Flexion algebra meets tree algebra:
a tale of asymmetric cross-fertilisation.

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Abstract *As mathematical objects, finite trees would seem to be nearly as basic and ubiquitous as the natural integers, were it not for their apparent 'chemical inertness', by which we mean the paucity of natural operations (of any given arity) defined on them. The present paper tries to redress this state of affairs by bringing trees into close relation with* $\mathrm{Flex}(\mathfrak{E})$ *— the flexion polyalgebra generated by a so-called flexion unit* $\mathfrak{E}$*, and by uploading the rich structure of that polyalgebra onto trees. The rapprochement also benefits* $\mathrm{Flex}(\mathfrak{E})$*, leading in particular*
(i) to a neat filtration by depth and alternality codegree,
(ii) to exact formulae for the dimensions that go with that filtration,
(iii) to remarkable expansions for all the main elements of $\mathrm{Flex}(\mathfrak{E})$*.*
We conclude by introducing the notion of pre-associative algebra, parallel to that of pre-Lie algebra and potentially capable of rendering roughly the same services.

Contents

1 Introduction.

1.1 Flexion algebra.

Flexion (poly)-algebra has been around for some 22 years. It deals with *bimoulds*, i.e. functions of double strings of variables (u_i's and v_i's) of any length, and with a host of bimould operations that typically add the u_i's (clusterwise) and subtract the v_i's (pairwise). It possesses a central involtion, *swap*, and excels at handling double symmetries, i.e. symmetries that simultaneously affect a bimould and its *swappee*. Flexion algebra originally arose in Analysis, to describe some intricate resurgence patterns. It was later applied to multizeta algebra to elucidate the fascinating subject of *arithmetical dimorphy*. But it is also a subject with definite contours and great inner unity, well deserving of being studied for its own sake.

1.2 Tree algebra.

Trees, of course, make out an important and sprawling chapter of graph theory, but it is only recently that they entered algebra proper, thanks in particular to seminal papers by F. Chapoton, M. Livernet, D. Manchon (see §11.6). These authors equipped trees with a natural pre-Lie product, and derived therefrom, among other results, a precise description of free pre-Lie algebras.

1.3 The encounter.

Even prior to this paper, trees and moulds were no strangers to one another. Indeed, in [E-V], we had developed the so-called arborification-coarborification technique, which in many situations of Analysis restores convergence in expansions of mould-comould[1] type $\sum_\bullet M^\bullet B_\bullet$ by changing them[2] to expansions $\sum_{\bullet^<} M^{\bullet^<} B_{\bullet^<}$, formally equivalent but no longer divergent.[3]

The present paper, however, is about a quite distinct mathematical encounter, namely the match between
(i) the polyalgebra $\mathit{Flex}(\mathfrak{E})$ generated by a flexion unit $\mathfrak{E}$
(ii) the free pre-Lie algebra $\mathbb{UT}$ on unordered trees, equipped with its standard pre-Lie product, and its natural extension $\mathbb{OT}$ to ordered trees.

As it happens, when equipped with a suitable pre-Lie product and a suitable basis, $\mathit{Flex}(\mathfrak{E})$ maps naturally onto $\mathbb{OT}$ and the alternal part $\mathit{Flex}^{al}(\mathfrak{E})$ of $\mathit{Flex}(\mathfrak{E})$ maps naturally onto $\mathbb{UT}$.

Like with most such mathematical rapprochements, both sides stand to benefit, but here it is tree algebra that gains most. In very rough terms:
(i) Flexion algebra gains very convenient, tree indexed bases, which in turn lead

[1] Here, $M^\bullet$ denotes a scalar *mould*, and $B_\bullet$ a operatorial *bimould* – typically, a string of differential operators.

[2] by means of the dual tranforms $M^{\boldsymbol{\omega}^<} = \sum_{\boldsymbol{\omega}^< < \boldsymbol{\omega}} M^{\boldsymbol{\omega}}$, $B_{\boldsymbol{\omega}} = \sum_{\boldsymbol{\omega}^< < \boldsymbol{\omega}} B^{\boldsymbol{\omega}^<}$ that turns ordered sequences $\boldsymbol{\omega}$ into partially ordered ones $\boldsymbol{\omega}^<$.

[3] The reason the magic works is that, in most instances, the change hardly affects the size of the mould part but drastically shrinks the comould part: $|M^{\boldsymbol{\omega}^<}| \sim |M^{\boldsymbol{\omega}}|$, $|B_{\boldsymbol{\omega}^<}| \ll |B_{\boldsymbol{\omega}}|$.

to a resolution of the long-standing problem of 'alternality stratification': calculating the dimensions of the subspaces of $Flex(\mathfrak{E})$ of depth r and alternality co-degree d.
(ii) Tree algebra experiences a massive influx of structure, with the full array of operations (inflected or non-inflected) defined on $Flex(\mathfrak{E})$ and $Flex^{al}(\mathfrak{E})$ automatically carrying over to $\mathbb{OT}$ and $\mathbb{UT}$.

1.4 Articulation of the paper.

- **§2:** We begin with some reminders on pre-Lie calculus. We then show how to improve the usual pre-Lie formulae (for group composition, group inversion, group iteration, group-to-algebra logarithm) by relying on just two types of bracketings – *backward* inside *forward* – rather than on the general bracket combinations commonly used. Besides being more economical (they carry far fewer terms), the new formulae have the added merit of uniqueness and expliciteness.

- **§3:** We illustrate the technique on the group of identity-tangent diffeomorphisms, and investigate in passing two 'exotic' pre-Lie products alongside the 'exotic' group laws that go with them.

- **§4:** To set the stage for the encounter between flexion and tree algebra, we give a short introduction to the lesser known of the two: flexion algebra (a subcategory of mould and bimould algebra), with special emphasis on the prototypal case of the algebra $Flex(\mathfrak{E})$ generated by a 'flexion unit' $\mathfrak{E}$.

- **§5:** From the very start, flexion algebra has made liberal use of pre-Lie products, but here we introduce yet another type, the so-called semi-inflected pre-Lie product *dle*, which commends itself on at least three grounds:
  (i) it preserves the alternality of bimoulds, which the earlier pre-Lie products (whether uninflected or fully inflected) did not
  (ii) it generates the whole algebra $Flex(\mathfrak{E})$ from $\mathfrak{E}$ alone, which none of the ealier products did
  (iii) it shall prove admirably suited for the future link-up with tree algebra.

  We then use the pre-Lie product *dle* and a kindred, 'pre-associative' operation *dme* to construct multilinear operators:

  $$\text{câlt} : (A_0^\bullet, A_1^\bullet, \ldots, A_r^\bullet) \mapsto B^\bullet = \text{câlt}_{A_0^\bullet}(A_1^\bullet, \ldots, A_r^\bullet)$$

  with the unexpected property that the alternality of $B^\bullet$ as a bimould *decreases*, in an exactly quantifiable way, when the alternality of *câlt*, as a function of its arguments $A_i^\bullet$, *increases.* We actually construct three such operators, *câlt*, *călt*, *cālt*. Though all three share 'counter-alternality' and seem equally promising, on closer examination *câlt* proves the best choice by a long stretch.

- **§6:** There exists on $Flex(\mathfrak{E})$ an appealing basis, naturally indexed by *binary trees* and quite simple to construct, but with the downside that none of the flexion operations possesses a transparent expression in that basis. To remedy this, we harness the counter-alternators to construct two new bases. The simpler of the two, indexed by *ordered trees*, will facilitate the link-up with tree algebra (§8). The other one, indexed by *stacked trees* (which are carefully crafted linear combinations of ordered trees), will help clarify the structure of $Flex(\mathfrak{E})$ through a filtration according to alternality (§7). The upshot is that we must juggle three distinct bases, but this isn't nearly as bad as it sounds, since the matrices connecting these bases admit remarkable expressions, directly in terms of the underlying trees (§6.8).

- **§7:** This section, technically the most demanding, solves the delicate problem of splitting each space $Flex_r(\mathfrak{E})$ into subspaces $\oplus_d Flex_{r,d}(\mathfrak{E})$ of alternality co-degree d, i.e. consisting of bimoulds which, when contracted with differential operators, yield a result of differential degree d. Rather than directly calculating $dim(Flex_{r,d}(\mathfrak{E}))$, for which there exist no closed formulae, we form the generating series of these dimensions with the help of some remarkable special polynomials, the so-called 'pilot polynomials'.

