{Q}_{[{u_2 \atop \check{v}_0}]} \\ -\sum_{u_1+u_2=u_0} \mathbb{Q}_{[{u_1 \atop \check{\alpha}_0}]}\, \mathbb{P}_{\omega,[{u_2 \atop \check{\alpha}_0}]} \\ +\mathbb{P}_{\omega,[{u_0 \atop \check{\alpha}_0}]} \end{cases} \tag{208}$$

Remark4: the meromorphic Riccati case.

Let us return to the equation (200) but from the point of view of coequational resurgence.

$$(\mathbf{BE2}) \qquad \Delta_\omega Y^{\epsilon,\eta}(z,x) \;:=\; P^{\nu}_{\omega\|z}(x)\, Y^{\bar{\tau},\eta}(z,x) \tag{209}$$

$$
\begin{aligned}
Y^{\epsilon,\eta}(z,x) &:= 1_{\epsilon,\eta} + \sum \mathrm{wemu}^{\left(\begin{smallmatrix} u_\epsilon & , & u_{\bar\epsilon} & ,\dots, & u_{\bar\eta} & , & u_\eta \\ c_\epsilon & , & c_{\bar\epsilon} & ,\dots, & c_{\bar\eta} & , & c_\eta \end{smallmatrix}\right)}(x) \\
Q^{+}_{||i}(x) &:= \sum\sum \mathrm{welu}^{\left(\begin{smallmatrix} u_+ & , & u_- & ,\dots, & \left(\begin{smallmatrix} u_+ \\ \rho_{+\|i}\end{smallmatrix}\right)^{\dagger} & ,\dots, & u_- & , & u_+ \\ c_{+\|i} & , & c_{-\|i} & ,\dots, & & ,\dots, & c_{-\|i} & , & c_{+\|i} \end{smallmatrix}\right)}(x) \\
Q^{-}_{||i}(x) &:= \sum\sum \mathrm{welu}^{\left(\begin{smallmatrix} u_- & , & u_+ & ,\dots, & \left(\begin{smallmatrix} u_- \\ \rho_{-\|i}\end{smallmatrix}\right)^{\dagger} & ,\dots, & u_+ & , & u_- \\ c_{-\|i} & , & c_{+\|i} & ,\dots, & & ,\dots, & c_{+\|i} & , & c_{-\|i} \end{smallmatrix}\right)}(x) \\
P^{+}_{\omega\|z}(x) &:= \sum \mathrm{tes}^{\left(\begin{smallmatrix} u_+ & u_- & ,\dots, & u_- & u_+ \\ v_{i_1} & v_{i_2} & ,\dots, & v_{i_{r-1}} & v_{i_r} \end{smallmatrix}\right)} Q^{+}_{\||i_1}(x) Q^{-}_{\||i_2}(x) \dots Q^{-}_{\||i_{r-1}}(x) Q^{+}_{\||i_r}(x) \\
P^{-}_{\omega\|z}(x) &:= \sum \mathrm{tes}^{\left(\begin{smallmatrix} u_- & u_+ & ,\dots, & u_+ & u_- \\ v_{i_1} & v_{i_2} & ,\dots, & v_{i_{r-1}} & v_{i_r} \end{smallmatrix}\right)} Q^{-}_{\||i_1}(x) Q^{+}_{\||i_2}(x) \dots Q^{+}_{\||i_{r-1}}(x) Q^{-}_{\||i_r}(x)
\end{aligned}
$$

$$
(\mathbf{BE3}) \qquad \Delta_\omega Q^{\epsilon,\tau,\eta}_{\|i}(x) \;:=\; \begin{cases} +P^{\tau}_{\omega\|i}(x)\, Q^{\bar\tau,\eta}_{\|i}(x) \\ -Q^{\bar\tau,\eta}_{\|i}(x)\, P^{\tau}_{\omega\|i}(x) \end{cases} \tag{210}
$$

$$
\begin{aligned}
Q^{\epsilon,\tau,\eta}_{||i}(x) &:= \sum\sum \mathrm{welu}^{\left(\begin{smallmatrix} u_\epsilon & ,\dots, & \left(\begin{smallmatrix} u_\tau \\ \rho_{\tau\|i}\end{smallmatrix}\right)^{\dagger} & ,\dots, & u_\eta \\ c_{\epsilon\|i} & ,\dots, & & ,\dots, & c_{\eta\|i} \end{smallmatrix}\right)}(x) \\
P^{+}_{\omega\|i}(x) &:= \sum \mathrm{tes}^{\left(\begin{smallmatrix} u_+ & u_- & ,\dots, & u_- & u_+ \\ v_{i_1:i} & v_{i_2:i} & ,\dots, & v_{i_{r-1}:i} & v_{i_r:i} \end{smallmatrix}\right)} Q^{+}_{\||i_1:i}(x) Q^{-}_{\||i_2:i}(x) \dots Q^{-}_{\||i_{r-1}:i}(x) Q^{+}_{\||i_r:i}(x) \\
P^{-}_{\omega\|i}(x) &:= \sum \mathrm{tes}^{\left(\begin{smallmatrix} u_- & u_+ & ,\dots, & u_+ & u_- \\ v_{i_1:i} & v_{i_2:i} & ,\dots, & v_{i_{r-1}:i} & v_{i_r:i} \end{smallmatrix}\right)} Q^{-}_{\||i_1:i}(x) Q^{+}_{\||i_2:i}(x) \dots Q^{+}_{\||i_{r-1}:i}(x) Q^{-}_{\||i_r:i}(x)
\end{aligned}
$$

Remark 5: the hyperelliptic Riccati case.

This is again the case $\partial_z Y = x\,Y + b_-(z) + b_+(z)\,Y^2$ but with

$$
b_\pm(z) := \pm H(z) \;\; \textit{with} \;\; \begin{cases} H(z) = \frac{1}{2}\frac{q''(z)}{q'(z)} \\ z = z(q) = \int_0^q (W(q')^{\frac{1}{2}} dq' \\ W(q) := q^\nu + \alpha_1 q^{\nu-1} + \dots + \alpha_\nu \end{cases} \tag{211}
$$

This Riccati equation is of course in relation with the much investigated Schrodinger equation $\partial_q^2 \psi(q) = \frac{x^2}{4} W(q)\psi(q)$ $(x = \frac{2}{\hbar})$. It is also one of those instances where, due to the self-reproduction properties of $b_\pm(z)$ under shifts, the relation between the $\mathbb{P}_\omega$'s and the $\mathbb{Q}_\omega$'s simplifies dramatically.

Before winding up this section, let us mention two elementary applications and sketch a more interesting third one.

Application 1: Finding the singularities in the ξ-plane.

(i) *In the Second Bridge equation:* all the singularities always lie *over* some linear combination of *frequencies* and *singularities* $v_i := z - \alpha_i$. Since the

weights u_i may add up to zero[56], the corresponding combinations $\sum u_i\, v_i$ will be independent of z. But a proper determination[57] of $weco^\bullet(\xi)$ will always eliminate these parasitical, z-independent singularities from BE2.
(ii) *In the Third Bridge equation:* the singularities always lie *over* some linear combination of *frequencies* and *singularities* $v_i - v_j := \alpha_j - \alpha_i$ of the individual coefficients.

Application 2: Establishing the convergence in the ξ-plane.

It can (very easily) be established, first in the star of holomorphy; and then gradually extended to the adjacent sheets by using the alien derivatives. Multi-path deformations would be unfeasible here.

Remark 3: Finding 'interesting' instances, with finitely many generators and/or simple $\mathbb{Q}_\omega$-to-$\mathbb{P}_\omega$ relations .

Since BE2 and BE3 give the alien derivatives of the $\mathbb{Q}_\omega$'s in terms of the $\mathbb{P}_\omega$'s, and these in turn are expressible as sums of multibrackets of $\mathbb{Q}_\omega$'s, BE2 and BE3 amount to a *closed, indefinitely iterable system* that contains all the information about the x-resurgence. Together with the information about *weco* and *welo*, BE2 and BE3 also give us a systematic tool for identifying the situations that may narrow, or altogether remove, the gap between the $\mathbb{Q}_\omega$'s and the $\mathbb{P}_\omega$'s. The Schrodinger-related Riccati equation (211) is an important case in point. But it also tells us something else: namely, that when spectacular simplifications occur, they may point to the existence of a change of variable $z \to q$ that renders the equation's coefficients polynomial or rational or otherwise elementary. In such situations, working directly in the q-plane may well prove more expedient. But as tools for systematic exploration and as vehicles of in-depth understanding, the z- and x-planes, with their Borel counterparts ζ and ξ, remain irreplaceable.

By way of conclusion.

At the end of this tour of coequational resurgence, we find a clear four level stratification:

- *The atomic level*, populated by objects such as simple poles or hyperlogarithms.

[56]at least in the general or *sesquilateral* case. See preceding footnote.

[57]As we saw, each vanishing partial sum $u_1 + ... + u_i$ introduces a ramification in the determination of $weco^\bullet(\xi)$, but there is always a privileged choice.

- *The molecular level*, consisting of huge clusters of atoms, with unsuspected emergent properties.

- *The microscopic level*, consisting of derivation operators $\mathbb{Q}_\omega$, usually infinite chains of molecules contracted by elementary derivation operators.

- *The macroscopic level*, consisting of new derivation operators $\mathbb{P}_\omega$ assembled from the earlier $\mathbb{Q}_\omega$.

- The passage from the atomic to the molecular level is mediated on the Analysis side by *weighted convolution* and on the combinatorial side by the *scrambling transform*.

- The passage from the molecular to the microsopic level is rather mechanical – mere growth by accumulation.

- The passage from the microscopic to the macroscopic level, arguably the most interesting of the three, is mediated by the *tessellation coefficients*. While much is known about them, it would seem that just as much remains to be discovered.

3 Multizeta algebra: the independence theorem for bicolours.

This brief chapter is devoted to
(i) some sketchy reminders about the flexion structure and multizetas
(ii) a discussion of the phenomenon of *retro-action* – the central difficulty which complicates the decomposition of multizetas into irreducibles but assumes quite distinct forms for monocolours and bicolours and calls for different strategies.
(iii) the proof of the independence conjecture for the basic generators for bicolours.

3.1 Reminders about the flexion structure.

Elementary flexions.

Bimoulds $M^\bullet$ have a two-tier indexation $\bullet = \boldsymbol{w} = \binom{u_1\ ,\ldots,\ u_r}{v_1\ ,\ldots,\ v_r}$ with upper u_i's and lower v_i's that interact in a very special way, through four basic flexions $\rfloor, \lceil$ and $\rceil, \lfloor$.

Thus, if $\boldsymbol{w} = \boldsymbol{w}'.\boldsymbol{w}''$ with $\boldsymbol{w}' = \binom{u_1\ ,\ u_2}{v_1\ ,\ v_2}$ *and* $\boldsymbol{w}'' = \binom{u_3\ ,\ u_4\ ,\ u_5}{v_3\ ,\ v_4\ ,\ v_5}$, we set:

$$\boldsymbol{w}'\rfloor = \begin{pmatrix} u_1 & , & u_2 \\ v_{1:3} & , & v_{2:3} \end{pmatrix} \qquad \lceil \boldsymbol{w}'' = \begin{pmatrix} u_{1,2,3} & , & u_4 & , & u_5 \\ v_3 & , & v_4 & , & v_5 \end{pmatrix}$$

$$\boldsymbol{w}'\rceil = \begin{pmatrix} u_1 & , & u_{2,3,4,5} \\ v_1 & , & v_2 \end{pmatrix} \qquad \lfloor \boldsymbol{w}'' = \begin{pmatrix} u_3 & , & u_4 & , & u_5 \\ v_{3:2} & , & v_{4:2} & , & v_{5:2} \end{pmatrix}$$

Throughout, we shall use the shorthand:

$$u_{i,j,k\ldots} := u_i + u_j + u_k \ldots\ ,\ v_{i:j} := v_i - v_j$$

The products of upper and lower indices remain invariant:

$$\boldsymbol{w} = \boldsymbol{w}'\boldsymbol{w}''\ ,\ \boldsymbol{w}^* = \boldsymbol{w}'\rfloor\,\lceil\boldsymbol{w}''\ ,\ \boldsymbol{w}^{**} = \boldsymbol{w}'\rceil\,\lfloor\boldsymbol{w}'' \quad \Rightarrow$$

$$\sum u_i\, v_i \equiv \sum u_i^*\, v_i^* \equiv \sum u_i^{**}\, v_i^{**}$$

$$\sum du_i \wedge dv_i \equiv \sum du_i^* \wedge dv_i^* \equiv \sum du_i^{**} \wedge dv_i^{**}$$

The core involution *swap*.

$$\{B^\bullet = \mathrm{swap}\, A^\bullet\} \quad \Longleftrightarrow \quad \{B^{\binom{u_1\ ,\ldots,\ u_r}{v_1\ ,\ldots,\ v_r}} = A^{\binom{v_r\quad ,\ldots,\ v_{3:4}\ ,\ v_{2:3}\ ,\ v_{1:2}}{u_{1,\ldots,r}\ ,\ldots,\ u_{1,2,3}\ ,\ u_{1,2}\ ,\ u_1}}\} \tag{212}$$

Once again, the invariance holds: $\sum_i u_i\, v_i = \sum_i v_{i:i+1}\, u_{1,\ldots,i}$

• The *swap* transform ($swap^2 = id$) is as central to flexion theory as the Fourier transform ($\mathcal{F}^4 = id$) is to Analysis. There are even contexts where the two coincide.

• Interesting bimoulds $M^\bullet$ tend to possess a *double symmetry*: one for $M^\bullet$, another for the *swappee* ($swap.M^\bullet$).

Basic flexion operations: ***ari, gari.***

Very loosely speaking, the flexion structure is the sum total of all *interesting operations* that may be constructed from the four afore-mentioned flexions. More specifically, one can show that, up to isomorphisms, there exist exactly seven pairs {*Lie algebra, Lie group*} obtainable in this way. Of these substructures, four have the added distinction of preserving *double symmetries.* Moreover, when restricted to doubly symmetric bimoulds, these four substructures actually coincide. So we choose to work with the simplest of the four pairs: the Lie algebra *ARI* and the Lie Group *GARI.*

The Lie bracket *ari* and the pre-Lie law *preari* are defined as follows:

$$
\begin{aligned}
N^\bullet = \mathrm{arit}(B^\bullet)M^\bullet \;\Leftrightarrow\; N^{\boldsymbol{w}} &= \sum^{\boldsymbol{w}=\boldsymbol{abc}} M^{\boldsymbol{a}\lceil\boldsymbol{c}}B^{\boldsymbol{b}\rfloor} - \sum^{\boldsymbol{w}=\boldsymbol{abc}} M^{\boldsymbol{a}\rceil\boldsymbol{c}}B^{\lfloor\boldsymbol{b}} \\
\mathrm{ari}(A^\bullet,B^\bullet) &:= \mathrm{arit}(B^\bullet).A^\bullet - \mathrm{arit}(A^\bullet).B^\bullet + \mathrm{lu}(A^\bullet,B^\bullet) \\
\mathrm{preari}(A^\bullet,B^\bullet) &:= \mathrm{arit}(B^\bullet).A^\bullet + \mathrm{mu}(A^\bullet,B^\bullet)
\end{aligned}
$$

The corresponding associative law is denoted *gari*. It is linear in $A^\bullet$ but severely non-linear in $B^\bullet$:

$$
\begin{aligned}
N^\bullet = \mathrm{garit}(B^\bullet)M^\bullet \;\Leftrightarrow\; N^{\boldsymbol{w}} &= \sum^{\boldsymbol{w}=\prod \boldsymbol{a^i b^i c^i}} M^{\lceil \boldsymbol{b^1}\rceil..\lceil \boldsymbol{b^s}\rceil} B^{\boldsymbol{a^1}\rfloor}..B^{\boldsymbol{a^s}\rfloor} B_*^{\lfloor \boldsymbol{c^1}}..B_*^{\lfloor \boldsymbol{c^s}} \\
\mathrm{gari}(A^\bullet,B^\bullet) &:= \mathrm{mu}(\mathrm{garit}(B^\bullet).A^\bullet,B^\bullet) \quad (B_*^\bullet := \mathit{invmu}\, B^\bullet)
\end{aligned}
$$

The exponential from ARI to $GARI$, denoted *expari*[58], admits an analytical expression in terms of *preari*, with pre-bracketting from left to right:

$$
\mathit{expari}A^\bullet \;:=\; A^\bullet + \sum_{2\leqslant r} \overrightarrow{\mathit{preari}}\,(\overbrace{A^\bullet,\ldots,A^\bullet}^{(r\ times)}) \tag{213}
$$

$$
\overrightarrow{\mathit{preari}}\,(A_1^\bullet,\ldots,A_r^\bullet) \;:=\; \mathit{preari}(...(\mathit{preari}(A_1^\bullet,A_2^\bullet),\ldots,A_r^\bullet) \tag{214}
$$

3.2 Multizetas and their generating series.

The coloured multizetas $wa^\bullet$ and $ze^\bullet$.

We first define the scalar multizetas in the *convergent* or *regular* case. The underlining signals convergence.

- *Polylogarithmic integrals* $\big(\alpha_j = 0$ or unit root; $\binom{\alpha_1 \neq 0}{\alpha_s \neq 1}\big)$:

$$
\underline{\mathrm{wa}}^{\alpha_1,\ldots,\alpha_s} := (-1)^{s_0}\int_0^1 \frac{dt_s}{\alpha_s - t_s}\cdots\int_0^{t_3}\frac{dt_2}{\alpha_2-t_2}\int_0^{t_2}\frac{dt_1}{\alpha_1-t_1} \tag{215}
$$

- *Harmonic sums* $\big(e_j = e^{2\pi i\epsilon_j}$=unit root; $s_j \in \mathbb{N}^*$; $\binom{e_1}{s_1} \neq \binom{1}{1}\big)$:

$$
\underline{\mathrm{ze}}^{\binom{\epsilon_1,\ldots,\epsilon_r}{s_1,\ldots,s_r}} := \sum_{n_1>\ldots>n_r>0} n_1^{-s_1}e_1^{-n_1}\ldots n_r^{-s_r}e_r^{-n_r} \quad (e_j = e^{2\pi i\epsilon_j}) \tag{216}
$$

- *Conditional conversion rule* $\big($assuming convergence, i.e. $\binom{e_1}{s_1} \neq \binom{1}{1}\big)$:

$$
\underline{\mathrm{ze}}^{\binom{\epsilon_1\ \epsilon_2,\ldots,\epsilon_r}{s_1\ s_2,\ldots,s_r}} \;\equiv\; \underline{\mathrm{wa}}^{e_1\ldots e_r,0^{[s_r-1]},\ldots,e_1e_2,0^{[s_2-1]},e_1,0^{[s_1-1]}} \tag{217}
$$

- s = *weight* , r = *length* (or *depth*) , $d := s-r$ = *degree*.

[58]to distinguish it from the ordinary mould exponential *expmu* and from the other exponentials attached to the seven flexion substructures previously alluded to.

Algebraic constraints on the scalar multizetas.

(i) *First symmetry:* $\underline{wa}^\bullet$ is symmetral, with a unique symmetral extension $\underline{wa}^\bullet \to wa^\bullet$ such that $wa^0 = wa^1 = 0$.

(ii) *Second symmetry:* $\underline{ze}^\bullet$ is symmetrel, with a unique symmetrel extension $\underline{ze}^\bullet \to ze^\bullet$ such that $ze^{\binom{0}{1}} = 0$.

(iii) *Conversion rule:* The conversion formula $\underline{wa}^\bullet \leftrightarrow \underline{ze}^\bullet$ has a non-trivial extension $wa^\bullet \leftrightarrow ze^\bullet$, best expressed in terms of the generating series $zag^\bullet$ and $zig^\bullet$. Cf *infra.*

(iv) *Colour-consistency:* If $p \in \mathbb{N}$, $\mathbb{Q}_\infty := \mathbb{Q}/\mathbb{Z}$, $\mathbb{Q}_p := (\frac{1}{p}\mathbb{Z})/\mathbb{Z}$

$$\sum_{\tau_j \in \mathbb{Q}_p} \underline{\mathrm{ze}}^{\binom{\epsilon_1+\tau_1\ ,\dots,\ \epsilon_r+\tau_r}{s_1\ ,\dots,\ s_r}} \equiv p^{-d}\, \underline{\mathrm{ze}}^{\binom{p\,\epsilon_1\ ,\dots,\ p\,\epsilon_r}{s_1\ ,\dots,\ s_r}} \quad \textit{with} \quad d := s - r \tag{218}$$

(v) **Conjecture:** *the above system of algebraic constraints is exhaustive.*

Attached to each of the two encodings $\underline{wa}^\bullet$ and $\underline{ze}^\bullet$ there is a specific *symmetry type*, which amounts to a specific way of multiplying the scalar multizetas. This is the essence of *arithmetical dimorphy* — a phenomenon that extends far beyond the multizeta landscape, but finds there its most striking manifestion.

Dropping the convergence assumption while preserving the symmmetries, i.e. extending $\underline{wa}^\bullet, \underline{ze}^\bullet$ to $wa^\bullet, ze^\bullet$, is a purely formal-algebraic affair, but it comes at the cost of a slight complication in the *conversion rule* and *colour consistency* constraints. The modified constraints are best expressed in terms of the generating functions $zag^\bullet, zig^\bullet$ and of two suitable elements in $centre(GARI)$: see (500),(226) *infra.*

The generating series/functions $zag^\bullet$ and $zig^\bullet$.

The first way of defining $zag^\bullet$ and $zig^\bullet$ is as generating series of the *extended* scalar multizetas:

$$\mathrm{zag}^{\binom{u_1\ ,\dots,\ u_r}{\epsilon_1\ ,\dots,\ \epsilon_r}} := \sum_{1 \leqslant s_j} \mathrm{wa}^{e_1, 0^{[s_1-1]},\dots,e_r,0^{[s_r-1]}} u_1^{s_1-1} u_{1,2}^{s_2-1} \dots u_{1,..,r}^{s_r-1} \tag{219}$$

$$\mathrm{zig}^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{v_1\ ,\dots,\ v_r}} := \sum_{1 \leqslant s_j} \mathrm{ze}^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{s_1\ ,\dots,\ s_r}} v_1^{s_1-1} \dots v_r^{s_r-1} \tag{220}$$

Here $\epsilon_j \in \mathbb{Q}_p = \frac{1}{p}\mathbb{Z}/doZ$ and $e_j := \exp(2\pi i\, \epsilon_j)$.

A second, equivalent definition introduces $zag^\bullet$ and $zig^\bullet$ directly as multivariate meromorphic functions of the u_i's and v_i's respectively: Setting $P(t) := \frac{1}{t}$

and using the usual abbreviations, that second definition reads:

$$\mathrm{zag}^\bullet = \lim_{k\to} \left(\mathrm{dozag}^\bullet_k \times \mathrm{cozag}^\bullet_k\right) \tag{221}$$

$$\mathrm{zig}^\bullet = \lim_{k\to} \left(\mathrm{dozig}^\bullet_k \times \mathrm{cozig}^\bullet_k\right) \tag{222}$$

$$\mathrm{dozag}^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ \epsilon_1 & ,\dots, & \epsilon_r \end{smallmatrix}\right)} = \sum_{1\leqslant m_j\leqslant k} \quad \prod_{1\leqslant j\leqslant r} e_j^{-m_j}\, P(m_{1,..,j} - u_{1,..,j}) \tag{223}$$

$$\mathrm{dozig}^{\left(\begin{smallmatrix} \epsilon_1 & ,\dots, & \epsilon_r \\ s_1 & ,\dots, & \epsilon_r \end{smallmatrix}\right)} = \sum_{k\geqslant n_1>..>n_r>0} \quad \prod_{1\leqslant j\leqslant r} e_j^{-n_j}\, P(n_j - v_j) \tag{224}$$

The dominant factors *dozag*$^\bullet$, *dozig*$^\bullet$ require the corrective terms *cozag*$^\bullet$, *cozig*$^\bullet$ to ensure convergence.

Algebraic constraints on the generating series.

(i) *First symmetry: zag*$^\bullet$ is symmetral.

(ii) *Second symmetry: zig*$^\bullet$ is symmetril.
$\mathrm{zig}^{\dots,w_i+w_j,\dots} \to \mathrm{zig}^{\left(\begin{smallmatrix} \dots, & u_{i,j} & ,\dots \\ \dots, & v_i & ,\dots \end{smallmatrix}\right)} P(v_{i:j}) + \mathrm{zig}^{\left(\begin{smallmatrix} \dots, & u_{i,j} & ,\dots \\ \dots, & v_j & ,\dots \end{smallmatrix}\right)} P(v_{j:i})$

(iii) *Conversion rule:* It reads

$$\mathrm{swap.zig}^\bullet \begin{cases} = \mathrm{gari}(\mathrm{zag}^\bullet, \mathrm{man}^\bullet) = \mathrm{gari}(\mathrm{man}^\bullet, \mathrm{zag}^\bullet) \\ = \mathrm{mu}(\mathrm{zag}^\bullet, \mathrm{man}^\bullet) \end{cases} \tag{225}$$

for a well-defined element *man*$^\bullet$ of $GARI_{centre}$: see (230) below.

(iv) *Colour-consistency:* It reads

$$\mu_p\, \mathrm{zag}^\bullet \begin{cases} = \mathrm{gari}(\delta_\mathrm{p}\, \mathrm{zag}^\bullet, \mathrm{lag}^\bullet_\mathrm{p}) = \mathrm{gari}(\mathrm{lag}^\bullet_\mathrm{p}, \delta_\mathrm{p}\, \mathrm{zag}^\bullet) \\ = \mathrm{mu}(\delta_\mathrm{p}\, \mathrm{zag}^\bullet, \mathrm{lag}^\bullet_\mathrm{p}) \qquad\qquad (\forall p \in \mathbb{N}) \end{cases} \tag{226}$$

for operators μ_p and δ_p defined as follows:

$$\mu_p\, \mathrm{zag}^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ \epsilon_1 & ,\dots, & \epsilon_r \end{smallmatrix}\right)} := p^{-r} \sum_{p\epsilon'_j \equiv p\epsilon_j} \mathrm{zag}^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ \epsilon'_1 & ,\dots, & \epsilon'_r \end{smallmatrix}\right)} \quad (p\textit{-averaging}) \tag{227}$$

$$\delta_p\, \mathrm{zag}^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ \epsilon_1 & ,\dots, & \epsilon_r \end{smallmatrix}\right)} := p^{-r}\, \mathrm{zag}^{\left(\begin{smallmatrix} u_1/p & ,\dots, & u_r/p \\ \epsilon_1\, p & ,\dots, & \epsilon_r\, p \end{smallmatrix}\right)} \quad (p\textit{-dilation}) \tag{228}$$

and for a well-defined elements $lag^\bullet_p$ of $GARI_{centre}$: see (232) below.

The centre of GARI.

The elements *ca*$^\bullet$ of $GARI_{centre}$ are all of the form:

$$\mathrm{ca}^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ v_1 & ,\dots, & v_r \end{smallmatrix}\right)} = \begin{cases} \mathrm{ca}_r \in \mathbb{C} & \textit{if} \quad (v_1, \dots, v_r) = (0, \dots, 0) \\ 0 & \textit{otherwise} \end{cases} \tag{229}$$

and verify for all $Ma^\bullet \in GARI$:

$$\mathrm{gari}(\mathrm{ca}^\bullet, \mathrm{Ma}^\bullet) \equiv \mathrm{gari}(\mathrm{Ma}^\bullet, \mathrm{ca}^\bullet) \equiv \mathrm{mu}(\mathrm{Ma}^\bullet, \mathrm{ca}^\bullet)$$

The central elements $man^\bullet$, $mane^\bullet$, $lag_p^\bullet$ featuring in the conversion rules (500), (236) and in the colour consistency constraints (226) correspond to constants man_r, $mane_r$, $lag_{p,r}$ so defined:

$$\sum_{1\leqslant r} \mathrm{man}_r\, t^r \;:=\; \exp\Big(\sum_{2\leqslant s}(-1)^{s-1}\zeta(s)\frac{t^s}{s}\Big) \tag{230}$$

$$\sum_{1\leqslant r} \mathrm{mane}_r\, t^r \;:= \tag{231}$$

$$\mathrm{lag}_{p,r} \;:=\; \frac{(-\log p)^r}{r!} = \frac{(-1)^r}{r!}\Big(\sum_{a^p=1\,,\,a\neq 1}\log(1-a)\Big)^r \tag{232}$$

The parity condition for length-one components.

The sets $GARI^{\underline{as}/\underline{as}}$ resp. $GARI^{\underline{as}/\underline{is}}$ consisting of all bimoulds of type *symmetral/symmetral*[59] resp. *symmetral/symmetril*[60] and with length-one components *even* in w_1 (i.e. $S^{w_1} \equiv S^{-w_1}$) are two important subgroups of *GARI*.

The sets $GARI^{as/as}$ resp. $GARI^{as/is}$ whose elements display the double symmetry but whose length-1 components are not constrained by the parity condition, *are no* subgroups of *GARI*, but they admit a right action of the above subgroups:

$$GARI^{as/as}.GARI^{\underline{as}/\underline{as}} \;=\; GARI^{as/as} \tag{233}$$

$$GARI^{as/is}.GARI^{\underline{as}/\underline{is}} \;=\; GARI^{as/is} \tag{234}$$

The same applies to the sets $ARI^{\underline{al}/\underline{al}}$ resp. $ARI^{\underline{al}/\underline{il}}$ consisting of all bimoulds of type *alternal/alternal* resp. *alternal/alternil* and with length-one components *even* in w_1: they are subalgebras of *ARI*, whereas the sets $ARI^{al/al}$ resp. $ARI^{al/il}$ are not.[61]

Our generating series $zag^\bullet$ is in $GARI^{as/is}$, not in $GARI^{\underline{as}/\underline{is}}$. However, it can be factored into a three-term *GARI*-product, with one exceptional factor in $GARI^{as/is}$ and two main factors in $GARI^{\underline{as}/\underline{is}}$

[59] i.e. symmetral and with a symmetral *swappee.*

[60] i.e. symmetral and with a symmetril *swappee.*

[61] It should be noted that, for the components of length $r \geqslant 2$, bialternality *implies* global parity, i.e. invariance under a simultaneous sign change of all w_i's. For $r = 1$, on the other hand, the bialternality condition, being empty, implies nothing.

Adequation of the flexion structure to multizeta arithmetics.

(i) Moving from the scalar multizetas $wa^\bullet/ze^\bullet$ to the generating series $zag^\bullet/zig^\bullet$ simplifies and *compactifies* everything.

(ii) The series $zag^\bullet/zig^\bullet$ clarify the expression of the double symmetry, conversion rule ('dimorphy'), colour consistency etc.

(iii) *GARI* contains, alone of all competing frameworks, such basic, even downright indispensible objects as the bimoulds $pal^\bullet/pil^\bullet$ and $tal^\bullet/til^\bullet$.

(iv) The series $zag^\bullet/zig^\bullet$ can also be viewed as *meromorphic functions* in $\boldsymbol{u}$ or $\boldsymbol{v}$ respectively, with *simple multivariate poles*. This makes them ideally suited for disentangling the algebraic identities between multizetas, which seem to be wholly derivable from (iterated) polar identities of the form:

$$\frac{1}{n_1^{s_1}\, n_2^{s_2}} = \sum_{\sigma_1,\sigma_2} \Big(\frac{\alpha^{s_1,s_2}_{\sigma_1,\sigma_2}}{n_1^{\sigma_1}\, n_{1,2}^{\sigma_2}} + \frac{\beta^{s_1,s_2}_{\sigma_1,\sigma_2}}{n_2^{\sigma_1}\, n_{2,1}^{\sigma_2}}\Big) = \sum_{\sigma_1,\sigma_2} \Big(\frac{\gamma^{s_1,s_2}_{\sigma_1,\sigma_2}}{n_1^{\sigma_1}\, n_{2:1}^{\sigma_2}} + \frac{\delta^{s_1,s_2}_{\sigma_1,\sigma_2}}{n_2^{\sigma_1}\, n_{1:2}^{\sigma_2}}\Big)$$

3.3 The basic polar/trigonometric bisymmetrals.

Set $P(t) := \frac{1}{t}$ and $Q(t) := \frac{\pi}{\tan(\pi\, t)}$. Then there exists

($*$) an essentially unique pair of 'polar' bimoulds $pal^\bullet/pil^\bullet \in GARI^{\mathrm{as/as}}$ with $pal^{w_1,\dots,w_r}$ r-homogeneous in the terms $P(u_i)$ and $P(u_1{+}\dots{+}u_{2\,i})$.

($**$) an essentially unique pair of 'trigonometric' $tal^\bullet/til^\bullet \in GARI^{\mathrm{as/as}}$ with $tal^{w_1,\dots,w_r}$ r-homogeneous[62] in the terms π^2, $Q(u_i)$ and $Q(u_1{+}\dots{+}u_{2\,i})$.

These two bisymmetrals $pal^\bullet/pil^\bullet$ and $tal^\bullet/til^\bullet$
(i) admit several equivalent definitions/characterisations,
(ii) possess no end of remarkable properties,
(iii) are key to the understanding of multizetas (many times over!),
(iv) cannot be defined in any of the alternative frameworks.

For the pair $tal^\bullet/til^\bullet$, the conversion formula involves the central bimould $mane^\bullet$ in $GARI_{centre}$ (see (231)):

$$\mathrm{swap.pil}^\bullet \;=\; \mathrm{pal}^\bullet \tag{235}$$
$$\mathrm{swap.til}^\bullet \;=\; \mathrm{gari}(\mathrm{tal}^\bullet,\mathrm{mane}^\bullet) = \mathrm{gari}(\mathrm{mane}^\bullet,\mathrm{tal}^\bullet) \tag{236}$$

Since their length-1 components are *odd* functions of w_1:

$$\mathrm{pal}^{w_1} = -\frac{1}{2}\,P(u_1) \qquad ; \qquad \mathrm{tal}^{w_1} = -\frac{1}{2}\,Q(u_1) \tag{237}$$

[62] π and $Q(.)$ are both assigned degree 1, but π occurs only through its *even* powers.

the bimoulds $pal^\bullet$ and $tal^\bullet$ are in $GARI^{as/as}$, not $GARI^{\underline{as}/\underline{as}}$. That prevents their *gari*-inverses $ripal^\bullet$ and $rital^\bullet$ from being bisymmetral. These are remarkable nonetheless. Thus, $ripal^\bullet$ is in $GARI^{as/is}$.

The double symmetry exchanger $adari(pal^\bullet)$.

In multizeta algebra, the double symmetries that count most are $\underline{al}/\underline{il}$ and $\underline{as}/\underline{is}$, but we must also resort to the double symmetries $\underline{al}/\underline{al}$ and $\underline{as}/\underline{as}$ which have the signal advantage of being *iso-length*, i.e. of involving only bimould components of the same length. Hence the need for *double symmetry exchangers*, assembled from the bisymmetral $pal^\bullet$:

$$\begin{array}{ccc} \mathrm{GARI}^{\underline{as}/\underline{as}} & \stackrel{\mathrm{adgari(pal^\bullet)}}{\longrightarrow} & \mathrm{GARI}^{\underline{as}/\underline{is}} \\ \uparrow{\scriptstyle \mathrm{expari}} & & \uparrow{\scriptstyle \mathrm{expari}} \\ \mathrm{ARI}^{\underline{al}/\underline{al}} & \stackrel{\mathrm{adari(pal^\bullet)}}{\longrightarrow} & \mathrm{ARI}^{\underline{al}/\underline{il}} \end{array}$$

and operating through adjoint action:

$$\mathrm{adgari}(A^\bullet)\, B^\bullet \ := \ \mathrm{gari}(A^\bullet, B^\bullet, \mathrm{invgari}\, A^\bullet) \tag{238}$$
$$\mathrm{adari}(A^\bullet) \ := \ \mathrm{logari.adgari}(A^\bullet)\mathrm{.expari} \tag{239}$$

Mark here the first occurrence of $pal^\bullet/pil^\bullet$.

3.4 The double trifactorisation of $zag^\bullet/zig^\bullet$.

The basic trifactorisation.

We have the π^2-isolating, parity-splitting identity:

$$\mathrm{zag}^\bullet = \mathrm{gari}(\mathrm{zag}^\bullet_{\mathrm{I}}, \mathrm{zag}^\bullet_{\mathrm{II}}, \mathrm{zag}^\bullet_{\mathrm{III}}) \tag{240}$$

with $\mathrm{zag}^\bullet_{\mathrm{I}} \in GARI^{as/is}, \mathrm{zag}^\bullet_{\mathrm{II}} \in GARI^{\underline{as}/\underline{is}}_{\mathrm{even}}, \mathrm{zag}^\bullet_{\mathrm{III}} \in GARI^{\underline{as}/\underline{is}}_{\mathrm{odd}}$.

$$\mathrm{zag}^\bullet_{\mathrm{I}} = \mathrm{gari}(\mathrm{tal}^\bullet, \mathrm{invgari}\,.\,\mathrm{pal}^\bullet, \mathrm{expari}\,.\,\mathrm{røma}^\bullet) \tag{241}$$
$$\mathrm{zag}^\bullet_{\mathrm{II}} = \mathrm{expari}\Big(\sum_{k\ even} \rho_{*\mathrm{II}}^{s_1,..,s_k}\, \vec{\mathrm{preari}}(\mathrm{løma}^\bullet_{s_1}, ..., \mathrm{løma}^\bullet_{s_k})\Big) \tag{242}$$
$$\mathrm{zag}^\bullet_{\mathrm{III}} = \mathrm{expari}\Big(\sum_{k\ odd} \rho_{*\mathrm{III}}^{s_1,..,s_k}\, \vec{\mathrm{preari}}(\mathrm{løma}^\bullet_{s_1}, ..., \mathrm{løma}^\bullet_{s_k})\Big) \tag{243}$$

Mark here the second consecutive occurrence of $pal^\bullet/pil^\bullet$ and the first appearance of $tal^\bullet/til^\bullet$.

$\rho^\bullet_{*\mathrm{II}}$ and $\rho^\bullet_{*\mathrm{III}}$ denote two alternal moulds with values in the $\mathbb{Q}$-ring of multizeta irreducibles.

The bimoulds $røma^\bullet$ and $løma^\bullet$ are both in $ARI^{\underline{al}/\underline{il}}$, but intervene in very different capacities. As a $\boldsymbol{u}$-function, $røma^\bullet$ must carry singularities at the origin to cancel those of $tal^\bullet$ and $pal^\bullet$ and produce a singularity-free $zag^\bullet_I$. The bimould $løma^\bullet$, on the other hand, and its components $løma^\bullet_s$ of total weight s, should from the start be free of poles at the origin, again to produce singularity-free factors $zag^\bullet_{II}$ and $zag^\bullet_{III}$.

In the above formulae, *preari* denotes the pre-Lie product (214) behind *ari*, and *expari* the natural exponential (213) from *ARI* to *GARI*.

An alternative expression for $zag^\bullet_{II}$, $zag^\bullet_{III}$ would be

$$\mathrm{zag}^\bullet_{\mathrm{II}} = 1^\bullet + \sum_{k\ even} \rho_{\mathrm{II}}^{s_1,..,s_k} \overrightarrow{\mathrm{preari}}(\mathrm{løma}^\bullet_{\mathrm{s}_1}, ..., \mathrm{løma}^\bullet_{\mathrm{s}_\mathrm{k}}) \tag{244}$$

$$\mathrm{zag}^\bullet_{\mathrm{III}} = 1^\bullet + \sum_{k\ odd} \rho_{\mathrm{III}}^{s_1,..,s_k} \overrightarrow{\mathrm{preari}}(\mathrm{løma}^\bullet_{\mathrm{s}_1}, ..., \mathrm{løma}^\bullet_{\mathrm{s}_\mathrm{k}}) \tag{245}$$

with two symmetral moulds $\rho^\bullet_{\mathrm{II}}$, $\rho^\bullet_{\mathrm{III}}$ that are none other than the mould-exponentials of the alternal moulds $\rho^\bullet_{*\mathrm{II}}$, $\rho^\bullet_{*\mathrm{III}}$.

Note that whereas separating $zag^\bullet_{III}$ from the first two factors is easy (a simple flexion formula takes care of that), disentangling $zag^\bullet_{II}$ from $zag^\bullet_I$ is arduous and calls for the construction of an auxiliary bimould $røma^\bullet/rømi^\bullet$ analogous to $løma^\bullet/l{ø}mi^\bullet$.

3.5 Singulators, singulates, singulands.

Bimoulds like $løma^\bullet$ are elements of $ARI^{\underline{al}/\underline{il}}_{\underline{ent}}$, i.e. of type $\underline{al}/\underline{il}$ with values in the ring of $\boldsymbol{u}$-polynomials. To construct such bimoulds, we require a machinery for singularity compensation: we must not only shuttle back and forth between $ARI^{\underline{al}/\underline{il}}_{\underline{ent}}$ and $ARI^{\underline{al}/\underline{al}}_{\underline{ent}}$ but also, at every second induction step, remove unwanted singular parts of type $\underline{al}/\underline{al}$. This, however, is easier said than done. It calls for sophisticated operators capable of producing, from regular bimoulds, any given bisymmetral singularity at the origin of the $\boldsymbol{u}$-multiplane.

(i) The operators in question are the *singulators*.
(ii) The regular inputs are the *singulands*.
(iii) The singular, bisymmetral outputs are the *singulates*.

Here again, for the third time, the pair $pal^\bullet/pil^\bullet$ turns out to be the construction's essential ingredient, in combination with the elementary operators $leng_r$, $neginvar$, $pushinvar$, mut. Here are the bare definitions.[63].

We begin with the elementary singulators:

[63]For details, see [...]. Regarding the inadequacy of *ari*-composition by u_1^{-2} for the purpose of correcting bialternal singularities, see [...] on our homepage.

• Singulator $slank_r$: linear operator, turns $S^\bullet$ into $\Sigma^\bullet$
• Singuland $S^\bullet$: regular, length-1 bimould (parity opposed to r)
• Singulate $\Sigma^\bullet$: singular bialternal with polarity of order $r-1$

$$\begin{aligned}
\mathrm{slank_r} : \quad & S^\bullet \in \mathrm{BIMU}_{1,regular} \mapsto \Sigma^\bullet \in \mathrm{ARI}^{\underline{\mathrm{al}}/\underline{\mathrm{al}}}_{r,singular} && (246)\\
2\,\mathrm{slank}_r.S^\bullet \;=\; & \mathrm{leng}_r.\mathrm{neginvar}.(\mathrm{adari}(\mathrm{pal}^\bullet))^{-1}.\mathrm{mut}(\mathrm{pal}^\bullet).S^\bullet && (247)\\
=\; & \mathrm{leng}_r.\mathrm{pushinvar}.\mathrm{mut}(\mathrm{neg}.\mathrm{pal}^\bullet).\mathrm{garit}(\mathrm{pal}^\bullet).S^\bullet && (248)
\end{aligned}$$

with

$$\begin{aligned}
\mathrm{mut}(A^\bullet).M^\bullet \;&:=\; \mathrm{mu}(\mathrm{invmu}.A^\bullet, M^\bullet, A^\bullet) && (249)\\
\mathrm{neginvar} \;&:=\; \mathrm{id} + \mathrm{neg} && (250)\\
\mathrm{pushinvar} \;&:=\; \sum_{0\leqslant r}(\mathrm{id} + \mathrm{push} + \mathrm{push}^2 + \cdots + \mathrm{push^r}).\mathrm{leng_r} && (251)
\end{aligned}$$

By taking multiple *ari*-brackets (from left to right) of elementary singulators $slank_{r_i}$, we easily arrive at the compsite singulators:

$$\mathrm{slank_{r_1,\ldots,r_n}} : \quad S^\bullet \in \mathrm{BIMU}_{n,regular} \mapsto \Sigma^\bullet \in \mathrm{ARI}^{\underline{\mathrm{al}}/\underline{\mathrm{al}}}_{r,singular} \tag{252}$$

• Singulator $slank_{r_1,\ldots,r_n}$: linear operator, turns $S^\bullet$ into $\Sigma^\bullet$.
• Singuland $S^\bullet$: regular bimould of length n bimould, with partial parities in each w_i opposed to r_i.
• Singulate $\Sigma^\bullet$: singular bialternal bimould with total polarity at the origin of order $r-n = \sum(r_i-1)$.

Symmetry-respecting singularity removal.

We are now in a position to construct elements $løma^\bullet/lømi^\bullet$ of $ARI^{\underline{al}/\underline{il}}$ inductively on the *length* r (also known as *depth*). Start from length 1, where the condition $\underline{al}/\underline{il}$ reduces to *parity in* w_1. Assume we have already reached some higher *odd* length r. Apply the double symmetry exchanger $adari(pal^\bullet)^{-1} = adari(ripal^\bullet)$ so as to get into the more congenial environment $ARI^{\underline{al}/\underline{al}}$. Then leave the component of length $r+1$ as it is but add a *suitable singulate*[64] to the component of length $r+2$. Lastly, apply $adari(pal^\bullet)$ to return to $ARI^{\underline{al}/\underline{il}}$, where $løma^\bullet/lømi^\bullet$ is now *defined and regular at* $\boldsymbol{u} = \mathbf{0}$

[64] i.e. a singulate that verifies the *desingularisation equations* of **19**.

up to length $r+2$ inclusively.

$$
\begin{array}{lll}
\mathrm{løma}^\bullet\|_r & \in \mathrm{ARI}^{\underline{al}/\underline{il}} & \text{and regular at } 0 \\
\downarrow \mathrm{adari}(\mathrm{pal}^\bullet)^{-1} & & \\
\mathrm{viløma}^\bullet\|_r & \in \mathrm{ARI}^{\underline{al}/\underline{al}} & \text{and singular at } 0 \\
\downarrow \text{trivial extension} & & \\
\mathrm{viløma}^\bullet\|_{r+1} & \in \mathrm{ARI}^{\underline{al}/\underline{al}} & \text{and singular at } 0 \\
\downarrow \mathrm{adari}(\mathrm{pal}^\bullet) & & \overset{(\textit{desingularisation})}{\text{with correction if } r \text{ even}} \\
\mathrm{løma}^\bullet\|_{r+1} & \in \mathrm{ARI}^{\underline{al}/\underline{il}} & \text{and regular at } 0
\end{array}
$$

So much for the general scheme, of which there exist three main specialisations, denoted by the vowels *u, o, a* in place of the unassigned, all-purpose vowel *ø*. See §5.6 and §5.7.

Constructing *løma*$^\bullet$ by desingularisation.

The first and simplest desingularisation occurs at length $r = 3$ with a composite singuland $S_{1,2}^{w_1,w_2}$:

$$\mathrm{slank}_{1,2}.S_{1,2}^\bullet = \mathrm{ari}(\mathrm{slank}_1.S_1^\bullet, \mathrm{slank}_2.S_2^\bullet) \quad \textit{with} \quad S_{1,2}^\bullet = S_1^\bullet \otimes S_2^\bullet$$

For $S_{1,2}^\bullet$, the *desingularisation equation* reads:

$$S_{1,2}^{\binom{u_1\ ,\ u_2}{\epsilon_1\ ,\ \epsilon_2}} + S_{1,2}^{\binom{u_2\ ,\ u_{1,2}}{\epsilon_{2:1}\ ,\ \epsilon_1}} - S_{1,2}^{\binom{u_1\ ,\ u_{1,2}}{\epsilon_{1:2}\ ,\ \epsilon_2}} - S_{1,2}^{\binom{u_{1,2}\ ,\ u_2}{\epsilon_1\ ,\ \epsilon_{2:1}}} = \textit{earlier terms}$$

For uncoloureds and with conventional notations, we get:

$$S_{1,2}^{u_1,u_2} + S_{1,2}^{u_2,u_1+u_2} - S_{1,2}^{u_1,u_1+u_2} - S_{1,2}^{u_1+u_2,u_2} = \textit{earlier terms}$$

For the general singuland $\mathrm{S}_{r_1,\ldots,r_k}^{u_1,\ldots,u_r}$, the desingularisation equation reads:

$$\sum_\sigma \epsilon_\sigma S_{r_1,..,r_k}^{\sigma(u_1,..,u_k)} = \textit{earlier terms} \quad \big(\sigma \in \mathrm{SL}_k(\mathbb{Z}), \epsilon_r \in \{0, \pm 1\}\big)$$

More generally, to proceed from length r to length $r+2$ (r odd) in the inductive construction of *løma*$^\bullet$, composite singulands $S_{r_1,\ldots,r_k}^\bullet$ are required, with $2 \leqslant k \leqslant r+1, 1 \leqslant r_i, \sum r_i = r+2$. The corresponding singulates $\Sigma_{r_1,\ldots,r_k}^\bullet$ are obtained as *ari*-products of the simple singulates $\Sigma_{r_i}^\bullet$ and have polarity of order $2+r-k$ at the origin of the $\boldsymbol{u}$-space. The step $r \to r+2$ actually resolves itself into a sub-induction on k, from $k = 2$ (polarity of order r) to $k = r+1$ (polarity of order 1).

3.6 General difficulty: infinitude underlying the double symmetry.

For any given length r, the *first* resp. *second* symmetry amounts to a set of relations between $A^{\boldsymbol{w}}$ and the various $A^{\sigma.\boldsymbol{w}}$ resp. between $A^{\boldsymbol{w}}$ and the various $A^{\tau.\boldsymbol{w}}$, where $\sigma \in \mathfrak{S}_r$ and $\tau \in \mathfrak{S}_r^* := swap.\mathfrak{S}_r.swap$. Combining the two symmetries forces us to work with the group $< \mathfrak{S}_r, \mathfrak{S}_r^* >$ generated by the classical symmetric group $\mathfrak{S}_r$ and its copy $\mathfrak{S}_r^*$. That larger group is *infinite* as soon as $r \geqslant 3$.

This complicates matters, e.g. by precluding the existence of *functional* projectors of ARI onto $ARI^{\underline{al}/\underline{al}}$ or $ARI^{\underline{al}/\underline{il}}$.

For $r = 2$, $< \mathfrak{S}_2, \mathfrak{S}_2^* >$ essentially reduces (modulo parity) to the anharmonic group. This explains why length-2 multizetas are quite elementary and decidedly untypical.

3.7 Difficulties proper to the monocolours and bicolours.

Generators and irreducibles.

It should be clear by now that the construction of a system $\{\rho^{s_1,\dots,s_r}\}$ of irreducibles involves two very distinct steps:

(i) The construction of a system of generators $\{lø ma_s^\bullet, s\ odd\}$, according to the general scheme of §3.5.

(ii) The expression of elements of $ARI^{\underline{al}/\underline{il}}$ in terms of these generators.

All known algebraic relations between multizetas respect the s-gradation, but the multizetas of a given weight s soon become too numerous for practical handling. Hence the need to work with the finer grained (s, r)-filtration. Here, however, the nuisance of *retro-action* raises its head – a nuisance which assumes two distinct, almost opposed forms for the monocolours and bicolours, and call for distinct remedies.

Retro-action for monocolours.

(i) The *construction* of a generating system $\{lø ma_s^\bullet, s = 3, 5, 7...\}$ of $ARI_{mono}^{\underline{al}/\underline{il}}$ can be carried out in accordance with the (s, r)-filtration. This means that once all the relations implied by the two symmetries have been taken into account up to length r, there is no retro-action to expect: the symmetry relations for higher lengths r' induce no further constraints on the length-r component.[65]

[65]This might a priori have been the case, since an alternality relation relative to two partial sequence $\boldsymbol{w^1}, \boldsymbol{w^2}$ of lengths r_1, r_2 contrains all the sequences of length between

(ii) However, the *decomposition* of an element of $ARI^{al/il}_{mono}$ into multibrackets of $l\o ma^{\bullet}_{s}$ cannot proceed entirely within the (s,r)-filtration. This is due the well-known relations which exist between the length-1 bialternals, and which induce on $ARI^{al/il}_{mono}$ non-trivial relations of type

$$\sum_{s_1+\dots+s_n=s} c_{s_1,..,s_r}\ \overrightarrow{ari}\ (\mathrm{l\o ma}^{\bullet}_{s_1},...,\mathrm{l\o ma}^{\bullet}_{s_r}) \equiv 0 \qquad \mathrm{mod}\ \mathit{length}\ r+2 \tag{253}$$

As a consequence, when decomposing $ARI^{al/il}_{mono}$ into multibrackets of $l\o ma^{\bullet}_{s}$ according to the (s,r) filtration, parasitical degrees of liberty are liable to appear at length r that will be removed only at length $r+2$.

(iii) The remedy lies in *perinomal analysis*.

Retro-action for bicolours.

With bicolours, the position is exactly the reverse.

(i) Once we get hold of any system of generators $\{l\o ma^{\bullet}_{s}, s = 1,3,5...\}$ (with one generator for any odd weight and with nonzero length-1 component, the *decomposition* of an element of $ARI^{al/il}_{bico}$ into multibrackets can proceed smoothly in accordance with the (s,r)-filtration, because of an independence lemma (see next section) that precludes any relation of *ari*-dependence between the $l\o ma^{\bullet}_{s}$ in $ARI^{al/il}_{bico}$.

(ii) However, the *construction* of such a system cannot proceed entirely within the (s,r)-filtration. At each odd length $r < s/3$, we are saddled with (abundant) parasitical degrees of freedom which manifest in the construction of the length-r component of $l\o ma^{\bullet}_{s}$, and these won't be removed until we proceed to much higher lengths (not just $r+2$). A glaring manifestation of this phonomenon already occurs at length $r = 1$. The double symmetry condition there is empty and therefore any choice of type

$$l\o ma^{\bullet\binom{u_1}{0}}_{s} := \alpha\, u_1^{s_1-1} \quad , \quad l\o ma^{\bullet\binom{u_1}{0}}_{s} := \beta\, u_1^{s_1-1} \quad (\alpha,\beta \in \mathbb{C}) \tag{254}$$

would seem to be acceptable — which of course it is not, given that the *colour consistency* relation (226) implies

$$\alpha + \beta = 2^{1-s_1}.\alpha \tag{255}$$

Since the *colour consistency* constraints are themselves an algebraic consequence of the double symmetry, (255) is a spectacular instance of retro-action.

$\sup(r_1, r_2)$ and $r_1 + r_2$

(iii) Even *adding* the colour consistency constraints would not salvage the (s,r)-scheme by ridding it of retro-action. At length $r=3$, for instance, a large number of parasitical degrees of freedom would remain. So we must look elsewhere for a remedy – namely to the technique of *satellisation*, to which the entire §4 will be devoted.

