# Resurgence's two main types and their signature complications: tessellation, isography, autarchy.

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## Contents.

## 1. The barest basics about resurgence.

- **The three models: formal, convolutive, geometric.**

Resurgent functions live in three models:
(i) In the *formal model*, as formal power series or transseries $\widetilde{\varphi}(z)$ of $z^{-1}$.
(ii) In the *convolution model* or *Borel plane*, as analytic germs $\widehat{\varphi}(\zeta)$ at 0, endlessly continuable (laterally along any finitely broken line).
(iii) In the *geometric models*, as sectorial germs $\varphi_\theta(z)$ at $\infty$ in $z$.

$(i)$ $\quad \widetilde{\varphi}(z) = \sum a_n z^{-n}$ *multiplicative*

$\downarrow$ *Borel*

$(ii)$ $\quad \widehat{\varphi}(\zeta) = \sum a_n \frac{\zeta^{n-1}}{(n-1)!}$ *convolutive* $\begin{cases} (\widehat{\varphi}_1 * \widehat{\varphi}_2)(\zeta) := \\ \int_0^\zeta \widehat{\varphi}_1(\zeta_1)\, \widehat{\varphi}_2(\zeta - \zeta_1) d\zeta_1 \end{cases}$

$\downarrow$ *Laplace*

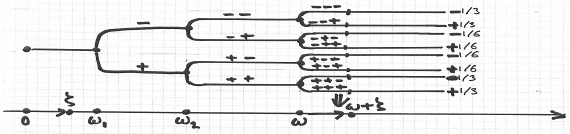
$(iii)$ $\quad \varphi_\theta(z) = \int_{\arg \zeta = \theta} e^{-z\zeta} \widehat{\varphi}(\zeta) d\zeta$ *multiplicative*

The singularities of $\widehat{\varphi}(\zeta)$ carry the Stokes constants and are responsible for the divergence of $\widetilde{\varphi}(z)$. So they deserve close attention.
The tools for measuring them are the so-called *alien derivations* $\Delta_\omega$.

# 1. The barest basics about resurgence.

## • Standard alien derivations.

The one oustanding fact about resurgent functions is the existence on them of a huge array of exotic derivations – the so-called *alien derivations* $\Delta_\omega$ $(\omega \in \mathbb{C}_\bullet = \widetilde{\mathbb{C} - \{0\}})$. They are bound by no *a priori* constraints.




$$\widehat{\Delta}_\omega \widehat{\varphi}(\zeta) = \sum_{\epsilon_i=\pm} \delta^{p,q} \left( \widehat{\varphi}^{\left(\begin{smallmatrix}\epsilon_1 \\ \omega_1\end{smallmatrix};\begin{smallmatrix}\epsilon_2 \\ \omega_2\end{smallmatrix};\begin{smallmatrix}+ \\ \omega\end{smallmatrix}\right)}(\zeta+\omega) - \widehat{\varphi}^{\left(\begin{smallmatrix}\epsilon_1 \\ \omega_1\end{smallmatrix};\begin{smallmatrix}\epsilon_2 \\ \omega_2\end{smallmatrix};\begin{smallmatrix}- \\ \omega\end{smallmatrix}\right)}(\zeta+\omega) \right)$$

$$\widehat{\Delta}_\omega \widehat{\varphi}(\zeta) = \frac{1}{2\pi i} \sum_{\epsilon_i \in \{+,-\}} \frac{p!\, q!}{(p+q+1)!} \begin{cases} +\widehat{\varphi}^{\left(\begin{smallmatrix}\epsilon_1 & ,\dots, & \epsilon_{r-1} & , & + \\ \omega_1 & ,\dots, & \omega_{r-1} & , & \omega\end{smallmatrix}\right)}(\zeta+\omega) \\ -\widehat{\varphi}^{\left(\begin{smallmatrix}\epsilon_1 & ,\dots, & \epsilon_{r-1} & , & - \\ \omega_1 & ,\dots, & \omega_{r-1} & , & \omega\end{smallmatrix}\right)}(\zeta+\omega) \end{cases}$$

$$\widehat{\Delta}_\omega \ (convol.model) \leftrightarrow \Delta_\omega \ (mult.models) \rightarrow \begin{cases} \mathbb{\Delta}_\omega := e^{-\omega z} \Delta_\omega \\ [\partial_z, \mathbb{\Delta}_\omega] \equiv 0 \end{cases}$$

The $\mathbb{\Delta}_\omega$ (with double-struck $\Delta$) are the *invariant alien derivations*.

## 1. The barest basics about resurgence.

• **Active alien algebras.**
Let $\mathcal{A}$ be an algebra of resurgent functions.
Let $\overline{\mathcal{A}}$ be its closure under (ordinary and alien) differentiation.
Let $\mathbb{I}_{\mathcal{A}}$ be the bilateral ideal of $\mathbf{\Delta}$ that annihilates $\overline{\mathcal{A}}$.
The quotient $\mathbf{\Delta}_{\mathcal{A}} := \mathbf{\Delta}/\mathbb{I}_{\mathcal{A}}$ is known as $\mathcal{A}$'s *active alien algebra*.

• **Displays.** The display of a resurgent $\widetilde{\varphi}$ is defined by:

$$\mathrm{dpl}\,\widetilde{\varphi} := \widetilde{\varphi} + \sum_{1\leq r}\sum_{\omega_i\in\mathbb{C}^*}\mathbb{Z}^{\omega_1,\ldots,\omega_r}\,\mathbf{\Delta}_{\omega_r}\ldots\mathbf{\Delta}_{\omega_1}\,\widetilde{\varphi}$$

with symmetral, $z$-constant symbols $\mathbb{Z}^{\boldsymbol{\omega}}$:

$$[\partial,\mathbb{Z}^{\boldsymbol{\omega}}]\equiv 0 \ \ , \ \ \mathbb{Z}^{\boldsymbol{\omega}^1}\,\mathbb{Z}^{\boldsymbol{\omega}^2} = \sum_{\boldsymbol{\omega}\in\mathrm{sha}(\boldsymbol{\omega}^1,\boldsymbol{\omega}^2)}\mathbb{Z}^{\boldsymbol{\omega}}$$

Main property: $\mathcal{R}(\widetilde{\varphi}_1,\ldots,\widetilde{\varphi}_s)=0 \Longrightarrow \mathcal{R}(\mathrm{dpl}\widetilde{\varphi}_1,\ldots,\mathrm{dpl}\widetilde{\varphi}_s)=0$

## 2. Equational vs coequational resurgence.

### Equational resurgence

- *Relative to a critical variable.*
- *Active alien algebras* $\sim$ *One-piece algebras of ordinary diff. operators.*
- *One Bridge Equation.*
- *Complex valued Stokes constants.*
- *Governed by ordinary convolution.*
- *Diff. operators : unconstrained.*
- *Sums ramified at* $\infty$.
- *Straightforward proofs & statements.*

### Coequational resurgence

- *Relative to a critical parameter.*
- *Active alien algebras* $\sim$ *Two-piece algebras of ordinary diff. operators.*
- *Two Bridge Equations.*
- *Discrete tessellation coefficients.*
- *Governed by weighted convolution.*
- *Diff. oper. constrained by isography.*
- *Autark sums unramified at* $\infty$.
- *Much higher levels of complexity.*

## 3. Equational resurgence: a brief review.

- **Critical variables (or critical 'times').**

Start from an equation $E(\varphi) = 0$ (differential, difference, functional etc). Form its *full (parameter saturated) solution* $\widetilde{\varphi}(z, \boldsymbol{t})$ with $\boldsymbol{t} = (t_1, ..., t_s)$. Rule of thumb: there are as many *critical times* $z_\alpha = z^\alpha$ as there are exponential blocks $e^{-\omega z^\alpha}$ copresent with negative powers of $z$ in $\widetilde{\varphi}(z, \boldsymbol{t})$. Resurgence equations: $\{E(\varphi) = 0\} \Rightarrow \{E_\omega(\varphi, \mathbf{\Delta}_\omega \varphi) = 0\}$. Formally solvable, up to the integration constants (Stokes constants).

- **Bridge equation.**

$$\mathbf{\Delta}_\omega \varphi(z, \boldsymbol{t}) = \mathbb{A}_\omega \varphi(z, \boldsymbol{t}) \quad \textit{with} \quad \begin{cases} \omega \in \Omega \quad (\textit{res. support}) \\ \mathbb{A}_\omega \;\; \textit{ordinary diff. operators in} \, (z, \boldsymbol{t}). \end{cases}$$

The operators $A_\omega$ carry the Stokes contants as coefficients. Otherwise, they are subject to no other constraints than 'making sense', i.e. sensibly pairing off the exponentials on both sides of the Bridge equation.

B.E. keeps the part of Analysis down to a minimum. B.E. also covers a huge ground, succeeding in situations where all competing methods fail.

## 3. Equational resurgence: a brief review.

• **Display.** Despite looking like a magnified version of the full solution $\varphi(z, \boldsymbol{t})$, the display $\mathrm{dpl}\,\widetilde{\varphi} := \widetilde{\varphi} + \sum_{1\leq r} \sum_{\omega_i \in \mathbb{C}^*} \mathbb{Z}^{\omega_1,\ldots,\omega_r} \boldsymbol{\Delta}_{\omega_r} \ldots \boldsymbol{\Delta}_{\omega_1} \widetilde{\varphi}$ carries far more information: it has far more components and also encodes the Stokes constants. In fact, it amounts to more than even the full solution *plus* the Stokes contants, due to transport property $\mathcal{R}(\varphi_1, ..., \varphi_s) = 0 \Rightarrow \mathcal{R}(dpl.\varphi_1, ..., dpl.\varphi_s) = 0$ with the independence relations and transcendence properties that flow therefrom.

• **Resurgence and self-coherence: the part implies the whole.** The resurgent solutions of a singular equation cohere in a way that convergent solutions do not and cannot: the full solution (nay, the original equation) can be recovered from a particular solution, and that too fully constructively (through alien differentiation).

Analogy with irreducible polynomials (recoverable from a single root).

## 4. Model problem: singular & singularly perturbed system.

Consider this model instance of a *doubly singular* differential system:

$$0 = \epsilon\, t^2\, \partial_t\, y^i + \lambda_i\, y^i + b^i(t, \epsilon, y^1, \ldots, y^\nu) \quad (1 \le i \le \nu) \begin{cases} t \sim 0 \ (\textit{variable}) \\ \epsilon \sim 0 \ (\textit{parameter}) \end{cases}$$

It is advisable, both technically and theoretically, to change to the problem's *'critical variables'* $z$ and *'critical parameter'* $x$, i.e. to set

$$z := 1/t \sim \infty \qquad , \qquad x := 1/\epsilon \sim \infty$$

so as to prepare for working in the conjugate Borel planes $\zeta$ and $\xi$. This leads to the system:

$$\partial_z Y^i \ = \ Y^i \left( \lambda_i\, x + \sum_{n_j \ge 0 \text{ if } j \ne i}^{1+n_i \ge 0} B^i_{\boldsymbol{n}}(z)\, Y^{\boldsymbol{n}} \right) \qquad (1 \le i \le \nu)$$

with coefficients $B^i_{\boldsymbol{n}}(z) \in \mathbb{C}\{z^{-1}\}$ analytic at infinity and $x$-free.

## 4. Model problem: loose duality equational/coequational.

We assume that the multipliers $\lambda_i$ are neither resonant and nor quasi-resonant (meaning that the combinations $-\lambda_i + \sum_{n_j \geq 0} n_j \lambda_j$) are all $\neq 0$ and do not approximate 0 abnormally fast). The general solution, with its full set $\{\tau_1, \ldots, \tau_\nu\}$ of integration parameters, may be formally expanded in powers of either $z^{-1}$ or $x^{-1}$:

$$\widetilde{Y} = \widetilde{Y}(z, x, \boldsymbol{\tau}) \in \mathbb{C}[[z^{-1} \text{ or } x^{-1}]] \otimes \mathbb{C}\{z^{\rho_1} \tau_1 e^{\lambda_1 z x}, \ldots, z^{\rho_\nu} \tau_\nu e^{\lambda_\nu z x}\}$$

The "residues" $\rho_i \in \mathbb{C}$ are the coefficient of $z^{-1}$ in $B^i_{\mathbf{0}}(z) = B^i_{0,\ldots,0}(z)$. To get rid of the ramifications $z^{\rho_i}$ (which complicate the formal expansions without adding anything of substance to the Analysis) we shall set not only $\rho_i \equiv 0$ but also $B^i_{\mathbf{0}}(z) \equiv 0$.

There is bound to be a certain kinship between the $z$- and $x$-resurgence, since in the special case when $B^i_{\boldsymbol{n}}(z) = \beta^i_{\boldsymbol{n}}/z$ with $\beta^i_{\boldsymbol{n}}$ scalar, the variable $z$ and the perturbation parameter $x$ coalesce:

$$\widetilde{Y}^i(z, x, \boldsymbol{\tau}) \;=\; \widetilde{Y}^i(z\,x) + \sum_{n_j \geq 0}^{j \neq i} \sum_{n_i \geq -1} \widetilde{Y}^i_{\boldsymbol{n}}(z\,x)\, \tau_i \boldsymbol{\tau}^{\boldsymbol{n}}\, e^{(\lambda_i + <\boldsymbol{n}, \boldsymbol{\lambda}>)\, z\, x} \qquad (1)$$

with $\widetilde{Y}^i(zx)$ and $\widetilde{Y}^i_{\boldsymbol{n}}(zx) \in \mathbb{C}[[(zx)^{-1}]]$. A loose kinship, or lax 'duality', survives even in the general case, and justifies the label *equational* for the $z$-resurgence ($z$ being the variable with respect to which we differentiate in our model system) and *co-equational* for the $x$-resurgence.