- **§8:** This section returns to the basis indexed by ordered trees to show how all useful operations on the polygebra $Flex(\mathfrak{E})$ admit transparent interpretations in that basis. Thus, the *uninflected* Lie bracket *lu* reduces to *attaching* two trees to each other, while the *inflected* Lie bracket *ari* reduces to *grafting* two trees onto each other. Strictly speaking, this applies only to the basis that relies on the 'good' counter-alternator $c\hat{a}lt$. But the parallel constructions relying on $c\breve{a}lt$ or $c\bar{a}lt$ also have their uses, especially for expanding the *push*-invariant or bialternal elements of $Flex(\mathfrak{E})$.

- **§9:** As it happens, the monogenous polyalgebra $Flex(\mathfrak{E})$, which is in every way the core and marrow of the polyalgebra *BIMU*, possesses its own core and marrow, consisting of the bisymmetral bimoulds $pal^\bullet$ and $pil^\bullet$, and it is truly gratifying to observe that these two bimoulds, along with the numerous alternal or bialternal bimoulds naturally attached to them, tend to admit surprisingly explicit expansions in the new bases of $Flex(\mathfrak{E})$.

- **§10:** The interpretation of all flexion operations in terms of the ordered tree basis of $Flex(\mathfrak{E})$, apart from leading to a massive enrichment of 'tree algebra', also acts as an invitation to put the whole thing on a clean axiomatic basis to unleash its full potential. The result is the two notions of *pre-associative*[4] and *Janus* algebra.

  Pre-associative algebras relate to associative algebras as pre-Lie to Lie. But pre-associative algebras also 'enfold' pre-Lie algebras in much the same way as associative algebras 'envelop' Lie algebras. The upshot is an elegant four-fold 'unfolding' of Lie algebras, which helps clarify a host of

[4]Mark that it has nothing to with what sometimes goes by that name in the literature.

notions such as degree, co-degree, counter-alternator, μ-generator etc. We describe the structure of free pre-associative algebras, their various stratifications and dimensions, and provide some examples of pre-associative algebras, some free, some not.

Whereas pre-associatice algebra, strictly speaking, merely extends the uninflected or semi-inflected operations on $Flex(\mathfrak{E})$, the richer (but still inchoate) notion of *Janus* or *bifrons* algebra purports to take on board the fully inflected operations as well, beginning with the *ari* bracket. However, it deliberately leaves out the involution *swap* and the whole 'dimorphy' aspect, because adding these would impose far too many constraints and practically shrink the construction to something isomorphic to bimould algebra.

- **§11:** For illustration, and also to palliate what may be an excessive conciseness in parts of the exposition, we provide extensive tables, notably on the connection matrices; the pilot polynomials; the co-degee dimensions and their generating functions.

- **§12:** We wind up with a list of the most salient results, and hint at some open questions and possible developments.

2 Pre-Lie calculus: optimal formulae.

2.1 Some auxiliary moulds.

Let us settle some notations:
Let $\mathbb{G}$ be a Lie group with elements $A, B, ...$ and the group law: $A, B \mapsto A \circ B$.
Let $\mathbb{L}$ be its Lie algebra with elements $A_*, B_*, ...$ and the Lie bracket: $A_*, B_* \mapsto [A_*, B_*]$
Let there be a pre-Lie bracket[5]: $A_*, B_* \mapsto \langle A_*, B_*\rangle$ or[6] $A, B \mapsto \langle A, B\rangle$
$\langle ..\langle\langle A_1, A_2\rangle, A_3\rangle, ..., A_r\rangle$ will get abbreviated as $\langle A_1, \overrightarrow{...}, A_r\rangle$ *or* $\langle A_1, ..., A_r\rangle^{\rightarrow}$
$\langle A_1, .., \langle A_{r-2}, \langle A_{r-1}, A_r\rangle\rangle ..\rangle$ will get abbreviated as $\langle A_1, \overleftarrow{...}, A_r\rangle$ *or* $\langle A_1, ..., A_r\rangle^{\leftarrow}$
$\overbrace{\langle ..\langle\langle A, A\rangle, A\rangle, ..., A\rangle}^{n\ times}$ will get abbreviated as $A_{\overrightarrow{n}}$
$\overbrace{\langle A, ..., \langle A, \langle A, A\rangle\rangle ..\rangle}^{n\ times}$ will get abbreviated as $A_{\overleftarrow{n}}$

We take as our starting point the fact that each Lie group exactly determines its Lie algebra, but that a Lie algebra determines its Lie group only up to isomorphism. However, when there exists a pre-Lie operation behind the Lie

[5] Actually, a *right* pre-Lie bracket, i.e. one whose commutator $\langle A_*, B_*\rangle - \langle B_*, A_*\rangle$ coincides with the Lie bracket $[A_*, B_*]$ and whose associator $\langle A, \langle B, C\rangle\rangle - \langle\langle A, B\rangle, C\rangle$ is symmetrical in the last two arguments B, C.

[6] The pre-Lie bracket will act mainly on elements $A_*, B_*...$ of the Lie algebra, and occasionally on elements $A, B...$ of the group itself.

bracket, with it goes a privileged realisation of the Lie group, induced by the mapping:

$$algebra \to group : \qquad A_* \mapsto A = \sum_{1\leqslant r} \frac{1}{r!} \, \langle \overbrace{A_*, ..., A_*}^{r\ times} \rangle^{\to} \tag{1}$$

That mapping can then be reversed by brute force, that is to say, by treating the pre-Lie bracket as if it were an arbitrary binary operation subject to no other constraint than bilinearity. The result is what we may call a *raw* expansion. Via transfer of the Campbell-Hausdorff formula from algebra to group, it eases the way for two new raw expansions — one for group composition, another for continuous iteration. These expansions are *raw* not just on account of their rough mode of derivation, but also in the sense of involving close to the maximum number of summands, namely $\frac{(2\,r-2)!}{r!\,(r-1)!}$ at each order r — far more, as it happens, than strictly necessary. To improve on that, we must take into account the functional identities of the pre-Lie brackets. The result will be *optimal* expansions, that involve at most 2^{r-1} summands at order r, all of the form *leftward within rightward*,[7] with the added bonus of explicit coefficients (absent from the raw expansions).

Let us first introduce a few moulds essential for the sequel:

Lemma 2.1 (The auxiliary moulds $San^\bullet, Zan^\bullet, Lit^\bullet, It^\bullet_w$**)** .
The moulds $San^\bullet, Zan^\bullet$ *on* N *defined by* $San^{\varnothing} = Zan^{\varnothing} = 1$ *and*

$$\mathrm{San}^{n_1,\dots,n_r} \quad := \quad (-1)^{r+\sum n_j} \prod_{1\leqslant j\leqslant r} \frac{1}{n_1+\ldots+n_j} \tag{2}$$

$$\mathrm{Zan}^{n_1,\dots,n_r} \quad := \quad (-1)^{\sum n_j} \prod_{1\leqslant j\leqslant r} \frac{1}{n_j+\ldots+n_r} \tag{3}$$

are symmetral and mutually inverse (for mould multiplication).
The mould $It^\bullet_w$ *on* N *equivalently defined by (4) or (5)*

$$\mathrm{It}^\bullet_w \quad := \quad w.\mathbf{1}^\bullet + \sum_{1\leqslant k\leqslant r(\bullet)} \quad \frac{\prod_{j=0}^{k} (w-j)}{(k+1)!} \;\; \big(\mathrm{San}^\bullet - \mathbf{1}^\bullet\big)^{\times k} \tag{4}$$

$$= \quad w.\mathbf{1}^\bullet - \sum_{1\leqslant k\leqslant r(\bullet)} (-1)^k \frac{\prod_{j=-1}^{k-1} (w+j)}{(k+1)!} \big(\mathrm{Zan}^\bullet - \mathbf{1}^\bullet\big)^{\times k} \tag{5}$$

is not symmetral, but verifies the difference equations

$$\mathrm{It}^\bullet_{w+1} \quad := \quad \mathrm{It}^\bullet_w \times \mathrm{San}^\bullet + \mathbf{1}^\bullet \tag{6}$$

$$\mathrm{It}^\bullet_{w-1} \quad := \quad \mathrm{It}^\bullet_w \times \mathrm{Zan}^\bullet - \mathrm{Zan}^\bullet \tag{7}$$

and the reflection equation

$$\mathrm{It}^\bullet_w + anti.pari.\mathrm{It}^\bullet_{1-w} \equiv \mathbf{1}^\bullet \tag{8}$$

[7] i.e. of the form $\langle A_{\overleftarrow{1+n_1}}, A_{\overleftarrow{n_r}}, ..., A_{\overleftarrow{n_r}} \rangle^{\to}$.