3.8 The independence theorem for bicolours.

Consider the homogeneous, length-1 elements of $ARI^{\underline{al}/\underline{al}}$ that verify the colour consistency condition (226). They are all of the form $b^\bullet_{d_1}$ with

$$b_{d_1}^{\binom{u_1}{\epsilon_1}} = \begin{cases} u_1^{d_1} & \textit{if} \quad \epsilon_1 = 0\ ,\ \forall d_1 \in 2\mathbb{N}^* \\ u_1^{d_1}\,(2^{-d_1}-1) & \textit{if} \quad \epsilon_1 = \frac{1}{2}\ ,\ \forall d_1 \in 2\mathbb{N}^* \end{cases} \tag{256}$$

$$b_{0}^{\binom{u_1}{\epsilon_1}} = \begin{cases} 0 & \textit{if} \quad \epsilon_1 = 0 \\ 1 & \textit{if} \quad \epsilon_1 = \frac{1}{2} \end{cases} \tag{257}$$

Proving the independence of these $b^\bullet_{d_1}$ under the *ari*-bracket is the same as proving that of the following $B^\bullet_{d_1}$

$$B_{d_1}^{\binom{u_1}{\epsilon_1}} = \begin{cases} u_1^{d_1}\,x^{d_1} & \textit{if} \quad \epsilon_1 = 0\ ,\ \forall d_1 \in \mathbb{N}^* \\ u_1^{d_1}\,(1-x^{d_1}) & \textit{if} \quad \epsilon_1 = \frac{1}{2}\ ,\ \forall d_1 \in \mathbb{N}^* \end{cases} \tag{258}$$

$$B_{0}^{\binom{u_1}{\epsilon_1}} = \begin{cases} 0 & \textit{if} \quad \epsilon_1 = 0 \\ 1 & \textit{if} \quad \epsilon_1 = \frac{1}{2} \end{cases} \tag{259}$$

for $x=2$ and *even* degrees d_1, since $2^{d_1}\,b^\bullet_{d_1} \equiv B^\bullet_{d_1}\|_{x=2}$. It is actually no harder to prove the independence for all integers $x \geqslant 2$ and all degrees d_1, even or odd. To do that, it suffixes to consider, for bimoulds $M^\bullet$ with lower indices $v_i = \epsilon_i \in \frac{1}{2}\mathbb{Z}/\mathbb{Z}$, the 'parts' $sa^*_0.M^\bullet$ and $sa^*_{\frac{1}{2}}.M^\bullet$ so defined:

$$\{\mathcal{M}^\bullet_0 = \mathrm{sa}^*_0.M^\bullet\} \iff \{\mathcal{M}_0^{u_1,\dots,u_r} = M^{\binom{u_1\ ,\dots,\ u_r}{0\ ,\dots,\ 0}}\} \tag{260}$$

$$\{\mathcal{M}^\bullet_{\frac{1}{2}} = \mathrm{sa}^*_{\frac{1}{2}}.M^\bullet\} \iff \{\mathcal{M}_{\frac{1}{2}}^{u_1,\dots,u_r} = M^{\binom{u_1\ ,\dots,\ u_r}{\frac{1}{2}\ ,\dots,\ \frac{1}{2}}}\} \tag{261}$$

and to note how they behave under the *ari*-bracket:[66]

$$\mathrm{sa}^*_0.\mathrm{ari}(A^\bullet,B^\bullet) = \mathrm{ari}(\mathrm{sa}^*_0.A^\bullet,\mathrm{sa}^*_0.B^\bullet) \tag{262}$$

$$\mathrm{sa}^*_{\frac{1}{2}}.\mathrm{ari}(A^\bullet,B^\bullet) = \begin{cases} +\mathrm{arit}(\mathrm{sa}^*_0.B^\bullet).(\mathrm{sa}^*_{\frac{1}{2}}.A^\bullet) - \mathrm{arit}(\mathrm{sa}^*_0.A^\bullet).(\mathrm{sa}^*_{\frac{1}{2}}.B^\bullet) \\ +\mathrm{lu}(\mathrm{sa}^*_{\frac{1}{2}}.A^\bullet,\mathrm{sa}^*_{\frac{1}{2}}.B^\bullet) \end{cases} \tag{263}$$

[66] see §4.2, where the procedure is systematised. Though the 'parts' $sa_0.M^\bullet$ and $sa_{\frac{1}{2}}.M^\bullet$ are *moulds*, not bimoulds, we can subject them to all the flexion operation by regarding them as *bimoulds* independent of the lower indices.

The idea is to introduce the moulds

$$\mathcal{A}_{d_1}^{u_1} \quad := \quad u_1^{d_1} \qquad\qquad \forall d_1 \in \mathbb{N} \tag{264}$$

and to compare the *lu*-brackets of the $\mathcal{A}_{d_i}^{\bullet}$ with the *ari*-brackets of the $B_{d_i}^{\bullet}$, or rather with the $sa_{\frac{1}{2}}$ part of these *ari*-brackets.

Let us fix a length r and a total degree $d := d_1 + \cdots + d_r$. For any sequence $\boldsymbol{d} = (d_1, ..., d_r)$ of non-negative integers d_i, let us set

$$\mathcal{A}_{\boldsymbol{d}}^{\bullet} \quad := \quad \overrightarrow{\mathrm{lu}}\,(\mathcal{A}_{d_1}^{\bullet}, \ldots, \mathcal{A}_{d_r}^{\bullet}) \tag{265}$$

$$\mathcal{B}_{\boldsymbol{d}}^{\bullet} \quad := \quad \mathrm{sa}_{\frac{1}{2}}.\,\overrightarrow{\mathrm{ari}}\,(B_{d_1}^{\bullet}, \ldots, B_{d_r}^{\bullet}) \tag{266}$$

Let $\mathcal{E}_{r,d} = \{\mathcal{A}_{\boldsymbol{d}^1}^{\bullet}, \mathcal{A}_{\boldsymbol{d}^2}^{\bullet}, \ldots, \mathcal{A}_{\boldsymbol{d}^{n(r,d)}}^{\bullet}\}$ be a basis of all alternal, polynomial-valued moulds of length r and total degree d. The alternal, polynomial-valued mould $\mathcal{B}_{\boldsymbol{d}}^{\bullet}$ can be expressed in that basis. We find:

$$\mathcal{B}_{\boldsymbol{d}}^{\bullet} = \sum_{\boldsymbol{d}'} c_{\boldsymbol{d}}^{\boldsymbol{d}'}(x)\,\mathcal{A}_{\boldsymbol{d}'}^{\bullet} \quad \textit{with} \quad c_{\boldsymbol{d}}^{\boldsymbol{d}'}(x) \in \mathbb{Z}[x] \ \ \textit{and} \ \begin{cases} c_{\boldsymbol{d}}^{\boldsymbol{d}}(0) = 1 \\ c_{\boldsymbol{d}}^{\boldsymbol{d}'}(0) = 0 \ \ \textit{if} \ \ \boldsymbol{d} \neq \boldsymbol{d}' \end{cases} \tag{267}$$

The reason is quite simply that, according to formula (263), the x-constant terms in $\mathcal{B}_{\boldsymbol{d}}^{\bullet}$ can only come from the *lu*-bracketting. As a consequence, the corresponding determinant, independent of the basis choice

$$\mathrm{det}_{r,d}(x) := \mathrm{Det}\big[c_{\boldsymbol{d}}^{\boldsymbol{d}'}(x); \boldsymbol{d}, \boldsymbol{d}'\big] = 1 + \sum \gamma_{r,r,k} x^k \qquad \Big(\sum \gamma_{r,r,k} \in \mathbb{Z}\Big) \tag{268}$$

is a polynomial in x, with integer coefficients and with 1 as constant term. It is therefore $\neq 0$ for all integer values of x larger than 1. This establishes, for all such values of x and in particular for $x = 2$, the *ari*-independence of the bimoulds $\mathcal{B}_{d}^{\bullet}$.

Remark 1: The above argument would collapse if we were to work with the swappees $C_{d_1}^{\bullet} := swap.B_{d_1}^{\bullet}$:

$$C_{d_1}^{\binom{\epsilon_1}{v_1}} \quad = \quad \begin{cases} v_1^{d_1}\, x^{d_1} & \textit{if} \quad \epsilon_1 = 0\ ,\ \forall d_1 \in \mathbb{N}^* \\ v_1^{d_1}\,(1 - x^{d_1}) & \textit{if} \quad \epsilon_1 = \frac{1}{2}\ ,\ \forall d_1 \in \mathbb{N}^* \end{cases} \tag{269}$$

$$C_{0}^{\binom{\epsilon_1}{v_1}} \quad = \quad \begin{cases} 0 & \textit{if} \quad \epsilon_1 = 0 \\ 1 & \textit{if} \quad \epsilon_1 = \frac{1}{2} \end{cases} \tag{270}$$

and their monocolour 'parts' $si_0^*.M^\bullet$ and $si_{\frac{1}{2}}^*.M^\bullet$:

$$\{\mathcal{M}_0^{\bullet} = \mathrm{si}_0^*.M^\bullet\} \iff \{\mathcal{M}_0^{v_1,\ldots,v_r} = M^{\binom{0}{v_1}\,\begin{smallmatrix},\ldots,\\,\ldots,\end{smallmatrix}\,\binom{0}{v_r}}\} \tag{271}$$

$$\{\mathcal{M}_{\frac{1}{2}}^{\bullet} = \mathrm{si}_{\frac{1}{2}}^*.M^\bullet\} \iff \{\mathcal{M}_{\frac{1}{2}}^{v_1,\ldots,v_r} = M^{\binom{\frac{1}{2}}{v_1}\,\begin{smallmatrix},\ldots,\\,\ldots,\end{smallmatrix}\,\binom{\frac{1}{2}}{v_r}}\} \tag{272}$$

For one thing, there would be no *closed* identities like (262)-(263) to describe the *ari*-action on the new 'parts'. Then we would find that there exist, even for $x = 2$ and even degrees d_i, non-trivial dependence relations of the form:

$$\sum_{d_1+\cdots+d_r=d} c_0^{d_1,\ldots,d_r}\ si_0^*.\mathrm{ari}(C^\bullet_{d_1},\ldots,C^\bullet_{d_r}) \equiv 0 \qquad (c_0^{\boldsymbol{d}} \in \mathbb{Z}) \tag{273}$$

$$\sum_{d_1+\cdots+d_r=d} c_{\frac{1}{2}}^{d_1,\ldots,d_r}\ si_{\frac{1}{2}}^*.\mathrm{ari}(C^\bullet_{d_1},\ldots,C^\bullet_{d_r}) \equiv 0 \qquad (c_{\frac{1}{2}}^{\boldsymbol{d}} \in \mathbb{Z}) \tag{274}$$

though of course none of the form

$$\sum_{d_1+\cdots+d_r=d} c^{d_1,\ldots,d_r}\ \mathrm{ari}(C^\bullet_{d_1},\ldots,C^\bullet_{d_r}) \equiv 0 \qquad (c^{\boldsymbol{d}} \in \mathbb{Z}) \tag{275}$$

Remark 2: The *ari*-independence of the $\underline{al}/\underline{al}$ bimoulds $b^\bullet_{d_i}$ of (256)-(257) automatically implies the independence of every possible $\underline{al}/\underline{il}$ extension ${}^*b^\bullet_{d_i}$ of these $b^\bullet_{d_i}$, since the length-r component of any dependence relation

$$\sum_{d_1+\cdots+d_r=d} c^{d_1,\ldots,d_r}\ \mathrm{ari}({}^*b^\bullet_{d_1},\ldots,{}^*b^\bullet_{d_r}) \equiv 0 \qquad (c^{\boldsymbol{d}} \in \mathbb{Z}) \tag{276}$$

would amount to a dependence relation between the $b^\bullet_{d_i}$. The situation is quite different for the monocolour generators of $ARI^{\underline{al}/\underline{il}}_{\underline{en}}$: they too are conjectured to be independent, but their length-1 components are not independent in $ARI^{\underline{al}/\underline{al}}$.

Remark 3: The only case relevant to multizeta algebra is when $x = 2$ and all degrees d_i are even.[67] Remarkably, the case $x = 2$ is also the only one when the prime factor decomposition of the integers $det_{r,d}(x)$ is arithmetically 'special': it systematically displays (large) prime factors coming from the Bernoulli numbers. Moreover, to take into account the exclusive presence of *even* degrees d_i and isolate the interesting part of $det_{r,d}(x)$, one should change the expansion (267) to

$$\mathcal{B}^\bullet_{\boldsymbol{d}}\|_{\mathrm{even}} = \sum_{\boldsymbol{d'}} c^{\boldsymbol{d'}}_{\boldsymbol{d}}(x)\, \mathcal{A}^\bullet_{\boldsymbol{d'}} \ \ \textit{with}\ \ c^{\boldsymbol{d'}}_{\boldsymbol{d}}(x) \in \mathbb{Z}[x] \ \textit{and} \ \begin{cases} c^{\boldsymbol{d}}_{\boldsymbol{d}}(0) = 1 \\ c^{\boldsymbol{d'}}_{\boldsymbol{d}}(0) = 0 \ \ \textit{if} \ \ \boldsymbol{d} \neq \boldsymbol{d'} \end{cases} \tag{277}$$

where $\mathcal{B}^{\boldsymbol{w}}_{\boldsymbol{d}}\|_{\mathrm{even}}$ denotes the part of $\mathcal{B}^{\boldsymbol{w}}_{\boldsymbol{d}}$ even in each u_i, and where $\mathcal{A}^\bullet_{\boldsymbol{d'}}$ runs through a basis of all alternate, polynomial-valued moulds also *even* in each u_i. The corresponding determinant $det^*_{r,d}(x)$, defined as (268) but with all sequences $\boldsymbol{d}$, $\boldsymbol{d'}$ consisting only of *even* integers, is also an *even* function of

[67] The case when x is an integer $\geqslant 3$ is of no direct relevance to the x-coloured multizetas.

x. These more basic determinants $det^*_{r,d}(t)$ have been tabulated in §6.3 (in terms of $t := x^2$) and the reader may check on these tables how 'special' the case $x = 2$ (i.e. $t = 4$) really is, arithmetically speaking.

- $det^*_{2,d}(2)$ carries all large prime factors of Ber_{d+2} with multiplicity one.
- $det^*_{3,d}(2)$ carries all large prime factors of $Ber_d, Ber_{d-2}, Ber_{d-4}...$ with multiplicity one.
- $det^*_{r,d}(2)$ carries all large prime factors of all $\prod_{\delta \leqslant d+6-2\,r} Ber_\delta$, usually with higher multiplicities, as soon as $r \geqslant 4$.

Remark 4: Replacing *ari, lu* by *preari, mu* in the previous argument, i.e. setting:

$$\mathcal{A}^\bullet_{\boldsymbol{d}} \quad := \quad \vec{\text{mu}}\,(\mathcal{A}^\bullet_{d_1}, \ldots, \mathcal{A}^\bullet_{d_r}) \tag{278}$$

$$\mathcal{B}^\bullet_{\boldsymbol{d}} \quad := \quad \text{sa}_{\frac{1}{2}}.\,\vec{\text{preari}}\,(B^\bullet_{d_1}, \ldots, B^\bullet_{d_r}) \tag{279}$$

and using the identities that describe the behavior of *preari* on $sa_0, sa_{\frac{1}{2}}$:

$$\text{sa}^*_0.\text{preari}(A^\bullet, B^\bullet) = \text{preari}(\text{sa}^*_0.A^\bullet, \text{sa}^*_0.B^\bullet) \tag{280}$$

$$\text{sa}^*_{\frac{1}{2}}.\text{preari}(A^\bullet, B^\bullet) = \text{arit}(\text{sa}^*_0.B^\bullet).(\text{sa}^*_{\frac{1}{2}}.A^\bullet) + \text{mu}(\text{sa}^*_{\frac{1}{2}}.A^\bullet, \text{sa}^*_{\frac{1}{2}}.B^\bullet) \tag{281}$$

we can easily establish the *preari*- indepence of the generators $B^\bullet_{r,d_i}$. However, we find that the determinants $predet_{r,d}(x)$ resp. $predet^*_{r,d}(x)$ calculated from the coefficients $c^{\boldsymbol{d}'}_{\boldsymbol{d}}(x)$ of the re-interpreted expansions (268) resp. (277) carry no new information: they turn out, unsurprisingly, to be entirely reducible to the previous determinants $det_{r,d}(x)$ resp. $det^*_{r,d}(x)$. Concretely:

$$\text{predet}_{r,d}(x) = \prod_{2 \leqslant \delta \leqslant d}^{\delta\ even} \text{predet}_{r-1,\delta}(x) \prod_{\rho | r\,,\, \rho | \frac{d}{2}}^{1 \leqslant \rho} \text{det}_{\frac{r}{\rho}, \frac{d}{\rho}}(x) \quad (\forall d\ even \geqslant 2) \tag{282}$$

$$\text{predet}^*_{r,d}(x) = \prod_{2r \leqslant \delta \leqslant d-2}^{\delta\ even} \text{predet}^*_{r-1,\delta}(x) \prod_{\rho | r\,,\, \rho | \frac{d}{2}}^{1 \leqslant \rho \leqslant \frac{d}{2} - r} \text{det}^*_{\frac{r}{\rho}, \frac{d}{\rho}}(x) \quad (\forall d\ even \geqslant 2r) \tag{283}$$

4 Multizeta algebra: the satellisation technique for bicolours.

Introduction.

The present chapter is devoted to the task of *data reduction* for bicolours. As usual, rather than directly handling the scalar multizetas, we deal with their generating functions $A^\bullet, S^\bullet$, at home in either $ARI^{\underline{al}/\underline{il}}_{bico}$ or $GARI^{\underline{as}/\underline{is}}_{bico}$:

$$\begin{array}{rcccllll} ARI^{\underline{al}/\underline{il}}_{bico} & \ni & A^\bullet & = & \{A^{\binom{u_1\ ,\dots,\ u_r}{\epsilon_1\ ,\dots,\ \epsilon_r}} & , \ u_i \in \mathbb{C} & , \ \epsilon_i \in \{0, 1/2\}\} \\ GARI^{\underline{as}/\underline{is}}_{bico} & \ni & S^\bullet & = & \{S^{\binom{u_1\ ,\dots,\ u_r}{\epsilon_1\ ,\dots,\ \epsilon_r}} & , \ u_i \in \mathbb{C} & , \ \epsilon_i \in \{0, 1/2\}\} \end{array}$$

- We successively define three 'satellites' sa, sa^*, sa^{**}, consisting each of a small number of *boundary data*.

- The *lower* or *root* satellite sa retains only the lower indices ϵ_i, i.e. the colours 0 (*white*) and 1/2 (*black*) while discarding all multizetas with partial weights s_i larger than 1.

- The *first upper* satellite sa^* does the opposite: it retains only the upper indices u_i and sets all colours ϵ_i equal to either 0 (*'all-whites'*) or 1/2 (*'all-blacks'*).

- The *second upper* satellite sa^{**} is deduced from sa under a construction known as *mould amplification*, but in outward shape and behaviour under *ari/gari*, it closely resembles sa^*.

- All these constructions, initially performed in $ARI^{\underline{al}}_{bico}$ or $GARI^{\underline{as}}_{bico}$, acquire new significance when we move to $ARI^{\underline{al}/\underline{il}}_{bico}$ or $GARI^{\underline{as}/\underline{is}}_{bico}$. The adjunction of the second symmetry *rigidifies* everything: each satellite contains all the information, and the challenge is now to extract that information.

- One of the first consequences is the existence of quite remarkable formulae expressing all mould components of *odd degree* in terms of those of *even degree.*[68]

- Another consequence is the existence of an explicit procedure, based on the operators *discram* and *viscram*, for recovering the whole of a mould $M^\bullet$ in $ARI^{\underline{al}/\underline{il}}_{bico}$ or $GARI^{\underline{as}/\underline{is}}_{bico}$ from the sole knowledge of its first upper satellite $sa^*.M^\bullet$.

- Yet another consequence is the existence of a remarkably explicit correspondence between the two upper satellites, so similar in shape yet so different in origin. For the *all-whites* (correctly defined), we have identity pure and simple, while for the *all-blacks* the correspondence assumes the form of an involution $\mathfrak{K}$ whose definition, unexpectedly, requires us to perform a *length* $\leftrightarrow$ *degree* exchanging isomorphism.

[68]and that too in every meaningful setting, i.e. in both upper satellites as well as in the whole of $ARI^{\underline{al}/\underline{il}}_{bico}$ or $GARI^{\underline{as}/\underline{is}}_{bico}$.

That said, it should be borne in mind that the whole business of *satellisation*, fascinating though it may appear, is not an end in itself. It is there only to pave the way for the real task: the explicit decomposition of bicolours into irreducibles. But this is another story, to be told some other time.

4.1 The lower or root satellisation sa: zero-degree bicolours.

Zero-degree elements.

In the *lower* or *root* satellisation (noted *"sa"*), the only extremal data we retain are the scalar multizetas $Ze^{\binom{\epsilon_1,\ \dots,\ \epsilon_r,}{s_1\ ,\dots,\ ,s_r}}$ whose partial weights s_i are all equal to 1 or, what amounts to the same, whose total degree $d := s - r$ is 0. In terms of generating series, this amounts to setting all u_i-variables equal to 0.

$$A^\bullet \in \mathrm{ARI}^{\underline{al}}_{bico} \quad\mapsto\quad \mathcal{A}^\bullet = \mathrm{sa}.A^\bullet \quad \textit{with} \quad \mathcal{A}^{\epsilon_1,\dots,\epsilon_r} := A^{\binom{0\ ,\dots,\ 0}{\epsilon_1\ ,\dots,\ \epsilon_r}} \tag{284}$$

$$S^\bullet \in \mathrm{GARI}^{\underline{as}}_{bico} \quad\mapsto\quad \mathcal{S}^\bullet = \mathrm{sa}.S^\bullet \quad \textit{with} \quad \mathcal{S}^{\epsilon_1,\dots,\epsilon_r} := S^{\binom{0\ ,\dots,\ 0}{\epsilon_1\ ,\dots,\ \epsilon_r}} \tag{285}$$

The extremal and penextremal algebra.

Needless to say, the extremal data $sa.ARI^{\underline{al}}_{bico}$ and $sa.GARI^{\underline{as}}_{bico}$ provide no information at all regarding the – totally independent – rest of $ARI^{\underline{al}}_{bico}$ and $GARI^{\underline{as}}_{bico}$. Things change completely, however, if we adduce a second symmetry. We shall see in the sequel that the whole of $ARI^{\underline{al}/\underline{il}}_{bico}$ (resp. $GARI^{\underline{as}/\underline{is}}_{bico}$) can be recovered from the extremal algebra $sa.ARI^{\underline{al}/\underline{il}}_{bico}$ (resp. from the extremal group $sa.GARI^{\underline{as}/\underline{is}}_{bico}$). This may sound improbable, if only because only the first symmetry of, say, $ARI^{\underline{al}/\underline{il}}_{bico}$, i.e. alternality, can be expressed internally in $sa.ARI^{\underline{al}/\underline{il}}_{bico}$. The second one, i.e. alternility, necessarily takes us beyond the range of 0-degree elements. However, we shall see that by considering the *penextremal* algebra, that is to say by retaining all terms of degree 0 *or* 1, we can overcome the deadlock:
(i) a *fraction* of the alternility relations becomes expressible *within* the penextremal algebra
(ii) this fraction turns out to be equivalent to the full alternility
(iii) the alternility relations so obtained can, after elimination of the degree-1 elements, be re-phrased purely in terms of the degree-0 elements, that is to say, within the extremal algebra.

The colour-switch ideal.

For the moment we may note a simple but consequential – and easy to check – fact: *Those elements of the extremal algebra that are invariant under the white ↔ black colour switch*

$$A^{\binom{0\ ,\dots,\ 0}{\epsilon_1\ ,\dots,\ \epsilon_r}} \quad \equiv \quad A^{\binom{0\ ,\dots,\ 0}{\bar{\epsilon}_1\ ,\dots,\ \bar{\epsilon}_r}} \qquad \textit{with} \quad \bar{\epsilon} := \frac{1}{2} - \epsilon \tag{286}$$

constitute an ideal of the extremal algebra.

In the trans-satellite equivalences yet to emerge, this *colour-switch* ideal in the root satellite shall correspond to the ideals of *vanishing all-whites* in the first and second satellites.

4.2 The first upper satellisation sa^*: all-whites and all-blacks

The *first upper satellisation* (noted sa^*), or *first satellisation* for short, proceeds in exactly the opposite direction. Instead of retaining the sole *colours*, as in the root satellisation, we now nearly completely eliminate them, and retain only monochrome multizetas, either fully painted in the colour 0 (*'all-whites'*) or in the colour $\frac{1}{2}$ (*'all-blacks'*):

$$A^\bullet \in \mathrm{ARI}^{\underline{al}}_{bico} \quad \mapsto \quad \mathrm{sa}^*.A^\bullet \quad \textit{with} \begin{cases} (\mathrm{sa}^*_0.A)^{u_1,\dots,u_r} := A^{\binom{u_1\ ,\dots,\ u_r}{0\ ,\dots,\ 0}} \\ (\mathrm{sa}^*_{\frac{1}{2}}.A)^{u_1,\dots,u_r} := A^{\binom{u_1\ ,\dots,\ u_r}{\frac{1}{2}\ ,\dots,\ \frac{1}{2}}} \end{cases} \tag{287}$$

$$S^\bullet \in \mathrm{GARI}^{\underline{as}}_{bico} \quad \mapsto \quad \mathrm{sa}^*.S^\bullet \quad \textit{with} \begin{cases} (\mathrm{sa}^*_0.S)^{u_1,\dots,u_r} := S^{\binom{u_1\ ,\dots,\ u_r}{0\ ,\dots,\ 0}} \\ (\mathrm{sa}^*_{\frac{1}{2}}.S)^{u_1,\dots,u_r} := S^{\binom{u_1\ ,\dots,\ u_r}{\frac{1}{2}\ ,\dots,\ \frac{1}{2}}} \end{cases} \tag{288}$$

The real justification for this drastic data restriction will emerge in the sequel. But right now we may observe that it has at least the merit of respecting the *ari/gari* operations, in the sense that these remain expressible entirely *within* the new framework.[69] Indeed:

[69]This is obvious enough for sa^*_0, much less so for $sa^*_{\frac{1}{2}}$. And it wouldn't be true at all if we had defined satellites $si^*.A^\bullet$, $si^*.S^\bullet$ based on the *swappes*, by setting:

$$\begin{aligned} (\mathrm{si}^*_0.A)^{v_1,\dots,v_r} \quad &:= \quad (\mathrm{swap}.A)^{\binom{0\ ,\dots,\ 0}{v_1\ ,\dots,\ v_r}} \\ (\mathrm{si}^*_{\frac{1}{2}}.A)^{v_1,\dots,v_r} \quad &:= \quad (\mathrm{swap}.A)^{\binom{\frac{1}{2}\ ,\dots,\ \frac{1}{2}}{v_1\ ,\dots,\ v_r}} \end{aligned}$$

Proposition 4.1 (Impact of the first satellisation on *ari/gari*) .
Let as usual $A^\bullet, B^\bullet$ etc stand for elements of ARI^{al}_{bico} and $S^\bullet, T^\bullet$ etc stand for elements of $\mathrm{GARI}^{as}_{bico}$. Then:

$$\mathrm{sa}^*_0\,\mathrm{ari}(A^\bullet,B^\bullet) = \mathrm{ari}(\mathrm{sa}^*_0\,A^\bullet,\mathrm{sa}^*_0\,B^\bullet) \tag{289}$$
$$\mathrm{sa}^*_0\,\mathrm{preari}(A^\bullet,B^\bullet) = \mathrm{preari}(\mathrm{sa}^*_0\,A^\bullet,\mathrm{sa}^*_0\,B^\bullet) \tag{290}$$
$$\mathrm{sa}^*_0\,\mathrm{gari}(S^\bullet,T^\bullet) = \mathrm{gari}(\mathrm{sa}^*_0\,S^\bullet,\mathrm{sa}^*_0\,T^\bullet) \tag{291}$$

$$\mathrm{sa}^*_{\frac12}\,\mathrm{ari}(A^\bullet,B^\bullet) = \begin{cases} +\mathrm{lu}(\mathrm{sa}^*_{\frac12}\,A^\bullet,\mathrm{sa}^*_{\frac12}\,B^\bullet) \\ +\mathrm{arit}(\mathrm{sa}^*_0\,B^\bullet)\,\mathrm{sa}^*_{\frac12}\,A^\bullet \\ -\mathrm{arit}(\mathrm{sa}^*_0\,A^\bullet)\,\mathrm{sa}^*_{\frac12}\,B^\bullet \end{cases} \tag{292}$$
$$\mathrm{sa}^*_{\frac12}\,\mathrm{preari}(A^\bullet,B^\bullet) = \begin{cases} +\mathrm{mu}(\mathrm{sa}^*_{\frac12}\,A^\bullet,\mathrm{sa}^*_{\frac12}\,B^\bullet) \\ +\mathrm{arit}(\mathrm{sa}^*_0 B^\bullet)\,\mathrm{sa}^*_{\frac12}\,A^\bullet \end{cases} \tag{293}$$
$$\mathrm{sa}^*_{\frac12}\,\mathrm{gari}(S^\bullet,T^\bullet) = \mathrm{mu}\Big(\big(\mathrm{garit}(\mathrm{sa}^*_0\,T^\bullet)\,\mathrm{sa}^*_{\frac12}\,S^\bullet\big)\,,\,\mathrm{sa}^*_{\frac12}\,T^\bullet\Big) \tag{294}$$

4.3 The second upper satellisation sa^{**}: amplification.

The amplification technique.

We have already used mould amplification to go from $wa^\bullet$ to $zag^\bullet$. We shall now use it once more to construct the *second satellisation.* Here are the basic facts about the *amplification* transform amp_{ω_*}:
(i) It acts on ordinary moulds $M^\bullet$.
(ii) It singles out the index ω_* for special treatment,
(iii) It adds a new indexation layer (here, the u_i indices),
(iv) It preserves simple symmetries (alternality/symmetrality).
(v) It act according to the formula[70]:

$$\big(\mathrm{amp}_{\omega_*}\,M\big)^{\binom{u_1\ ,\dots,\ u_r}{\omega_1\ ,\dots,\ \omega_r}} := \sum_{0\leqslant n_r} M^{\,\omega_1,\omega_*^{[n_1]},\dots,\omega_r,\omega_*^{[n_r]}}\; u_1^{n_1} u_{1,2}^{n_2}\dots u_{1,..,r}^{n_r} \tag{295}$$

(vi) If $M^\bullet$ possesses no particular symmetry, the passage $M^\bullet \to amp_{\omega_*} M^\bullet$ entails a loss of information, since the right-hand side of (295) 'ignores' all terms $M^{\boldsymbol\omega}$ with sequences $\boldsymbol\omega$ beginning with a string of ω_*'s.
(vii) If M$^\bullet$ is alternal or symmetral, so is $amp_{\omega_*} M^\bullet$, and there is no loss of information, since in that case any $M^{\boldsymbol\omega}$ can be expressed in terms of M^{ω_*}

[70]Here, $\omega_*^{[n]} := \overbrace{\omega_*, .., \omega_*}^{n\ times}$ and $u_{1,..,j} := u_1 + .. + u_j$ as usual.

and of other $M^{\boldsymbol{\omega}'}$, for indices $\boldsymbol{\omega}'$ without initial ω_*.
(viii) Mould amplification nearly commutes with mould multiplication, but with a corrective term which involves the special index ω_* and whose form depends only on the symmetry type of the second factor. Thus, for $B^\bullet$ alternal and $T^\bullet$ symmetral, we get the identities:

$$\mathrm{amp}_{\omega_*}(S^\bullet \times T^\bullet) = \big(\exp(T^{\omega_*}\mathfrak{D}_{\boldsymbol{u}})\,\mathrm{amp}_{\omega_*}S^\bullet\big) \times \big(\mathrm{amp}_{\omega_*}T^\bullet\big) \tag{296}$$

$$\mathrm{amp}_{\omega_*}(A^\bullet \times B^\bullet) = \big(\mathrm{amp}_{\omega_*}A^\bullet\big) \times \big(\mathrm{amp}_{\omega_*}B^\bullet\big) + B^{\omega_*}\,\mathfrak{D}_{\boldsymbol{u}}\,\big(\mathrm{amp}_{\omega_*}A^\bullet\big) \tag{297}$$

with $(\mathfrak{D}_{\boldsymbol{u}}M)^{\binom{u_1\ ,\dots,\ u_r}{\omega_1\ ,\dots,\ \omega_r}} := (u_1+\dots+u_r)\,M^{\binom{u_1\ ,\dots,\ u_r}{\omega_1\ ,\dots,\ \omega_r}}$

The amplification of elements of $sa.ARI^{al}_{bico}$ **or** $sa.GARI^{as}_{bico}$**.**

We shall now *amplify* elements $M^\bullet$ of the extremal algebra or group. These are bimoulds, but for the circumstance we may treat them as plain moulds, with indices either $\binom{0}{0}$ or $\binom{0}{\frac{1}{2}}$. That leaves only two possible amplications, namely $amp_{\binom{0}{0}}$ and $amp_{\binom{0}{\frac{1}{2}}}$. Since, in either case, all the lower indices on the right-hand side of (295) will be the same, $\frac{1}{2}$ or 0 respectively, we can ignore them as contributing no information. So, for any bimould $M^\bullet$ in ARI^{al}_{bico} or $GARI^{as}_{bico}$, we are justified in setting:

$$\mathrm{am}_0\, M^\bullet := \mathrm{amp}_{\binom{0}{0}}\,\mathrm{sa}\,M^\bullet \quad , \quad \mathrm{am}_{\frac{1}{2}}\, M^\bullet := \mathrm{amp}_{\binom{0}{\frac{1}{2}}}\,\mathrm{sa}\,M^\bullet \tag{298}$$

or more explicitely:

$$(\mathrm{am}_0.M)^{u_1,\dots,u_r} := \sum_{0\leqslant n_r} M^{\left(\begin{smallmatrix} 0 & \overleftrightarrow{,0\ ,\dots,\ 0,}^{n_1} & ,\dots, & 0 & \overleftrightarrow{,0\ ,\dots,\ 0}^{n_r} \\ 1/2 & ,0\ ,\dots,\ 0, & ,\dots, & 1/2 & ,0\ ,\dots,\ 0 \end{smallmatrix}\right)}\, u_1^{n_1} u_{1,2}^{n_2} \dots u_{1,..,r}^{n_r} \tag{299}$$

$$(\mathrm{am}_{\frac{1}{2}}.M)^{u_1,\dots,u_r} := \sum_{0\leqslant n_r} M^{\left(\begin{smallmatrix} 0, & \overleftrightarrow{0\ ,\dots,\ 0,}^{n_1} & ,\dots, & 0, & \overleftrightarrow{0\ ,\dots,\ 0}^{n_r} \\ 0, & 1/2\ ,\dots,\ 1/2, & ,\dots, & 0, & 1/2\ ,\dots,\ 1/2 \end{smallmatrix}\right)}\, u_1^{n_1} u_{1,2}^{n_2} \dots u_{1,..,r}^{n_r} \tag{300}$$

The impact on $ari/gari$**.**

For $M^\bullet$ in ARI^{al}_{bico} (resp. $GARI^{as}_{bico}$), the amplifications $am_0.M^\bullet$ and $am_{\frac{1}{2}}.M^\bullet$ automatically inherit alternality (resp. symmetrality). The real question is: how will amplification impact lu/mu and $ari/gari$? For the uninflected operations lu/mu, the answer is provided by the earlier formulae (296), (297). Not so for $ari/gari$. In fact, to get manageable formulae, we must work, not directly with $am_0.M^\bullet$ and $am_{\frac{1}{2}}.M^\bullet$, but with suitable combinations of the two. This, together with the proposition immediately to follow, motivates our definition of the *second satellisation*, under the simplifying (and provisional) assumption that the length-1 component of $M^\bullet$ vanish[71]:

Definition 4.1 (The second satellisation $M^\bullet \mapsto sa^{**}.M^\bullet$**)** .
For any $A^\bullet$ *in* $\mathrm{ARI}^{\underline{al}}_{bico}$ *and any* $S^\bullet$ *in* $\mathrm{GARI}^{\underline{as}}_{bico}$ *such that*

$$A^{\binom{0}{0}} = A^{\binom{0}{\frac{1}{2}}} = 0\ ,\ S^{\binom{0}{0}} = S^{\binom{0}{\frac{1}{2}}} = 0 \tag{301}$$

we set:

$$\mathrm{sa}^{**}_0\, A^\bullet \ := \ -\mathrm{neg.am}_0\, A^\bullet + \mathrm{neg.am}_{\frac{1}{2}}\, A^\bullet \tag{302}$$

$$\mathrm{sa}^{**}_{\frac{1}{2}}\, A^\bullet \ := \ -\mathrm{neg.am}_0\, A^\bullet \tag{303}$$

$$\mathrm{sa}^{**}_0\, S^\bullet \ := \ \mathrm{mu}\Big(\mathrm{invmu}(\mathrm{neg.am}_0\, S^\bullet), \mathrm{neg.am}_{\frac{1}{2}}\, S^\bullet\Big) \tag{304}$$

$$\mathrm{sa}^{**}_{\frac{1}{2}}\, S^\bullet \ := \ \mathrm{invmu}(\mathrm{neg.am}_0\, S^\bullet) \tag{305}$$

Here *neg* denotes the sign reversal of all indices, and *invmu* the inversion (relative to the mould multiplication *mu*), which for symmetral moulds (– as is the case here –) reduces to a sequence reversal with or without sign change in front, depending on parity:

$$(\mathrm{neg}.\mathcal{M})^{u_1,\ldots,u_r} \ := \ \mathcal{M}^{-u_1,\ldots,-u_r} \tag{306}$$

$$(\mathrm{invmu}.\mathcal{M})^{u_1,\ldots,u_r} \ \equiv \ (-1)^r\, \mathcal{M}^{u_r,\ldots,u_1} \quad \textit{if} \ \ \mathcal{M}^\bullet \ \ \textit{symmetral} \tag{307}$$

Proposition 4.2 (Impact of the second satellisation on *ari/gari***)** .
Let as usual $A^\bullet, B^\bullet$ *stand for elements of* $\mathrm{ARI}^{\underline{al}}_{bico}$ *and* $S^\bullet, T^\bullet$ *for elements of* $\mathrm{GARI}^{\underline{as}}_{bico}$ *. Then*

$$\mathrm{sa}^{**}_0\,\mathrm{ari}(A^\bullet, B^\bullet) = \mathrm{ari}(\mathrm{sa}^{**}_0\, A^\bullet, \mathrm{sa}^{**}_0\, B^\bullet) \tag{308}$$

$$\mathrm{sa}^{**}_0\,\mathrm{preari}(A^\bullet, B^\bullet) = \mathrm{preari}(\mathrm{sa}^{**}_0\, A^\bullet, \mathrm{sa}^{**}_0\, B^\bullet) \tag{309}$$

$$\mathrm{sa}^{**}_0\,\mathrm{gari}(S^\bullet, T^\bullet) = \mathrm{gari}(\mathrm{sa}^{**}_0\, S^\bullet, \mathrm{sa}^{**}_0\, T^\bullet) \tag{310}$$

Moreover, provided that

$$A^{\binom{0}{0}} = A^{\binom{0}{\frac{1}{2}}} = B^{\binom{0}{0}} = B^{\binom{0}{\frac{1}{2}}} = 0\ ,\ S^{\binom{0}{0}} = T^{\binom{0}{\frac{1}{2}}} = S^{\binom{0}{0}} = T^{\binom{0}{\frac{1}{2}}} = 0 \tag{311}$$

[71]It is mainly the relations (312)-(314) that require this simplifying assumption. It will be removed in the next section.

we have the further identities:

$$\mathrm{sa}^{**}_{\frac{1}{2}}\,\mathrm{ari}(A^\bullet,B^\bullet) \;=\; \begin{cases} +\mathrm{lu}(\mathrm{sa}^{**}_{\frac{1}{2}}\,A^\bullet,\mathrm{sa}^{**}_{\frac{1}{2}}\,B^\bullet) \\ +\mathrm{arit}(\mathrm{sa}^{**}_{0}\,B^\bullet)\,\mathrm{sa}^{**}_{\frac{1}{2}}\,A^\bullet \\ -\mathrm{arit}(\mathrm{sa}^{**}_{0}\,A^\bullet)\,\mathrm{sa}^{**}_{\frac{1}{2}}\,B^\bullet \end{cases} \tag{312}$$

$$\mathrm{sa}^{**}_{\frac{1}{2}}\,\mathrm{preari}(A^\bullet,B^\bullet) \;=\; \begin{cases} +\mathrm{mu}(\mathrm{sa}^{**}_{\frac{1}{2}}\,A^\bullet,\mathrm{sa}^{**}_{\frac{1}{2}}\,B^\bullet) \\ +\mathrm{arit}(\mathrm{sa}^{**}_{0}B^\bullet)\,\mathrm{sa}^{**}_{\frac{1}{2}}\,A^\bullet \end{cases} \tag{313}$$

$$\mathrm{sa}^{**}_{\frac{1}{2}}\,\mathrm{gari}(S^\bullet,T^\bullet) \;=\; \mathrm{mu}\Big(\big(\mathrm{garit}(\mathrm{sa}^{**}_{0}\,T^\bullet)\,\mathrm{sa}^{**}_{\frac{1}{2}}\,S^\bullet\big)\,,\,\mathrm{sa}^{**}_{\frac{1}{2}}\,T^\bullet\Big) \tag{314}$$

In other words, under the (essential) assumption that all lenght-1 components vanish, the second satellisation sa^{**} *affects* ari/gari *in exactly the same way as does the first satellisation* sa^{*}.

Despite the formal similarity, the identities of Proposition 4.2 are completely different in nature from those of Proposition 4.1, and much deeper. They also have this uncanny feature of relating the *ari/gari* operations on $sa.M^\bullet$, which bear on the lower indices ϵ_i, to the utterly different *ari/gari* operations on $sa^{**}.M^\bullet$, which bear on the upper indices u_i.

4.4 The mischief potential of $\log 2$.

We are already familiar with the (mild) difficulties attendant on the divergence of $Ze^{\binom{0}{0}} \sim \sum n^{-1}$. They merely introduce a correcting factor $man^\bullet$ in the identity (500) connecting $zag^\bullet$ and $zig^\bullet$.

We are also familiar with the (more serious) difficulties related to the scalar multizetas that belong $\mathbb{C}[[\pi^2]]$. These are responsible for the presence of an irregular first factor $zag^\bullet_I$ in the trifactorisation (240) of $zag^\bullet$. That first factor belongs to $GARI^{al/il}$ but not $GARI^{\underline{al/il}}$, which causes no end of difficulties.

We must now prepare ourselves for the difficulties (of intermediate severity) resulting from $Ze^{\binom{0}{1/2}} = \sum(-1)^{n-1}\,n^{-1} = \log 2$, or in other words, from the presence of non-zero length-1 components $M^{\binom{0}{1/2}}$.(Let us recall that, taking our stand on the normalisation $zag^{\binom{0}{0}} = zig^{\binom{0}{0}} = 0$, we have already, once and for all, ruled out any non-zero components $M^{\binom{0}{0}}$).

Definition 4.2 (The second satellisation $M^\bullet \mapsto sa^{**}.M^\bullet$ (bis)) .
In presence of a nonzero length-1 component $M^{\binom{0}{\frac{1}{2}}}$, *the earlier definition of*

sa^{**} *should be modified to:*

$$\begin{aligned}
\mathrm{sa}_0^{**}\, A^\bullet &:= -\mathrm{neg.am}_0\, A^\bullet + \mathrm{neg.am}_{\frac{1}{2}}\, A^\bullet {\color{red}+ A^{\binom{0}{\frac{1}{2}}}\, I^\bullet} \\
\mathrm{sa}_{\frac{1}{2}}^{**}\, A^\bullet &:= -\mathrm{neg.am}_0\, A^\bullet
\end{aligned} \tag{315}$$

$$\begin{aligned}
\mathrm{sa}_0^{**}\, S^\bullet &:= \mathrm{mu}\Big({\color{red}e^{-S^{\binom{0}{\frac{1}{2}}}\mathfrak{D}}}.\,\mathrm{invmu}(\mathrm{neg.am}_0\, S^\bullet), \mathrm{neg.am}_{\frac{1}{2}}\, S^\bullet, {\color{red}e^{S^{\binom{0}{\frac{1}{2}}}}\, I^\bullet}\Big) \\
\mathrm{sa}_{\frac{1}{2}}^{**}\, S^\bullet &:= \mathrm{invmu}(\mathrm{am}_0\, S^\bullet)
\end{aligned} \tag{316}$$

with $\mathfrak{D}$ *denoting the elementary mould derivation:*

$$(\mathfrak{D}\mathcal{A})^{u_1,\ldots,u_r} \;:=\; (u_1 + \cdots + u_r)\, \mathcal{A}^{u_1,\ldots,u_r} \tag{317}$$

In order to fittingly describe the interaction of sa^{**} with *ari/gari* in the most general situation, we must now introduce two mould operators:

$$\mathrm{ut}(\mathcal{A}^\bullet)\, \mathcal{B}^\bullet \;:=\; -\mathcal{A}^{(0)}\, \mathfrak{D}\, \mathcal{B}^\bullet \tag{318}$$

$$\mathrm{gut}(\mathcal{S}^\bullet)\, \mathcal{B}^\bullet \;:=\; \exp(-\mathcal{S}^{(0)}\, \mathfrak{D})\, \mathcal{B}^\bullet \tag{319}$$

$ut(\mathcal{A}^\bullet)$ is clearly a derivation relative to the *mu*-product, and $gut(\mathcal{S}^\bullet)$ an automorphism, again relative to *mu*.

In view of (315)-(316) and given that $(sa_{\frac{1}{2}}^{**}.M)^{(0)} \equiv M^{\binom{0}{\frac{1}{2}}}$ for $M^\bullet$ in $ARI_{\underline{bico}}^{\underline{al}}$ or $GARI_{\underline{bico}}^{\underline{as}}$, the relevance of the operators $\mathrm{ut}(\mathcal{A}^\bullet)$ and $\mathrm{gut}(\mathcal{S}^\bullet)$ is rather obvious, and we are now in a position to remove the restrictive assumption of Proposition 4.2.

Proposition 4.3 (Impact of the second satellisation on *ari/gari* (bis))
.
For general elements $A^\bullet, B^\bullet$ *in* $\mathrm{ARI}_{\underline{bico}}^{\underline{al}}$ *and* $S^\bullet, T^\bullet$ *in* $\mathrm{GARI}_{\underline{bico}}^{\underline{as}}$*, the earlier identities (308)-(314) have to be supplemented by the following terms in red to account for the presence of non-vanishing length-1 components:*

$$\mathrm{sa}_0^{**}\, \mathrm{ari}(A^\bullet, B^\bullet) = \mathrm{ari}(\mathrm{sa}_0^{**}\, A^\bullet, \mathrm{sa}_0^{**}\, B^\bullet) \tag{320}$$

$$\mathrm{sa}_0^{**}\, \mathrm{preari}(A^\bullet, B^\bullet) = \mathrm{preari}(\mathrm{sa}_0^{**}\, A^\bullet, \mathrm{sa}_0^{**}\, B^\bullet) \tag{321}$$

$$\mathrm{sa}_0^{**}\, \mathrm{gari}(S^\bullet, T^\bullet) = \mathrm{gari}(\mathrm{sa}_0^{**}\, S^\bullet, \mathrm{sa}_0^{**}\, T^\bullet) \tag{322}$$

$$\mathrm{sa}^{**}_{\frac12}\,\mathrm{ari}(A^\bullet,B^\bullet) = \begin{cases} +\mathrm{lu}(\mathrm{sa}^{**}_{\frac12}\,A^\bullet,\mathrm{sa}^{**}_{\frac12}\,B^\bullet) \\ +\mathrm{arit}(\mathrm{sa}^{**}_0\,B^\bullet)\,\mathrm{sa}^{**}_{\frac12}\,A^\bullet + \mathrm{ut}(\mathrm{sa}^{**}_{\frac12}\,B^\bullet)\,\mathrm{sa}^{**}_{\frac12}\,A^\bullet \\ -\mathrm{arit}(\mathrm{sa}^{**}_0\,A^\bullet)\,\mathrm{sa}^{**}_{\frac12}\,B^\bullet - \mathrm{ut}(\mathrm{sa}^{**}_{\frac12}\,A^\bullet)\,\mathrm{sa}^{**}_{\frac12}\,B^\bullet \end{cases} \tag{323}$$

$$\mathrm{sa}^{*}_{\frac12}\,\mathrm{preari}(A^\bullet,B^\bullet) = \begin{cases} +\mathrm{mu}(\mathrm{sa}^{**}_{\frac12}\,A^\bullet,\mathrm{sa}^{**}_{\frac12}\,B^\bullet) \\ +\mathrm{arit}(\mathrm{sa}^{**}_0 B^\bullet)\,\mathrm{sa}^{**}_{\frac12}\,A^\bullet + \mathrm{ut}(\mathrm{sa}^{**}_{\frac12} B^\bullet)\,\mathrm{sa}^{**}_{\frac12}\,A^\bullet \end{cases} \tag{324}$$

$$\mathrm{sa}^{**}_{\frac12}\,\mathrm{gari}(S^\bullet,T^\bullet) = \mathrm{mu}\Big(\big(\mathrm{garit}(\mathrm{sa}^{**}_0\,T^\bullet).\mathrm{gut}(\mathrm{sa}^{**}_{\frac12}\,T^\bullet).\mathrm{sa}^{**}_{\frac12}\,S^\bullet\big)\,,\,\mathrm{sa}^{**}_{\frac12}\,T^\bullet\Big) \tag{325}$$

$$= \mathrm{mu}\Big(\big(\mathrm{gut}.(\mathrm{sa}^{**}_{\frac12}\,T^\bullet).\mathrm{garit}(\mathrm{sa}^{**}_0\,T^\bullet).\mathrm{sa}^{**}_{\frac12}\,S^\bullet\big)\,,\,\mathrm{sa}^{**}_{\frac12}\,T^\bullet\Big) \tag{326}$$

Proposition 4.4 (Impact of the second satellisation on *ari/gari* **(ter))**
.
The relations

$$\mathrm{lu}^*(\mathcal{A}^\bullet,\mathcal{B}^\bullet) := \mathrm{lu}(\mathcal{A}^\bullet,\mathcal{B}^\bullet) + \mathcal{A}^0\,\mathfrak{D}\,\mathcal{B}^\bullet - \mathcal{B}^0\,\mathfrak{D}\,\mathcal{A}^\bullet \tag{327}$$

$$= \mathrm{lu}(\mathcal{A}^\bullet,\mathcal{B}^\bullet) + \mathrm{ut}(\mathcal{B}^\bullet)\,\mathcal{A}^\bullet - \mathrm{ut}(\mathcal{A}^\bullet)\,\mathcal{B}^\bullet \tag{328}$$

$$\mathrm{mu}^*(\mathcal{S}^\bullet,\mathcal{T}^\bullet) := \mathrm{mu}(\exp(-\mathcal{T}^0\,\mathfrak{D})\,\mathcal{S}^\bullet,\mathcal{T}^\bullet) \tag{329}$$

$$= \mathrm{mu}(\mathrm{gut}(\mathcal{T}^\bullet)\,\mathcal{S}^\bullet,\mathcal{T}^\bullet) \tag{330}$$

define a modified Lie bracket lu^* *and a modified associative product* mu^*. *With them, the identities (323)-(326) simplify:*

$$\mathrm{sa}^{**}_{\frac12}\,\mathrm{ari}(A^\bullet,B^\bullet) = \begin{cases} +\mathrm{lu}^*(\mathrm{sa}^{**}_{\frac12}\,A^\bullet,\mathrm{sa}^{**}_{\frac12}\,B^\bullet) \\ +\mathrm{arit}(\mathrm{sa}^{**}_0\,B^\bullet)\,\mathrm{sa}^{**}_{\frac12}\,A^\bullet \\ -\mathrm{arit}(\mathrm{sa}^{**}_0\,A^\bullet)\,\mathrm{sa}^{**}_{\frac12}\,B^\bullet \end{cases} \tag{331}$$

$$\mathrm{sa}^{**}_{\frac12}\,\mathrm{preari}(A^\bullet,B^\bullet) = \begin{cases} +\mathrm{mu}^*(\mathrm{sa}^{**}_{\frac12}\,A^\bullet,\mathrm{sa}^{***}_{\frac12}\,B^\bullet) \\ +\mathrm{arit}(\mathrm{sa}^{**}_0 B^\bullet)\,\mathrm{sa}^{***}_{\frac12}\,A^\bullet \end{cases} \tag{332}$$

$$\mathrm{sa}^{**}_{\frac12}\,\mathrm{gari}(S^\bullet,T^\bullet) = \mathrm{mu}^*\big(\mathrm{garit}(\mathrm{sa}^{**}_0\,T^\bullet)\,\mathrm{sa}^{**}_{\frac12}\,S^\bullet\,,\,\mathrm{sa}^{**}_{\frac12}\,T^\bullet\big) \tag{333}$$

4.5 The double symmetry and the even-to-odd-degree extrapolation.

So far, we have reviewed the properties of sa, sa^*, sa^{**} as defined on ARI^{al}_{bico} and $GARI^{as}_{bico}$. Let us now move on to $ARI^{al/il}_{bico}$ and $GARI^{as/is}_{bico}$. The introduction of a second symmetry has momentous consequences, the first of which is the possibility of deducing all *odd-degree* components of a bimould $M^\bullet$ from its *even-degree* components.

Even-to-odd extrapolation in $ARI^{\underline{al}/\underline{il}}_{bico}$.

Let us work in the algebra $ARI^{\underline{al}/\underline{il}}_{bico}$ for simplicity[72] and consider there some homogeneous element $A^\bullet$ of total weight s, with its various components $A^\bullet_{|r}$ of length r ($1 \leqslant r \leqslant s$) and total degree $d = s - r$. For the non-vanishing component $A^\bullet_{|r_0}$ of lowest length, the symmetry $(\underline{al}/\underline{il})$ actually implies $(\underline{al}/\underline{al})$, i.e. bialternality. That component is therefore[73] necessarily of *even* degree d_0. Let us now search for an explicit even-to-odd extrapolation formula:

$$(0,..,0,A^\bullet_{|r_0},A^\bullet_{|r_0+2},..,A^\bullet_{|r_0+2n}..)\mapsto(0,..,0,A^\bullet_{|r_0+1},A^\bullet_{|r_0+3},..,A^\bullet_{|r_0+2n+1},..) \quad (334)$$

based on the five-step induction already mentioned in §3.5:

Step 1: Calculate $A^\bullet_{\|r_0+2\,n} := \sum_{r\leqslant r_0+2\,n} A_r \in ARI^{\underline{al}/\underline{il}}$

Step 2: Calculate $^*A^\bullet_{r_0+2\,n} := \mathrm{adari}(\mathrm{ripal}^\bullet).A_{\|r_0+2\,n} \in ARI^{\underline{al}/\underline{al}}$

Step 3: Define $^*A^\bullet_{\|r_0+2\,n}$ as $^*A^\bullet_{r_0+2\,n}$ truncated at length r_0+2n+1 (included!)

Step 4: Calculate $^{**}A^\bullet_{r_0+2\,n} := \mathrm{adari}(\mathrm{pal}^\bullet).^*A_{\|r_0+2\,n} \in ARI^{\underline{al}/\underline{il}}$

Step 5: Define $A^\bullet_{\|r_0+2\,n+1}$ as the component of length r_0+2n+1 of $^{**}A^\bullet_{r_0+2\,n}$

If we now denote by $trunc_r$ the linear operator which acts on moulds by retaining only their components of length $\leqslant r$ and if further we set

$$\theta_r \quad := \quad \mathrm{trunc_{r+1}adari}(pal^\bullet)\,.\,\mathrm{trunc_r}\,.\,\mathrm{adari}(ripal^\bullet) \quad (335)$$

the above induction can be summarised as

$$A^\bullet_{r_0+2n+1} = \begin{cases} \theta_{r_0+2n}(A^\bullet_{r_0+2n}+\theta_{r_0+2n-2}(A^\bullet_{r_0+2n-2}+\cdots\cdots \\ \cdots\cdots+\theta_{r_0+2}(A^\bullet_{r_0+2}+\theta_{r_0}A^\bullet_{r_0}))...)) \end{cases} \quad (336)$$

In theory, (336) could qualify as an even-to-odd extrapolation formula of type (334). In practice, though, it is no good: $pal^\bullet$ and its *gari*-inverse $ripal^\bullet$ are very complex bimoulds; the adjoint action *adari* is itself a highly complex operation; and as $2n$ grows, the number of terms on the right-hand side of (336) becomes, *prior to simplifications*, fantastically large. The miracle, however, is that sweeping simplification *do occur*, leading in the end to a formula that is both practical and beautiful.

But before enuntiating it we need to get a few definitions out of way.

