Guided tour through resurgence theory.

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1 The starting point.

The initial nudge that got the whole thing rolling[1] came serendipitously, almost in text-book fashion. Back in the early 1970s I was, just for the heck of it,[2] looking at the continuous iterates $f^{\circ t}$ of identity-tangent, local analytic mappings f, with the fixed point taken to ∞ for technical convenience:

$$f(z) \quad = \quad z+1+\sum a_n z^{-n} \qquad\qquad (z \sim \infty) \tag{1}$$

$$f^{\circ t}(z) \quad = \quad z+t+\sum a_n(t)\, z^{-n} \qquad (t \in \mathbb{C}, a_n(t) \textit{ polynomial in } t) \tag{2}$$

I knew that others had already pondered the question and established such basic facts as the generic divergence of (2); the impossibility for entire functions to admit a full set of analytic iterates (I.N.Baker); also this interesting, vaguely

[1] As far as I was concerned, that is. I am telling here my side of the story, and in no way implying the absence of parallel approaches to asymptotics.

[2] I was, in effect, wondering what strange beasts the fractional iterates of familiar mappings such as $x \mapsto x + x^2$ or $x \mapsto x\exp(x)$ might be.

'Tauberian' dichotomy: for any given f, the set W_f of all iteration powers t that keep $f^{\circ t}$ analytic is either $\mathbb{C}$ or $\frac{1}{p}\mathbb{Z}$ for some $p \in \mathbb{N}$ (E.Jabotinsky).

To get beyond that, given the Gevrey-type behaviour $\limsup \frac{1}{n}|a_n(t)|^{\frac{1}{n}} > 0$ clearly displayed by the coefficients of divergent iterates, it was tempting to consider the Borel transform of $f^{\circ 0}$:

$$\widehat{f}^{\circ t}(\zeta) \quad = \quad \delta' + t\,\delta + \sum a_n(t)\,\frac{\zeta^{n-1}}{(n-1)!} \qquad\qquad (\delta = \mathit{dirac\ at\ } 0) \tag{3}$$

From there it was but a small step to show that $\widehat{f}^{\circ t}(\sigma)$ was not only convergent at 0, but analytically continuable to the entire universal covering of $\mathbb{C} - 2\pi i\mathbb{Z}$, with at most exponential growth along each non-vertical axis. This automatically allowed Laplace integration on the two real half-axes, and yielded two distinct germs, $f_+^{\circ t}(z)$ and $f_-^{\circ t}(z)$, respectively defined on $\{C_1 < \pm\Re z\} \cup \{C_2 < |\Im z|\}$, and commuting there with f. This made them *bona fide* iterates of f, albeit only 'sectorial' ones. But on closer examination, it also became clear that, for $f_+^{\circ t}(z)$ and $f_-^{\circ t}(z)$ to relate the way they should on their common domains of definition $\{C_2 < |\Im z|\}$, the *singularities of* $\widehat{f}(\zeta)$ *at the points* $\omega := 2\pi i m$ $(m \in \mathbb{Z}^*)$ *had to be of a very specific type, namely closely related to each other and to* $\widehat{f}(\zeta)$ *itself, viewed as a germ at the origin.* To illuminate these elusive relations, it became imperative to have linear operators $\widehat{\Delta}_\omega$ capable of measuring singularities at ω or rather, due to multivaluedness, *over* ω. Moreover, in view of the non-linear nature of the problem at hand, these $\widehat{\Delta}_\omega$ had to act as *derivations*:

$$\widehat{\Delta}_\omega(\widehat{\varphi}_1 * \widehat{\varphi}_2) \equiv \widehat{\Delta}_\omega(\widehat{\varphi}_1) * \widehat{\varphi}_2 + \widehat{\varphi}_1 * \widehat{\Delta}_\omega(\widehat{\varphi}_2) \tag{4}$$

relative to the natural product in the Borel plane. That product, of course, is the convolution $*$, which is first defined locally at the origin:

$$(\widehat{\varphi}_1 * \widehat{\varphi}_2)(\zeta) := \int_0^\zeta \widehat{\varphi}_1(\zeta_1) * \widehat{\varphi}_2(\zeta - \zeta_1)\, d\zeta_1 \tag{5}$$

and then extended by analytic continuation in the large.

As for the functions that like $\widehat{f}^{\circ t}$ exhibit 'self-replicating singularities', let us call them *resurgent*, pending the more precise definitions of §3.

2 Alien derivations and alien calculus.

Alien derivations.

The sought-after derivations $\widehat{\Delta}_\omega$ have to be of the form:

$$\widehat{\Delta}_\omega\, \widehat{\varphi}\,(\zeta) := \sum_{\epsilon_i = \pm} \frac{\epsilon_r}{2\pi i}\, \delta^{\left(\begin{smallmatrix}\epsilon_1 & ,\dots, & \epsilon_r \\ \omega_1 & ,\dots, & \omega_r\end{smallmatrix}\right)}\, \widehat{\varphi}^{\left(\begin{smallmatrix}\epsilon_1 & ,\dots, & \epsilon_r \\ \omega_1 & ,\dots, & \omega_r\end{smallmatrix}\right)}(\zeta + \omega) \qquad\quad \omega_r := \omega) \tag{6}$$

with the ω_i's denoting the singular points successively encountered when moving from 0 to ω; with the ϵ_i's indicating the mode of circumvention, right or left;

and with $\widehat{\varphi}^{\left(\begin{smallmatrix}\epsilon_1 & ,\dots, & \epsilon_r\\ \omega_1 & ,\dots, & \omega_r\end{smallmatrix}\right)}$ standing for the corresponding branch of $\widehat{\varphi}$. We first take ζ on the interval $[0,\omega]$ but close to 0, then extend the definition in the large by analytic continuation.

For $\widehat{\Delta}_\omega$ to be an actual derivation, the scalar weights must verify certain algebraic relations, and if we want them to depend only on the signs ϵ_i (and not on the points ω_i), these weights are unambiguously determined: we then get the so-called standard system of *alien derivations* $\{\widehat{\Delta}_\omega\}$, with indices ω running through $\mathbb{C}_\bullet = \widetilde{\mathbb{C}-\{0\}}$ rather than $\mathbb{C}-\{0\}$ to account for possible ramifications at the origin.

This bounty of *free*[3] alien derivations[4] arising *ex nihilo* from a *one variable context* immediately opens exhilarating vistas, conjuring up as it does an *alien calculus* of extreme richness, and naturally endowed with two faces: differential and integral. We shall in due course encounter applications galore, but here let us give right away, as a first appetizer, three examples that stand on their own.

Resurgence monomials.

To deserve that name, resurgence monomials $\widehat{\mathcal{W}}^\bullet$ should be defined for ω-strings $\bullet$ of any length, and behave as simply as possible under convolution and alien differentiation. Concretely, that means:[5]

$$(\widehat{\mathcal{W}}^{\boldsymbol{\omega^1}} * \widehat{\mathcal{W}}^{\boldsymbol{\omega^2}})(\zeta) \equiv \sum_{\boldsymbol{\omega}\in \mathrm{sha}(\boldsymbol{\omega^1},\boldsymbol{\omega^2})} \widehat{\mathcal{W}}^{\boldsymbol{\omega}}(\zeta) \qquad (\forall\, \boldsymbol{\omega^1},\boldsymbol{\omega^2}) \tag{7}$$

$$\Delta_{\omega_0}\widehat{\mathcal{W}}^{\omega_1,\dots,\omega_r} \equiv \delta^{\omega_1}_{\omega_0}\,\widehat{\mathcal{W}}^{\omega_2,\dots,\omega_r} \qquad (\delta = \textit{Kronecker symbol}) \tag{8}$$

Desingularisators.

Assuming convergence, the relation:

$$E^{\widehat{\mathcal{W}}}(\widehat{\varphi}) := \widehat{\varphi} + \sum_{1\leqslant r}\sum_{\omega_i}(-1)^r\,\widehat{\mathcal{W}}^{\omega_1,\dots,\omega_r} * (\Delta_{\omega_1}\dots\Delta_{\omega_r}\widehat{\varphi}) \tag{9}$$

defines a desingularisator $E^{\widehat{\mathcal{W}}}$, i.e. a convolution-respecting projector of the algebra of resurgent function into that of resurgence constants:

$$E^{\widehat{\mathcal{W}}}E^{\widehat{\mathcal{W}}} \equiv E^{\widehat{\mathcal{W}}} \tag{10}$$

$$E^{\widehat{\mathcal{W}}}(\widehat{\varphi}_1 * \widehat{\varphi}_2) \equiv E^{\widehat{\mathcal{W}}}(\varphi_1) * E^{\widehat{\mathcal{W}}}(\varphi_2) \tag{11}$$

$$\Delta_{\omega_0}E^{\widehat{\mathcal{W}}}(\widehat{\varphi}) \equiv 0 \qquad \forall\omega_0 \tag{12}$$

[3]The Δ_ω are mutually independent.

[4]Together with another system of operators ∇_ω also acting as derivations, but relative to ordinary multiplication and with a much more restricted domain of definition, the Δ_ω are the only natural instance of an infinite Lie algebra acting in complex Analysis.

[5]In (7) $\boldsymbol{\omega^1}$ and $\boldsymbol{\omega^2}$ are two ω-strings, and $\boldsymbol{\omega}$ runs through all their shuffle products.

Alien Taylor expansions.