Lastly, the mould $Lit^\bullet$ on N equivalently defined by (9) or (10) or (11)

$$\begin{aligned}
\mathrm{Lit}^\bullet \;&:=\; \mathbf{1}^\bullet + \sum_{1\leqslant k\leqslant r(\bullet)} \frac{(-1)^k}{(k+1)} \left(\mathrm{San}^\bullet - \mathbf{1}^\bullet\right)^{\times k} &(9)\\
&:=\; \mathbf{1}^\bullet - \sum_{1\leqslant k\leqslant r(\bullet)} \frac{(-1)^k}{k\,(k+1)} \left(\mathrm{Zan}^\bullet - \mathbf{1}^\bullet\right)^{\times k} &(10)\\
&:=\; \partial_w \mathrm{It}^\bullet_w \|_{w=0} &(11)
\end{aligned}$$

verifies only 'traces' of the reflection equation (see below).

Proof: The moulds $Hit^\bullet_w$ and $varHit^\bullet_w$ defined by (4) and (5) respectively are clearly the only solutions of (6) resp. (7) that vanish for $w = 0$. But since $Hit^\bullet_w$ and $varHit^\bullet_w$, by construction, commute with $San^\bullet$ and $Zan^\bullet$, it follows that the relations (6) and (7) are equivalent. Hence the identity $Hit^\bullet_w = varHit^\bullet_w$, and the equivalence of (9), (10), (11).

Note that $It^\bullet_0 = 0^\bullet, It^\bullet_1 = 1^\bullet, It^\bullet_2 = San^\bullet, It^\bullet_{-1} = -Zan^\bullet$. Note, too, that for w in $\mathbb{Z}$ (resp. not in it) the mould $It^\bullet_w$, along with its arborified and anti-arborified variants, grows exponentially (resp. super-exponentially) as the depth $r(\bullet)$ increases.

Remark: some properties of $Lit^\bullet$.

(i) *Zeros of $Lit^\bullet$.* Setting $x^{[n]} := (x, ..., x)$ (n times), we have:

$$\begin{aligned}
\mathrm{Lit}^{1^{[n]}} &\equiv 0 \quad \textit{if } n \textit{ odd} &(12)\\
\mathrm{Lit}^{1^{[n]},2} &\equiv 0 \quad \textit{if } n \textit{ odd} &(13)\\
\mathrm{Lit}^{1^{[n]},2,1^{[n]}} &\equiv 0 \quad \forall\, n \geqslant 1 &(14)\\
\mathrm{Lit}^{1^{[2^n-1]},2^n,1} &\equiv 0 \quad \forall\, n \geqslant 1 &(15)
\end{aligned}$$

(ii) *Link with the Bernoulli numbers:*

$$\begin{aligned}
\mathrm{Lit}^{1^{[2\,n]}} &= \frac{B_{2n}}{(2\,n)!} \qquad \forall n \geqslant 1 &(16)\\
\mathrm{Lit}^{1^{[p]},2,1^{[q]}} &= -\frac{1}{2}\frac{B_{p+q+1}}{(p+q+1)!}\left(1 + (-1)^p \frac{(p+q)!}{p!\;q!}\right) \;\; \textit{if } p+q \textit{ odd} &(17)\\
\mathrm{Lit}^{1^{[p]},3,1^{[q]}} + \mathrm{Lit}^{1^{[q]},3,1^{[p]}} &= 2\,(-1)^p \frac{B_{p+q+2}}{(p+q+2)!}\frac{(p+q)!}{p!\;q!} \qquad \textit{if } p+q \textit{ even} &(18)\\
\mathrm{Lit}^{1^{[p]},3,1^{[q]}} - \mathrm{Lit}^{1^{[q]},3,1^{[p]}} &= (-1)^p \frac{B_{p+q+1}}{(p+q+1)!}\frac{(p+q)!}{p!\;q!} \qquad \textit{if } p+q \textit{ even} &(19)
\end{aligned}$$

(iii) *Faint traces of the reflection equation (8) :*
On average, $Lit^\bullet + anti.pari.Lit^\bullet$ is much smaller than $Lit^\bullet - anti.pari.Lit^\bullet$. In particular $Lit^{1^{[p]},2,1^{[q]}} + Lit^{1^{[q]},2,1^{[p]}} \equiv 0$ for $p+q$ even. For $p+q$ odd, see (17).

(iv) *Stability under arborification.* As a special instance, we get the identities:

$$\sum_{\sigma\in\mathfrak{S}(r)} \mathrm{It}_w^{n_{\sigma(r)},\ldots,n_{\sigma(1)}} = \frac{1}{r!}\,\frac{(-1)^{r+\sum n_j}}{n_1\ldots n_r}\,\mathrm{It}_w^{\overbrace{1,\ldots,1}^{r\ times}} \tag{20}$$

$$\sum_{\sigma\in\mathfrak{S}(r)} \mathrm{Lit}^{n_{\sigma(r)},\ldots,n_{\sigma(1)}} = \frac{1}{r!}\,\frac{(-1)^{r+\sum n_j}}{n_1\ldots n_r}\,\mathrm{Lit}^{\overbrace{1,\ldots,1}^{r\ times}} \tag{21}$$

2.2 Optimal formulae.

Let us now enuntiate straightaway the *optimal* formulae, using the notations and special moulds of the previous section.

Proposition 2.1 .
Algebra-to-group exponential:

$$A = \sum_{1\leqslant r}\frac{1}{r!}A_{*\vec{r}} = \sum_{1\leqslant r}\frac{1}{r!}\,\big\langle\overbrace{A_*,\ldots,A_*}^{r\ times}\big\rangle^{\rightarrow} \tag{22}$$

Group-to-algebra logarithm:

$$\begin{aligned} A_* &= A + \sum_{1\leqslant r, 1\leqslant n_i}\mathrm{Lit}^{n_1,\ldots,n_r}\big\langle A, A_{\overleftarrow{n_1}},\ldots,A_{\overleftarrow{n_r}}\big\rangle^{\rightarrow} && (23)\\ &= A + \sum_{1\leqslant r, 1\leqslant n_i}\mathrm{Lit}^{n_1,\ldots,n_r}\big\langle A_{\overleftarrow{1+n_1}},\ldots,A_{\overleftarrow{n_r}}\big\rangle^{\rightarrow} && (24)\end{aligned}$$

Group law of $\mathbb{G}$*:*

$$A\circ B = A + B + \sum_{1\leqslant r, 1\leqslant n_i}\mathrm{San}^{n_1,\ldots,n_r}\big\langle A, B_{\overleftarrow{n_1}},\ldots,B_{\overleftarrow{n_r}}\big\rangle^{\rightarrow} \tag{25}$$

Inversion in $\mathbb{G}$*:*

$$\begin{aligned} A^{-1} &= -A - \sum_{1\leqslant r, 1\leqslant n_i}\mathrm{Zan}^{n_1,\ldots,n_r}\big\langle A, A_{\overleftarrow{n_1}},\ldots,A_{\overleftarrow{n_r}}\big\rangle^{\rightarrow} && (26)\\ &= -A - \sum_{1\leqslant r, 1\leqslant n_i}\mathrm{Zan}^{n_1,\ldots,n_r}\big\langle A_{\overleftarrow{1+n_1}},\ldots,A_{\overleftarrow{n_r}}\big\rangle^{\rightarrow} && (27)\end{aligned}$$

Continuous iteration in $\mathbb{G}$*:*

$$\begin{aligned} A^{w} &= w.A + \sum_{1\leqslant r, 1\leqslant n_i}\mathrm{It}_w^{n_1,\ldots,n_r}\big\langle A, A_{\overleftarrow{n_1}},\ldots,A_{\overleftarrow{n_r}}\big\rangle^{\rightarrow} && (28)\\ &= w.A + \sum_{1\leqslant r, 1\leqslant n_i}\mathrm{It}_w^{n_1,\ldots,n_r}\big\langle A_{\overleftarrow{1+n_1}},\ldots,A_{\overleftarrow{n_r}}\big\rangle^{\rightarrow} && (29)\end{aligned}$$

Proof of formula (25) for the group law.
We first check it in a case of special interest to us: $A = a(x), B = b(x), \{A, B\} = a'(x)\,b(x)$. Let us denote by E_n the n-linear term on the left-hand side of (25)

and let us check inductively that $E_n = \frac{1}{n!}\, a^{(n)}$. If the relation (clearly true for $n = 1$) holds up to $n-1$, it also holds for n, due to the identity:

$$E_n = \sum_{1\leqslant k\leqslant n} \frac{(-1)^{k-1}}{n} E_{n-k}\, B_k \quad \textit{with} \quad B_k := (b')^{k-1}\, b \tag{30}$$

Indeed, in the above sum, the term of index $k=1$ contributes the required result $\frac{b^n}{n!}\, a^{(n)}$, while all other contributions, of type $a^{(n-k)}\, b^{n-k}\, (b')^k$ and stemming from the terms of index k and $k+1$, cancel out pairwise.