## 4. Model problem: normalisers and resurgence monomials.

We replace the general solution $\widetilde{Y}$ by the information-equivalent but more flexible *normalising* operators $\Theta$ and $\Theta^{-1}$. These are (mutually inverse) formal automorphisms of $\mathbb{C}[[\boldsymbol{\tau}]] := \mathbb{C}[[\tau_1, \ldots, \tau_\nu]]$:

$$\Theta^{\pm 1}\Big(\widetilde{\varphi}_1(\boldsymbol{\tau}).\widetilde{\varphi}_2(\boldsymbol{\tau})\Big) \equiv \Big(\Theta^{\pm 1}\widetilde{\varphi}_1(\boldsymbol{\tau})\Big)\Big(\Theta^{\pm 1}\widetilde{\varphi}_2(\boldsymbol{\tau})\Big) \quad (\widetilde{\varphi}_i \in \mathbb{C}[[\boldsymbol{\tau}]])$$

They exchange the general solution $Y$ of our model system and the elementary solution $Y_{\mathrm{nor}}$ of the corresponding (linear) normal system:

$$\partial_z Y^i = Y^i \left(\lambda_i x + \sum B^i_{\boldsymbol{n}}(z)\, Y^{\boldsymbol{n}}\right) \;;\; Y^i(z,x,\boldsymbol{\tau}) \in \mathbb{C}[[z^{-1}]] \otimes \mathbb{C}\{\cup_i \tau_i\, e^{\lambda_i x z}\}$$

$$\partial_z Y^i_{\mathrm{nor}} = \lambda_i\, x\, Y^i_{nor} \;;\; Y^i_{\mathrm{nor}}(z,x,\boldsymbol{\tau}) = \tau_i\, e^{\lambda_i x z}$$

$$\begin{cases} \Theta \;\; Y^i(z,x,\boldsymbol{\tau}) \equiv Y^i_{\mathrm{nor}}(z,x,\boldsymbol{\tau}) \\ \Theta^{-1}\, Y^i_{\mathrm{nor}}(z,x,\boldsymbol{\tau}) \equiv Y^i(z,x,\boldsymbol{\tau}) \end{cases}$$

## 4. Model problem: normalisers and resurgence monomials.

The normalisers $\Theta^{\pm 1}$ result from the *contraction* of ordinary differential operators $\mathbb{D}_\bullet$ and biresurgent monomials $\mathcal{W}^\bullet(z,x)$. The latter *absorb* all the $z$- and $x$-divergence, hence all our problem's difficulties.

$$\Theta = 1 + \sum_{i_k, \boldsymbol{n}_k}^{1\leq r} e^{|\boldsymbol{u}|xz}\, \widetilde{\mathcal{W}}^{\binom{u_1 \;\; ,\ldots, \;\; u_r}{B^{i_1}_{\boldsymbol{n}_1} ,\ldots, B^{i_r}_{\boldsymbol{n}_r}}}(z,x)\; \mathbb{D}^{i_r}_{\boldsymbol{n}_r} \ldots \mathbb{D}^{i_1}_{\boldsymbol{n}_1}$$

$$\Theta^{-1} = 1 + \sum_{i_k, \boldsymbol{n}_k}^{1\leq r} (-1)^r e^{|\boldsymbol{u}|xz}\, \widetilde{\mathcal{W}}^{\binom{u_1 \;\; ,\ldots, \;\; u_r}{B^{i_1}_{\boldsymbol{n}_1} ,\ldots, B^{i_r}_{\boldsymbol{n}_r}}}(z,x)\; \mathbb{D}^{i_1}_{\boldsymbol{n}_1} \ldots \mathbb{D}^{i_r}_{\boldsymbol{n}_r}$$

$$\text{with} \quad \begin{cases} u_k := < \boldsymbol{n}_k, \boldsymbol{\lambda} > \;\; , & \mathbb{D}^{i_k}_{\boldsymbol{n}_k} := \boldsymbol{\tau}^{\boldsymbol{n}_k}\, \tau^{i_k}\, \partial_{\tau_{i_k}} \\ 1 \leq i_k \leq \nu \;\; , & \boldsymbol{\tau}^{\boldsymbol{n}}_k\, \tau_{i_k} \in \boldsymbol{\tau}^{\mathbb{N}} \end{cases}$$

and with 'monomials' $\widetilde{\mathcal{W}}^\bullet$ inductively defined by

$$(\partial_z + |\boldsymbol{u}|x)\widetilde{\mathcal{W}}^{\binom{u_1 \;\; ,\ldots, \;\; u_r}{B^{i_1}_{\boldsymbol{n}_1} ,\ldots, B^{i_r}_{\boldsymbol{n}_r}}}(z,x) = -\widetilde{\mathcal{W}}^{\binom{u_1 \;\; ,\ldots, \;\; u_{r-1}}{B^{i_1}_{\boldsymbol{n}_1} ,\ldots, B^{i_{r-1}}_{\boldsymbol{n}_{r-1}}}}(z,x)\, B^{i_r}_{\boldsymbol{n}_r}(z)$$

Or to lighten notations:

$$(\partial_z + |\boldsymbol{u}|x)\widetilde{\mathcal{W}}^{\binom{u_1 ,\ldots, u_r}{b_1 ,\ldots, b_r}}(z,x) = -\widetilde{\mathcal{W}}^{\binom{u_1 ,\ldots, u_{r-1}}{b_1 ,\ldots, b_{r-1}}}(z,x)\, b_r(z)$$

## 4. Model problem: symmetral/alternal moulds.

$$\{S^\bullet \ \ \textit{symmetral}\} \iff \{\sum_{\boldsymbol{\omega} \in \textit{shuffle}(\boldsymbol{\omega}', \boldsymbol{\omega}'')} S^{\boldsymbol{\omega}} \equiv S^{\boldsymbol{\omega}'} S^{\boldsymbol{\omega}''} \quad \forall \boldsymbol{\omega}', \boldsymbol{\omega}''\}$$

$$\{A^\bullet \ \ \textit{alternal}\} \iff \{\sum_{\boldsymbol{\omega} \in \textit{shuffle}(\boldsymbol{\omega}', \boldsymbol{\omega}'')} A^{\boldsymbol{\omega}} \equiv 0 \qquad\qquad \forall \boldsymbol{\omega}', \boldsymbol{\omega}''\}$$

Let the $D_{\omega_i}$'s be (ordinary) formal derivations. Then:

$$\{S^\bullet \ \ \textit{symmetral}\} \iff \begin{cases} 1 + \sum_{1 \leq r} \sum_{\omega_1, \ldots, \omega_r} S^{\omega_1, \ldots, \omega_r} D_{\omega_r} \ldots D_{\omega_1} \\ \textit{is a formal automorphism} \end{cases}$$

$$\{A^\bullet \ \ \textit{alternal}\} \iff \begin{cases} \sum_{1 \leq r} \sum_{\omega_1, \ldots, \omega_r} A^{\omega_1, \ldots, \omega_r} D_{\omega_r} \ldots D_{\omega_1} \\ \textit{is a formal derivation} \end{cases}$$

## 4. Model problem: equational resurgence.

$$(\partial_z + |\boldsymbol{u}|\, x)\ \mathcal{W}^{\binom{u_1\ ,\dots,\ u_r}{b_1\ ,\dots,\ b_r}}(z,x) \ = \ -\mathcal{W}^{\binom{u_1\ ,\dots,\ u_{r-1}}{b_1\ ,\dots,\ b_{r-1}}}(z,x)\, b_r(z) \qquad (2)$$

Under the $z$-Borel tranform $\quad \mathcal{B}_z \,:\, z^{-n} \mapsto \frac{\zeta^{n-1}}{(n-1)!} \quad , \quad b(z) \mapsto \widehat{b}(\zeta) \quad , \quad \mathcal{W}^{\bullet}(z,x) \mapsto \widehat{\mathcal{W}}^{\bullet}(\zeta,x)$

the induction rule (2) becomes

$$(-\zeta+|\boldsymbol{u}|\, x)\, \widehat{\mathcal{W}}^{\binom{u_1\ ,\dots,\ u_r}{b_1\ ,\dots,\ b_r}}(\zeta,x) = -\int_0^{\zeta} \widehat{\mathcal{W}}^{\binom{u_1\ ,\dots,\ u_{r-1}}{b_1\ ,\dots,\ \widehat{b}_{r-1}}}(\zeta_1,x)\, \widehat{b}_r(\zeta-\zeta_1)\, dz_1 \qquad (3)$$

and readily yields all the information we need: location of singularities, Stokes constants, pattern of $z$-resurgence:

$$\Delta_{u\,x}\mathcal{W}^{\binom{u_1\ ,\dots,\ u_r}{b_1\ ,\dots,\ b_r}}(z,x) = \sum_{u_1+\ldots+u_i=u} W^{\binom{u_1\ ,\dots,\ u_i}{b_1\ ,\dots,\ b_i}}(x)\, \mathcal{W}^{\binom{u_{i+1}\ ,\dots,\ u_r}{b_{i+1}\ ,\dots,\ b_r}}(z,x) \quad (4)$$

$$\textit{with} \quad \begin{cases} \textit{monomials} \ \ \mathcal{W}^{\bullet}(z,x) & \textit{symmetral} \ \& \ \textit{resurgent in } z \\ \quad \textit{monics} \qquad W^{\bullet}(x) & \textit{alternal} \ \& \ \textit{entire function of } x \end{cases}$$

## 4. Model problem: co-equational resurgence.

Under the *x-Borel tranform* $\mathcal{B}_x$ : $\begin{cases} x^{-n} \mapsto \frac{\xi^{n-1}}{(n-1)!} \\ \mathcal{W}^{\bullet}(z,x) \mapsto \mathcal{B}_x \mathcal{W}^{\bullet}(z,\xi) \end{cases}$

things are incomparably more complex than under $\mathcal{B}_z$. The induction rule now assumes the form of a partial differential equation in $z$ and $\xi$:

$$(\partial_z + |\boldsymbol{u}|\,\partial_\xi)\ \mathcal{B}_x \mathcal{W}^{\binom{u_1\ ,\dots,\ u_r}{b_1\ ,\dots,\ b_r}}(z,\xi) \ = \ -\mathcal{B}_x \mathcal{W}^{\binom{u_1\ ,\dots,\ u_{r-1}}{b_1\ ,\dots,\ b_{r-1}}}(z,\xi)\, b_r(z) \quad (5)$$

with for $r \geq 2$ the limit condition : $\mathcal{B}_x \mathcal{W}^{\binom{u_1\ ,\dots,\ u_r}{b_1\ ,\dots,\ b_r}}(z,0) = 0$ (5 *bis*)

For $r = 1$, solving (5) in decreasing powers of $x$ and then applying the Borel transform $x \to \xi$, we find:

$$\mathcal{B}_x \mathcal{W}^{\binom{u_1}{b_1}}(z,\xi) \ = \ -\sum_{n\geq 0} \frac{1}{u_1} \frac{(-\xi/u_1)^n}{n!}\,\partial_z^n\, b_1(z) \ = \ -\frac{1}{u_1}\, b_1\left(z - \frac{\xi}{u_1}\right)$$

But for $r \geq 2$ we must resort to a suitably defined *weighted convolution*.

## 4. Model problem: four requirements.

Our approach is unabashedly *analytical*, in that it strives to identify and resolve the difficulties at the **most basic level**, i.e. at the level of the monomials $\mathcal{W}^{\binom{u_1\ ,\dots,\ u_r}{b_1\ ,\dots,\ b_r}}(z,x)$. But **even at that level**, co-equational resurgence is a hard nut to crack. To completely master it, we shall require four things:

(i) a symmetral *weighted convolution* product *weco*$^\bullet$.
(ii) an alternal *weighted convolution* product *welo*$^\bullet$.
(iii) the (closed) rules for *alien-differentiating* *weco*$^\bullet$ and *welo*$^\bullet$.
(iv) the discrete-valued *tessellation coefficients*, which in this new context shall take the place of the continuous-valued Stokes constants.