Much like the classical Taylor formula, which expresses a function $f(z)$ in terms of simple *universtal monomials* $z^n/n!$ and f-dependent constants $f^{(n)}(0)$, the following formula:

$$\widehat{\varphi} \quad \equiv \quad E^{\widehat{\mathcal{W}}}(\widehat{\varphi}) + \sum_{1\leqslant r}\sum_{\omega_i} \left(E^{\widehat{\mathcal{W}}}(\Delta_{\omega_r}\dots\Delta_{\omega_1}\widehat{\varphi})\right) * \widehat{\mathcal{W}}^{\omega_1,\dots,\omega_r} \tag{13}$$

expresses[6] the resurgent function $\widehat{\varphi}$ in terms of elementary resurgence monomials $\widehat{\mathcal{W}}^{\bullet}$ and resurgence constants $E^{\widehat{\mathcal{W}}}(\widehat{\Delta}_{\omega_r}\dots\widehat{\Delta}_{\omega_1}\widehat{\varphi})$.

3 Algebra of resurgent functions.

Three models: formal, convolutive, geometric.

Resurgent 'functions' live simultaneously in three models:

- in the *formal model*, typically as formal power series $\widetilde{\varphi}(z) = \sum_{s\uparrow} a_s z^{-s}$, with Gevey-type bounds on the coefficients.
- in the *convolutive model*, as analytic germs $\widehat{\varphi}(\zeta)$ at the origin $0_\bullet$ of $\mathbb{C}_\bullet$, with the property of endless analytic continuation[7] on $\mathbb{C}_\bullet$ and exponentially bounded growth at infinity.
- in the *geometric (or sectorial) model*,[8] as analytic germs $\varphi_\theta(z)$ defined in sectorial neighbourhoods of infinity: $\arg(e^{i\theta}z) < \pi/2 + Const.$

Pictorially, we get this triangle:

$$\begin{array}{lcccccl} & \widetilde{\varphi}(z) & & \dots & & \varphi_\theta(z) & z\text{-}plane\ (multiplication) \\ \text{Fig 1} & \mathcal{B} & \searrow & & \nearrow & \mathcal{L}_\theta & \\ & & & \widehat{\varphi}(\zeta) & & & \zeta\text{-}plane\ \ (convolution) \end{array}$$

The first arrow denotes the formal, term-wise Borel transform $\mathcal{B}$

$$\mathcal{B} \quad : \quad \begin{cases} z^{-\sigma} \mapsto \zeta^{\sigma-1}/\Gamma(\sigma) & (\sigma \notin -\mathbb{N}) \\ z^n \mapsto \delta^{(n)} & (n \in \mathbb{N}\,,\ \delta = Dirac) \end{cases} \tag{14}$$

The second arrow denotes the Laplace transform $\mathcal{L}$ or, for distinctiveness, $\mathcal{L}_\theta$:

$$\mathcal{L}_\theta : \ \widehat{\varphi}(\zeta) \quad \mapsto \quad \varphi_\theta(z) = \int_0^{e^{i\theta}\infty} \widehat{\varphi}(\zeta)\, e^{-z\,\zeta}\, d\zeta \qquad (\arg\zeta \equiv \theta) \tag{15}$$

[6] *formally* and, under suitable conditions, *actually*.

[7] under avoidance of a discrete configuration of singular points (their projections on $\mathbb{C}$, however, may well be dense – somewhere or everywhere).

[8] or, strictly speaking, *models*, since they depend on a polarisation angle θ.

The magic of resurgence equations.

The signature, name-giving property of resurgent functions – namely, the self-replication, echo-like, of $\widehat{\varphi}(\zeta)$'s singularities – is too protean, too elusive a feature to be made part of the formal definition. Yet it is something that we should always keep at the back of our minds. In any case, the alien derivations $\widehat{\Delta}_\omega$, along with their pull-backs Δ_ω in the multiplicative models (*formal* or *geometric*) are the pliant tools that, in each particular instance, allow an accurate description of the phenomenon, by means of so-called *resurgence equations*:

$$R_\omega(\varphi, \Delta_\omega \varphi) = 0 \qquad (\textit{multiplicative models}) \tag{16}$$

$$\widehat{R}_\omega(\widehat{\varphi}, \widehat{\Delta}_\omega \widehat{\varphi}) = 0 \qquad (\textit{convolutive model}) \tag{17}$$

The form (16), being the simpler one, is often given preference in statements, although it is the form (17) that makes concrete, tangible analytic sense.

Nonetheless – and we are touching here on the magic of resurgence – we can often work *directly* at the level of (16), with a minimum of Analysis or sometimes none at all. Indeed, suppose we are dealing with some differential or functional equation[9] $R(\varphi) = 0$, linear or not, but with divergent, formal power series solutions. If these are resurgent (there exist simple criteria for deciding that) and if the singularity locus in the Borel plane is a certain point set Ω (again, there exist simple methods for determining Ω), we can find, for $\omega \in \Omega$ and by purely formal means, the equation $R_\omega(\varphi, \Delta_\omega\varphi) = 0$ verified by $\Delta_\omega\varphi$, just as we would form the equation $R'(\varphi, \varphi') = 0$ verified by φ'. Now, since R_ω is automatically linear homogeneous in $\Delta_\omega\varphi$, we can calculate the general solution, usually of the form $\Delta_\omega\varphi = A_\omega\varphi_\omega$, with A_ω scalar and φ_ω some normalized power series. Here, the substantive factor φ_ω results from a strictly formal calculation. It is only to calculate the scalar A_ω (for the *actual* $\Delta_\omega\varphi$, that scalar is a well-defined, usually transcendental number – essentially a Stokes constant) that some Analysis may be called for, though not always: even A_ω can sometimes be had 'on the cheap'. Needless to say, the procedure can be repeated to calculate all multiple alien derivatives $\Delta_{\omega_n} ... \Delta_{\omega_1}\varphi$.

The minor/major duality.

To the *minor* $\widehat{\varphi}(\zeta)$, which is unambiguously defined, it is often useful to associate a *major* $\check{\varphi}(\zeta)$, defined up to regular germs at $0_\bullet$ and relating to the minor according to:

$$\widehat{\varphi}(\zeta) = -\frac{1}{2\pi i}\left(\check{\varphi}(e^{\pi i}\zeta) - \check{\varphi}(e^{-\pi i}\zeta)\right) \qquad (\zeta \textit{ near } 0_\bullet) \tag{18}$$

We can then remove all terms $\delta^{(n)}$ from $\widehat{\varphi}(\zeta)$ and replace them by terms $n!\,\zeta^{-n-1}$ in $\check{\varphi}(\zeta)$. With the jarring diracs thus whisked away, the quantity of missing information in *the* minor clearly equals the quantity of redundant information in *a* major. This is but the first manifestation of a minor/major duality that pervades the whole theory. See in particular the formulae in §4-§5.

[9]or, to be technicaly accurate, an *equation germ* at a given point (here: ∞).

Borel singularities: a hindrance *and* a mine of information.

Being responsible for the divergence of $\widetilde{\varphi}(z)$, the singularities of $\widehat{\varphi}(\zeta)$ may seem a nuisance, something that impedes the direct passage from *formal* solutions to *actual* ones. But they are also, and above all, a valuable source of imformation, because their leading terms (their residues, in the case of simple poles) carry crucial 'invariants' pertaining to $\widetilde{\varphi}(z)$. These invariants manifest as Stokes constants in the geometric model, where they govern the correspondence between neighbouring sectorial germs $\varphi_\theta(z)$.

Well-behaved convolution-preserving averages.

Physicists, who are mostly interested in Laplace integration along $\mathbb{R}^+$ (to preserve realness), often dismiss as 'non-resummable' functions $\widehat{\varphi}(\zeta)$ with singularities on $\mathbb{R}^+$. In actual fact, the presence of singular points ω_i on the integration axis is no obstacle, provided we apply Laplace to a suitable average $\mu\widehat{\varphi}$ of $\widehat{\varphi}(\zeta)$:

$$\widetilde{\varphi}(z) \xrightarrow{\mathcal{B}} \widehat{\varphi}(\zeta) \xrightarrow{\mu} \mu\widehat{\varphi}(\zeta) \xrightarrow{\mathcal{L}} \varphi(z) \tag{19}$$

Here $\mu\,\widehat{\varphi}(\zeta) := \sum_{\epsilon_i=\pm} \mu^{\left(\begin{smallmatrix}\epsilon_1 & ,\dots, & \epsilon_r\\ \omega_1 & ,\dots, & \omega_r\end{smallmatrix}\right)} \widehat{\varphi}^{\left(\begin{smallmatrix}\epsilon_1 & ,\dots, & \epsilon_r\\ \omega_1 & ,\dots, & \omega_r\end{smallmatrix}\right)}(\zeta)$ ($if\ \ \omega_r < \zeta < \omega_{r+1}$) and *suitable* means that the average must respect realness[10] as well as convolution:

$$\mu(\widehat{\varphi}_1 * \widehat{\varphi}_2) \equiv (\mu\widehat{\varphi}_1) * (\mu\widehat{\varphi}_2) \qquad\qquad (\textit{first} * \textit{local}\,,\ \textit{second} * \textit{global}) \tag{20}$$

This imposes simple algebraic conditions on its weights $\mu^{\left(\begin{smallmatrix}\epsilon\\ \omega\end{smallmatrix}\right)}$. In presence of infinitely many ω_i, an additionnal, subtler condition (non-algebraic in nature) must be added, to ensure exponential bounds on the growth of $\mu\,\widehat{\varphi}(\zeta)$. Such averages will be referred to as *well-behaved*.