However, this doesn't quite clinch the proof, as the bracket $\{a, b\} = a?(x)\, b(x)$ doesn't define a *free* pre-Lie algebra. So we now turn to local vector fields on $\mathbb{C}^s$ and to the pre-Lie bracket

$$\{A, B\} := \sum b_j(\partial_{x_j}\, a_i)\, \partial_{x_i} \quad \textit{with} \quad \begin{cases} A = \sum_{r=1}^{s} a_i(x)\, \partial_{x_i} \\ B = \sum_{r=1}^{s} b_i(x)\, \partial_{x_i} \end{cases} \tag{31}$$

which for $s=\infty$ (but for no finite s) does indeed define a free pre-Lie algebra. The proof is on the same lines as in the case $s=1$, with the induction-enabling identity (30) still in force, and the terms E_n, B_k re-interpreted as:

$$E_n := \sum e_{n,i}\partial_{x_i} \quad \textit{with} \quad e_{n,i} \; := \sum_{\sum n_j = n}^{n_j\geqslant 0} \Big(\prod_j \frac{b_j^{n_j}}{n_j!}\Big)\, \partial_{x_1}^{n_1}...\partial_{x_s}^{n_s} a_i \tag{32}$$

$$B_k := \sum b_{k,i}\partial_{x_i} \quad \textit{with} \quad b_{k,i} \; := \sum_{j_1,\ldots,j_k} (\partial_{x_{j_k}} b_{j_{k-1}})...(\partial_{x_{j_2}} b_{j_1}).(\partial_{x_{j_1}} b_i) \tag{33}$$

Proof of formula (28) for continous iteration.
Apply the composition law (25) with $(A^{\circ w}, A)$ in place of (A, B) and use the functional equation (6)-(7) for the mould $It_w^{\bullet}$.

Proof of formula (26) for group inversion.
Treat this as a special case of continuous iteration for $w = -1$ and use the mould identity $It_{-1}^{\bullet} = -Zan^{\bullet}$.

Proof of formula (23) for group-to-algebra logarithm.
Differentiate in w the formula (28) for continuous iteration; then set $w = 0$ and use the mould identity (11).

2.3 'Optimal' vs 'raw' formulae:

Let us compare the *raw* and *optimal* formulae up to order 5.

raw	*expansion*	*optimal*	*expansion*
$(A\circ B)_{\boldsymbol{raw}} =$	$A+B$	$(A\circ B)_{\boldsymbol{opt}} =$	$A+B$
$+1$	$\langle A,B\rangle$	$+1$	$\langle A,B\rangle$
$+\frac{1}{2}$	$\langle\langle A,B\rangle,B\rangle$	$+\frac{1}{2}$	$\langle\langle A,B\rangle,B\rangle$
$-\frac{1}{2}$	$\langle A,\langle B,B\rangle\rangle$	$-\frac{1}{2}$	$\langle A,\langle B,B\rangle\rangle$
$+\frac{1}{6}$	$\langle\langle\langle A,B\rangle,B\rangle,B\rangle$	$+\frac{1}{6}$	$\langle\langle\langle A,B\rangle,B\rangle,B\rangle$
$-\frac{1}{4}$	$\langle\langle A,\langle B,B\rangle\rangle,B\rangle$	$-\frac{1}{6}$	$\langle\langle A,\langle B,B\rangle\rangle,B\rangle$
$-\frac{1}{4}$	$\langle\langle A,B\rangle,\langle B,B\rangle\rangle$	$-\frac{1}{3}$	$\langle\langle A,B\rangle,\langle B,B\rangle\rangle$
$+\frac{1}{4}$	$\langle A,\langle B,\langle B,B\rangle\rangle\rangle$	$+\frac{1}{3}$	$\langle A,\langle B,\langle B,B\rangle\rangle\rangle$
$+\frac{1}{12}$	$\langle A,\langle\langle B,B\rangle,B\rangle\rangle$		
$+\frac{1}{24}$	$\langle\langle\langle\langle A,B\rangle,B\rangle,B\rangle,B\rangle$	$+\frac{1}{24}$	$\langle\langle\langle\langle A,B\rangle,B\rangle,B\rangle,B\rangle$
$-\frac{1}{12}$	$\langle\langle\langle A,\langle B,B\rangle\rangle,B\rangle,B\rangle$	$-\frac{1}{24}$	$\langle\langle\langle A,\langle B,B\rangle\rangle,B\rangle,B\rangle$
$-\frac{1}{12}$	$\langle\langle\langle A,B\rangle,\langle B,B\rangle\rangle,B\rangle$	$-\frac{1}{12}$	$\langle\langle\langle A,B\rangle,\langle B,B\rangle\rangle,B\rangle$
$+\frac{1}{8}$	$\langle\langle A,\langle B,\langle B,B\rangle\rangle\rangle,B\rangle$	$+\frac{1}{12}$	$\langle\langle A,\langle B,\langle B,B\rangle\rangle\rangle,B\rangle$
$-\frac{1}{12}$	$\langle\langle\langle A,B\rangle,B\rangle,\langle B,B\rangle\rangle$	$-\frac{1}{8}$	$\langle\langle\langle A,B\rangle,B\rangle,\langle B,B\rangle\rangle$
$+\frac{1}{8}$	$\langle\langle A,\langle B,B\rangle\rangle,\langle B,B\rangle\rangle$	$+\frac{1}{8}$	$\langle\langle A,\langle B,B\rangle\rangle,\langle B,B\rangle\rangle$
$+\frac{1}{8}$	$\langle\langle A,B\rangle,\langle B,\langle B,B\rangle\rangle\rangle$	$+\frac{1}{4}$	$\langle\langle A,B\rangle,\langle B,\langle B,B\rangle\rangle\rangle$
$-\frac{1}{8}$	$\langle A,\langle B,\langle B,\langle B,B\rangle\rangle\rangle\rangle$	$-\frac{1}{4}$	$\langle A,\langle B,\langle B,\langle B,B\rangle\rangle\rangle\rangle$
$+\frac{1}{24}$	$\langle\langle A,\langle\langle B,B\rangle,B\rangle\rangle,B\rangle$		
$+\frac{1}{24}$	$\langle\langle A,B\rangle,\langle\langle B,B\rangle,B\rangle\rangle$		
$+0$	$\langle A,\langle\langle\langle B,B\rangle,B\rangle,B\rangle\rangle$		
$-\frac{1}{24}$	$\langle A,\langle\langle B,\langle B,B\rangle\rangle,B\rangle\rangle$		
$-\frac{1}{24}$	$\langle A,\langle\langle B,B\rangle,\langle B,B\rangle\rangle\rangle$		
$-\frac{1}{24}$	$\langle A,\langle B,\langle\langle B,B\rangle,B\rangle\rangle\rangle$		
	$+\mathcal{O}(A\,B^5)$		$+\mathcal{O}(A\,B^5)$