## 5. Weighted products: symmetral weighted convolution.

For $u_i \in \mathbb{C}$ and inputs $\widehat{c}_i(\xi) \in \mathbb{C}\{x\}$, the following integrals

$$\mathrm{weco}^{\binom{u_1\ ,\dots,\ u_r}{\widehat{c}_1\ ,\dots,\ \widehat{c}_r}}(\xi) \ = \ \begin{cases} \int_0^{\theta_*} \widehat{c}_r(\xi_r)\, d\xi_r \int_{\xi_r}^{\theta_r} \widehat{c}_{r-1}(\xi_{r-1})\, d\xi_{r-1} \dots \\ \cdots \int_{\xi_4}^{\theta_4} \widehat{c}_3(\xi_3)\, d\xi_3 \int_{\xi_3}^{\theta_3} \widehat{c}_2(\xi_2)\, d\xi_2\, \widehat{c}_1(\xi_1)\, \frac{1}{u_1} \end{cases}$$

$$\textit{with} \qquad \begin{cases} u_1\,\xi_1 + \cdots + u_r\,\xi_r = \xi \\ \theta_i := (\xi - (u_i\,\xi_i + \cdots + u_r\,\xi_r))(u_1 + \cdots + u_{i-1})^{-1} \\ \theta_* := \xi\,(u_1 + \cdots + u_r)^{-1} \end{cases}$$

unambiguously define germs $weco^{\binom{u_1\ ,\dots,\ u_r}{\widehat{c}_1\ ,\dots,\ \widehat{c}_r}}(\xi) \in \mathbb{C}\{\xi\}$ provided $u_1 + \ldots + u_i \neq 0$. The mould $weco^{\bullet}$ is symmetral relative to the (ordinary) convolution product. If the inputs $\widehat{c}_i(\xi)$ extend to ramified functions defined on the whole Borel plane $\xi$, so does the total output $weco^{\bullet}(\xi)$.

N.B. At depth 1, the formula reduces to $\mathrm{weco}^{\binom{u_1}{\widehat{c}_1}}(\xi) = \frac{1}{u_1}\,\widehat{c}_1(\frac{\xi}{u_1})$

## 5. Weighted products: symmetral weighted multiplication.

Just as ordinary convolution is the Borel image of ordinary multiplication, the weighted convolution *weco* is the Borel image of a well-defined weighted multiplication *wemu* corresponding to a simple integral kernel:

$$c_1(x),\dots,c_r(x) \overset{\text{Borel}}{\longrightarrow} \widehat{c}_1(\xi),\dots,\widehat{c}_r(\xi)$$

$$\mathrm{wemu}\binom{u_1\,,\dots,\,u_r}{c_1\,,\dots,\,c_r}(x) \overset{\text{Borel}}{\longrightarrow} \mathrm{weco}\binom{u_1\,,\dots,\,u_r}{\widehat{c}_1\,,\dots\,\widehat{c}_r}(\xi)$$

For $u_i > 0$ and $\Re x$ positive and large, weighted multiplication is defined by the integrals:

$$\mathrm{wemu}\binom{u_1\,,\dots,\,u_r}{c_1\,,\dots,\,c_r}(x) := \frac{1}{(2\pi i)^r}\int_{-i\infty}^{+i\infty}\frac{c_1(x_1)\dots c_r(x_r)\; dx_1\dots dx_r}{\prod_{i=1}^{i=r}\big((u_1+\dots+u_i)\,x-(x_1+\dots+x_i)\big)} \quad (6)$$

Integration is along vertical axes $\Im x_j = \alpha_j < u_j\Re x$ but with $\alpha_j$ large enough for $c_j(x_j)$ to be holomorphic on $\alpha_j \leq \Re x_j$. The definition is then extended to the case of general weights $u_i$ by continuous contour deformation, which is always feasible provided the partial sums $u_1 + \cdots + u_j$ remain $\neq 0$.

## 5.Weighted products: alternal variants.

• Alternal marking. It is a mould operation $M^\bullet \to \underline{M}^\bullet$:

$$\underline{\mathrm{M}}^{\boldsymbol{\omega}',\omega_i^\sharp,\boldsymbol{\omega}''} := (-1)^{r''} \sum_{\boldsymbol{\omega}'''\in \mathrm{sha}(\boldsymbol{\omega}',\widetilde{\boldsymbol{\omega}}'')} M^{\boldsymbol{\omega}''',\,\omega_i^\sharp} \qquad \left(\tfrac{(r'+r'')!}{r'!\ r''!}\ \textit{summands}\right)$$

that turns *any* mould $M^\bullet$ into a $\sharp$-*marked* mould $\underline{M}^\bullet$ of alternal type.

• Alternal convolution $welo^\bullet$ : $welo^\bullet$ derives from the symmetral $weco^\bullet$ under alternal marking and is given by similar integrals.

• Alternal multiplication $welu^\bullet$: The symmetral and alternal variants $wemu^\bullet$ and $welu^\bullet$ have rather similar kernels

$$\begin{cases} \mathrm{wemu}^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_i & ,\dots, & u_r \\ c_1 & ,\dots, & c_i & ,\dots, & c_r \end{smallmatrix}\right)}(x) = \frac{1}{(2\pi i)^r} \int S^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_i & ,\dots, & u_r \\ x_1 & ,\dots, & x_i & ,\dots, & x_r \end{smallmatrix}\right)}(x) \prod c_i(x_i) dx_i \\ \mathrm{welu}^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_i^\dagger & ,\dots, & u_r \\ c_1 & ,\dots, & c_i\dagger & ,\dots, & c_r \end{smallmatrix}\right)}(x) = \frac{1}{(2\pi i)^r} \int \underline{S}^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_i\dagger & ,\dots, & u_r \\ x_1 & ,\dots, & x_i\dagger & ,\dots, & x_r \end{smallmatrix}\right)}(x) \prod c_i(x_i) dx_i \end{cases}$$

$$\begin{cases} S^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_i & ,\dots, & u_r \\ x_1 & ,\dots, & x_i & ,\dots, & x_r \end{smallmatrix}\right)}(x) = \prod_{i=1}^{i=r} \Big( (u_1 + .. + u_i)\, x - (x_1 + .. + x_i) \Big)^{-1} \\ \underline{S}^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_i\dagger & ,\dots, & u_r \\ x_1 & ,\dots, & x_i\dagger & ,\dots, & x_r \end{smallmatrix}\right)}(x) = \begin{cases} (-1)^{r-j} S^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_{i-1} \\ x_1 & ,\dots, & x_{i-1} \end{smallmatrix}\right)}(x)\ S^{\left(\begin{smallmatrix} u_r & ,\dots, & u_{i+1} \\ x_r & ,\dots, & x_{i+1} \end{smallmatrix}\right)}(x) \times \\ \Big( (u_1 + .. + u_r)\, x - (x_1 + .. + x_r) \Big)^{-1} \end{cases} \end{cases}$$

## 5. Relevance of the weighted convolutions.

• Relevance of *weco*. The $x$-Borel transforms $\mathcal{W}^\bullet \to \mathcal{B}_x\mathcal{W}^\bullet$ of the biresurgent monomials can be expressed in terms of *weco* products.

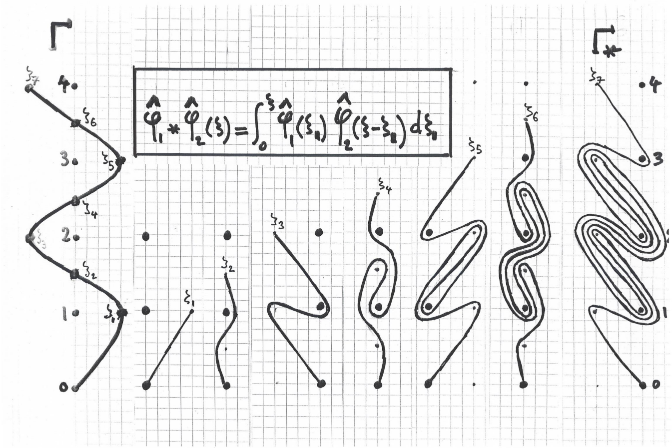
$$\mathcal{B}_x\mathcal{W}^{\binom{u_1}{b_1}\,\substack{,\ldots,\\,\ldots,}\,\binom{u_r}{b_r}}(z,\xi) = \mathrm{weco}^{\binom{u_1}{\widehat{c}_1}\,\substack{,\ldots,\\,\ldots,}\,\binom{u_r}{\widehat{c}_r}}(\xi) \quad \textit{with} \quad \widehat{c}_i(\xi) := -b_i(z-\xi)$$

with $z$ chosen close enough to $\infty$ for $\widehat{c}_i(\xi)$ to be regular at $\xi = 0$. Since $\widehat{c}_i(\xi) := -b_i(z-\xi)$, the singularities of the $b_i(z)$ are going to dominate co-equational resurgence. We note here the characteristic interference of the multiplicative $z$-plane and the convolutive $\xi$-plane.

• Relevance of *welo*. The alien derivatives of $\mathcal{B}_x\mathcal{W}^\bullet$ can be expressed as *welo* products of the inputs $\widehat{c}_i(\xi)$ and their own alien derivatives $\widehat{\Delta}_\omega\widehat{c}_i(\xi)$ with a third crucial ingredient: the universal tessellation coefficients.

## 6 The detour through combinatorics.

- The radical impracticability of integration multipaths.




Even for ordinary convolution we get impossibly contorted paths. The position is still worse with the *weighted multipaths*. Hence the need for a combinatorial approach.

# 6. The detour through combinatorics.

## • Hyperlogarithms: stability and density.

We are facing here a highly unusual but inescapable interference of two structures:
(i) the *multiplicative* structure, which leaves the singularities in place,
(ii) the *convolutive* structure, which *adds* singularities, in the sense that:
(singularity over $\omega_1$)$*$(singularity over $\omega_2$)$\Rightarrow$ (singularities over $\omega_1 + \omega_2$).

Then, messing up things still further, we must contend with the *weighted* convolution *weco*, which also *adds* singularities, but via weighted rather than straightforward sums. This forces us to juggle two systems of notation:

- *incremental*, with sequences $(\omega_1, \ldots, \omega_r)$ $\quad (\omega_i = \alpha_i - \alpha_{i-1})$
- *positional*, with sequences $[\alpha_1, \ldots, \alpha_r]$ $\quad (\alpha_i = \omega_1 + \ldots + \omega_i)$

The ideal tool for understanding this hybrid structure is the *hyperlogarithms* with

- their *two encodings* (*positional* and *incremental*)
- their stability under *two products* : *pointwise multiplication* and *convolution*, simple and weighted.
- their stability under *alien differentiation*
- their *density* property: any given resurgent function in the Borel plane is the limit, uniformly on any compact set of its Riemann surface, of a suitable series of hyperlogarithms.

## 6. The detour through combinatorics.

- Hyperlogarithmic monomials: dimorphy.

$$\text{(positional)} \quad \widehat{\mathcal{V}}^{[\alpha_1,\dots,\alpha_r]}(\tau) := \int_0^\tau \frac{d\tau_r}{\tau_r-\alpha_r} \cdots \int_0^{\tau_3} \frac{d\tau_2}{\tau_2-\alpha_2} \int_0^{\tau_2} \frac{d\tau_1}{\tau_1-\alpha_1}$$

$$\text{(incremental)} \quad \widehat{\mathcal{V}}^{\omega_1,\dots,\omega_r}(\tau) \equiv \widehat{\mathcal{V}}^{[\alpha_1,\dots,\alpha_r]}(\tau) \quad \textit{with} \quad \alpha_i \equiv \omega_1+\ldots+\omega_i \quad (\forall i)$$

To express the multiplication-convolution dimorphy we require the *upper convolution* $\widehat{*}$, which has the same unit 1 as pointwise multiplication. Its definition is: $(\widehat{\varphi}_1 \;\widehat{*}\; \widehat{\varphi}_2)(\tau) := \int_0^\tau \widehat{\varphi}_1(\tau_1)\, \widehat{\varphi}_1(\tau-\tau_1) d\tau_1$

$$\times\text{-}\textbf{symmetrality}: \quad (\widehat{\mathcal{V}}^{[\boldsymbol{\alpha'}]} . \widehat{\mathcal{V}}^{[\boldsymbol{\alpha''}]})(\tau) \equiv \sum_{\boldsymbol{\alpha}\in \mathrm{sha}(\boldsymbol{\alpha'},\boldsymbol{\alpha''})} \widehat{\mathcal{V}}^{[\boldsymbol{\alpha}]}(\tau) \tag{7}$$

$$\widehat{*}\text{-}\textbf{symmetrality}: \quad (\widehat{\mathcal{V}}^{\boldsymbol{\omega'}} \;\widehat{*}\; \widehat{\mathcal{V}}^{\boldsymbol{\omega''}})(\tau) \equiv \sum_{\boldsymbol{\omega}\in \mathrm{sha}(\boldsymbol{\omega'},\boldsymbol{\omega''})} \widehat{\mathcal{V}}^{\boldsymbol{\omega}}(\tau) \tag{8}$$

(7) says that $\widehat{\mathcal{V}}^{[\bullet]}$ is symmetral relative to pointwise multiplication.

(8) says that $\widehat{\mathcal{V}}^{\bullet}$ is symmetral relative to the convolution $\widehat{*}$.