4 Critical variables and acceleration transforms.

Ascending the ladder of critical times.

Set $z_i := z^{\sigma_i}$ $(0 < \sigma_1 < \dots < \sigma_r)$. Next, take r resurgent functions $\widetilde{\psi}_i(z_i)$, each resummable according to the scheme of Fig 1 but relative to its own variable z_i. Lastly, let $\widetilde{\varphi}(z)$ be some inextricable, non-linear superposition of the various $\widetilde{\psi}_i(z_i)$. Ordinary equations $R(\varphi) = 0$, differential or functional, often enough present us with superpositions of this type. Clearly, $\widetilde{\varphi}(z)$ ought to be resummable. Just as clearly, each of the z_i's ought to play its part in the process, although the scheme of Fig 1 isn't directly applicable to any of the series $\widetilde{\varphi}_i(z_i) \equiv \widetilde{\varphi}(z)$. So how are we to proceed? The short answer is this:

- We should order our *'critical variables'* z_i, also called *'critical times'*, from slow- to fast-flowing: $z_1 \lll z_2 \lll \dots \lll z_r$.

[10] $\mu\widehat{\varphi}(\zeta)$ must be real on the whole axis $\arg\zeta = 0$ if $\widehat{\varphi}(\zeta)$ itself is real there, for ζ small.

- We should take the Borel transform $\widehat{\varphi}_1(\zeta_1)$ of $\widetilde{\varphi}_1(z_1)$, since it is the only one that converges at $0_\bullet$. The ramified germ $\widehat{\varphi}_1(\zeta_1)$ duly retains the property of endless continuation, but loses exponential boundedness at infinity. This prohibits Laplace integration. Instead, we should apply, as in (21) *infra*, a so-called *acceleration transform*, whose integral kernel has exactly the right rate of decrease at infinity to accommodate the superexponential growth rate of $\widehat{\varphi}_1(\zeta_1)$.

- This results in a new ramified germ at $0_\bullet$, denoted $\widehat{\varphi}_2(\zeta_2)$. Though unobtainable as a direct Borel transform, $\widehat{\varphi}_2(\zeta_2)$ is morally the Borel counterpart of $\widetilde{\varphi}_2(z_2)$. It still has endless analytic continuation, but once again with superexponential growth at infinity.

- We can then continue the acceleration process to successively construct all germs $\widehat{\varphi}_i(\zeta_i)$ up to $\widehat{\varphi}_r(\zeta_r)$.

- Only at the last stage do we get a function $\widehat{\varphi}_r(\zeta_r)$ with (at most) exponential growth at infinity. This allows Laplace integration and produces at last a geometric germ $\varphi_r(z_r) \equiv \varphi_{(}z)$.

- Since each of the steps we went through is an algebra isomorphism[11], our hard-won germ $\varphi(z)$ is automatically a proper solution of whatever equation $R(\varphi)=0$, linear or non-linear, the initial formal series $\widetilde{\varphi}(z)$ happened to be a solution.

- But since each Borel plane ζ_i is saddled with its own singular locus Ω_i, the end result $\varphi(z)$ is also highly polarized: it depends on the choice of r integration axes $\arg(\zeta_i) = \theta_i$.

The accelero-summation scheme.

The process I have just sketched (I call it *accelero-summation*) can be represented pictorially as follows:[12]

Fig 2

$$\begin{array}{ccccccc}
\widetilde{\varphi}_1(z_1) & \leftarrow & \widetilde{\varphi}(z) & & \varphi(z) & \leftarrow & \varphi_r(z_r) \\
\downarrow \mathcal{B} & & & & & & \mathcal{L} \uparrow \\
\widehat{\varphi}_1(\zeta_1) & \underset{\mathcal{C}_{1,2}}{\rightarrow} & \widehat{\varphi}_2(\zeta_2) & \rightarrow \cdots \rightarrow & \widehat{\varphi}_{r-1}(\zeta_{r-1}) & \underset{\mathcal{C}_{r-1,r}}{\rightarrow} & \widehat{\varphi}_r(\zeta_r)
\end{array}$$

[11] First from multiplicative to convolutive, then from convolutive to convolutive over and over again, and eventually from convolutive to multiplicative.

[12] For simplicity, I left the *majors* $\check{\varphi}_i(\zeta_i)$ out of the picture. Actually, as shown in §5 *infra*, by selecting suitably *slow times* $z_{i\_} \sim z_i$ in each critical time class $[z_i]$, one can ensure the smoothness of the minors $\widehat{\varphi}_{i\_}$ and all their alien derivatives $\widehat{\Delta}_{\omega_r}.\ldots.\widehat{\Delta}_{\omega_1}.\widehat{\varphi}_{i\_}$ $(\omega_k \in \mathbb{R}^+)$ and, by the same token, render the corresponding majors redundant.

Acceleration/deceleration kernels.

Time now to construct the acceleration operators and their inverses, and to describe their action on both minors and majors.

A single pair C_F , C^F of integral kernels does duty for the four combinations of minor and major, acceleration and decelerations, but with a characteristic diagonal 'flip':

$$\text{Fig 3} \quad \left[\begin{array}{lcc} & \textit{acceleration} & \textit{deceleration} \\ \textit{minor} & C_F & C^F \\ \textit{major} & C^F & C_F \end{array} \right] \Big(z_1 \prec z_2 \ , \ z_1 = F(z_2) \Big)$$

These kernels depend on the germ F that expresses the *slower* 'time' z_1 in terms of the *faster* z_2.

$$C_F(\zeta_2, \zeta_1) \quad := \quad \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{z_2\zeta_2 - z_1\zeta_1} \, dz_2 \quad \textit{with} \ \ z_1 \equiv F(z_2) \tag{21}$$

$$C^F(\zeta_2, \zeta_1) \quad := \quad \int_{+u}^{+\infty} e^{-z_2\zeta_2 + z_1\zeta_1} \, dz_2 \quad \textit{with} \ \ z_1 \equiv F(z_2) \ \textit{and} \ 1 \prec u \tag{22}$$

Acceleration from ζ_1 to ζ_2 obeys the formula:

$$\widehat{\varphi}_2(\zeta_2) \quad = \quad \int_{+0}^{+\infty} C_F(\zeta_2, \zeta_1) \, \widehat{\varphi}_1(\zeta_1) \, d\zeta_1 \tag{23}$$

$$\check{\varphi}_2(\zeta_2) \quad = \quad \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} C^F(\zeta_2, \zeta_1) \, \check{\varphi}_1(\zeta_1) \, d\zeta_1 \tag{24}$$

Deceleration from ζ_2 to ζ_1 goes like this:

$$\zeta_1 \, \widehat{\varphi}_1(\zeta_1) \quad = \quad \frac{1}{2\pi i} \int_{0_1}^{0_2} \zeta_2 \, \widehat{\varphi}_2(\zeta_2) \, C^F(\zeta_2, \zeta_1) \, d\zeta_2 \quad (u > 0) \tag{25}$$

$$\zeta_1 \, \check{\varphi}_1(\zeta_1) \quad = \quad \int_{+0}^{+v} \zeta_2 \, \check{\varphi}_2(\zeta_2) \, C_F(\zeta_2, \zeta_1) \, d\zeta_2 \tag{26}$$

Here again, we notice a double flip between finite/infinite, path/loop integrals. Integration in (23) is along an infinite path, in (26) along a finite one. Integration in (24) is along an infinite loop that encircles 0 anticlockwise, in (25) along a finite loop from 0 to 0 that encircles $\zeta_1 > 0$ anticlockwise.

The basic, really indispensable transform is *minor acceleration* (23), and the crucial point to note here is that the lower kernel $C_F(\zeta_2, \zeta_1)$, through some minor miracle of pre-established harmony, has exactly the right faster-than-exponential rate of decrease (as $\zeta_1 \to +\infty$) to make the acceleration integral (23) convergent for small enough values of $\zeta_2 > 0$. It therefore defines a germ $\widehat{\varphi}_2(\zeta_2)$ which then must, and can, be continued in the large.

One last remark: a 'time' z_i is declared 'critical' if passage through the corresponding Borel plane ζ_i is *mandatory.* But there is some latitude in the choice of these critical times z_i. They are actually defined up to an equivalence

$\lim z_i/z'_i = c > 0$, so that one should properly speak of critical time classes $[z_i]$. The next section is devoted to the corresponding transformations $\zeta_i \to \zeta'_i$ on the Borel side, known as pseudo-accelerations and pseudo-deceleration.

5 The Great Divide: cohesive/loose.

The present section may be skipped at first, but is highly recommended in a second reading. It is there ostensibly to guide us is the choice of proper representatives z_i in the various critical time classes $[z_i]$. But it also sheds a sharp and unexpected light on what is arguably the central watershed in Analysis: the Great Divide between those smooth functions that are 'of one piece' (analytic or quasi-analytic) and the functions not smooth enough to enjoy that property (the 'loose' functions). First, though, we must define *cohesiveness*.

Notion of cohesive function.

We get the class *COHES* of *cohesive functions* by extending the classical Denjoy classes ${}^{\alpha}DEN$ to all transfinite orders $\alpha < \omega^{\omega}$ and then going to the limit: [13]

$$
\begin{aligned}
{}^{\alpha}\mathrm{DEN} \quad &:= \quad \{f \ ; \ |f^{(n)}(t)| < c_{0,f}\,(c_{1,f})^n \left(\log'_{\alpha+1}(n)\right)^{-n}\} && (27)\\
\mathrm{COHES} \quad &:= \quad \cup_{\alpha<\omega^{\omega}}\ {}^{\alpha}\mathrm{DEN} && (28)
\end{aligned}
$$

Like each ${}^{\alpha}$DEN, the limit *COHES* is stable under $+, \times, \circ, \partial$ and most other operations. Crucially, it is also *quasi-analytic*: two cohesive functions defined on a real interval J coincide as soon they coincide on a subinterval $I \subset J$.