Starting from the terms of order 4, the *optimal* expansion differs from the *raw* expansion: it carries only 2^{r-1} non-zero summands of order $r+1$ whereas the raw expansion has nearly all possible $\frac{(2r-2)!}{(r-1)!\,r!}$ summands affected with non-zero coefficients. Denoting by

$$li(A,B,C) := \begin{cases} +\langle\langle A,B\rangle,C\rangle - \langle A,\langle B,C\rangle\rangle \\ -\langle\langle A,C\rangle,B\rangle + \langle A,\langle C,B\rangle\rangle \end{cases} \tag{34}$$

the generators of the pre-Lie ideal[8] $\mathbb{I}$ and defining $A_{\overleftarrow{n}}$ as in (**??**), we find for the terms of order 4 and 5 the identities

$$\big((A\circ B)_{\boldsymbol{raw}} - (A\circ B)_{\boldsymbol{eco}}\big)_4 \;=\; \frac{1}{12}\, li(A, B_{\overleftarrow{1}}, B_{\overleftarrow{2}})$$

$$\big((A\circ B)_{\boldsymbol{raw}} - (A\circ B)_{\boldsymbol{eco}}\big)_5 \;=\; \frac{1}{24} \begin{cases} +2\, li(A, B_{\overleftarrow{3}}, B_{\overleftarrow{1}}) + \langle A, li(B_{\overleftarrow{1}}, B_{\overleftarrow{2}}, B_{\overleftarrow{1}})\rangle \\ +li(\langle A, B_{\overleftarrow{1}}\rangle, B_{\overleftarrow{1}}, B_{\overleftarrow{2}}) + \langle li(A, B_{\overleftarrow{1}}, B_{\overleftarrow{2}}), B_{\overleftarrow{1}}\rangle \end{cases}$$

[8] i.e. the ideal $\mathbb{I}$ such that .

which confirm that the two expansions do coincide modulo $\mathbb{I}$.

raw	*expansion*	‖	*optimal*	*expansion*
$A^{-1}_{\boldsymbol{raw}} =$	$-A$	‖	$A^{-1}_{\boldsymbol{opt}} =$	$-A$
$+1$	$\langle A,A\rangle$	‖	$+1$	$\langle A,A\rangle$
$-\frac{1}{2}$	$\langle\langle A,A\rangle,A\rangle$	‖	$-\frac{1}{2}$	$\langle\langle A,A\rangle,A\rangle$
$-\frac{1}{2}$	$\langle A,\langle A,A\rangle\rangle$	‖	$-\frac{1}{2}$	$\langle A,\langle A,A\rangle\rangle$
$+\frac{1}{6}$	$\langle\langle\langle A,A\rangle,A\rangle,A\rangle$	‖	$+\frac{1}{6}$	$\langle\langle\langle A,A\rangle,A\rangle,A\rangle$
$+\frac{1}{4}$	$\langle\langle A,\langle A,A\rangle\rangle,A\rangle$	‖	$-\frac{1}{3}$	$\langle\langle A,\langle A,A\rangle\rangle,A\rangle$
$+\frac{1}{4}$	$\langle\langle A,A\rangle,\langle A,A\rangle\rangle$	‖	$+\frac{1}{6}$	$\langle\langle A,A\rangle,\langle A,A\rangle\rangle$
$+\frac{1}{4}$	$\langle A,\langle A,\langle A,A\rangle\rangle\rangle$	‖	$+\frac{1}{3}$	$\langle A,\langle A,\langle A,A\rangle\rangle\rangle$
$+\frac{1}{12}$	$\langle A,\langle\langle A,A\rangle,A\rangle\rangle$	‖		
$-\frac{1}{24}$	$\langle\langle\langle\langle A,A\rangle,A\rangle,A\rangle,A\rangle$	‖	$-\frac{1}{24}$	$\langle\langle\langle\langle A,A\rangle,A\rangle,A\rangle,A\rangle$
$-\frac{1}{12}$	$\langle\langle\langle A,\langle A,A\rangle\rangle,A\rangle,A\rangle$	‖	$-\frac{1}{8}$	$\langle\langle\langle A,\langle A,A\rangle\rangle,A\rangle,A\rangle$
$-\frac{1}{12}$	$\langle\langle\langle A,A\rangle,\langle A,A\rangle\rangle,A\rangle$	‖	$-\frac{1}{12}$	$\langle\langle\langle A,A\rangle,\langle A,A\rangle\rangle,A\rangle$
$-\frac{1}{8}$	$\langle\langle A,\langle A,\langle A,A\rangle\rangle\rangle,A\rangle$	‖	$-\frac{1}{4}$	$\langle\langle A,\langle A,\langle A,A\rangle\rangle\rangle,A\rangle$
$-\frac{1}{12}$	$\langle\langle\langle A,A\rangle,A\rangle,\langle A,A\rangle\rangle$	‖	$-\frac{1}{24}$	$\langle\langle\langle A,A\rangle,A\rangle,\langle A,A\rangle\rangle$
$-\frac{1}{8}$	$\langle\langle A,\langle A,A\rangle\rangle,\langle A,A\rangle\rangle$	‖	$-\frac{1}{8}$	$\langle\langle A,\langle A,A\rangle\rangle,\langle A,A\rangle\rangle$
$-\frac{1}{8}$	$\langle\langle A,A\rangle,\langle A,\langle A,A\rangle\rangle\rangle$	‖	$-\frac{1}{12}$	$\langle\langle A,A\rangle,\langle A,\langle A,A\rangle\rangle\rangle$
$-\frac{1}{8}$	$\langle A,\langle A,\langle A,\langle A,A\rangle\rangle\rangle\rangle$	‖	$-\frac{1}{4}$	$\langle A,\langle A,\langle A,\langle A,A\rangle\rangle\rangle\rangle$
$-\frac{1}{24}$	$\langle\langle A,\langle\langle A,A\rangle,A\rangle\rangle,A\rangle$	‖		
$-\frac{1}{24}$	$\langle\langle A,A\rangle,\langle\langle A,A\rangle,A\rangle\rangle$	‖		
$+0$	$\langle A,\langle\langle\langle A,A\rangle,A\rangle,A\rangle\rangle$	‖		
$-\frac{1}{24}$	$\langle A,\langle\langle A,\langle A,A\rangle\rangle,A\rangle\rangle$	‖		
$-\frac{1}{24}$	$\langle A,\langle\langle A,A\rangle,\langle A,A\rangle\rangle\rangle$	‖		
$-\frac{1}{24}$	$\langle A,\langle A,\langle\langle A,A\rangle,A\rangle\rangle\rangle$	‖		
	$+\mathcal{O}(A^6)$	‖		$+\mathcal{O}(A^6)$

Starting from the terms of order 4, the *optimal* expansion starts differing from, and improving on, the *raw* expansion: at order r it carries only 2^{r-1} non-zero summands, whereas the raw expansion has nearly all possible summands affected with non-zero coefficients. Denoting by

$$li(A,B,C) := \begin{cases} +\langle\langle A,B\rangle,C\rangle - \langle A,\langle B,C\rangle\rangle \\ -\langle\langle A,C\rangle,B\rangle + \langle A,\langle C,B\rangle\rangle \end{cases} \tag{35}$$

the generators of the pre-Lie ideal[9] $\mathbb{I}$ and defining $A_{\overset{\leftarrow}{n}}$ as in (**??**), we find for the terms of order 4 and 5 the identities

$$\begin{aligned} \left(A^{-1}_{\boldsymbol{raw}} - A^{-1}_{\boldsymbol{eco}}\right)_4 &= \frac{1}{12}\, li(A_{\overset{\leftarrow}{1}}, A_{\overset{\leftarrow}{1}}, A_{\overset{\leftarrow}{2}}) \\ \left(A^{-1}_{\boldsymbol{raw}} - A^{-1}_{\boldsymbol{eco}}\right)_5 &= \frac{1}{24}\,(t^2 - \frac{1}{4}) \begin{cases} +2\, li(A_{\overset{\leftarrow}{1}}, A_{\overset{\leftarrow}{3}}, A_{\overset{\leftarrow}{1}}) + \langle A_{\overset{\leftarrow}{1}}, li(A_{\overset{\leftarrow}{1}}, A_{\overset{\leftarrow}{2}}, A_{\overset{\leftarrow}{1}})\rangle \\ -\, li(A_{\overset{\leftarrow}{2}}, A_{\overset{\leftarrow}{1}}, A_{\overset{\leftarrow}{2}}) - \langle li(A_{\overset{\leftarrow}{1}}, A_{\overset{\leftarrow}{1}}, A_{\overset{\leftarrow}{2}}), A_{\overset{\leftarrow}{1}}\rangle \end{cases} \end{aligned}$$

[9]i.e. the ideal $\mathbb{I}$ such that .

which confirm that the two expansions do coincide modulo $\mathbb{I}$.