# 6. The detour through combinatorics.

## • Hyperlogarithmic monomials and monics.

The **hyperlogarithmic monomials** $\widehat{\mathcal{V}}^{\bullet}$ (symmetral) relevant to the present context are defined by:

$$\widehat{\mathcal{V}}^{\,\omega_1,\dots,\omega_r}(\zeta) \;:=\; \frac{1}{\zeta-(\omega_1+\dots+\omega_r)}\int_0^{\zeta}\frac{d\zeta_{r-1}}{\zeta_{r-1}-(\omega_1+\dots+\omega_{r-1})}\dots\int_0^{\zeta_2}\frac{d\zeta_1}{\zeta_1-\omega_1}$$

and verify

$$\begin{cases} \widetilde{\mathcal{V}}^{\,\boldsymbol{\omega}'}(z)\,.\,\widetilde{\mathcal{V}}^{\,\boldsymbol{\omega}''}(z) & \equiv \sum_{\boldsymbol{\omega}\in \mathrm{sha}(\boldsymbol{\omega}',\boldsymbol{\omega}'')} \widetilde{\mathcal{V}}^{\,\boldsymbol{\omega}}(z) \qquad \textit{(convolutive model)} \\ (\widehat{\mathcal{V}}^{\,\boldsymbol{\omega}'} * \widehat{\mathcal{V}}^{\,\boldsymbol{\omega}''})(\tau) & \equiv \sum_{\boldsymbol{\omega}\in \mathrm{sha}(\boldsymbol{\omega}',\boldsymbol{\omega}'')} \widehat{\mathcal{V}}^{\,\boldsymbol{\omega}}(\zeta) \qquad \textit{(formel model)} \end{cases}$$

The corresponding **hyperlogarithmic monics** $V^{\bullet}$ (alternal) are inductively defined (for $\omega_0 = \omega_1 + .. + \omega_r$) by:

$$\Delta_{\omega_0}\mathcal{V}^{\omega_1,\dots,\omega_r}(z) = \sum_{\omega_1+\dots+\omega_i=\omega_0} V^{\omega_1,\dots,\omega_i}\;\mathcal{V}^{\omega_{i+i},\dots,\omega_r}(z) \tag{9}$$

The alternility relations read $\sum_{\boldsymbol{\omega}\in \mathrm{sha}(\boldsymbol{\omega}',\boldsymbol{\omega}'')} V^{\boldsymbol{\omega}} \equiv 0$.

They monics $V^{\bullet}$ univalued, piecewise analytic functions of their indices $\omega_i$.

## 6. The detour through combinatorics.

- Index differentiation for the hyperlogarithmic monomials:

$$
\begin{aligned}
\omega_1(\partial_{\omega_1}+z)\,\mathcal{V}^{\omega_1,\ldots,\omega_r}(z) &= -\mathcal{V}^{\omega_1+\omega_2,\ldots,\omega_r}(z)\\
\omega_j(\partial_{\omega_j}+z)\,\mathcal{V}^{\omega_1,\ldots,\omega_r}(z) &= +\mathcal{V}^{\omega_1,\ldots,\omega_{j-1}+\omega_j,\ldots,\omega_r}(z)-\mathcal{V}^{\omega_1,\ldots,\omega_j+\omega_{j+1},\ldots,\omega_r}(z)\\
\omega_r(\partial_{\omega_r}+z)\,\mathcal{V}^{\omega_1,\ldots,\omega_r}(z) &= +\mathcal{V}^{\omega_1,\ldots,\omega_{r-1}+\omega_r}(z)-\mathcal{V}^{\omega_1,\ldots,\omega_{r-1}}(z)\\
z(\partial_z+|\boldsymbol{\omega}|)\,\mathcal{V}^{\omega_1,\ldots,\omega_r}(z) &= -\mathcal{V}^{\omega_1,\ldots,\omega_{r-1}}(z)
\end{aligned}
$$

- Index differentiation for the hyperlogarithmic monics:

$$
\begin{aligned}
\omega_1\partial_{\omega_1}V^{\omega_1,\ldots,\omega_r} &= -V^{\omega_1+\omega_2,\ldots,\omega_r}\\
\omega_j\partial_{\omega_j}V^{\omega_1,\ldots,\omega_r} &= +V^{\omega_1,\ldots,\omega_{j-1}+\omega_j,\ldots,\omega_r}-V^{\omega_1,\ldots,\omega_j+\omega_{j+1},\ldots,\omega_r}\\
\omega_r\partial_{\omega_r}V^{\omega_1,\ldots,\omega_r} &= +V^{\omega_1,\ldots,\omega_{r-1}+\omega_r}
\end{aligned}
$$

- Jump rules for the hyperlogarithmic monics:

The monics $V^{\bullet}$ are univalued, piecewise analytic functions with cuts along the hypersurfaces $\frac{\omega_1+\cdots+\omega_i}{\omega_{i+1}+\cdots+\omega_r}V^{\omega_1,\ldots,\omega_r}\in\mathbb{R}^+$ and determination discontinuities given by the *jump formula*:

$$
\begin{cases}
D_{\frac{\omega_1+\cdots+\omega_i}{\omega_{i+1}+\cdots+\omega_r}}V^{\omega_1,\ldots,\omega_r}\equiv 2\pi i\;V^{\omega_1,\ldots,\omega_i}\;V^{\omega_{i+1},\ldots,\omega_r}\\
D_x\,F(x):=\lim_{\epsilon\to 0}(F(x+i\,\epsilon)-F(x-i\,\epsilon))\qquad\qquad (t,\epsilon\in\mathbb{R}^+)
\end{cases}
$$

# 7 Weighted convolution: polar inputs.

Setting $\widehat{\mathcal{S}}^{\binom{u_1\ ,\dots,\ u_r}{v_1\ ,\dots,\ v_r}}(\xi) := weco^{\binom{u_1\ ,\dots,\ u_r}{\widehat{c}_1\ ,\dots,\ \widehat{c}_r}}(\xi)$ with $\widehat{c}_i(\xi) := \frac{1}{\xi - v_i}$, we get

$$\mathcal{S}^{\binom{u_1}{v_1}}(x) \quad := \quad \mathcal{V}^{u_1\, v_1}(x)$$

$$\mathcal{S}^{\binom{u_1\ ,\ u_2}{v_1\ ,\ v_2}}(x) \quad := \quad \begin{cases} +\mathcal{V}^{u_1\ v_1\ ,\ u_2\ v_2}(x) \\ -\mathcal{V}^{(u_1+u_2)\ v_1\ ,\ u_2\ (v_2-v_1)}(x) \\ +\mathcal{V}^{(u_1+u_2)\ v_2\ ,\ u_1\ (v_1-v_2)}(x) \end{cases}$$

$$\mathcal{S}^{\binom{u_1\ ,\ u_2\ ,\ u_3}{v_1\ ,\ v_2\ ,\ v_3}}(x) \quad := \quad \begin{cases} +\mathcal{V}^{u_1\ v_1\ ,\ u_2\ v_2\ ,\ u_3\ v_3}(x) \\ +\mathcal{V}^{u_1\ v_1\ ,\ (u_2+u_3)\ v_3\ ,\ u_2\ (v_2-v_3)}(x) \\ -\mathcal{V}^{u_1\ v_1\ ,\ (u_2+u_3)\ v_2\ ,\ u_3\ (v_3-v_2)}(x) \\ +\mathcal{V}^{(u_1+u_2)\ v_2\ ,\ u_1\ (v_1-v_2)\ ,\ u_3\ v_3}(x) \\ -\mathcal{V}^{(u_1+u_2)\ v_1\ ,\ u_2\ (v_2-v_1)\ ,\ u_3\ v_3}(x) \\ +\mathcal{V}^{(u_1+u_2)\ v_2\ ,\ u_3\ v_3\ ,\ u_1\ (v_1-v_2)}(x) \\ -\mathcal{V}^{(u_1+u_2)\ v_1\ ,\ u_3\ v_3\ ,\ u_2\ (v_2-v_1)}(x) \\ +\mathcal{V}^{(u_1+u_2+u_3)\ v_1\ ,\ (u_2+u_3)\ (v_2-v_1)\ ,\ u_3\ (v_3-v_2)}(x) \\ -\mathcal{V}^{(u_1+u_2+u_3)\ v_1\ ,\ (u_2+u_3)\ (v_3-v_1)\ ,\ u_2\ (v_2-v_3)}(x) \\ +\mathcal{V}^{(u_1+u_2+u_3)\ v_1\ ,\ u_3\ (v_3-v_1)\ ,\ u_2\ (v_2-v_1)}(x) \\ -\mathcal{V}^{(u_1+u_2+u_3)\ v_2\ ,\ u_1\ (v_1-v_2)\ ,\ u_3\ (v_3-v_2)}(x) \\ -\mathcal{V}^{(u_1+u_2+u_3)\ v_2\ ,\ u_3\ (v_3-v_2)\ ,\ u_1\ (v_1-v_2)}(x) \\ +\mathcal{V}^{(u_1+u_2+u_3)\ v_3\ ,\ u_1\ (v_1-v_3)\ ,\ u_2\ (v_2-v_3)}(x) \\ -\mathcal{V}^{(u_1+u_2+u_3)\ v_3\ ,\ (u_1+u_2)\ (v_1-v_3)\ ,\ u_2\ (v_2-v_1)}(x) \\ +\mathcal{V}^{(u_1+u_2+u_3)\ v_3\ ,\ (u_1+u_2)\ (v_2-v_3)\ ,\ u_1\ (v_1-v_2)}(x) \end{cases}$$

$\widehat{\mathcal{S}}^{\binom{u_1\ ,\dots,\ u_r}{v_1\ ,\dots,\ v_r}}(\xi)$ has $r!! := 1.3.5...(2r-1)$ hyperlogarithmic summands.

## 7 Weighted convolution: hyperlogarithmic inputs.

• The *inputs* $\widehat{c}_i(\xi)$ are now general hyperlogs of depth $s_i$, but taken in *positional notation*: $\widehat{c}_i(\xi) = \widehat{\mathcal{V}}^{[\underline{v}_i]}(\xi) = \widehat{\mathcal{V}}^{[v_{i,1},v_{i,2},v_{i,3},\ldots]}(\xi)$

• Accordindingly, the *lower indices* $v_i$ become sequences $\underline{v}_i := (v_{i,1}, v_{i,2}, v_{i,3} \ldots)$ of arbitrary length $s_i$.

• The outputs $\widehat{\mathcal{S}}^{\left( {u_1 \atop \underline{v}_1} {, \ldots, \atop , \ldots,} {u_r \atop \underline{v}_r} \right)}(\xi) := weco^{\left( {u_1 \atop \widehat{c}_1} {, \ldots, \atop , \ldots,} {u_r \atop \widehat{c}_r} \right)}(\xi)$, as before, get expanded into sums of hyperlogarithms $\widehat{\mathcal{V}}^{\omega_1,\ldots,\omega_s}(\xi)$ taken in *incremental notation*. They all have depth $s := s_1 + \ldots + s_r$.

• As before, the $\omega_i$'s in the output are bilinear in the $u_i$'s and $v_i$'s.

• As before, there are two recursion rules (forward/backward) behind the expansion formula, only twice more complex.

# 7. Weighted convolution: hyperlogarithmic inputs.

The weighted convolution of $r$ hyperlogs of depths $d_1, \ldots, d_r$ is a sum of $\mu(d_1, \ldots, d_r)$ distinct hyperlogs, each of depth $\sum d_i$. That number $\mu(d_1, \ldots, d_r) = \frac{(d_1+\cdots+d_r-1)!}{(d_1-1)!\ldots(d_r-1)!} \prod_{2\leq i\leq r} \left(2 + \frac{1}{d_i+\cdots+d_r}\right)$ tends to be huge. Thus:

| | | | | | | |
|---|---|---|---|---|---|---|
| $\mu(\overbrace{1,\ldots,1}^{r \text{ times}})$ | $=$ | $1.3.5\ldots(2r-1)$ | $=$ | $r!!$ | | *polar inputs* |
| $\mu(5,5,5)$ | $=$ | 29 135 106 | $\sim$ | 29 | $10^6$ | *hyperlog. inputs* |
| $\mu(4,4,4,4)$ | $=$ | 10 050 665 625 | $\sim$ | 10 | $10^9$ | |
| $\mu(1,3,5,7)$ | $=$ | 349 098 750 | $\sim$ | 0.4 | $10^9$ | |
| $\mu(7,5,3,1)$ | $=$ | 539 188 650 | $\sim$ | 0.5 | $10^9$ | |
| $\mu(3,3,3,3,3)$ | $=$ | 60 575 515 000 | $\sim$ | 60 | $10^9$ | |
| $\mu(1,2,3,4,5)$ | $=$ | 6 067 061 000 | $\sim$ | 6 | $10^9$ | |
| $\mu(5,4,3,2,1)$ | $=$ | 9 641 071 440 | $\sim$ | 10 | $10^9$ | |

Thus, for a linear system as simple as (*),we have just 4 singularities in the $\zeta$-plane, but $\sim 10^{10}$ in the $\xi$-plane.

$$(*) \quad (\partial_z + \omega_i\, x)\, Y_i(z,x) = Y_{i-1}(z,x)\, b_i(z) \quad \begin{cases} (1 \leq i \leq 4,\ Y_0 \equiv 1) \\ b_i \text{ hyperlog. of depth } 4 \end{cases}$$