First, we require the constants ξ_n:

$$\xi_n := \begin{cases} \frac{2\,(1-2^{n+1})}{n+1}\ \mathrm{Ber}_{n+1} & \textit{if}\ \ n\ \ \textit{odd} \quad (\mathrm{Ber}_\bullet = \textit{Bernoulli number}) \\ \quad 0 & \textit{if}\ \ n\ \ \textit{even} \end{cases} \quad (337)$$

---

[72]analogous results hold for $GARI^{\underline{as}/\underline{is}}_{bico}$

[73]See [...] §....

Thus $\quad \xi_1=-\frac{1}{2}\ ,\ \xi_3=\frac{1}{4}\ ,\ \xi_5=-\frac{1}{2}\ ,\ \xi_7=\frac{17}{8}\ ,\ \xi_9=-\frac{31}{2}\ ,\ \xi_{11}=\frac{691}{4}\ ,\ \xi_{13}=-\frac{5461}{2}\ ,\ \xi_{15}=\frac{929569}{16}$

Next, we require two elementary symmetral bimoulds:

$$S_x^{\varnothing} := 1 \quad , \quad S_x^{\binom{u_1\ ,\dots,\ u_r}{v_1\ ,\dots,\ v_r}} :=(-x)^r\, P(u_1)P(u_1{+}u_2)\dots P(u_1{+}\dots u_r) \tag{338}$$

$$T_x^{\varnothing} := 1 \quad , \quad T_x^{\binom{u_1\ ,\dots,\ u_r}{v_1\ ,\dots,\ v_r}} := \quad x^r\, P(u_r)\, P(u_{r-1}{+}u_r)\dots P(u_1{+}\dots u_r) \tag{339}$$

Lastly, we require operators $\mathfrak{H}_x$ constructed from these ingredients:

$$\mathfrak{H}_x \quad : \quad M^\bullet \mapsto \widetilde{M}^\bullet \tag{340}$$

$$\widetilde{M}^\bullet := (id - x\,\mathfrak{P}_L + x\,\mathfrak{P}_R).\big(S_x^{\bullet\,-1} \times (\mathrm{garit}(S_x^\bullet).M^\bullet) \times S_x^\bullet\big) \tag{341}$$

$$\textit{with} \qquad \begin{cases} (\mathfrak{P}_R\, M)^{\binom{u_1\ ,\dots,\ u_r}{\epsilon_1\ ,\dots,\ \epsilon_r}} & := M^{\binom{u_1\ \ ,\dots,\ \ u_{r-1}}{\epsilon_1-\epsilon_r\ ,\dots,\ \epsilon_{r-1}-\epsilon_r}}\, P(u_1+\dots+u_r) \\ (\mathfrak{P}_L\, M)^{\binom{u_1\ ,\dots,\ u_r}{\epsilon_1\ ,\dots,\ \epsilon_r}} & := M^{\binom{u_2\ \ ,\dots,\ \ u_r}{\epsilon_2-\epsilon_1\ ,\dots,\ \epsilon_{r-1}-\epsilon_1}}\, P(u_1+\dots+u_r) \end{cases}$$

We may note in passing the operators $\mathfrak{H}_x$ form a group:

$$\mathfrak{H}_0 = id \qquad \textit{and} \qquad \mathfrak{H}_x\, \mathfrak{H}_y \equiv \mathfrak{H}_{x+y} \tag{342}$$

The proof relies mainly on identities such as

$$(\mathfrak{P}_R - \mathfrak{P}_L)\, M^\bullet \;=\; \mathrm{arit}(M^\bullet)\, P^\bullet \;\; \forall M^\bullet \tag{343}$$

$$S_x^\bullet \;=\; \mathrm{expari}(-x\, Pa^\bullet) \tag{344}$$

with the elementary mould $Pa^\bullet$:

$$Pa^{\binom{u_1\ ,\dots,\ u_r}{v_1\ ,\dots,\ v_r}} = \begin{cases} P(u_1) & \textit{if } r=1 \\ 0 & \textit{otherwise} \end{cases} \tag{345}$$

Proposition 4.5 (Even-to-odd extrapolation on $ARI^{al/il}_{bico}$) .
Let $A^\bullet$ be a homogeneous element of $ARI^{al/il}_{bico}$ of weight s and let $A^\bullet_{even}$ (resp. $A^\bullet_{odd}$) the sum of its components of even (resp. odd) degree. These components have of course lengths of opposite parities, and the extrapolation formula reads:

$$A^\bullet_{odd} \;=\; \big(\mathfrak{H}_x.A^\bullet_{even}\big)\|_{x^n=\xi_n} \tag{346}$$

In other words, we expand $\big(\mathfrak{H}_x.A^\bullet_{even}\big)$ as a formal power of x and then replace each x^n by ξ_n. Given that $\xi_{2n} \equiv 0$, this leaves in $A^\bullet_{odd}$ only components with lengths of the right parity. Moreover, and though this is non-obvious, all components of length $r > s$ automatically vanish, as indeed they should.

Even-to-odd extrapolation in the first upper satellite.

The change $\mathfrak{H}_x : M^\bullet \to \widetilde{M}^\bullet$ admits an *internal* restriction to the first upper satellite.[74] Indeed, one easily checks that:

$$\mathrm{sa}_0^* \widetilde{M}^\bullet \;:=\; (id + x\,\mathfrak{P}_R - x\,\mathfrak{P}_L).\big(S_x^{\bullet -1} \times (\mathrm{garit}(S_x^\bullet).\mathrm{sa}_0^* M^\bullet) \times S_x^\bullet\big) \tag{347}$$

$$\mathrm{sa}_{\frac12}^* \widetilde{M}^\bullet \;:=\; \begin{cases} +\big(S_x^{\bullet -1} \times (\mathrm{garit}(S_x^\bullet).\mathrm{sa}_{\frac12}^* M^\bullet) \times S_x^\bullet\big) \\ +x\,(\mathfrak{P}_R - \mathfrak{P}_L).\big(S_x^{\bullet -1} \times (\mathrm{garit}(S_x^\bullet).\mathrm{sa}_0^* M^\bullet) \times S_x^\bullet\big) \end{cases} \tag{348}$$

$$(\mathcal{M}_0^\bullet, \mathcal{M}_{\frac12}^\bullet) \;:=\; (\mathrm{sa}^*{}_0.M^\bullet, \mathrm{sa}^*{}_{\frac12}.M^\bullet) \tag{349}$$

$$(\mathfrak{P}.M)^{u_1,\dots,u_r} \;:=\; (u_1 + \ldots + u_r)^{-1}\, M^{u_1,\dots,u_r} \tag{350}$$

$$(\mathfrak{D}.M)^{u_1,\dots,u_r} \;:=\; (u_1 + \ldots + u_r)\; M^{u_1,\dots,u_r} \tag{351}$$

and denoting for uniformity the bimoulds $S_x^\bullet, T_x^\bullet$ as simple moulds $\mathcal{S}_x^\bullet, \mathcal{T}_x^\bullet$ (which is legitimate, since the former depend only on their upper indices), the identitities (340)-(341) can be brought into more explicit shape:

$$\widetilde{\mathcal{M}}_0^\bullet \;=\; \begin{cases} +\mathcal{T}_x^\bullet \times \mathcal{M}_0^\bullet \circ (\mathcal{S}_x^\bullet \times \mathcal{I}^\bullet \times \mathcal{T}_x^\bullet) \times \mathcal{S}_x^\bullet \\ -x\,\mathfrak{P}.\Big(\mathcal{I}^\bullet \times \mathcal{T}_x^\bullet \times \mathcal{M}_0^\bullet \circ (\mathcal{S}_x^\bullet \times \mathcal{I}^\bullet \times \mathcal{T}_x^\bullet) \times \mathcal{S}_x^\bullet\Big) \\ +x\,\mathfrak{P}.\Big(\mathcal{T}_x^\bullet \times \mathcal{M}_0^\bullet \circ (\mathcal{S}_x^\bullet \times \mathcal{I}^\bullet \times \mathcal{T}_x^\bullet) \times \mathcal{S}_x^\bullet \times \mathcal{I}^\bullet\Big) \end{cases} \tag{352}$$

$$\widetilde{\mathcal{M}}_{\frac12}^\bullet \;=\; \begin{cases} +\mathcal{T}_x^\bullet \times \mathcal{M}_{\frac12}^\bullet \circ (\mathcal{S}_x^\bullet \times \mathcal{I}^\bullet \times \mathcal{T}_x^\bullet) \times \mathcal{S}_x^\bullet \\ -x\,\mathfrak{P}.\Big(\mathcal{I}^\bullet \times \mathcal{T}_x^\bullet \times \mathcal{M}_0^\bullet \circ (\mathcal{S}_x^\bullet \times \mathcal{I}^\bullet \times \mathcal{T}_x^\bullet) \times \mathcal{S}_x^\bullet\Big) \\ +x\,\mathfrak{P}.\Big(\mathcal{T}_x^\bullet \times \mathcal{M}_0^\bullet \circ (\mathcal{S}_x^\bullet \times \mathcal{I}^\bullet \times \mathcal{T}_x^\bullet) \times \mathcal{S}_x^\bullet \times \mathcal{I}^\bullet\Big) \end{cases} \tag{353}$$

where $\mathcal{I}^\bullet$ denotes the identity mould.[75]

Proposition 4.6 (Even-to-odd extrapolation in the first upper satellite.) *Let $A^\bullet$ be a homogeneous element of $ARI^{\underline{al}/\underline{il}}_{bico}$ of weight s. Let $\mathcal{A}_0^\bullet := \mathrm{sa}_0^*.A^\bullet$ and $\mathcal{A}_{\frac12}^\bullet := \mathrm{sa}_{\frac12}^*.A^\bullet$ be its all-white and all-black parts. Then, to perform the even-to-odd extrapolation, it suffices*
(i) to substitute the pair $(\mathcal{A}_{0,\mathrm{even}}^\bullet, \mathcal{A}_{\frac12,\mathrm{even}}^\bullet)$ for $(\mathcal{M}_0^\bullet, \mathcal{M}_{\frac12}^\bullet)$ in (352)-(353),
(ii) to set $x^n := \xi_n$ in the corresponding pair $(\widetilde{\mathcal{M}}_0^\bullet, \widetilde{\mathcal{M}}_{\frac12}^\bullet)$.

[74] The fact is non trivial: it wouldn't be true if we had defined that satellisation based on *swap*.$M^\bullet$ rather than $M^\bullet$.

[75] $\mathcal{T}^{u_1} \equiv 1$ and $\mathcal{T}^{u1,\dots,u_r} \equiv 0$ if $r \neq 1$.

Remark 1: Using the identities

$$\mathcal{S}^\bullet_x \times \mathcal{T}^\bullet_x = 1^\bullet \;,\;\; \mathfrak{D}.\mathfrak{P} = id \;,\; \mathfrak{D}.\mathcal{S}^\bullet_x = -x\,\mathcal{S}^\bullet_x \times \mathcal{I}^\bullet \;,\; \mathfrak{D}.\mathcal{T}^\bullet_x = x\,\mathcal{I}^\bullet \times \mathcal{T}^\bullet_x \tag{354}$$

together with the fact that $\mathfrak{D}$ is a *derivation* relative to mould multiplication, we can recast the correspondence $(\mathcal{A}^\bullet_0, \mathcal{A}^\bullet_{\frac{1}{2}}) \mapsto (\widetilde{\mathcal{A}}^\bullet_0, \widetilde{\mathcal{A}}^\bullet_{\frac{1}{2}})$ into an almost involutive form:

$$\mathcal{S}^\bullet_x \times (\mathfrak{D}.\widetilde{\mathcal{M}}^\bullet_0) \times \mathcal{T}^\bullet_x = (\mathcal{D}.\mathcal{M}^\bullet_0) \circ (\mathcal{S}^\bullet_x \times \mathcal{I}^\bullet \times \mathcal{T}^\bullet_x) \tag{355}$$

$$\mathcal{S}^\bullet_x \times (\mathfrak{D}.\widetilde{\mathcal{M}}^\bullet_{\frac{1}{2}}) \times \mathcal{T}^\bullet_x = \begin{cases} +(\mathfrak{D}.\mathcal{M}^\bullet_{\frac{1}{2}}) \circ (\mathcal{S}^\bullet_x \times \mathcal{I}^\bullet \times \mathcal{T}^\bullet_x) \\ +(\mathcal{S}^\bullet_x \times \mathcal{I}^\bullet \times \mathcal{T}^\bullet_x) \times \big(\mathcal{M}^\bullet_{\frac{1}{2}} \circ (\mathcal{S}^\bullet_x \times \mathcal{I}^\bullet \times \mathcal{T}^\bullet_x)\big) \\ -\big(\mathcal{S}^\bullet_x \times \mathcal{I}^\bullet \times \mathcal{T}^\bullet_x\big) \times \big(\mathcal{M}^\bullet_0 \circ (\mathcal{S}^\bullet_x \times \mathcal{I}^\bullet \times \mathcal{T}^\bullet_x)\big) \\ -\big(\mathcal{M}^\bullet_{\frac{1}{2}} \circ (\mathcal{S}^\bullet_x \times \mathcal{I}^\bullet \times \mathcal{T}^\bullet_x)\big) \times (\mathcal{S}^\bullet_x \times \mathcal{I}^\bullet \times \mathcal{T}^\bullet_x) \\ +\big(\mathcal{M}^\bullet_0 \circ (\mathcal{S}^\bullet_x \times \mathcal{I}^\bullet \times \mathcal{T}^\bullet_x)\big) \times (\mathcal{S}^\bullet_x \times \mathcal{I}^\bullet \times \mathcal{T}^\bullet_x) \end{cases} \tag{356}$$

If we then set $\mathcal{M}^\bullet_{\frac{1}{2}:0} := \mathcal{M}^\bullet_{\frac{1}{2}} - \mathcal{M}^\bullet_0$, $\widetilde{\mathcal{M}}^\bullet_{\frac{1}{2}:0} := \widetilde{\mathcal{M}}^\bullet_{\frac{1}{2}} - \widetilde{\mathcal{M}}^\bullet_0$, the above system further simplifies

$$\mathcal{S}^\bullet_x \times (\mathfrak{D}.\widetilde{\mathcal{M}}^\bullet_0) \times \mathcal{T}^\bullet_x = (\mathfrak{D}.\mathcal{M}^\bullet_0) \circ (\mathcal{S}^\bullet_x \times \mathcal{I}^\bullet \times \mathcal{T}^\bullet_x) \tag{357}$$

$$\mathcal{S}^\bullet_x \times (\mathfrak{D}.\widetilde{\mathcal{M}}^\bullet_{\frac{1}{2}:0}) \times \mathcal{T}^\bullet_x = \begin{cases} +(\mathfrak{D}.\mathcal{M}^\bullet_{\frac{1}{2}:0}) \circ (\mathcal{S}^\bullet_x \times \mathcal{I}^\bullet \times \mathcal{T}^\bullet_x) \\ +(\mathcal{S}^\bullet_x \times \mathcal{I}^\bullet \times \mathcal{T}^\bullet_x) \times \big(\mathcal{M}^\bullet_{\frac{1}{2}:0} \circ (\mathcal{S}^\bullet_x \times \mathcal{I}^\bullet \times \mathcal{T}^\bullet_x)\big) \\ -\big(\mathcal{M}^\bullet_{\frac{1}{2}:0} \circ (\mathcal{S}^\bullet_x \times \mathcal{I}^\bullet \times \mathcal{T}^\bullet_x)\big) \times (\mathcal{S}^\bullet_x \times \mathcal{I}^\bullet \times \mathcal{T}^\bullet_x) \end{cases} \tag{358}$$

Remark 2: organic moulds. The group identity $\mathfrak{H}_x\ \mathfrak{H}_y \equiv \mathfrak{H}_{x+y}$ is intimately connected with the strong stability – mainly under mould composition, but not only – of the so-called *organic* mould family generated by $S^\bullet_x$ and $T^\bullet_x$:

$$\begin{array}{rcll} \mathcal{S}^\bullet_x \times \mathcal{T}^\bullet_x & \equiv & 1^\bullet & \\ \mathcal{SIT}^\bullet_x \circ \mathcal{SIT}^\bullet_y & \equiv & \mathcal{SIT}^\bullet_{x+y} & \textit{with} \quad \mathcal{SIT}^\bullet_x := \mathcal{S}^\bullet_x \times \mathcal{I}^\bullet \times \mathcal{T}^\bullet_x \\ \mathcal{SIT}^\bullet_{x,x'} \circ \mathcal{SIT}^\bullet_{y,y'} & \equiv & \mathcal{SIT}^\bullet_{xy'+y,x'y'} & \textit{with} \quad \mathcal{SIT}^\bullet_{x,x'} := x'\,\mathcal{S}^\bullet_x \times \mathcal{I}^\bullet \times \mathcal{T}^\bullet_x \end{array}$$

The organic moulds occur in various other contexts, notably in alien calculus: they crucially enter the construction of the so-called *organic* derivations Δ^{org}_ω which, unlike the *standard* derivations Δ_ω, are *well-behaved*, that is to say, possess optimal growth properties in ω as $|\omega| \to \infty$.

4.6 Recovering the general bicolours from the all-blacks: the operators *discram* and *viscram*.

The formulae we are going to enuntiate now may be thought of as *Green-like*, in the sense that they express the 'whole picure' (here: the whole of $ARI^{al/il}_{bico}$) from 'boundary data' (here: any of the three satellite systems).

We shall start from the first upper satellite sa^* and show how to recover everything from it (next proposition). Then, in the next two sections, we shall show how to go *directly* from the second upper satellite sa^{**} to the first, and back. Since the lower satellite sa was, from the very start, in biconstructive correspondence with sa^{**}, that will automatically provide *indirect* paths from sa and sa^{**} to $ARI^{al/il}_{bico}$. But to arrive at a truly satisfying picture, we shall also sketch *direct* paths from sa and sa^{**} to $ARI^{al/il}_{bico}$

Proposition 4.7 (Recovering $ARI^{al/il}_{bico}$ from $sa^*.ARI^{al/il}_{bico}$) .
Let $A^\bullet$ be an element of $ARI^{al/il}_{bico}$ with $(\mathcal{A}^\bullet_0, \mathcal{A}^\bullet_{\frac{1}{2}}) = (sa^*_0.A^\bullet, sa^*_{\frac{1}{2}}.A^\bullet)$ as usual. Then the whole of $A^\bullet$ is constructively determined by its all-black part $\mathcal{A}^\bullet_{\frac{1}{2}}$, and even by the sole even-degreed components of $\mathcal{A}^\bullet_{\frac{1}{2}}$. Roughly speaking, the all-white part $\mathcal{A}^\bullet_0$ can be recovered from $\mathcal{A}^\bullet_{\frac{1}{2}}$ via the operator viscram*, and the terms of mixed colour via the operator* discram*. The exact procedure, rather involved but entirely constructive and formula-based, is set forth in detail below.*

Explicit procedure: To ease the exposition, we shall slightly depart from the previous notations. We now decompose $A^\bullet$ and its image ${}^*A^\bullet$ under $adari(pal^\bullet)$ into all-white parts $W^\bullet, {}^*W^\bullet$, all-black parts $B^\bullet, {}^*B^\bullet$, and (strictly) mixed-colour parts $M^\bullet, {}^*M^\bullet$.

$$A^\bullet \quad = \quad W^\bullet + M^\bullet + B^\bullet \quad \in ARI^{al/il}_{bico} \tag{359}$$

$$ {}^*A^\bullet \quad = \quad {}^*W^\bullet + {}^*M^\bullet + {}^*B^\bullet \quad \in ARI^{al/al}_{non\text{-}entire} \tag{360}$$

For each mould, the length-r component is marked by a lower index r. We can assume $A^\bullet$ to be of weight s. The moulds of the upper series (476) have at most s non-vanishing components (polynomial in $\boldsymbol{u}$) while the moulds of the lower series (476) usually have infinitely many components (rational in $\boldsymbol{u}$ rather than polynomial).

Let $A^\bullet_{r_0}$ be the lowest component of $A^\bullet$. It coincides with the lowest component of ${}^*A^\bullet_{r_0}$ of ${}^*A^\bullet$, has even degree d_0, and is automatically bialternal.[76]

The aim is to construct the whole of $A^\bullet$ from the data $B^\bullet_{r_0}, B^\bullet_{r_0+2}, B^\bullet_{r_0+4}\dots$

[76]That lowest length r_0 has the same parity as the weight s.

Let us recall/introduce the operators *trunc*$_r$ and *viscram**:[77]

$$\begin{aligned}
\mathrm{trunc_r}\, S^\bullet \quad &:= \quad S_0^\bullet + S_1^\bullet + S_2^\bullet + \cdots + S_r^\bullet & (361)\\
\mathrm{viscram}^* S_r^\bullet \quad &:= \quad (2^{-d}-1)^{-1}\, \mathrm{viscram}\, S_r^\bullet \quad \mathit{if} \quad \deg(S_r^\bullet) = d & (362)
\end{aligned}$$

Starting the induction: from $B^\bullet_{r_0}$ to $A^\bullet_{r_0}$ and $A^\bullet_{r_0+1}$

These three steps enlarge the even-degreed $B^\bullet_{r_0}$ to the even-degreed $A^\bullet_{r_0}$:

$$\begin{aligned}
B^\bullet_{r_0} \quad &\overset{\mathrm{viscram}^*}{\longrightarrow} \quad W^\bullet_{r_0} & (363)\\
B^\bullet_{r_0} \quad &\overset{\mathrm{discram}}{\longrightarrow} \quad M^\bullet_{r_0} + B^\bullet_{r_0} & (364)\\
B^\bullet_{r_0} \quad &\longrightarrow \quad A^\bullet_{r_0} := W^\bullet_{r_0} + M^\bullet_{r_0} + B^\bullet_{r_0} & (365)
\end{aligned}$$

This one step takes us from the even-degreed $A^\bullet_{r_0}$ to the odd-degreed $A^\bullet_{r_0+1}$:

$$ {}^*\!A^\bullet_{r_0} \overset{\mathrm{trunc_{r_0+1}}\, \mathrm{adari(pal^\bullet)}}{\longrightarrow} A^\bullet_{r_0+1} \qquad \big(A^\bullet_{r_0} = {}^*\!A^\bullet_{r_0} \;\; \mathit{but} \;\; A^\bullet_{r_0+1} \neq {}^*\!A^\bullet_{r_0+1}\big) \quad (366)$$

Continuing the induction: from $B^\bullet_{2n+r_0}$ to $A^\bullet_{2n+r_0}$ and $A^\bullet_{2n+r_0+1}$

The following step takes us from $\mathit{trunc}_{2n+r_0-1}\, A^\bullet$ (already known) to ${}^*\!B^\bullet_{2n+r_0}$ (not yet known). It also produces parasitical terms ${}^{**}W^\bullet_{2n+r_0}$ and ${}^{**}M^\bullet_{2n+r_0}$ which bear no relation to ${}^*W^\bullet_{2n+r_0}$ and ${}^*M^\bullet_{2n+r_0}$.

$$\begin{aligned}
&A^\bullet_{r_0} + \cdots + A^\bullet_{2n+r_0-1} + B^\bullet_{2n+r_0} \overset{\mathrm{trunc_{2n+r_0}}\, \mathrm{adari(ripal^\bullet)}}{\longrightarrow} & (367)\\
&{}^*\!A^\bullet_{r_0} + \cdots + {}^*\!A^\bullet_{2n+r_0-1} + {}^{**}W^\bullet_{2n+r_0} + {}^{**}M^\bullet_{2n+r_0} + {}^*B^\bullet_{2n+r_0} & (368)
\end{aligned}$$

The genuine ${}^*W^\bullet_{2n+r_0}$ and ${}^*M^\bullet_{2n+r_0}$ are produced by the next steps:

$$\begin{aligned}
{}^*B^\bullet_{2n+r_0} \quad &\overset{\mathrm{viscram}^*}{\longrightarrow} \quad {}^*W^\bullet_{2n+r_0} & (369)\\
{}^*B^\bullet_{2n+r_0} \quad &\overset{\mathrm{discram}}{\longrightarrow} \quad {}^*M^\bullet_{2n+r_0} + {}^*B^\bullet_{2n+r_0} & (370)\\
{}^*B^\bullet_{2n+r_0} \quad &\longrightarrow \quad {}^*\!A^\bullet_{2n+r_0} := {}^*W^\bullet_{2n+r_0} + {}^*M^\bullet_{2n+r_0} + {}^*B^\bullet_{2n+r_0} & (371)
\end{aligned}$$

We are now in full possession of $\mathit{trunc}_{2n+r_0}\, {}^*\!A^\bullet$ and can proceed in one step to $\mathit{trunc}_{2n+r_0+1}\, A^\bullet$:

$$ {}^*\!A^\bullet_{r_0} + \cdots + {}^*\!A^\bullet_{2n+r_0} \overset{\mathrm{trunc_{2n+r_0+1}}\, \mathrm{adari(pal^\bullet)}}{\longrightarrow} A^\bullet_{r_0} + \cdots + A^\bullet_{2n+r_0+1} \quad (372)$$

This completes the induction □

[77] *viscram** is a normalised variant of *viscram*. The normalising factor $(2^{-d}-1)^{-1}$ stems from the constraints of colour consistency. See (476).

4.7 The double symmetry's reflection in the extremal algebra.

Introduction. The extremal and penextremal algebras.

The extremal algebra $ARI^{al/il}_{bico.ext}$ consists of bimoulds of degree $d = 0$ and therefore $r = s$. Since all alternility relations commingle components of various lengths and degrees, there seems to be no way of expressing these relations *within* $ARI^{al/il}_{bico,ext}$, at least directly so. If however we consider the slightly larger 'penextremal' algebra $ARI^{al/il}_{bico.penext}$, consisting of all bimoulds of degree 0 or 1, we can at least express *weak alternility* (see below) there, since weak alternility involves only two consecutive component lengths, e.g. $r = s$, $r = s - 1$. Improbable though it may sound, this in fact implies *full* alternility. Moreover, in the constraints so obtained, we shall find that the components of length 1 can be easily eliminated. This shall leave us with a *complete* system of constraints, fully *internal* to the extremal algebra.

The dimorphy constraints in the extremal algebra.

Definition 4.3 (Weak symmetries) .
A bimould $A^\bullet$ is said to be weakly alternal *if it verifies only the alternality relations $\sum_{\boldsymbol{w}\in \mathrm{sha}(\boldsymbol{w}',\boldsymbol{w}'')} A^{\boldsymbol{w}} \equiv 0$ with $\boldsymbol{w}'$ of length 1 and $\boldsymbol{w}''$ of any length. The same applies for* weakly alternil.

Lemma 1: *In a double symmetry, either symmetry may be weakened, but not both simultaneously*

$$\begin{array}{ccccccc}
\{\mathrm{al/al}\} & \Leftrightarrow & \{\mathrm{al^{weak}/al}\} & \Leftrightarrow & \{\mathrm{al/al^{weak}}\} & \not\Leftrightarrow & \{\mathrm{al^{weak}/al^{weak}}\} \\
\{\mathrm{al/il}\} & \Leftrightarrow & \{\mathrm{al^{weak}/il}\} & \Leftrightarrow & \{\mathrm{al/il^{weak}}\} & \not\Leftrightarrow & \{\mathrm{al^{weak}/il^{weak}}\}
\end{array}$$

Lemma 2: *A bimould $A^\bullet$ of weight* s *in* $\mathrm{ARI}^{\mathrm{al/il}}_{\mathrm{bico}}$ *is enterily determined by its restriction to the extremal algebra* $\mathrm{ARI}^{\mathrm{al/il}}_{\mathrm{bico.ext}}$ *, that is to say by its values $A^{\binom{0\ ,\ldots,\ 0}{\epsilon_1\ ,\ldots,\ \epsilon_s}}$ for all ϵ_i in $\{0, \frac{1}{2}\}$.*

Let us now express the dimorphy constraints first within the penextremal, then the extremal algebra. Any element $A^\bullet \in ARI^{al}_{bico.penext}$ may be expanded in the form:

$$A^\bullet = \sum b^{\epsilon_1,..,\epsilon_s}\ \vec{\mathrm{lu}}(\lambda^\bullet_{0,\epsilon_1}, \lambda^\bullet_{0,\epsilon_2}, \ldots, \lambda^\bullet_{0,\epsilon_s}) \qquad \textit{if}\ \ r = s \tag{373}$$

$$A^\bullet = \sum c^{\epsilon_1,..,\epsilon_{s-1}}\ \vec{\mathrm{lu}}(\lambda^\bullet_{1,\epsilon_1}, \lambda^\bullet_{0,\epsilon_2}, \ldots, \lambda^\bullet_{0,\epsilon_{s-1}}) \quad \textit{if}\ \ r = s-1 \tag{374}$$

$$\text{with} \qquad \lambda^{\binom{u_1}{\epsilon_1}}_{d_0,\epsilon_0} := \begin{cases} u_1^{d_0} & \textit{if}\ \ \epsilon_0 = \epsilon_1 \\ 0 & \textit{otherwise} \end{cases} \tag{375}$$

We must of course take *all* the multibrackets $\vec{lu}(\lambda^\bullet_{1,\epsilon_1}, ..., \lambda^\bullet_{0,\epsilon_{s-1}})$ to get a basis for the degree-1 alternals, but only *some* of the $\vec{lu}(\lambda^\bullet_{1,\epsilon_0}, ..., \lambda^\bullet_{0,\epsilon_s})$ to generate the degree-0 alternals. Let us now express the weak alternality relations for such a bimould $A^\bullet$. They read:

$$(\mathrm{swap.Wil.swap}\ A)^{\binom{0\ ,\dots,\ 0}{\epsilon_1\ ,\dots,\ \epsilon_s}} = \sum^{*} A^{\boldsymbol{w}^*} + \sum^{**} A^{\boldsymbol{w}^{**}} P(u_{**}) \tag{376}$$

Here *Wil* denotes the linearisation (resp. annihilation) operator for *symmetril* (resp. *alternil*) bimoulds, relative to the sequence splitting

$$\boldsymbol{w} = \boldsymbol{w}'\boldsymbol{w}'' \qquad \textit{with} \qquad \boldsymbol{w} = (w_1, .., w_r), \boldsymbol{w}' = (w_1), \boldsymbol{w}'' = (w_2, .., w_r)$$

Explicitly:

$$(\mathrm{Wil}.A)^{\boldsymbol{w}} \;=\; \sum_{\boldsymbol{w}^* \in \mathrm{sha}(\boldsymbol{w}',\boldsymbol{w}'')} A^{\boldsymbol{w}'} + \sum_{2\leqslant i \leqslant r} P(v_1 - v_i)\left(A^{\boldsymbol{w}^{\boldsymbol{1}*\boldsymbol{i}}} - A^{\boldsymbol{w}^{\boldsymbol{i}*\boldsymbol{1}}}\right) \tag{377}$$

$$\textit{with} \qquad \boldsymbol{w}^{\boldsymbol{1}*\boldsymbol{i}} = \left(\begin{smallmatrix} \dots\ ,u_{i-1}, & u_1+u_i & ,u_{i+1}, \dots \\ \dots\ ,v_{i-1}, & v_1 & ,v_{i+1}, \dots \end{smallmatrix}\right) \ ,\ \boldsymbol{w}^{\boldsymbol{i}*\boldsymbol{1}} = \left(\begin{smallmatrix} \dots\ ,u_{i-1}, & u_1+u_i & ,u_{i+1}, \dots \\ \dots\ ,v_{i-1}, & v_i & ,v_{i+1}, \dots \end{smallmatrix}\right)$$

We now plug (373) into $\sum^*$ of (376) and (374) into $\sum^{**}$ of (376). Simplifications occur, leading to the disappearance of the u_i variables from both numerators and denominators. Eventually, for sequences $(\epsilon_1, ..., \epsilon_s)$ ending with $\epsilon_s = 0$ and $\epsilon_s = \frac{1}{2}$, we find respectively

$$0 = \sum H^{\epsilon_1,\dots,\epsilon_{s-1}}_{\epsilon'_1,\dots,\epsilon'_s} b^{\epsilon'_1,\dots,\epsilon'_s} + c^{\epsilon_1,\dots,\epsilon_{s-1}} \tag{378}$$

$$0 = \sum K^{\epsilon_1,\dots,\epsilon_{s-1}}_{\epsilon'_1,\dots,\epsilon'_s} b^{\epsilon'_1,\dots,\epsilon'_s} + \sum L^{\epsilon_1,\dots,\epsilon_{s-1}}_{\epsilon''_1,\dots,\epsilon''_s} c^{\epsilon''_1,\dots,\epsilon''_{s-1}} \tag{379}$$

with coefficients $H^\bullet_\bullet, K^\bullet_\bullet, L^\bullet_\bullet$ in $\mathbb{Z}$. Eliminating the coefficients $c^\bullet$ between (378) and (379), we get the following 2^{s-1} structure constraints which characterise the subalgebra $ARI^{\underline{al}/\underline{il}}_{bico.ext}$ of $ARI^{\underline{al}}_{bico.ext}$:

$$\mathcal{R}^{\epsilon_1,\dots,\epsilon_{s-1}} : \qquad 0 = \sum_{\epsilon'_i \in \{0,\frac{1}{2}\}} R^{\epsilon_1,\dots,\epsilon_{s-1}}_{\epsilon'_1,\dots,\epsilon'_s} b^{\epsilon'_1,\dots,\epsilon'_s} \qquad (\textit{with} \quad R^\bullet_\bullet \in \mathbb{Z}) \tag{380}$$

The 2^{s-1} relations $\mathcal{R}^{\epsilon_1,\dots,\epsilon_{s-1}}$ are clearly not independent. However:

Conjecture: *The* first ρ_s *relations* $\mathcal{R}^{\epsilon_1,\dots,\epsilon_{s-1}}$ *are independent and imply all others. Here, 'first' is relative to the order induced by* $n(\boldsymbol{\epsilon}) := \sum \epsilon_i 2^i$ *and* $\rho_s := 1 + d_s - d^*_s$*, where* d_s *resp.* d^*_s *denotes the dimension of the component of weight* s *in the free Lie algebra* $\mathfrak{L}[e_1, e_2, e_3, e_4 \dots]$ *resp.* $\mathfrak{L}[e_1, e_3, e_5, e_7 \dots]$ *(*e_s *is assigned weight* s*).*

Subalgebras: keeping track of *push*-invariance.

One can in similar fashion express the symmetry *alternality+ pushu-invariance*[78] first in the penextremal algebra and then, after elimination of the components of degree 1, purely in the extremal algebra. This leads to an important algebra $ARI^{al/pushu}_{bico.ext}$ halfway between $ARI^{\underline{al}/\underline{il}}_{bico.ext}$ and $ARI^{al}_{bico.ext}$. Here, however, bimoulds in $ARI^{al/pushu}_{bico}$ are *not* fully determined by their restriction to $ARI^{al/pushu}_{bico.ext}$: it takes the full dimorphy – alternality (of the bimould itself) *and* alternility (of the *swappee*) – to ensure complete rigidity.

4.8 The degree-length exchanger *dre*. Co-satellites.

This section's object is to prepare for one of our main results – the correspondence between the first and second upper satellites. As it happens, the correspondence in question is best understood following the $(d\uparrow, r\downarrow)$ filtration, i.e. starting from low degrees d and correspondingly large lengths r. But r being the number of u_i-variables, that filtration is rather unwieldy. So, to fall back on the more familiar and tractable filtration $(d\downarrow, r\uparrow)$, we shall resort to a suitable $d \leftrightarrow r$ exchanging isomorphism.

The Hoffman duality.

The classical Hoffman duality for monocolours

$$\mathrm{Ze}^{d_1+1,1^{\{r_1-1\}},\ldots,d_n+1,1^{\{r_n-1\}}} = \mathrm{Ze}^{r_n+1,1^{\{d_n-1\}},\ldots,r_1+1,1^{\{d_1-1\}}} \qquad (\forall d_j, r_j \geqslant 1) \quad (381)$$

easily follows from the integral representation (216) and does indead exchange d and r, but it possesses no simple extension to bicolours. So we must come up with something else.

The $d \leftrightarrow r$ exchanger *dre*.

In analogy with the situation in $ARI^{\underline{al}/\underline{il}}_{bico}$, we say that a polynomial-valued mould is of weight s if each component of length $r \leqslant s$ is a homogeneous polynomials in $u_1, \ldots, u_r$ of total degree $d = s - r$, and each component of length $r > s$ vanishes. Any *alternal* polynomial-valued mould $\mathcal{A}^\bullet$ of weight s can be uniquely expressed as the 0-amplification of an *alternal*, scalar-valued mould $X^\bullet$ of length s with discrete binary indices $\eta_j \in \{0, 1\}$. If we now take the 1-amplification of that same $X^\bullet$, we get a new alternal mould $\mathcal{B}$ of weight s. Since the involution $\mathcal{A}^\bullet \leftrightarrow \mathcal{B}^\bullet$ so-defined exchanges the degree d

[78] *pushu*-invariance is the tweaked form of *push*-invariance induced by the classical isomorphism $adari(pal^\bullet) : ARI^{\underline{al}\,\underline{al}} \to ARI^{\underline{al}\underline{il}}$.

and length r of mould components, we call it the $d \leftrightarrow r$-exchanger, or *dre* for short. The same construction applies without modification to *symmetral* moulds. Graphically:

$$\begin{array}{lclcllcl}
\mathcal{A}^\bullet & = & \mathrm{am}_0\, X^\bullet & \xleftrightarrow{dre} & \mathcal{B}^\bullet & = & \mathrm{am}_1\, X^\bullet & \quad (X^\bullet \textit{ binary alternal}) \\
\mathcal{A}^\bullet & \in & \mathrm{MU}^{al}_{r,d} & \xleftrightarrow{dre} & \mathcal{B}^\bullet & \in & \mathrm{MU}^{al}_{d,r} & \\
\mathcal{S}^\bullet & = & \mathrm{am}_0\, Y^\bullet & \xleftrightarrow{dre} & \mathcal{T}^\bullet & = & \mathrm{am}_1\, Y^\bullet & \quad (Y^\bullet \textit{ binary symmetral}) \\
\mathcal{S}^\bullet & \in & \mathrm{MU}^{as}_{r,d} & \xleftrightarrow{dre} & \mathcal{T}^\bullet & \in & \mathrm{MU}^{as}_{d,r} &
\end{array}$$

4.9 Correspondence of the two upper satellite systems.

We are now in a position to address this chapter's last remaining challenge, i.e. finding a direct connection between the first and second upper satellites:

$$\begin{array}{ccccccc}
(\mathcal{A}^\bullet_{*0}, \mathcal{A}^\bullet_{*\frac{1}{2}}) & := & (sa^*_0.A^\bullet, sa^*_{\frac{1}{2}}.A^\bullet) & \qquad & (\mathcal{S}^\bullet_{*0}, \mathcal{S}^\bullet_{*\frac{1}{2}}) & := & (sa^*_0.S^\bullet, sa^*_{\frac{1}{2}}.S^\bullet) \\
\updownarrow & & & & \updownarrow & & \\
(\mathcal{A}^\bullet_{**0}, \mathcal{A}^\bullet_{**\frac{1}{2}}) & := & (sa^{**}_0.A^\bullet, sa^{**}_{\frac{1}{2}}.A^\bullet) & & (\mathcal{S}^\bullet_{**0}, \mathcal{S}^\bullet_{**\frac{1}{2}}) & := & (sa^{**}_0.S^\bullet, sa^{**}_{\frac{1}{2}}.S^\bullet) \\
A^\bullet & \in & ARI^{\underline{al}/\underline{il}}_{bico} & & S^\bullet & \in & GARI^{\underline{as}/\underline{is}}_{bico}
\end{array}$$

Equivalence of the all-whites.

Proposition 4.8 (Coincidence of sa^*_0 and sa^{**}_0) .
Provided we adopt for sa^{**}_0 the correct definitions (315)-(316) that take into account the perturbations introduced by length-1 components, we find that the all-whites of both upper satellites exactly coincide:

$$\mathcal{A}^\bullet_{*0} = \mathcal{A}^\bullet_{**0} \qquad \forall A^\bullet \in ARI^{\underline{al}/\underline{il}}_{bico} \tag{382}$$

$$\mathcal{S}^\bullet_{*0} = \mathcal{S}^\bullet_{**0} \qquad \forall S^\bullet \in GARI^{\underline{as}/\underline{is}}_{bico} \tag{383}$$

Involutive correspondence between the all-blacks.

The correspondence between the all-blacks is more recondite. To express it, we require a mould derivation $\mathcal{K}$ and an involutive mould automorphism $\mathfrak{K}$. Here are the definitions:

$$\mathcal{K}\,\mathcal{M}^\bullet = \mathrm{arit}(\mathcal{P}a^\bullet).\mathcal{M}^\bullet - \mathrm{lu}(\mathcal{P}a^\bullet, \mathcal{M}^\bullet) \quad \textit{with} \quad \begin{cases} \mathcal{P}a^{u_1} := P(u_1) = \frac{1}{u_1} \\ \mathcal{P}a^{u_1,\ldots,u_r} := 0 \ \ \textit{if } r \neq 1 \end{cases} \tag{384}$$

$$\mathfrak{K} = \mathrm{dre}\,.\,e^{\mathcal{K}}\,.\,\mathrm{dre}\,.\,\mathrm{pari} \tag{385}$$

A more explicit formula for $\mathcal{K}$'s action reads:

$$(\mathcal{K}\,\mathcal{M})^{u_1,\dots,u_r} = \begin{cases} +\sum_{1\leqslant j<r} \left(\mathcal{M}^{\dots,u_{j-1},u_j+u_{j+1},\dots} - \mathcal{M}^{\dots,u_{j-1},u_{j+1},\dots}\right) P(u_j) \\ -\sum_{1<j\leqslant r} \left(\mathcal{M}^{\dots,u_{j-1}+u_j,u_{j+1},\dots} - \mathcal{M}^{\dots,u_{j-1},u_{j+1},\dots}\right) P(u_j) \end{cases}$$

As for the involutive character of $\mathfrak{K}$, it results from:

$$\mathrm{dre}\,.\,e^{\mathcal{L}}\,.\,\mathrm{dre}\,.\,\mathrm{pari} = \mathrm{dre}\,.\,e^{\mathcal{K}}\,.\,\mathrm{neg}\,.\,\mathrm{dre} = \mathrm{dre}\,.\,\mathrm{neg}\,.\,e^{-\mathcal{K}}\,.\,\mathrm{dre} = \mathrm{pari}\,.\,\mathrm{dre}\,.\,e^{-\mathcal{K}}\,.\,\mathrm{dre}$$

Proposition 4.9 (Involutive correspondence between $\mathrm{sa}^*_{\frac{1}{2}}$ and $\mathrm{sa}^{**}_{\frac{1}{2}}$) . *Provided we adopt for $\mathrm{sa}^{**}_{\frac{1}{2}}$ the correct definitions (315)-(316) that take into account the perturbations introduced by length-1 components, we find that the all-blacks of both upper satellites correspond under the involution $\mathfrak{K}$:*

$$\mathcal{A}^{\bullet}_{*\frac{1}{2}} \xleftrightarrow{\mathfrak{K}} \mathcal{A}^{\bullet}_{**\frac{1}{2}} \qquad \forall A^{\bullet} \in ARI^{al/il}_{bico} \tag{386}$$

$$\mathcal{S}^{\bullet}_{*\frac{1}{2}} \xleftrightarrow{\mathfrak{K}} \mathcal{S}^{\bullet}_{**\frac{1}{2}} \qquad \forall S^{\bullet} \in GARI^{as/is}_{bico} \tag{387}$$

Remark 1: Given that each upper satellite contains 'all the information', the existence of a more or less explicit correspondence between the two was a foregone conclusion. The surprise, though, is that the correspondence should operate, not between the *pairs* $(\mathcal{A}_{*0}, \mathcal{A}_{*\frac{1}{2}}) \leftrightarrow (\mathcal{A}_{**0}, \mathcal{A}_{**\frac{1}{2}})$, but *separately* between the all-whites and all-blacks: $\mathcal{A}_{*0} \leftrightarrow \mathcal{A}_{**0}$, $\mathcal{A}_{*\frac{1}{2}} \leftrightarrow \mathcal{A}_{**\frac{1}{2}}$.

Remark 2: The identity $sa^*_0 = sa^{**}_0$ is easy to spot (less so to prove) in the algrebra $ARI^{al/il}_{bico}$, because there the presence of a length-1 component $A^{\binom{u_1}{1/2}}$ hardly affects the shape of $sa^{**}.A^{\bullet}$. See (315). This is no longer the case in the group $GARI^{as/is}_{bico}$, where the presence of a length-1 component $S^{\binom{u_1}{1/2}}$ upsets everything, as obvious from the formula (316). This must be the reason why so remarkable and so fundamental an identity as $sa^*_0.zag^{\bullet} = sa^{**}_0.zag^{\bullet}$ had so long escaped notice.

Remark 3: The involutive correspondence $\mathfrak{K} : sa^*_{1/2} \leftrightarrow sa^{**}_{1/2}$ was even less conspicuous and we confess that it took us quite some time to figure it out. The thing is that the low-length components (- on which one tends to focus -) hardly bear any resemblance in $\mathcal{A}_{*1/2}$ and $\mathcal{A}_{**1/2}$. It is only when we go to the low-degree components that a pattern begins to emerge.

4.10 Recapitulation: the circulation of information.

A telling analogy.

To appreciate the minor miracles of bicolour satellisation, which begin – but do not end – with the recoverability of the *whole* from *small parts*, the analogy with functions defined on the closed unit disk may not be out of place. The two, largely self-explanatory pictures below show how the *whole* (in blue) and the *three systems of boundary data* (in black) relate to each other in both situations. The black arrows depict the circulation of information under the weaker assumptions (- one single symmetry for bicolours; mere smoothness for functions -), while the red arrows show what new channels of communication suddenly open under the stronger assumptions (- dimorphy i.e. a double symmetry for bicolours; harmonicity for functions-).

$\{\mathrm{zag}^\bullet$ *symmetral*$\}$ *Fig. 1* $\{\mathrm{zig}^\bullet$ *symmetril*$\}$

$$\boldsymbol{sa}^* = \left\{ \begin{array}{l} \mathrm{zag}^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ 0 & ,\dots, & 0 \end{smallmatrix}\right)} \\ \mathrm{zag}^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ \frac{1}{2} & ,\dots, & \frac{1}{2} \end{smallmatrix}\right)} \end{array} \right\}$$

$$\nearrow\!\swarrow \qquad\qquad \uparrow\downarrow \qquad\qquad \nwarrow\!\searrow$$

$$\nearrow\!\swarrow \qquad \left\{ \mathrm{zag}^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ \epsilon_1 & ,\dots, & \epsilon_r \end{smallmatrix}\right)} \right\} \qquad \nwarrow\!\searrow$$

$$\nearrow\!\swarrow \qquad\qquad \searrow\!\nwarrow$$

$$\boldsymbol{sa} = \left\{ \mathrm{zag}^{\left(\begin{smallmatrix} 0 & ,\dots, & 0 \\ \epsilon_1 & ,\dots, & \epsilon_r \end{smallmatrix}\right)} \right\} \qquad \longleftarrow\!\longrightarrow \qquad \boldsymbol{sa}^{**} = \left\{ \begin{array}{l} (\mathrm{am}_0.\mathrm{zag})^{(u_1,\dots,u_r)} \\ (\mathrm{am}_{\frac{1}{2}}.\mathrm{zag})^{(u_1,\dots,u_r)} \end{array} \right\}$$

$\{\mathrm{F}$ *smooth*$\}$ *Fig. 2* $\{\mathrm{F}$ *harmonic*$\}$

$$\boldsymbol{sa}^* = \left\{ \mathrm{F}^{(n_1,n_2)}(0,0)\,;\, n_i \in \mathbb{N} \right\}$$

$$\nearrow\!\swarrow \qquad\qquad \uparrow\downarrow \qquad\qquad \nwarrow\!\searrow$$

$$\nearrow\!\swarrow \qquad \left\{ \mathrm{F}(x_1,x_2)\,;\, x_1^2+x_2^2 \leqslant 1 \right\} \qquad \nwarrow\!\searrow$$

$$\nearrow\!\swarrow \qquad\qquad \searrow\!\nwarrow$$

$$\boldsymbol{sa} = \left\{ \mathrm{F}(x_1,x_2)\,;\, x_1^2+x_2^2=1 \right\} \qquad \longleftarrow\!\longrightarrow \qquad \boldsymbol{sa}^{**} = \left\{ \mathrm{P}\!\!\!\!-_n \,;\, n \in \mathbb{Z} \right\}$$

Let us now collect in one place, for easier survey, all the main formulae pertaining to satellisation and co-satellisation .

Lower satellisation of bicolours.

$$\mathrm{ARI}^{\mathrm{al/il}}_{\mathrm{bico}} \ni A^\bullet \ \overset{\mathrm{sa}}{\to}\ \mathcal{A}^\bullet \qquad\qquad \mathrm{GARI}^{\mathrm{as/is}}_{\mathrm{bico}} \ni S^\bullet \ \overset{\mathrm{sa}}{\to}\ \mathcal{S}^\bullet$$

$$\mathcal{A}^{\epsilon_1,\dots,\epsilon_r} := A^{\left(\begin{smallmatrix} 0 & ,\dots, & 0 \\ \epsilon_1 & ,\dots, & \epsilon_r \end{smallmatrix}\right)} \quad , \quad \mathcal{S}^{\epsilon_1,\dots,\epsilon_r} := S^{\left(\begin{smallmatrix} 0 & ,\dots, & 0 \\ \epsilon_1 & ,\dots, & \epsilon_r \end{smallmatrix}\right)}$$

First (upper) satellisation of bicolours.

$$\mathrm{ARI}^{\mathrm{al/il}}_{\mathrm{bico}} \ni A^\bullet \ \overset{\mathrm{sa}^*}{\to}\ \underline{\mathcal{A}}^\bullet \qquad\qquad \mathrm{GARI}^{\mathrm{as/is}}_{\mathrm{bico}} \ni S^\bullet \ \overset{\mathrm{sa}^*}{\to}\ \underline{\mathcal{S}}^\bullet$$

$$\underline{\mathcal{A}}_0^{\epsilon_1,\dots,\epsilon_r} := A^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ 0 & ,\dots, & 0 \end{smallmatrix}\right)} \quad , \quad \underline{\mathcal{S}}_0^{\epsilon_1,\dots,\epsilon_r} := S^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ 0 & ,\dots, & 0 \end{smallmatrix}\right)}$$

$$\underline{\mathcal{A}}_{\frac{1}{2}}^{\epsilon_1,\dots,\epsilon_r} := A^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ \frac{1}{2} & ,\dots, & \frac{1}{2} \end{smallmatrix}\right)} \quad , \quad \underline{\mathcal{S}}_{\frac{1}{2}}^{\epsilon_1,\dots,\epsilon_r} := S^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ \frac{1}{2} & ,\dots, & \frac{1}{2} \end{smallmatrix}\right)}$$

Second (upper) satellisation of bicolours.

$$\mathrm{ARI}^{\mathrm{al/il}}_{\mathrm{bico}} \ni A^\bullet \ \overset{\mathrm{sa}^{**}}{\to}\ \underline{\underline{\mathcal{A}}}^\bullet \qquad\qquad \mathrm{GARI}^{\mathrm{as/is}}_{\mathrm{bico}} \ni S^\bullet \ \overset{\mathrm{sa}^{**}}{\to}\ \underline{\underline{\mathcal{S}}}^\bullet$$

$$\begin{aligned}
\underline{\underline{\mathcal{A}}}_0^\bullet &:= -\mathrm{neg.am}_0\, A^\bullet + \mathrm{neg.am}_{\frac{1}{2}}\, A^\bullet + A^{\binom{0}{\frac{1}{2}}}\, I^\bullet \\
\underline{\underline{\mathcal{A}}}_{\frac{1}{2}}^\bullet &:= -\mathrm{neg.am}_0\, A^\bullet \\
\underline{\underline{\mathcal{S}}}_0^\bullet &:= \mathrm{mu}\Big(e^{-S^{\binom{0}{\frac{1}{2}}}\mathfrak{D}}.\,\mathrm{invmu}(\mathrm{neg.am}_0\, S^\bullet), \mathrm{neg.am}_{\frac{1}{2}}\, S^\bullet, e^{S^{\binom{0}{\frac{1}{2}}}I^\bullet}\Big) \\
\underline{\underline{\mathcal{S}}}_{\frac{1}{2}}^\bullet &:= \mathrm{invmu}(\mathrm{am}_0\, S^\bullet)
\end{aligned}$$

with the mould derivation $\mathfrak{D}$:

$$(\mathfrak{D}\mathcal{A})^{u_1,\dots,u_r} \ := \ (u_1 + \dots + u_r)\,\mathcal{A}^{u_1,\dots,u_r}$$

and the amplification operators am_0, $am_{\frac{1}{2}}$:

$$(\mathrm{am}_0.M)^{u_1,\dots,u_r} := \sum\nolimits_{0\leqslant n_r} M^{\left(\begin{smallmatrix} 0 & \overbrace{,0 \ ,\dots, \ 0,}^{\leftarrow n_1 \rightarrow} & ,\dots, & 0 & \overbrace{,0 \ ,\dots, \ 0}^{\leftarrow n_r \rightarrow} \\ 1/2 & ,0 \ ,\dots, \ 0, & ,\dots, & 1/2 & ,0 \ ,\dots, \ 0 \end{smallmatrix}\right)} u_1^{n_1} u_{1,2}^{n_2} \dots u_{1,..,r}^{n_r}$$

$$(\mathrm{am}_{\frac{1}{2}}.M)^{u_1,\dots,u_r} := \sum\nolimits_{0\leqslant n_r} M^{\left(\begin{smallmatrix} 0, & \overbrace{0 \ ,\dots, \ 0,}^{\leftarrow n_1 \rightarrow} & ,\dots, & 0, & \overbrace{0 \ ,\dots, \ 0}^{\leftarrow n_r \rightarrow} \\ 0, & 1/2 \ ,\dots, \ 1/2, & ,\dots, & 0, & 1/2 \ ,\dots, \ 1/2 \end{smallmatrix}\right)} u_1^{n_1} u_{1,2}^{n_2} \dots u_{1,..,r}^{n_r}$$

First and second co-satellisation of bicolours.

$$\mathrm{ARI}^{\mathrm{al/il}}_{\mathrm{bico}} \ni A^\bullet \ \overset{\mathrm{sa}^\sharp}{\to}\ \underline{\mathfrak{A}}^\bullet := \mathrm{dre}\,.\,\underline{\mathcal{A}}^\bullet \quad , \quad \mathrm{GARI}^{\mathrm{as/is}}_{\mathrm{bico}} \ni S^\bullet \ \overset{\mathrm{sa}^\sharp}{\to}\ \underline{\mathfrak{S}}^\bullet := \mathrm{dre}\,.\,\underline{\mathcal{S}}^\bullet$$

$$\mathrm{ARI}^{\mathrm{al/il}}_{\mathrm{bico}} \ni A^\bullet \ \overset{\mathrm{sa}^{\sharp\sharp}}{\to}\ \underline{\underline{\mathfrak{A}}}^\bullet := \mathrm{dre}\,.\,\underline{\underline{\mathcal{A}}}^\bullet \quad , \quad \mathrm{GARI}^{\mathrm{as/is}}_{\mathrm{bico}} \ni S^\bullet \ \overset{\mathrm{sa}^{\sharp\sharp}}{\to}\ \underline{\underline{\mathfrak{S}}}^\bullet := \mathrm{dre}\,.\,\underline{\underline{\mathcal{S}}}^\bullet$$

with the $d \leftrightarrow r$-exchanger *dre* introduced in §4.8.