$$
\begin{array}{rl|rl}
\textit{raw} & \textit{expansion} & \textit{optimal} & \textit{expansion} \\
A_{*,\boldsymbol{raw}} = & A & A_{*,\boldsymbol{opt}} = & A \\
-\frac{1}{2} & \langle A,A\rangle & -\frac{1}{2} & \langle A,A\rangle \\
+\frac{1}{12} & \langle\langle A,A\rangle,A\rangle & +\frac{1}{12} & \langle\langle A,A\rangle,A\rangle \\
+\frac{1}{4} & \langle A,\langle A,A\rangle\rangle & +\frac{1}{4} & \langle A,\langle A,A\rangle\rangle \\
+0 & \langle\langle\langle A,A\rangle,A\rangle,A\rangle & +0 & \langle\langle\langle A,A\rangle,A\rangle,A\rangle \\
-\frac{1}{24} & \langle\langle A,\langle A,A\rangle\rangle,A\rangle & -\frac{1}{12} & \langle\langle A,\langle A,A\rangle\rangle,A\rangle \\
-\frac{1}{24} & \langle\langle A,A\rangle,\langle A,A\rangle\rangle & +0 & \langle\langle A,A\rangle,\langle A,A\rangle\rangle \\
-\frac{1}{8} & \langle A,\langle A,\langle A,A\rangle\rangle\rangle & -\frac{1}{6} & \langle A,\langle A,\langle A,A\rangle\rangle\rangle \\
-\frac{1}{24} & \langle A,\langle\langle A,A\rangle,A\rangle\rangle & & \\
-\frac{1}{720} & \langle\langle\langle\langle A,A\rangle,A\rangle,A\rangle,A\rangle & -\frac{1}{720} & \langle\langle\langle\langle A,A\rangle,A\rangle,A\rangle,A\rangle \\
+0 & \langle\langle\langle A,\langle A,A\rangle\rangle,A\rangle,A\rangle & +\frac{1}{144} & \langle\langle\langle A,\langle A,A\rangle\rangle,A\rangle,A\rangle \\
+0 & \langle\langle\langle A,A\rangle,\langle A,A\rangle\rangle,A\rangle & +0 & \langle\langle\langle A,A\rangle,\langle A,A\rangle\rangle,A\rangle \\
+\frac{1}{48} & \langle\langle A,\langle A,\langle A,A\rangle\rangle\rangle,A\rangle & +\frac{5}{72} & \langle\langle A,\langle A,\langle A,A\rangle\rangle\rangle,A\rangle \\
-0 & \langle\langle\langle A,A\rangle,A\rangle,\langle A,A\rangle\rangle & -\frac{1}{144} & \langle\langle\langle A,A\rangle,A\rangle,\langle A,A\rangle\rangle \\
+\frac{1}{48} & \langle\langle A,\langle A,A\rangle\rangle,\langle A,A\rangle\rangle & +\frac{1}{48} & \langle\langle A,\langle A,A\rangle\rangle,\langle A,A\rangle\rangle \\
+\frac{1}{48} & \langle\langle A,A\rangle,\langle A,\langle A,A\rangle\rangle\rangle & -\frac{1}{72} & \langle\langle A,A\rangle,\langle A,\langle A,A\rangle\rangle\rangle \\
+\frac{1}{16} & \langle A,\langle A,\langle A,\langle A,A\rangle\rangle\rangle\rangle & +\frac{1}{8} & \langle A,\langle A,\langle A,\langle A,A\rangle\rangle\rangle\rangle \\
+\frac{1}{144} & \langle\langle A,\langle\langle A,A\rangle,A\rangle\rangle,A\rangle & & \\
+\frac{1}{144} & \langle\langle A,A\rangle,\langle\langle A,A\rangle,A\rangle\rangle & & \\
+0 & \langle A,\langle\langle\langle A,A\rangle,A\rangle,A\rangle\rangle & & \\
+\frac{1}{48} & \langle A,\langle\langle A,\langle A,A\rangle\rangle,A\rangle\rangle & & \\
+\frac{1}{48} & \langle A,\langle\langle A,A\rangle,\langle A,A\rangle\rangle\rangle & & \\
+\frac{1}{48} & \langle A,\langle A,\langle\langle A,A\rangle,A\rangle\rangle\rangle & & \\
 & +\mathcal{O}(A^6) & & +\mathcal{O}(A^6)
\end{array}
$$

Here, we find for the terms of order 4 and 5 the identities

$$
\begin{aligned}
\big(A_{*,\boldsymbol{raw}} - A_{*,\boldsymbol{eco}}\big)_4 &= -\frac{1}{24}\, li(A_{\overset{\leftarrow}{1}}, A_{\overset{\leftarrow}{1}}, A_{\overset{\leftarrow}{2}}) \\
\big(A_{*,\boldsymbol{raw}} - A_{*,\boldsymbol{eco}}\big)_5 &= -\frac{1}{144} \begin{cases} +6\, li(A_{\overset{\leftarrow}{1}}, A_{\overset{\leftarrow}{3}}, A_{\overset{\leftarrow}{1}}) + 3\, \big\langle A_{\overset{\leftarrow}{1}}, li(A_{\overset{\leftarrow}{1}}, A_{\overset{\leftarrow}{2}}, A_{\overset{\leftarrow}{1}})\big\rangle \\ - li(A_{\overset{\leftarrow}{2}}, A_{\overset{\leftarrow}{1}}, A_{\overset{\leftarrow}{2}}) - \big\langle li(A_{\overset{\leftarrow}{1}}, A_{\overset{\leftarrow}{1}}, A_{\overset{\leftarrow}{2}}), A_{\overset{\leftarrow}{1}}\big\rangle \end{cases}
\end{aligned}
$$

which confirm that the two expansions do coincide modulo $\mathbb{I}$.