Pseudoaccelerations/pseudodecelerations.

Here, the change is between two *equivalent* 'times', denoted for distinction by $z_{1\_}$ and z_1 with $z_1 = z_{1\_} + F(z_{1\_})$ and $1 \prec F(x) \prec x$ as above.[14] The new transforms serve a totally different purpose, but their integral kernels C_{id+F} , C^{id+F} are closely related to the old ones:

$$
\begin{aligned}
C_{id+F}(\zeta_{1\_},\zeta_1) \quad &= \quad C_F(\zeta_{1\_} - \zeta_1, \zeta_1) && (29)\\
C^{id+F}(\zeta_{1\_},\zeta_1) \quad &= \quad C^F(\zeta_{1\_} - \zeta_1, \zeta_1) && (30)
\end{aligned}
$$

In keeping with the more elementary character of the new transforms, all integration paths/loops now become finite.
Pseudodeceleration from ζ_1 to $\zeta_{1\_}$ obeys the formulae:

$$
\begin{aligned}
\widehat{\varphi}_{1\_}(\zeta_{1\_}) \quad &= \quad \int_{+0}^{\zeta_1} C_{id+F}(\zeta_{1\_},\zeta_1)\,\widehat{\varphi}_1(\zeta_1)\,d\zeta_1 && (31)\\
\check{\varphi}_{1\_}(\zeta_{1\_}) \quad &= \quad \frac{1}{2\pi i}\int_{v_1}^{v_2} C^{id+F}(\zeta_{1\_},\zeta_1)\,\check{\varphi}_1(\zeta_1)\,d\zeta_1 && (32)
\end{aligned}
$$

[13] Despite the latitude in the analytic incarnation of the transfinite iterates $\log_{\alpha+1}$, each class ${}^{\alpha}$DEN is unambiguously defined: the indeterminacy in $\log_{\alpha+1}$ is absorbed by the constant $c_{1,f}$ in (27).

[14] The case when $z_{1\_}$ and z_1 are too close, i.e. when $F(x) = o(1)$, is uninteresting.

Pseudoacceleration from ζ_{1_-} to ζ_1 goes like this:

$$\zeta_1\,\widehat{\varphi}_1(\zeta_1) \quad = \quad \frac{1}{2\pi i}\int_{0_1}^{0_2} \zeta_{1_-}\,\widehat{\varphi}_{1_-}(\zeta_{1_-})\,C^{id+F}(\zeta_{1_-},\zeta_1)\,d\zeta_{1_-} \tag{33}$$

$$\zeta_1\,\check{\varphi}_1(\zeta_1) \quad = \quad \int_{\zeta_0}^{v} \zeta_{1_-}\,\check{\varphi}_{1_-}(\zeta_{1_-})\,C_{id+F}(\zeta_{1_-},\zeta_1)\,d\zeta_{1_-} \tag{34}$$

The most useful transforms are, paradoxically, the *accelerations* and *pseudo-decelerations*. Despite going 'in opposite directions', both share a common regularising effect, albeit of crucially different force. To adequately describe that common effect together with the discrepancy in regularising potency, we must distinguish three sub-classes for each :

strong accelerations	$\log z_1/\log z_2 \to 0$	e.g. $z_1 = \log z_2$
medium accelerations	$\log z_1/\log z_2 \to \alpha \in]0,1[$	e.g. $z_1 = (z_2)^\alpha$
weak accelerations	$\log z_1/\log z_2 \to 1$	e.g $z_1 = \frac{z_2}{\log z_2}$
strong pseudodeceler.	$\log z_1/\log(z_{1_-} - z_1) \to 1$	e.g. $z_1 = z_{1_-} + \frac{z_{1_-}}{\log z_{1_-}}$
medium pseudodeceler.	$\log z_1/\log(z_{1_-} - z_1) \to \alpha$	e.g. $z_1 = z_{1_-} + (z_{1_-})^\alpha$
weak pseudodeceler.	$\log z_1/\log(z_{1_-} - z_1) \to 0$	e.g. $z_1 = z_{1_-} + \log z_{1_-}$

Whatever the nature of the accelerand $\widehat{\varphi}_1$ (provided it has the proper accelerable growth at infinity), the corresponding accelerate $\widehat{\varphi}_2$ is automatically guaranteed a minimum of quasi-analytic smoothness – the weaker the accelaration, the less the smoothness.

- Strong accelerates are always analytic in a spiralling neighbourhood of $0_\bullet$ with infinite aperture.
- Medium accelerates are always analytic in a neighbourhood of $0_\bullet$ with at least finite aperture.
- Weak accelerates are always cohesive in a real right-neighbourhood $]0, ...[$ of $0_\bullet$, but may lack an extension to the complex domain.

With pseudo-decelerations, the picture is the same, but on the other side of the Great Divide — on the side of *looseness*: whatever the nature of the pseudo-decelerand $\widehat{\varphi}_1$, one can always, by suitably strenghtening the pseudo-deceleration, ensure in the pseudo-decelerate $\widehat{\varphi}_{1_-}$ *any given* degree of smoothness, *short of cohesive.*

Another difference is this : *accelerations* completely upset the singularity landscape (they remove the old singular points and may create new ones) whereas *pseudo-decelerations* keep all singular points ω in place and merely smoothen the singularities there.

Cohesive continuation.

Any cohesive function given on an interval $]0, \zeta[$ (viewed for the circumstance as part of the axis $arg\zeta = 0$ in some Borel plane) can be constructively continued to its maximal interval of cohesiveness $]0,\zeta_*[$ by a suitably weak deceleration followed by the reverse weak acceleration.

Cohesive singularities and their circumvention.

In some contexts like the Dulac problem[15], accelero-summation may produce strictly cohesive germs on $arg\zeta = 0$ in some Borel planes, with any number of cohesive singularities there. To proceed with accelero-summation, the germs in question have to be cohesively continued (multivaluedly so) up to $+\infty$, which means bypassing all intervening singularities to the right or to the left, while never leaving the real axis! This sounds an impossibility, but is not.

6 Equational resurgence and the Bridge equation.

After describing, in its main outlines, the apparatus of resurgence theory, we can now proceed to the applications. But let us first get some terminology straight. By *equational resurgence* we mean resurgence that affects formal expansions in the variable of some equation (of whatever nature: differential or functional). *Coequational resurgence*, by contrast, affects expansions in some 'inert' parameter. Both types, of course, rely on *resurgence equations* for their description. Quite often, these resurgence equations are of a very specific type – they unexpectedly connect *alien* and *ordinary* derivations, these denizens of two distinct worlds – and are therefore called *Bridge equations.* So let us keep these diverse notions cleanly apart.

Shape of the Bridge equation.

- In the present context of *equational resurgence*, the Bridge equation is always of the form :

$$\mathbf{\Delta}_\omega \, Y(z,\boldsymbol{u}) \;=\; \mathbf{A}_\omega \, Y(z,\boldsymbol{u}) \qquad\qquad (\forall \omega \in \Omega) \tag{35}$$

- $Y(z,\boldsymbol{u})$ is the full (i.e. parameter-saturated) formal solution of, say, some singular functional equation $\mathcal{R}(Y) = 0$. To unclutter the notations, we drop the tildas indicative of formalness. Here $\boldsymbol{u} := (u_1, .., u_d)$ denotes a maximal system of independent parameters, and z denotes *the* 'critical time' of the problem or, if there be several, any *one* of them.

- $\mathbf{\Delta}_\omega$ is an alien derivation taken in invariant form, i.e. with an exponential factor that ensures commutation with ∂_z:

$$\mathbf{\Delta}_\omega := e^{-\omega\, z}\, \Delta_\omega \quad \Longrightarrow \quad [\mathbf{\Delta}_\omega, \partial_z] = 0 \tag{36}$$

  The index ω runs through a discrete set $\Omega \subset \mathbb{C}_\bullet$ deducible from $\mathcal{R}$.

[15]See at the end of §10.

- $\mathbf{A}_\omega$ is an ordinary, first order differential operator in z and in the parameters $u_1, ..., u_d$. We may refer to it as the *Stokes operator*, because its coefficients are indeed Stokes constants. They are also 'holomorphic' *and* 'analytic' invariants, since they depend *holomorphically* on the Taylor coefficients of the equation R *and* remain unchanged when $\mathcal{R}$ undergoes an *analytic* change affecting both the unknown and the variable.

- The outward shape of the differential operators $\mathbf{A}_\omega$ is quite simply the most general of all possible shapes that makes formal sense. This means that, when identifying the terms in front of any given monomial $\boldsymbol{u}^{\boldsymbol{n}}$ on both sides of (36), the exponential factors $e^{-\omega z}$ should match.

- $\{\mathbf{A}_\omega\,;\, \omega \in \Omega\}$ is a *complete* system of holomorphic-analytic invariants. In the multicritical case, one should of course take all operators $\mathbf{A}_\omega$ relative to all critical times z_i, with ω running through the corresponding Ω_i.

- The entire divergence of $Y(z, \boldsymbol{u})$ is concentrated in the power series of z^{-1}. Once those have been resummed, there is no divergence left.