First and second (upper) satellisation of *ari/gari*.

$$
\begin{array}{ccccccc}
(A^\bullet, B^\bullet) & \xrightarrow{\text{ari}} & C^\bullet & \qquad & (S^\bullet, T^\bullet) & \xrightarrow{\text{gari}} & R^\bullet \\
\text{sa}^* \downarrow \text{sa}^* & & \downarrow \text{sa}^* & & \text{sa}^* \downarrow \text{sa}^* & & \downarrow \text{sa}^* \\
(\{\underline{\mathcal{A}}^\bullet_0, \underline{\mathcal{A}}^\bullet_{\frac12}\}, \{\underline{\mathcal{B}}^\bullet_0, \underline{\mathcal{B}}^\bullet_{\frac12}\}) & \xrightarrow{\text{ari}^*} & \{\underline{\mathcal{C}}^\bullet_0, \underline{\mathcal{C}}^\bullet_{\frac12}\} & & (\{\underline{\mathcal{S}}^\bullet_0, \underline{\mathcal{S}}^\bullet_{\frac12}\}, \{\underline{\mathcal{T}}^\bullet_0, \underline{\mathcal{T}}^\bullet_{\frac12}\}) & \xrightarrow{\text{gari}^*} & \{\underline{\mathcal{R}}^\bullet_0, \underline{\mathcal{R}}^\bullet_{\frac12}\}
\end{array}
$$

$$\underline{\mathcal{C}}^\bullet_0 = \text{lu}(\underline{\mathcal{A}}^\bullet_0, \underline{\mathcal{B}}^\bullet_0) + \text{arit}(\underline{\mathcal{B}}^\bullet_0).\underline{\mathcal{A}}^\bullet_0 - \text{arit}(\underline{\mathcal{A}}^\bullet_0).\underline{\mathcal{B}}^\bullet_0 \equiv \text{ari}(\underline{\mathcal{A}}^\bullet_0, \underline{\mathcal{B}}^\bullet_0) \tag{388}$$

$$\underline{\mathcal{C}}^\bullet_{\frac12} = \text{lu}(\underline{\mathcal{A}}^\bullet_{\frac12}, \underline{\mathcal{B}}^\bullet_{\frac12}) + \text{arit}(\underline{\mathcal{B}}^\bullet_0).\underline{\mathcal{A}}^\bullet_{\frac12} - \text{arit}(\underline{\mathcal{A}}^\bullet_0).\underline{\mathcal{B}}^\bullet_{\frac12} \tag{389}$$

$$\underline{\mathcal{R}}^\bullet_0 = \text{mu}(\text{garit}(\underline{\mathcal{T}}^\bullet_0).\underline{\mathcal{S}}^\bullet_0, \underline{\mathcal{T}}^\bullet_0) \equiv \text{gari}(\underline{\mathcal{S}}^\bullet_0, \underline{\mathcal{T}}^\bullet_0) \tag{390}$$

$$\underline{\mathcal{R}}^\bullet_{\frac12} = \text{mu}(\text{garit}(\underline{\mathcal{T}}^\bullet_0).\underline{\mathcal{S}}^\bullet_{\frac12}, \underline{\mathcal{T}}^\bullet_{\frac12}) \tag{391}$$

$$
\begin{array}{ccccccc}
(A^\bullet, B^\bullet) & \xrightarrow{\text{ari}} & C^\bullet & \qquad & (S^\bullet, T^\bullet) & \xrightarrow{\text{gari}} & R^\bullet \\
\text{sa}^{**} \downarrow \text{sa}^{**} & & \downarrow \text{sa}^{**} & & \text{sa}^{**} \downarrow \text{sa}^{**} & & \downarrow \text{sa}^{**} \\
(\{\underline{\underline{\mathcal{A}}}^\bullet_0, \underline{\underline{\mathcal{A}}}^\bullet_{\frac12}\}, \{\underline{\underline{\mathcal{B}}}^\bullet_0, \underline{\underline{\mathcal{B}}}^\bullet_{\frac12}\}) & \xrightarrow{\text{ari}^{**}} & \{\underline{\underline{\mathcal{C}}}^\bullet_0, \underline{\underline{\mathcal{C}}}^\bullet_{\frac12}\} & & (\{\underline{\underline{\mathcal{S}}}^\bullet_0, \underline{\underline{\mathcal{S}}}^\bullet_{\frac12}\}, \{\underline{\underline{\mathcal{T}}}^\bullet_0, \underline{\underline{\mathcal{T}}}^\bullet_{\frac12}\}) & \xrightarrow{\text{gari}^{**}} & \{\underline{\underline{\mathcal{R}}}^\bullet_0, \underline{\underline{\mathcal{R}}}^\bullet_{\frac12}\}
\end{array}
$$

$$\underline{\underline{\mathcal{C}}}^\bullet_0 = \text{lu}(\underline{\underline{\mathcal{A}}}^\bullet_0, \underline{\underline{\mathcal{B}}}^\bullet_0) + \text{arit}(\underline{\underline{\mathcal{B}}}^\bullet_0).\underline{\underline{\mathcal{A}}}^\bullet_0 - \text{arit}(\underline{\underline{\mathcal{A}}}^\bullet_0).\underline{\underline{\mathcal{B}}}^\bullet_0 \equiv \text{ari}(\underline{\underline{\mathcal{A}}}^\bullet_0, \underline{\underline{\mathcal{B}}}^\bullet_0) \tag{392}$$

$$\underline{\underline{\mathcal{C}}}^\bullet_{\frac12} = \text{lu}^*(\underline{\underline{\mathcal{A}}}^\bullet_{\frac12}, \underline{\underline{\mathcal{B}}}^\bullet_{\frac12}) + \text{arit}(\underline{\underline{\mathcal{B}}}^\bullet_0).\underline{\underline{\mathcal{A}}}^\bullet_{\frac12} - \text{arit}(\underline{\underline{\mathcal{A}}}^\bullet_0).\underline{\underline{\mathcal{B}}}^\bullet_{\frac12} \tag{393}$$

$$\underline{\underline{\mathcal{R}}}^\bullet_0 = \text{mu}(\text{garit}(\underline{\underline{\mathcal{T}}}^\bullet_0).\underline{\underline{\mathcal{S}}}^\bullet_0, \underline{\underline{\mathcal{T}}}^\bullet_0) \equiv \text{gari}(\underline{\underline{\mathcal{S}}}^\bullet_0, \underline{\underline{\mathcal{T}}}^\bullet_0) \tag{394}$$

$$\underline{\underline{\mathcal{R}}}^\bullet_{\frac12} = \text{mu}^*(\text{garit}(\underline{\underline{\mathcal{T}}}^\bullet_0).\underline{\underline{\mathcal{S}}}^\bullet_{\frac12}, \underline{\underline{\mathcal{T}}}^\bullet_{\frac12}) \tag{395}$$

with

$$\text{lu}^*(\mathcal{A}^\bullet, \mathcal{B}^\bullet) := \text{lu}(\mathcal{A}^\bullet, \mathcal{B}^\bullet) + \mathcal{A}^0 \,\mathfrak{D}\, \mathcal{B}^\bullet - \mathcal{B}^0 \,\mathfrak{D}\, \mathcal{A}^\bullet \tag{396}$$

$$\text{mu}^*(\mathcal{S}^\bullet, \mathcal{T}^\bullet) := \text{mu}(\exp(-\mathcal{T}^0 \,\mathfrak{D}).\mathcal{S}^\bullet, \mathcal{T}^\bullet) \tag{397}$$

First and second (upper) co-satellisation of *ari/gari***.**

$$
\begin{array}{ccc}
(A^\bullet, B^\bullet) & \stackrel{\mathrm{ari}}{\longrightarrow} & C^\bullet \\
\mathrm{sa}^\sharp \downarrow \mathrm{sa}^\sharp & & \downarrow \mathrm{sa}^\sharp \\
(\{\underline{\mathfrak{A}}^\bullet_0, \underline{\mathfrak{A}}^\bullet_{\frac{1}{2}}\}, \{\underline{\mathfrak{B}}^\bullet_0, \underline{\mathfrak{B}}^\bullet_{\frac{1}{2}}\}) & \stackrel{\mathrm{ari}^\sharp}{\longrightarrow} & \{\underline{\mathfrak{C}}^\bullet_0, \underline{\mathfrak{C}}^\bullet_{\frac{1}{2}}\}
\end{array}
\qquad
\begin{array}{ccc}
(S^\bullet, T^\bullet) & \stackrel{\mathrm{gari}}{\longrightarrow} & R^\bullet \\
\mathrm{sa}^\sharp \downarrow \mathrm{sa}^\sharp & & \downarrow \mathrm{sa}^\sharp \\
(\{\underline{\mathfrak{S}}^\bullet_0, \underline{\mathfrak{S}}^\bullet_{\frac{1}{2}}\}, \{\underline{\mathfrak{T}}^\bullet_0, \underline{\mathfrak{T}}^\bullet_{\frac{1}{2}}\}) & \stackrel{\mathrm{gari}^\sharp}{\longrightarrow} & \{\underline{\mathfrak{R}}^\bullet_0, \underline{\mathfrak{R}}^\bullet_{\frac{1}{2}}\}
\end{array}
$$

$$\underline{\mathfrak{C}}^\bullet_0 = \mathrm{lu}(\underline{\mathfrak{A}}^\bullet_0, \underline{\mathfrak{B}}^\bullet_0) + \mathrm{arit}(\underline{\mathfrak{B}}^\bullet_0).\underline{\mathfrak{A}}^\bullet_0 - \mathrm{arit}(\underline{\mathfrak{A}}^\bullet_0).\underline{\mathfrak{B}}^\bullet_0 \equiv \mathrm{ari}(\underline{\mathfrak{A}}^\bullet_0, \underline{\mathfrak{B}}^\bullet_0) \tag{398}$$

$$\underline{\mathfrak{C}}^\bullet_{\frac{1}{2}} = \begin{cases} -\mathrm{lu}^\sharp(\underline{\mathfrak{A}}^\bullet_{\frac{1}{2}}, \underline{\mathfrak{B}}^\bullet_{\frac{1}{2}}) + \underline{\mathfrak{A}}^0_{\frac{1}{2}} \,.\, \mathrm{lu}(\mathcal{I}^\bullet, \underline{\mathfrak{B}}^\bullet_{\frac{1}{2}}) - \underline{\mathfrak{B}}^0_{\frac{1}{2}} \,.\, \mathrm{lu}(\mathcal{I}^\bullet, \underline{\mathfrak{A}}^\bullet_{\frac{1}{2}}) \\ +\mathrm{lu}^\sharp(\underline{\mathfrak{A}}^\bullet_0, \underline{\mathfrak{B}}^\bullet_{\frac{1}{2}}) + \mathrm{lu}^\sharp(\underline{\mathfrak{A}}^\bullet_{\frac{1}{2}}, \underline{\mathfrak{B}}^\bullet_0) \\ +\mathrm{arit}(\underline{\mathfrak{B}}^\bullet_0).\underline{\mathfrak{A}}^\bullet_{\frac{1}{2}} - \mathrm{arit}(\underline{\mathfrak{A}}^\bullet_0).\underline{\mathfrak{B}}^\bullet_{\frac{1}{2}} \end{cases} \tag{399}$$

with the composition unit $\mathcal{I}^\bullet$ and the tweaked Lie bracket $lu^\sharp \neq lu^*$:

$$\mathcal{I}^{u_1} := 1 \;\; \forall u_1 \quad , \quad \mathcal{I}^{u_1,\dots,u_r} := 0 \;\; \forall r \neq 1 \tag{400}$$

$$\mathrm{lu}^\sharp(\mathcal{A}^\bullet, \mathcal{B}^\bullet) \quad := \quad \mathrm{lu}(\mathcal{A}^\bullet, \mathcal{B}^\bullet) - \mathcal{A}^0 \,\mathfrak{D}\, \mathcal{B}^\bullet + \mathcal{B}^0 \,\mathfrak{D}\, \mathcal{A}^\bullet \tag{401}$$

$$
\begin{array}{ccc}
(A^\bullet, B^\bullet) & \stackrel{\mathrm{ari}}{\longrightarrow} & C^\bullet \\
\mathrm{sa}^{\sharp\sharp} \downarrow \mathrm{sa}^{\sharp\sharp} & & \downarrow \mathrm{sa}^{\sharp\sharp} \\
(\{\underline{\underline{\mathfrak{A}}}^\bullet_0, \underline{\underline{\mathfrak{A}}}^\bullet_{\frac{1}{2}}\}, \{\underline{\underline{\mathfrak{B}}}^\bullet_0, \underline{\underline{\mathfrak{B}}}^\bullet_{\frac{1}{2}}\}) & \stackrel{\mathrm{ari}^{\sharp\sharp}}{\longrightarrow} & \{\underline{\underline{\mathfrak{C}}}^\bullet_0, \underline{\underline{\mathfrak{C}}}^\bullet_{\frac{1}{2}}\}
\end{array}
\qquad
\begin{array}{ccc}
(S^\bullet, T^\bullet) & \stackrel{\mathrm{gari}}{\longrightarrow} & R^\bullet \\
\mathrm{sa}^{\sharp\sharp} \downarrow \mathrm{sa}^{\sharp\sharp} & & \downarrow \mathrm{sa}^{\sharp\sharp} \\
(\{\underline{\underline{\mathfrak{S}}}^\bullet_0, \underline{\underline{\mathfrak{S}}}^\bullet_{\frac{1}{2}}\}, \{\underline{\underline{\mathfrak{T}}}^\bullet_0, \underline{\underline{\mathfrak{T}}}^\bullet_{\frac{1}{2}}\}) & \stackrel{\mathrm{gari}^{\sharp\sharp}}{\longrightarrow} & \{\underline{\underline{\mathfrak{R}}}^\bullet_0, \underline{\underline{\mathfrak{R}}}^\bullet_{\frac{1}{2}}\}
\end{array}
$$

$$\underline{\underline{\mathfrak{C}}}^\bullet_0 = \mathrm{lu}(\underline{\underline{\mathfrak{A}}}^\bullet_0, \underline{\underline{\mathcal{B}}}^\bullet_0) + \mathrm{arit}(\underline{\underline{\mathfrak{B}}}^\bullet_0).\underline{\underline{\mathfrak{A}}}^\bullet_0 - \mathrm{arit}(\underline{\underline{\mathfrak{A}}}^\bullet_0).\underline{\underline{\mathfrak{B}}}^\bullet_0 \equiv \mathrm{ari}(\underline{\underline{\mathfrak{A}}}^\bullet_0, \underline{\underline{\mathfrak{B}}}^\bullet_0) \tag{402}$$

$$\underline{\underline{\mathfrak{C}}}^\bullet_{\frac{1}{2}} = \begin{cases} -\mathrm{lu}^\sharp(\underline{\underline{\mathfrak{A}}}^\bullet_{\frac{1}{2}}, \underline{\underline{\mathfrak{B}}}^\bullet_{\frac{1}{2}}) + 2 \,.\, \underline{\underline{\mathfrak{A}}}^0_{\frac{1}{2}} \,.\, \mathrm{lu}(\mathcal{I}^\bullet, \underline{\underline{\mathfrak{B}}}^\bullet_{\frac{1}{2}}) - 2 \,.\, \underline{\underline{\mathfrak{B}}}^0_{\frac{1}{2}} \,.\, \mathrm{lu}(\mathcal{I}^\bullet, \underline{\underline{\mathfrak{A}}}^\bullet_{\frac{1}{2}}) \\ +\mathrm{lu}^\sharp(\underline{\underline{\mathfrak{A}}}^\bullet_0, \underline{\underline{\mathfrak{B}}}^\bullet_{\frac{1}{2}}) + \mathrm{lu}^\sharp(\underline{\underline{\mathfrak{A}}}^\bullet_{\frac{1}{2}}, \underline{\underline{\mathfrak{B}}}^\bullet_0) \\ +\mathrm{arit}(\underline{\underline{\mathfrak{B}}}^\bullet_0).\underline{\underline{\mathfrak{A}}}^\bullet_{\frac{1}{2}} - \mathrm{arit}(\underline{\underline{\mathfrak{A}}}^\bullet_0).\underline{\underline{\mathfrak{B}}}^\bullet_{\frac{1}{2}} \end{cases} \tag{403}$$

Thus, the formulae for $ari^\sharp$ and $ari^{\sharp\sharp}$ differ only by the presence of a factor 2 in front of the two corrective terms $\underline{\underline{\mathfrak{A}}}^0_{\frac{1}{2}} \,.\, \mathrm{lu}(\mathcal{I}^\bullet, \underline{\underline{\mathfrak{B}}}^\bullet_{\frac{1}{2}})$ and $\underline{\underline{\mathfrak{B}}}^0_{\frac{1}{2}} \,.\, \mathrm{lu}(\mathcal{I}^\bullet, \underline{\underline{\mathfrak{A}}}^\bullet_{\frac{1}{2}})$. There exist similar formulae for $gari^\sharp$ and $gari^{\sharp\sharp}$.

Counting our luck and listing our gains.

Satellisation succeeds only thanks to an improbable string of good luck:

Fluke 1: The drastic restriction sa to the extremal algebra ($d = 0$) does not involve any loss of information, nor does the equally drastic restriction sa^* to the all-whites and all-blacks .

Fluke 2: The *amplification*, which takes us from sa to sa^{**}, turns the subtractive ϵ_i-flexions into additive u_i-flexions.

Fluke 3: All the constraints flowing from the double symmetry (*'dimorphy'*) can be expressed internally within each satellite system.

Fluke 4: The *ari/gari* operations can also be expressed internally within each satellite system.

Fluke 5: Despite their completely different origin, the two upper satellisations sa^* and sa^{**} are easily convertible into each other: the all-whiles sa^*_0 and sa^{**}_0 simply coincide, while the all-blacks $sa^*_{\frac{1}{2}}$ and $sa^{**}_{\frac{1}{2}}$ get exchanged under a remarkable involution $\mathfrak{K}$.

Fluke 6: There is an effective procedure, based on the operators *discram* and *viscram*, for recovering the whole of $ARI^{al/il}_{bico}$ or $GARI^{al/il}_{bico}$ from each satellite.

Satellisation also brings huge rewards:

Gain 1: It makes possible a dramatic data reduction, by showing how to recover all the information from the *all-whites+all-blacks*, or even from the sole *all-blacks*, or even from the *all-blacks* of even degree.

Gain 2: In combination with the $d \leftrightarrow r$ exchanger, satellisation, or rather the dual 'co-satellisation', enables one to work entirely within the (s, d)-filtration, and thus to overcome the *'curse of retro-action'*.

Gain 3: Satellisation extends *'perinomal'* irreducible analysis ($luma^\bullet$-based) to the case of bicolours, and it eases *'arithmetical'* irreducible analysis ($loma^\bullet$- or $lama^\bullet$-based) for both monocolours and bicolours.

5 Multizeta algebra: decomposing the monocolours into irreducibles.

In this brief section, we return to the monocolours. Since the independence theorem for length-1 bicolour bialternals has no exact equivalent for monocolours, we are led to explore various alternative settings in search of 'rigidity', so as to ensure the uniqueness of decomposition.
We shall compare here four main settings:

(i) $\mathbb{Z}/p\mathbb{Z}$-supported bialternals,
(ii) $\mathbb{Z}$-supported bialternals,
(iii) polynomial-valued bialternals.
(iv) perinomal bialternals,
and we shall attempt to show how deeply they differ in regard to 'rigidity' by comparing the strikingly different forms which the *ari-oddari*-conversion formulae[79] assume in each case.

Lastly (– and briefly, because this doesn't fall within the purview of this investigation and will be treated at length in a follow-up paper –), we shall sketch the two main strategies for the decomposition of monocolours into remarkable ('canonical') systems of irreducibles, and examine in great detail how this works out up to length $r = 4$.

5.1 Polynomial bialternals.

This subsection is purely for perspective and contains no new information.
(i) It gives, subject (for $r \geqslant 4$) to the Broadhurst-Kreimer conjectures, the dimensions $dim_{r,d}$ of the polynomial bialternals (for monocolours).
(ii) It gives, subject to a further classical conjecture saying that all bialternals are semi-freely[80] generated by the so-called $ekma^{\bullet}_{2d}$ (length-1) and $carma^{\bullet}_{2d,k}$ (length-4), the dimensions $dimelem_{r,d}$ of the 'elementary' bialternals (generated by the $ekma^{\bullet}_{2d}$), and the complementary dimensions of the 'exceptional' bialternals $dimexcep_{r,d} := dim_{r,d} - dimelem_{r,d}$.
(iii) For comparison, it also give the dimensions $dimfree_{r,d}$ of all alternals freely genrated under the *lu*-bracket by the $ekma^{\bullet}_{2d}$ $(1 \leqslant d)$, or again the dimensions of all bicolour bialternals generated by the length-1 bicolour generators (leaving out the one of degree 0).

[79] i.e. the formulae for mutual conversion of the length-2 bialternals generated, in each setting, by the bracket *ari* and the pseudo-bracket *oddari*.

[80] i.e. without other relations between the $ekma^{\bullet}_{2\ d}$ than the well-known relations in length 2, and all those generated by them.

In each case the dimensions are given via the generating series.[81]

$$\begin{aligned}
\mathrm{dimfree}_1(t) &= \frac{t^2}{(1-t^2)} \\
\mathrm{dimfree}_2(t) &= \frac{t^6}{(1-t^2)\,(1-t^4)} \\
\mathrm{dimfree}_3(t) &= \frac{t^8}{(1-t^2)^2\,(1-t^6)} \\
\mathrm{dimfree}_4(t) &= \frac{t^{10}}{(1-t^2)^2\,(1-t^4)^2} \\
\mathrm{dimfree}_5(t) &= \frac{t^{12}\,(1+t^6)}{(1-t^2)^3\,(1-t^4)\,(1-t^{10})} \\
\mathrm{dimfree}_6(t) &= \frac{t^{14}\,(1+t^2+2\,t^4+2\,t^6+3\,t^8+2\,t^{12}+t^{14})}{(1-t^2)^2\,(1-t^4)^2\,(1-t^6)\,(1-t^{12})}
\end{aligned}$$

$$\begin{aligned}
\mathrm{dim}_1(t) &:= \frac{t^2}{(1-t^2)} \\
\mathrm{dim}_2(t) &:= \frac{t^6}{(1-t^2)\,(1-t^6)} \\
\mathrm{dim}_3(t) &:= \frac{t^8\,(1+t^2-t^4)}{(1-t^2)\,(1-t^4)\,(1-t^6)} \\
\mathrm{dim}_4(t) &:= \frac{t^8\,(1+2\,t^4+t^6+t^8+2\,t^{10}+t^{14}-t^{16})}{(1-t^2)\,(1-t^6)\,(1-t^8)\,(1-t^{12})} \\
\mathrm{dim}_5(t) &:= \frac{t^{10}\,(1+2\,t^2+3\,t^4+3\,t^6+2\,t^8)}{(1-t^4)^2\,(1-t^6)^2\,(1-t^{10})} \\
\mathrm{dim}_6(t) &:= \frac{t^{12}\,(1+2\,t^2+3\,t^4+\cdots+2\,t^{24}-t^{32}+t^{34})}{(1-t^2)\,(1-t^4)\,(1-t^6)\,(1-t^8)\,(1-t^{12})\,(1-t^{18})}
\end{aligned}$$

[81] Thus $dim_r(t) = \sum dim_{r,d}\, t^d$ etc.

$$\begin{aligned}
\mathrm{dimelem}_1(t) &= \frac{t^2}{(1-t^2)}\\
\mathrm{dimelem}_2(t) &= \frac{t^6}{(1-t^2)\,(1-t^6)}\\
\mathrm{dimelem}_3(t) &= \frac{t^8\,(1-t^2+t^4)}{(1-t^2)\,(1-t^4)\,(1-t^6)}\\
\mathrm{dimelem}_4(t) &= \frac{t^{10}\,(1+t^2+2\,t^4+t^6+2\,t^8+t^{10}-t^{16})}{(1-t^2)\,(1-t^6)\,(1-t^8)\,(1-t^{12})}\\
\mathrm{dimelem}_5(t) &= \frac{t^{12}\,(1+2\,t^2+t^4-t^6-2\,t^8-t^{12}-t^{14}+t^{18})}{(1-t^2)\,(1-t^4)^2\,(1-t^6)^2\,(1-t^{10})}\\
\mathrm{dimelem}_6(t) &= \frac{t^{14}\,(1+2\,t^2+4\,t^4+\cdots-t^{28}-t^{30}-t^{32})}{(1-t^2)\,(1-t^4)\,(1-t^6)\,(1-t^8)\,(1-t^{12})\,(1-t^{18})}
\end{aligned}$$

$$\begin{aligned}
\mathrm{dimexcep}_i(t) &= 0 \quad \textit{for}\;\; i=1,2,3\\
\mathrm{dimexcep}_4(t) &= \frac{t^8}{(1-t^4)\,(1-t^6)}\\
\mathrm{dimexcep}_5(t) &= \frac{t^{10}}{(1-t^2)\,(1-t^4)\,(1-t^6)}\\
\mathrm{dimexcep}_6(t) &= \frac{t^{12}\,(1-t^4-2\,t^6+2\,t^8)}{(1-t^2)^2\,(1-t^4)^2\,(1-t^6)^2}
\end{aligned}$$

The exact numerators in $dim_6(t)$ and $dimelem_6(t)$ are respectively

$$t^{12}\;.\;(1+2\,t^2+3\,t^4+4\,t^6+6\,t^8+6\,t^{10}+6\,t^{12}+7\,t^{14}+4\,t^{16}+5\,t^{18}+4\,t^{20}+2\,t^{22}+2\,t^{24}-t^{32}+t^{34})$$

$$t^{14}\;.\;(1+2\,t^2+4\,t^4+5\,t^6+7\,t^8+7\,t^{10}+7\,t^{12}+6\,t^{14}+6\,t^{16}+5\,t^{18}+3\,t^{20}+2\,t^{22}+t^{24}-t^{26}-t^{28}-t^{30}-t^{32})$$

$$\mathrm{dimfree}_2(t)-\mathrm{dim}_2(t)=t^2\,\mathrm{dimexcep}_4(t)=\frac{t^{10}}{(1-t^4)\,(1-t^6)} \tag{404}$$

To each missing (elementary) bialternal of depth 2 there corresponds a supernumerary (non-elementary) bialternal of depth 4, with an explicit formula giving the latter in terms of the former.[82]

5.2 Discrete-periodical bialternals.

We have a somewhat similar situation on $\mathbb{Z}/p.\mathbb{Z}$. There, the length-1 bialternals $eda_n^{\bullet}$:

$$\mathrm{eda}_n^{\binom{u_1}{v_1}}=\begin{cases}1 & \textit{if}\;\; u_1=\pm\,n \mod p\\ 0 & \textit{otherwise}\end{cases} \tag{405}$$

[82] See .... or ....

are not free under *ari*, and do not generate all bialternals. As in the polynomial case, there are 'missing bialternals' in depth 2 and 'exceptional' bialternals in depth 4. Here, however, there is no known procedure for generating the exceptional, depth-4 bialternals from the missing, depth-2 bialternals.

Moreover, when counting the dependence relations between the *ari*-brackets of the $eda_n^\bullet$, one should rule out two semi-trivial instances, involving:
(i) elements of type $eda_0^\bullet$ or $\sum_{n\neq 0} eda_n^\bullet$, which belong to the centre of ARI
(ii) for non-prime values of p, relations induced by 'earlier' relations in $\mathbb{Z}/q\,\mathbb{Z}$, with $q|p$.

The following generating series $reldisc_2^*(t)$ resp. $reldisc_2^*(t)$ enumerates the independent relations involving the all the generators $eda_n^\bullet$ with n in the interval $[1,..,[\frac{p}{2}]]$ resp. $[1,..,[\frac{p}{2}]-1]$.

$$\mathrm{reldisc}_2(t) := \frac{t^6}{(1-t)\,(1-t^2)\,(1-t^3)} \tag{406}$$

$$\mathrm{reldisc}_2^*(t) := \frac{t^8}{(1-t^2)^2\,(1-t^3)} \tag{407}$$

The first exceptional bialternal of depth 4 appears for $p=5$. It is necessarily exceptional since for $p=5$ there exist no depth-2 bialternals.

Remark: There is a distinct notion of discrete periodic bialternals, namely with indices u_i/v_i in $\mathbb{Z}/p\mathbb{Z}$ and with bimoulds *also* taking their values in $\mathbb{Z}/p\mathbb{Z}$. The bialternals there are strictly more numerous than when the bimoulds take the values in $\mathbb{Q}$ (or, what amounts to the same, $\mathbb{R}$ or $\mathbb{C}$.) but they are all induced by restricting on $\mathbb{Z}/p\mathbb{Z}$ the polynomial bialternals (see preceding section).

For p prime, though, there is no difference. Thus, in either case, for $p=2$ or 3, there are no depth-4 bialternals. For $p=5$, there is only one (-exceptional-) depth-4 bialternal. For $p=7$, there are three regular and three exceptional bialternals. Etc.

5.3 General discrete bialternals.

Finitely-supported bialternals.

Here, the picture changes. The suitably redefined elementary eda_n

$$\mathrm{eda}_n^{\binom{u_1}{v_1}} = \begin{cases} 1 & \mathit{if}\ \ u_1 = \pm\, n \\ 0 & \mathit{otherwise} \end{cases} \tag{408}$$

are *ari*-, even *preari*-independent provided we restrict ourselves to finite combinations (409).

$$S^\bullet_r = \sum_{n_1+..+n_r\leqslant Const}^{n_i>0} c^{n_1,\dots,n_r}\, \mathrm{pr\vec{e}ari}(\mathrm{eda}^\bullet_{n_1},\dots,\mathrm{eda}^\bullet_{n_r}) \tag{409}$$

Let us show that $S^\bullet_r \equiv 0$ implies $c^{\boldsymbol{n}} \equiv 0$. Assume the opposite and set $n_* = \sup_{c^{\boldsymbol{n}}\neq 0} |\boldsymbol{n}|$. Then let $\boldsymbol{n}$ be a particular sequence of length r with $|\boldsymbol{n}| = n_*$. For any j in $[1,r]$, any factorisation $\boldsymbol{n} = (\boldsymbol{n}', n_j, \boldsymbol{n}'')$, and $\boldsymbol{w}$ of the form

$$\boldsymbol{w} = (\begin{smallmatrix}\boldsymbol{u}\\ \boldsymbol{v}\end{smallmatrix}) \quad \textit{with} \quad \boldsymbol{u} = (\boldsymbol{n}', -n_*, \tilde{\boldsymbol{n}}'')$$

the identity holds

$$S^{\boldsymbol{w}} = (-1)^{r-j} \sum_{\boldsymbol{n}'''\in \mathrm{sha}(\boldsymbol{n}',\boldsymbol{n}'')} c^{n_j,\boldsymbol{n}'''} \tag{410}$$

with $\tilde{\boldsymbol{n}}''$ denoting $\boldsymbol{n}''$ in reverse order. For $j = 1$ this reduces to

$$S^{\boldsymbol{w}} = (-1)^{r-j} c^{n_1,n_2,\dots,n_r} \quad \textit{with} \quad \boldsymbol{u} = (-n_*, n_r, \dots, n_2, n_1) \tag{411}$$

implying $S^\bullet_r \neq 0$. Contradiction.

Remark: The above independence statement no longer holds if we replace the di-atomic $eda^\bullet_n$ by the mono-atomic $da^\bullet_n$ defined as in (408) but with "$u_1 = n$" in place of "$u_1 = \pm n$". Indeed, take the *ari* $\leftrightarrow$ *oddari* conversion formulae (437) or (438) *infra* and re-write them in terms of the atoms $da^\bullet_n$. They yield non-trivial *finite* sums $S^\bullet = \sum_{n_1,n_2} c_{n_1,n_2}\, ari(da^\bullet_{n_1}, da^\bullet_{n_2})$ with some non-vanishing coefficients $c_{\boldsymbol{n}}$ but an identically vanishing $S^\bullet$. The same would apply with $pr\vec{e}ari$ in place of $\vec{ari}$.

Bialternals with unbounded support.

The examples of the preceding section (with $u_i \in \mathbb{Z}/p.\mathbb{Z}$) immediately yield, for any depth $r \geqslant 2$, sums of type $S^\bullet = \sum_{n_j\in\mathbb{Z}} c_{n_1,\dots,n_r}\, \vec{ari}(da^\bullet_{n_1},\dots,da^\bullet_{n_r})$ with infinitely many non-zero coefficients $c_{n_1,\dots,n_r}$, p-periodical in each n_j, but with $S^{\boldsymbol{w}} \equiv 0$.

Bialternals with unbounded support but decreasing at infinity.

If we impose a sufficient rate of decrease on the coefficients $c_{\boldsymbol{n}}$ as $\boldsymbol{n}$ increases[83] and corresponding bounds on $|S^{\boldsymbol{w}}|$ as $\boldsymbol{w}$ increases, we recover the unicity of decomposition of $\mathbb{Z}^r$-supported bialternals as multibrackets of elementary generators $eda^\bullet_{n_j}$.

[83] Bounds of type $|c_{\boldsymbol{n}}| < \mathrm{Const.}|\boldsymbol{n}|^{-1}$ are more than enough.

5.4 Perinomal bialternals.

Standard and symbolic expansions for perinomals.

Perinomal bimoulds are meromorphic functions of either $\boldsymbol{u}$ or $\boldsymbol{v}$, but with a very peculiar pole structure: their poles lie over $\mathbb{Z}^r$ and are of *eupolar type*, i.e. they admit *standard* expansions of the form

$$S^{\binom{u_1,\dots,u_r}{v_1,\dots,v_r}} = \sum_{m_j,n_j\in\mathbb{Z}}^{1\leqslant k\leqslant\kappa_r} \mathfrak{P}_{r,k}^{\binom{u_1-m_1,\dots,u_r-m_r}{v_1-n_1,\dots,v_r-n_r}} c_{r,k}^{\binom{m_1,\dots,m_r}{n_1,\dots,n_r}} \quad \Big(\kappa_r := \frac{(2\,r)!}{r!\,(r+1)!}\Big) \tag{412}$$

Here, $\mathfrak{P}$ denotes a polar *flexion unit*, necessarily of the form:

$$\mathfrak{P}^{\binom{u_1}{v_1}} = \alpha P(u_1) + \beta P(v_1) \qquad \big(\textit{usually}\ \ \mathfrak{P}^{\binom{u_1}{v_1}} = P(u_1)\ \textit{or}\ P(v_1)\big) \tag{413}$$

and $\{\mathfrak{P}^\bullet_{r,k};1\leqslant k\leqslant\kappa_r\}$ denotes the standard basis of the length-r component $\mathit{Flex}_r(\mathfrak{P})$ of the monogenous flexion algebra generated by $\mathfrak{P}$.

The *standard* expansions (412), with their infinite sums, are rather unwieldy, especially when it comes to performing flexion operation on them. So we often replace them by the information-equivalent *symbolic* forms (414), which carry only a finite number of summands:

$$\bar{S}^{\binom{\bar{u}_1,\dots,\bar{u}_r}{\bar{v}_1,\dots,\bar{v}_r}} = \sum^{1\leqslant k\leqslant\kappa_r} \mathfrak{P}_{r,k}^{\binom{\bar{u}_1,\dots,\bar{u}_r}{\bar{v}_1,\dots,\bar{v}_r}} \mathfrak{C}_{r,k}^{\binom{\bar{u}_1,\dots,\bar{u}_r}{\bar{v}_1,\dots,\bar{v}_r}} \tag{414}$$

The change from *standard* to *symbolic* ('encoding') has the advantage of commuting with all flexion operations[84] and of being reversible ('decoding'):

$$\begin{array}{lccl}
\textit{standard}: & S^\bullet_1,S^\bullet_2 & \longrightarrow & S^\bullet_3=\mathrm{ari}(S^\bullet_1,S^\bullet_2)\ \ \textit{or}\ \ \mathrm{preari}(S^\bullet_1,S^\bullet_2)\\
 & {\scriptstyle encoding}\downarrow\uparrow{\scriptstyle decoding} & & {\scriptstyle encoding}\downarrow\uparrow{\scriptstyle decoding}\\
\textit{symbolic}: & \bar{S}^\bullet_1,\bar{S}^\bullet_2 & \longrightarrow & \bar{S}^\bullet_3=\mathrm{ari}(\bar{S}^\bullet_1,\bar{S}^\bullet_2)\ \ \textit{or}\ \ \mathrm{preari}(\bar{S}^\bullet_1,\bar{S}^\bullet_2)
\end{array}$$

Symbolic expansions for the perinomal bialternals.

Let us apply the procedure to calculate the length-r perinomal bialternals

$$\mathrm{Rai}^\bullet_r := \sum_{m_i,n_i\in\mathbb{Z}} \gamma_r^{\binom{m_1,\dots,m_r}{n_1,\dots,n_r}}\ \vec{\mathrm{ari}}\,(\mathrm{epai}^\bullet_{\binom{m_1}{n_1}},\dots,\mathrm{epai}^\bullet_{\binom{m_r}{n_r}}) \tag{415}$$

generated by the elementary bialternals

$$\mathrm{epai}^{\binom{u_1}{v_1}}_{\binom{m_1}{n_1}} := +\mathfrak{P}^{\binom{u_1-m_1}{v_1-n_1}} - \mathfrak{P}^{\binom{u_1-m_1}{v_1-n_1}} \tag{416}$$

[84] *lu/mu, swap, ari/gari, arit/garit, preari* etc. It also commutes with the full set of flexion unit identities. All these, in turn, derive from the basic (characteristic) identity: $\mathfrak{P}^{\binom{u_1}{v_1}}\mathfrak{P}^{\binom{u_2}{v_2}} \equiv \mathfrak{P}^{\binom{u_{1,2}}{v_1}}\mathfrak{P}^{\binom{u_2}{v_{2:1}}} + \mathfrak{P}^{\binom{u_{1,2}}{v_2}}\mathfrak{P}^{\binom{u_1}{v_{1:2}}}$.

Setting

$$c_r^{\binom{m_1\,,\dots,\,m_r}{n_1\,,\dots,\,n_r}} := \sum_{\epsilon_i\in\{\pm 1\}} := \epsilon_1 \dots \epsilon_r \gamma_r^{\binom{\epsilon_1 m_1\,,\dots,\,\epsilon_r m_r}{\epsilon_1 n_1\,,\dots,\,\epsilon_r n_r}} \tag{417}$$

we find the symbolic, easily decodable expansions $\bar{\mathrm{R}}\mathrm{ai}_r^{\bullet} = \sum \mathfrak{P}_{r,k}^{\bullet}\ \mathfrak{E}_{r,k}^{\bullet}$:

$$(r=1) \qquad \mathfrak{P}_{1,1}^{\binom{u_1}{v_1}} = \mathfrak{P}^{\binom{u_1}{v_1}} \quad ; \quad \mathfrak{E}_{1,1}^{\binom{u_1}{v_1}} = c_1^{\binom{u_1}{v_1}}$$

$$(r=2) \qquad \begin{cases} \mathfrak{P}_{2,1}^{\binom{u_1\,,\,u_2}{v_1\,,\,v_2}} = \mathfrak{P}^{\binom{u_{1,2}}{v_2}}\mathfrak{P}^{\binom{u_1}{v_{1:2}}} \;\; ; \;\; \mathfrak{E}_{2,1}^{\binom{m_1\,,\,m_2}{n_1\,,\,n_2}} = c_2^{\binom{m_1\,,\,m_2}{n_1\,,\,n_2}} + c_2^{\binom{m_{1,2}\,,\,m_1}{n_2\,,\,n_{1:2}}} \\ \mathfrak{P}_{2,2}^{\binom{u_1\,,\,u_2}{v_1\,,\,v_2}} = \mathfrak{P}^{\binom{u_{1,2}}{v_1}}\mathfrak{P}^{\binom{u_2}{v_{2:1}}} \;\; ; \;\; \mathfrak{E}_{2,2}^{\binom{m_1\,,\,m_2}{n_1\,,\,n_2}} = c_2^{\binom{m_1\,,\,m_2}{n_1\,,\,n_2}} - c_2^{\binom{m_{1,2}\,,\,m_2}{n_1\,,\,n_{2:1}}} \end{cases}$$

For $r=3$, the standard basis of *Flex₃* has got five elements:

$$(r=3) \qquad \begin{cases} \mathfrak{P}_{3,1}^{\binom{u_1\,,\,u_2\,,\,u_3}{v_1\,,\,v_2\,,\,v_3}} = \mathfrak{P}^{\binom{u_{1,2,3}}{v_3}}\mathfrak{P}^{\binom{u_{1,2}}{v_{2:3}}}\mathfrak{P}^{\binom{u_1}{v_{1:2}}} \\ \mathfrak{P}_{3,2}^{\binom{u_1\,,\,u_2\,,\,u_3}{v_1\,,\,v_2\,,\,v_3}} = \mathfrak{P}^{\binom{u_{1,2,3}}{v_3}}\mathfrak{P}^{\binom{u_{1,2}}{v_{1:3}}}\mathfrak{P}^{\binom{u_2}{v_{2:1}}} \\ \mathfrak{P}_{3,3}^{\binom{u_1\,,\,u_2\,,\,u_3}{v_1\,,\,v_2\,,\,v_3}} = \mathfrak{P}^{\binom{u_{1,2,3}}{v_2}}\mathfrak{P}^{\binom{u_1}{v_{1:2}}}\mathfrak{P}^{\binom{u_3}{v_{3:2}}} \\ \mathfrak{P}_{3,4}^{\binom{u_1\,,\,u_2\,,\,u_3}{v_1\,,\,v_2\,,\,v_3}} = \mathfrak{P}^{\binom{u_{1,2,3}}{v_1}}\mathfrak{P}^{\binom{u_{2,3}}{v_{3:1}}}\mathfrak{P}^{\binom{u_2}{v_{2:3}}} \\ \mathfrak{P}_{3,5}^{\binom{u_1\,,\,u_2\,,\,u_3}{v_1\,,\,v_2\,,\,v_3}} = \mathfrak{P}^{\binom{u_{1,2,3}}{v_1}}\mathfrak{P}^{\binom{u_{2,3}}{v_{2:1}}}\mathfrak{P}^{\binom{u_3}{v_{3:2}}} \end{cases}$$

and the corresponding coefficients $\mathfrak{E}_{3,k}^{\bullet}$ have got six summands each:

$$\begin{aligned}
\mathfrak{E}_{3,1}^{\binom{m_1\,,\,m_2\,,\,m_3}{n_1\,,\,n_2\,,\,n_3}} &= c_3^{\binom{m_1\,,\,m_2\,,\,m_3}{n_1\,,\,n_2\,,\,n_3}} + c_3^{\binom{m_1\,,\,m_{2,3}\,,\,m_2}{n_1\,,\,n_3\,,\,n_{2:3}}} + c_3^{\binom{m_{1,2}\,,\,m_1\,,\,m_3}{n_2\,,\,n_{1:2}\,,\,n_3}} \\
&+ c_3^{\binom{m_{1,2}\,,\,m_3\,,\,m_1}{n_2\,,\,n_3\,,\,n_{1:2}}} + c_3^{\binom{m_{1,2,3}\,,\,m_1\,,\,m_2}{n_3\,,\,n_{1:3}\,,\,n_{2:3}}} + c_3^{\binom{m_{1,2,3}\,,\,m_{1,2}\,,\,m_1}{n_3\,,\,n_{2:3}\,,\,n_{1:2}}} \\
\mathfrak{E}_{3,2}^{\binom{m_1\,,\,m_2\,,\,m_3}{n_1\,,\,n_2\,,\,n_3}} &= c_3^{\binom{m_1\,,\,m_2\,,\,m_3}{n_1\,,\,n_2\,,\,n_3}} + c_3^{\binom{m_1\,,\,m_{2,3}\,,\,m_2}{n_1\,,\,n_3\,,\,n_{2:3}}} - c_3^{\binom{m_{1,2}\,,\,m_2\,,\,m_3}{n_1\,,\,n_{2:1}\,,\,n_3}} \\
&- c_3^{\binom{m_{1,2}\,,\,m_3\,,\,m_2}{n_1\,,\,n_3\,,\,n_{2:1}}} + c_3^{\binom{m_{1,2,3}\,,\,m_1\,,\,m_2}{n_3\,,\,n_{1:3}\,,\,n_{2:3}}} - c_3^{\binom{m_{1,2,3}\,,\,m_{1,2}\,,\,m_2}{n_3\,,\,n_{1:3}\,,\,n_{2:1}}} \\
\mathfrak{E}_{3,3}^{\binom{m_1\,,\,m_2\,,\,m_3}{n_1\,,\,n_2\,,\,n_3}} &= c_3^{\binom{m_1\,,\,m_2\,,\,m_3}{n_1\,,\,n_2\,,\,n_3}} + c_3^{\binom{m_{1,2}\,,\,m_1\,,\,m_3}{n_2\,,\,n_{1:2}\,,\,n_3}} + c_3^{\binom{m_{1,2}\,,\,m_3\,,\,m_1}{n_2\,,\,n_3\,,\,n_{1:2}}} \\
&- c_3^{\binom{m_1\,,\,m_{2,3}\,,\,m_3}{n_1\,,\,n_2\,,\,n_{3:2}}} - c_3^{\binom{m_{1,2,3}\,,\,m_1\,,\,m_3}{n_2\,,\,n_{1:2}\,,\,n_{3:2}}} - c_3^{\binom{m_{1,2,3}\,,\,m_3\,,\,m_1}{n_2\,,\,n_{3:2}\,,\,n_{1:2}}} \\
\mathfrak{E}_{3,4}^{\binom{m_1\,,\,m_2\,,\,m_3}{n_1\,,\,n_2\,,\,n_3}} &= c_3^{\binom{m_1\,,\,m_2\,,\,m_3}{n_1\,,\,n_2\,,\,n_3}} + c_3^{\binom{m_1\,,\,m_{2,3}\,,\,m_2}{n_1\,,\,n_3\,,\,n_{2:3}}} - c_3^{\binom{m_{1,2}\,,\,m_2\,,\,m_3}{n_1\,,\,n_{2:1}\,,\,n_3}} \\
&- c_3^{\binom{m_{1,2}\,,\,m_3\,,\,m_2}{n_1\,,\,n_3\,,\,n_{2:1}}} + c_3^{\binom{m_{1,2,3}\,,\,m_3\,,\,m_2}{n_1\,,\,n_{3:1}\,,\,n_{2:1}}} - c_3^{\binom{m_{1,2,3}\,,\,m_{2,3}\,,\,m_2}{n_1\,,\,n_{3:1}\,,\,n_{2:3}}} \\
\mathfrak{E}_{3,5}^{\binom{m_1\,,\,m_2\,,\,m_3}{n_1\,,\,n_2\,,\,n_3}} &= c_3^{\binom{m_1\,,\,m_2\,,\,m_3}{n_1\,,\,n_2\,,\,n_3}} - c_3^{\binom{m_{1,2}\,,\,m_2\,,\,m_3}{n_1\,,\,n_{2:1}\,,\,n_3}} - c_3^{\binom{m_{1,2}\,,\,m_3\,,\,m_2}{n_1\,,\,n_3\,,\,n_{2:1}}} \\
&- c_3^{\binom{m_1\,,\,m_{2,3}\,,\,m_3}{n_1\,,\,n_2\,,\,n_{3:2}}} + c_3^{\binom{m_{1,2,3}\,,\,m_3\,,\,m_2}{n_1\,,\,n_{3:1}\,,\,n_{2:1}}} + c_3^{\binom{m_{1,2,3}\,,\,m_{2,3}\,,\,m_3}{n_1\,,\,n_{2:1}\,,\,n_{3:2}}}
\end{aligned}$$

Perinomal rigidity.

In practice, the important perinomal bialternals depend only on one set of variables ($\boldsymbol{u}$ or $\boldsymbol{v}$):

$$\mathrm{Ra}^\bullet_r := \sum_{m_i \in \mathbb{N}^*} \gamma_r^{(m_1,\dots,m_r)} \ \vec{\mathrm{ari}}\,(\mathrm{epa}^\bullet_{m_1},\dots,\mathrm{epa}^\bullet_{m_r}) \tag{418}$$

$$\mathrm{Ri}^\bullet_r := \sum_{n_i \in \mathbb{N}^*} \gamma_r^{(n_1,\dots,n_r)} \ \vec{\mathrm{ari}}\,(\mathrm{epi}^\bullet_{n_1},\dots,\mathrm{epi}^\bullet_{n_r}) \tag{419}$$

They correspond to the one-variable flexion units, and are generated by the elementary $epa^\bullet_m$ or $epi^\bullet_n$:

$$\mathrm{epa}_{m_1}^{\binom{u_1}{v_1}} := P(u_1 - m_1) - P(u_1 + m_1) \tag{420}$$

$$\mathrm{epi}_{n_1}^{\binom{u_1}{v_1}} := P(v_1 - n_1) - P(v_1 + n_1) \tag{421}$$

One obtains their symbolic (and standard) expansions by specialising the earlier formulae for $\mathfrak{Rai}^\bullet$. One simply replaces $c^\bullet_r$ by $ca^\bullet$ or $ci^\bullet$:

$$\begin{array}{rcll} ca_r^{m_1,\dots,m_r} & := & \mathrm{sgn}(m_1)...\mathrm{sgn}(m_r) \ \ \gamma_r^{|m_1|,\dots,|m_r|} & \quad (m_i \in \mathbb{Z}^*) \\ ci_r^{n_1,\dots,n_r} & := & \mathrm{sgn}(n_1)...\mathrm{sgn}(n_r) \ \ \gamma_r^{|n_1|,\dots,|n_r|} & \quad (n_i \in \mathbb{Z}^*) \end{array}$$

and neglects in $\mathfrak{E}^\bullet_{r,k}$ the irrelevant sequence of indices (either $\boldsymbol{n}$ or $\boldsymbol{m}$).

The main fact about the expansions (418) or (419) is their uniqueness:

$$\{\mathrm{Ra}^\bullet_r = 0\} \Leftrightarrow \{\gamma_r^{m_1,\dots,m_r} \equiv 0\} \quad , \quad \{\mathrm{Ri}^\bullet_r = 0\} \Leftrightarrow \{\gamma_r^{n_1,\dots,n_r} \equiv 0\} \tag{422}$$

There even exists an effective algorithm for deducing the $\gamma^\bullet_r$ from the $\mathfrak{E}^\bullet_{r,k}$. These facts are central to the perinomal decomposition of multizetas into irreducibles.

5.5 Comparing various flexion settings.

Two operations producing depth-2 bialternality: *ari* **and** *oddari* **.**

By suitably modifying the signs in front of the six summands of $ari(A^\bullet, B^\bullet)$ for length-1 bimoulds $A^\bullet, B^\bullet$, we can define a pseudo-bracket[85] *oddari* that

[85] *pseudo* because *oddari* cannot be extended to a genuine Lie bracket for factors $A^\bullet, B^\bullet$ of arbitrary lengths.

turns each pair $(A^\bullet, B^\bullet)$ of *odd*[86], length-1 bimoulds into a length-2 bialternal – exactly as *ari* does with pairs of *even* bimoulds.

$$\begin{array}{lccc} \text{ari}: & \text{ARI}_1^{\text{even}} \times \text{ARI}_1^{\text{even}} & \longrightarrow & \text{ARI}_2^{\underline{\text{al}}/\underline{\text{al}}} \\ \text{oddari}: & \text{ARI}_1^{\text{odd}} \times \text{ARI}_1^{\text{odd}} & \longrightarrow & \text{ARI}_2^{\underline{\text{al}}/\underline{\text{al}}} \end{array}$$

Here are the definitions, with $C^\bullet := ari(A^\bullet, B^\bullet)$ and $D^\bullet := oddari(A^\bullet, B^\bullet)$:

$$C^{\binom{u_1, u_2}{v_1, v_2}} = \begin{cases} +A^{\binom{u_1}{v_1}} B^{\binom{u_2}{v_2}} + A^{\binom{u_{1,2}}{v_2}} B^{\binom{u_1}{v_{1:2}}} - A^{\binom{u_{1,2}}{v_1}} B^{\binom{u_2}{v_{2:1}}} \\ -B^{\binom{u_1}{v_1}} A^{\binom{u_2}{v_2}} - B^{\binom{u_{1,2}}{v_2}} A^{\binom{u_1}{v_{1:2}}} + B^{\binom{u_{1,2}}{v_1}} A^{\binom{u_2}{v_{2:1}}} \end{cases} \tag{423}$$

$$D^{\binom{u_1, u_2}{v_1, v_2}} = \begin{cases} +A^{\binom{u_1}{v_1}} B^{\binom{u_2}{v_2}} - A^{\binom{u_{1,2}}{v_2}} B^{\binom{u_1}{v_{1:2}}} + A^{\binom{u_{1,2}}{v_1}} B^{\binom{u_2}{v_{2:1}}} \\ -B^{\binom{u_1}{v_1}} A^{\binom{u_2}{v_2}} + B^{\binom{u_{1,2}}{v_2}} A^{\binom{u_1}{v_{1:2}}} - B^{\binom{u_{1,2}}{v_1}} A^{\binom{u_2}{v_{2:1}}} \end{cases} \tag{424}$$

Due to the rigidity statements of the preceding sections, there must exist, in each setting, precise formulae for converting *oddari*-brackets into sums of *ari*-brackets, and vice versa. Even when there is no rigidity and therefore no uniqueness, as with polynomial-valued bialternals, there exist *privileged* formulae. In any case, the conversion formulae bring the specificity of each setting into sharp relief.

The *ari-oddari* conversion for polynomial-valued bialternals.

Consider the elementary bialternals

$$\text{esa}_{d_1}^{\binom{u_1}{v_1}} := u_1^{d_1} \qquad (\textit{for } d_1 \textit{ even} \geqslant 2) \tag{425}$$

$$\text{osa}_{\delta_1}^{\binom{u_1}{v_1}} := u_1^{\delta_1} \qquad (\textit{for } \delta_1 \textit{ even} \geqslant 1) \tag{426}$$

$$\text{eesa}_{d_1,d_2}^\bullet := \text{ari}(\text{esa}_{d_1}^\bullet, \text{esa}_{d_2}^\bullet) \qquad (d_1, d_2 \;\; \textit{even}) \tag{427}$$

$$\text{oosa}_{\delta_1,\delta_2}^\bullet := \text{oddari}(\text{osa}_{\delta_1}^\bullet, \text{osa}_{\delta_2}^\bullet) \qquad (\delta_1, \delta_2 \;\; \textit{odd}) \tag{428}$$

and let χ_{2k} (resp. τ_{2k}, θ_{2k}) be the integers (resp. rationals) defined by:

$$\frac{t^6}{(1-t^2)(1-t^6)} = \sum \chi_{2k}\, t^{2k} \tag{429}$$

$$-\frac{t}{\tanh(t/2)} = \sum_{0 \leqslant k} \tau_{2k}\, t^{2k} \quad , \quad -\frac{\tanh(t/2)}{t} = \sum_{0 \leqslant k} \theta_{2k}\, t^{2k} \tag{430}$$

[86] i.e. with A^{w_1}, B^{w_1} odd functions of w_1.

Proposition 5.1 (First *ari-oddari* conversion law.)

$$\frac{1}{\delta_1!}\,\mathrm{oosa}^{\bullet}_{\delta_1,\delta_2} \;:=\; \sum_{1+\delta_1\leqslant d_1}^{\delta_1+\delta_2=d_1+d_2} \tau_{1+\delta_1-d_1}\,\frac{1}{d_1!}\,\mathrm{eesa}^{\bullet}_{d_1,d_2} \tag{431}$$

$$\frac{1}{d_1!}\,\mathrm{eesa}^{\bullet}_{d_1,d_2} \;:=\; \sum_{d_1\leqslant 1+\delta_1}^{d_1+d_2=\delta_1+\delta_2} \theta_{d_1-1-\delta_1}\,\frac{1}{\delta_1!}\,\mathrm{oosa}^{\bullet}_{\delta_1,\delta_2} \tag{432}$$

Remarkably, the above identities are valid for all pairs (δ_1,δ_2) (resp. (d_1,d_2)), not just when $\frac{1+\delta_1}{2}\leqslant\chi_{\delta_1+\delta_2}$ (resp. $\frac{d_1}{2}\leqslant\chi_{d_1+d_2}$). Simply, in this case the expansions on the right-hand sides of (431) and (432) are unique.[87]

The *ari-oddari* conversion for discrete bialternals.