$$
\begin{array}{rl|rl}
\textit{raw} & \textit{expansion} & \textit{optimal} & \textit{expansion} \\
A^{t+\frac{1}{2}}_{\boldsymbol{raw}} = & (t+\frac{1}{2})\,A & A^{t+\frac{1}{2}}_{\boldsymbol{opt}} = & (t+\frac{1}{2})\,A \\
+\frac{1}{2}\,(t^2-\frac{1}{4}) & \langle A,A\rangle & +\frac{1}{2}\,(t^2-\frac{1}{4}) & \langle A,A\rangle \\
+\frac{1}{6}\,t\,(t^2-\frac{1}{4}) & \langle\langle A,A\rangle,A\rangle & +\frac{1}{6}\,t\,(t^2-\frac{1}{4}) & \langle\langle A,A\rangle,A\rangle \\
-\frac{1}{4}\,(t^2-\frac{1}{4}) & \langle A,\langle A,A\rangle\rangle & -\frac{1}{4}\,(t^2-\frac{1}{4}) & \langle A,\langle A,A\rangle\rangle \\
+\frac{1}{24}\,(t^2-\frac{1}{4})^2 & \langle\langle\langle A,A\rangle,A\rangle,A\rangle & +\frac{1}{24}\,(t^2-\frac{1}{4})^2 & \langle\langle\langle A,A\rangle,A\rangle,A\rangle \\
-\frac{1}{12}\,t\,(t^2-\frac{1}{4}) & \langle\langle A,\langle A,A\rangle\rangle,A\rangle & -\frac{1}{12}\,(t^2-\frac{1}{4})\,(t-\frac{1}{2}) & \langle\langle A,\langle A,A\rangle\rangle,A\rangle \\
-\frac{1}{12}\,t\,(t^2-\frac{1}{4}) & \langle\langle A,A\rangle,\langle A,A\rangle\rangle & -\frac{1}{12}\,(t^2-\frac{1}{4})\,(t+\frac{1}{2}) & \langle\langle A,A\rangle,\langle A,A\rangle\rangle \\
+\frac{1}{8}\,(t^2-\frac{1}{4}) & \langle A,\langle A,\langle A,A\rangle\rangle\rangle & +\frac{1}{6}\,(t^2-\frac{1}{4}) & \langle A,\langle A,\langle A,A\rangle\rangle\rangle \\
+\frac{1}{24}\,(t^2-\frac{1}{4}) & \langle A,\langle\langle A,A\rangle,A\rangle\rangle & & \\
+\frac{1}{120}t(t^2-\frac{1}{4})(t^2-\frac{7}{12}) & \langle\langle\langle\langle A,A\rangle,A\rangle,A\rangle,A\rangle & +\frac{1}{120}t(t^2-\frac{1}{4})(t^2-\frac{7}{12}) & \langle\langle\langle\langle A,A\rangle,A\rangle,A\rangle,A\rangle \\
-\frac{1}{48}\,(t^2-\frac{1}{4})^2 & \langle\langle\langle A,\langle A,A\rangle\rangle,A\rangle,A\rangle & -\frac{1}{48}(t^2-\frac{1}{4})(t^2-\frac{2}{3}\,t-\frac{1}{4}) & \langle\langle\langle A,\langle A,A\rangle\rangle,A\rangle,A\rangle \\
-\frac{1}{48}\,(t^2-\frac{1}{4})^2 & \langle\langle\langle A,A\rangle,\langle A,A\rangle\rangle,A\rangle & -\frac{1}{48}\,(t^2-\frac{1}{4})^2 & \langle\langle\langle A,A\rangle,\langle A,A\rangle\rangle,A\rangle \\
+\frac{1}{24}\,t\,(t^2-\frac{1}{4}) & \langle\langle A,\langle A,\langle A,A\rangle\rangle\rangle,A\rangle & +\frac{1}{18}\,(t^2-\frac{1}{4})\,(t-\frac{3}{4}) & \langle\langle A,\langle A,\langle A,A\rangle\rangle\rangle,A\rangle \\
-\frac{1}{48}\,(t^2-\frac{1}{4})^2 & \langle\langle\langle A,A\rangle,A\rangle,\langle A,A\rangle\rangle & -\frac{1}{48}(t^2-\frac{1}{4})(t^2+\frac{2}{3}\,t-\frac{1}{4}) & \langle\langle\langle A,A\rangle,A\rangle,\langle A,A\rangle\rangle \\
+\frac{1}{24}\,t\,(t^2-\frac{1}{4}) & \langle\langle A,\langle A,A\rangle\rangle,\langle A,A\rangle\rangle & +\frac{1}{24}\,t\,(t^2-\frac{1}{4}) & \langle\langle A,\langle A,A\rangle\rangle,\langle A,A\rangle\rangle \\
+\frac{1}{24}\,t\,(t^2-\frac{1}{4}) & \langle\langle A,A\rangle,\langle A,\langle A,A\rangle\rangle\rangle & +\frac{1}{18}\,(t^2-\frac{1}{4})\,(t+\frac{3}{4}) & \langle\langle A,A\rangle,\langle A,\langle A,A\rangle\rangle\rangle \\
-\frac{1}{16}\,(t^2-\frac{1}{4}) & \langle A,\langle A,\langle A,\langle A,A\rangle\rangle\rangle\rangle & -\frac{1}{8}\,(t^2-\frac{1}{4}) & \langle A,\langle A,\langle A,\langle A,A\rangle\rangle\rangle\rangle \\
+\frac{1}{72}\,t\,(t^2-\frac{1}{4}) & \langle\langle A,\langle\langle A,A\rangle,A\rangle\rangle,A\rangle & & \\
+\frac{1}{72}\,t\,(t^2-\frac{1}{4}) & \langle\langle A,A\rangle,\langle\langle A,A\rangle,A\rangle\rangle & & \\
+0 & \langle A,\langle\langle\langle A,A\rangle,A\rangle,A\rangle\rangle & & \\
-\frac{1}{48}\,(t^2-\frac{1}{4}) & \langle A,\langle\langle A,\langle A,A\rangle\rangle,A\rangle\rangle & & \\
-\frac{1}{48}\,(t^2-\frac{1}{4}) & \langle A,\langle\langle A,A\rangle,\langle A,A\rangle\rangle\rangle & & \\
-\frac{1}{48}\,(t^2-\frac{1}{4}) & \langle A,\langle A,\langle\langle A,A\rangle,A\rangle\rangle\rangle & & \\
 & +\mathcal{O}(A^6) & & +\mathcal{O}(A^6)
\end{array}
$$

Starting from the terms of order 4, the *optimal* expansion starts differing from, and improving on, the *raw* expansion: at order r it carries only 2^{r-1} non-zero summands, whereas the raw expansion has nearly all possible summands affected with non-zero coefficients. Denoting by

$$
li(A,B,C) := \begin{cases} +\langle\langle A,B\rangle,C\rangle - \langle A,\langle B,C\rangle\rangle \\ -\langle\langle A,C\rangle,B\rangle + \langle A,\langle C,B\rangle\rangle \end{cases} \tag{36}
$$

the generators of the pre-Lie ideal[10] $\mathbb{I}$ and defining $A_{\overleftarrow{n}}$ as in (**??**), we find for the terms of order 4 and 5 the identities

$$
\big(A^{t+\frac{1}{2}}_{\boldsymbol{raw}} - A^{t+\frac{1}{2}}_{\boldsymbol{eco}}\big)_4 \;=\; \frac{1}{24}\,(t^2-\frac{1}{4})\,li(A_{\overleftarrow{1}},A_{\overleftarrow{1}},A_{\overleftarrow{2}})
$$

$$
\big(A^{t+\frac{1}{2}}_{\boldsymbol{raw}} - A^{t+\frac{1}{2}}_{\boldsymbol{eco}}\big)_5 \;=\; \frac{1}{144}\,(t^2-\frac{1}{4}) \begin{cases} +6\,li(A_{\overleftarrow{1}},A_{\overleftarrow{3}},A_{\overleftarrow{1}}) + 3\,\langle A_{\overleftarrow{1}}, li(A_{\overleftarrow{1}},A_{\overleftarrow{2}},A_{\overleftarrow{1}})\rangle \\ +2\,t\,li(A_{\overleftarrow{2}},A_{\overleftarrow{1}},A_{\overleftarrow{2}}) + 2\,t\,\langle li(A_{\overleftarrow{1}},A_{\overleftarrow{1}},A_{\overleftarrow{2}}),A_{\overleftarrow{1}}\rangle \end{cases}
$$

which confirm that the two expansions do coincide modulo $\mathbb{I}$.

[10] i.e. the ideal $\mathbb{I}$ generated by the two functional identities of the pre-Lie bracket $\langle .,.\rangle$.

2.4 Alternative expansions.

The expansions we have just constructed are optimal only in the sense of minimizing the number of elementary summands at each order r. But there exist alternative expansions that sometimes *look* simpler, and more appealing, because involving *fewer* basis elements[11] in this or that standard basis of $PreLie_r(A)$, by which we denote the set spanned by all multiple pre-Lie brackets of arity r. We shall give here three instances of such alternative expansions, using three distinct bases of $PreLie_r(A)$. The corresponging basis elements being naturally indexed by unordered, rooted trees, let us say a few words about these.

Unordered, rooted trees $ut_{r,k}$ admit a decomposition

$$\mathrm{ut}_{r,k} = h(\mathrm{ut}_{r_1,k_1}, \ldots, \mathrm{ut}_{r_d,k_d}) \quad \textit{with} \quad \begin{cases} r_1 + \ldots + r_d = r - 1 \\ \mathrm{ut}_{r_j,k_j} \leqslant \mathrm{ut}_{r_{j+1},k_{j+1}} \end{cases} \tag{37}$$

with r the total node number; with ut_{r_j,k_j} the subtrees attached to the root node; and with an indexation $k \in [1, \varkappa(r)]$ reflecting the tree ordering. They also admit a full ordering $<$ inductively defined in this way: we set $ut_{r,k} < ut_{r',k'}$ iff either of the three following relations hold

$$\begin{array}{ll} (i) & r < r' \\ (ii) & r = r' \;\; but \;\; d < d' \\ (iii) & [r, d, k_1, ..., k_{j-1}] = [r', d', k'_1, ..., k'_{j-1}] \;\; but \;\; k_j < k'_j \; for\ some\ j \end{array}$$

This ordering has the advantage that many functions attached to unordered trees $ut_{r,k}$ turn out to be independent of r, or elementarily dependent on it. One such function is the *multiplicity* $\mu^{r,k} \equiv \mu_k$ of $ut_{r,k}$, i.e. the number of rooted, ordered trees which reduce to $ut_{r,k}$ after obliteration of the order.[12]

Assuming at least a fleeting acquaintance with the three bases $\{\bar{u}t_{r,k}\}$, $\{\hat{u}t_{r,k}\}$, $\{\check{u}t_{r,k}\}$ of $PreLie_r(A)$ to be constructed in §8.3 as images of the bases $\{\bar{\mathfrak{u}}\mathfrak{te}_{r,k}\}$, $\{\hat{\mathfrak{u}}\mathfrak{te}_{r,k}\}$, $\{\check{\mathfrak{u}}\mathfrak{te}_{r,k}\}$ of $Flex^{al}(\mathfrak{E})$, we can now proceed to present our alternative expansions, first in the basis $\{\bar{u}t_{r,k}\}$, then in $\{\hat{u}t_{r,k}\}$ and $\{\check{u}t_{r,k}\}$.