Scope and derivation of the Bridge equation.

The scope of the Bridge equation is breathtaking. It covers practically all singular analytic equations[16] of all main types: differential, difference, mixed, general functional. It also covers the so-called 'local objects', chiefly the singular vector fields X on $\mathbb{C}^\nu$ at 0, and the local self-mappings f of $\mathbb{C}^\nu$ with 0 as fixed points. The main source of divergence for local objects is *resonance*, broadly understood.[17] That divergence manifests when we attempt to bring the 'local objects' to some normal form by means of a change of coordinates, or when we solve the equations of motion:

$$\partial_z x_i(z, \boldsymbol{u}) \;=\; x_i(x_1(z, \boldsymbol{u}), \ldots, x_\nu(z, \boldsymbol{u})) \qquad (1 \leqslant i \leqslant \nu) \tag{37}$$

$$f_i(z{+}1, \boldsymbol{u}) \;=\; f_i(x_1(z, \boldsymbol{u}), \ldots, x_\nu(z, \boldsymbol{u})) \qquad (1 \leqslant i \leqslant \nu) \tag{38}$$

As it happens, such divergence born of resonance is always amenable to resurgent Analysis and fully describable by the Bridge equation, even in cases that are completely out of bounds for alternative methods (geometric or others). A bulky book and numerous papers have been devoted to the subject, and there can be no question here of summarizing them. Let us just emphasize two points.

The first point is the relative ease of derivation of the Bridge equation, at least in the monocritical case. Provided we judiciously decompose our local object Ob as $Ob_0{+}Ob_1$ with a main part Ob_0 and a complement Ob_1 treated as a perturbation, the analysis in the Borel plane becomes pretty straightforward: it amounts to repeatedly dividing by expressions like $(\zeta - \omega)$ or $(e^{-\omega\,\zeta} - 1)$, and to calculating scores of convolution products that all have a strong regularizing

[16] Properly speaking, *local equations* or *germs of equation*, with analytical germs as coefficients and analytical dependence on the unknown.

[17] See §9. It includes the vanishing of one or several 'multipliers'.

effect. As a result, convergence outside the singularity locus Ω is not overly difficult to establish.

The second point to make is the unexpected *unity* that resurgence tends to impart upon divergent objects. Let for instance $Y(z, \boldsymbol{u}) = \sum_{\boldsymbol{n}} \boldsymbol{u}^{\boldsymbol{n}} Y_{\boldsymbol{n}}(z)$ be the formal solution of a singular differential equation $\mathcal{R}(Y) = 0$, expressed in terms of its critical time z and expanded as a power series of the parameters $\boldsymbol{u}$. In the generic case, i.e. if none of the $\mathbf{A}_\omega$ vanishes, the Bridge equation allows one to recover (constructively so!) *all* components $Y_{\boldsymbol{n}}$ from *any one* of them. This would be clearly impossible in case of convergence, i.e. when all $\mathbf{A}_\omega$ vanish. A rather apt simile may help bring the phenomenon into perspective. Let $P(x)$ be a polynomial with integer coefficients. If P is totally irreducible on $\mathbb{Q}$, then each root x_i contains *in spe* all the others, but if P is fully reducible, these roots become strangers to one another.

7 Co-equational resurgence.

The WKB or semi-classical approach.

My involvement with coequational resurgence dates back to my personal encounter with Yasutaka Sibuya and Andre Voros in the early 80s, and their work on the one-dimensional, time-independent Schrödinger equation with polynomial potential:

$$\partial_q^2 \psi(q,x) = \frac{x^2}{4} W(q) \psi(q,x) \qquad \textit{with} \quad \begin{cases} W(q) = q^\nu + \alpha_1 q^{v-1} + \ldots + \alpha_\nu \\ x = \frac{2}{\hbar b} \ ; \ \alpha = -E \ (\textit{energy}) \end{cases} \tag{39}$$

Sibuya had studied the q-dependence of the solutions in the spirit of Stirling analysis, while Voros had tackled the x-dependence along the WKB approach and formulated remarkable conjectures regarding what he called a 'bootstrap' phenomenon in the conjugate Borel plane – essentially a resurgence phenomenon. On my part, I interpreted Sibuya's result as a case of resurgence in the 'critical variable' $z = z(q) = \int_0^q \sqrt{W(q')}\, dq'$; proved the missing link in Voros' conjectures; and observed an intriguing interplay between the two patterns of resurgence, in z and in x, that seemed to point to a much wider phenomenon — a duality of sorts between what I came to call *equational* and *coequational* resurgence.

The second and third Bridge equations.

Coequational resurgence is distinctly trickier than the equational sort. For one thing, it relies in the Borel plane on an intricate, 'weighted' convolution product which is harder to handle than plain convolution, lacks its regularising quality, and tends to generate singular points in far greater numbers. The main difference, though, is the appearance of two new Bridge equations where there was only one, and of universal 'tesselation coefficients'.

Let x be the new multiplicative, resurgence-carrying variable, and ξ the conjugate Borel variable. Our resurgent function $\psi(x)$ still verifies a Bridge equation of familiar form:[18]

$$\boldsymbol{\Delta}_\omega\, \psi(x,\boldsymbol{u}) = \mathbf{B}_\omega(x)\, \psi(x,\boldsymbol{u}) \qquad (\omega \in \Omega_2) \tag{40}$$

except that now x reappears in the differential operators $\mathbf{B}_\omega(x)$, so that a third Bridge equation is needed to describe their resurgent behaviour:

$$\boldsymbol{\Delta}_\omega\, \mathbf{B}_{\omega_0}(x) = h_{\omega,\omega_0}(\mathbf{B}_{\omega_1}, \mathbf{B}_{\omega_2}, ...) \qquad (\omega \in \Omega_3) \tag{41}$$

The exact shape of the right-hand side in (41) varies from case to case, but essentially comprises two things:
(i) multiple Lie brackets of the $\mathbf{B}_{\omega_i}$'s.
(ii) universal scalars $tes^{\binom{u_1\ ,\dots,\ u_1}{v_1\ ,\dots,\ v_1}}$ – the so-called tesselation coefficients – that depend only on two strings $\boldsymbol{u}$ anf $\boldsymbol{v}$ of complex numbers.

Tesselation coefficients.

Let us just mention two of their many fascinating properties:
(i) although they are piece-wise constant in each u_i and v_i, the only way of expressing them without breaking their natural symmetries is as finite sums of $r!!$ hyperlogarithms, with $r!! := 1.3.5...(2r-1)$.
(ii) they are the most basic object exhibiting the important double symmetry technically known as *bialternality*: see §12 towards the end.

Notion of autark function.

Many entire functions naturally occurring in Analysis, notably at the interface of equational and coequational resurgence,[19] display remarkable 'finiteness' and 'closure' properties: their behaviour at ∞ depends on the sector, but in each sector it can be modelled by divergent-resurgent asymptotic expansions, with a closed system of alien derivatives. Let us call them *autark* functions, a name suitably evocative of self-closure and self-sufficiency.

Autark functions have a quality of finiteness about them, and a predictability of behaviour, that sets them apart from the wilder transcendental functions. In fact, the dichotomy autark/non-autark is arguably no less basic than the dichotomy algebraic/transcendent. The paradigmatic example of an autark function is $\frac{1}{\Gamma(z)}$. The paradigmatic example of a non-autark function is $\Xi(s)$, the entire fonction classically attached to Riemann's zeta function. This of course is due to the erratic behaviour of $\Xi(s)$ in the vertical strip $|\Re(s)| < \frac{1}{2}$, which completely defies formalisation.

8 Object synthesis.

In the Bridge equation (35) of §6, we start from a singular equation $\mathcal{R}(Y) = 0$ and derive its complete system $\{\mathbf{A}_\omega\}$ of holomorphic invariants. But what about

[18] As with the earlier Bridge equation, we drop the tildas indicative of formalness.
[19] e.g. the Stokes constants attached to singular ODE's, when viewed as functions of one of the ODE's coefficients.

the reverse problem: *starting from an admissible collection*[20] *of differential operators* $\{\mathbf{A}_\omega\}$, *search for an equation* $\mathcal{R}(Y) = 0$ *admitting that collection as its system of invariants?* This is a problem of 'object synthesis', and solving it is clearly a matter for 'alien integration'. On the formal side, it reduces to a rather mechanical exercise, over which there is no need to detain ourselves. It is simply a question of iterating the Bridge equation by taking advantage of the commutation $[\mathbf{\Delta}_{\omega_1}, \mathbf{A}_{\omega_2}] = 0$ and of expanding $Y(z, \boldsymbol{u})$ into highly multiple series whose general term is of the form $\boldsymbol{\mathcal{U}}^{\omega_1,\dots,\omega_r}(z)\, \mathbf{A}_{\omega_1}...\mathbf{A}_{\omega_r}$ for some system $\{\boldsymbol{\mathcal{U}}^\bullet(z)\}$ of resurgence monomials. The crux, of course, is to get these expansions to *converge* in the space of resurgent functions.

Here, run-of-the-mill monomials won't do. What is called for is a very special type, the so-called *paralogarithmic* or *spherical* monomials. They are best defined in the geometric model by means of the integrals:

$$\boldsymbol{\mathcal{U}}_c^{\omega_1,..,\omega_r}(z) := SPA\int_0^\infty \frac{e^{\sum_j \omega_j (z-y_j) + c^2 \sum_j \bar{\omega}_j (z^{-1}-y_j^{-1})} dy_1...dy_r}{(y_r - y_{r-1})\ \dots\ (y_2 - y_1)\,(y_1 - z)} \tag{42}$$

The acronym SPA stands for 'standard path averaging' (a way of specifying how the integration variables y_i bypass each other). The crucial ingredient, however, is the real parameter c which, if taken large enough[21], enforces convergence in the 'synthesis expansions'.