Let δ be the discrete dirac ($\delta(0):=1,\delta(n):=0$ *if* $n\neq 0$) and consider the elementary bialternals

$$\mathrm{eda}_{n_1}^{\binom{u_1}{v_1}} \;:=\; \delta(u_1-n_1)+\delta(u_1+n_1) \qquad \big(\ \textit{or}\quad \sinh(n_1\,u_1)\ \big) \tag{433}$$

$$\mathrm{oda}_{n_1}^{\binom{u_1}{v_1}} \;:=\; \delta(u_1-n_1)-\delta(u_1+n_1) \qquad \big(\ \textit{or}\quad \cosh(n_1\,u_1)\ \big) \tag{434}$$

$$\mathrm{eeda}^{\bullet}_{n_1,n_2}:=\mathrm{ari}(\mathrm{eda}^{\bullet}_{n_1},\mathrm{ena}^{\bullet}_{n_2})\ ,\ \mathrm{ooda}^{\bullet}_{n_1,n_2}:=\mathrm{oddari}(\mathrm{oda}^{\bullet}_{n_1},\mathrm{oda}^{\bullet}_{n_2}) \tag{435}$$

together with the operator $\mathfrak{f}$:

$$(\mathfrak{f}M)_{n_1,n_2} \;:=\; \begin{cases} 0 & \textit{if}\ n_1=n_2 \\ M_{n_1,n_2-n_1} & \textit{if}\ \ n_2>n_1 \\ M_{n_1-n_2,n_2} & \textit{if}\ \ n_1>n_2 \end{cases} \tag{436}$$

Due to the statements of §5.3, the conversion law here is rigidly determined:

Proposition 5.2 (Second *ari-oddari* conversion law.)

$$\mathrm{ooda}^{\bullet}_{n_1,n_2} \;=\; \mathrm{eeda}^{\bullet}_{n_1,n_2}+2\sum_{1\leqslant k}(\mathfrak{f}^k\,\mathrm{eeda})^{\bullet}_{n_1,n_2} \tag{437}$$

$$\mathrm{eeda}^{\bullet}_{n_1,n_2} \;=\; \mathrm{ooda}^{\bullet}_{n_1,n_2}+2\sum_{1\leqslant k}(-1)^k(\mathfrak{f}^k\,\mathrm{ooda})^{\bullet}_{n_1,n_2} \tag{438}$$

The two sums $\sum_{1\leqslant k}$ are actually finite.

[87]When we don‘t have $\frac{1+\delta_1}{2}\leqslant\chi_{\delta_1+\delta_2}$ (resp. $\frac{d_1}{2}\leqslant\chi_{d_1+d_2}$), the conversion formula is not rigidly determined, but the simplest expansions are still given by (431) (resp. (432)).

The *ari-oddari* **conversion for perinomal bialternals.**

Consider now the polar-perinomal bialternals

$$\mathrm{epa}_{n_1}^{\binom{u_1}{v_1}} \;:=\; P(u_1-n_1) - P(u_1+n_1) \tag{439}$$
$$\mathrm{opa}_{n_1}^{\binom{u_1}{v_1}} \;:=\; P(u_1-n_1) + P(u_1+n_1) \tag{440}$$

$$\mathrm{eepa}^{\bullet}_{n_1,n_2} := \mathrm{ari}(\mathrm{epa}^{\bullet}_{n_1},\mathrm{epa}^{\bullet}_{n_2}) \quad , \quad \mathrm{oopa}^{\bullet}_{n_1,n_2} := \mathrm{oddari}(\mathrm{opa}^{\bullet}_{n_1},\mathrm{opa}^{\bullet}_{n_2}) \tag{441}$$

together with the operator $\mathfrak{g}$.

$$(\mathfrak{g}M)_{n_1,n_2} \;:=\; M_{n_1,n_2+n_1} + M_{n_1+n_2,n_2} \tag{442}$$

Here again, the conversion formulae are rigidly determined:

Proposition 5.3 (Third *ari-oddari* **conversion law)** .

$$\mathrm{oopa}^{\bullet}_{n_1,n_2} \;=\; -\mathrm{eepa}^{\bullet}_{n_1,n_2} - 2\sum_{1\leqslant k}(\mathfrak{g}^{\,k}\,\mathrm{eepa})^{\bullet}_{n_1,n_2} \tag{443}$$
$$\mathrm{eepa}^{\bullet}_{n_1,n_2} \;=\; -\mathrm{oopa}^{\bullet}_{n_1,n_2} - 2\sum_{1\leqslant k}(-1)^k(\mathfrak{g}^{\,k}\,\mathrm{oopa})^{\bullet}_{n_1,n_2} \tag{444}$$

The two sums $\sum_{1\leqslant k}$ *are always infinite.*

Remark 1: The conversion formulae for the swappees

$$(\mathrm{epa}^{\bullet}_{n},\mathrm{opa}^{\bullet}_{n}) \xrightarrow{\text{swap}} (\mathrm{epi}^{\bullet}_{n},\mathrm{opi}^{\bullet}_{n})$$

retain their form, but with a sign change in the structure constants.

Remark 2: The change from δ to *exp* also involves a sign change in the structure constants, because it amounts to a Fourier transform, which itself amounts to a *swap* transform. This explains why in (433)-(434) $eda^{\bullet}_{n_1}$ may be replaced by $sinh(n_1u_1)$ and $oda^{\bullet}_{n_1}$ may be replaced by $cosh(n_1u_1)$, despite opposite parities.

5.6 'Arithmetical' or 'perinomal' generators.

According to the scheme of §3.4. any given system of generators $\{l\o ma^{\bullet}_{\|s}\}$ of $ARI^{al/il}_{ent}$ leads to a systems $\{\rho^{s_1,\dots,s_r}\}$ of multizeta irreducibles. In the case of monocolours, the best way to overcome the nuisance of 'retro-action' is to resort to the well-defined system of *perinomal* generators $\{luma^{\bullet}_{\|s}\}$, whose characteristic property is that they sum to a bimould $luma^\bullet = \sum luma^\bullet_{\|s}$, each component of which is meromorphic in $\boldsymbol{u}$, with perinomal multi-poles over the multi-integers. We can then take full advantage of the strong *rigidity* properties of these functions, of which we caught a glimpse in §5.4.

But two parallel systems of generators, $\{lama^\bullet_{\|s}\}$ and $\{loma^\bullet_{\|s}\}$, also recommend themselves to attention on account of their arithmetical simplicity: they possess only small prime factors on their denominators. Of the two, $\{loma^\bullet_{\|s}\}$ is (slightly) arithmetically less simple, but it carries a fewer number of distinct coefficients, as a result of sharing the basic symmetry properties[88] of $\{luma^\bullet_{\|s}\}$.

We shall now describe in detail all three systems up to length 4 inclusively[89] – not just for their own sake, but also to derive three parallel systems of exceptional bialternals of lentgth 4 (they are presumed to be the only ones).

The alternative aritmetical/perinomal.

The $l\o ma^\bullet$ denerators up to length 4.

Following the general scheme of §3.5 and setting

$$\mathrm{slang}_{r_1,\dots,r_n} \quad := \quad \mathrm{adari}(\mathrm{pal}^\bullet)\,\mathrm{slank}_{r_1,\dots,r_n} \tag{445}$$

we can express the first four components of the generic element $l\o ma^\bullet$ of $ARI^{\underline{al}/\underline{il}}$ with the help of just two singulands $S\o^\bullet_1$ and $S\o^\bullet_{1,2}$. We find:

$$\mathrm{l\o ma}^{u_1} \quad := \quad (\mathrm{slank}_1.\mathrm{S\o}_1)^{u_1} \;=\; \mathrm{S\o}_1^{u_1} \tag{446}$$

$$\begin{aligned}\mathrm{l\o ma}^{u_1,u_2} \quad &:= \quad (\mathrm{slank}_1.\mathrm{S\o}_1)^{u_1,u_2} \\ &= \quad \frac{1}{2}\begin{cases}\mathrm{S\o}_1^{u_1}\,P(u_2) - \mathrm{S\o}_1^{u_1}\,P(u_{12}) - \mathrm{S\o}_1^{u_2}\,P(u_1)\\ \mathrm{S\o}_1^{u_2}\,P(u_{12}) - \mathrm{S\o}_1^{u_{12}}\,P(u_2) + \mathrm{S\o}_1^{u_{12}}\,P(u_1)\end{cases}\end{aligned} \tag{447}$$

$$\mathrm{l\o ma}^{u_1,u_2,u_3} \quad := \quad (\mathrm{slank}_1.\mathrm{S\o}_1)^{u_1,u_2,u_3} + (\mathrm{slank}_{1,2}.\mathrm{S\o}_{1,2})^{u_1,u_2,u_3} \tag{448}$$

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[88]

[89] We already gave a cursory treatment of these question in [...], but it seems to have been misunderstood in some quarters. In any case, the detailed arithmetical description of the singulands $Sa^\bullet_{1,2}$ and $So^\bullet_{1,2}$ and of their coefficients given towards the end of this section is new.

Or explicitly:

$$
\begin{aligned}
&\mathrm{løma}^{u_1,u_2,u_3} := \\
&+\mathrm{Sø}_1^{u_1}\begin{cases} \frac{1}{3}\,P(u_2)\,P(u_{23}) - \frac{1}{4}\,P(u_2)\,P(u_{123}) - \frac{1}{12}\,P(u_{23})\,P(u_3) \\ -\frac{1}{12}\,P(u_{12})\,P(u_{123}) + \frac{1}{12}\,P(u_3)\,P(u_{123}) \end{cases} \\
&+\mathrm{Sø}_1^{u_2}\begin{cases} \frac{1}{6}\,P(u_{12})\,P(u_3) - \frac{1}{4}\,P(u_1)\,P(u_3) + \frac{1}{12}\,P(u_1)\,P(u_{23}) \\ -\frac{1}{12}\,P(u_{12})\,P(u_{123}) + \frac{1}{12}\,P(u_1)\,P(u_{123}) \end{cases} \\
&+\mathrm{Sø}_1^{u_3}\begin{cases} \frac{1}{3}\,P(u_2)\,P(u_{12}) - \frac{1}{4}\,P(u_2)\,P(u_{123}) + \frac{1}{6}\,P(u_1)\,P(u_{123}) \\ -\frac{1}{12}\,P(u_1)\,P(u_{12}) - \frac{1}{12}\,P(u_1)\,P(u_{23}) \end{cases} \\
&+\mathrm{Sø}_1^{u_{123}}\begin{cases} \frac{1}{3}\,P(u_1)\,P(u_{12}) + \frac{1}{3}\,P(u_{23})\,P(u_3) - \frac{1}{4}\,P(u_1)\,P(u_3) \\ -\frac{1}{12}\,P(u_2)\,P(u_{23}) - \frac{1}{12}\,P(u_2)\,P(u_{12}) \end{cases} \\
&+\mathrm{Sø}_1^{u_{12}}\begin{cases} \frac{1}{4}\,P(u_1)\,P(u_3) - \frac{1}{4}\,P(u_2)\,P(u_3) \\ +\frac{1}{4}\,P(u_2)\,P(u_{123}) - \frac{1}{4}\,P(u_1)\,P(u_{123}) \end{cases} \\
&+\mathrm{Sø}_1^{u_{23}}\begin{cases} \frac{1}{4}\,P(u_1)\,P(u_3) - \frac{1}{4}\,P(u_1)\,P(u_2) \\ +\frac{1}{4}\,P(u_2)\,P(u_{123}) - \frac{1}{4}\,P(u_3)\,P(u_{123}) \end{cases} \\
&-\tfrac{1}{2}\,\mathrm{Sø}_{1,2}^{u_1,u_2}\big(P(u_3)+P(u_{23})\big) + \tfrac{1}{2}\,\mathrm{Sø}_{1,2}^{u_2,u_1}\big(P(u_{23})+P(u_{123})\big) \\
&+\tfrac{1}{2}\,\mathrm{Sø}_{1,2}^{u_2,u_3}\big(P(u_{12})+P(u_{123})\big) - \tfrac{1}{2}\,\mathrm{Sø}_{1,2}^{u_3,u_2}\big(P(u_1)+P(u_{12})\big) \\
&+\tfrac{1}{2}\,\mathrm{Sø}_{1,2}^{u_1,u_{12}}\big(P(u_3)+P(u_{123})\big) + \tfrac{1}{2}\,\mathrm{Sø}_{1,2}^{u_1,u_{23}}\big(P(u_2)-P(u_3)\big) \\
&-\tfrac{1}{2}\,\mathrm{Sø}_{1,2}^{u_2,u_{12}}\big(P(u_3)+P(u_{123})\big) - \tfrac{1}{2}\,\mathrm{Sø}_{1,2}^{u_2,u_{23}}\big(P(u_1)+P(u_{123})\big) \\
&-\tfrac{1}{2}\,\mathrm{Sø}_{1,2}^{u_3,u_{12}}\big(P(u_1)-P(u_2)\big) + \tfrac{1}{2}\,\mathrm{Sø}_{1,2}^{u_3,u_{23}}\big(P(u_1)+P(u_{123})\big) \\
&-\tfrac{1}{2}\,\mathrm{Sø}_{1,2}^{u_{123},u_1}\big(P(u_2)+P(u_{12})\big) + \tfrac{1}{2}\,\mathrm{Sø}_{1,2}^{u_1,u_{123}}\big(P(u_3)-P(u_{12})\big) \\
&-\tfrac{1}{2}\,\mathrm{Sø}_{1,2}^{u_{123},u_3}\big(P(u_2)+P(u_{23})\big) + \tfrac{1}{2}\,\mathrm{Sø}_{1,2}^{u_3,u_{123}}\big(P(u_1)-P(u_{23})\big) \\
&+\tfrac{1}{2}\,\mathrm{Sø}_{1,2}^{u_{123},u_{12}}\big(P(u_1)-P(u_2)\big) + \tfrac{1}{2}\,\mathrm{Sø}_{1,2}^{u_{123},u_{23}}\big(P(u_3)-P(u_2)\big) \\
&+\tfrac{1}{2}\,\mathrm{Sø}_{1,2}^{u_1,u_3}\big(P(u_2)-P(u_{12})+P(u_{23})-P(u_{123})\big) \\
&+\tfrac{1}{2}\,\mathrm{Sø}_{1,2}^{u_3,u_1}\big(P(u_2)+P(u_{12})-P(u_{23})-P(u_{123})\big) \\
&+\tfrac{1}{2}\,\mathrm{Sø}_{1,2}^{u_{123},u_2}\big(P(u_1)+P(u_{12})+P(u_3)+P(u_{23})\big) \\
&-\tfrac{1}{2}\,\mathrm{Sø}_{1,2}^{u_2,u_{123}}\big(P(u_1)+P(u_3)-P(u_{12})-P(u_{23})\big)
\end{aligned}
$$

$$\mathrm{løma}^{u_1,u_2,u_3,u_4} \;:=\; (\mathrm{slank}_1.\mathrm{Sø}_1)^{u_1,u_2,u_3,u_4} + (\mathrm{slank}_{1,2}.\mathrm{Sø}_{1,2})^{u_1,u_2,u_3,u_4} \tag{449}$$

We don't mention the expanded expression for $løma^{u_1,u_2,u_3,u_4}$ as it involves several hundred terms.

For any input $Sø_1^{u_1}$ even in u_1, the second component $løma^{u_1,u_2}$ as defined by the above formula is automatically polynomial in u_1, u_2. It is also an easy matter to check that the third component $l\o ma^{u_1,u_2,u_3}$ is polynomial in u_1, u_2, u_3 if and only if the singuland $S\o^{\bullet}_{1,2}$ verifies the *desingularisation criterion*:

$$0 = \begin{cases} +\mathrm{S\o}^{u_1,u_2}_{1,2} + \mathrm{S\o}^{u_2,u_{12}}_{1,2} - \mathrm{S\o}^{u_1,u_{12}}_{1,2} - \mathrm{S\o}^{u_{12},u_2}_{1,2} \\ +\frac{1}{12}\Big(P(u_2)\,\mathrm{S\o}^{u_{12}} - P(u_{12})\,\mathrm{S\o}^{u_2} - P(u_2)\,\mathrm{S\o}^{u_1} + P(u_{12})\,\mathrm{S\o}^{u_1}\Big) \end{cases} \tag{450}$$

Note that despite the presence of poles $P(.)$, the second line in (450) is automatically polynomial in u_1, u_2. Of course, when fulfilled, the desingularisation criterion (450) ensures the polynomialness not just of $l\o ma^{u_1,u_2,u_3}$ but of $l\o ma^{u_1,u_2,u_3,u_4}$ as well. To make the components of length 5 and 6 polynomial, five higher-order singulands[90] must be added, each subject to their own desingularisation criteria. And so on, for each pair $(2\,r', 2\,r'+1)$.

The first arithmetical generators $lama^{\bullet}_{\|s}/lami^{\bullet}_{\|s}$**.**

They correspond to 'lacunary' singulands $Sa^{\bullet}_{1,2}$.

Proposition 5.4 (Best aritmetical singuland $Sa^{\bullet}_{1,2}$**)** .
For any odd weight $s \geqslant 5$ *there exists a unique singuland of the form*[91]

$$\mathrm{Sa}^{u_1,u_2}_{1,2\|s} = \sum_{1\leqslant\delta\leqslant[\frac{s-1}{2}]-[\frac{s+1}{6}]} \mathrm{sa}_{2\,\delta,s-2-2\delta} \;\; u_1^{2\,\delta}\, u_2^{s-2-2\,\delta} \tag{451}$$

that verifies the desingularisation criterion (450). The largest prime factor pa_s *on the denominators of the coefficients* sa *is always* $\mathrm{pa}_s \leqslant \frac{s-1}{3}$.

Proof: It relies on the formulae:

$$\begin{aligned}
\mathrm{sa}_{4k-2m,2k+2m-1} &= \mathrm{la}_{1,m}\,\frac{2^{2m}(4k+1)!}{(2k+2m+1)!(2k)!(2m+2)!}\;\frac{(6k+1)(2k+1)}{(4k-2m)(4k-2m-1)}\;\mathrm{pa}_{1,m}(k)\\
\mathrm{sa}_{4k-2m+2,2k+2m-1} &= \mathrm{la}_{3,m}\,\frac{2^{2m}(4k+3)!}{(2k+2m+1)!(2k+1)!(2m+2)!}\frac{1}{(4k-2m+2)(4k-2m+3)}\;\mathrm{pa}_{3,m}(k)\\
\mathrm{sa}_{4k-2m+2,2k+2m+1} &= \mathrm{la}_{5,m}\,\frac{2^{2m}(4k+3)!}{(2k+2m+3)!(2k+1)!(2m+2)!}\frac{(6k+5)(k+1)}{(4k-2m+2)(4k-2m+3)}\;\mathrm{pa}_{5,m}(k)
\end{aligned}$$

(i) with simple rational coefficients $la_{i,m}$
(ii) with polynomials $pa_{i,m}(x)$ in $\mathbb{Z}[x]$[92]
(iii) of degrees: $\deg(\mathrm{pa}_{1,m}) = 4m-1$, $\deg(\mathrm{pa}_{3,m}) = 4m$, $\deg(\mathrm{pa}_{5,m}) = 4m$
(iv) and determined inductively on m by difference equations.

[90]to wit: $S\o^{\bullet}_{1,4}, S\o^{\bullet}_{2,3}, S\o^{\bullet}_{1,1,3}, S\o^{\bullet}_{1,2,2}, S\o^{\bullet}_{1,1,1,2}$.
[91]the case $s=3$ does not arise, since $l\o ma^{u_1,u_2,u_3}_{\|3} \equiv 0$.
[92]except for the term $\mathrm{pa}_{1,0}(k) = \frac{1}{2k+1}$.

The second arithmetical generators $loma^{\bullet}_{\|s}/lomi^{\bullet}_{\|s}$

They correspond to singulands $So^{\bullet}_{1,2}$ even more 'lacunary' than the earlier $Sa^{\bullet}_{1,2}$ but they are marginally less simple, arithmetically speaking. Their main feature, though, is that of sharing the fundamental symmetry of the perinomal singulands $Su^{\bullet}_{1,2}$ (see *infra*):

$$So^{u_1,u_2}_{1,2}\,u_2 \equiv So^{u_2,u_1}_{1,2}\,u_1 \quad , \quad Su^{u_1,u_2}_{1,2}\,u_2 \equiv Su^{u_2,u_1}_{1,2}\,u_1 \tag{452}$$

Proposition 5.5 (Second best arithmetical singuland $So^{\bullet}_{1,2}$**)** .
For any odd weight $s \geqslant 5$ there exists a unique singuland of the form

$$\mathrm{So}^{u_1,u_2}_{1,2\|s} = u_1^2\,u_2 \sum_{1\leqslant\delta\leqslant[\frac{s-3}{6}]} \mathrm{so}_{2\,\delta,s-2-2\delta}\;\left(u_1^{2\,\delta}\,u_2^{s-5-2\,\delta} + u_2^{2\,\delta}\,u_1^{s-5-2\,\delta}\right) \tag{453}$$

that verifies the desingularisation criterion (450). The largest prime factor po_s *on the denominators of the coefficients* so *is always* $\mathrm{po}_s \leqslant \frac{2s-5}{3}$.

Proof: Similar as in the case of $Sa^{\bullet}_{1,2}$ but based on the formulae:

$$\begin{aligned}
\mathrm{so}_{2k-2m-2,4k+2m+1} &= \mathrm{lo}_{1,m}\,\frac{2^m(6k+1)!(2k+m)!(k-1)!}{(4k+2m+1)!(4k-1)!(k-m)!(2m+2)!}\;\frac{(2k+1)}{(2k-2m-1)}\,\mathrm{po}_{1,m}(k)\\
\mathrm{so}_{4k-2m,2k+2m+1} &= \mathrm{lo}_{3,m}\,\frac{2^m(6k+1)!(2k+m)!(k-1)!}{(4k+2m+1)!(4k-1)!(k-m+1)!(2m+2)!}\frac{1}{(2k-2m+1)}\,\mathrm{po}_{3,m}(k)\\
\mathrm{so}_{2k-2m,2k+2m+3} &= \mathrm{lo}_{5,m}\,\frac{2^m(6k+3)!(2k+m+1)!(k)!}{(4k+2m+3)!(4k+2)!(k-m+1)!(2m+2)!}\frac{(6k+5)}{(2k-2m+1)}\,\mathrm{po}_{5,m}(k)
\end{aligned}$$

with $\deg(\mathrm{po}_{1,m}) = 2m-1$, $\deg(\mathrm{po}_{3,m}) = 2m+1$, $\deg(\mathrm{po}_{5,m}) = 2m+1$ and the exceptional term $\mathrm{po}_{1,0}(k) = \frac{1}{2k+1}$.

Remark about the arithmetical singulands.

If we were to look for solutions $\underline{Sa}^{\bullet}_{1,2\|s}$ of the desingularisation criterion (450) similar to $Sa^{\bullet}_{1,2\|s}$ in (451), with δ running through a support set $Da^{\bullet}_{1,2\|s}$ of the same cardinality, for instance with $Da^{\bullet}_{1,2\|s} = [1+n, [\frac{s-1}{2}] - [\frac{s+1}{6}] + n]$ for n small, we would in nearly all cases get a unique solution, *but without the bonus of small prime numbers in the denominators.*

Likewise, if we were to look for solutions $\underline{So}^{\bullet}_{1,2\|s}$ of the desingularisation criterion (450) similar to $So^{\bullet}_{1,2\|s}$ in (453), with the same symmetry constraint $\underline{So}^{u_1,u_2}_{1,2\|s}\,u_2 \equiv \underline{So}^{u_2,u_1}_{1,2\|s}\,u_1$ and with δ running through a support set $Do^{\bullet}_{1,2\|s}$ of the same cardinality, for instance with $Do^{\bullet}_{1,2\|s} = [1+n, [\frac{s+3}{6}]+n]$ for n small, we would also in nearly all cases get a unique solution, *but again without the bonus of small prime numbers in the denominators.*

The perinomal generators $luma^\bullet_{\|s}/lumi^\bullet_{\|s}$**.**

The characteristic feature of the $luma^\bullet_{\|s}$'s is that they, as also the underlying singulands, add up to meromorphic functions with *perinomal poles.*

Proposition 5.6 (Perinomal singuland $Su^\bullet_{1,2}$**)** .
Both the global meromorphic singuland $\mathrm{Su}^\bullet_{1,2}$

$$\mathrm{Su}^{u_1,u_2}_{1,2} := \sum_{n_i \in \mathbb{Z}^*} n_1 P(u_1+n_1)P(u_2+n_2) = \sum_{s\ odd} \mathrm{Su}^{u_1,u_2}_{1,2\|s} \tag{454}$$

and its homogenous components $\mathrm{Su}^\bullet_{1,2\|s}$

$$\mathrm{Su}^{u_1,u_2}_{1,2\|s} \quad = \quad \frac{1}{12} \sum_{1\leqslant \delta_1,\delta_2 \leqslant \frac{s-3}{2}}^{\delta_1+\delta_2=\frac{s-3}{2}} \mathrm{su}_{2\,\delta_1,2\delta_2+1}\ \ u_1^{2\,\delta_1}\, u_2^{2\,\delta_2+1} \tag{455}$$

$$\textit{with} \quad \begin{cases} \mathrm{su}_{2\,\delta_1,2\,\delta_2+1} := \frac{\beta_{2\,\delta_1}\ \beta_{2\,\delta_2}}{\beta_{2\,\beta_1+2\,\beta_2}} = \frac{\beta_{2\,\delta_1}\ \beta_{2\,\delta_2}}{\beta_{s-3}} \\ \beta_{2\,\delta} := \frac{\mathrm{Bernoulli}(2\,\delta)}{(2\,\delta)!} \ \Leftrightarrow\ \sum_{0\leqslant\delta} \beta_{2\,\delta}\, t^{2\,\delta} := \frac{1}{2}\frac{e^t+1}{e^t-1} \end{cases} \tag{456}$$

verify the desingularisation criterion (450). They cannot be beaten for explicitness, but the denominators β_{s-3} *in their coefficients sets them apart from the 'arithmetical' singulates.*

The associated exceptional bialnernals.

For any system $\{l\o ma^\bullet_{\|s}; s = 3, 5...\}$, a combination of type

$$\mathrm{h\o}^\bullet \quad := \sum_{s_1+s_2=s}^{s_i \geqslant 3} c_{s_1,s_2}\ \mathrm{ari}(\mathrm{l\o ma}^\bullet_{\|s_1}, \mathrm{l\o ma}^\bullet_{\|s_2}) \tag{457}$$

has a length-4 component $h\o^\bullet_4$ that is *bialternal* if and only if its length-2 component $h\o^\bullet_2$ (and *therefore* $h\o^\bullet_3$ too) vanish. That condition in turn is equivalent to:

$$0 \ \equiv \sum_{s_1+s_2=s}^{s_i\geqslant 3} c_{s_1,s_2}\mathrm{ari}(\mathrm{ekma}^\bullet_{\|s_1}, \mathrm{ekma}^\bullet_{\|s_2}) \tag{458}$$

$$\textit{with} \quad \begin{cases} \mathrm{ekma}^{w_1}_{\|s} := u_1^{s-1} \\ \mathrm{ekma}^{w_1,\dots,w_r}_{\|s} := 0 \quad \textit{if} \ \ r>1 \end{cases} \tag{459}$$

Proposition 5.7 (Distinguished pre-*cørma* relations) . *Let*

$$\sigma_2(s) := [\frac{s+4}{12}] + [\frac{s-2}{12}] \quad , \quad \sigma_4^*(s) := [\frac{s+4}{12}] - [\frac{s-2}{12}] + [\frac{s-4}{12}] \tag{460}$$

For any even weight $s \geqslant 8$ there exist $\sigma_2(s)$ independent bialternals of weight 2, and for any even weight $s \geqslant 16$ and $\neq 14$, there exist exactly $\sigma_4^*(s)$ dependence relations between the bialternals of weight s. Amongst these, we have an arithmetically privileged system. Indeed, for $1 \leqslant k \leqslant \sigma^*(s)$, we find

$$0 = \begin{cases} +\mathrm{ari}\big(\mathrm{ekma}^{\bullet}_{\|1+2\sigma_2(s)+k}, \mathrm{ekma}^{\bullet}_{\|s-1-2\sigma_2(s)+k}\big) \\ +\sum_{1\leqslant\delta\leqslant\sigma_2(s)} c^k_{1+2\delta,s-1-2\delta} \ \ \mathrm{ari}\big(\mathrm{ekma}^{\bullet}_{\|1+2\delta}, \mathrm{ekma}^{\bullet}_{\|s-1-2\delta}\big) \end{cases} \tag{461}$$

with rational coefficients $c^k_{1+2\delta,s-1-2\delta}$ that are arithmetically regular in the sense that the largest prime factor p *on their denominators is always $\leqslant s-5$.*

Proof: It relies on formulae closely parallel to those mentioned *supra* for the coefficients of the singulands $Sa^{\bullet}_{1,2}$, $So^{\bullet}_{1,2}$ and their coefficients.

The bottom-line is that to any system $\{l\o ma^{\bullet}_{\|s}; s = 3, 5..\}$ there corresponds a system $\{c\o rma^{\bullet}_{\|s,k}; 1 \leqslant k \leqslant \sigma_4^*(s)\}$ of exceptional bialternals:

$$\mathrm{c\o rma}^{w_1,\dots,w_4}_{\|s,k} := \mathrm{h\o}^{w_1,\dots,w_4}_{\|s,k} \quad , \quad \mathrm{c\o rma}^{w_1,\dots,w_r}_{\|s,k} := \ 0 \qquad \textit{if} \quad r \neq 4 \tag{462}$$

$$\textit{with} \ \ \mathrm{h\o}^{\bullet}_{\|s,k} := \begin{cases} +\mathrm{ari}\big(\mathrm{l\o ma}^{\bullet}_{\|1+2\sigma_2(s)+k}, \mathrm{l\o ma}^{\bullet}_{\|s-1-2\sigma_2(s)+k}\big) \\ +\sum_{1\leqslant\delta\leqslant\sigma_2(s)} c^k_{1+2\delta,s-1-2\delta} \ \ \mathrm{ari}\big(\mathrm{l\o ma}^{\bullet}_{\|1+2\delta}, \mathrm{l\o ma}^{\bullet}_{\|s-1-2\delta}\big) \end{cases} \tag{463}$$

In particular, to the three systems $\{l\o ma^{\bullet}_{\|s};\ \o = a, o, u\}$ there correspond the three systems $\{c\o rma^{\bullet}_{\|s,k};\ \o = a, o, u\}$. The first two (with a or o) are arithmetically simple (no prime factors larger than $s-5$ on the denominators) and the last one is particularly explicit.

Thus, while the *elementary* length-4 bialternals (i.e. those generated by the $ekma^{\bullet}_{\|s}$'s) do not appear to possess really privileged bases, the conceptually more complex *exceptional* bialternals, strangely, do. Moreover, as we shall see in §6.4, at any given weight s, they are, though independent, yet connected by a mysterious dependence relation modulo β^*_s, where β^*_s denotes the *essential* part of the Bernoulli numerators, i.e. these numerators pruned of all their small prime factors (those less than s).

6 Complements and tables.

6.1 Basic reminders about moulds and bimoulds.

6.2 Basic reminders.

This brief subsection serves no other purpose than recalling some elementary definitions and fixing the corresponding notations.

6.1.1. Alien derivations and displays.

Alien derivations are noted Δ_ω (resp. $\hat{\Delta}_\omega$) in the multiplicative (resp. convolutive) models. In the multiplicative model, we also have the ∂_z-commuting variant $\mathbb{\Delta}_\omega$ and the corresponding z-constant pseudovariables $\mathbb{Z}^\omega$:

$$\mathbb{\Delta}_\omega := e^{-\omega z}\Delta_\omega \quad ; \quad \begin{cases} [\partial_z, \mathbb{\Delta}_\omega] = 0 \\ \ \partial_z\, \mathbb{Z}_\omega = 0 \end{cases} \tag{464}$$

From these are formed the 'displays' $dpl(\widetilde{\varphi})$, which automatically extend relations $\mathcal{R}$ involving resurgent functions $\widetilde{\varphi}_i$ and the operations $(+, \times, \circ)$:

$$\mathrm{dpl.}(\widetilde{\varphi}) := \widetilde{\varphi} + \sum_{1\leqslant r}\sum_{\omega_i} \mathbb{Z}^{\omega_1,\dots,\omega_r}\, \mathbb{\Delta}_{\omega_r}\dots\mathbb{\Delta}_{\omega_1}\widetilde{\varphi} \tag{465}$$

$$\{\mathcal{R}(\widetilde{\varphi}_1, \widetilde{\varphi}_1, \dots) \equiv 0\} \Longrightarrow \{\mathcal{R}(\mathrm{dpl}(\widetilde{\varphi}_1), \mathrm{dpl}(\widetilde{\varphi}_1), \dots) \equiv 0\} \tag{466}$$

6.1.2. Basic symmetry types for moulds and bimoulds.

$$\begin{array}{lcccll}
A^\bullet \ \textit{alternal} & \Leftrightarrow & 0 & \equiv & \sum_{\boldsymbol{\omega}\in \mathrm{sha}(\boldsymbol{\omega}',\boldsymbol{\omega}'')} A^{\boldsymbol{\omega}} & \forall \boldsymbol{\omega}',\boldsymbol{\omega}'' \\
S^\bullet \ \textit{symmetral} & \Leftrightarrow & S^{\boldsymbol{\omega}'}S^{\boldsymbol{\omega}''} & \equiv & \sum_{\boldsymbol{\omega}\in \mathrm{sha}(\boldsymbol{\omega}',\boldsymbol{\omega}'')} S^{\boldsymbol{\omega}} & \forall \boldsymbol{\omega}',\boldsymbol{\omega}'' \\
A^\bullet \ \textit{alternel} & \Leftrightarrow & 0 & \equiv & \sum_{\boldsymbol{\omega}\in \mathrm{she}(\boldsymbol{\omega}',\boldsymbol{\omega}'')} A^{\boldsymbol{\omega}} & \forall \boldsymbol{\omega}',\boldsymbol{\omega}'' \\
S^\bullet \ \textit{symmetrel} & \Leftrightarrow & S^{\boldsymbol{\omega}'}S^{\boldsymbol{\omega}''} & \equiv & \sum_{\boldsymbol{\omega}\in \mathrm{she}(\boldsymbol{\omega}',\boldsymbol{\omega}'')} S^{\boldsymbol{\omega}} & \forall \boldsymbol{\omega}',\boldsymbol{\omega}'' \\
A^\bullet \ \textit{alternil} & \Leftrightarrow & 0 & \equiv & \sum_{\boldsymbol{w}\in \mathrm{shi}(\boldsymbol{w}',\boldsymbol{w}'')} A^{\boldsymbol{w}} & \forall \boldsymbol{w}',\boldsymbol{w}'' \\
S^\bullet \ \textit{symmetril} & \Leftrightarrow & S^{\boldsymbol{w}'}S^{\boldsymbol{w}''} & \equiv & \sum_{\boldsymbol{w}\in \mathrm{shi}(\boldsymbol{w}',\boldsymbol{w}'')} S^{\boldsymbol{w}} & \forall \boldsymbol{w}',\boldsymbol{w}''
\end{array}$$

(i) $sha(\boldsymbol{\omega}', \boldsymbol{\omega}'')$ is the set of all shufflings of the sequences $\boldsymbol{\omega}'$,$\boldsymbol{\omega}'$.
(ii) $sha(\boldsymbol{\omega}', \boldsymbol{\omega}'')$ allows order-compatible contractions $\omega_i' + \omega_j''$
(iii) $sha(\boldsymbol{w}', \boldsymbol{w}'')$ allows order-compatible contractions $w_i' \oplus w_j''$ and to each such contraction (multilinearly) associates a pair:

$$\Big(A^{\binom{\dots,\, u_i'+u_j'' \,,\dots}{\dots,\ \ v_i' \ \ ,\dots}} - A^{\binom{\dots,\, u_i'+u_j'' \,,\dots}{\dots,\ \ v_j'' \ \ ,\dots}}\Big) P(v_i' - v_j'') \qquad \textit{with} \ \ P(t) := \frac{1}{t}$$

6.1.3. Basic mould operations.

$$C^\bullet = \mathrm{mu}(A^\bullet, B^\bullet) = A^\bullet \times B^\bullet \Leftrightarrow C^{\boldsymbol{u}} = \sum^{\boldsymbol{u}=\boldsymbol{u}'\boldsymbol{u}''} A^{\boldsymbol{u}'} B^{\boldsymbol{u}''}$$

$$C^\bullet = \mathrm{ko}(A^\bullet, B^\bullet) = A^\bullet \circ B^\bullet \Leftrightarrow C^{\boldsymbol{u}} = \sum_{1\leqslant s}^{\boldsymbol{u}=\boldsymbol{u}^1..\boldsymbol{u}^s} A^{|\boldsymbol{u}^1|,..,|\boldsymbol{u}^s|} B^{\boldsymbol{u}^1}..B^{\boldsymbol{u}^s}$$

$$\mathrm{lu}(A^\bullet, B^\bullet) := \mathrm{mu}(A^\bullet, B^\bullet) - \mathrm{mu}(B^\bullet, A^\bullet)$$

6.1.4. Basic bimould operations.

Systematic abbreviations: $u_{i,j,k...} := u_i + u_j + u_k \dots$, $v_{i:j} := v_i - v_j$

Main unary operations:

$$\left\{B^\bullet = \mathrm{pari}\, A^\bullet\right\} \implies \left\{B^{(w_1,\dots,w_r)} = (-1)^r A^{(w_1,\dots,w_r)}\right\} \tag{467}$$

$$\left\{B^\bullet = \mathrm{neg}\, A^\bullet\right\} \implies \left\{B^{(w_1,\dots,w_r)} = A^{(-w_1,\dots,-w_r)}\right\} \tag{468}$$

$$\left\{B^\bullet = \mathrm{anti}\, A^\bullet\right\} \implies \left\{B^{(w_1,\dots,w_r)} = A^{(w_r,\dots,w_1)}\right\} \tag{469}$$

$$\left\{B^\bullet = \mathrm{swap}\, A^\bullet\right\} \implies \left\{B^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ v_1 & ,\dots, & v_r \end{smallmatrix}\right)} = A^{\left(\begin{smallmatrix} v_r & ,\dots, & v_{3:4} & , & v_{2:3} & , & v_{1:2} \\ u_{1,\dots,r} & ,\dots, & u_{1,2,3} & , & u_{1,2} & , & u_1 \end{smallmatrix}\right)}\right\} \tag{470}$$

$$\left\{B^\bullet = \mathrm{push}\, A^\bullet\right\} \implies \left\{B^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ v_1 & ,\dots, & v_r \end{smallmatrix}\right)} = A^{\left(\begin{smallmatrix} -u_{1,\dots,r} & u_1 & , & u_2 & ,\dots, & u_{r-1} \\ -v_r & v_{1:r} & , & v_{2:r} & ,\dots, & v_{r-1:r} \end{smallmatrix}\right)}\right\} \tag{471}$$

push = neg.anti.swap.anti.swap

The four basic flexions $\rfloor, \lceil$ *and* $\rceil, \lfloor$.
They are always defined relative to a factorisation of $\boldsymbol{w}$. Thus, if $\boldsymbol{w} = \boldsymbol{w}'.\boldsymbol{w}''$ with $\boldsymbol{w}' = \left(\begin{smallmatrix} u_1 & , & u_2 \\ v_1 & , & v_2 \end{smallmatrix}\right)$ *and* $\boldsymbol{w}'' = \left(\begin{smallmatrix} u_3 & , & u_4 & , & u_5 \\ v_3 & , & v_4 & , & v_5 \end{smallmatrix}\right)$, we set:

$$\boldsymbol{w}'\rfloor = \begin{pmatrix} u_1 & , & u_2 \\ v_{1:3} & , & v_{2:3} \end{pmatrix} \qquad \lceil\boldsymbol{w}'' = \begin{pmatrix} u_{1,2,3} & , & u_4 & , & u_5 \\ v_3 & , & v_4 & , & v_5 \end{pmatrix}$$

$$\boldsymbol{w}'\rceil = \begin{pmatrix} u_1 & , & u_{2,3,4,5} \\ v_1 & , & v_2 \end{pmatrix} \qquad \lfloor\boldsymbol{w}'' = \begin{pmatrix} u_3 & , & u_4 & , & u_5 \\ v_{3:2} & , & v_{4:2} & , & v_{5:2} \end{pmatrix}$$

The ari/gari *structure.* The Lie bracket *ari*, the pre-Lie law *preari*, and the *mu*-derivation *arit*$(A^\bullet)$ are defined by:

$$N^\bullet = \mathrm{arit}(B^\bullet)M^\bullet \Leftrightarrow N^{\boldsymbol{w}} = \sum^{\boldsymbol{w}=\boldsymbol{abc}} M^{\boldsymbol{a}\lceil\boldsymbol{c}} B^{\boldsymbol{b}\rfloor} - \sum^{\boldsymbol{w}=\boldsymbol{abc}} M^{\boldsymbol{a}\rceil\boldsymbol{c}} B^{\lfloor\boldsymbol{b}}$$

$$\mathrm{ari}(A^\bullet, B^\bullet) := \mathrm{arit}(B^\bullet).A^\bullet - \mathrm{arit}(A^\bullet).B^\bullet + \mathrm{lu}(A^\bullet, B^\bullet)$$

$$\mathrm{preari}(A^\bullet, B^\bullet) := \mathrm{arit}(B^\bullet).A^\bullet + \mathrm{mu}(A^\bullet, B^\bullet)$$

The associative law *gari* and *mu*-automorphisms $garit(A^\bullet)$ are defined by:

$$\begin{aligned} N^\bullet = \mathrm{garit}(B^\bullet) M^\bullet \;\Leftrightarrow\; N^{\boldsymbol{w}} &= \sum_{\boldsymbol{w} = \prod \boldsymbol{a}^i \boldsymbol{b}^i \boldsymbol{c}^i} M^{\lceil \boldsymbol{b}^1 \rceil .. \lceil \boldsymbol{b}^s \rceil} B^{\boldsymbol{a}^1 \rfloor} .. B^{\boldsymbol{a}^s \rfloor} B_*^{\lfloor \boldsymbol{c}^1} .. B_*^{\lfloor \boldsymbol{c}^s} \\ \mathrm{gari}(A^\bullet, B^\bullet) &:= \mathrm{mu}(\mathrm{garit}(B^\bullet).A^\bullet, B^\bullet) \quad (B_*^\bullet := \mathit{invmu}\, B^\bullet) \end{aligned}$$

6.3 The operations *lu/mu* and *ari/gari*: so different, yet so close.

Despite the sharp differences – in shape, complexity, sophistication, properties – between the homely, uninflected operations *lu/mu* and their inflected counterparts *ari/gari*, there is no lack of pathways and correspondences between the two domains. Let us mention but four such pathways.

6.2.1. Origin of the flexion structure in mould algebra.

Moulds of the form $\mathcal{M}_A^\bullet = A^\bullet \times Id^\bullet \times A_*^\bullet$ with $A^\bullet \times A_*^\bullet \equiv \mathbf{1}^\bullet$ are stable under (mould) composition, and the equivalence holds:

$$\{\mathcal{M}_C^\bullet = \mathcal{M}_A^\bullet \circ \mathcal{M}_B^\bullet\} \iff \{C^\bullet = \mathrm{gari}(A^\bullet, B^\bullet)\} \tag{472}$$

Interpretation: the left identity in (476) involves $\boldsymbol{u}$-indexed *moulds* $A^{\boldsymbol{u}}$, $B^{\boldsymbol{u}}$, $C^{\boldsymbol{u}}$; the right identity re-uses those same moulds, but viewed as *bimoulds* $A^{\binom{\boldsymbol{u}}{\boldsymbol{v}}}$, $B^{\binom{\boldsymbol{u}}{\boldsymbol{v}}}$, $C^{\binom{\boldsymbol{u}}{\boldsymbol{v}}}$ constant in $\boldsymbol{v}$.

Strictly speaking, (472) derives *gari* only for $\boldsymbol{u}$-dependent bimoulds, but *once a flexion operation is defined on the* u_i*'s, it uniquely extends to the* v_i*'s, and vice versa.*

By the way, the quickest way to check the associativity of *gari* is actually by using the mould-to-bimould correspondence of formula (472).

The *ari*-bracket, needless to say, is capable of a similar derivation, from purely uninflected mould operations.

6.2.2. *scram/viscram* as bridges between *non-inflected* and *inflected*.

As already noted in §1, *scram* and *viscram* turn *lu/mu* into *ari/gari* when acting on *alternals/symmetrals*. In the case of *viscram*, one must also assume the *neg*-invariance [93] of the arguments $A^\bullet, B^\bullet, R^\bullet, S^\bullet$.

$$\mathrm{scram}\,.\,\mathrm{lu}(A^\bullet, B^\bullet) \;\equiv\; \mathit{ari}(\mathrm{scram}.A^\bullet, \mathrm{scram}.B^\bullet) \tag{473}$$

$$\mathrm{scram}\,.\,\mathrm{mu}(R^\bullet, S^\bullet) \;\equiv\; \mathit{gari}(\mathrm{scram}.R^\bullet, \mathrm{scram}.S^\bullet) \tag{474}$$

$$\mathrm{viscram}\,.\,\mathrm{lu}(A^\bullet, B^\bullet) \;\equiv\; \mathit{ari}(\mathrm{viscram}.A^\bullet, \mathrm{viscram}.B^\bullet) \tag{475}$$

$$\mathrm{viscram}\,.\,\mathrm{mu}(R^\bullet, S^\bullet) \;\equiv\; \mathit{gari}(\mathrm{viscram}.R^\bullet, \mathrm{viscram}.S^\bullet) \tag{476}$$

[93] i.e. invariance under the change $\boldsymbol{w} \to -\boldsymbol{w}$.

6.2.3. Internal flexion substructures where $ari \sim lu$ and $gari \sim mu$.

A bimould $A^\bullet$ is said to be *internal* if, for all r, it verifies two dual properties:

$$\{u_1 + \ldots u_r \neq 0\} \implies \{A^{\binom{u_1\ ,\ldots,\ u_r}{v_1\ ,\ldots,\ v_r}} \equiv 0\} \tag{477}$$

$$\{v_i - v_i' = \text{const}\,;\ \forall i\} \implies \{A^{\binom{u_1\ ,\ldots,\ u_r}{v_1\ ,\ldots,\ v_r}} \equiv A^{\binom{u_1\ ,\ldots,\ u_r}{v_1'\ ,\ldots,\ v_r'}}\} \tag{478}$$

Internals constitute an ideal ARI_{intern} of ARI resp. a normal subgroup $GARI_{intern}$ of $GARI$. The elements of the corresponding quotients are referred to as *externals*:

$$\mathrm{ARI}_{\mathrm{extern}} := \mathrm{ARI}/\mathrm{ARI}_{\mathrm{intern}} \tag{479}$$

$$\mathrm{GARI}_{\mathrm{extern}} := \mathrm{GARI}/\mathrm{GARI}_{\mathrm{intern}} \tag{480}$$

The crux, however, at least from this section's viewpoint, is this: when restricted to internals, the *ari* bracket reduces (up to order) to the *lu* bracket, and the *gari* product reduces (again up to order) to the *mu* product:

$$\mathrm{ari}(A^\bullet, B^\bullet) \equiv \mathrm{lu}(B^\bullet, A^\bullet) \quad , \quad \forall A^\bullet, B^\bullet \in \mathrm{ARI}_{\mathrm{intern}} \tag{481}$$

$$\mathrm{gari}(A^\bullet, B^\bullet) \equiv \mathrm{mu}(B^\bullet, A^\bullet) \quad , \quad \forall A^\bullet, B^\bullet \in \mathrm{GARI}_{\mathrm{intern}} \tag{482}$$

The identity (482) is particularly striking, as it connects the *gari*-product, which is linear in its first argument but highly non-linear in the second, to the bilinear *mu*-product.

6.2.4. Another flexion substructure where $ari \sim lu$ and $gari \sim mu$

Let $l\varnothing^\bullet_{\|1}$ be the weight-1 generator of $ARI^{al/il}_{bico}$:

$$l\varnothing_{\|1}^{\binom{u_1\ ,\ldots,\ u_r}{\epsilon_1\ ,\ldots,\ \epsilon_r}} := 0 \quad if \quad r \neq 1 \quad , \quad l\varnothing_{\|1}^{\binom{u_1}{\epsilon_1}} := \begin{cases} 0 & if \quad \epsilon_1 := 0 \\ 1 & if \quad \epsilon_1 := \frac{1}{2} \end{cases} \tag{483}$$

The so-called 'colour-swicth' ideal $ARI^{al/il}_{bico*} := ari(l\varnothing^\bullet_{\|1}, ARI^{al/il}_{bico})$ it generates is characterised by any of the three following properties:

(*i*) $sa.A^\bullet$ *is invariant under the switch* $\epsilon_i \leftrightarrow \frac{1}{2} - \epsilon_i$ $\quad \forall A^\bullet \in ARI^{al/il}_{bico*}$
(*ii*) $sa^*_0.A^\bullet \equiv 0$ $\quad \forall A^\bullet \in ARI^{al/il}_{bico*}$
(*iii*) $sa^*_{\frac{1}{2}}.ari(A^\bullet, B^\bullet) \equiv lu(sa^*_{\frac{1}{2}}A^\bullet, sa^*_{\frac{1}{2}}B^\bullet)$ $\quad \forall A^\bullet, B^\bullet \in ARI^{al/il}_{bico*}$

The last identity is yet another instance when *ari* reduces to *lu*.

6.4 The non-vanishing determinants behind the independence of the bicolour generators.

Here are the first determinants $det^*_{2,d}(x), det^*_{3,d}(x), det^*_{4,d}(x)$ related to the expansions (247) and the independence theorem for bicolour generators. To simplify, we give their expression in terms of $t := x^2$ and after factorisation. The properties mentioned at the end of §3.8, Remark 3 (regarding the systematic occurence of Bernoulli numbers when $x = 2$ i.e. $t = 4$) are easy to check on these polynomials.

$$
\begin{aligned}
\mathrm{det}^*_{2,6} &= (1-t)\,(1+5\,t+3\,t^2)\\
\mathrm{det}^*_{2,8} &= (1-t)\,(1+14\,t+14\,t^2+12\,t^3)\\
\mathrm{det}^*_{2,10} &= (1-t)^2\,(1+28\,t+68\,t^2-186\,t^3-242\,t^4-335\,t^5-388\,t^6-132\,t^7-78\,t^8)\\
\mathrm{det}^*_{2,12} &= (1-t)\,(1-t^2)\,(1+44\,t+113\,t^2-1540\,t^3-1473\,t^4-2224\,t^5-2266\,t^6-2404\,t^7-682\,t^8-816\,t^9)\\
\mathrm{det}^*_{2,14} &= (1-t)^3\,(1+67\,t+406\,t^2-4949\,t^3-26348\,t^4-63628\,t^5-172470\,t^6-195653\,t^7-126185\,t^8\\
&\quad -46598\,t^9-10837\,t^{10}+148108\,t^{11}+293092\,t^{12}+338388\,t^{13}+272508\,t^{14}+198298\,t^{15}\\
&\quad +177792\,t^{16}+58188\,t^{17}+21996\,t^{18})\\
\mathrm{det}^*_{2,16} &= (1-t)^2\,(1-t^2)\,(1+91\,t+675\,t^2-14627\,t^3-101013\,t^4-280923\,t^5-1435701\,t^6-2666839\,t^7\\
&\quad -2584726\,t^8-2527926\,t^9-2320040\,t^{10}-3326922\,t^{11}-1668990\,t^{12}-411564\,t^{13}+1053724\,t^{14}\\
&\quad +971728\,t^{15}+979812\,t^{16}+721968\,t^{17}+1802856\,t^{18}+337212\,t^{19}+234072\,t^{20})\\
\mathrm{det}^*_{2,18} &= (1-t)^3\,(1-t^3)\,(1+121\,t+1359\,t^2-32180\,t^3-399947\,t^4-1835023\,t^5-11185716\,t^6-52269321\,t^7\\
&\quad -137804883\,t^8-244724288\,t^9-120412367\,t^{10}-385583935\,t^{11}-1034912118\,t^{12}-651619915\,t^{13}\\
&\quad -441792167\,t^{14}-569706696\,t^{15}-571598493\,t^{16}-140742595\,t^{17}-172000763\,t^{18}+435966682\,t^{19}\\
&\quad +991769202\,t^{20}+785612744\,t^{21}+620751262\,t^{22}+813401872\,t^{23}+877320078\,t^{24}+580476302\,t^{25}\\
&\quad +487631332\,t^{26}+111355464\,t^{27}+232438932\,t^{28}+59619348\,t^{29}+24120828\,t^{30})
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{det}^*_{3,8} &= (1-t)\,(1+9\,t+23\,t^2+7\,t^3)\\
\mathrm{det}^*_{3,10} &= (1-t)^2\,(1+5\,t+t^2-15\,t^3-11\,t^4)\,(1+27\,t+196\,t^2+194\,t^3+142\,t^4)\\
\mathrm{det}^*_{3,12} &= (1-t)^3\,(1+72\,t+1836\,t^2+19479\,t^3+75638\,t^4+58044\,t^5+421323\,t^6+2091202\,t^7-2919364\,t^8\\
&\quad -12020401\,t^9-23718680\,t^{10}-29632044\,t^{11}-27041474\,t^{12}-18620272\,t^{13}-6653096\,t^{14}-2356984\,t^{15})
\end{aligned}
$$

$$
\begin{aligned}
\det^*_{4,10} &= (1-t)\,(1+13\,t+59\,t^2+99\,t^3+3\,t^4)\\
\det^*_{4,12} &= (1-t)^2\,(1+5\,t+3\,t^2)\,(1+4\,t-2\,t^2-13\,t^3)\,(1+40\,t+547\,t^2+2742\,t^3+2664\,t^4+1650\,t^5)\\
\det^*_{4,14} &= (1-t)^5\,(1+5\,t-t^2-25\,t^3-13\,t^4+35\,t^5+27\,t^6)\,(1+133\,t+7564\,t^2+240867\,t^3+4727566\,t^4+59397187\,t^5\\
&\quad +481146696\,t^6+2469970604\,t^7+7500150554\,t^8+7969894970\,t^9-44183297627\,t^{10}-248885402276\,t^{11}\\
&\quad -796111962965\,t^{12}-4021650070796\,t^{13}-11629580824379\,t^{14}+1023971816277\,t^{15}+49784572223508\,t^{16}\\
&\quad +139955874257862\,t^{17}+228311239164350\,t^{18}+271152533003464\,t^{19}+246093900307300\,t^{20}\\
&\quad +165974984510692\,t^{21}+84693433549488\,t^{22}+26943862007448\,t^{23}+6658284781512\,t^{24})
\end{aligned}
$$

6.5 Unexpected arithmetical interdependence of the length-4 bialternals.

Let B_{2n} be the Bernoulli number, and let β^*_{2n} be the *essential part* of its numerator, that is to say, $numer(B_{2n})$ deprived of its small prime factors p (of all $p \leqslant 2n-5$ to be precise).