# 7. Weighted convolution: exit Stokes, enter Tes.

Applying the rules $\begin{cases} \Delta_{\omega_0} \mathcal{V}^{\omega_1,\dots,\omega_r}(x) = \sum_{\omega_1+\dots+\omega_i=\omega_0} V^{\omega_1,\dots,\omega_i} \mathcal{V}^{\omega_{i+1},\dots,\omega_r}(x) \\ V^{\omega_1} \equiv 1 \ , \ V^{\omega_1,\omega_2} = \textit{suitable determination of } \log \frac{\omega_2}{\omega_1} \end{cases}$

to the weighted convolution product: $\mathcal{S}^{\left(\begin{smallmatrix} u_1 & , & u_2 \\ v_1 & , & v_2 \end{smallmatrix}\right)}(x) := \begin{cases} +\mathcal{V}^{u_1\, v_1 \ , \ u_2\, v_2}(x) \\ -\mathcal{V}^{(u_1+u_2)\, v_1 \ , \ u_2\, (v_2-v_1)}(x) \\ +\mathcal{V}^{(u_1+u_2)\, v_2 \ , \ u_1\, (v_1-v_2)}(x) \end{cases}$

we find that the continuous-valued Stokes constants disappear. Indeed:

$$\begin{aligned}
\Delta_{u_1\, v_1} \mathcal{S}^{\left(\begin{smallmatrix} u_1 & , & u_2 \\ v_1 & , & v_2 \end{smallmatrix}\right)}(x) &= \mathcal{V}^{u_2\, v_2}(x) &&= \mathcal{S}^{\left(\begin{smallmatrix} u_2 \\ v_2 \end{smallmatrix}\right)}(x) \\
\Delta_{(u_1+u_2)\, v_1} \mathcal{S}^{\left(\begin{smallmatrix} u_1 & , & u_2 \\ v_1 & , & v_2 \end{smallmatrix}\right)}(x) &= -\mathcal{V}^{u_2\, (v_2-v_1)}(x) &&= -\mathcal{S}^{\left(\begin{smallmatrix} u_2 \\ v_2-v_1 \end{smallmatrix}\right)}(x) \\
\Delta_{(u_1+u2)\, v_2} \mathcal{S}^{\left(\begin{smallmatrix} u_1 & , & u_2 \\ v_1 & , & v_2 \end{smallmatrix}\right)}(x) &= \mathcal{V}^{u_1\, (v_1-v_2)}(x) &&= \mathcal{S}^{\left(\begin{smallmatrix} u_1 \\ v_1-v_2 \end{smallmatrix}\right)}(x) \\
\Delta_{u_1\, v_1+u_2\, v_2} \mathcal{S}^{\left(\begin{smallmatrix} u_1 & , & u_2 \\ v_1 & , & v_2 \end{smallmatrix}\right)}(x) &= \mathrm{tes}\left(\begin{smallmatrix} u_1 & , & u_2 \\ v_1 & , & v_2 \end{smallmatrix}\right) &&= \log \frac{u_2 v_2}{u_1 v_1} - \log \frac{u_2(v_2-v_1)}{(u_1+u2)v_1} + \log \frac{u_1(v_1-v_2)}{(u_1+u2)v_2}
\end{aligned}$$

with a locally constant *tessellation coefficient* $\mathrm{tes}\left(\begin{smallmatrix} u_1 & , & u_2 \\ v_1 & , & v_2 \end{smallmatrix}\right) \in \{0, \pm 2\pi i\}$.

The phenomenon is general and holds for all values of $r$.

**Caveat:** The disappearance of Stokes constants is incomplete in the case of $v_i$-repetitions.

# 8. Tessellation coefficients: hyperlogarithmic expansions.

At depths $r \geq 3$, local constancy still holds: differentiate the following $tes^{\bullet}$ in any $u_i$ or any $v_i$, and you get ... 0.

Taking the expansion $\mathcal{S}^{\boldsymbol{w}}(x) = \sum \pm \mathcal{V}^{\boldsymbol{\omega}}$, changing $\mathcal{V}^{\boldsymbol{\omega}}(x)$ to $V^{\boldsymbol{\omega}}$,
we get $\Delta_{u_1\,v_1+u_2\,v_2+u_3\,v_3} = tes^{\left(\begin{smallmatrix} u_1 & , & u_2 & , & u_3 \\ v_1 & , & v_2 & , & v_3 \end{smallmatrix}\right)}$ with

$$
\mathrm{tes}^{\left(\begin{smallmatrix} u_1 & , & u_2 & , & u_3 \\ v_1 & , & v_2 & , & v_3 \end{smallmatrix}\right)} := \begin{cases}
+V^{u_1\,v_1\,,\,u_2\,v_2\,,\,u_3\,v_3} \\
+V^{u_1\,v_1\,,\,(u_2+u_3)\,v_3\,,\,u_2\,(v_2-v_3)} \\
-V^{u_1\,v_1\,,\,(u_2+u_3)\,v_2\,,\,u_3\,(v_3-v_2)} \\
+V^{(u_1+u_2)\,v_2\,,\,u_1\,(v_1-v_2)\,,\,u_3\,v_3} \\
-V^{(u_1+u_2)\,v_1\,,\,u_2\,(v_2-v_1)\,,\,u_3\,v_3} \\
+V^{(u_1+u_2)\,v_2\,,\,u_3\,v_3\,,\,u_1\,(v_1-v_2)} \\
-V^{(u_1+u_2)\,v_1\,,\,u_3\,v_3\,,\,u_2\,(v_2-v_1)} \\
+V^{(u_1+u_2+u_3)\,v_1\,,\,(u_2+u_3)\,(v_2-v_1)\,,\,u_3\,(v_3-v_2)} \\
-V^{(u_1+u_2+u_3)\,v_1\,,\,(u_2+u_3)\,(v_3-v_1)\,,\,u_2\,(v_2-v_3)} \\
+V^{(u_1+u_2+u_3)\,v_1\,,\,u_3\,(v_3-v_1)\,,\,u_2\,(v_2-v_1)} \\
-V^{(u_1+u_2+u_3)\,v_2\,,\,u_1\,(v_1-v_2)\,,\,u_3\,(v_3-v_2)} \\
-V^{(u_1+u_2+u_3)\,v_2\,,\,u_3\,(v_3-v_2)\,,\,u_1\,(v_1-v_2)} \\
+V^{(u_1+u_2+u_3)\,v_3\,,\,u_1\,(v_1-v_3)\,,\,u_2\,(v_2-v_3)} \\
-V^{(u_1+u_2+u_3)\,v_3\,,\,(u_1+u_2)\,(v_1-v_3)\,,\,u_2\,(v_2-v_1)} \\
+V^{(u_1+u_2+u_3)\,v_3\,,\,(u_1+u_2)\,(v_2-v_3)\,,\,u_1\,(v_1-v_2)}
\end{cases}
$$

# 8. Tessellation coefficients: elementary induction.

Local constancy is an invitation to search for a more elementary expression of $tes^{\bullet}$.

**Limiting hypersurfaces** $\mathcal{H}^{+}_{i,j} = \{\boldsymbol{w} \in \mathbb{C}^{2r}\ ;\ H_{i,j}(\boldsymbol{w}) \in \mathbb{R}^{+}\}$ (there are $r^2 - 1$ of them):

$$\begin{aligned}
H_{i,j}(\boldsymbol{w}) &:= Q^{*}_{i,j}(\boldsymbol{w})/Q^{**}_{i,j}(\boldsymbol{w}) \qquad (i-j\neq 0;\, i,j\in\mathbb{Z}_{r+1})\\
Q^{*}_{i,j}(\boldsymbol{w}) &:= \sum_{\mathrm{circ}(i<q\leq j)} u^{\sharp}_{q}\,(v^{\sharp}_{q} - v^{\sharp}_{j})\\
Q^{**}_{i,j}(\boldsymbol{w}) &:= \sum_{\mathrm{circ}(j<q\leq j)} u^{\sharp}_{q}\,(v^{\sharp}_{q} - v^{\sharp}_{j}) \ = \ <\boldsymbol{u},\boldsymbol{v}> - Q^{*}_{i,j}(\boldsymbol{w})
\end{aligned}$$

**The jump rule for $tes^{\boldsymbol{w}}$:** It is only when $\boldsymbol{w}$ crosses a hypersurface $\mathcal{H}^{+}_{i,j}$ that $\mathrm{tes}^{\boldsymbol{w}}$ can change its value. Let $\boldsymbol{w}$ be any point on $\mathcal{H}^{+}_{i,j}$ and let $\boldsymbol{w}^{+}, \boldsymbol{w}^{-}$ be two points close by, with $\pm\Im H_{i,j}(\boldsymbol{w}^{\pm}) > 0$. Then

$$tes^{\boldsymbol{w}^{+}} - tes^{\boldsymbol{w}^{-}} = tes^{\boldsymbol{w}^{*}}\ tes^{\boldsymbol{w}^{**}}$$

$$with \quad \begin{cases}
\boldsymbol{w}^{*} := \begin{pmatrix} u_{i+1} & ,\dots, & u_p & ,\dots, & u_j \\ v_{i+1}-v_i & ,\dots, & v_p - v_i & ,\dots, & v_j - v_i \end{pmatrix} & (\mathrm{circ}(i<p\leq j)\ \in \mathbb{Z}_{r+1})\\
\boldsymbol{w}^{**} := \begin{pmatrix} u_{j+1} & ,\dots, & u_q & ,\dots, & u_{i-1} \\ v_{j+1}-v_i & ,\dots, & v_q - v_i & ,\dots, & v_{i-1} - v_i \end{pmatrix} & (\mathrm{circ}(j<q<i)\ \in \mathbb{Z}_{r+1})
\end{cases}$$

# 8. The tessellation coefficients: elementary expression.

We fix some $c \in \mathbb{C}^*$ and set $\Re_c : z \in \mathbb{C} \mapsto \Re(c\,z) \in \mathbb{R}$. Then we define:

$$f_{\boldsymbol{w}}^{\boldsymbol{w}'} := <\boldsymbol{u}', \boldsymbol{v}'><\boldsymbol{u}, \boldsymbol{v}>^{-1} \quad , \quad g_{\boldsymbol{w}}^{\boldsymbol{w}'} := <\boldsymbol{u}', \Re_\theta \boldsymbol{v}'><\boldsymbol{u}, \Re_\theta \boldsymbol{v}>^{-1} \tag{10}$$

From these scalars we construct the crucial sign factor *sig* which takes its values in $\{-1, 0, 1\}$. Here, the abbreviation *si*(.) stands for *sign*($\Im$(.)).

$$\mathrm{sig}^{\boldsymbol{w}',\boldsymbol{w}''} = \mathrm{sig}_c^{\boldsymbol{w}',\boldsymbol{w}''} := \frac{1}{8} \begin{cases} (\mathrm{si}(f_{\boldsymbol{w}}^{\boldsymbol{w}'} - f_{\boldsymbol{w}}^{\boldsymbol{w}''}) - \mathrm{si}(g_{\boldsymbol{w}}^{\boldsymbol{w}'} - g_{\boldsymbol{w}}^{\boldsymbol{w}''})) \times \\ (1 + \mathrm{si}(f_{\boldsymbol{w}}^{\boldsymbol{w}'}/g_{\boldsymbol{w}}^{\boldsymbol{w}'}) \;\; \mathrm{si}(f_{\boldsymbol{w}}^{\boldsymbol{w}'} - g_{\boldsymbol{w}}^{\boldsymbol{w}'})) \times \\ (1 + \mathrm{si}(f_{\boldsymbol{w}}^{\boldsymbol{w}''}/g_{\boldsymbol{w}}^{\boldsymbol{w}''}) \; \mathrm{si}(f_{\boldsymbol{w}}^{\boldsymbol{w}''} - g_{\boldsymbol{w}}^{\boldsymbol{w}''})) \end{cases} \tag{11}$$

Next, from the pair $(\boldsymbol{w}', \boldsymbol{w}'')$ we derive a pair $(\boldsymbol{w}^*, \boldsymbol{w}^{**})$ by setting:

$$\boldsymbol{u}^* := \boldsymbol{u}' \quad , \quad \boldsymbol{v}^* := \boldsymbol{v}' \; <\boldsymbol{u}, \boldsymbol{v}>^{-1} \Im g_{\boldsymbol{w}}^{\boldsymbol{w}'} \; - \; \Re_c \boldsymbol{v}' \; <\boldsymbol{u}, \Re_c \boldsymbol{v}>^{-1} \Im f_{\boldsymbol{w}}^{\boldsymbol{w}'} \tag{12}$$

$$\boldsymbol{u}^{**} := \boldsymbol{u}'' \quad , \quad \boldsymbol{v}^{**} := \boldsymbol{v}'' \; <\boldsymbol{u}, \boldsymbol{v}>^{-1} \Im g_{\boldsymbol{w}}^{\boldsymbol{w}''} \; - \; \Re_c \boldsymbol{v}'' \; <\boldsymbol{u}, \Re_c \boldsymbol{v}>^{-1} \; \Im f_{\boldsymbol{w}}^{\boldsymbol{w}''} \tag{13}$$

or more symmetrically:

$$\boldsymbol{v}^* := \det \begin{pmatrix} \frac{\boldsymbol{v}'}{<\boldsymbol{u},\boldsymbol{v}>} & \frac{\Re_c \boldsymbol{v}'}{<\boldsymbol{u},\Re_c \boldsymbol{v}>} \\ \Im \frac{<\boldsymbol{u}',\boldsymbol{v}'>}{<\boldsymbol{u},\boldsymbol{v}>} & \Im \frac{<\boldsymbol{u}',\Re_c \boldsymbol{v}'>}{<\boldsymbol{u},\Re_c \boldsymbol{v}>} \end{pmatrix} \quad , \quad \boldsymbol{v}^{**} := \det \begin{pmatrix} \frac{\boldsymbol{v}''}{<\boldsymbol{u},\boldsymbol{v}>} & \frac{\Re_c \boldsymbol{v}''}{<\boldsymbol{u},\Re_c \boldsymbol{v}>} \\ \Im \frac{<\boldsymbol{u}'',\boldsymbol{v}''>}{<\boldsymbol{u},\boldsymbol{v}>} & \Im \frac{<\boldsymbol{u}'',\Re_c \boldsymbol{v}''>}{<\boldsymbol{u},\Re_c \boldsymbol{v}>} \end{pmatrix}$$