Another peculiarity of these monomials, which is almost obvious on their definition and justifies the label *spherical*, is their broadly similar behaviour at the antipodes $z = \infty$ and $z = 0$ of the Riemann sphere. It is indeed a strange feature of 'spherical object synthesis' that, while the avowed aim is to produce a local object at $z = \infty$, it automatically creates an 'antipodal shadow' at $z = 0$.

9 Causes of divergence and sources of resurgence.

Let us briefly discuss two important causes of divergence, which commonly go by the name of *small denominators*. The divergence they call into being is non-resurgent in nature, but may interfere with resurgence proper; and depending on the situation, it may yield to resummation, or resist it absolutely.

The Liouvillian small denominators and the compensation technique.

An irrational number is *Liouvillian* if it is approximated by rational numbers abnormally fast. Liouvillian small denominators typically occur with local vector fields or mappings whose *multipliers* (the eigenvalues of their linear part) combine to produce Liouvillian numbers. As a rule of thumb, 'Liouvillian divergence' is resummable whenever there stands a definite geometric object behind the divergent series. In the opposite case, it is intrinsically unsurmountable.

[20] *admissible* here means that the operators $\mathbf{A}_\omega$ should have the proper shape, and that their norms $\|\mathbf{A}_\omega\|$ should be properly bounded when ω increases.

[21] Quite often, it is enough to take $c > 0$.

Consider for example a local analytic vector field on $\mathbb{R}^2$, of the form:

$$X = \sum_{1\leqslant i\leqslant 2} (\lambda_i\, x_i + \dots)\, \partial_{x_i} \quad \textit{with}\ \frac{\lambda_1}{\lambda_2}\ \textit{negative Liouvillian} \tag{43}$$

X is formally conjugate to its linear part $X_{\text{lin}} = \sum_{1\leqslant i\leqslant 2} \lambda_i\, x_i\, \partial_{x_i}$, but the corresponding change of coordinates, having no geometric reality at its back, is fated to remain formal. By contrast, fix two small numbers $a_1, a_2 > 0$ and consider the well-defined geometric correspondence $x_1 \leftrightarrow x_2$ such that (a_1, x_2) and (x_1, a_2) lie on the same 'hyperbolic' branch of X. The formal power series that express x_1 in terms or x_2 (or x_2 in terms of x_1) are again clear cases of Liouvillian divergence, but underwritten this time by geometric warranty, and therefore resummable. How so?

The answer is that all abnormally large terms in these expansions can be aggregated into clusters that guarantee mutual sign compensation. These clusters, the so-called *compensators*, are of the form:[22]

$$z^{-\{\sigma_0,\sigma_1,\dots,\sigma_r\}} := \sum_{0\leqslant i\leqslant r} z^{-\sigma_i} \prod_{j\neq i} (\sigma_i - \sigma_j)^{-1} \quad (z \in \mathbb{C}_\bullet, \sigma_i \in \mathbb{R}^+) \tag{44}$$

No matter how close the σ_i get to each other, the bound holds:

$$\left| z^{-\{\sigma_0,\sigma_1,\dots,\sigma_r\}} \right| < \frac{1}{r!} \left| \log \frac{1}{z} \right|^r \left| \frac{1}{z} \right|^{\sigma_*} \quad (\sigma_* := \inf \sigma_i) \tag{45}$$

Now, the beauty is that this second type of Liouvillian divergence can amicably coexist with resurgent divergence, and naturally fits into the general scheme of accelero-summation, but with three essential nuances:

- Instead of the narrow critical time classes $[z_i]$ associated with resurgence and corresponding to the equivalence relation $\mathit{lim}\, z_i'/z_i = c > 0$, Liouvillian divergence gives rise to much wider critical classes $[\![z_i]\!]$, corresponding to the looser equivalence $\mathit{lim}\, \mathit{log} z_i'/\mathit{log} z_i = c > 0$.
- On its own, Liouvillian divergence produces no singularities in the Borel plane, and begets only analytic (as opposed to holomorphic) invariants.[23]
- In the geometric model, i.e. after resummation, Liouvillian divergence leads to spiral-like neighbourhoods of $\infty_\bullet$ instead of sectorial ones.

The small denominators in Celestial Mechanics.

The small denominators associated with Celestial Mechanics and, more generally, with Hamiltonian systems, are of a quite distinct nature, and a cause for much confusion.

The so-called Lindstedt-Poincare series – formally quasi-periodic – that describe the motion in the Many Body problem are, depending on the initial

[22]For consistency, we revert to the large variable $z \sim \infty$.
[23]Regarding the difference between *holomorphic* and *analytic* invariants, see §6.

conditions, sometimes convergent, sometimes divergent. In either case, there is nothing to 'compensate'. The confusion arises from the fact that even in the convergent case, the coefficients c_ω of the L-P series that emerge from the calculations involve finite sums of the form:

$$c_{\omega,r} = \sum_{1\leqslant k\leqslant K} \frac{a_{\omega,k}}{\omega_{k,1}\,\omega_{k,2}\ldots\omega_{k,r}} \quad \textit{with} \quad \begin{cases} r=r(\omega)\,,\ K=K(\omega) \\ \lambda_1,\ldots,\lambda_\nu \ \ \textit{fixed} \\ \omega_{k,i}=\langle \boldsymbol{n^i},\boldsymbol{\lambda}\rangle \quad \textit{with} \quad \boldsymbol{n^i}\in\mathbb{Z}^\nu \end{cases} \tag{46}$$

which may carry prohibitively small denominators, and therefore call for some 'compensation' mechanism. However, the finite expansion (46) for $c_{\omega,r}$ is by no means unique, and if we conduct the calculations (inductively on r) with deftness, by observing a neat set of rules, we can keep these abnormally small denominators at bay.

More interesting is the case of Hamiltonian vector fields that, on top of the intrinsic resonance $\lambda_i+\lambda_{i+\nu}=0$, exhibit some extrinsic resonance, say $\lambda_1=0$ or $\sum n_i\lambda_i=0$. Attached to this extrinsic resonance, we have *resurgence*, carried by a critical variable z and described as usual by the Bridge equation (35), but with two eye-catching peculiarities:

- Each differential operator $\mathbf{A}_\omega$ now derives from a potential $\mathcal{A}_\omega$ – an 'alien potential', so to speak.
- The shortest cut for calculating the potentials $\mathcal{A}_\omega$ involves writing the original Hamiltonian H as $H_2+\mathcal{H}$, with a quadradic part H_2 and a 'perturbation' $\mathcal{H}$, and subjecting $\mathcal{H}$ to a remarkable involution $\mathcal{H}\mapsto\mathcal{K}$:

$$\mathcal{K} = -\mathcal{H} - \frac{1}{2!}\{z,\mathcal{H}\}_{P} - \frac{1}{3!}\{z,\{z,\mathcal{H}\}_{P}\}_{P} - \frac{1}{4!}\{z,\{z,\{z,\mathcal{H}\}_{P}\}_{P}\}_{P}\ldots \tag{47}$$

  where $\{.,.\}_{P}$ denotes the Poisson bracket, expressed in any map that isolates the critical variable z.

Sources of resurgence.

By no means do *equational* and *coequational* exhaust the types of resurgence that nature — physical nature or mathematical nature — may force on us. To keep with mathematics for the moment,[24] there is the wealth of power series whose Taylor coefficients a_n have a pre-assigned form. Given sufficient regularity in the make up of these a_n, resurgence is as good as guaranteed. To name but one example, here is the case of coefficients with a so-called sum-product syntax:

$$\widehat{\varphi}(\zeta) := \sum a_n\,\zeta^n \quad \textit{with} \quad a_n := \sum_{0<m<n}\ \prod_{0<k<m} F\big(\frac{k}{n}\big) \quad (F\ \textit{meromorphic}) \tag{48}$$

The corresponding series $\widehat{\varphi}(\zeta)$ arise naturally in Knot theory, and lead to resurgence equations entirely *sui generis*.

[24]About physics, see §13 below.

10 Transseries and analyzable germs.

Transseries.

Very roughly, the algebra $\widetilde{\mathbb{T}}$ of transseries can be thought of as the natural closure of $\mathbb{R}[x]$ under $\{+, \times, \partial, \circ\}$ and the inverse operations, with x living in the real neighbourhood of $+\infty$. $\widetilde{\mathbb{T}}$ obviously contains $E := exp$, $L := log$ and the finite iterates E_n, L_n. After expulsion of all formal infinitesimals from the exponentials, and of all terms other than x from L_n, each transseries $\widetilde{T}(x)$ decomposes into a sum of irreducible *transmonomials*, ordered from larger to smaller. We crucially impose well-orderedness (each subseries of each $T(x)$ should have a first element), plus bounded logarithmic depth, plus, depending on the context, various simplifying assumptions.

Analyzable germs.