The *exceptional* bialernals, or $c\o rma^\bullet$ bialternals, have length 4, and three systems $\{carma^\bullet_{\|s,k}\}$, $\{corma^\bullet_{\|s,k}\}$, $\{curma^\bullet_{\|s,k}\}$ have been constructed at the end of §5.7. The first such bialternal occurs at weight $s = 12$ and in that particular instance all three constructions coincide:

$$carma^\bullet_{\|12,1} = corma^\bullet_{\|12,1} = curma^\bullet_{\|12,1}$$

This is but natural, since they could only differ by *natural* bialternals, which do not yet exist at weight $s = 12$. But the surprise is that *all* the (rational) coefficients of this unique $c\o rma^\bullet_{\|12,1}$ have numerators divisible by $\beta^*_{12} = 691$, although nothing in the way they are constructed would lead one to expect such improbable divisibility.[94] This makes one wonder whether the phenomenon, in some form or other, extends to higher weights. Well, the empirical data suggest, overwhelmingly, that it does: *for all weights s up to $s = 60$, we found that, given any basis $\{e^\bullet_{s_1,s_2,s_3,s_4}\}$ of natural, length-4,*

[94] This applies even to $curma^\bullet_{\|12,1}$: the $lurma^\bullet_{\|3}$, $lurma^\bullet_{\|5}$, $lurma^\bullet_{\|7}$ and $lurma^\bullet_{\|9}$ that enter its construction do involve Bernoulli numbers, but smaller ones.

weight-s bialternals,[95] *there exist unique relations*[96] *of the form:*

$$\sum_{k\leqslant\sigma_4^*(s)} \mathrm{ba}_{s,k}\,\mathrm{carma}^\bullet_{||s,k} + \sum_{\sum s_i=s} \mathrm{ca}_{s_1,s_2,s_3,s_4}\; e^\bullet_{s_1,s_2,s_3,s_4} = 0 \qquad \mathrm{mod}\ \beta_s^* \tag{484}$$

$$\sum_{k\leqslant\sigma_4^*(s)} \mathrm{bo}_{s,k}\,\mathrm{corma}^\bullet_{||s,k} + \sum_{\sum s_i=s} \mathrm{co}_{s_1,s_2,s_3,s_4}\; e^\bullet_{s_1,s_2,s_3,s_4} = 0 \qquad \mathrm{mod}\ \beta_s^* \tag{485}$$

$$\sum_{k\leqslant\sigma_4^*(s)} \mathrm{bu}_{s,k}\,\mathrm{curma}^\bullet_{||s,k} + \sum_{\sum s_i=s} \mathrm{cu}_{s_1,s_2,s_3,s_4}\; e^\bullet_{s_1,s_2,s_3,s_4} = 0 \qquad \mathrm{mod}\ \beta_s^* \tag{486}$$

Remark 1: The identities (484) and (485) make full sense, since by construction, all the denominators in $\mathrm{carma}^\bullet_{||s,k}$ or $\mathrm{corma}^\bullet_{||s,k}$ are invertible mod β_s^*. But the third identity (486) also makes sense when the denominators $\beta_{s_0}^*, s_0 \leqslant s-2$ of the $luma^\bullet_{s_0}$ entering the construction of $\mathrm{curma}^\bullet_{||s,k}$, are coprime with β_s. That appears to be almost always the case: the large prime factors of a given Bernoulli number do not seem to recur in the next consecutive numbers.
Remark 2: Clearly, the existence (resp. uniqueness) of the relation (484) is equivalent to the existence (resp. uniqueness) of (485) – and also to that of (486), modulo the caveat of Remark 1. But we prefer to consider all three systems to help identify hidden patterns, also for guidance in the search for a series of 'remarkable' and exact (as opposed to reduced mod β_s^*) bialternals standind 'behind' these relations. But so far no such pattern and no such back-stage bialternals have emerged.
Remark 3: All the numerical data show that (with the trivial exception of $s = 12$), the identities (484),(485),(486) always involve a *non-zero* second sum consisting of natural bialternals. Again based on empirical evidence, this still holds true if, taking advantage of the latitude allowed in the construction of the exceptional bialternals,[97] we replace the first sums (consisting of $\sigma_4^*(s) = \mathcal{O}(s)$ terms) by larger sums (consisting of $\sigma_4^{**}(s) = \mathcal{O}(s^2)$ terms) and correspondingly shrink the second sums (which still retains $\mathcal{O}(s^3)$ terms).

---

[95] with $e^\bullet_{s_1,s_2,s_3,s_4} := \overleftarrow{ari}\ (ekma^\bullet_{\|s_1}, ekma^\bullet_{\|s_2}, ekma^\bullet_{\|s_3}, ekma^\bullet_{\|s_4})$. We must of course pick the basis elements $e^\bullet_{s_1,s_2,s_3,s_4}$ that themselves verify no trivial dependence relations mod β_s^*, but that poses no difficulty.

[96] unique, of course, up to multiplication by any invertible factor modulo β_s.

[97] Indeed, for any given odd weight s, there exist exactly $[\frac{s+1}{6}]$ degrees of liberty in the construction of the singuland-based $l\o ma^\bullet_s$, since the general solution of the desingularisation equation (476) for $S\o^\bullet_{1,2}$ depends on exactly that number of parameters. As a consequence, the latitude in the determination of the corresponding $c\o rma^\bullet_{\|s,k}$ bialternals is $\sigma_4^{**}(s) \leqslant \sum_{3\leqslant s_1\leqslant s-3}^{s_1\,odd}[\frac{s_1+1}{6}] = \mathcal{O}(s^2)$ and definitely of order $\mathcal{O}(s^2)$. Note that the relevant sum here is $\sum[\frac{s_1+1}{6}]$, not $\sum[\frac{s_1+1}{6}][\frac{s_2+1}{6}]$, since in the construction (476) of $c\o rma^\bullet_{\|s,k}$ the length-3 components of $l\o ma^\bullet_{\|s_1}$ get bracketed with the length-1 components of $l\o ma^\bullet_{\|s_2}$.

Some examples.

The first dependence relations with $\sigma_4^*(s) = 1$ is for $s = 16$, $\beta_{16}^* = 3617$.

$$\begin{aligned}
&\mathrm{corma}^\bullet_{\|16,1} + 1805\,\mathrm{e}^\bullet_{3,3,3,7} + 1115\,\mathrm{e}^\bullet_{3,3,3,7} && \equiv 0 \quad \bmod 3617\\
&\mathrm{carma}^\bullet_{\|16,1} + 2675\,\mathrm{e}^\bullet_{3,3,3,7} + 518\,\mathrm{e}^\bullet_{3,3,3,7} && \equiv 0 \quad \bmod 3617\\
&\mathrm{curma}^\bullet_{\|16,1} + 1111\,\mathrm{e}^\bullet_{3,3,3,7} + 3436\,\mathrm{e}^\bullet_{3,3,3,7} && \equiv 0 \quad \bmod 3617
\end{aligned}$$

For $s = 18$, we get the following relations $\bmod \beta_{18}^* = 43867$:

$$\begin{aligned}
&\mathrm{corma}^\bullet_{\|18,1} + 38314\,\mathrm{e}^\bullet_{3,3,3,9} + 413\,\mathrm{e}^\bullet_{3,5,3,7} + 41405\,\mathrm{e}^\bullet_{5,3,3,7} + 11781\,\mathrm{e}^\bullet_{5,5,3,5} && \equiv 0\\
&\mathrm{carma}^\bullet_{\|18,1} + 27081\,\mathrm{e}^\bullet_{3,3,3,9} + 16590\,\mathrm{e}^\bullet_{3,5,3,7} + 2381\,\mathrm{e}^\bullet_{5,3,3,7} + 5152\,\mathrm{e}^\bullet_{5,5,3,5} && \equiv 0\\
&\mathrm{curma}^\bullet_{\|18,1} + 38314\,\mathrm{e}^\bullet_{3,3,3,9} + 413\,\mathrm{e}^\bullet_{3,5,3,7} + 16938\,\mathrm{e}^\bullet_{5,3,3,7} + 37406\,\mathrm{e}^\bullet_{5,5,3,5} && \equiv 0
\end{aligned}$$

For $s = 20$, we get these relations, $\bmod \beta_{20}^* = 174611 = 283 \times 617$:

$$\begin{aligned}
&\mathrm{corma}^\bullet_{\|20,1} + 21797\,\mathrm{e}^\bullet_{3,3,3,11} + 6686\,\mathrm{e}^\bullet_{3,3,5,9} + 80152\,\mathrm{e}^\bullet_{3,5,3,9}\\
&+154426\,\mathrm{e}^\bullet_{3,7,3,7} + 55432\,\mathrm{e}^\bullet_{5,3,3,9} + 170246\,\mathrm{e}^\bullet_{5,5,3,7} \equiv 0 && \bmod 283 \times 617\\
&\mathrm{carma}^\bullet_{\|20,1} + 93615\,\mathrm{e}^\bullet_{3,3,3,11} + 106745\,\mathrm{e}^\bullet_{3,3,5,9} + 150715\,\mathrm{e}^\bullet_{3,5,3,9}\\
&+123787\,\mathrm{e}^\bullet_{3,7,3,7} + 12924\,\mathrm{e}^\bullet_{5,3,3,9} + 16025\,\mathrm{e}^\bullet_{5,5,3,7} \equiv 0 && \bmod 283 \times 617\\
&\mathrm{curma}^\bullet_{\|20,1} + 50086\,\mathrm{e}^\bullet_{3,3,3,11} + 69114\,\mathrm{e}^\bullet_{3,3,5,9} + 65057\,\mathrm{e}^\bullet_{3,5,3,9}\\
&+61841\,\mathrm{e}^\bullet_{3,7,3,7} + 153912\,\mathrm{e}^\bullet_{5,3,3,9} + 22526\,\mathrm{e}^\bullet_{5,5,3,7} \equiv 0 && \bmod 283 \times 617
\end{aligned}$$

The first relations with $\sigma_4^*(s) = 2$ come with $s = 28$. Neglecting the second sum (i.e. the natural bialternals), we find:

$$\begin{aligned}
&3148968694\,\mathrm{corma}_{\|28,1} + 522158523\,\mathrm{corma}_{\|28,2} + \ldots \equiv 0 && \bmod 9349 \times 362903\\
&325201091\,\mathrm{carma}_{\|28,1} + 2689482059\,\mathrm{carma}_{\|28,2} + \ldots \equiv 0 && \bmod 9349 \times 362903\\
&933645869\,\mathrm{curma}_{\|28,1} + 1708525547\,\mathrm{curma}_{\|28,2} + \ldots \equiv 0 && \bmod 9349 \times 362903
\end{aligned}$$

The reason behind these extraordinary relations (which have no equivalent modulo any number m_s of the form $\prod_{s\leqslant p_i} p_i^{n_i}$ but other than β_s^*) is totally unclear to us. Nor could we find any *privileged* and *uniformly defined* series $\{bial_s^\bullet\}$ of bialternals which, after reduction modulo β_s^*, would produce these relations.

6.6 Spectral analysis of the *push* operator acting on the eupolars.

Eigenspaces of *push* and their dimensions $DP_{r,d}$.

Let $Flex = Flex(\mathfrak{E})$ be the monogenous flexion structure generated by a flexion unit $\mathfrak{E}$ (all such $Flex(\mathfrak{E})$ are isomorphic) and let $Flex_r$ be its component of length r (i.e. the component containing the bimoulds of length r). The *push*-operator, when restricted to $Flex_r$, has order $r{+}1$. For any $d|r{+}1$, let $Flex_{r,d}$ be the subspace of $Flex_r$ spanned by all *push* eigenvectors with eigenvalues that are exactly unit roots of order d. Lastly, let $DP_{r,d} = \dim(Flex_r)$.

Main conjecture.

The dimensions of the push*'s eigenspaces are given by:*

$$DP_{r,\delta} = 2\,\frac{(2\,r)!}{r!\,(r{+}1)!} - \frac{1}{2r+2}\sum_{d|(r+1)} \frac{(2\,d)!}{d!\,d!}\,\Phi(\frac{r{+}1}{d},\frac{r+1}{\delta}) \tag{487}$$

Here, the one-argument $\Phi(.)$ is Euler's classical totient function:

$$\Phi(d) \quad := \quad \prod_{n_i\geqslant 1}(p_i^{n_i} - p_i^{n_i-1}) \quad \textit{if} \quad d = \prod_{n_i\geqslant 1} p_i^{n_i} \tag{488}$$

and the two-argument $\Phi(.,.)$ admits these two equivalent definitions:

$$\Phi(d,\delta) \quad := \quad \Phi(d)\|_{p_i^{1+\nu_i}=p_i^{2+\nu_i}=\cdots=0} \quad \textit{if} \quad \delta = \prod_{\nu_i\geqslant 0} p_i^{\nu_i} \tag{489}$$

$$\Phi(d,\delta) \quad := \quad \prod_{n_i\geqslant 1\,,\,\nu_i\geqslant 0}\Big([\nu_i{-}n_i]^+\,p_i^{n_i} - [\nu_i{-}n_i{+}1]^+\,p_i^{n_i-1}\Big) \tag{490}$$

with the sign function $[m]^+ := 1$ if $m \geqslant 0$ and $[m]^+ := 0$ if $m < 0$. If the prime factor p_i occurs in the decomposition of d but not in that of δ, we have to set $\nu_i := 0$ in formula (490).
Clearly:

$$\begin{aligned}\Phi(d,1) &= \mu(d) = \textit{Mobius function}\\ \Phi(d,d) &= \Phi(d) = \textit{Euler's totient function}\end{aligned}$$

The following easy-to-check identities shall also prove useful:

$$\Phi(d,\delta) = \sum_{\delta_*|d\,,\,\delta_*|\delta} \mu(\frac{d}{\delta_*})\,\delta_* \tag{491}$$

$$\begin{aligned}\forall n \qquad \sum_{\delta|n}\Phi(d,\delta)\,\Phi(n/\delta) &= n \quad \textit{if} \quad d = 1\\ &= 0 \quad \textit{if} \quad d \neq 1 \quad \textit{and} \quad d|n\end{aligned} \tag{492}$$

Properties of the dimensions $DP_{r,d}$.

Property 1: The formulae (487) holds true for all pair (r, d) up to $r = 10$.

Property 2: It yields previously conjectured formulae in the special cases $d = 1$ (since $\Phi(d, 1) = \mu(d)$) and $d = r+1$ (since $\Phi(r+1, r+1) = \Phi(r+1)$) while preserving the general expression of $DP_{r,d}$ as a pondered sum of median binomial coefficients $\frac{(2\,d)!}{d!\,d!}$.

Property 3: It also yields the proper dimension $\frac{(2\,r)!}{r!\,r+1)!}$ for the component $Flex_r(\mathfrak{E})$ of the monogenous flexion algebra. Indeed, due to the above identity (39), the sum $\sum_{\delta|(r+1)} PD_{r,\delta}\,\Phi(\delta)$ reduces to the difference $2\,\frac{(2\,r)!}{r!\,r!} - \frac{1}{2}\frac{(2\,r+2)!}{(r+1)!\,(r+1)!}$, which is equal to the expected dimension $\frac{(2\,r)!}{r!\,(r+1)!}$.

Property 4: Lastly, and even more convincingly, it yields an *integer* for each eigenspace of *push*, despite expressing $DP_{r,\delta}$ as a sum of *fractional terms* $\frac{1}{2r+2}\frac{(2\,d)!}{d!\,d!}\,\Phi(\frac{r+1}{d}, \frac{r+1}{\delta})$.

Remark 1: (34) easily implies $\delta_1|\delta_2 \Rightarrow DP_{r,\delta_1} < DP_{r,\delta_2}$

Remark 2: There is an alternative, simpler expression for $DP_{r,d}$. Let $\chi_{\text{push}}(r, t)$ be the characteristic polynomial of the *push* operator restricted to $Flex_r(\mathfrak{E})$. Then (34) amounts to saying that

$$\chi_{\text{push}}(r, t) = \prod_{\delta'|r+1} (1 - t^{\delta'})^{DP^*_{r,\delta'}} \tag{493}$$

with

$$DP^*_{r,\delta'} = \sum_{\delta'|\delta|r+1} DP_{r,\delta}\;\mu(\delta/\delta') \tag{494}$$

The remarkable thing, though, is that, for any given value of δ, the coefficients $DP^*_{r,\delta}$, unlike the earlier $DP_{r,\delta}$ assume ony *two* distinct values. In fact, r is necessarily of the form $n\delta - 1$ and we have

$$DP^*_{\delta-1,\delta} \;=\; +\alpha_\delta > 0 \tag{495}$$

$$DP^*_{n\,\delta-1,\delta} \;=\; -\beta_\delta < 0 \qquad \forall n > 1 \tag{496}$$

with

$$\alpha_n = 2\frac{(2\,n-2)!}{n!\,(n-1)!} - \frac{1}{2\,n}\sum_{d|n}\mu(n/d)\,\frac{(2\,d)!}{d!\,d!} \tag{497}$$

$$\beta_n = \frac{1}{2\,n}\sum_{d|n}\mu(n/d)\,\frac{(2\,d)!}{d!\,d!} \tag{498}$$

Thus

$$[\alpha_1, \alpha_2, \ldots] = [1, 1, 1, 2, 3, 9, 19, 58, 160, 499, 1527, 4940 \ldots]$$
$$[\beta_1, \beta_2, \ldots] = [1, 1, 3, 8, 25, 75, 245, 800, 2700, 9225, 32065, 112632 \ldots]$$

The factorisation (493) therefore becomes

$$\chi_{\mathrm{push}}(r,t) = (1-t^{r+1})^{\alpha_{r+1}} \prod_{\delta|r+1}^{\delta<r+1} (1-t^{\delta})^{-\beta_\delta} \tag{499}$$

which implies for the dimensions $DP_{r,\delta}$ the alternative expression:

$$DP_{r,\delta} = \alpha_{r+1} - \sum_{\delta|\delta'|r+1}^{\delta'<r+1} \beta_{\delta'} \qquad (\textit{in particular } DP_{r,r+1} = \alpha_{r+1}) \tag{500}$$

To show that 500) with α_n and β_n as in (497)-(498) is truly equivalent to the earlier expression (34), it is enough to plug the identity (38) into (34).

6.7 The *lifted* variants of the *ari* bracket.

To each *flexion unit* $\mathfrak{E}$ there corresponds a flexion algebra *Flex* a lift operator $\mathfrak{le}$ acting on

$$\mathfrak{le}\, A^\bullet := \mathrm{arit}(A^\bullet)\, \mathfrak{E}^\bullet \qquad\qquad \mathfrak{le} : \begin{cases} \mathrm{Flex} \to \mathrm{Flex} \\ \mathrm{ARI} \to \mathrm{ARI} \end{cases} \tag{501}$$

The lift $\mathfrak{le}$ and its powers clearly preserve alternality. More significantly:

Proposition 6.1 *Although* $\mathfrak{le}^n$.Flex *and* $\mathfrak{le}^n$.ARI *are but small subspaces of* Flex *and* ARI*, these subspaces are stable under the* ari*-bracket.*

$$\mathrm{ari} : \quad \begin{cases} (\mathfrak{le}^n.Flex_{r_1}, \mathfrak{le}^n.Flex_{r_2}) & \to \quad \mathfrak{le}^n.Flex_{r_1+r_2+n} \\ (\mathfrak{le}^n.ARI_{r_1}, \mathfrak{le}^n.ARI_{r_2}) & \to \quad \mathfrak{le}^n.ARI_{r_1+r_2+n} \end{cases} \tag{502}$$

This induces a series of lifted *Lie brackets* $\mathrm{arile_n}$*:*

$$\mathrm{arile_n} : \quad \begin{cases} (Flex_{r_1}, Flex_{r_2}) & \to \quad Flex_{r_1+r_2+n} \\ (ARI_{r_1}, ARI_{r_2}) & \to \quad ARI_{r_1+r_2+n} \end{cases} \tag{503}$$

characterised by

$$\mathrm{ari}(\mathfrak{le}^n A^\bullet, \mathfrak{le}^n B^\bullet) \;\equiv\; \mathfrak{le}^n\, \mathrm{arile_n}(A^\bullet, B^\bullet) \tag{504}$$

and acting according to the formula

$$\mathrm{arile_n}(A^\bullet, B^\bullet) \;:=\; \begin{cases} -\mathrm{arit}(\mathfrak{le}^n\, A^\bullet)\, B^\bullet + \mathrm{arit}(\mathfrak{le}^n\, B^\bullet)\, A^\bullet \\ + \sum_{n_1+n_2=n}^{n_1,n_2\geqslant 0} \mathrm{lu}(\mathfrak{le}^{n_1} A^\bullet, \mathfrak{le}^{n_2} B^\bullet) \end{cases} \tag{505}$$

For $n=0$, $arile_0 = ari$ and we recover the usual definition of the ari bracket:

$$\mathrm{ari}(A^\bullet, B^\bullet) = -\mathrm{arit}(A^\bullet)\, B^\bullet + \mathrm{arit}(B^\bullet)\, A^\bullet + \mathrm{lu}(A^\bullet, B^\bullet) \tag{506}$$

For the polar flexion units $\mathfrak{E}^\bullet = Pa^\bullet$ resp. $Pi^\bullet$ with $Pa^{w_1} = P(u_1) = 1/u_1$ and $Pi^{w_1} = P(v_1) = 1/v_1$, the pair $(\mathfrak{le}, arile_n)$ is denoted $(la, arila_n)$ resp. $(li, arili_n)$. Only this second pair of operations is of practical importance, because it alone preserves entireness, and that too only when the bimoulds depend on the sole lower indices v_j's: $arili_n : ARI^{\boldsymbol{u}\text{-}const}_{ent} \to ARI^{\boldsymbol{u}\text{-}const}_{ent}$.

6.8 Tables: the satellites sa, sa^*, sa^{**} up to weight 9.

We tabulate here, for the first 11 linear generators of $ARI^{a/il}_{bico}$ up to weight 7:

$$M^\bullet_{\|s_1,s_2,\ldots,s_k} := \overrightarrow{ari}\,(M^\bullet_{\|s_1}, M^\bullet_{\|s_2}, \ldots, M^\bullet_{\|s_k})$$

all three satellites sa, sa^*, sa^{**} with the following abbreviations:

$$\mathrm{sa}.M^\bullet =: \mathcal{C}^\bullet \qquad \begin{cases} \mathrm{sa}^*_0 M^\bullet =: \underline{\mathcal{A}}^\bullet \quad , \quad \mathrm{sa}^*_{\frac{1}{2}} M^\bullet =: \underline{\mathcal{B}}^\bullet \quad , \quad \mathrm{sa}^\sharp_{\frac{1}{2}} M^\bullet =: \underline{\mathfrak{B}}^\bullet \\ \mathrm{sa}^{**}_0 M^\bullet =: \underline{\underline{\mathcal{A}}}^\bullet \quad , \quad \mathrm{sa}^{**}_{\frac{1}{2}} M^\bullet =: \underline{\underline{\mathcal{B}}}^\bullet \quad , \quad \mathrm{sa}^{\sharp\sharp}_{\frac{1}{2}} M^\bullet =: \underline{\underline{\mathfrak{B}}}^\bullet \end{cases}$$

(i)For the lower satellite $saM^\bullet$, we give the list of values $\left\{\mathcal{C}^{\epsilon_1,\ldots,\epsilon_s}, \epsilon_i \in \{0, \frac{1}{2}\}\right\}$ in lexicographic order.
(ii) We tabulate the all-white upper satellites $sa^*_0 M^\bullet \equiv sa^{**}_0 M^\bullet$ only for $M^\bullet_{\|1}$, $M^\bullet_{\|3}$, $M^\bullet_{\|5}$, $M^\bullet_{\|7}$ since in all other cases they are $\equiv 0$.
(iii) For a given weight s, the all-black upper satellites $sa^*_{\frac{1}{2}}.M^\bullet$ and $sa^{**}_{\frac{1}{2}}.M^\bullet$ differ more and more as the degree d increases.
(iv) Dually, for a given weight s, the co-satellites $sa^\sharp_{\frac{1}{2}}.M^\bullet$ and $sa^{\sharp\sharp}_{\frac{1}{2}}.M^\bullet$ differ more and more as the length $r = s - d$ increases.
(v) The lowest-degree non-vanishing satellites $sa^*_{\frac{1}{2}}.M^\bullet$ and $sa^{**}_{\frac{1}{2}}.M^\bullet$ coincide up to sign, and so do the lowest-length non-vanishing co-satellites $sa^\sharp_{\frac{1}{2}}.M^\bullet$ and $sa^{\sharp\sharp}_{\frac{1}{2}}.M^\bullet$. In fact:

$$\begin{aligned} \mathrm{sa}^*_{\frac{1}{2}}.M^\bullet &\equiv (-1)^d\, \mathrm{sa}^{**}_{\frac{1}{2}}.M^\bullet \qquad \textit{for lowest degree } d \\ \mathrm{sa}^\sharp_{\frac{1}{2}}.M^\bullet &\equiv (-1)^r\, \mathrm{sa}^{\sharp\sharp}_{\frac{1}{2}}.M^\bullet \qquad \textit{for lowest length } r \end{aligned}$$

(vi) The lowest-degree non-vanishing satellites $sa^*_{\frac{1}{2}}.M^\bullet$ and $sa^{**}_{\frac{1}{2}}.M^\bullet$ are marked in red when they coincide; in blue when they have opposite signs.

(vii) The lowest-length non-vanishing co-satellites $sa^{\sharp}_{\frac{1}{2}}.M^{\bullet}$ and $sa^{\sharp\sharp}_{\frac{1}{2}}.M^{\bullet}$ are marked in red when they coincide; in blue when they carry opposite signs.
(viii) For easier comparison, we resisted factorising the degree-1 components; nor did we factor out the prime 7 common to all components of all satellites of $M^{*}_{||3,1}, M^{*}_{||3,1,1}, M^{*}_{||3,1,3}, M^{*}_{||3,1,1,1,1}$.

. . . .

$$\mathcal{C}_{[1]} = \{0, 1\}, \quad \underline{\mathcal{A}}^{u_1}_{[1]} = 0, \quad \underline{\mathcal{B}}^{u_1}_{[1]} = 1, \underline{\underline{\mathcal{B}}}^{u_1}_{[1]} = -1, \ \underline{\mathfrak{B}}^{u_1}_{[1]} = 1, \underline{\underline{\mathfrak{B}}}^{u_1}_{[1]} = -1$$

. . . .

$$\mathcal{C}_{[3]} = \{0, -\tfrac{7}{8}, \tfrac{7}{4}, \tfrac{1}{8}, -\tfrac{7}{8}, -1\,4, \tfrac{1}{8}, 0\}$$

$\underline{\mathcal{A}}^{u_1}_{[3]} = \underline{\underline{\mathcal{A}}}^{u_1}_{[3]} = +u_1^2$
$\underline{\mathcal{B}}^{u_1}_{[3]} = -\frac{3}{4}u_1^2$
$\underline{\underline{\mathcal{B}}}^{u_1}_{[3]} = +\frac{7}{8}u_1^2$
$\underline{\mathfrak{B}}^{u_1}_{[3]} = +\frac{1}{8}u_1^2$
$\underline{\underline{\mathfrak{B}}}^{u_1}_{[3]} = +\frac{1}{8}u_1^2$

$\underline{\mathcal{A}}^{u_1,u_2}_{[3]} = \underline{\underline{\mathcal{A}}}^{u_1,u_2}_{[3]} = -u_1 + u_2$
$\underline{\mathcal{B}}^{u_1,u_2}_{[3]} = -\frac{1}{8}u_1 + \frac{1}{8}u_2$
$\underline{\underline{\mathcal{B}}}^{u_1,u_2}_{[3]} = -\frac{1}{8}u_1 + \frac{1}{8}u_2$
$\underline{\mathfrak{B}}^{u_1,u_2}_{[3]} = +\frac{3}{4}u_1 - \frac{3}{4}u_2$
$\underline{\underline{\mathfrak{B}}}^{u_1,u_2}_{[3]} = -\frac{7}{8}u_1 + \frac{7}{8}u_2$

. . . .

$$\mathcal{C}_{[3,1]} = \{0, \tfrac{7}{8}, -\tfrac{21}{8}, 0, \tfrac{21}{8}, 0, 0, -\tfrac{7}{8}, -\tfrac{7}{8}, 0, 0, \tfrac{21}{8}, 0, -\tfrac{21}{8}, \tfrac{7}{8}, 0\}$$

$\underline{\mathcal{B}}^{u_1}_{[3,1]} = 0$
$\underline{\underline{\mathcal{B}}}^{u_1}_{[3,1]} = +\frac{7}{8}u_1^3$
$\underline{\mathfrak{B}}^{u_1}_{[3,1]} = +\frac{7}{8}u_1^3$
$\underline{\underline{\mathfrak{B}}}^{u_1}_{[3,1]} = -\frac{7}{8}u_1^3$
$\underline{\mathcal{B}}^{u_1,u_2}_{[3,1]} = -\frac{7}{4}u_1^2 + \frac{7}{4}u_2^2$
$\underline{\underline{\mathcal{B}}}^{u_1,u_2}_{[3,1]} = 0$
$\underline{\mathfrak{B}}^{u_1,u_2}_{[3,1]} = -\frac{7}{4}u_1^2 + \frac{7}{4}u_2^2$
$\underline{\underline{\mathfrak{B}}}^{u_1,u_2}_{[3,1]} = 0$
$\underline{\mathcal{B}}^{u_1,u_2,u_3}_{[3,1]} = +\frac{7}{8}u_1 - \frac{7}{4}u_2 + \frac{7}{8}u_3$
$\underline{\underline{\mathcal{B}}}^{u_1,u_2,u_3}_{[3,1]} = -\frac{7}{8}u_1 + \frac{7}{4}u_2 - \frac{7}{8}u_3$
$\underline{\mathfrak{B}}^{u_1,u_2,u_3}_{[3,1]} = 0$
$\underline{\underline{\mathfrak{B}}}^{u_1,u_2,u_3}_{[3,1]} = +\frac{7}{8}u_1 - \frac{7}{4}u_2 + \frac{7}{8}u_3$

$$\mathcal{C}_{[5]} = \{0\,,0\,,0\,,\tfrac{1}{32},0\,,\tfrac{23}{64},-\tfrac{29}{64},-\tfrac{63}{32},0\,,-\tfrac{81}{64},\tfrac{29}{16},\tfrac{375}{64},-\tfrac{29}{64},-\tfrac{369}{64},\tfrac{123}{32},1\,,$$
$$0\,,\tfrac{27}{32},-\tfrac{81}{64},\tfrac{3}{64},\tfrac{23}{64},-\tfrac{3}{16},-\tfrac{369}{64},-4\,,\tfrac{1}{32},\tfrac{3}{64},\tfrac{375}{64},6\,,-\tfrac{63}{32},-4\,,1\,,0\}$$

$$\underline{\mathcal{A}}^{u_1}_{[5]} = \underline{\underline{\mathcal{A}}}^{u_1}_{[5]} = +u_1^4$$
$$\underline{\mathcal{B}}^{u_1}_{[5]} = -\tfrac{15}{16}u_1^4$$
$$\underline{\underline{\mathcal{B}}}^{u_1}_{[5]} = 0$$
$$\underline{\mathfrak{B}}^{u_1}_{[5]} = +u_1^4$$
$$\underline{\underline{\mathfrak{B}}}^{u_1}_{[5]} = +u_1^4$$
$$\underline{\mathcal{A}}^{u_1,u_2}_{[5]} = \underline{\underline{\mathcal{A}}}^{u_1,u_2}_{[5]} = -2\,u_1^3 - \tfrac{1}{2}u_1^2u_2 + \tfrac{1}{2}u_1u_2^2 + 2\,u_2^3$$
$$\underline{\mathcal{B}}^{u_1,u_2}_{[5]} = +\tfrac{29}{32}u_1^3 + \tfrac{23}{16}u_1^2u_2 - \tfrac{23}{16}u_1u_2^2 - \tfrac{29}{32}u_2^3$$
$$\underline{\underline{\mathcal{B}}}^{u_1,u_2}_{[5]} = -\tfrac{1}{32}u_1^3 - \tfrac{29}{64}u_1^2u_2 + \tfrac{29}{64}u_1u_2^2 + \tfrac{1}{32}u_2^3$$
$$\underline{\mathfrak{B}}^{u_1,u_2}_{[5]} = -\tfrac{33}{32}u_1^3 - \tfrac{125}{64}u_1^2u_2 + \tfrac{125}{64}u_1u_2^2 + \tfrac{33}{32}u_2^3$$
$$\underline{\underline{\mathfrak{B}}}^{u_1,u_2}_{[5]} = -\tfrac{63}{32}u_1^3 - \tfrac{3}{64}u_1^2u_2 + \tfrac{3}{64}u_1u_2^2 + \tfrac{63}{32}u_2^3$$
$$\underline{\mathcal{A}}^{u_1,u_2,u_3}_{[5]} = \underline{\underline{\mathcal{A}}}^{u_1,u_2,u_3}_{[5]} = +2\,u_1^2 - \tfrac{3}{2}u_1u_2 - 4\,u_2^2 + 3\,u_1u_3 - \tfrac{3}{2}u_2u_3 + 2\,u_3^2$$
$$\underline{\mathcal{B}}^{u_1,u_2,u_3}_{[5]} = +\tfrac{33}{32}u_1^2 + \tfrac{59}{64}u_1u_2 - \tfrac{33}{16}u_2^2 - \tfrac{59}{32}u_1u_3 + \tfrac{59}{64}u_2u_3 + \tfrac{33}{32}u_3^2$$
$$\underline{\underline{\mathcal{B}}}^{u_1,u_2,u_3}_{[5]} = +\tfrac{63}{32}u_1^2 - \tfrac{123}{64}u_1u_2 - \tfrac{63}{16}u_2^2 + \tfrac{123}{32}u_1u_3 - \tfrac{123}{64}u_2u_3 + \tfrac{63}{32}u_3^2$$
$$\underline{\mathfrak{B}}^{u_1,u_2,u_3}_{[5]} = -\tfrac{29}{32}u_1^2 - \tfrac{17}{32}u_1u_2 + \tfrac{29}{16}u_2^2 + \tfrac{17}{16}u_1u_3 - \tfrac{17}{32}u_2u_3 - \tfrac{29}{32}u_3^2$$
$$\underline{\underline{\mathfrak{B}}}^{u_1,u_2,u_3}_{[5]} = +\tfrac{1}{32}u_1^2 + \tfrac{27}{64}u_1u_2 - \tfrac{1}{16}u_2^2 - \tfrac{27}{32}u_1u_3 + \tfrac{27}{64}u_2u_3 + \tfrac{1}{32}u_3^2$$
$$\underline{\mathcal{A}}^{u_1,\ldots,u_4}_{[5]} = \underline{\underline{\mathcal{A}}}^{u_1,\ldots,u_4}_{[5]} = -u_1 + 3\,u_2 - 2\,u_3u_4$$
$$\underline{\mathcal{B}}^{u_1,\ldots,u_4}_{[5]} = -u_1 + 3\,u_2 - 3\,u_3u_4$$
$$\underline{\underline{\mathcal{B}}}^{u_1,\ldots,u_4}_{[5]} = -u_1 + 3\,u_2 - 3\,u_3u_4$$
$$\underline{\mathfrak{B}}^{u_1,\ldots,u_4}_{[5]} = +\tfrac{15}{16}u_1 - \tfrac{45}{16}u_2 + \tfrac{45}{16}u_3 - \tfrac{15}{16}u_4$$
$$\underline{\underline{\mathfrak{B}}}^{u_1,\ldots,u_4}_{[5]} = 0$$

$$\mathcal{C}_{[3,1,1]} = \{+0\,, -\tfrac{7}{8}, \tfrac{7}{2}, \tfrac{7}{8}, -\tfrac{21}{4}\,, -\tfrac{21}{8}\,, 0\,, \tfrac{7}{8}, \tfrac{7}{2}, \tfrac{21}{8}\,, 0\,, -\tfrac{21}{8}\,, 0\,, \tfrac{21}{8}\,, -\tfrac{7}{4}, -\tfrac{7}{8},$$
$$-\tfrac{7}{8}, -\tfrac{7}{4}, \tfrac{21}{8}\,, 0\,, -\tfrac{21}{8}\,, 0\,, \tfrac{21}{8}\,, \tfrac{7}{2}, \tfrac{7}{8}, 0\,, -\tfrac{21}{8}\,, -\tfrac{21}{4}\,, \tfrac{7}{8}, \tfrac{7}{2}, -\tfrac{7}{8}, 0\,\}$$

$$\underline{\mathcal{B}}^{u_1}_{[3,1,1]} = 0$$
$$\underline{\underline{\mathcal{B}}}^{u_1}_{[3,1,1]} = +\tfrac{7}{8}u_1^4$$
$$\underline{\mathfrak{B}}^{u_1}_{[3,1,1]} = -\tfrac{7}{8}u_1^4$$
$$\underline{\underline{\mathfrak{B}}}^{u_1}_{[3,1,1]} = -\tfrac{7}{8}u_1^4$$
$$\underline{\mathcal{B}}^{u_1,u_2}_{[3,1,1]} = 0$$
$$\underline{\underline{\mathcal{B}}}^{u_1,u_2}_{[3,1,1]} = -\tfrac{7}{8}u_1^3 + \tfrac{7}{8}u_2^3$$
$$\underline{\mathfrak{B}}^{u_1,u_2}_{[3,1,1]} = +\tfrac{7}{4}u_1^3 + \tfrac{7}{4}u_1^2u_2 - \tfrac{7}{4}u_1u_2^2 - \tfrac{7}{4}u_2^3$$
$$\underline{\underline{\mathfrak{B}}}^{u_1,u_2}_{[3,1,1]} = +\tfrac{7}{8}u_1^3 - \tfrac{7}{8}u_2^3$$
$$\underline{\mathcal{B}}^{u_1,u_2,u_3}_{[3,1,1]} = -\tfrac{7}{4}u_1^2 + \tfrac{7}{2}u_2^2 - \tfrac{7}{4}u_3^2$$
$$\underline{\underline{\mathcal{B}}}^{u_1,u_2,u_3}_{[3,1,1]} = -\tfrac{7}{8}u_1^2 + \tfrac{7}{8}u_1u_2 + \tfrac{7}{4}u_2^2 - \tfrac{7}{4}u_1u_3 + \tfrac{7}{8}u_2u_3 - \tfrac{7}{8}u_3^2$$
$$\underline{\mathfrak{B}}^{u_1,u_2,u_3}_{[3,1,1]} = 0$$
$$\underline{\underline{\mathfrak{B}}}^{u_1,u_2,u_3}_{[3,1,1]} = +\tfrac{7}{8}u_1^2 - \tfrac{7}{8}u_1u_2 - \tfrac{7}{4}u_2^2 + \tfrac{7}{4}u_1u_3 - \tfrac{7}{8}u_2u_3 + \tfrac{7}{8}u_3^2$$
$$\underline{\mathcal{B}}^{u_1,\dots,u_4}_{[3,1,1]} = +\tfrac{7}{8}u_1 - \tfrac{21}{8}u_2 + \tfrac{21}{8}u_3 - \tfrac{7}{8}u_4$$
$$\underline{\underline{\mathcal{B}}}^{u_1,\dots,u_4}_{[3,1,1]} = +\tfrac{7}{8}u_1 - \tfrac{21}{8}u_2 + \tfrac{21}{8}u_3 - \tfrac{7}{8}u_4$$
$$\underline{\mathfrak{B}}^{u_1,\dots,u_4}_{[3,1,1]} = 0$$
$$\underline{\underline{\mathfrak{B}}}^{u_1,\dots,u_4}_{[3,1,1]} = -\tfrac{7}{8}u_1 + \tfrac{21}{8}u_2 - \tfrac{21}{8}u_3 + \tfrac{7}{8}u_4$$

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$$\mathcal{C}_{[5,1]} = \{0\,,0\,,0\,,\tfrac{31}{32},0\,,-\tfrac{279}{64},\tfrac{31}{64},0\,,0\,,\tfrac{465}{64},-\tfrac{93}{64},0\,,0\,,\tfrac{93}{16},-\tfrac{93}{16},-\tfrac{31}{32},0\,,-\tfrac{155}{32},0\,,-\tfrac{93}{16},\tfrac{93}{64},$$
$$0\,,\tfrac{93}{8},\tfrac{279}{64},-\tfrac{31}{64},0\,,-\tfrac{93}{8},-\tfrac{465}{64},\tfrac{93}{16},\tfrac{155}{32},0\,,0\,,0\,,0\,,\tfrac{155}{32},\tfrac{93}{16},-\tfrac{465}{64},-\tfrac{93}{8},0\,,-\tfrac{31}{64},\tfrac{279}{64},\tfrac{93}{8},$$
$$0\,,\tfrac{93}{64},-\tfrac{93}{16},0\,,-\tfrac{155}{32},0\,,-\tfrac{31}{32},-\tfrac{93}{16},\tfrac{93}{16},0\,,0\,,-\tfrac{93}{64},\tfrac{465}{64},0\,,0\,,\tfrac{31}{64},-\tfrac{279}{64},0\,,\tfrac{31}{32},0\,,0\,,0\}$$

$$\underline{\mathcal{B}}^{u_1}_{[5,1]} = \underline{\underline{\mathcal{B}}}^{u_1}_{[5,1]} = \underline{\mathfrak{B}}^{u_1}_{[5,1]} = \underline{\underline{\mathfrak{B}}}^{u_1}_{[5,1]} = 0$$
$$\underline{\mathcal{B}}^{u_1,u_2}_{[5,1]} = -\tfrac{31}{16}u_1^4 + \tfrac{31}{16}u_2^4$$
$$\underline{\underline{\mathcal{B}}}^{u_1,u_2}_{[5,1]} = +\tfrac{31}{32}u_1^4 - \tfrac{31}{64}u_1^3u_2 + \tfrac{31}{64}u_1u_2^3 - \tfrac{31}{32}u_2^4$$
$$\underline{\mathfrak{B}}^{u_1,u_2}_{[5,1]} = -\tfrac{31}{32}u_1^4 + \tfrac{31}{64}u_1^3u_2 - \tfrac{31}{64}u_1u_2^3 + \tfrac{31}{32}u_2^4$$
$$\underline{\underline{\mathfrak{B}}}^{u_1,u_2}_{[5,1]} = -\tfrac{31}{32}u_1^4 + \tfrac{31}{64}u_1^3u_2 - \tfrac{31}{64}u_1u_2^3 + \tfrac{31}{32}u_2^4$$
$$\underline{\mathcal{B}}^{u_1,u_2,u_3}_{[5,1]} = +\tfrac{93}{32}u_1^3 + \tfrac{31}{16}u_1^2u_2 - \tfrac{31}{16}u_1u_2^2 - \tfrac{93}{16}u_2^3 - \tfrac{31}{16}u_2^2u_3 + \tfrac{31}{16}u_2u_3^2 + \tfrac{93}{32}u_3^3$$
$$\underline{\underline{\mathcal{B}}}^{u_1,u_2,u_3}_{[5,1]} = -\tfrac{93}{16}u_1u_2^2 + \tfrac{93}{16}u_1^2u_3 - \tfrac{93}{16}u_2^2u_3 + \tfrac{93}{16}u_1u_3^2$$
$$\underline{\mathfrak{B}}^{u_1,u_2,u_3}_{[5,1]} = +\tfrac{93}{32}u_1^3 + \tfrac{31}{16}u_1^2u_2 - \tfrac{217}{32}u_1u_2^2 - \tfrac{93}{16}u_2^3 + \tfrac{155}{32}u_1^2u_3 - \tfrac{217}{32}u_2^2u_3 + \tfrac{155}{32}u_1u_3^2 + \tfrac{31}{16}u_2u_3^2 + \tfrac{93}{32}u_3^3$$
$$\underline{\underline{\mathfrak{B}}}^{u_1,u_2,u_3}_{[5,1]} = +\tfrac{93}{16}u_1u_2^2 - \tfrac{93}{16}u_1^2u_3 + \tfrac{93}{16}u_2^2u_3 - \tfrac{93}{16}u_1u_3^2$$
$$\underline{\mathcal{B}}^{u_1,\ldots,u_4}_{[5,1]} = -\tfrac{31}{32}u_1^2 + \tfrac{155}{64}u_1u_2 + \tfrac{93}{32}u_2^2 - \tfrac{155}{32}u_1u_3 - \tfrac{93}{32}u_3^2 + \tfrac{155}{32}u_2u_4 - \tfrac{155}{64}u_3u_4 + \tfrac{31}{32}u_4^2$$
$$\underline{\underline{\mathcal{B}}}^{u_1,\ldots,u_4}_{[5,1]} = -\tfrac{31}{32}u_1^2 + \tfrac{155}{64}u_1u_2 + \tfrac{93}{32}u_2^2 - \tfrac{155}{32}u_1u_3 - \tfrac{93}{32}u_3^2 + \tfrac{155}{32}u_2u_4 - \tfrac{155}{64}u_3u_4 + \tfrac{31}{32}u_4^2$$
$$\underline{\mathfrak{B}}^{u_1,\ldots,u_4}_{[5,1]} = -\tfrac{31}{16}u_1^2 + \tfrac{31}{8}u_1u_2 + \tfrac{93}{16}u_2^2 - \tfrac{31}{4}u_1u_3 - \tfrac{93}{16}u_3^2 + \tfrac{31}{4}u_2u_4 - \tfrac{31}{8}u_3u_4 + \tfrac{31}{16}u_4^2$$
$$\underline{\underline{\mathfrak{B}}}^{u_1,\ldots,u_4}_{[5,1]} = +\tfrac{31}{32}u_1^2 - \tfrac{155}{64}u_1u_2 - \tfrac{93}{32}u_2^2 + \tfrac{155}{32}u_1u_3 + \tfrac{93}{32}u_3^2 - \tfrac{155}{32}u_2u_4 + \tfrac{155}{64}u_3u_4 - \tfrac{31}{32}u_4^2$$
$$\underline{\mathcal{B}}^{u_1,\ldots,u_4}_{[5,1]} = \underline{\underline{\mathcal{B}}}^{u_1,\ldots,u_4}_{[5,1]} = \underline{\mathfrak{B}}^{u_1,\ldots,u_4}_{[5,1]} = \underline{\underline{\mathfrak{B}}}^{u_1,\ldots,u_4}_{[5,1]} = 0$$

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$$\mathcal{C}_{[3,1,1,1]} = \{0, \tfrac{7}{8}, -\tfrac{35}{8}, -\tfrac{7}{4}, \tfrac{35}{4}, \tfrac{49}{8}, \tfrac{7}{8}, 0, -\tfrac{35}{4}, -\tfrac{63}{8}, -\tfrac{21}{8}, 0, 0, -\tfrac{21}{8}, \tfrac{21}{8}, \tfrac{7}{4}, \tfrac{35}{8}, \tfrac{21}{4}, 0, \tfrac{21}{8}, \tfrac{21}{8}, 0, -\tfrac{21}{4}, -\tfrac{49}{8}, -\tfrac{7}{8}, 0, \tfrac{21}{4}, \tfrac{63}{8}, -\tfrac{21}{8}, -\tfrac{21}{4}, 0, -\tfrac{7}{8}, -\tfrac{7}{8}, 0, -\tfrac{21}{4}, -\tfrac{21}{8}, \tfrac{63}{8}, \tfrac{21}{4}, 0, -\tfrac{7}{8}, -\tfrac{49}{8}, -\tfrac{21}{4}, 0, \tfrac{21}{8}, \tfrac{21}{8}, 0, \tfrac{21}{4}, \tfrac{35}{8}, \tfrac{7}{4}, \tfrac{21}{8}, -\tfrac{21}{8}, 0, 0, -\tfrac{21}{8}, -\tfrac{63}{8}, -\tfrac{35}{4}, 0, \tfrac{7}{8}, \tfrac{49}{8}, \tfrac{35}{4}, -\tfrac{7}{4}, -\tfrac{35}{8}, \tfrac{7}{8}, 0\}$$

$$\underline{\mathcal{B}}^{u_1}_{[3,1,1,1]} = 0$$
$$\underline{\underline{\mathcal{B}}}^{u_1}_{[3,1,1,1]} = +\tfrac{7}{8}u_1^5$$
$$\underline{\mathfrak{B}}^{u_1}_{[3,1,1,1]} = +\tfrac{7}{8}u_1^5$$
$$\underline{\underline{\mathfrak{B}}}^{u_1}_{[3,1,1,1]} = -\tfrac{7}{8}u_1^5$$

$$\underline{\mathcal{B}}^{u_1,u_2}_{[3,1,1,1]} = 0$$
$$\underline{\underline{\mathcal{B}}}^{u_1,u_2}_{[3,1,1,1]} = -\tfrac{7}{4}u_1^4 - \tfrac{7}{8}u_1^3u_2 + \tfrac{7}{8}u_1u_2^3 + \tfrac{7}{4}u_2^4$$
$$\underline{\mathfrak{B}}^{u_1,u_2}_{[3,1,1,1]} = -\tfrac{7}{4}u_1^4 - \tfrac{7}{2}u_1^3u_2 + \tfrac{7}{2}u_1u_2^3 + \tfrac{7}{4}u_2^4$$
$$\underline{\underline{\mathfrak{B}}}^{u_1,u_2}_{[3,1,1,1]} = +\tfrac{7}{4}u_1^4 + \tfrac{7}{8}u_1^3u_2 - \tfrac{7}{8}u_1u_2^3 - \tfrac{7}{4}u_2^4$$

$$\underline{\mathcal{B}}^{u_1,u_2,u_3}_{[3,1,1,1]} = 0$$
$$\underline{\underline{\mathcal{B}}}^{u_1,u_2,u_3}_{[3,1,1,1]} = +\tfrac{21}{8}u_1u_2^2 - \tfrac{21}{8}u_1^2u_3 + \tfrac{21}{8}u_2^2u_3 - \tfrac{21}{8}u_1u_3^2$$
$$\underline{\mathfrak{B}}^{u_1,u_2,u_3}_{[3,1,1,1]} = 0$$
$$\underline{\underline{\mathfrak{B}}}^{u_1,u_2,u_3}_{[3,1,1,1]} = -\tfrac{21}{8}u_1u_2^2 + \tfrac{21}{8}u_1^2u_3 - \tfrac{21}{8}u_2^2u_3 + \tfrac{21}{8}u_1u_3^2$$

$$\underline{\mathcal{B}}^{u_1,\ldots,u_4}_{[3,1,1,1]} = -\tfrac{7}{4}u_1^2 + \tfrac{21}{4}u_2^2 - \tfrac{21}{4}u_3^2\tfrac{7}{4}u_4^2$$
$$\underline{\underline{\mathcal{B}}}^{u_1,\ldots,u_4}_{[3,1,1,1]} = +\tfrac{7}{4}u_1^2 - \tfrac{21}{8}u_1u_2 - \tfrac{21}{4}u_2^2 + \tfrac{21}{4}u_1u_3 + \tfrac{21}{4}u_3^2 - \tfrac{21}{4}u_2u_4 + \tfrac{21}{8}u_3u_4 - \tfrac{7}{4}u_4^2$$
$$\underline{\mathfrak{B}}^{u_1,\ldots,u_4}_{[3,1,1,1]} = 0$$
$$\underline{\underline{\mathfrak{B}}}^{u_1,\ldots,u_4}_{[3,1,1,1]} = -\tfrac{7}{4}u_1^2 + \tfrac{21}{8}u_1u_2 + \tfrac{21}{4}u_2^2 - \tfrac{21}{4}u_1u_3 - \tfrac{21}{4}u_3^2 + \tfrac{21}{4}u_2u_4 - \tfrac{21}{8}u_3u_4 + \tfrac{7}{4}u_4^2$$

$$\underline{\mathcal{B}}^{u_1,\ldots,u_5}_{[3,1,1,1]} = +\tfrac{7}{8}u_1 - \tfrac{7}{2}u_2 + \tfrac{21}{4}u_3 - \tfrac{7}{2}u_4 + \tfrac{7}{8}u_5$$
$$\underline{\underline{\mathcal{B}}}^{u_1,\ldots,u_5}_{[3,1,1,1]} = -\tfrac{7}{8}u_1 + \tfrac{7}{2}u_2 - \tfrac{21}{4}u_3 + \tfrac{7}{2}u_4 - \tfrac{7}{8}u_5$$
$$\underline{\mathfrak{B}}^{u_1,\ldots,u_5}_{[3,1,1,1]} = 0$$
$$\underline{\underline{\mathfrak{B}}}^{u_1,\ldots,u_5}_{[3,1,1,1]} = +\tfrac{7}{8}u_1 - \tfrac{7}{2}u_2 + \tfrac{21}{4}u_3 - \tfrac{7}{2}u_4 + \tfrac{7}{8}u_5$$

. . . .