Lastly, from all these ingredients, we construct an auxilliary bimould $\mathrm{urtes}_{\mathrm{nor}}^{\bullet}$ by setting:

$$\mathrm{urtes}_{\mathrm{nor}}^{\boldsymbol{w}} = \sum_{\boldsymbol{w}'\boldsymbol{w}''=\boldsymbol{w}} \mathrm{sig}^{\boldsymbol{w}'\boldsymbol{w}''} \; \mathrm{tes}_{\mathrm{nor}}^{\boldsymbol{w}^*} \; \mathrm{tes}_{\mathrm{nor}}^{\boldsymbol{w}^{**}} \qquad \Big((\boldsymbol{w}', \boldsymbol{w}'') \neq (\boldsymbol{w}^*, \boldsymbol{w}^{**})\Big) \tag{14}$$

Then the tessellation bimould can be inductively calculated from:

$$\mathrm{tes}_{\mathrm{nor}}^{\bullet} = \sum_{0 \le n \le r(\bullet)} \mathrm{push}^n \, \mathrm{urtes}_{\mathrm{nor}}^{\bullet} \qquad (\forall c \in \mathbb{C}^*) \tag{15}$$

# 8. Tessellation coefficients: main properties.

$\boldsymbol{P_0}$: Double homogeneity: $tes\binom{\lambda\, u_1\ ,\dots,\ \lambda\, u_r}{\mu\, v_1\ ,\dots,\ \mu\, v_r} \equiv tes\binom{u_1\ ,\dots,\ u_r}{v_1\ ,\dots,\ v_r}\quad \forall\lambda,\mu.$

$\boldsymbol{P_1}$: $tes^{\bullet}$ is invariant under the involution *swap* and the iden-potent *push*:

$$\mathrm{swap}.A\binom{u_1\ ,\dots,\ u_r}{v_1\ ,\dots,\ v_r} = A\binom{v_r\ ,\dots,\ v_3-v_4\ ,\ v_2-v_3\ ,\ v_1-v_2}{u_1+\dots+u_r\ ,\dots,\ u_1+u_2+u_3\ ,\ u_1+u_2\ ,\ u_1}\qquad (\mathrm{swap}^2=\mathrm{iden})$$

$$\mathrm{push}.A\binom{u_1\ ,\dots,\ u_r}{v_1\ ,\dots,\ v_r} = A\binom{-u_1\dots-u_r\ ,\ u_1\ ,\ u_2\ ,\dots,\ u_{r-1}}{-v_r\ ,\ v_1-v_r\ ,\ v_2-v_r\ ,\dots,\ v_{r-1}-v_r}\qquad (\mathrm{push}^{r+1}=\mathrm{iden})$$

$\boldsymbol{P_2}$: the bimould $tes^{\bullet}$ is *bialternal*, i.e. alternal and of alternal *swappee*.

$\boldsymbol{P_3}$: $tes^{\bullet}_{nor}$ assumes all its sole values in $\mathbb{Z}$ and $|tes^{w_1,\dots,w_r}| < (r-1)!(r+1)!$ (far from sharp)

$\boldsymbol{P_4}$: As $r$ increases, the set where $\mathrm{tes}^{\bullet} \neq 0$ has surprisingly small Lebesgue measure.

$$\begin{array}{lll}
\mathrm{tes}^{w_1} \equiv 1 & & \\
\mathrm{tes}^{w_1,w_2} \in \{0,\pm1\} & \mathcal{P}(\mathrm{tes}^{w_1,w_2}=\pm1)\sim 0.21 & \\
\mathrm{tes}^{w_1,w_2,w_3} \in \{0,\pm1\} & \mathcal{P}(\mathrm{tes}^{w_1,w_2,w_3}=\pm1)\sim 0.026 & \\
\mathrm{tes}^{w_1,\dots,w_4} \in \{0,\pm1,\pm2\} & \mathcal{P}(\mathrm{tes}^{w_1,\dots,w_4}=\pm1)\sim 0.0037 & \mathcal{P}(\mathrm{tes}^{w_1,\dots,w_4}=\pm2)\sim 0.0000037
\end{array}$$

$\boldsymbol{P_5}$: in presence of vanishing $u_i$-sums, we no longer have local constancy in the $v_j$'s.

$\boldsymbol{P_6}$: conversely, in presence of $v_i$-repetitions, we no longer have local constancy in the $u_j$'s.

$\boldsymbol{P_7}$: in the *semi-real* case, i.e. when *either* all $u_i$'s *or* all $v_i$'s are aligned with the origin, the tessellation coefficients altogether exit the picture, since in that case $tes^{w_1,\dots,w_r} \equiv 0$ as soon as $2 \leq r$.

# 9. Weighted convolution under alien differentiation.

The only alien derivatives $\Delta_{\omega_0}$ acting effectively on $\mathrm{wemu}^{\binom{u_1\ ,\dots,\ u_r}{c_1\ ,\dots,\ c_r}}(x)$ correspond either to simple ($s = 1$) or composite ($s > 1$) indices $\omega_0$ of the form

$$\omega_0 = |\boldsymbol{u^1}|\, v^1_{i_1} + \cdots + |\boldsymbol{u^s}|\, v^s_{i_s} \quad \textit{with} \begin{cases} \boldsymbol{u^1}\,\boldsymbol{u^2}\dots\boldsymbol{u^{s-1}}\,\boldsymbol{u^s}\,\boldsymbol{u^*} = \boldsymbol{u} \\ \Delta_{v^k_{i_k}}\, c^k_{i_k} \neq 0 \textit{ and } \binom{u^k_{i_k}}{c^k_{i_k}} \in \binom{\boldsymbol{u^k}}{\boldsymbol{c^k}} \end{cases}$$

with each factor sequence $\binom{\boldsymbol{u^k}}{\boldsymbol{c^k}}$ re-indexed for convenience as $\binom{u^k_1\ ,\dots,\ u^k_{r_k}}{c^k_1\ ,\dots,\ c^k_{r_k}}$. The corresponding alien derivative is given by:

$$\Delta_{\omega_0}\mathrm{wemu}^{\binom{u_1\ ,\dots,\ u_r}{c_1\ ,\dots,\ c_r}}(x) = \begin{cases} \sum_{\check{v}^k_j \text{ over } v^k_{i_k}} \mathrm{Tes}^{\binom{|\boldsymbol{u^1}|\ ;\dots;\ |\boldsymbol{u^s}|}{\check{v}^1_1,\dots,\check{v}^1_{r_1}\ ;\dots;\ \check{v}^s_1,\dots,\check{v}^s_{r_s}}} \times \\ \prod_{1\le k\le s} \mathrm{welu}^{\binom{u^k_1}{\check{v}^k_1\ c^k_1}\ ,\dots,\ \binom{u^k_{i_k}}{\Delta_{\check{v}^k_{i_k}} c^k_{i_k}}^{\dagger}\ ,\dots,\ \binom{u^k_{r_k}}{\check{v}^k_{r_k}\ c^k_{r_k}}}(x) \times \\ \mathrm{wemu}^{\binom{u^*_1\ ,\dots,\ u^*_{r_*}}{c^*_1\ ,\dots,\ c^*_{r_*}}}(x) \end{cases}$$

# 9. Weighted convolution under alien differentiation.

The only alien derivatives $\Delta_{\omega_0}$ acting effectively on $\mathrm{welu}^{\binom{u_1}{c_1},\ldots,\binom{u_j}{c_j}^{\dagger},\ldots,\binom{u_r}{c_r}}(x)$ correspond either to simple ($s=1$) or composite ($s>1$) indices $\omega_0$ of three possible types – initial, final, global. Respectively:

$$\omega_0^{ini} = |\boldsymbol{u^1}|\, v^1_{i_1} + \cdots + |\boldsymbol{u^s}|\, v^s_{i_s} \ \ \textit{with} \ \begin{cases} \boldsymbol{u^1} \ \ldots \ \boldsymbol{u^s}\, \boldsymbol{u^*} = \boldsymbol{u} \ \ ; \ \ \binom{u_j}{c_j}^{\dagger} \in \binom{\boldsymbol{u^*}}{\boldsymbol{c^*}} \\ \Delta_{v^k_{i_k}}\, c^k_{i_k} \neq 0 \ \textit{and} \ \binom{u^k_{i_k}}{c^k_{i_k}} \in \binom{\boldsymbol{u^k}}{\boldsymbol{c^k}} \end{cases} \tag{16}$$

$$\omega_0^{fin} = |\boldsymbol{u^1}|\, v^1_{i_1} + \cdots + |\boldsymbol{u^s}|\, v^s_{i_s} \ \ \textit{with} \ \begin{cases} {}^*\boldsymbol{u}\, \boldsymbol{u^1} \ \ldots \ \boldsymbol{u^s} = \boldsymbol{u} \ \ ; \ \ \binom{u_j}{c_j}^{\dagger} \in \binom{{}^*\boldsymbol{u}}{{}^*\boldsymbol{c}} \\ \Delta_{v^k_{i_k}}\, c^k_{i_k} \neq 0 \ \textit{and} \ \binom{u^k_{i_k}}{c^k_{i_k}} \in \binom{\boldsymbol{u^k}}{\boldsymbol{c^k}} \end{cases} \tag{17}$$

$$\omega_0^{glo} = |\boldsymbol{u^1}|\, v^1_{i_1} + \cdots + |\boldsymbol{u^s}|\, v^s_{i_s} \ \ \textit{with} \ \begin{cases} \boldsymbol{u^1} \ \ldots \ \boldsymbol{u^s} = \boldsymbol{u} \\ \Delta_{v^k_{i_k}}\, c^k_{i_k} \neq 0 \ \textit{and} \ \binom{u^k_{i_k}}{c^k_{i_k}} \in \binom{\boldsymbol{u^k}}{\boldsymbol{c^k}} \end{cases} \tag{18}$$

with each factor sequence $\binom{\boldsymbol{u^k}}{\boldsymbol{c^k}}$ re-indexed for convenience as $\binom{u^k_1\ ,\ldots,\ u^k_{r_k}}{c^k_1\ ,\ldots,\ c^k_{r_k}}$. The corresponding alien derivatives are given by:

## 9. Weighted convolution under alien differentiation.

$$\Delta_{\omega_0^{ini}} \mathrm{welu}^{\left(\begin{smallmatrix} u_1 & \dots, & (u_j)^\dagger & \dots, & u_r \\ c_1 & ,\dots, & (c_j) & ,\dots, & c_r \end{smallmatrix}\right)}(x) = \begin{cases} + \sum_{\check{v}_j^k \text{ over } v_{i_k}^k} \mathrm{Tes}^{\left(\begin{smallmatrix} |\boldsymbol{u^1}| & ;\dots; & |\boldsymbol{u^s}| \\ \check{v}_1^1,\dots,\check{v}_{r_1}^1 & ;\dots; & \check{v}_1^s,\dots,\check{v}_{r_s}^s \end{smallmatrix}\right)} \times \\ \prod_{1\le k\le s} \mathrm{welu}^{\left(\begin{smallmatrix} u_1^k & ,\dots, & (u_{i_k}^k)^\dagger & ,\dots, & u_{r_k}^k \\ \check{v}_1^k\, c_1^k & ,\dots, & (\Delta_{\check{v}_{i_k}^k}\, c_{i_k}^k) & ,\dots, & \check{v}_{r_k}^k\, c_{r_k}^k \end{smallmatrix}\right)}(x) \times \\ \mathrm{welu}^{\left(\begin{smallmatrix} u_1^* & \dots, & (u_j)^\dagger & \dots, & u_{r_*}^* \\ c_1^* & ,\dots, & (c_j) & ,\dots, & c_{r_*}^* \end{smallmatrix}\right)}(x) \end{cases}$$

$$\Delta_{\omega_0^{fin}} \mathrm{welu}^{\left(\begin{smallmatrix} u_1 & \dots, & (u_j)^\dagger & \dots, & u_r \\ c_1 & ,\dots, & (c_j) & ,\dots, & c_r \end{smallmatrix}\right)}(x) = \begin{cases} - \sum_{\check{v}_j^k \text{ over } v_{i_k}^k} \mathrm{Tes}^{\left(\begin{smallmatrix} |\boldsymbol{u^1}| & ;\dots; & |\boldsymbol{u^s}| \\ \check{v}_1^1,\dots,\check{v}_{r_1}^1 & ;\dots; & \check{v}_1^s,\dots,\check{v}_{r_s}^s \end{smallmatrix}\right)} \times \\ \prod_{1\le k\le s} \mathrm{welu}^{\left(\begin{smallmatrix} u_1^k & ,\dots, & (u_{i_k}^k)^\dagger & ,\dots, & u_{r_k}^k \\ \check{v}_1^k\, c_1^k & ,\dots, & (\Delta_{\check{v}_{i_k}^k}\, c_{i_k}^k) & ,\dots, & \check{v}_{r_k}^k\, c_{r_k}^k \end{smallmatrix}\right)}(x) \times \\ \mathrm{welu}^{\left(\begin{smallmatrix} {}^*u_1 & \dots, & (u_j)^\dagger & \dots, & {}^*u_{r_*} \\ {}^*c_1 & ,\dots, & (c_j) & ,\dots, & {}^*c_{r_*} \end{smallmatrix}\right)}(x) \end{cases}$$

$$\Delta_{\omega_0^{glo}} \mathrm{welu}^{\left(\begin{smallmatrix} u_1 & \dots, & (u_j)^\dagger & \dots, & u_r \\ c_1 & ,\dots, & (c_j) & ,\dots, & c_r \end{smallmatrix}\right)}(x) = \begin{cases} + \sum_{\check{v}_j^k \text{ over } v_{i_k}^k} \mathrm{Tes}^{\left(\begin{smallmatrix} |\boldsymbol{u^1}| & ;\dots; & |\boldsymbol{u^s}| \\ \check{v}_1^1,\dots,\check{v}_{r_1}^1 & ;\dots; & \check{v}_1^s,\dots,\check{v}_{r_s}^s \end{smallmatrix}\right)} \times \\ \prod_{1\le k\le s} \mathrm{welu}^{\left(\begin{smallmatrix} u_1^k & ,\dots, & (u_{i_k}^k)^\dagger & ,\dots, & u_{r_k}^k \\ \check{v}_1^k\, c_1^k & ,\dots, & (\Delta_{\check{v}_{i_k}^k}\, c_{i_k}^k) & ,\dots, & \check{v}_{r_k}^k\, c_{r_k}^k \end{smallmatrix}\right)}(x) \end{cases}$$