$\widetilde{\mathbb{T}}$ possesses a subsalgebra $\widetilde{\mathbb{T}}^{\mathrm{cv}}$ of *directly* convergent transseries (for x large enough) but $\widetilde{\mathbb{T}}^{\mathrm{cv}}$ is radically unstable under integration, since even the simplest transmonomials (think of $L^\alpha(x)$ or $x^\alpha/E(x)$ for $\alpha \notin \mathbb{Z}$) have primitives that are divergent (and resurgent). If we want both summability and stability, the proper framework is $\widetilde{\mathbb{T}}^{\mathrm{as}}$, defined as the algebra of all transseries $\widetilde{T}(x)$ that may be resummed by *accelero-synthesis*. The corresponding sums $T(x)$, analytic or cohesive on a real neighbourhood of $+\infty$, are dubbed *analyzable germs*.

Accelero-synthesis.

Accelero-synthesis is closely patterned on *accelero-summation* (see Fig 2 in §4), but with four significant differences:

- In the early stages of the process, it is only the subexponential parts of the transseries $\widetilde{T}_i(x_i) \equiv \widetilde{T}(x)$ and of their individual transmonomials that incarnate as analytic or cohesive germs in the Borel planes ξ_i. The other parts provisionally retain their status as formal transseries.

- Since realness has to be preserved, integration must always take place on the real axis $\arg \xi_i = 0$ of each critical Borel plane. When that axis contains singular points, one must therefore resort to a well-behaved convolution average μ_i.

- Whenever two consecutive critical times $x_{i-1} \ll x_i$ are 'close'[25], the germ $\widehat{T}_i(\xi_i)$ is liable to be non-analytic and merely cohesive. In the presence of singularities on the real axis $\arg \xi_i = 0$, we therefore face the challenge, in order to calculate the average $\mu_i \widehat{T}_i(\xi_i)$, of having to bypass these singularities, to the right and to the left, without leaving the real axis! Fortunately, that seeming impossibility can be overcome through some delicate 'cohesive acrobatics'.

[25] e.g. when $\lim \log x_i / \log x_{i-1} = 1$.

- The final sum $T(x)$ is always cohesive[26] though not necessarily analytic.[27]

Decelero-analysis.

Being of the nature of construction, accelero-synthesis is a slow, arduous, stepwise process, with mandatory passage through each critical Borel plane – or Borel line, as the case may be. The reverse process, decelero-analysis, being of the nature of destruction, is faster and easier, though not instantaneous: it too may necessitate the passage through *some* critical Borel planes, and bypass others. Nor need we always resort to the deceleration integrals: there exist faster alternatives.

The Dulac problem and the return map.

The impetus behind the introduction of transseries and analyzable germs came from the so-called Dulac problem. Given a polynomial vector field on $\mathbb{R}^2$, proving the finiteness of isolated cycles reduces to proving the existence of only finitely many isolated fixed points for the return map $T(x)$ attached to any given polycycle $\mathcal{C}$. Once $T(x)$ has been formalized to a transseries $\widetilde{T}(x)$, the property becomes self-evident.

11 Pseudo-variables and display.

The notion of pseudovariable is dual to that of alien derivation $\mathbf{\Delta}_\omega$ of the boldface, invariant sort: see (36). Pseudovarialbles carry as upper indices sequences $\boldsymbol{\omega} := (\omega_1, ..., \omega_r)$ of arbitrary length r. Multiplication for them reduces to sequence shuffling, while differentiation (ordinary or alien) and post-composition obey the predictable rules:

$$
\begin{aligned}
\mathbf{Z}^{\boldsymbol{\omega}'} . \mathbf{Z}^{\boldsymbol{\omega}''} &= \sum \mathbf{Z}^{\boldsymbol{\omega}} && \big(\boldsymbol{\omega} \in \text{shuffle}(\boldsymbol{\omega}', \boldsymbol{\omega}'')\big) && (49)\\
\mathbf{\Delta}_{\omega_0} \mathbf{Z}^{\omega_1,\dots,\omega_r} &= \delta^{\omega_1}_{\omega_0} \mathbf{Z}^{\omega_2,\dots,\omega_r} && (\delta \ = \ \textit{Kronecker symbol}) && (50)\\
\partial_z \mathbf{Z}^{\omega_1,\dots,\omega_r} &= 0 && && (51)\\
\mathbf{Z}^{\boldsymbol{\omega}} \circ g &= \mathbf{Z}^{\boldsymbol{\omega}} \qquad \textit{if} && g(z) = z + o(z) && (52)
\end{aligned}
$$

The display in the monocritical case.

The display *Dpl* is best thought of as some sort of ‘alien Taylor expansion’, though of a radically different sort than the one mentioned in (13) of §2.

$$
\mathrm{Dpl}\,\widetilde{\varphi} := \widetilde{\varphi} + \sum_r \sum_{\omega_j} \mathbf{Z}^{\omega_1,\dots,\omega_r} \mathbf{\Delta}_{\omega_r} \dots \mathbf{\Delta}_{\omega_1} \widetilde{\varphi} \tag{53}
$$

[26] Think of sums like $\sum_n 1/E_s(x+n)$ $(s \geqslant 2)$ that always converge to cohesive sums, on the strength of what we might call a ‘transserial Abel’s Lemma’.

[27] Whereas with accelero-summation, the sum $\varphi_\theta(z)$ is always analytic in some sectorial neighbourhood of ∞.

The expansion (53) being formal in nature, the question of convergence does not arise. The display has a double character − both local, via its z-dependence, and global, via its $\mathbf{Z}$-dependence. It encodes (*displays*, as it were) in ultra-compact and user-friendly form, a huge amount of information about the function $\widehat{\varphi}(\zeta)$, describing as it does the behaviour of $\widehat{\varphi}(\zeta)$ at each ω and on each of its various Riemann sheets.

What is more, thanks to the rules (50)-(52), any relation $\mathcal{R}$ between resurgent functions automatically extends to their displays:

$$\{\ \mathcal{R}(\widetilde{\varphi}_1,\ldots,\widetilde{\varphi}_s)\equiv 0\ \}\quad\Longrightarrow\quad\{\ \mathcal{R}(\mathrm{Dpl}\,\widetilde{\varphi}_1,\ldots,\mathrm{Dpl}\,\widetilde{\varphi}_s)\equiv 0\ \}\tag{54}$$

which can be extremely helpful for establishing transcendence or independence results.

The display in the multicritical case.

The display is still defined in the multicritical case, but with two-tiered indices $\varpi=(\begin{smallmatrix}\varpi\\ z_i\end{smallmatrix})$ to specify the critical time:[28]

$$\mathrm{Dpl}.\widetilde{\varphi}\quad=\quad\widetilde{\varphi}+\sum_{1\leqslant r}\ \sum_{\varpi_1,\ldots,\varpi_r}\mathbf{Z}^{\varpi_1,\ldots,\varpi_r}\,\boldsymbol{\Delta}_{\varpi_r}\ldots\boldsymbol{\Delta}_{\varpi_1}\,\widetilde{\varphi}\tag{55}$$

Any operation on the displays, like applying some alien derivation Δ_{ϖ_0} or going from one multipolarised sum[29]

$$(\mathrm{Dpl}.\varphi)_{\boldsymbol{\tau}}\quad=\quad S_{\boldsymbol{\tau}}+\sum_{1\leqslant r}\ \sum_{\varpi_1,\ldots,\varpi_r}\mathbf{Z}^{\varpi_1,\ldots,\varpi_r}\,(\boldsymbol{\Delta}_{\varpi_r}\ldots\boldsymbol{\Delta}_{\varpi_1}\,\varphi)_{\boldsymbol{\tau}}\tag{56}$$

to another multipolarised sum $(\mathrm{Dpl}.\varphi)_{\boldsymbol{\tau}'}$, reduces to *purely formal manipulations on the pseudovariables – munipulations that involve universal constants depending on the pair* $(\boldsymbol{\tau},\boldsymbol{\tau}')$, *but strictly independent of* φ. In that sense, the display can be said to contain "everything there is to know" about the object φ. In fact, it is only at the level of displays that the correspondence *formal* $\leftrightarrow$ *geometric* attains perfection.

12 Resurgence as the impetus behind mould calculus.

Mould symmetries and mould operations.

Alien calculus presents us at every step with resurgence *monomials* $\mathcal{U}^{\omega_1,\ldots,\omega_r}(z)$ or scalar *monics* $U^{\omega_1,\ldots,\omega_r}$ that have their own symmetries (four main types, scores of secondary ones) and undergo numerous operations that either preserve or exchange these symmetries. The systematization of these operations naturally led to *mould calculus* (*moulds* are simply objects $M^{\omega_1,\ldots,\omega_r}$ indexed

[28] In the general transserial context, it is convenient to represent ϖ by the leading transmonomial of $\omega\,x_i$, with x_i expressed as a transseries in x.

[29] Any multipolarised sum depends on the choice, in each critical Borel plane ζ_i, of an integration axis $\arg\zeta_i=\theta_i$ plus, if need be, a well-behaved average μ_i (see at the end of §3).

by scalar sequences of any length), which later found many applications outside its native context, in such fields as differential geometry, Lie or pre-Lie algebra, etc. While at one level moulds with all their wherewithal may be dismissed as just a glorified system of notations, the fact is that they often allow us to make fully explicit what would otherwise remain implicit, and to go beyond mere 'existence theorems' (that all too often are sterile dead-ends) by illuminating the innards of the object whose bare 'existence' has been proved.

Bimoulds, double symmetries, and the flexion structure.