$$\mathcal{C}_{[7]} = \Big\{0, \tfrac{30663}{512}, -\tfrac{91989}{256}, -\tfrac{29123}{512}, \tfrac{459945}{512}, \tfrac{184363}{512}, -\tfrac{9687}{128}, -5, -\tfrac{153315}{128}, -\tfrac{187217}{256}, \tfrac{1427}{64}, \tfrac{221}{16}, \tfrac{17947}{128}, -\tfrac{103217}{512}, \tfrac{106385}{512}, 0, \tfrac{459945}{512}, \tfrac{118283}{256}, \tfrac{103401}{128}, \tfrac{64703}{512}, -\tfrac{53841}{64}, \tfrac{97855}{512}, -\tfrac{248477}{512}, 0, \tfrac{17947}{128}, \tfrac{119}{512}, \tfrac{105779}{256}, \tfrac{68607}{512}, -\tfrac{70559}{256}, -\tfrac{34317}{512}, -\tfrac{17145}{256}, -\tfrac{30659}{512}, -\tfrac{91989}{256}, \tfrac{19585}{512}, -\tfrac{34467}{32}, -\tfrac{30829}{512}, \tfrac{103401}{128}, -\tfrac{36681}{256}, \tfrac{36443}{512}, -\tfrac{102897}{512}, \tfrac{1427}{64}, -\tfrac{24969}{512}, \tfrac{119}{64}, \tfrac{102879}{512}, \tfrac{105779}{256}, -\tfrac{34281}{256}, \tfrac{171477}{512}, \tfrac{189995}{512}, -\tfrac{9687}{128}, -\tfrac{61}{8}, \tfrac{36443}{512}, -\tfrac{9}{512}, -\tfrac{248477}{512}, -\tfrac{34281}{512}, -\tfrac{102897}{256}, -\tfrac{191569}{256}, \tfrac{106385}{512}, 0, \tfrac{171477}{512}, \tfrac{122891}{256}, -\tfrac{17145}{256}, \tfrac{14465}{512}, -\tfrac{2893}{256}, \tfrac{31175}{512}, \tfrac{30663}{512}, -\tfrac{3917}{256}, \tfrac{19585}{512}, -\tfrac{16633}{256}, \tfrac{118283}{256}, \tfrac{167797}{512}, -\tfrac{61}{8}, \tfrac{102897}{512}, -\tfrac{187217}{256}, -\tfrac{101265}{256}, -\tfrac{24969}{512}, -\tfrac{240093}{512}, \tfrac{119}{512}, \tfrac{34299}{512}, 0, -\tfrac{9175}{128}, \tfrac{184363}{512}, \tfrac{167797}{512}, -\tfrac{36681}{256}, \tfrac{102915}{256}, \tfrac{97855}{512}, -\tfrac{9}{64}, -\tfrac{34281}{512}, \tfrac{787}{64}, -\tfrac{103217}{512}, \tfrac{34299}{512}, -\tfrac{34281}{256}, \tfrac{103017}{128}, -\tfrac{34317}{512}, -\tfrac{34339}{32}, \tfrac{14465}{512}, -\tfrac{93525}{256}, -\tfrac{29123}{512}, -\tfrac{16633}{256}, -\tfrac{30829}{512}, -\tfrac{68607}{256}, \tfrac{64703}{512}, \tfrac{102915}{256}, -\tfrac{9}{512}, \tfrac{17563}{128}, \tfrac{221}{16}, -\tfrac{240093}{512}, \tfrac{102879}{512}, -\tfrac{52689}{64}, \tfrac{68607}{512}, \tfrac{103017}{128}, \tfrac{122891}{256}, \tfrac{467625}{512}, -5, \tfrac{102897}{512}, -\tfrac{102897}{512}, \tfrac{17563}{128}, 0, \tfrac{787}{64}, -\tfrac{191569}{256}, -\tfrac{155875}{128}, 0, -\tfrac{9175}{128}, \tfrac{189995}{512}, \tfrac{467625}{512}, -\tfrac{30659}{512}, -\tfrac{93525}{256}, \tfrac{31175}{512}, 0\Big\}$$

$$\underline{\mathcal{A}}^{u_1}_{[7]} = \underline{\underline{\mathcal{A}}}^{u_1}_{[7]} = u_1^6$$
$$\underline{\mathcal{B}}^{u_1}_{[7]} = -\tfrac{63}{64} u_1^6$$
$$\underline{\underline{\mathcal{B}}}^{u_1}_{[7]} = -\tfrac{30663}{512} u_1^6$$
$$\underline{\mathfrak{B}}^{u_1}_{[7]} = +\tfrac{31175}{512} u_1^6$$
$$\underline{\underline{\mathfrak{B}}}^{u_1}_{[7]} = +\tfrac{31175}{512} u_1^6$$
$$\underline{\mathcal{A}}^{u_1,u_2}_{[7]} = \underline{\underline{\mathcal{A}}}^{u_1,u_2}_{[7]} = -3\, u_1^5 - 4\, u_1^4 u_2 - 3\, u_1^3 u_2^2 + 3\, u_1^2 u_2^3 + 4\, u_1 u_2^4 + 3\, u_2^5$$
$$\underline{\mathcal{B}}^{u_1,u_2}_{[7]} = +\tfrac{251}{128} u_1^5 + \tfrac{631}{128} u_1^4 u_2 + \tfrac{251}{128} u_1^3 u_2^2 - \tfrac{251}{128} u_1^2 u_2^3 - \tfrac{631}{128} u_1 u_2^4 - \tfrac{251}{128} u_2^5$$
$$\underline{\underline{\mathcal{B}}}^{u_1,u_2}_{[7]} = +\tfrac{29123}{512} u_1^5 - \tfrac{9687}{128} u_1^4 u_2 - \tfrac{17947}{128} u_1^3 u_2^2 + \tfrac{17947}{128} u_1^2 u_2^3 + \tfrac{9687}{128} u_1 u_2^4 - \tfrac{29123}{512} u_2^5$$
$$\underline{\mathfrak{B}}^{u_1,u_2}_{[7]} = -\tfrac{3913}{16} u_1^5 - \tfrac{317275}{512} u_1^4 u_2 - \tfrac{226127}{512} u_1^3 u_2^2 + \tfrac{226127}{512} u_1^2 u_2^3 + \tfrac{317275}{512} u_1 u_2^4 + \tfrac{3913}{16} u_2^5$$
$$\underline{\underline{\mathfrak{B}}}^{u_1,u_2}_{[7]} = -\tfrac{30659}{512} u_1^5 + \tfrac{9175}{128} u_1^4 u_2 + \tfrac{17563}{128} u_1^3 u_2^2 - \tfrac{17563}{128} u_1^2 u_2^3 - \tfrac{9175}{128} u_1 u_2^4 + \tfrac{30659}{512} u_2^5$$

$$\underline{\mathcal{A}}^{u_1,u_2,u_3}_{[7]} = \underline{\underline{\mathcal{A}}}^{u_1,u_2,u_3}_{[7]} = \begin{cases} +5\,u_1^4 + \frac{99}{16}u_1^3u_2 - \frac{61}{16}u_1^2u_2^2 - 13\,u_1u_2^3 - 10\,u_2^4 + \frac{109}{16}u_1^3u_3 \\ +\frac{65}{16}u_1^2u_2u_3 - \frac{65}{8}u_1u_2^2u_3 - 13\,u_2^3u_3 + \frac{61}{8}u_1^2u_3^2 \\ +\frac{65}{16}u_1u_2u_3^2 - \frac{61}{16}u_2^2u_3^2 + \frac{109}{16}u_1u_3^3 + \frac{99}{16}u_2u_3^3 + 5\,u_3^4 \end{cases}$$

$$\underline{\mathcal{B}}^{u_1,u_2,u_3}_{[7]} = \begin{cases} +\frac{31309}{128}u_1^4 + \frac{78111}{256}u_1^3u_2 + \frac{34969}{128}u_1^2u_2^2 - \frac{38213}{128}u_1u_2^3 - \frac{31309}{64}u_2^4 \\ -\frac{1685}{256}u_1^3u_3 - \frac{1119}{256}u_1^2u_2u_3 + \frac{1119}{128}u_1u_2^2u_3 - \frac{38213}{128}u_2^3u_3 - \frac{34969}{64}u_1^2u_3^2 \\ -\frac{1119}{256}u_1u_2u_3^2 + \frac{34969}{128}u_2^2u_3^2 - \frac{1685}{256}u_1u_3^3 + \frac{78111}{256}u_2u_3^3 + \frac{31309}{128}u_3^4 \end{cases}$$

$$\underline{\underline{\mathcal{B}}}^{u_1,u_2,u_3}_{[7]} = \begin{cases} +5\,u_1^4 + \frac{99}{16}u_1^3u_2 - \frac{70559}{512}u_1^2u_2^2 - \frac{109553}{512}u_1u_2^3 - 10\,u_2^4 + \frac{106385}{512}u_1^3u_3 \\ +\frac{35339}{256}u_1^2u_2u_3 - \frac{35339}{128}u_1u_2^2u_3 - \frac{109553}{512}u_2^3u_3 + \frac{70559}{256}u_1^2u_3^2 \\ +\frac{35339}{256}u_1u_2u_3^2 - \frac{70559}{512}u_2^2u_3^2 + \frac{106385}{512}u_1u_3^3 + \frac{99}{16}u_2u_3^3 + 5\,u_3^4 \end{cases}$$

$$\underline{\mathfrak{B}}^{u_1,u_2,u_3}_{[7]} = \begin{cases} +\frac{94557}{256}u_1^4 + \frac{192291}{256}u_1^3u_2 - \frac{56943}{128}u_1^2u_2^2 - \frac{332229}{256}u_1u_2^3 - \frac{94557}{128}u_2^4 \\ +\frac{69969}{128}u_1^3u_3 + \frac{285377}{512}u_1^2u_2u_3 - \frac{285377}{256}u_1u_2^2u_3 - \frac{332229}{256}u_2^3u_3 + \frac{56943}{64}u_1^2u_3^2 \\ +\frac{285377}{512}u_1u_2u_3^2 - \frac{56943}{128}u_2^2u_3^2 + \frac{69969}{128}u_1u_3^3 + \frac{192291}{256}u_2u_3^3 + \frac{94557}{256}u_3^4 \end{cases}$$

$$\underline{\underline{\mathfrak{B}}}^{u_1,u_2,u_3}_{[7]} = \begin{cases} +\frac{68607}{512}u_1^2u_2^2 + \frac{102897}{512}u_1u_2^3 - \frac{102897}{512}u_1^3u_3 - \frac{34299}{256}u_1^2u_2u_3 \\ +\frac{34299}{128}u_1u_2^2u_3 + \frac{102897}{512}u_2^3u_3 - \frac{68607}{256}u_1^2u_3^2 - \frac{34299}{256}u_1u_2u_3^2 \\ +\frac{68607}{512}u_2^2u_3^2 - \frac{102897}{512}u_1u_3^3 \end{cases}$$

$$\underline{\mathcal{A}}_{[7]}^{u_1,\ldots,u_4} = \underline{\underline{\mathcal{A}}}_{[7]}^{u_1,\ldots,u_4} = \begin{cases} -5\,u_1^3 - \frac{19}{16}u_1^2u_2 + 12\,u_1u_2^2 + 15\,u_2^3 - \frac{141}{16}u_1^2u_3 \\ -\frac{17}{16}u_1u_2u_3 + \frac{179}{16}u_2^2u_3 - \frac{205}{16}u_1u_3^2 - \frac{179}{16}u_2u_3^2 \\ -15\,u_3^3 - 2\,u_1^2u_4 + \frac{51}{16}u_1u_2u_4 + \frac{205}{16}u_2^2u_4 - \frac{51}{16}u_1u_3u_4 \\ +\frac{17}{16}u_2u_3u_4 - 12\,u_3^2u_4 + 2\,u_1u_4^2 + \frac{141}{16}u_2u_4^2 + \frac{19}{16}u_3u_4^2 + 5\,u_4^3 \end{cases}$$

$$\underline{\mathcal{B}}_{[7]}^{u_1,\ldots,u_4} = \begin{cases} -\frac{94557}{256}u_1^3 - \frac{48867}{128}u_1^2u_2 + \frac{35139}{64}u_1u_2^2 + \frac{283671}{256}u_2^3 + \frac{25683}{256}u_1^2u_3 \\ +\frac{853}{512}u_1u_2u_3 + \frac{169785}{256}u_2^2u_3 - \frac{111327}{256}u_1u_3^2 - \frac{169785}{256}u_2u_3^2 - \frac{283671}{256}u_3^3 \\ -\frac{68505}{256}u_1^2u_4 - \frac{2559}{512}u_1u_2u_4 + \frac{111327}{256}u_2^2u_4 + \frac{2559}{512}u_1u_3u_4 - \frac{853}{512}u_2u_3u_4 \\ -\frac{35139}{64}u_3^2u_4 + \frac{68505}{256}u_1u_4^2 - \frac{25683}{256}u_2u_4^2 + \frac{48867}{128}u_3u_4^2 + \frac{94557}{256}u_4^3 \end{cases}$$

$$\underline{\underline{\mathcal{B}}}_{[7]}^{u_1,\ldots,u_4} = \begin{cases} +\frac{102897}{512}u_1u_2^2 - \frac{68607}{512}u_1^2u_3 - \frac{34299}{512}u_1u_2u_3 + \frac{68607}{512}u_2^2u_3 - \frac{137187}{512}u_1u_3^2 \\ -\frac{68607}{512}u_2u_3^2 - \frac{17145}{256}u_1^2u_4 + \frac{102897}{512}u_1u_2u_4 + \frac{137187}{512}u_2^2u_4 - \frac{102897}{512}u_1u_3u_4 \\ +\frac{34299}{512}u_2u_3u_4 - \frac{102897}{512}u_3^2u_4 + \frac{17145}{256}u_1u_4^2 + \frac{68607}{512}u_2u_4^2 \end{cases}$$

$$\underline{\mathfrak{B}}_{[7]}^{u_1,\ldots,u_4} = \begin{cases} -\frac{31309}{128}u_1^3 - \frac{15493}{256}u_1^2u_2 + \frac{29583}{32}u_1u_2^2 + \frac{93927}{128}u_2^3 - \frac{113403}{128}u_1^2u_3 \\ -\frac{47691}{256}u_1u_2u_3 + 1007\,u_2^2u_3 - \frac{13471}{16}u_1u_3^2 - 1007\,u_2u_3^2 - \frac{93927}{128}u_3^3 \\ +\frac{5635}{256}u_1^2u_4 + \frac{143073}{256}u_1u_2u_4 + \frac{13471}{16}u_2^2u_4 - \frac{143073}{256}u_1u_3u_4 + \frac{47691}{256}u_2u_3u_4 \\ -\frac{29583}{32}u_3^2u_4 - \frac{5635}{256}u_1u_4^2 + \frac{113403}{128}u_2u_4^2 + \frac{15493}{256}u_3u_4^2 + \frac{31309}{128}u_4^3 \end{cases}$$

$$\underline{\underline{\mathfrak{B}}}_{[7]}^{u_1,\ldots,u_4} = \begin{cases} -5\,u_1^3 - \frac{19}{16}u_1^2u_2 - \frac{96753}{512}u_1u_2^2 + 15\,u_2^3 + \frac{64095}{512}u_1^2u_3 \\ +\frac{33755}{512}u_1u_2u_3 - \frac{62879}{512}u_2^2u_3 + \frac{130627}{512}u_1u_3^2 + \frac{62879}{512}u_2u_3^2 - 15\,u_3^3 \\ +\frac{16633}{256}u_1^2u_4 - \frac{101265}{512}u_1u_2u_4 - \frac{130627}{512}u_2^2u_4 + \frac{101265}{512}u_1u_3u_4 \\ -\frac{33755}{512}u_2u_3u_4 + \frac{96753}{512}u_3^2u_4 - \frac{16633}{256}u_1u_4^2 - \frac{64095}{512}u_2u_4^2 + \frac{19}{16}u_3u_4^2 + 5\,u_4^3 \end{cases}$$

$$\underline{\mathcal{A}}_{[7]}^{u_1,\dots,u_5} = \underline{\underline{\mathcal{A}}}_{[7]}^{u_1,\dots,u_5} = \begin{cases} +3\,u_1^2 - 5\,u_1u_2 - 12\,u_2^2 + 12\,u_1u_3 + 3\,u_2u_3 \\ +18\,u_3^2 - 6\,u_1u_4 - 12\,u_2u_4 + 3\,u_3u_4 - 12\,u_4^2 \\ +4\,u_1u_5 - 6\,u_2u_5 + 12\,u_3u_5 - 5\,u_4u_5 + 3\,u_5^2 \end{cases}$$

$$\underline{\mathcal{B}}_{[7]}^{u_1,\dots,u_5} = \begin{cases} +\frac{3913}{16}u_1^2 - \frac{58373}{512}u_1u_2 - \frac{3913}{4}u_2^2 + \frac{75407}{256}u_1u_3 + \frac{24305}{512}u_2u_3 + \frac{11739}{8}u_3^2 \\ -\frac{25551}{128}u_1u_4 - \frac{24305}{128}u_2u_4 + \frac{24305}{512}u_3u_4 - \frac{3913}{4}u_4^2 + \frac{8517}{64}u_1u_5 \\ -\frac{25551}{128}u_2u_5 + \frac{75407}{256}u_3u_5 - \frac{58373}{512}u_4u_5 + \frac{3913}{16}u_5^2 \end{cases}$$

$$\underline{\underline{\mathcal{B}}}_{[7]}^{u_1,\dots,u_5} = \begin{cases} +\frac{30659}{512}u_1^2 - \frac{128677}{512}u_1u_2 - \frac{30659}{128}u_2^2 + \frac{254461}{512}u_1u_3 + \frac{65785}{256}u_2u_3 \\ +\frac{91977}{256}u_3^2 + \frac{8679}{512}u_1u_4 - \frac{65785}{64}u_2u_4 + \frac{65785}{256}u_3u_4 - \frac{30659}{128}u_4^2 \\ -\frac{2893}{256}u_1u_5 + \frac{8679}{512}u_2u_5 + \frac{254461}{512}u_3u_5 - \frac{128677}{512}u_4u_5 + \frac{30659}{512}u_5^2 \end{cases}$$

$$\underline{\mathfrak{B}}_{[7]}^{u_1,\dots,u_5} = \begin{cases} -\frac{251}{128}u_1^2 + \frac{61}{64}u_1u_2 + \frac{251}{32}u_2^2 - \frac{115}{128}u_1u_3 - \frac{251}{128}u_2u_3 - \frac{753}{64}u_3^2 \\ -\frac{387}{128}u_1u_4 + \frac{251}{32}u_2u_4 - \frac{251}{128}u_3u_4 + \frac{251}{32}u_4^2 + \frac{129}{64}u_1u_5 \\ -\frac{387}{128}u_2u_5 - \frac{115}{128}u_3u_5 + \frac{61}{64}u_4u_5 - \frac{251}{128}u_5^2 \end{cases}$$

$$\underline{\underline{\mathfrak{B}}}_{[7]}^{u_1,\dots,u_5} = \begin{cases} -\frac{29123}{512}u_1^2 + \frac{126117}{512}u_1u_2 + \frac{29123}{128}u_2^2 - \frac{248317}{512}u_1u_3 - \frac{65017}{256}u_2u_3 \\ -\frac{87369}{256}u_3^2 - \frac{11751}{512}u_1u_4 + \frac{65017}{64}u_2u_4 - \frac{65017}{256}u_3u_4 + \frac{29123}{128}u_4^2 \\ +\frac{3917}{256}u_1u_5 - \frac{11751}{512}u_2u_5 - \frac{248317}{512}u_3u_5 + \frac{126117}{512}u_4u_5 - \frac{29123}{512}u_5^2 \end{cases}$$

$$\underline{\mathcal{A}}_{[7]}^{u_1,\dots,u_6} = \underline{\underline{\mathcal{A}}}_{[7]}^{u_1,\dots,u_6} = -u_1 + 5\,u_2 - 110\,u_3 + 10\,u_4 - 5\,u_5 + u_6$$

$$\underline{\mathcal{B}}_{[7]}^{u_1,\dots,u_6} = -\frac{31175}{512}u_1 + \frac{155875}{512}u_2 - \frac{155875}{256}u_3 + \frac{155875}{256}u_4 - \frac{155875}{512}u_5 + \frac{31175}{512}u_6$$

$$\underline{\underline{\mathcal{B}}}_{[7]}^{u_1,\dots,u_6} = -\frac{31175}{512}u_1 + \frac{155875}{512}u_2 - \frac{155875}{256}u_3 + \frac{155875}{256}u_4 - \frac{155875}{512}u_5 + \frac{31175}{512}u_6$$

$$\underline{\mathfrak{B}}_{[7]}^{u_1,\dots,u_6} = +\frac{63}{64}u_1 - \frac{315}{64}u_2 + \frac{315}{32}u_3 - \frac{315}{32}u_4 + \frac{315}{64}u_5 - \frac{63}{64}u_6$$

$$\underline{\underline{\mathfrak{B}}}_{[7]}^{u_1,\dots,u_6} = +\frac{30663}{512}u_1 - \frac{153315}{512}u_2 + \frac{153315}{256}u_3 - \frac{153315}{256}u_4 + \frac{153315}{512}u_5 - \frac{30663}{512}u_6$$

$$\mathcal{C}_{[5,1,1]} = \{0,0,0,-\tfrac{31}{32},0,\tfrac{279}{64},\tfrac{31}{64},\tfrac{31}{32},0,-\tfrac{465}{64},-\tfrac{93}{32},-\tfrac{279}{64},\tfrac{31}{64},-\tfrac{341}{64},\tfrac{93}{16},\tfrac{31}{32},0,\tfrac{155}{32},$$
$$\tfrac{465}{64},\tfrac{837}{64},-\tfrac{93}{32},-\tfrac{93}{64},-\tfrac{93}{8},-\tfrac{279}{64},\tfrac{31}{64},0,\tfrac{279}{16},\tfrac{837}{64},-\tfrac{93}{8},-\tfrac{341}{32},-\tfrac{31}{32},-\tfrac{31}{32},0,0,-\tfrac{155}{16},$$
$$-\tfrac{341}{32},\tfrac{465}{64},\tfrac{93}{8},-\tfrac{93}{16},-\tfrac{341}{64},-\tfrac{93}{32},-\tfrac{651}{64},0,-\tfrac{93}{64},\tfrac{279}{16},\tfrac{93}{8},\tfrac{589}{64},\tfrac{279}{64},\tfrac{31}{64},\tfrac{341}{64},-\tfrac{93}{16},0,-\tfrac{93}{8},$$
$$-\tfrac{651}{64},-\tfrac{465}{32},-\tfrac{465}{64},\tfrac{93}{16},\tfrac{341}{64},\tfrac{589}{64},\tfrac{155}{32},-\tfrac{31}{32},0,0,0,0,0,0,-\tfrac{31}{32},\tfrac{155}{32},\tfrac{589}{64},\tfrac{341}{64},\tfrac{93}{16},-\tfrac{465}{64},$$
$$-\tfrac{465}{32},-\tfrac{651}{64},-\tfrac{93}{8},0,-\tfrac{93}{16},\tfrac{341}{64},\tfrac{31}{64},\tfrac{279}{64},\tfrac{589}{64},\tfrac{93}{8},\tfrac{279}{16},-\tfrac{93}{64},0,-\tfrac{651}{64},-\tfrac{93}{32},-\tfrac{341}{64},-\tfrac{93}{16},$$
$$\tfrac{93}{8},\tfrac{465}{64},-\tfrac{341}{32},-\tfrac{155}{16},0,0,-\tfrac{31}{32},-\tfrac{31}{32},-\tfrac{341}{32},-\tfrac{93}{8},\tfrac{837}{64},\tfrac{279}{16},0,\tfrac{31}{64},-\tfrac{279}{64},-\tfrac{93}{8},-\tfrac{93}{64},-\tfrac{93}{32},$$
$$\tfrac{837}{64},\tfrac{465}{64},\tfrac{155}{32},0,\tfrac{31}{32},\tfrac{93}{16},-\tfrac{341}{64},\tfrac{31}{64},-\tfrac{279}{64},-\tfrac{93}{32},-\tfrac{465}{64},0,\tfrac{31}{32},\tfrac{31}{64},\tfrac{279}{64},0,-\tfrac{31}{32},0,0,0\}$$

$$\underline{\mathcal{B}}^{u_1}_{[5,1,1]} = \underline{\underline{\mathcal{B}}}^{u_1}_{[5,1,1]} = \underline{\mathfrak{B}}^{u_1}_{[5,1,1]} = \underline{\underline{\mathfrak{B}}}^{u_1}_{[5,1,1]} = 0$$

$$\underline{\mathcal{B}}^{u_1,u_2}_{[5,1,1]} = 0$$

$$\underline{\underline{\mathcal{B}}}^{u_1,u_2}_{[5,1,1]} = 0 + \tfrac{31}{32}u_1^5 + \tfrac{31}{64}u_1^4u_2 - \tfrac{31}{64}u_1^3u_2^2 + \tfrac{31}{64}u_1^2u_2^3 - \tfrac{31}{64}u_1u_2^4 - \tfrac{31}{32}u_2^5$$

$$\underline{\mathfrak{B}}^{u_1,u_2}_{[5,1,1]} = +\tfrac{31}{32}u_1^5 + \tfrac{31}{64}u_1^4u_2 - \tfrac{31}{64}u_1^3u_2^2 + \tfrac{31}{64}u_1^2u_2^3 - \tfrac{31}{64}u_1u_2^4 - \tfrac{31}{32}u_2^5$$

$$\underline{\underline{\mathfrak{B}}}^{u_1,u_2}_{[5,1,1]} = -\tfrac{31}{32}u_1^5 - \tfrac{31}{64}u_1^4u_2 + \tfrac{31}{64}u_1^3u_2^2 - \tfrac{31}{64}u_1^2u_2^3 + \tfrac{31}{64}u_1u_2^4 + \tfrac{31}{32}u_2^5$$

$$\underline{\mathcal{B}}^{u_1,u_2,u_3}_{[5,1,1]} = -\tfrac{31}{16}u_1^4 + \tfrac{31}{8}u_2^4 - \tfrac{31}{16}u_3^4$$

$$\underline{\underline{\mathcal{B}}}^{u_1,u_2,u_3}_{[5,1,1]} = \begin{cases} -\tfrac{31}{32}u_1^4 + \tfrac{31}{64}u_1^3u_2 - \tfrac{93}{16}u_1^2u_2^2 - \tfrac{403}{64}u_1u_2^3 + \tfrac{31}{16}u_2^4 + \tfrac{93}{16}u_1^3u_3 \\ +\tfrac{93}{16}u_1^2u_2u_3 - \tfrac{93}{8}u_1u_2^2u_3, -\tfrac{403}{64}u_2^3u_3 + \tfrac{93}{8}u_1^2u_3^2 + \tfrac{93}{16}u_1u_2u_3^2 \\ -\tfrac{93}{16}u_2^2u_3^2 + \tfrac{93}{16}u_1u_3^3 + \tfrac{31}{64}u_2u_3^3 - \tfrac{31}{32}u_3^4 \end{cases}$$

$$\underline{\mathfrak{B}}^{u_1,u_2,u_3}_{[5,1,1]} = \begin{cases} -\tfrac{93}{32}u_1^4 - \tfrac{155}{32}u_1^3u_2 + \tfrac{155}{32}u_1^2u_2^2 + \tfrac{403}{32}u_1u_2^3 + \tfrac{93}{16}u_2^4 - \tfrac{31}{4}u_1^3u_3 \\ -\tfrac{217}{32}u_1^2u_2u_3 + \tfrac{217}{16}u_1u_2^2u_3 + \tfrac{403}{32}u_2^3u_3 - \tfrac{155}{16}u_1^2u_3^2 - \tfrac{217}{32}u_1u_2u_3^2 \\ +\tfrac{155}{32}u_2^2u_3^2 - \tfrac{31}{4}u_1u_3^3 - \tfrac{155}{32}u_2u_3^3 - \tfrac{93}{32}u_3^4 \end{cases}$$

$$\underline{\underline{\mathfrak{B}}}^{u_1,u_2,u_3}_{[5,1,1]} = \begin{cases} +\tfrac{31}{32}u_1^4 - \tfrac{31}{64}u_1^3u_2 + \tfrac{93}{16}u_1^2u_2^2 + \tfrac{403}{64}u_1u_2^3 - \tfrac{31}{16}u_2^4 - \tfrac{93}{16}u_1^3u_3 \\ -\tfrac{93}{16}u_1^2u_2u_3 + \tfrac{93}{8}u_1u_2^2u_3 + \tfrac{403}{64}u_2^3u_3 - \tfrac{93}{8}u_1^2u_3^2 - \tfrac{93}{16}u_1u_2u_3^2 \\ +\tfrac{93}{16}u_2^2u_3^2 - \tfrac{93}{16}u_1u_3^3 - \tfrac{31}{64}u_2u_3^3 + \tfrac{31}{32}u_3^4 \end{cases}$$

$$\underline{\mathcal{B}}^{u_1,\dots,u_4}_{[5,1,1]} = \begin{cases} +\tfrac{93}{32}u_1^3 + \tfrac{31}{16}u_1^2u_2 - \tfrac{31}{16}u_1u_2^2 - \tfrac{279}{32}u_2^3 - \tfrac{31}{8}u_2^2u_3 \\ +\tfrac{31}{8}u_2u_3^2 + \tfrac{279}{32}u_3^3 + \tfrac{31}{16}u_3^2u_4 - \tfrac{31}{16}u_3u_4^2 - \tfrac{93}{32}u_4^3 \end{cases}$$

$$\underline{\underline{\mathcal{B}}}^{u_1,\dots,u_4}_{[5,1,1]} = \begin{cases} -\tfrac{31}{32}u_1^3 + \tfrac{93}{64}u_1^2u_2 + \tfrac{713}{64}u_1u_2^2 + \tfrac{93}{32}u_2^3 - \tfrac{93}{8}u_1^2u_3 - \tfrac{155}{64}u_1u_2u_3 \\ +\tfrac{279}{32}u_2^2u_3 - \tfrac{217}{16}u_1u_3^2 - \tfrac{279}{32}u_2u_3^2 - \tfrac{93}{32}u_3^3 - \tfrac{31}{32}u_1^2u_4 + \tfrac{465}{64}u_1u_2u_4 \\ +\tfrac{217}{16}u_2^2u_4 - \tfrac{465}{64}u_1u_3u_4 + \tfrac{155}{64}u_2u_3u_4 - \tfrac{713}{64}u_3^2u_4 + \tfrac{31}{32}u_1u_4^2 \\ +\tfrac{93}{8}u_2u_4^2 - \tfrac{93}{64}u_3u_4^2 + \tfrac{31}{32} * u_4^3 \end{cases}$$

$$\underline{\mathfrak{B}}^{u_1,\dots,u_4}_{[5,1,1]} = \begin{cases} +\tfrac{31}{16}u_1^3 - \tfrac{31}{16}u_1^2u_2 - \tfrac{155}{16}u_1u_2^2 - \tfrac{93}{16}u_2^3 + \tfrac{155}{16}u_1^2u_3 + \tfrac{31}{8}u_1u_2u_3 \\ -\tfrac{93}{16}u_2^2u_3 + \tfrac{217}{16}u_1u_3^2 + \tfrac{93}{16}u_2u_3^2 + \tfrac{93}{16}u_3^3 + \tfrac{31}{16}u_1^2u_4 - \tfrac{93}{8}u_1u_2u_4 \\ -\tfrac{217}{16}u_2^2u_4 + \tfrac{93}{8}u_1u_3u_4 - \tfrac{31}{8}u_2u_3u_4 + \tfrac{155}{16}u_3^2u_4 \\ -\tfrac{31}{16}u_1u_4^2 - \tfrac{155}{16}u_2u_4^2 + \tfrac{31}{16}u_3u_4^2 - \tfrac{31}{16}u_4^3 \end{cases}$$

$$\underline{\underline{\mathfrak{B}}}^{u_1,\dots,u_4}_{[5,1,1]} = \begin{cases} +\tfrac{31}{32}u_1^3 - \tfrac{93}{64}u_1^2u_2 - \tfrac{713}{64}u_1u_2^2 - \tfrac{93}{32}u_2^3 + \tfrac{93}{8}u_1^2u_3 + \tfrac{155}{64}u_1u_2u_3 \\ -\tfrac{279}{32}u_2^2u_3 + \tfrac{217}{16}u_1u_3^2 + \tfrac{279}{32}u_2u_3^2 + \tfrac{93}{32}u_3^3 + \tfrac{31}{32}u_1^2u_4 - \tfrac{465}{64}u_1u_2u_4 \\ -\tfrac{217}{16}u_2^2u_4 + \tfrac{465}{64}u_1u_3u_4 - \tfrac{155}{64}u_2u_3u_4 + \tfrac{713}{64}u_3^2u_4 - \tfrac{31}{32}u_1u_4^2 \\ -\tfrac{93}{8}u_2u_4^2 + \tfrac{93}{64}u_3u_4^2 - \tfrac{31}{32}u_4^3 \end{cases}$$

$$\underline{\mathcal{B}}^{u_1,\dots,u_5}_{[5,1,1]} = \begin{cases} -\frac{31}{32}u_1^2 + \frac{155}{64}u_1u_2 + \frac{31}{8}u_2^2 - \frac{155}{32}u_1u_3 - \frac{155}{64}u_2u_3 - \frac{93}{16}u_3^2 \\ +\frac{155}{16}u_2u_4 - \frac{155}{64}u_3u_4 + \frac{31}{8}u_4^2 - \frac{155}{32}u_3u_5 + \frac{155}{64}u_4u_5 - \frac{31}{32}u_5^2 \end{cases}$$

$$\underline{\underline{\mathcal{B}}}^{u_1,\dots,u_5}_{[5,1,1]} = \begin{cases} +\frac{31}{32}u_1^2 - \frac{155}{64}u_1u_2 - \frac{31}{8}u_2^2 + \frac{155}{32}u_1u_3 + \frac{155}{64}u_2u_3 + \frac{93}{16}u_3^2 \\ -\frac{155}{16}u_2u_4 + \frac{155}{64}u_3u_4 - \frac{31}{8}u_4^2 + \frac{155}{32}u_3u_5 - \frac{155}{64}u_4u_5 + \frac{31}{32}u_5^2 \end{cases}$$

$$\underline{\mathfrak{B}}^{u_1,\dots,u_5}_{[5,1,1]} = 0$$

$$\underline{\underline{\mathfrak{B}}}^{u_1,\dots,u_5}_{[5,1,1]} = \begin{cases} -\frac{31}{32}u_1^2 + \frac{155}{64}u_1u_2 + \frac{31}{8}u_2^2 - \frac{155}{32}u_1u_3 - \frac{155}{64}u_2u_3 - \frac{93}{16}u_3^2 \\ +\frac{155}{16}u_2u_4 - \frac{155}{64}u_3u_4 + \frac{31}{8}u_4^2 - \frac{155}{32}u_3u_5 + \frac{155}{64}u_4u_5 - \frac{31}{32}u_5^2 \end{cases}$$

$$\underline{\mathcal{B}}^{u_1,\dots,u_6}_{[5,1,1]} = \underline{\underline{\mathcal{B}}}^{u_1,\dots,u_6}_{[5,1,1]} = \underline{\mathfrak{B}}^{u_1,\dots,u_6}_{[5,1,1]} = \underline{\underline{\mathfrak{B}}}^{u_1,\dots,u_6}_{[5,1,1]} = 0$$

. . . .

$$\begin{aligned}\mathcal{C}_{[3,1,3]} = \{&0, 0, 0, -\tfrac{7}{8}, 0, \tfrac{7}{4}, \tfrac{21}{8}, \tfrac{7}{8}, 0, -\tfrac{49}{64}, -\tfrac{175}{32}, -\tfrac{161}{64}, -\tfrac{161}{64}, \tfrac{77}{32}, -\tfrac{217}{64}, \tfrac{7}{8}, 0, \tfrac{49}{16},\\ &-\tfrac{441}{64}, -\tfrac{21}{64}, \tfrac{483}{32}, \tfrac{21}{32}, \tfrac{483}{64}, -\tfrac{161}{64}, -\tfrac{161}{64}, 0, -\tfrac{63}{8}, -\tfrac{21}{64}, \tfrac{21}{4}, \tfrac{21}{64}, -\tfrac{7}{64}, -\tfrac{7}{8}, 0, -\tfrac{245}{64}, \tfrac{147}{16},\\ &\tfrac{21}{64}, -\tfrac{441}{64}, -\tfrac{21}{32}, \tfrac{21}{64}, \tfrac{77}{32}, -\tfrac{175}{32}, 0, 0, \tfrac{21}{32}, -\tfrac{63}{8}, -\tfrac{21}{32}, \tfrac{7}{32}, \tfrac{7}{4}, \tfrac{21}{8}, -\tfrac{7}{64}, \tfrac{21}{64}, 0, \tfrac{483}{64}, 0, 0,\\ &-\tfrac{49}{64}, -\tfrac{217}{64}, -\tfrac{7}{64}, \tfrac{7}{32}, \tfrac{49}{16}, -\tfrac{7}{64}, -\tfrac{245}{64}, \tfrac{49}{32}, 0, 0, \tfrac{49}{32}, -\tfrac{245}{64}, -\tfrac{7}{64}, \tfrac{49}{16}, \tfrac{7}{32}, -\tfrac{7}{64}, -\tfrac{217}{64},\\ &-\tfrac{49}{64}, 0, 0, \tfrac{483}{64}, 0, \tfrac{21}{64}, -\tfrac{7}{64}, \tfrac{21}{8}, \tfrac{7}{4}, \tfrac{7}{32}, -\tfrac{21}{32}, -\tfrac{63}{8}, \tfrac{21}{32}, 0, 0, -\tfrac{175}{32}, \tfrac{77}{32}, \tfrac{21}{64}, -\tfrac{21}{32}, -\tfrac{441}{64},\\ &\tfrac{21}{64}, \tfrac{147}{16}, -\tfrac{245}{64}, 0, -\tfrac{7}{8}, -\tfrac{7}{64}, \tfrac{21}{64}, \tfrac{21}{4}, -\tfrac{21}{64}, -\tfrac{63}{8}, 0, -\tfrac{161}{64}, -\tfrac{161}{64}, \tfrac{483}{64}, \tfrac{21}{32}, \tfrac{483}{32}, -\tfrac{21}{64},\\ &-\tfrac{441}{64}, \tfrac{49}{16}, 0, \tfrac{7}{8}, -\tfrac{217}{64}, \tfrac{77}{32}, -\tfrac{161}{64}, -\tfrac{161}{64}, -\tfrac{175}{32}, -\tfrac{49}{64}, 0, \tfrac{7}{8}, \tfrac{21}{8}, \tfrac{7}{4}, 0, -\tfrac{7}{8}, 0, 0, 0\}\end{aligned}$$

$$\underline{\mathcal{B}}^{u_1}_{[3,1,3]} = \underline{\underline{\mathcal{B}}}^{u_1}_{[3,1,3]} = \underline{\mathfrak{B}}^{u_1}_{[3,1,3]} = \underline{\underline{\mathfrak{B}}}^{u_1}_{[3,1,3]} = 0$$

$$\underline{\mathcal{B}}^{u_1,u_2}_{[3,1,3]} = 0$$

$$\underline{\underline{\mathcal{B}}}^{u_1,u_2}_{[3,1,3]} = +\frac{7}{8}u_1^5 + \frac{21}{8}u_1^4u_2 + \frac{161}{64}u_1^3u_2^2 - \frac{161}{64}u_1^2u_2^3 - \frac{21}{8}u_1u_2^4 - \frac{7}{8}u_2^5$$

$$\underline{\mathfrak{B}}^{u_1,u_2}_{[3,1,3]} = +\frac{7}{8}u_1^5 + \frac{21}{8}u_1^4u_2 + \frac{161}{64}u_1^3u_2^2 - \frac{161}{64}u_1^2u_2^3 - \frac{21}{8}u_1u_2^4 - \frac{7}{8}u_2^5$$

$$\underline{\underline{\mathfrak{B}}}^{u_1,u_2}_{[3,1,3]} = -\frac{7}{8}u_1^5 - \frac{21}{8}u_1^4u_2 - \frac{161}{64}u_1^3u_2^2 + \frac{161}{64}u_1^2u_2^3 + \frac{21}{8}u_1u_2^4 + \frac{7}{8}u_2^5$$

$$\underline{\mathcal{B}}^{u_1,u_2,u_3}_{[3,1,3]} = \begin{cases} -\frac{7}{4}u_1^4 - \frac{7}{2}u_1^3u_2 - \frac{49}{16}u_1^2u_2^2 + \frac{7}{2}u_1u_2^3 + \frac{7}{2}u_2^4 + \frac{7}{2}u_2^3u_3 \\ +\frac{49}{8}u_1^2u_3^2 - \frac{49}{16}u_2^2u_3^2 - \frac{7}{2}u_2u_3^3 - \frac{7}{4}u_3^4 \end{cases}$$

$$\underline{\underline{\mathcal{B}}}^{u_1,u_2,u_3}_{[3,1,3]} = \begin{cases} -\frac{7}{8}u_1^4 - \frac{63}{64}u_1^3u_2 + \frac{21}{8}u_1^2u_2^2 + \frac{35}{8}u_1u_2^3 + \frac{7}{4}u_2^4 - \frac{217}{64}u_1^3u_3 - \frac{21}{8}u_1^2u_2u_3 \\ +\frac{21}{4}u_1u_2^2u_3 + \frac{35}{8}u_2^3u_3 - \frac{21}{4}u_1^2u_3^2 - \frac{21}{8}u_1u_2u_3^2 + \frac{21}{8}u_2^2u_3^2 \\ -\frac{217}{64}u_1u_3^3 - \frac{63}{64}u_2u_3^3 - \frac{7}{8}u_3^4 \end{cases}$$

$$\underline{\mathfrak{B}}^{u_1,u_2,u_3}_{[3,1,3]} = \begin{cases} -\frac{21}{8}u_1^4 - \frac{189}{32}u_1^3u_2 + \frac{77}{32}u_1^2u_2^2 + \frac{63}{8}u_1u_2^3 + \frac{21}{4}u_2^4 - \frac{63}{32}u_1^3u_3 \\ -\frac{21}{8}u_1^2u_2u_3 + \frac{21}{4}u_1u_2^2u_3 + \frac{63}{8}u_2^3u_3 - \frac{77}{16}u_1^2u_3^2 - \frac{21}{8}u_1u_2u_3^2 \\ +\frac{77}{32}u_2^2u_3^2 - \frac{63}{32}u_1u_3^3 - \frac{189}{32}u_2u_3^3 - \frac{21}{8}u_3^4 \end{cases}$$

$$\underline{\underline{\mathfrak{B}}}^{u_1,u_2,u_3}_{[3,1,3]} = \begin{cases} +\frac{7}{8}u_1^4 + \frac{63}{64}u_1^3u_2 - \frac{21}{8}u_1^2u_2^2 - \frac{35}{8}u_1u_2^3 - \frac{7}{4}u_2^4 + \frac{217}{64}u_1^3u_3 \\ +\frac{21}{8}u_1^2u_2u_3 - \frac{21}{4}u_1u_2^2u_3 - \frac{35}{8}u_2^3u_3 + \frac{21}{4}u_1^2u_3^2 + \frac{21}{8}u_1u_2u_3^2 \\ -\frac{21}{8}u_2^2u_3^2 + \frac{217}{64}u_1u_3^3 + \frac{63}{64}u_2u_3^3 + \frac{7}{8}u_3^4 \end{cases}$$

$$\underline{\mathcal{B}}^{u_1,\ldots,u_4}_{[3,1,3]} = \begin{cases} +\frac{21}{8}u_1^3 + \frac{105}{32}u_1^2u_2 - \frac{21}{4}u_1u_2^2 - \frac{63}{8}u_2^3 - \frac{35}{32}u_1^2u_3 - \frac{175}{32}u_2^2u_3 \\ +\frac{161}{32}u_1u_3^2 + \frac{175}{32}u_2u_3^2 + \frac{63}{8}u_3^3 + \frac{49}{16}u_1^2u_4 - \frac{161}{32}u_2^2u_4 + \frac{21}{4}u_3^2u_4 \\ -\frac{49}{16}u_1u_4^2 + \frac{35}{32}u_2u_4^2 - \frac{105}{32}u_3u_4^2 - \frac{21}{8}u_4^3 \end{cases}$$

$$\underline{\underline{\mathcal{B}}}^{u_1,\ldots,u_4}_{[3,1,3]} = \begin{cases} -\frac{7}{8}u_1^3 - \frac{7}{64}u_1^2u_2 + \frac{21}{8}u_2^3 + \frac{7}{32}u_1^2u_3 - \frac{21}{8}u_3^3 - \frac{7}{64}u_1^2u_4 \\ +\frac{7}{64}u_1u_4^2 - \frac{7}{32}u_2u_4^2 + \frac{7}{64}u_3u_4^2 + \frac{7}{8}u_4^3 \end{cases}$$

$$\underline{\mathfrak{B}}^{u_1,\ldots,u_4}_{[3,1,3]} = \begin{cases} +\frac{7}{4}u_1^3 + \frac{7}{4}u_1^2u_2 - \frac{21}{4}u_1u_2^2 - \frac{21}{4}u_2^3 + \frac{77}{16}u_1^2u_3 - \frac{133}{16}u_2^2u_3 \\ +\frac{35}{16}u_1u_3^2 + \frac{133}{16}u_2u_3^2 + \frac{21}{4}u_3^3 - \frac{21}{16}u_1^2u_4 - \frac{35}{16}u_2^2u_4 + \frac{21}{4}u_3^2u_4 \\ +\frac{21}{16}u_1u_4^2 - \frac{77}{16}u_2u_4^2 - \frac{7}{4}u_3u_4^2 - \frac{7}{4}u_4^3 \end{cases}$$

$$\underline{\underline{\mathfrak{B}}}^{u_1,\ldots,u_4}_{[3,1,3]} = \begin{cases} +\frac{7}{8}u_1^3 + \frac{7}{64}u_1^2u_2 - \frac{21}{8}u_2^3 - \frac{7}{32}u_1^2u_3 + \frac{21}{8}u_3^3 \\ +\frac{7}{64}u_1^2u_4 - \frac{7}{64}u_1u_4^2 + \frac{7}{32}u_2u_4^2 - \frac{7}{64}u_3u_4^2 - \frac{7}{8}u_4^3 \end{cases}$$

$$\underline{\mathcal{B}}^{u_1,\ldots,u_5}_{[3,1,3]} = \begin{cases} -\frac{7}{8}u_1^2 + \frac{7}{2}u_2^2 - \frac{49}{64}u_1u_3 + \frac{49}{64}u_2u_3 - \frac{21}{4}u_3^2 + \frac{147}{64}u_1u_4 - \frac{49}{16}u_2u_4 \\ +\frac{49}{64}u_3u_4 + \frac{7}{2}u_4^2 - \frac{49}{32}u_1u_5 + \frac{147}{64}u_2u_5 - \frac{49}{64}u_3u_5 - \frac{7}{8}u_5^2 \end{cases}$$

$$\underline{\underline{\mathcal{B}}}^{u_1,\ldots,u_5}_{[3,1,3]} = \begin{cases} +\frac{7}{8}u_1^2 - \frac{7}{2}u_2^2 + \frac{49}{64}u_1u_3 - \frac{49}{64}u_2u_3 + \frac{21}{4}u_3^2 - \frac{147}{64}u_1u_4 + \frac{49}{16}u_2u_4 \\ -\frac{49}{64}u_3u_4 - \frac{7}{2}u_4^2 + \frac{49}{32}u_1u_5 - \frac{147}{64}u_2u_5 + \frac{49}{64}u_3u_5 + \frac{7}{8} * u_5^2 \end{cases}$$

$$\underline{\mathfrak{B}}^{u_1,\ldots,u_5}_{[3,1,3]} = 0$$

$$\underline{\underline{\mathfrak{B}}}^{u_1,\ldots,u_5}_{[3,1,3]} = \begin{cases} -\frac{7}{8}u_1^2 + \frac{7}{2}u_2^2 - \frac{49}{64}u_1u_3 + \frac{49}{64}u_2u_3 - \frac{21}{4}u_3^2 + \frac{147}{64}u_1u_4 - \frac{49}{16}u_2u_4 \\ +\frac{49}{64}u_3u_4 + \frac{7}{2}u_4^2 - \frac{49}{32}u_1u_5 + \frac{147}{64}u_2u_5 - \frac{49}{64}u_3u_5 - \frac{7}{8}u_5^2 \end{cases}$$

$$\underline{\mathcal{B}}^{u_1,\ldots,u_6}_{[3,1,3]} = \underline{\underline{\mathcal{B}}}^{u_1,\ldots,u_6}_{[3,1,3]} = \underline{\mathfrak{B}}^{u_1,\ldots,u_6}_{[3,1,3]} = \underline{\underline{\mathfrak{B}}}^{u_1,\ldots,u_6}_{[3,1,3]} = 0$$

. . . .

$$\mathcal{C}_{[3,1,1,1,1]} = \{0, -\frac{7}{8}, \frac{21}{4}, \frac{21}{8}, -\frac{105}{8}, -\frac{21}{2}, -\frac{21}{8}, -\frac{7}{4}, \frac{35}{2}, \frac{133}{8}, \frac{35}{4}, \frac{49}{8}, \frac{7}{8}, \frac{7}{2}, -\frac{21}{8}, -\frac{7}{4},$$
$$-\frac{105}{8}, -14, -\frac{63}{8}, -\frac{21}{2}, -\frac{21}{4}, -\frac{21}{8}, \frac{21}{4}, \frac{49}{8}, \frac{7}{8}, 0, -\frac{63}{8}, -\frac{21}{2}, \frac{21}{4}, \frac{63}{8}, \frac{7}{4}, \frac{21}{8}, \frac{21}{4}, \frac{35}{8}, \frac{21}{2},$$
$$\frac{63}{8}, -\frac{63}{8}, -\frac{21}{4}, \frac{21}{8}, \frac{7}{2}, \frac{35}{4}, \frac{63}{8}, 0, -\frac{21}{8}, -\frac{63}{8}, -\frac{21}{4}, -\frac{91}{8}, -\frac{21}{2}, -\frac{21}{8}, -\frac{7}{2}, \frac{21}{8}, 0, \frac{21}{4}, \frac{63}{8},$$
$$\frac{63}{4}, \frac{133}{8}, -\frac{21}{8}, -\frac{7}{2}, -\frac{91}{8}, -14, \frac{7}{4}, \frac{35}{8}, -\frac{7}{4}, -\frac{7}{8}, -\frac{7}{8}, -\frac{7}{4}, \frac{35}{8}, \frac{7}{4}, -14, -\frac{91}{8}, -\frac{7}{2}, -\frac{21}{8},$$
$$\frac{133}{8}, \frac{63}{4}, \frac{63}{8}, \frac{21}{4}, 0, \frac{21}{8}, -\frac{7}{2}, -\frac{21}{8}, -\frac{21}{2}, -\frac{91}{8}, -\frac{21}{4}, -\frac{63}{8}, -\frac{21}{8}, 0, \frac{63}{8}, \frac{35}{4}, \frac{7}{2}, \frac{21}{8}, -\frac{21}{4},$$
$$-\frac{63}{8}, \frac{63}{8}, \frac{21}{2}, \frac{35}{8}, \frac{21}{4}, \frac{21}{8}, \frac{7}{4}, \frac{63}{8}, \frac{21}{4}, -\frac{21}{2}, -\frac{63}{8}, 0, \frac{7}{8}, \frac{49}{8}, \frac{21}{4}, -\frac{21}{8}, -\frac{21}{4}, -\frac{21}{2}, -\frac{63}{8},$$
$$-14, -\frac{105}{8}, -\frac{7}{4}, -\frac{21}{8}, \frac{7}{2}, \frac{7}{8}, \frac{49}{8}, \frac{35}{4}, \frac{133}{8}, \frac{35}{2}, -\frac{7}{4}, -\frac{21}{8}, -\frac{21}{2}, -\frac{105}{8}, \frac{21}{8}, \frac{21}{4}, -\frac{7}{8}, 0\}$$

$$\underline{\mathcal{B}}^{u_1}_{[3,1,1,1,1]} = 0$$
$$\underline{\underline{\mathcal{B}}}^{u_1}_{[3,1,1,1,1]} = +\tfrac{7}{8}u_1^6$$
$$\underline{\mathfrak{B}}^{u_1}_{[3,1,1,1,1]} = -\tfrac{7}{8}u_1^6$$
$$\underline{\underline{\mathfrak{B}}}^{u_1}_{[3,1,1,1,1]} = -\tfrac{7}{8}u_1^6$$

$$\underline{\mathcal{B}}^{u_1,u_2}_{[3,1,1,1,1]} = 0$$
$$\underline{\underline{\mathcal{B}}}^{u_1,u_2}_{[3,1,1,1,1]} = -\tfrac{21}{8}u_1^5 - \tfrac{21}{8}u_1^4u_2 - \tfrac{7}{8}u_1^3u_2^2 + \tfrac{7}{8}u_1^2u_2^3 + \tfrac{21}{8}u_1u_2^4 + \tfrac{21}{8}u_2^5$$
$$\underline{\mathfrak{B}}^{u_1,u_2}_{[3,1,1,1,1]} = +\tfrac{7}{4}u_1^5 + \tfrac{21}{4}u_1^4u_2 + \tfrac{7}{2}u_1^3u_2^2 - \tfrac{7}{2}u_1^2u_2^3 - \tfrac{21}{4}u_1u_2^4 - \tfrac{7}{4}u_2^5$$
$$\underline{\underline{\mathfrak{B}}}^{u_1,u_2}_{[3,1,1,1,1]} = +\tfrac{21}{8}u_1^5 + \tfrac{21}{8}u_1^4u_2 + \tfrac{7}{8}u_1^3u_2^2 - \tfrac{7}{8}u_1^2u_2^3 - \tfrac{21}{8}u_1u_2^4 - \tfrac{21}{8}u_2^5$$

$$\underline{\mathcal{B}}^{u_1,u_2,u_3}_{[3,1,1,1,1]} = 0$$
$$\underline{\underline{\mathcal{B}}}^{u_1,u_2,u_3}_{[3,1,1,1,1]} = \begin{cases} +\tfrac{7}{4}u_1^4 + \tfrac{7}{8}u_1^3u_2 + \tfrac{21}{8}u_1^2u_2^2 + \tfrac{7}{4}u_1u_2^3 - \tfrac{7}{2}u_2^4 - \tfrac{21}{8}u_1^3u_3 - \tfrac{21}{8}u_1^2u_2u_3 \\ +\tfrac{21}{4}u_1u_2^2u_3 + \tfrac{7}{4}u_2^3u_3 - \tfrac{21}{4}u_1^2u_3^2 - \tfrac{21}{8}u_1u_2u_3^2 + \tfrac{21}{8}u_2^2u_3^2 \\ -\tfrac{21}{8}u_1u_3^3 + \tfrac{7}{8}u_2u_3^3 + \tfrac{7}{4}u_3^4 \end{cases}$$
$$\underline{\mathfrak{B}}^{u_1,u_2,u_3}_{[3,1,1,1,1]} = 0$$
$$\underline{\underline{\mathfrak{B}}}^{u_1,u_2,u_3}_{[3,1,1,1,1]} = \begin{cases} -\tfrac{7}{4}u_1^4 - \tfrac{7}{8}u_1^3u_2 - \tfrac{21}{8}u_1^2u_2^2 - \tfrac{7}{4}u_1u_2^3 + \tfrac{7}{2}u_2^4 + \tfrac{21}{8}u_1^3u_3 \\ +\tfrac{21}{8}u_1^2u_2u_3 - \tfrac{21}{4}u_1u_2^2u_3 - \tfrac{7}{4}u_2^3u_3 + \tfrac{21}{4}u_1^2u_3^2 + \tfrac{21}{8}u_1u_2u_3^2 - \tfrac{21}{8}u_2^2u_3^2 \\ +\tfrac{21}{8}u_1u_3^3 - \tfrac{7}{8}u_2u_3^3 - \tfrac{7}{4}u_3^4 \end{cases}$$

$$\underline{\mathcal{B}}^{u_1,\ldots,u_4}_{[3,1,1,1,1]} = 0$$
$$\underline{\underline{\mathcal{B}}}^{u_1,\ldots,u_4}_{[3,1,1,1,1]} = \begin{cases} +\tfrac{7}{4}u_1^3 - \tfrac{7}{8}u_1^2u_2 - \tfrac{21}{2}u_1u_2^2 - \tfrac{21}{4}u_2^3 + \tfrac{77}{8}u_1^2u_3 + \tfrac{21}{8}u_1u_2u_3 - \tfrac{63}{8}u_2^2u_3 \\ +\tfrac{105}{8}u_1u_3^2 + \tfrac{63}{8}u_2u_3^2 + \tfrac{21}{4}u_3^3 + \tfrac{7}{4}u_1^2u_4 - \tfrac{63}{8}u_1u_2u_4 - \tfrac{105}{8}u_2^2u_4 \\ +\tfrac{63}{8}u_1u_3u_4 - \tfrac{21}{8}u_2u_3u_4 + \tfrac{21}{2}u_3^2u_4 - \tfrac{7}{4}u_1u_4^2 - \tfrac{77}{8}u_2u_4^2 + \tfrac{7}{8}u_3u_4^2 - \tfrac{7}{4}u_4^3 \end{cases}$$
$$\underline{\mathfrak{B}}^{u_1,\ldots,u_4}_{[3,1,1,1,1]} = 0$$
$$\underline{\underline{\mathfrak{B}}}^{u_1,\ldots,u_4}_{[3,1,1,1,1]} = \begin{cases} -\tfrac{7}{4}u_1^3 + \tfrac{7}{8}u_1^2u_2 + \tfrac{21}{2}u_1u_2^2 + \tfrac{21}{4}u_2^3 - \tfrac{77}{8}u_1^2u_3 - \tfrac{21}{8}u_1u_2u_3 + \tfrac{63}{8}u_2^2u_3 \\ -\tfrac{105}{8}u_1u_3^2 - \tfrac{63}{8}u_2u_3^2 - \tfrac{21}{4}u_3^3 - \tfrac{7}{4}u_1^2u_4 + \tfrac{63}{8}u_1u_2u_4 + \tfrac{105}{8}u_2^2u_4 \\ -\tfrac{63}{8}u_1u_3u_4 + \tfrac{21}{8}u_2u_3u_4 - \tfrac{21}{2}u_3^2u_4 + \tfrac{7}{4}u_1u_4^2 + \tfrac{77}{8}u_2u_4^2 - \tfrac{7}{8}u_3u_4^2 + \tfrac{7}{4}u_4^3 \end{cases}$$

$$
\begin{aligned}
\underline{\mathcal{B}}^{u_1,\ldots,u_5}_{[3,1,1,1,1]} &= -\tfrac{7}{4}u_1^2 + 7\,u_2^2 - \tfrac{21}{2}u_3^2 + 7\,u_4^2 - \tfrac{7}{4}u_5^2 \\
\underline{\underline{\mathcal{B}}}^{u_1,\ldots,u_5}_{[3,1,1,1,1]} &= \begin{cases} -\tfrac{21}{8}u_1^2 + \tfrac{21}{4}u_1u_2 + \tfrac{21}{2}u_2^2 - \tfrac{91}{8}u_1u_3 - \tfrac{35}{8}u_2u_3 - \tfrac{63}{4}u_3^2 \\ +\tfrac{21}{8}u_1u_4 + \tfrac{35}{2}u_2u_4 - \tfrac{35}{8}u_3u_4 + \tfrac{21}{2}u_4^2 - \tfrac{7}{4}u_1u_5 + \tfrac{21}{8}u_2u_5 \\ -\tfrac{91}{8}u_3u_5 + \tfrac{21}{4}u_4u_5 - \tfrac{21}{8}u_5^2 \end{cases} \\
\underline{\mathfrak{B}}^{u_1,\ldots,u_5}_{[3,1,1,1,1]} &= 0 \\
\underline{\underline{\mathfrak{B}}}^{u_1,\ldots,u_5}_{[3,1,1,1,1]} &= \begin{cases} +\tfrac{21}{8}u_1^2 - \tfrac{21}{4}u_1u_2 - \tfrac{21}{2}u_2^2 + \tfrac{91}{8}u_1u_3 + \tfrac{35}{8}u_2u_3 + \tfrac{63}{4}u_3^2 \\ -\tfrac{21}{8}u_1u_4 - \tfrac{35}{2}u_2u_4 + \tfrac{35}{8}u_3u_4 - \tfrac{21}{2}u_4^2 + \tfrac{7}{4}u_1u_5 - \tfrac{21}{8}u_2u_5 \\ +\tfrac{91}{8}u_3u_5 - \tfrac{21}{4}u_4u_5 + \tfrac{21}{8}u_5^2 \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
\underline{\mathcal{B}}^{u_1,\ldots,u_6}_{[3,1,1,1,1]} &= +\tfrac{7}{8}u_1 - \tfrac{35}{8}u_2 + \tfrac{35}{4}u_3 - \tfrac{35}{4}u_4 + \tfrac{35}{8}u_5 - \tfrac{7}{8}u_6 \\
\underline{\underline{\mathcal{B}}}^{u_1,\ldots,u_6}_{[3,1,1,1,1]} &= +\tfrac{7}{8}u_1 - \tfrac{35}{8}u_2 + \tfrac{35}{4}u_3 - \tfrac{35}{4}u_4 + \tfrac{35}{8}u_5 - \tfrac{7}{8}u_6 \\
\underline{\mathfrak{B}}^{u_1,\ldots,u_6}_{[3,1,1,1,1]} &= 0 \\
\underline{\mathfrak{B}}^{u_1,\ldots,u_6}_{[3,1,1,1,1]} &= -\tfrac{7}{8}u_1 + \tfrac{35}{8}u_2 - \tfrac{35}{4}u_3 + \tfrac{35}{4}u_4 - \tfrac{35}{8}u_5 + \tfrac{7}{8}u_6
\end{aligned}
$$

. . . .