## 10. First, second, third Bridge equations.

• **Equational resurgence** (In all cases)

**First Bridge equation:** $[\boldsymbol{\Delta}_\omega, \Theta^{-1}] = \mathbb{A}_\omega\, \Theta^{-1}$

with $\boldsymbol{\Delta}_\omega := e^{-\omega z}\Delta_\omega$ ($z$-resurgence) and

$$\mathbb{A}_\omega = \sum_{(u_1+\dots+u_r)x=\omega} W^{\binom{u_1\ ,\dots,\ u_r}{b_1\ ,\dots,\ b_r}}(x)\mathbb{D}_{\|u_1}\dots\mathbb{D}_{\|u_r}$$

• **Coequational resurgence**: First in the case of meromorphic inputs and all-real weights or all-real singularities (which ensures trivial tessellation).

**Second Bridge equation:** $[\boldsymbol{\Delta}_\omega, \Theta^{-1}] = \mathbb{Q}_{[\substack{u_0\\ \alpha_0}]}\, \Theta^{-1}$

with $\boldsymbol{\Delta}_\omega := e^{-\omega x}\Delta_\omega$ ($x$-resurgence), $\omega = u_0\,(z-\alpha_0)$ and:

$$\mathbb{Q}_{[\substack{u_0\\ \alpha_0}]} := e^{u_0\alpha_0\, x} \sum_{\sum u_i=u_0} \mathrm{welu}^{\left(\substack{u_1\\ \bar{\alpha}_0\cdot c_1}\ \substack{,\dots,\\ ,\dots,}\ \left(\substack{u_i\\ \Delta_{\alpha_0} c_i}\right)^{\sharp}\ \substack{,\dots,\\ ,\dots,}\ \substack{u_r\\ \bar{\alpha}_0\cdot c_r}\right)}\mathbb{D}_{\|u_1}\dots\mathbb{D}_{\|u_r}$$

**Third Bridge equation:** $\boldsymbol{\Delta}_\omega \mathbb{Q}_{[\substack{u_0\\ \alpha_0}]} = \sum_{u_1+u_2=u_0}^{u_1(\alpha_0-\alpha_1)=\omega} \left[\mathbb{Q}_{[\substack{u_1\\ \alpha_1}]}, \mathbb{Q}_{[\substack{u_2\\ \alpha_0}]}\right]$

## 10. BE2 and BE3 in the general case.

**New layer of complexity:** We now require new operators $\mathbb{P}_\omega$ formed from the earlier operators $\mathbb{Q}_\omega$ and the tessellation coefficients.

$$\begin{cases} \mathbb{P}_\omega & := \sum_{\sum u_i(z-\alpha_i)=\omega} \mathrm{tes}\binom{u_1\ ,\dots,\ u_r}{z-\alpha_1\ ,\dots,\ z-\alpha_r} \mathbb{Q}_{[\,{u_1 \atop \alpha_1}\,]} \dots \mathbb{Q}_{[\,{u_r \atop \alpha_r}\,]} \\ \mathbb{Q}_{[\,{u_0 \atop \alpha_0}\,]} & := e^{u_0 \alpha_0 x} \sum_{\sum u_i = u_0} \mathrm{welu}\binom{u_1\ ,\dots,\ (\ u_i\ )^\sharp\ ,\dots,\ u_r}{\bar{\alpha}_0 . c_1\ ,\dots,\ (\Delta_{\alpha_0} c_i)\ ,\dots,\ \bar{\alpha}_0 . c_r} \mathbb{D}_{\| u_1} \dots \mathbb{D}_{\| u_r} \end{cases}$$

- **Second Bridge equation:** $[\boldsymbol{\Delta}_\omega, \Theta^{-1}] = \mathbb{P}_\omega\ \Theta^{-1}$

- **Third Bridge equation:** $$\boldsymbol{\Delta}_\omega \mathbb{Q}_{[\,{u_0 \atop \check{\alpha}_0}\,]} = \begin{cases} +\sum_{u_1+u_2=u_0} \mathbb{P}_{\omega,[\,{u_1 \atop \check{\alpha}_0}\,]}\ \mathbb{Q}_{[\,{u_2 \atop \check{v}_0}\,]} \\ -\sum_{u_1+u_2=u_0} \mathbb{Q}_{[\,{u_1 \atop \check{\alpha}_0}\,]}\ \mathbb{P}_{\omega,[\,{u_2 \atop \check{\alpha}_0}\,]} \\ +\ \ \mathbb{P}_{\omega,[\,{u_0 \atop \check{\alpha}_0}\,]} \end{cases}$$

with $u_1(\alpha_0 - \alpha_1) = \omega$.

## 10. Coequational resurgence: four levels of complexity.

We have thus a clear, four-level stratification:

- *The atomic level*, inhabited by objects such as simple poles or hyperlogarithms.
- *The molecular level*, consisting of huge clusters of atoms, with unsuspected emergent properties.
- *The microscopic level*, consisting of derivation operators $\mathbb{Q}_\omega$, usually infinite chains of molecules contracted by elementary derivation operators.
- *The macroscopic level*, consisting of new derivation operators $\mathbb{P}_\omega$ assembled from the earlier $\mathbb{Q}_\omega$.
- The passage from the atomic to the molecular level is mediated on the Analysis side by *weighted convolution* and on the combinatorial side by the *scrambling transform*.
- The passage from the molecular to the microsopic level is rather mechanical – mere growth by accumulation.
- The passage from the microscopic to the macroscopic level, arguably the most interesting of the three, is mediated by the *tessellation coefficients*. While much is known about them, it would seem that just as much remains to be discovered.

When we have both $z$- and $x$-resurgence, there can be no hesitation. But often (esp. in physics), the $x$-resurgence is all we have.

## 11 Mid-talk review: equational vs co-equational.

- To produce *equational resurgence*, the coefficients $b_i(z)$ need only be analytic germs at $\infty$ (and verify a uniformity condition).
- To produce *co-equational resurgence*, the $b_i(z)$ must be endlessly continuable over the Riemann sphere (with a uniformity condition).
- **BE1** : The index reservoir $\Omega_1$ is rigidly determined by the *multipliers* $\lambda_i$. The Stokes constants are entire functions of $x$.
- **BE2** : The index reservoir $\Omega_2$ depends linearly on $z$ and the singular points of the coefficients $b_i(z)$. The Stokes constants disappear (*caveat!*) and make way for discrete-valued tessellation coefficients.
- **BE3** : The index reservoir $\Omega_3$ and the tessellation coefficients cease to depend on $z$. BE2 involves $wemu^\bullet$ and $welu^\bullet$, BE3 only $welu^\bullet$

That leaves only two aspects to review:

- Isography and autarchy of the BE3 resurgence.

## 12. Isography and rotator idempotence.

• With equational and coequational alike, the *active alien algebras* are isomorphic to algebras $\mathcal{D}$ of ordinary differential operators.

• But whereas in the equational case the elements of $\mathcal{D}$ are a priori *constraint-free*, in the coequational case they are *constrained by isography*: each $\mathbb{D}$ in the core part of $\mathcal{D}$ (the part attached to BE3) annihilates a (case-specific) differential two-form $\varpi^{iso}$, the so-called isographic form.

• Isographic invariance tends to make the *one-turn rotator* $\mathbb{R}_{[\theta,\theta+2\pi[} = \mathbb{R}_{\theta_r} \ldots \mathbb{R}_{\theta_1}$ $(\theta_j \downarrow)$ *with* $\mathbb{R}_{\theta_i} := \exp(2\pi i \sum_{\arg\omega=\theta_i} \mathbf{\Delta}_\omega)$ either idempotent, or the identity itself, in the *active alien algebra*. As a consequence, *all* Laplace sums, as germs at $\infty$, are going to be either finitely ramifed, or not ramified at all.

**13. Autarchy vs anarchy.**

• Autark functions: roughly, they are entire functions whose asymptotic behaviour in the various sectors is fully described by resurgent asymptotic expansions, which in turn generate, under alien differentiation, *closed finite systems* ("autarchy relations"). Despite being 'transcendental', autark functions have a strong algebraic flavour. They are quite common, too: for instance, most Stokes constants are *autark* relative to their various parameters.

• Prototypal autarchy: $\frac{1}{\Gamma(1+x)} = \frac{1}{\sqrt{2\pi x}}(\frac{e}{x})^x\, H(x)$ with $\begin{cases} H(x) \text{ resurgent} \\ \Omega := 2\pi i\,\mathbb{Z} \end{cases}$.

• Prototypal anarchy: $\Xi(x) := -(\frac{1}{8} + \frac{x^2}{2})\pi^{-\frac{1}{4}-\frac{xi}{2}}\Gamma(\frac{1}{4}+\frac{xi}{2})\zeta(\frac{1}{4}+\frac{xi}{2})$

• Autarchy and isography: The two are intimately connected. *Isography* leads to *idempotent rotators*, which lead to *entireness.*

## 14. Isography and autarchy: first example.

$$\overset{\textcolor{red}{z\text{-resurgence}}}{\mathcal{W}^{\binom{u_1,\dots,u_r}{\alpha_1,\dots,\alpha_r}}(z,x)} = \overset{\textcolor{red}{x\text{-resurgence}}}{\mathcal{S}^{\binom{u_1,\dots,u_r}{v_1,\dots,v_r}}(x)} \quad \text{with} \quad \begin{cases} u_i = \textit{weights} \\ v_i = z - \alpha_i \end{cases}$$

---

**BE₁** $\sum_{1\leq j\leq 3} {}^{(z)}\mathbf{\Delta}_{u_{1,2,3}\,x}\mathcal{W}^{\binom{u_1\,,\,u_2\,,\,u_3}{v_1\,,\,v_2\,,\,v_3}}(z,x) = W_*^{\binom{u_1\,,\,u_2\,,\,u_3}{v_1\,,\,v_2\,,\,v_3}}(x)$

The monics $W_*^\bullet(x)$ neatly split into two parts:
(*) the universal hyperlog. monics $V^\bullet$ ($x$- and $\alpha$- independent).
(**) the monics $C^\bullet(x)$, entire in $x$ and $\alpha$ and recursively defined by simple integrals.
The relation reads $W_*^\bullet(x) = V^\bullet \circ C^\bullet(x)$. For instance:

$$W_*^{\binom{u_1\,,\,u_2\,,\,u_3}{\alpha_1\,,\,\alpha_2\,,\,\alpha_3}}(x) = \begin{cases} +V^{\,u_1+u_2+u_3}\; C^{\binom{u_1\,,\,u_2\,,\,u_3}{\alpha_1\,,\,\alpha_2\,,\,\alpha_3}}(x) \\ +V^{\,u_1+u_2,u_3}\; C^{\binom{u_1\,,\,u_2}{\alpha_1\,,\,\alpha_2}}(x)\,C^{\binom{u_3}{\alpha_3}}(x) + V^{\,u_1,u_2+u_3}\; C^{\binom{u_1}{\alpha_1}}(x)\,C^{\binom{u_2\,,\,u_3}{\alpha_2\,,\,\alpha_3}}(x) \\ +V^{\,u_1,u_2,u_3}\; C^{\binom{u_1}{\alpha_1}}(x)\,C^{\binom{u_2}{\alpha_2}}(x)\,C^{\binom{u_3}{\alpha_3}}(x) \end{cases}$$

## 14. Isography and autarchy: first example.

**BE₂** $\sum_{1\leq j\leq 3}{}^{(x)}\boldsymbol{\Delta}_{u_{1,2,3}\,v_j}\mathcal{S}^{\left(\begin{smallmatrix}u_1\,,\,u_2\,,\,u_3\\ v_1\,,\,v_2\,,\,v_3\end{smallmatrix}\right)}(x) = \mathcal{T}_*^{\left(\begin{smallmatrix}u_1\,,\,u_2\,,\,u_3\\ v_1\,,\,v_2\,,\,v_3\end{smallmatrix}\right)}(x)$