The bimoulds $M^{\binom{u_1\ ,\dots,\ u_r}{v_1\ ,\dots,\ v_r}}$ are a rather different proposition. They crystallized, together with their two-tier indexation, out of the intricate combinatorics that underpins coequational resurgence, and inherited therefrom a plethora of structure. The fact is that they can undergo an incredibly rich array of operations, unary and binary, resulting in what is known as *flexion polyalgebra.* But the most salient feature is perhaps the existence:

- of a central involution *swap*:

$$(\mathrm{swap.M})^{\binom{u_1\ ,\dots,\ u_r}{v_1\ ,\dots,\ v_r}} := \mathrm{M}^{\binom{v_r\ ,\dots,\ v_2-v_3\ ,\ v_1-v_2}{u_1+\dots+u_r\ ,\dots,\ u_1+u_2\ ,\ u_1}} \tag{57}$$

- of bimoulds $M^\bullet$ possessed of a double symmetry, e.g. *bialternality*, meaning that both $M^\bullet$ and its 'swappee' are simultaneously alternal:

$$\sum_{\boldsymbol{w}\in\mathrm{sha}(\boldsymbol{w^1},\boldsymbol{w^2})} \mathrm{M}^{\boldsymbol{w}} = \sum_{\boldsymbol{w}\in\mathrm{sha}(\boldsymbol{w^1},\boldsymbol{w^2})} \mathrm{swap.M}^{\boldsymbol{w}} = 0 \quad \begin{cases} \forall\ \boldsymbol{w^1},\boldsymbol{w^2} \\ \text{sha } \textit{for} \text{ shuffle} \end{cases} \tag{58}$$

- of binary operations that preserve these double symmetries.

This makes flexion algebra an ideal framework for unpicking *arithmetical dimorphy*, a phenomenon in no way confined to the ring of *multizetas*, but preeminently manifest there, due the two canonical encodings of multizetas and the two multiplication tables that go with them.

13 Resurgence in mathematical physics.

The 'Michael Berry principle'.

The well-known quantum physicist Michael Berry[30] is fond of saying that when a new physical theory departs from its classical model by the introduction of a small constant (like h for quantum mechanics), we should expect expansions in power series of that small constant to diverge: their divergence merely reflects the *non-trivial nature* of the transition from classical to non-classical. Withal,

[30] *Sir Michael*, to give him his full due. His PhD advisor, incidentally, was Robert B.Dingle, one of the pioneers of asymptotics. Other emblematic figures of that select club were Leonhard Euler, James Stirling, and (somewhat controversially) Henri Poincare.

these expansions ought somehow to be resummable, this time to reflect the *physical relevance* of the transition.

The Planck constant is an obvious case in point. We saw in §7 examples of this with the so-called WKB method, A.Voros' semi-classical treatment of the harmonic oscillator, and the whole sprawling field of *co-equational resurgence*.

There exist other candidates than h for serving as 'variable' in formal expansions, such as coupling constants. The trouble here is that only the very first terms are accessible to experimental verification, and that anything beyond the *dominant effects* (read: the *closest singularities* in the Borel plane) seems to lie hopelessly beyond its reach.

Front-line physics.

In somewhat different directions, I hear of resummation methods being brought to bear on such questions as tunneling effects in quantum mechanics, renormalons in quantum field theory, even some toy models in string theory, by authors such as Mithat Unsal, Ricardo Schiappa, Ines Aniceto, and others. The line of research I am most familiar with is Ovidiu Costin's and Gerald Dunne's. I know of their attempts to squeeze the maximum of information out of scant data, and I am greatly impressed by the odd-defying numerical accuracy their methods often achieve. In the nature of things, however, the overall resurgence picture seems to elude the current approaches. Missing, too, is the algebraic side (crucial, from my viewpoint) of resurgence theory. I am told, however, that efforts are afoot to fill this lacuna, and one certainly cannot rule out a conceptual breakthrough that would turn things around for good, and allow theory to outpace experiment.

Asymptotics beyond all orders.

Going by this name is an asymptotic doctrine, pioneered by Michael Berry, that has attracted wide attention over the last three decades and spawned a flood of publications. The idea, roughly, is to indefinitely iterate the classical – and numerically highly effective – trick known as *least term truncation* to achieve any order of accuracy. The idea is seductive at first sight, but only at first sight. In fact, a recent paper, coauthored by Ovidiu Costin and sir Michael himself, has taken a closer look at the cost-effectiveness of the method, and laid bare the somewhat pyrrhic nature of the gains it claims to achieve. Another drawback, no less serious in my view, is that in the successive 'models' on which the method relies, the resurgence picture (beginning with the resurgence equations) gets completely distorted.

14 Some loose thoughts by way of conclusion.

Mathematicians, while in hot pursuit of their prize, understandably won't let others see into their cards. But even after the event, with all work done and safely under their belt, many remain unwilling to open up about their initial motivations, or to explain candidly what exactly it is that they cherish in their brainchildren. Such reticence seems misplaced, for what merit can there be in hiding the vital part of the creative process under the bushel, and what profit in keeping the allure of a mathematical structure under burqa? Anyway, the reticent are welcome to their reticence, but here I feel free to deliver myself of a few thoughts about *resurgence*.

First to come to mind is the theory's breadth of application along with its unifying power – two traits best in evidence in the *Bridge equation*, which covers huge ground and corrals seemingly disparate phenomena into a single framework of compelling simplicity.

Next comes what we might call the analytical bent of the theory, which culminates in the twin notions of *transseries* and *analyzable germ*. It is here or nowhere that the dream of full formalization comes true – the dream of reducing opaque, seemingly untractable geometric entities (and the operations on them) to transparent formal objects (themselves subject to transparent operations).

Third (or should I say *first*?) comes the wonder of a precise, sharp-contoured structure – *alien calculus* – spontaneously arising from what would seem to be the most unpromising and amorphous of contexts: *divergence*. Here lies, at least to my subjective feeling, the core attraction of resurgence. It is indeed often the case that mathematicians harbour secret preferences, centered on quite concrete notions. Some, for instance, are in thrall to the identity $d^2 = 0$ and the cohomological marvels that flow from it. I for one confess to an innate liking for *derivations*, i.e. for operators that obey the Leibniz rule. I distinctly recall how, aged 17, one fine day during the summer recess I found myself sitting on the bank of a mountain stream, wondering: *"Might there not exist a class of smooth functions on which there operate* deep *derivations?"* By which I meant derivations that would somehow involve, all at once, the infinite string of ordinary derivatives at a given point. When a few years later, as a result of pursuing a seemingly unconnected line of investigation, it dawned on me that such critters actually existed, *in superabundant quantity and with no taint of pathology about them*, there was an immediate click of recognition, which comforted me in my sense of being on the right track, and kept me hooked to the subject for the next fifteen years.

Then there is – rarely acknowledged, but nonetheless essential – the symbolic charge, or if you prefer the aura of associations, which nearly all key mathematical notions carry with them, not just in the eye of the beholder, but also, I venture, in an almost objective sense. This is, however, a point that people are apt to misconstrue, so let us clarify it on a striking example: *analytic functions*. These functions[31] have the fantastic property of being 'of one piece' – if you know a little chunk, you know the whole thing. A truly magic quality, that infuses them with life, and turns them into natural similes for this eternal theme: the Whole in each of its Parts; the Macrocosm in the Microcosm; etc. We can find echoes of this everywhere – in Oriental or Hermetic Philosophy; in Biology (the full genome is encoded in each cell of a living organism); and, at an almost

[31] Together with the *cohesive functions* which we encountered in §5.

literal level, in Newtonian physics: if space were truly analytic, then by knowing the gravitational potential in a cubic inch of space to infinite accuracy, we could in theory infer the position of all massive particles in the world, and to that extent "know everything". Taken literally, this is nonsense, of course, and we should carefully avoid mistaking symbols for explanatory mechanisms. But this in no way detracts from their evocative power or their vivifying potency for the soul, not least the creative scientific soul.

So let me conclude by pointing to two such aspects in resurgence theory, both highly loaded, and both arresting.

There is this hierarchy of 'emergent' levels of organization in the universe – microphysics, chemistry, biology, history – each with its own laws and its own patterns of 'causality', the higher the looser. Resurgence to me is strongly evocative of these multi-track causalities (and others we are free to imagine). Indeed, here we have in the Borel plane, riding piggyback on a local causality (the step-by-step analytical continuation from one Weierstrass element to the next), a long-range causality that 'miraculously' transports the situation at the origin to distant singular points.[32] I am not suggesting, heaven forbid, that physical space-time might serve as a medium for such non-local transportation,[33] or even that resurgence might model some real-world mechanism. I am just saying that it provides a *simile* – nothing more, nothing less – for the peaceable coexistence of quite distinct levels of causation.

Then we have the grand scheme of accelero-synthesis and decelero-analysis, as sketched in §10, with its majestic double movement between a state of maximal dispersal (– the *formal transseries* $\widetilde{T}(x)$, with its disparate collection of isolated coefficients hanging forlornly on a sprawling tree-like structure –) and a state of maximal compactness-cum-cohesion (–the infrangible *geometric germ* $T(x)$, of *one piece* on account of its analyticity or quasi-analyticity–). That double movement – one of slow, arduous ascent; the other of sudden, precipitous fall – to me carries a compelling, almost self-evident symbolic charge.

[32]Meromorphic periodic functions also have that quality, but to a lesser degree, because here the singularities merely *repeat*. And if we quotient the underlying space by the period, the repetition disappears altogether. Not so with genuine resurgence: here we have creative innovation on top of repetition, and no artifice of definition or change of framework can whisk that aspect away.

[33]For one thing, the consensus at the moment, or should we say the general suspicion, seems to be that physical space-time is an emergent rather than a primary reality. In any case it is certainly not, at the finest resolution, a 'real analytic manifold' in the mathematical sense.