6.9 Tables: ordinary and generalised scramble.

For a double sequence $\underline{\boldsymbol{w}}$ as in (112), we set $\boldsymbol{m}(\underline{\boldsymbol{w}}) := (\#\underline{v}_1, \ldots, \#\underline{v}_r)$ as usual. The following table gives, for low signatures $\boldsymbol{m}(\underline{\boldsymbol{w}})$, the number $\mu = \mu^+ + \mu^-$ of terms on the right-hand side of (128), with $\mu^\pm$ denoting the number of summands preceded by the sign $\pm$.

$\boldsymbol{m}$	$\mu = \mu^+ + \mu^-$	$\boldsymbol{m}$	$\mu = \mu^+ + \mu^-$	$\boldsymbol{m}$	$\mu = \mu^+ + \mu^-$
$(1,1)$	$3 = 2+1$	$(1,1,1)$	$15 = 8+7$	$(1,1,1,1)$	$105 = 53+52$
$(1,2)$	$5 = 3+2$	$(1,1,2)$	$35 = 18+17$	$(1,1,1,2)$	$315 = 158+157$
$(2,1)$	$6 = 4+2$	$(1,2,1)$	$42 = 22+20$	$(1,1,2,1)$	$378 = 190+188$
$(1,3)$	$7 = 4+3$	$(2,1,1)$	$45 = 24+21$	$(1,2,1,1)$	$405 = 204+201$
$(2,2)$	$15 = 9+6$	$(1,1,3)$	$63 = 32+31$	$(2,1,1,1)$	$420 = 212+208$
$(3,1)$	$9 = 6+3$	$(1,3,1)$	$81 = 42+39$	$(1,1,1,3)$	$693 = 347+346$
$(1,4)$	$9 = 5+4$	$(3,1,1)$	$90 = 48+42$	$(1,1,3,1)$	$891 = 447+444$
$(2,3)$	$28 = 16+12$	$(1,2,2)$	$135 = 69+66$	$(1,3,1,1)$	$990 = 498+492$
$(3,2)$	$30 = 18+12$	$(2,1,2)$	$140 = 72+68$	$(3,1,1,1)$	$1050 = 530+520$
$(4,1)$	$12 = 8+4$	$(2,2,1)$	$168 = 88+80$		

The following tables give, for elementary signatures $\boldsymbol{m}(\underline{\boldsymbol{w}}) := (1, 1, ...)$, the scramble $SM^\bullet$ of $M^\bullet$.

$$\begin{aligned}
SM^{\left(\begin{smallmatrix} u_1 \\ v_1 \end{smallmatrix}\right)} &= +M^{\left(\begin{smallmatrix} u_1 \\ v_1 \end{smallmatrix}\right)} \\
SM^{\left(\begin{smallmatrix} u_1 , & u_2 \\ v_1 , & v_2 \end{smallmatrix}\right)} &= +M^{\left(\begin{smallmatrix} u_1 , & u_2 \\ v_1 , & v_2 \end{smallmatrix}\right)} + M^{\left(\begin{smallmatrix} u_{12} , & u_1 \\ v_2 , & v_{1:2} \end{smallmatrix}\right)} - M^{\left(\begin{smallmatrix} u_{12} , & u_2 \\ v_1 , & v_{2:1} \end{smallmatrix}\right)} \\
SM^{\left(\begin{smallmatrix} u_1 , & u_2 , & u_3 \\ v_1 , & v_2 , & v_3 \end{smallmatrix}\right)} &= +M^{\left(\begin{smallmatrix} u_1 , & u_2 , & u_3 \\ v_1 , & v_2 , & v_3 \end{smallmatrix}\right)} + M^{\left(\begin{smallmatrix} u_1 , & u_{23} , & u_2 \\ v_1 , & v_3 , & v_{2:3} \end{smallmatrix}\right)} - M^{\left(\begin{smallmatrix} u_1 , & u_{23} , & u_3 \\ v_1 , & v_2 , & v_{3:2} \end{smallmatrix}\right)} \\
&\quad +M^{\left(\begin{smallmatrix} u_{12} , & u_1 , & u_3 \\ v_2 , & v_{1:2} , & v_3 \end{smallmatrix}\right)} - M^{\left(\begin{smallmatrix} u_{12} , & u_2 , & u_3 \\ v_1 , & v_{2:1} , & v_3 \end{smallmatrix}\right)} \\
&\quad +M^{\left(\begin{smallmatrix} u_{12} , & u_3 , & u_1 \\ v_2 , & v_3 , & v_{1:2} \end{smallmatrix}\right)} - M^{\left(\begin{smallmatrix} u_{12} , & u_3 , & u_2 \\ v_1 , & v_3 , & v_{2:1} \end{smallmatrix}\right)} \\
&\quad +M^{\left(\begin{smallmatrix} u_{123} , & u_{23} , & u_3 \\ v_1 , & v_{2:1} , & v_{3:2} \end{smallmatrix}\right)} - M^{\left(\begin{smallmatrix} u_{123} , & u_{23} , & u_2 \\ v_1 , & v_{3:1} , & v_{2:3} \end{smallmatrix}\right)} + M^{\left(\begin{smallmatrix} u_{123} , & u_3 , & u_2 \\ v_1 , & v_{3:1} , & v_{2:1} \end{smallmatrix}\right)} \\
&\quad -M^{\left(\begin{smallmatrix} u_{123} , & u_1 , & u_3 \\ v_2 , & v_{1:2} , & v_{3:2} \end{smallmatrix}\right)} - M^{\left(\begin{smallmatrix} u_{123} , & u_3 , & u_1 \\ v_2 , & v_{3:2} , & v_{1:2} \end{smallmatrix}\right)} \\
&\quad +M^{\left(\begin{smallmatrix} u_{123} , & u_1 , & u_2 \\ v_3 , & v_{1:3} , & v_{2:3} \end{smallmatrix}\right)} - M^{\left(\begin{smallmatrix} u_{123} , & u_{12} , & u_2 \\ v_3 , & v_{1:3} , & v_{2:3} \end{smallmatrix}\right)} + M^{\left(\begin{smallmatrix} u_{123} , & u_{12} , & u_1 \\ v_3 , & v_{2:3} , & v_{1:2} \end{smallmatrix}\right)}
\end{aligned}$$

$$
\begin{array}{lll}
SM^{\binom{u_{1}, u_{2}, u_{3}, u_{4}}{v_{1}, v_{2}, v_{3}, v_{4}}} & = & \\
+M^{\binom{u_{1}, u_{2}, u_{3}, u_{4}}{v_{1}, v_{2}, v_{3}, v_{4}}} & -M^{\binom{u_{12}, u_{34}, u_{4}, u_{1}}{v_{2}, v_{3}, v_{4:3}, v_{1:2}}} & -M^{\binom{u_{1234}, u_{12}, u_{2}, u_{3}}{v_{4}, v_{1:4}, v_{2:1}, v_{3:4}}} \\
+M^{\binom{u_{12}, u_{1}, u_{3}, u_{4}}{v_{2}, v_{1:2}, v_{3}, v_{4}}} & -M^{\binom{u_{12}, u_{34}, u_{3}, u_{2}}{v_{1}, v_{4}, v_{3:4}, v_{2:1}}} & -M^{\binom{u_{1234}, u_{12}, u_{4}, u_{1}}{v_{3}, v_{2:3}, v_{4:3}, v_{1:2}}} \\
-M^{\binom{u_{12}, u_{2}, u_{3}, u_{4}}{v_{1}, v_{2:1}, v_{3}, v_{4}}} & +M^{\binom{u_{12}, u_{1}, u_{34}, u_{3}}{v_{2}, v_{1:2}, v_{4}, v_{3:4}}} & -M^{\binom{u_{1234}, u_{12}, u_{1}, u_{4}}{v_{3}, v_{2:3}, v_{1:2}, v_{4:3}}} \\
+M^{\binom{u_{12}, u_{3}, u_{1}, u_{4}}{v_{2}, v_{3}, v_{1:2}, v_{4}}} & +M^{\binom{u_{12}, u_{2}, u_{34}, u_{4}}{v_{1}, v_{2:1}, v_{3}, v_{4:3}}} & +M^{\binom{u_{1234}, u_{34}, u_{2}, u_{3}}{v_{1}, v_{4:1}, v_{2:1}, v_{3:4}}} \\
-M^{\binom{u_{12}, u_{3}, u_{2}, u_{4}}{v_{1}, v_{3}, v_{2:1}, v_{4}}} & -M^{\binom{u_{12}, u_{2}, u_{34}, u_{3}}{v_{1}, v_{2:1}, v_{4}, v_{3:4}}} & +M^{\binom{u_{1234}, u_{34}, u_{3}, u_{2}}{v_{1}, v_{4:1}, v_{3:4}, v_{2:1}}} \\
+M^{\binom{u_{12}, u_{3}, u_{4}, u_{1}}{v_{2}, v_{3}, v_{4}, v_{1:2}}} & -M^{\binom{u_{12}, u_{1}, u_{34}, u_{4}}{v_{2}, v_{1:2}, v_{3}, v_{4:3}}} & +M^{\binom{u_{1234}, u_{34}, u_{1}, u_{4}}{v_{2}, v_{3:2}, v_{1:2}, v_{4:3}}} \\
-M^{\binom{u_{12}, u_{3}, u_{4}, u_{2}}{v_{1}, v_{3}, v_{4}, v_{2:1}}} & +M^{\binom{u_{123}, u_{12}, u_{1}, u_{4}}{v_{3}, v_{2:3}, v_{1:2}, v_{4}}} & +M^{\binom{u_{1234}, u_{34}, u_{4}, u_{1}}{v_{2}, v_{3:2}, v_{4:3}, v_{1:2}}} \\
+M^{\binom{u_{1}, u_{23}, u_{4}, u_{2}}{v_{1}, v_{3}, v_{4}, v_{2:3}}} & -M^{\binom{u_{123}, u_{12}, u_{2}, u_{4}}{v_{3}, v_{1:3}, v_{2:1}, v_{4}}} & -M^{\binom{u_{1234}, u_{34}, u_{3}, u_{1}}{v_{2}, v_{4:2}, v_{3:4}, v_{1:2}}} \\
-M^{\binom{u_{1}, u_{23}, u_{4}, u_{3}}{v_{1}, v_{2}, v_{4}, v_{3:2}}} & +M^{\binom{u_{123}, u_{12}, u_{4}, u_{1}}{v_{3}, v_{2:3}, v_{4}, v_{1:2}}} & -M^{\binom{u_{1234}, u_{34}, u_{1}, u_{3}}{v_{2}, v_{4:2}, v_{1:2}, v_{3:4}}} \\
+M^{\binom{u_{1}, u_{23}, u_{2}, u_{4}}{v_{1}, v_{3}, v_{2:3}, v_{4}}} & -M^{\binom{u_{123}, u_{12}, u_{4}, u_{2}}{v_{3}, v_{1:3}, v_{4}, v_{2:1}}} & -M^{\binom{u_{1234}, u_{34}, u_{4}, u_{2}}{v_{1}, v_{3:1}, v_{4:3}, v_{2:1}}} \\
-M^{\binom{u_{1}, u_{23}, u_{3}, u_{4}}{v_{1}, v_{2}, v_{3:2}, v_{4}}} & +M^{\binom{u_{123}, u_{23}, u_{3}, u_{4}}{v_{1}, v_{2:1}, v_{3:2}, v_{4}}} & -M^{\binom{u_{1234}, u_{34}, u_{2}, u_{4}}{v_{1}, v_{3:1}, v_{2:1}, v_{4:3}}} \\
+M^{\binom{u_{1}, u_{2}, u_{34}, u_{3}}{v_{1}, v_{2}, v_{4}, v_{3:4}}} & -M^{\binom{u_{123}, u_{23}, u_{2}, u_{4}}{v_{1}, v_{3:1}, v_{2:3}, v_{4}}} & +M^{\binom{u_{1234}, u_{1}, u_{23}, u_{2}}{v_{4}, v_{1:4}, v_{3:4}, v_{2:3}}} \\
-M^{\binom{u_{1}, u_{2}, u_{34}, u_{4}}{v_{1}, v_{2}, v_{3}, v_{4:3}}} & +M^{\binom{u_{123}, u_{23}, u_{4}, u_{3}}{v_{1}, v_{2:1}, v_{4}, v_{3:2}}} & +M^{\binom{u_{1234}, u_{4}, u_{23}, u_{2}}{v_{1}, v_{4:1}, v_{3:1}, v_{2:3}}} \\
+M^{\binom{u_{123}, u_{3}, u_{2}, u_{4}}{v_{1}, v_{3:1}, v_{2:1}, v_{4}}} & -M^{\binom{u_{123}, u_{23}, u_{4}, u_{2}}{v_{1}, v_{3:1}, v_{4}, v_{2:3}}} & -M^{\binom{u_{1234}, u_{1}, u_{23}, u_{3}}{v_{4}, v_{1:4}, v_{2:4}, v_{3:2}}} \\
+M^{\binom{u_{123}, u_{1}, u_{2}, u_{4}}{v_{3}, v_{1:3}, v_{2:3}, v_{4}}} & +M^{\binom{u_{123}, u_{4}, u_{12}, u_{1}}{v_{3}, v_{4}, v_{2:3}, v_{1:2}}} & -M^{\binom{u_{1234}, u_{4}, u_{23}, u_{3}}{v_{1}, v_{4:1}, v_{2:1}, v_{3:2}}} \\
-M^{\binom{u_{123}, u_{1}, u_{3}, u_{4}}{v_{2}, v_{1:2}, v_{3:2}, v_{4}}} & -M^{\binom{u_{123}, u_{4}, u_{12}, u_{2}}{v_{3}, v_{4}, v_{1:3}, v_{2:1}}} & +M^{\binom{u_{1234}, u_{4}, u_{12}, u_{2}}{v_{3}, v_{4:3}, v_{1:3}, v_{2:1}}} \\
-M^{\binom{u_{123}, u_{3}, u_{1}, u_{4}}{v_{2}, v_{3:2}, v_{1:2}, v_{4}}} & +M^{\binom{u_{123}, u_{4}, u_{23}, u_{3}}{v_{1}, v_{4}, v_{2:1}, v_{3:2}}} & -M^{\binom{u_{1234}, u_{4}, u_{12}, u_{1}}{v_{3}, v_{4:3}, v_{2:3}, v_{1:2}}} \\
+M^{\binom{u_{123}, u_{3}, u_{4}, u_{2}}{v_{1}, v_{3:1}, v_{4}, v_{2:1}}} & -M^{\binom{u_{123}, u_{4}, u_{23}, u_{2}}{v_{1}, v_{4}, v_{3:1}, v_{2:3}}} & +M^{\binom{u_{1234}, u_{1}, u_{34}, u_{4}}{v_{2}, v_{1:2}, v_{3:2}, v_{4:3}}} \\
+M^{\binom{u_{123}, u_{1}, u_{4}, u_{2}}{v_{3}, v_{1:3}, v_{4}, v_{2:3}}} & +M^{\binom{u_{1}, u_{234}, u_{23}, u_{2}}{v_{1}, v_{4}, v_{3:4}, v_{2:3}}} & -M^{\binom{u_{1234}, u_{1}, u_{34}, u_{3}}{v_{2}, v_{1:2}, v_{4:2}, v_{3:4}}} \\
-M^{\binom{u_{123}, u_{1}, u_{4}, u_{3}}{v_{2}, v_{1:2}, v_{4}, v_{3:2}}} & -M^{\binom{u_{1}, u_{234}, u_{23}, u_{3}}{v_{1}, v_{4}, v_{2:4}, v_{3:2}}} & +M^{\binom{u_{1234}, u_{123}, u_{1}, u_{2}}{v_{4}, v_{3:4}, v_{1:3}, v_{2:3}}} \\
-M^{\binom{u_{123}, u_{3}, u_{4}, u_{1}}{v_{2}, v_{3:2}, v_{4}, v_{1:2}}} & +M^{\binom{u_{1}, u_{234}, u_{34}, u_{4}}{v_{1}, v_{2}, v_{3:2}, v_{4:3}}} & +M^{\binom{u_{1234}, u_{123}, u_{3}, u_{2}}{v_{4}, v_{1:4}, v_{3:1}, v_{2:1}}} \\
+M^{\binom{u_{123}, u_{4}, u_{1}, u_{2}}{v_{3}, v_{4}, v_{1:3}, v_{2:3}}} & -M^{\binom{u_{1}, u_{234}, u_{34}, u_{3}}{v_{1}, v_{2}, v_{4:2}, v_{3:4}}} & -M^{\binom{u_{1234}, u_{123}, u_{1}, u_{3}}{v_{4}, v_{2:4}, v_{1:2}, v_{3:2}}} \\
+M^{\binom{u_{123}, u_{4}, u_{3}, u_{2}}{v_{1}, v_{4}, v_{3:1}, v_{2:1}}} & +M^{\binom{u_{1234}, u_{1}, u_{2}, u_{3}}{v_{4}, v_{1:4}, v_{2:4}, v_{3:4}}} & -M^{\binom{u_{1234}, u_{123}, u_{3}, u_{1}}{v_{4}, v_{2:4}, v_{3:2}, v_{1:2}}} \\
-M^{\binom{u_{123}, u_{4}, u_{3}, u_{1}}{v_{2}, v_{4}, v_{3:2}, v_{1:2}}} & +M^{\binom{u_{1234}, u_{1}, u_{4}, u_{3}}{v_{2}, v_{1:2}, v_{4:2}, v_{3:2}}} & +M^{\binom{u_{1234}, u_{234}, u_{2}, u_{4}}{v_{1}, v_{3:1}, v_{2:3}, v_{4:3}}} \\
-M^{\binom{u_{123}, u_{4}, u_{1}, u_{3}}{v_{2}, v_{4}, v_{1:2}, v_{3:2}}} & +M^{\binom{u_{1234}, u_{4}, u_{1}, u_{3}}{v_{2}, v_{4:2}, v_{1:2}, v_{3:2}}} & +M^{\binom{u_{1234}, u_{234}, u_{4}, u_{2}}{v_{1}, v_{3:1}, v_{4:3}, v_{2:3}}} \\
+M^{\binom{u_{1}, u_{234}, u_{4}, u_{3}}{v_{1}, v_{2}, v_{4:2}, v_{3:2}}} & +M^{\binom{u_{1234}, u_{4}, u_{3}, u_{1}}{v_{2}, v_{4:2}, v_{3:2}, v_{1:2}}} & -M^{\binom{u_{1234}, u_{234}, u_{2}, u_{3}}{v_{1}, v_{4:1}, v_{2:4}, v_{3:4}}} \\
+M^{\binom{u_{1}, u_{234}, u_{2}, u_{3}}{v_{1}, v_{4}, v_{2:4}, v_{3:4}}} & -M^{\binom{u_{1234}, u_{4}, u_{1}, u_{2}}{v_{3}, v_{4:3}, v_{1:3}, v_{2:3}}} & -M^{\binom{u_{1234}, u_{234}, u_{4}, u_{3}}{v_{1}, v_{2:1}, v_{4:2}, v_{3:2}}} \\
-M^{\binom{u_{1}, u_{234}, u_{2}, u_{4}}{v_{1}, v_{3}, v_{2:3}, v_{4:3}}} & -M^{\binom{u_{1234}, u_{1}, u_{4}, u_{2}}{v_{3}, v_{1:3}, v_{4:3}, v_{2:3}}} & +M^{\binom{u_{1234}, u_{123}, u_{12}, u_{1}}{v_{4}, v_{3:4}, v_{2:3}, v_{1:2}}} \\
-M^{\binom{u_{1}, u_{234}, u_{4}, u_{2}}{v_{1}, v_{3}, v_{4:3}, v_{2:3}}} & -M^{\binom{u_{1234}, u_{4}, u_{3}, u_{2}}{v_{1}, v_{4:1}, v_{3:1}, v_{2:1}}} & -M^{\binom{u_{1234}, u_{123}, u_{12}, u_{2}}{v_{4}, v_{3:4}, v_{1:3}, v_{2:1}}} \\
+M^{\binom{u_{12}, u_{34}, u_{2}, u_{4}}{v_{1}, v_{3}, v_{2:1}, v_{4:3}}} & -M^{\binom{u_{1234}, u_{1}, u_{2}, u_{4}}{v_{3}, v_{1:3}, v_{2:3}, v_{4:3}}} & +M^{\binom{u_{1234}, u_{123}, u_{23}, u_{3}}{v_{4}, v_{1:4}, v_{2:1}, v_{3:2}}} \\
+M^{\binom{u_{12}, u_{34}, u_{3}, u_{1}}{v_{2}, v_{4}, v_{3:4}, v_{1:2}}} & +M^{\binom{u_{1234}, u_{12}, u_{3}, u_{1}}{v_{4}, v_{2:4}, v_{3:4}, v_{1:2}}} & -M^{\binom{u_{1234}, u_{123}, u_{23}, u_{2}}{v_{4}, v_{1:4}, v_{3:1}, v_{2:3}}} \\
+M^{\binom{u_{12}, u_{34}, u_{1}, u_{3}}{v_{2}, v_{4}, v_{1:2}, v_{3:4}}} & +M^{\binom{u_{1234}, u_{12}, u_{1}, u_{3}}{v_{4}, v_{2:4}, v_{1:2}, v_{3:4}}} & +M^{\binom{u_{1234}, u_{234}, u_{23}, u_{3}}{v_{1}, v_{4:1}, v_{2:4}, v_{3:2}}} \\
+M^{\binom{u_{12}, u_{34}, u_{4}, u_{2}}{v_{1}, v_{3}, v_{4:3}, v_{2:1}}} & +M^{\binom{u_{1234}, u_{12}, u_{4}, u_{2}}{v_{3}, v_{1:3}, v_{4:3}, v_{2:1}}} & -M^{\binom{u_{1234}, u_{234}, u_{23}, u_{2}}{v_{1}, v_{4:1}, v_{3:4}, v_{2:3}}} \\
-M^{\binom{u_{12}, u_{34}, u_{2}, u_{3}}{v_{1}, v_{4}, v_{2:1}, v_{3:4}}} & +M^{\binom{u_{1234}, u_{12}, u_{2}, u_{4}}{v_{3}, v_{1:3}, v_{2:1}, v_{4:3}}} & +M^{\binom{u_{1234}, u_{234}, u_{34}, u_{3}}{v_{1}, v_{2:1}, v_{4:2}, v_{3:4}}} \\
-M^{\binom{u_{12}, u_{34}, u_{1}, u_{4}}{v_{2}, v_{3}, v_{1:2}, v_{4:3}}} & -M^{\binom{u_{1234}, u_{12}, u_{3}, u_{2}}{v_{4}, v_{1:4}, v_{3:4}, v_{2:1}}} & -M^{\binom{u_{1234}, u_{234}, u_{34}, u_{4}}{v_{1}, v_{2:1}, v_{3:2}, v_{4:3}}}
\end{array}
$$

The following tables give, for general signatures $\boldsymbol{m}(\underline{\boldsymbol{w}}) := (m_1, m_2, ...)$, the generalised scramble $SM^\bullet$ of $M^\bullet$.

$$\boldsymbol{m} := (1,2) \quad , \quad \underline{v}_1 = (v_1) \quad , \quad \underline{v}_2 = (v_2, v_2')$$

$$\begin{aligned}
SM^{\left(\begin{smallmatrix} u_1 , & u_2 \\ \underline{v}_1 , & \underline{v}_2 \end{smallmatrix}\right)} = \; & +M^{\left(\begin{smallmatrix} u_1 , & u_2 , & u_2 \\ v_1 , & v_2 , & v_{2':2} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{12} , & u_2 , & u_2 \\ v_1 , & v_{2:1} , & v_{2':2} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{12} , & u_1 , & u_2 \\ v_2 , & v_{1:2} , & v_{2':2} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{12} , & u_{12} , & u_2 \\ v_2 , & v_{1:2} , & v_{2':1} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{12} , & u_{12} , & u_1 \\ v_2 , & v_{2':2} , & v_{1:2'} \end{smallmatrix}\right)}
\end{aligned}$$

$$\boldsymbol{m} := (2,1) \quad , \quad \underline{v}_1 = (v_1, v_1') \quad , \quad \underline{v}_2 = (v_2)$$

$$\begin{aligned}
SM^{\left(\begin{smallmatrix} u_1 , & u_2 \\ \underline{v}_1 , & \underline{v}_2 \end{smallmatrix}\right)} = \; & +M^{\left(\begin{smallmatrix} u_1 , & u_1 , & u_2 \\ v_1 , & v_{1':1} , & v_2 \end{smallmatrix}\right)} && +M^{\left(\begin{smallmatrix} u_{12} , & u_{12} , & u_1 \\ v_1 , & v_{2:1} , & v_{1':2} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_1 , & u_2 , & u_1 \\ v_1 , & v_2 , & v_{1':1} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{12} , & u_2 , & u_1 \\ v_1 , & v_{2:1} , & v_{1':1} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{12} , & u_1 , & u_1 \\ v_2 , & v_{1:2} , & v_{1':1} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{12} , & u_{12} , & u_2 \\ v_1 , & v_{1':1} , & v_{2:1'} \end{smallmatrix}\right)}
\end{aligned}$$

$$\boldsymbol{m} := (1,3) \quad , \quad \underline{v}_1 = (v_1) \quad , \quad \underline{v}_2 = (v_2, v_2', v_2'')$$

$$\begin{aligned}
SM^{\left(\begin{smallmatrix} u_1 , & u_2 \\ \underline{v}_1 , & \underline{v}_2 \end{smallmatrix}\right)} = \; & +M^{\left(\begin{smallmatrix} u_1 , & u_2 , & u_2 , & u_2 \\ v_1 , & v_2 , & v_{2':2} , & v_{2'':2'} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{12} , & u_2 , & u_2 , & u_2 \\ v_1 , & v_{2:1} , & v_{2':2} , & v_{2'':2'} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{12} , & u_{12} , & u_{12} , & u_1 \\ v_2 , & v_{2':2} , & v_{2'':2'} , & v_{1:2''} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{12} , & u_{12} , & u_2 , & u_2 \\ v_2 , & v_{1:2} , & v_{2':1} , & v_{2'':2'} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{12} , & u_{12} , & u_1 , & u_2 \\ v_2 , & v_{2':2} , & v_{1:2'} , & v_{2'':2'} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{12} , & u_{12} , & u_{12} , & u_2 \\ v_2 , & v_{2':2} , & v_{1:2'} , & v_{2'':1} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{12} , & u_1 , & u_2 , & u_2 \\ v_2 , & v_{1:2} , & v_{2':2} , & v_{2'':2'} \end{smallmatrix}\right)}
\end{aligned}$$

$$\boldsymbol{m} := (2,2) \quad , \quad \underline{v}_1 = (v_1, v_1') \quad , \quad \underline{v}_2 = (v_2, v_2')$$

$$\begin{aligned}
SM^{\left(\begin{smallmatrix} u_1 , & u_2 \\ \underline{v}_1 , & \underline{v}_2 \end{smallmatrix}\right)} = \; & +M^{\left(\begin{smallmatrix} u_1 , & u_2 , & u_2 , & u_1 \\ v_1 , & v_2 , & v_{2':2} , & v_{1':1} \end{smallmatrix}\right)} && +M^{\left(\begin{smallmatrix} u_{12} , & u_{12} , & u_{12} , & u_1 \\ v_2 , & v_{1:2} , & v_{2':1} , & v_{1':2'} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_1 , & u_2 , & u_1 , & u_2 \\ v_1 , & v_2 , & v_{1':1} , & v_{2':2} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{12} , & u_2 , & u_1 , & u_2 \\ v_1 , & v_{2:1} , & v_{1':1} , & v_{2':2} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_1 , & u_1 , & u_2 , & u_2 \\ v_1 , & v_{1':1} , & v_2 , & v_{2':2} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{12} , & u_{12} , & u_2 , & u_2 \\ v_1 , & v_{1':1} , & v_{2:1'} , & v_{2':2} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{12} , & u_{12} , & u_1 , & u_2 \\ v_1 , & v_{2:1} , & v_{1':2} , & v_{2':2} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{12} , & u_{12} , & u_{12} , & u_2 \\ v_2 , & v_{1:2} , & v_{1':1} , & v_{2':1'} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{12} , & u_{12} , & u_{12} , & u_1 \\ v_1 , & v_{2:1} , & v_{2':2} , & v_{1':2'} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{12} , & u_{12} , & u_{12} , & u_2 \\ v_1 , & v_{2:1} , & v_{1':2} , & v_{2':1'} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{12} , & u_{12} , & u_1 , & u_1 \\ v_2 , & v_{2':2} , & v_{1:2'} , & v_{1':1} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{12} , & u_{12} , & u_2 , & u_1 \\ v_2 , & v_{1:2} , & v_{2':1} , & v_{1':1} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{12} , & u_1 , & u_2 , & u_1 \\ v_2 , & v_{1:2} , & v_{2':2} , & v_{1':1} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{12} , & u_2 , & u_2 , & u_1 \\ v_1 , & v_{2:1} , & v_{2':2} , & v_{1':1} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{12} , & u_1 , & u_1 , & u_2 \\ v_2 , & v_{1:2} , & v_{1':1} , & v_{2':2} \end{smallmatrix}\right)}
\end{aligned}$$

$$\boldsymbol{m} := (3,1) \quad , \quad \underline{v}_1 = (v_1, v_1', v_1'') \quad , \quad \underline{v}_2 = (v_2)$$

$$\begin{aligned}
SM^{\left(\begin{smallmatrix} u_1 ,& u_2 \\ \underline{v}_1 ,& \underline{v}_2 \end{smallmatrix}\right)} = \; & +M^{\left(\begin{smallmatrix} u_1 ,& u_1 ,& u_1 ,& u_2 \\ v_1 ,& v_{1':1} ,& v_{1'':1'} ,& v_2 \end{smallmatrix}\right)} && +M^{\left(\begin{smallmatrix} u_1 ,& u_1 ,& u_2 ,& u_1 \\ v_1 ,& v_{1':1} ,& v_2 ,& v_{1'':1'} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_1 ,& u_2 ,& u_1 ,& u_1 \\ v_1 ,& v_2 ,& v_{1':1} ,& v_{1'':1'} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{12} ,& u_{12} ,& u_{12} ,& u_2 \\ v_1 ,& v_{1':1} ,& v_{1'':1'} ,& v_{2:1''} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{12} ,& u_{12} ,& u_{12} ,& u_1 \\ v_1 ,& v_{1':1} ,& v_{2:1'} ,& v_{1'':2} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{12} ,& u_{12} ,& u_2 ,& u_1 \\ v_1 ,& v_{1':1} ,& v_{2:1'} ,& v_{1'':1'} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{12} ,& u_{12} ,& u_1 ,& u_1 \\ v_1 ,& v_{2:1} ,& v_{1':2} ,& v_{1'':1'} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{12} ,& u_2 ,& u_1 ,& u_1 \\ v_1 ,& v_{2:1} ,& v_{1':1} ,& v_{1'':1'} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{12} ,& u_1 ,& u_1 ,& u_1 \\ v_2 ,& v_{1:2} ,& v_{1':1} ,& v_{1'':1'} \end{smallmatrix}\right)}
\end{aligned}$$

$$\boldsymbol{m} := (1,1,2) \quad , \quad \underline{v}_1 = (v_1) \quad , \quad \underline{v}_2 = (v_2) \quad , \quad \underline{v}_3 = (v_3, v_3')$$

$$\begin{aligned}
SM^{\left(\begin{smallmatrix} u_1 ,& u_2 ,& u_3 \\ \underline{v}_1 ,& \underline{v}_2 ,& \underline{v}_3 \end{smallmatrix}\right)} = \; & +M^{\left(\begin{smallmatrix} u_1 ,& u_2 ,& u_3 ,& u_3 \\ v_1 ,& v_2 ,& v_3 ,& v_{3':3} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{12} ,& u_2 ,& u_3 ,& u_3 \\ v_1 ,& v_{2:1} ,& v_3 ,& v_{3':3} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{12} ,& u_3 ,& u_1 ,& u_3 \\ v_2 ,& v_3 ,& v_{1:2} ,& v_{3':3} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{12} ,& u_3 ,& u_3 ,& u_2 \\ v_1 ,& v_3 ,& v_{3':3} ,& v_{2:1} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{12} ,& u_1 ,& u_3 ,& u_3 \\ v_2 ,& v_{1:2} ,& v_3 ,& v_{3':3} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_1 ,& u_{23} ,& u_3 ,& u_3 \\ v_1 ,& v_2 ,& v_{3:2} ,& v_{3':3} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{12} ,& u_3 ,& u_3 ,& u_1 \\ v_2 ,& v_3 ,& v_{3':3} ,& v_{1:2} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{12} ,& u_3 ,& u_2 ,& u_3 \\ v_1 ,& v_3 ,& v_{2:1} ,& v_{3':3} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_1 ,& u_{23} ,& u_2 ,& u_3 \\ v_1 ,& v_3 ,& v_{2:3} ,& v_{3':3} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_1 ,& u_{23} ,& u_{23} ,& u_3 \\ v_1 ,& v_3 ,& v_{2:3} ,& v_{3':2} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{123} ,& u_3 ,& u_3 ,& u_2 \\ v_1 ,& v_{3:1} ,& v_{3':3} ,& v_{2:1} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{123} ,& u_1 ,& u_3 ,& u_3 \\ v_2 ,& v_{1:2} ,& v_{3:2} ,& v_{3':3} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{123} ,& u_3 ,& u_2 ,& u_3 \\ v_1 ,& v_{3:1} ,& v_{2:1} ,& v_{3':3} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{123} ,& u_3 ,& u_3 ,& u_1 \\ v_2 ,& v_{3:2} ,& v_{3':3} ,& v_{1:2} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_1 ,& u_{23} ,& u_{23} ,& u_2 \\ v_1 ,& v_3 ,& v_{3':3} ,& v_{2:3'} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{123} ,& u_3 ,& u_1 ,& u_3 \\ v_2 ,& v_{3:2} ,& v_{1:2} ,& v_{3':3} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{123} ,& u_{123} ,& u_3 ,& u_2 \\ v_3 ,& v_{1:3} ,& v_{3':1} ,& v_{2:1} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{123} ,& u_{12} ,& u_2 ,& u_3 \\ v_3 ,& v_{1:3} ,& v_{2:1} ,& v_{3':3} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{123} ,& u_{12} ,& u_1 ,& u_3 \\ v_3 ,& v_{2:3} ,& v_{1:2} ,& v_{3':3} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{123} ,& u_{12} ,& u_3 ,& u_2 \\ v_3 ,& v_{1:3} ,& v_{3':3} ,& v_{2:1} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{123} ,& u_{23} ,& u_3 ,& u_3 \\ v_1 ,& v_{2:1} ,& v_{3:2} ,& v_{3':3} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{123} ,& u_{123} ,& u_{12} ,& u_2 \\ v_3 ,& v_{3':3} ,& v_{1:3'} ,& v_{2:1} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{123} ,& u_1 ,& u_{23} ,& u_2 \\ v_3 ,& v_{1:3} ,& v_{3':3} ,& v_{2:3'} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{123} ,& u_{123} ,& u_{23} ,& u_2 \\ v_3 ,& v_{1:3} ,& v_{3':1} ,& v_{2:3'} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{123} ,& u_{123} ,& u_{23} ,& u_3 \\ v_3 ,& v_{1:3} ,& v_{2:1} ,& v_{3':2} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{123} ,& u_{23} ,& u_{23} ,& u_2 \\ v_1 ,& v_{3:1} ,& v_{3':3} ,& v_{2:3'} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{123} ,& u_{123} ,& u_1 ,& u_2 \\ v_3 ,& v_{3':3} ,& v_{1:3'} ,& v_{2:3'} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{123} ,& u_{123} ,& u_3 ,& u_1 \\ v_3 ,& v_{2:3} ,& v_{3':2} ,& v_{1:2} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{123} ,& u_{12} ,& u_3 ,& u_1 \\ v_3 ,& v_{2:3} ,& v_{3':3} ,& v_{1:2} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{123} ,& u_{123} ,& u_1 ,& u_3 \\ v_3 ,& v_{2:3} ,& v_{1:2} ,& v_{3':2} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{123} ,& u_{123} ,& u_{12} ,& u_1 \\ v_3 ,& v_{3':3} ,& v_{2:3'} ,& v_{1:2)} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{123} ,& u_{23} ,& u_2 ,& u_3 \\ v_1 ,& v_{3:1} ,& v_{2:3} ,& v_{3':3} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{123} ,& u_1 ,& u_2 ,& u_3 \\ v_3 ,& v_{1:3} ,& v_{2:3} ,& v_{3':3} \end{smallmatrix}\right)} && -M^{\left(\begin{smallmatrix} u_{123} ,& u_1 ,& u_{23} ,& u_3 \\ v_3 ,& v_{1:3} ,& v_{2:3} ,& v_{3':2} \end{smallmatrix}\right)} \\
& +M^{\left(\begin{smallmatrix} u_{123} ,& u_{23} ,& u_{23} ,& u_3 \\ v_1 ,& v_{3:1} ,& v_{2:3} ,& v_{3':2} \end{smallmatrix}\right)}
\end{aligned}$$

$$\boldsymbol{m} := (1,2,1) \quad , \quad \underline{v}_1 = (v_1) \quad , \quad \underline{v}_2 = (v_2, v_2') \quad , \quad \underline{v}_3 = (v_3)$$

$$\begin{aligned}
SM^{\binom{u_1\,,\,u_2\,,\,u_3}{\underline{v}_1\,,\,\underline{v}_2\,,\,\underline{v}_3}} = \; & +M^{\binom{u_1, u_2, u_2, u_3}{v_1, v_2, v_{2':2}, v_3}} && +M^{\binom{u_1, u_2, u_3, u_2}{v_1, v_2, v_3, v_{2':2}}} \\
& +M^{\binom{u_1, u_{23}, u_2, u_2}{v_1, v_3, v_{2:3}, v_{2':2}}} && -M^{\binom{u_{12}, u_2, u_2, u_3}{v_1, v_{2:1}, v_{2':2}, v_3}} \\
& +M^{\binom{u_{12}, u_1, u_3, u_2}{v_2, v_{1:2}, v_3, v_{2':2}}} && -M^{\binom{u_{12}, u_2, u_3, u_2}{v_1, v_{2:1}, v_3, v_{2':2}}} \\
& +M^{\binom{u_1, u_{23}, u_{23}, u_2}{v_1, v_2, v_{3:2}, v_{2':3}}} && -M^{\binom{u_{12}, u_3, u_{12}, u_2}{v_2, v_3, v_{1:2}, v_{2':1}}} \\
& +M^{\binom{u_{12}, u_3, u_1, u_2}{v_2, v_3, v_{1:2}, v_{2':2}}} && -M^{\binom{u_1, u_{23}, u_3, u_2}{v_1, v_2, v_{3:2}, v_{2':2}}} \\
& +M^{\binom{u_{123}, u_1, u_2, u_2}{v_3, v_{1:3}, v_{2:3}, v_{2':2}}} && -M^{\binom{u_{12}, u_{12}, u_2, u_3}{v_2, v_{1:2}, v_{2':1}, v_3}} \\
& +M^{\binom{u_{12}, u_1, u_2, u_3}{v_2, v_{1:2}, v_{2':2}, v_3}} && -M^{\binom{u_{12}, u_{12}, u_3, u_2}{v_2, v_{1:2}, v_3, v_{2':1}}} \\
& +M^{\binom{u_{12}, u_{12}, u_1, u_3}{v_2, v_{2':2}, v_{1:2'}, v_3}} && -M^{\binom{u_{12}, u_3, u_2, u_2}{v_1, v_3, v_{2:1}, v_{2':2}}} \\
& +M^{\binom{u_{12}, u_3, u_{12}, u_1}{v_2, v_3, v_{2':2}, v_{1:2}}} && -M^{\binom{u_1, u_{23}, u_{23}, u_3}{v_1, v_2, v_{2':2}, v_{3:2'}}} \\
& +M^{\binom{u_{12}, u_{12}, u_3, u_1}{v_2, v_{2':2}, v_3, v_{1:2'}}} && -M^{\binom{u_{123}, u_{23}, u_2, u_2}{v_1, v_{3:1}, v_{2:3}, v_{2':2}}} \\
& +M^{\binom{u_{123}, u_3, u_2, u_2}{v_1, v_{3:1}, v_{2:1}, v_{2':2}}} && -M^{\binom{u_{123}, u_{123}, u_{23}, u_2}{v_2, v_{1:2}, v_{3:1}, v_{2':3}}} \\
& +M^{\binom{u_{123}, u_{23}, u_3, u_2}{v_1, v_{2:1}, v_{3:2}, v_{2':2}}} && -M^{\binom{u_{123}, u_{12}, u_{12}, u_2}{v_3, v_{2:3}, v_{1:2}, v_{2':1}}} \\
& +M^{\binom{u_{123}, u_{123}, u_{12}, u_1}{v_2, v_{3:2}, v_{2':3}, v_{1:2'}}} && -M^{\binom{u_{123}, u_1, u_3, u_2}{v_2, v_{1:2}, v_{3:2}, v_{2':2}}} \\
& +M^{\binom{u_{123}, u_{23}, u_{23}, u_3}{v_1, v_{2:1}, v_{2':2}, v_{3:2'}}} && -M^{\binom{u_{123}, u_{12}, u_2, u_2}{v_3, v_{1:3}, v_{2:1}, v_{2':2}}} \\
& +M^{\binom{u_{123}, u_{123}, u_{23}, u_3}{v_2, v_{1:2}, v_{2':1}, v_{3:2'}}} && -M^{\binom{u_{123}, u_{123}, u_3, u_1}{v_2, v_{2':2}, v_{3:2'}, v_{1:2'}}} \\
& +M^{\binom{u_{123}, u_{123}, u_1, u_2}{v_2, v_{3:2}, v_{1:3}, v_{2':3}}} && -M^{\binom{u_{123}, u_{123}, u_{12}, u_2}{v_2, v_{3:2}, v_{1:3}, v_{2':1}}} \\
& +M^{\binom{u_{123}, u_3, u_{12}, u_2}{v_2, v_{3:2}, v_{1:2}, v_{2':1}}} && -M^{\binom{u_{123}, u_1, u_{23}, u_3}{v_2, v_{1:2}, v_{2':2}, v_{3:2'}}} \\
& +M^{\binom{u_{123}, u_{123}, u_3, u_2}{v_2, v_{1:2}, v_{3:1}, v_{2':1}}} && -M^{\binom{u_{123}, u_{123}, u_1, u_3}{v_2, v_{2':2}, v_{1:2'}, v_{3:2'}}} \\
& +M^{\binom{u_{123}, u_{12}, u_{12}, u_1}{v_3, v_{2:3}, v_{2':2}, v_{1:2'}}} && -M^{\binom{u_{123}, u_3, u_1, u_2}{v_2, v_{3:2}, v_{1:2)}, v_{2':2}}} \\
& +M^{\binom{u_{123}, u_1, u_{23}, u_2}{v_2, v_{1:2}, v_{3:2}, v_{2':3}}} && -M^{\binom{u_{123}, u_3, u_{12}, u_1}{v_2, v_{3:2}, v_{2':2}, v_{1:2'}}} \\
& +M^{\binom{u_{123}, u_{12}, u_1, u_2}{v_3, v_{2:3}, v_{1:2}, v_{2':2}}} && -M^{\binom{u_{123}, u_{23}, u_{23}, u_2}{v_1, v_{2:1}, v_{3:2}, v_{2':3}}}
\end{aligned}$$

$$\boldsymbol{m} := (2,1,1) \quad , \quad \underline{v}_1 = (v_1, v_1') \quad , \quad \underline{v}_2 = (v_2) \quad , \quad \underline{v}_3 = (v_3)$$

$$\begin{aligned}
SM^{\binom{u_1\,,\,u_2\,,\,u_3}{\underline{v}_1\,,\,\underline{v}_2\,,\,\underline{v}_3}} = \; & +M^{\binom{u_1\,,\,u_1\,,\,u_2\,,\,u_3}{v_1\,,\,v_{1':1}\,,\,v_2\,,\,v_3}} && +M^{\binom{u_1\,,\,u_2\,,\,u_3\,,\,u_1}{v_1\,,\,v_2\,,\,v_3\,,\,v_{1':1}}} \\
& +M^{\binom{u_1\,,\,u_2\,,\,u_1\,,\,u_3}{v_1\,,\,v_2\,,\,v_{1':1}\,,\,v_3}} && -M^{\binom{u_1\,,\,u_1\,,\,u_{23}\,,\,u_3}{v_1\,,\,v_{1':1}\,,\,v_2\,,\,v_{3:2}}} \\
& +M^{\binom{u_1\,,\,u_{23}\,,\,u_1\,,\,u_2}{v_1\,,\,v_3\,,\,v_{1':1}\,,\,v_{2:3}}} && -M^{\binom{u_{12}\,,\,u_3\,,\,u_2\,,\,u_1}{v_1\,,\,v_3\,,\,v_{2:1'}\,,\,v_{1':1}}} \\
& +M^{\binom{u_{12}\,,\,u_3\,,\,u_1\,,\,u_1}{v_2\,,\,v_3\,,\,v_{1:2}\,,\,v_{1':1}}} && -M^{\binom{u_{12}\,,\,u_2\,,\,u_1\,,\,u_3}{v_1\,,\,v_{2:1}\,,\,v_{1':1}\,,\,v_3}} \\
& +M^{\binom{u_{12}\,,\,u_1\,,\,u_3\,,\,u_1}{v_2\,,\,v_{1:2}\,,\,v_3\,,\,v_{1':1}}} && -M^{\binom{u_1\,,\,u_{23}\,,\,u_3\,,\,u_1}{v_1\,,\,v_2\,,\,v_{3:2}\,,\,v_{1':1}}} \\
& +M^{\binom{u_1\,,\,u_1\,,\,u_{23}\,,\,u_2}{v_1\,,\,v_{1':1}\,,\,v_3\,,\,v_{2:3}}} && -M^{\binom{u_{12}\,,\,u_2\,,\,u_3\,,\,u_1}{v_1\,,\,v_{2:1}\,,\,v_3\,,\,v_{1':1}}} \\
& +M^{\binom{u_{12}\,,\,u_1\,,\,u_1\,,\,u_3}{v_2\,,\,v_{1:2}\,,\,v_{1':1}\,,\,v_3}} && -M^{\binom{u_1\,,\,u_{23}\,,\,u_1\,,\,u_3}{v_1\,,\,v_2\,,\,v_{1':1}\,,\,v_{3:2}}} \\
& +M^{\binom{u_1\,,\,u_{23}\,,\,u_2\,,\,u_1}{v_1\,,\,v_3\,,\,v_{2:3}\,,\,v_{1':1}}} && -M^{\binom{u_{12}\,,\,u_{12}\,,\,u_2\,,\,u_3}{v_1\,,\,v_{1':1}\,,\,v_{2:1'}\,,\,v_3}} \\
& +M^{\binom{u_{12}\,,\,u_{12}\,,\,u_1\,,\,u_3}{v_1\,,\,v_{2:1}\,,\,v_{1':2}\,,\,v_3}} && -M^{\binom{u_{12}\,,\,u_3\,,\,u_{12}\,,\,u_2}{v_1\,,\,v_3\,,\,v_{1':1}\,,\,v_{2:1'}}} \\
& +M^{\binom{u_{12}\,,\,u_3\,,\,u_{12}\,,\,u_1}{v_1\,,\,v_3\,,\,v_{2:1}\,,\,v_{1':2}}} && -M^{\binom{u_{12}\,,\,u_{12}\,,\,u_3\,,\,u_2}{v_1\,,\,v_{1':1}\,,\,v_3\,,\,v_{2:1'}}} \\
& +M^{\binom{u_{12}\,,\,u_{12}\,,\,u_3\,,\,u_1}{v_1\,,\,v_{2:1}\,,\,v_3\,,\,v_{1':2}}} && -M^{\binom{u_{123}\,,\,u_3\,,\,u_1\,,\,u_1}{v_2\,,\,v_{3:2}\,,\,v_{1:2}\,,\,v_{1':1}}} \\
& +M^{\binom{u_{123}\,,\,u_1\,,\,u_2\,,\,u_1}{v_3\,,\,v_{1:3}\,,\,v_{2:3}\,,\,v_{1':1}}} && -M^{\binom{u_{123}\,,\,u_1\,,\,u_3\,,\,u_1}{v_2\,,\,v_{1:2}\,,\,v_{3:2}\,,\,v_{1':1}}} \\
& +M^{\binom{u_{123}\,,\,u_1\,,\,u_1\,,\,u_2}{v_3\,,\,v_{1:3}\,,\,v_{1':1}\,,\,v_{2:3}}} && -M^{\binom{u_{123}\,,\,u_{23}\,,\,u_2\,,\,u_1}{v_1\,,\,v_{3:1}\,,\,v_{2:3}\,,\,v_{1':1}}} \\
& +M^{\binom{u_{123}\,,\,u_3\,,\,u_2\,,\,u_1}{v_1\,,\,v_{3:1}\,,\,v_{2:1}\,,\,v_{1':1}}} && -M^{\binom{u_{123}\,,\,u_3\,,\,u_{12}\,,\,u_1}{v_1\,,\,v_{3:1}\,,\,v_{2:1}\,,\,v_{1':2}}} \\
& +M^{\binom{u_{123}\,,\,u_{23}\,,\,u_3\,,\,u_1}{v_1\,,\,v_{2:1}\,,\,v_{3:2}\,,\,v_{1':1}}} && -M^{\binom{u_{123}\,,\,u_{123}\,,\,u_1\,,\,u_3}{v_1\,,\,v_{2:1}\,,\,v_{1':2}\,,\,v_{3:2}}} \\
& +M^{\binom{u_{123}\,,\,u_{23}\,,\,u_1\,,\,u_3}{v_1\,,\,v_{2:1}\,,\,v_{1':1}\,,\,v_{3:2}}} && -M^{\binom{u_{123}\,,\,u_1\,,\,u_1\,,\,u_3}{v_2\,,\,v_{1:2}\,,\,v_{1':1}\,,\,v_{3:2}}} \\
& +M^{\binom{u_{123}\,,\,u_{12}\,,\,u_1\,,\,u_1}{v_3\,,\,v_{2:3}\,,\,v_{1:2}\,,\,v_{1':1}}} && -M^{\binom{u_{123}\,,\,u_{12}\,,\,u_2\,,\,u_1}{v_3\,,\,v_{1:3}\,,\,v_{2:1}\,,\,v_{1':1)}}} \\
& +M^{\binom{u_{123}\,,\,u_{123}\,,\,u_{23}\,,\,u_3}{v_1\,,\,v_{1':1}\,,\,v_{2:1'}\,,\,v_{3:2}}} && -M^{\binom{u_{123}\,,\,u_{23}\,,\,u_1\,,\,u_2}{v_1\,,\,v_{3:1}\,,\,v_{1':1}\,,\,v_{2:3}}} \\
& +M^{\binom{u_{123}\,,\,u_{123}\,,\,u_{12}\,,\,u_1}{v_1\,,\,v_{3:1}\,,\,v_{2:3}\,,\,v_{1':2}}} && -M^{\binom{u_{123}\,,\,u_{123}\,,\,u_3\,,\,u_1}{v_1\,,\,v_{2:1}\,,\,v_{3:2}\,,\,v_{1':2}}} \\
& +M^{\binom{u_{123}\,,\,u_3\,,\,u_{12}\,,\,u_2}{v_1\,,\,v_{3:1}\,,\,v_{1':1}\,,\,v_{2:1'}}} && -M^{\binom{u_{123}\,,\,u_{123}\,,\,u_{23}\,,\,u_2}{v_1\,,\,v_{1':1}\,,\,v_{3:1'}\,,\,v_{2:3}}} \\
& +M^{\binom{u_{123}\,,\,u_{123}\,,\,u_1\,,\,u_2}{v_1\,,\,v_{3:1}\,,\,v_{1':3}\,,\,v_{2:3}}} && -M^{\binom{u_{123}\,,\,u_{12}\,,\,u_{12}\,,\,u_2}{v_3\,,\,v_{1:3}\,,\,v_{1':1}\,,\,v_{2:1'}}} \\
& +M^{\binom{u_{123}\,,\,u_{123}\,,\,u_3\,,\,u_2}{v_1\,,\,v_{1':1}\,,\,v_{3:1'}\,,\,v_{2:1'}}} && -M^{\binom{u_{123}\,,\,u_{123}\,,\,u_{12}\,,\,u_2}{v_1\,,\,v_{3:1}\,,\,v_{1':3}\,,\,v_{2:1'}}} \\
& +M^{\binom{u_{123}\,,\,u_{12}\,,\,u_{12}\,,\,u_1}{v_3\,,\,v_{1:3}\,,\,v_{2:1}\,,\,v_{1':2}}}
\end{aligned}$$