The one-turn rotator annihilates $\mathcal{T}_*^{\bullet}(x)$ via pairwise cancellations:

$$\mathcal{T}_*^{\left(\begin{smallmatrix}u_1\,,\,u_2\,,\,u_3\\ \alpha_1\,,\,\alpha_2\,,\,\alpha_3\end{smallmatrix}\right)}(x) = \begin{cases} +e^{-u_{1,2,3}\,v_1\,x}\,\mathcal{S}^{\left(\begin{smallmatrix}u_{2,3}\,,\,u_3\\ v_{2:1}\,,\,v_{3:2}\end{smallmatrix}\right)}(x) & (1)\\ -e^{-u_{1,2,3}\,v_1\,x}\,\mathcal{S}^{\left(\begin{smallmatrix}u_{2,3}\,,\,u_2\\ v_{3:1}\,,\,v_{2:3}\end{smallmatrix}\right)}(x) & (2)\\ +e^{-u_{1,2,3}\,v_1\,x}\,\mathcal{S}^{\left(\begin{smallmatrix}u_3\,,\,u_2\\ v_{3:1}\,,\,v_{2:1}\end{smallmatrix}\right)}(x) & (3)\\ -e^{-u_{1,2,3}\,v_2\,x}\,\mathcal{S}^{\left(\begin{smallmatrix}u_1\,,\,u_3\\ v_{1:2}\,,\,v_{3:2}\end{smallmatrix}\right)}(x) & (1)\\ -e^{-u_{1,2,3}\,v_2\,x}\,\mathcal{S}^{\left(\begin{smallmatrix}u_3\,,\,u_1\\ v_{3:2}\,,\,v_{1:2}\end{smallmatrix}\right)}(x) & (4)\\ +e^{-u_{1,2,3}\,v_3\,x}\,\mathcal{S}^{\left(\begin{smallmatrix}u_1\,,\,u_2\\ v_{1:3}\,,\,v_{2:3}\end{smallmatrix}\right)}(x) & (2)\\ -e^{-u_{1,2,3}\,v_3\,x}\,\mathcal{S}^{\left(\begin{smallmatrix}u_{1,2}\,,\,u_2\\ v_{1:3}\,,\,v_{2:1}\end{smallmatrix}\right)}(x) & (3)\\ +e^{-u_{1,2,3}\,v_3\,x}\,\mathcal{S}^{\left(\begin{smallmatrix}u_{1,2}\,,\,u_1\\ v_{2:3}\,,\,v_{1:2}\end{smallmatrix}\right)}(x) & (4)\end{cases}$$

(At depth $r = 10$, nearly a trillion such pairwise cancellations.)

# 15. Isography and autarchy: second example.

The time-independent Schrödinger equation with polynomial potential.

$$\frac{\hbar^2}{2\,m}\,\partial_q^2\Psi(q,\hbar) = W(q)\Psi(q,\hbar) \quad \textit{with} \quad W(q) = q^\nu + \sum_{i=0}^{\nu-1}\alpha_i q^i \quad \textit{with} \quad (\oint\sqrt{(W(q_0)}dq_0 = 0)$$

$$\begin{cases} z = z(q) = \int_0^q \sqrt{W(q_0)}dq_0 \Rightarrow q = q(z) \sim \left(\frac{\nu+2}{2}\right)^{\frac{2}{\nu+2}} z^{\frac{2}{\nu+2}} \quad ., \quad z = \frac{\sqrt{8\,m}}{\hbar} \\ \Psi(q,\hbar) = \psi(z,x) = C_+(x)e^{\frac{1}{2}xz}q'(z)^{\frac{1}{2}}\varphi_+(z,x) + C_-(x)e^{-\frac{1}{2}xz}q'(z)^{\frac{1}{2}}\varphi_-(z,x) \end{cases}$$

$$\mathbf{BE_2} \quad \Delta_{\pm z \pm \lambda_j}\varphi_\pm(z,x) = P_{j,\pm}(x)\,\varphi_\mp(z,x)\ ,\ P_{j,\pm} \in \mathbb{C}[[x^{-1}]]\ \lambda_j = \int_{\gamma_j}\sqrt{(W(q_0)}dq_0$$

The $P_{j,\pm}(x)$ are rational in $E_1(x), \ldots, E_\nu(x)$ with $E_1(x)E_2(x)\ldots E_\nu(x) \equiv 1$ and:

$$\mathbf{BE_3} \begin{cases} 2\pi i\,\Delta_{n\lambda_{i:j}}E_k(x) & (k \neq i,j \ ,\ \ \lambda_{i:j} := \lambda_i - \lambda_j) \\ 2\pi i\,\Delta_{n\lambda_{i:j}}E_i(x) = +\frac{1}{n}E_i(x)\,(-F_{i:j}(x))^n & n \in \mathbb{Z}^*\ ,\ F_{i:j} := \frac{E_{i+1}E_{i+2}\cdots E_{j-1}}{E_{j+1}E_{j+2}\cdots E_{i-1}} \end{cases}$$

The mapping $\mathbb{\Delta}_{n\lambda_{i:j}} \mapsto \mathrm{D}_{n;i,j} := \frac{1}{n}\left(\frac{t_{i+1}t_{i+2}\ldots t_{j-1}}{t_{j+1}t_{j+2}\ldots t_{i-1}}\right)^n \left(t_i\partial_{t_i} - t_j\partial_{t_j}\right)$ induces an isomorphism of the *active algebra* of $\{E_1, .., E_\nu\}$ into the algebra $\mathcal{D}$ generated by the ordinary differential operators $\mathrm{D}_{n;i,j}$, and all operators in $\mathcal{D}$ annihilate the isographic form (independent of k):

$$\varpi_\nu^{iso} := (-1)^{\nu\,k} \sum_{(k<i<j)_\nu^{circ}} \frac{dt_i}{t_i} \wedge \frac{dt_j}{t_j} \quad \forall k \qquad \mathrm{mod}\ (t_1 \ldots t_\nu - 1)$$

## 15. Isography and autarchy: second example.

Setting as above $\begin{cases} F_{i:j} := \frac{E_{i+1}E_{i+2}\dots E_{j-1}}{E_{j+1}E_{j+2}\dots E_{i-1}} \\ \mathbb{R}_{i:j} := \exp(2\pi i \sum_{\arg\omega=\arg\lambda_{i:j}} \mathbf{\Delta}_\omega) \end{cases}$
we get the axis crossing identities:

$$\begin{cases} \mathbb{R}_{i:j} E_k = E_k \quad \textit{if} \quad k \neq i, j \\ \mathbb{R}_{i:j} E_i = E_i \, (1 + e^{-\lambda_{i:j}} F_{i:j})^{-1} \\ \mathbb{R}_{i:j} E_j = E_j \, (1 + e^{-\lambda_{i:j}} F_{i:j}) \end{cases}$$

For $W(q)$ close to $q^\nu - 1$, the $\lambda_i$ form a near-regular star. The one-turn rotator $\mathbb{R}$ is then given by:

$$\mathbb{R} = \mathbb{R}^{**}_{\nu-1}\mathbb{R}^{*}_{\nu-1}\dots\mathbb{R}^{**}_{1}\mathbb{R}^{*}_{1}\mathbb{R}^{**}_{0}\mathbb{R}^{*}_{0} \; \textit{with} \; \begin{cases} \mathbb{R}^{*}_{j} := \prod_{1\leq k\leq \nu'} \mathbb{R}_{j+k:j+1+\nu-k} \\ \mathbb{R}^{**}_{j} := \prod_{2\leq k\leq \nu'} \mathbb{R}_{j+k:j+2+\nu-k} \end{cases}$$

and verifies the idempotence relation $\mathbb{R}^{2+\nu} = id$. (In fact it always does, even when we move far from the symmetric configuration).

## 15. Isography and autarchy: second example.

$$\frac{\hbar^2}{2\,m}\,\partial_q^2\Psi(q,\hbar)=W(q)\Psi(q,\hbar)\quad \textit{with}\quad \begin{cases}(q,\hbar)\mapsto(z,x)\\ \Psi(q,\hbar)\mapsto\Gamma_+(x)\varphi_+(z,x)+\Gamma_-(x)\varphi_-(z,x)\end{cases}$$

$$\mathbf{BE_1}\qquad \begin{cases}{}^{(z)}\Delta_{+x_i}\varphi_+(z,x)=A_i(x)\,\varphi_-(z,x)\quad, & (i=2,4,\ldots,\nu+2)\\ {}^{(z)}\Delta_{-x_i}\varphi_-(z,x)=A_i(x)\,\varphi_+(z,x)\quad, & (i=1,3,\ldots,\nu+1)\end{cases}$$

$$\mathbf{BE_2}\qquad \begin{cases}{}^{(x)}\Delta_{\;z+\lambda_j}\varphi_+(z,x)=P_{j,+}(x)\,\varphi_-(z,x)\;, & P_{j,\pm}\in\mathbb{C}[[x^{-1}]]\\ {}^{(x)}\Delta_{-z-\lambda_j}\varphi_-(z,x)=P_{j,-}(x)\,\varphi_+(z,x)\;, & \lambda_j=\int_{\gamma_j}\sqrt{(W(q_0)}dq_0\end{cases}$$

$$\mathbf{BE_3}\qquad \begin{cases}2\pi i\,\Delta_{n\lambda_{i:j}}E_k(x) & (k\neq i,j\;\;,\;\;\lambda_{i:j}:=\lambda_i-\lambda_j)\\ 2\pi i\,\Delta_{n\lambda_{i:j}}E_i(x)=+\frac{1}{n}E_i(x)\,(-F_{i:j}(x))^n & n\in\mathbb{Z}^*\;,\;F_{i:j}:=\frac{E_{i+1}E_{i+2}\cdots E_{j-1}}{E_{j+1}E_{j+2}\cdots E_{i-1}}\end{cases}$$

Up to simple algebraic changes, the $x^{\frac{1}{2+\nu}}$-*entire funct.* $A_i(x)$ and the resurgent funct. $P_{j,\pm}(x)\sim E_j(x)$ are the same. This makes them *autark*.

## 16. Isography and autarchy: third example.

Consider this special case of our model problem:

$$\partial_z Y(z) = x\, Y(z) + B_-(z) + B_+(z)\, Y^2(z) \tag{22}$$

with $B_\pm(z) = \sum_{i\in\mathcal{J}} \frac{\beta_i^\pm}{z-\lambda_i}$ meromorphic in $z$ and analytic at $\infty$. For this Riccati equation, the third Bridge equation involves resurgent functions $E_j(x)$ and alien derivations $\Delta_{\lambda_{i:j}}$ (with $\lambda_{i:j} := \lambda_i - \lambda_j$) The corresponding *active alien algebra* is isomorphic to an algebra $\mathcal{D}$ generated by the ordinary derivations $D_{i:j}$ (*infra*) which in turn annihilate an *isographic form* $\varpi^{iso}$ (*infra*):

$$\Delta_{\lambda_{i:j}} \;\mapsto\; D_{i:j} := t_i^*\, t_j\, \partial_{t_j^*} - t_j^{**}\, t_i\, \partial_{t_i^{**}} + \frac{1}{2} t_i^*\, t_j^{**} (\partial_{t_j} - \partial_{t_i}) \tag{23}$$

$$\varpi^{iso} := \sum_j \frac{1}{t_j}\, dt_j^* \wedge dt_j^{**} \qquad \text{mod } t_j^2 - t_j^* t_j^{**} = \mathrm{Const_j} \tag{24}$$

## 17. Further types/souces of resurgence.

• **Object synthesis.** Spherical vs standard synthesis. In standard synthesis, the form of the active alien algebra remains unchanged, but the resurgence equations assume a most unusual form.

• **Syntactic resurgence.** Taylor coefficients with a special syntax, e.g. sum-product coefficients.

• **Hyperasymptotics.** In each of the successive models, the active alien algebra remains unchanged, but the resurgence equations get ever more intricate and weird.

• **Physics.** Huge swaths of largely uncharted territory, but many pointers to a dominance of *coequational resurgence* (e.g. simplicity of the resurgence coefficients).

## 18. A spin-off from coeq. res.: the flexion (sic) structure.

• Through the rules for forming weighted convolutions; calculating their alien derivatives; handling the tessellation coefficients etc etc, *coequational resurgence* relies on two-tier indices $\boldsymbol{w} = (w_1, ..., w_r)$ $= \binom{u_1\ ,...,\ u_r}{v_1\ ,...,\ v_r}$, with $u_i$'s that get added clusterwise, $v_i$'s that get subtracted pairwise, under preservation of $< \boldsymbol{u}, \boldsymbol{v} >$ etc...

• These operations give rise to the *flexion structure*, which can be thought of as the constellation of *all interesting structures* formed from four basic "flexions": $\boldsymbol{w} \mapsto \lfloor \boldsymbol{w}\ ,\ \lceil \boldsymbol{w}\ ,\ \boldsymbol{w} \rfloor\ ,\ \boldsymbol{w} \rceil$

• Said *flexion structure* contains as its centre-piece the Lie algebra *ARI* and the group *GARI* which, owing to their preservation of *double symmetries* like $(alt^{al}/alt^{al})$ or $(sym^{al}/sym^{al})$ , prove extremely helpful for investigating *arithmetical dimorphy* in the $\mathbb{Q}$-rings of multizetas, hyperlogarithms etc. Since these, in turn, are the key transcendental ingredients of the Stokes constants of *equational resurgence*, we have come full circle...

## 19. Some references.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