Proposition 2.2 (Alternative expansions in $\{\bar{u}t_{r,k}\}$) .

$$A^{\circ t} = t\,A + \sum_{2\leqslant r} \sum_{1\leqslant k\leqslant\varkappa(r)} \big(B_{1+d}(t) - B_{1+d}(0)\big)\,\Gamma^{r,k}\,\bar{u}t_{r,k}(A) \tag{38}$$

$$A^{-1} = -A + \sum_{2\leqslant r} \sum_{1\leqslant k\leqslant\varkappa(r)} (d{+}1)\,B_d\,\Gamma^{r,k}\,\bar{u}t_{r,k}(A) \tag{39}$$

$$A_* = A + \sum_{2\leqslant r} \sum_{1\leqslant k\leqslant\varkappa(r)} (-1)^{d+1}\,(d{+}1)\,\Gamma^{r,k}\,\bar{u}t_{r,k}(A) \tag{40}$$

[11] No contradiction there: it is simply that each of these basis elements resolves itself into a sum of elementary summands. But this in no way diminishes the importance of the alternative expansions.

[12] Don't mix up the *ordering* on the set of *unordered trees* and the internal *order* of an *ordered tree*, i.e. the order on the edges issuing from a given node.

with the Bernoulli numbers B_d and polynomials $B_d(t)$. The rational coefficients $\Gamma^{r,k}$ are defined by the recursion:

$$\Gamma^{r,k} \quad \equiv \quad \frac{2^d}{(d+1)!}\Gamma^{r_1+1,k_1}\ldots\Gamma^{r_d+1,k_d} \tag{41}$$

$$\Gamma^{r+1,k} \quad \equiv \quad -\frac{d+1}{2}\,B_d\,\Gamma^{r,k} \tag{42}$$

with the initial conditions $\Gamma^{1,1}=1, \Gamma^{2,1}=-\frac{1}{2}$. If $\mathrm{d}>1$, then $\mathrm{r}_j{+}1<r$ for all j, so that (41) is properly inductive. If $\mathrm{d}=1$, (41) becomes tautological, but then (42) springs by to save the induction.

Remark 1. When the pre-Lie product derives from an associative product, i.e.

$$\langle A_1, A_2\rangle \equiv (A_1,A_2)-(A_2,A_1) \;\; \textit{with} \;\; (.,.) \;\; \textit{associative}$$

all r-linear terms $\bar{u}t_{r,k}(A)$ in A coincide [13] and the expansions (38)-(40) reduce to

$$\sum_{1\leqslant k\leqslant \varkappa(r)} \mu_k\,\Gamma^{r,k}\big(B_{d+1}(t)-B_{d+1}(0)\big) \quad \equiv \quad (-1)^{r-1}\frac{t!}{(t-r)!\,r!} \tag{43}$$

$$\sum_{1\leqslant k\leqslant \varkappa(r)} \mu_k\,\Gamma^{r+1,k} \quad \equiv \quad -\frac{1}{2r} \tag{44}$$

Remark 2. The link-up between the 'alternative' expansion (38) and its 'optimal' counterpart (28) is via the identities:

$$U_{r,d}(A) \;:=\; \sum_{stem(ut_{r,k})=d}\Gamma^{r,k}\,\bar{u}t_{r,k}(A) \equiv \sum_{d\leqslant\delta\leqslant r-1}^{r_1+\cdots+r_\delta=r-1} C_d^{r_1,\ldots,r_\delta}\big\langle A_{\overleftarrow{1+r_1}},\ldots,A_{\overleftarrow{r_\delta}}\big\rangle^{\rightarrow}$$

The less important coefficients $C_d^{\bullet}$ also verify their own induction, but let us simply mention the relations:

$$C_d^{r_1,\ldots,r_\delta} \quad \equiv \quad (-1)^{d-\delta}C_d^{r_\delta,\ldots,r_1} \tag{45}$$

$$C_d^{\overbrace{1,\ldots,1}^{r-1\; times}} \quad = \quad \begin{cases} \frac{1}{r!} & if \;\; d=r-1 \\ 0 & if \;\; d<r-1 \end{cases} \tag{46}$$

Remark 3. The coefficients involved in the expansions (38)-(40) are one more instance of scalars attached to trees $ut_{r,k}$ that depend essentially on k and trivially on r. Indeed, we may set:

$$\gamma_k := 2^{r-1}\Gamma^{r,k} \qquad \textit{for the integer } \mathrm{r} \textit{ such that } \varkappa(r-1)<k\leqslant\varkappa(r) \tag{47}$$

$$\gamma_k^* := 2^{r-1}\Gamma^{r,k} \qquad \textit{for all integers } \mathrm{r} \textit{ such that } k\leqslant\varkappa(r-1) \tag{48}$$

[13] with $\mu^{r,k}=\mu_k$ the multiplicity factor as above (cf beginning of §2.4).

The above induction for $\Gamma^{r,k}$ then simplifies to

$$\gamma_k \quad \equiv \quad \frac{2^d}{(d+1)!}\,\gamma^*_{k_1}\dots\gamma^*_{k_d} \qquad \textit{with} \qquad d = d_k \tag{49}$$

$$\gamma^*_k \quad \equiv \quad -(d+1)\,B_d\,\gamma_k \qquad \textit{with} \qquad d = d_k \tag{50}$$

with the attendant identities

$$\sum_{1\leqslant k\leqslant \varkappa(r)} \mu_k\,\gamma_k\,\big(B_{d+1}(t) - B_{d+1}(0)\big) \quad \equiv \quad (-1)^{r-1}\frac{t!}{(t-r)!\,r!} \tag{51}$$

$$\sum_{1\leqslant k\leqslant \varkappa(r)} \mu_k\,\gamma^*_k \quad \equiv \quad -\frac{2^{r-1}}{r} \tag{52}$$

In the four identities above, $d = d_k \geqslant 2$ is the number of edges issuing from the root node of the tree $ut_{r,k}$, for the only integer r such that $\varkappa(r-1) < k \leqslant \varkappa(r)$. The rule makes no sense for $k = 1$, but in that case we set $d_1 = 0$ and (50) then reduces to $\gamma^*_1 = -\gamma_1 = -1$.

Here are the sequences $\{\gamma_k\}$ and $\{\gamma^*_k\}$ up to $k = \varkappa(6) = 20$, along with the multiplicities μ_k and the stem numbers d_k involved in the relations (43)-(44):

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
γ_k	1	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{-1}{3}$	$\frac{2}{3}$	$\frac{2}{9}$	$\frac{2}{3}$	$\frac{-1}{3}$	$\frac{2}{15}$	$\frac{2}{3}$	$\frac{2}{9}$	$\frac{2}{9}$	0	$\frac{2}{3}$	$\frac{2}{9}$	$\frac{-1}{3}$	$\frac{-1}{9}$	$\frac{-1}{3}$	$\frac{2}{15}$	$\frac{-2}{45}$
γ^*_k	-1	$\frac{-1}{3}$	$\frac{-1}{3}$	0	$\frac{-1}{3}$	$\frac{-1}{9}$	$\frac{-1}{3}$	0	$\frac{1}{45}$	$\frac{-1}{3}$	$\frac{-1}{9}$	$\frac{-1}{9}$	0	$\frac{-1}{3}$	$\frac{-1}{9}$	0	0	0	$\frac{1}{45}$	0
μ_k	1	1	2	1	2	2	1	3	1	2	2	4	2	2	2	3	3	3	4	1
d_k	0	2	2	3	2	2	2	3	4	2	2	2	2	2	2	3	3	3	4	5

Proposition 2.3 (Alternative expansions in $\{\hat{u}t_{r,k}\}$ and $\{\check{u}t_{r,k}\}$) .
The simplest expansions in the bases $\{\hat{u}t_{r,k}\}$ and $\{\check{u}t_{r,k}\}$ are

$$A^{\circ 2} \quad = \quad 2\,A + \sum_{2\leqslant r}(-1)^{r+1}\hat{u}t_{r,\varkappa(r)}(A) \tag{53}$$

$$A^{\circ -1} \quad = \quad -A + \sum_{2\leqslant r}^{1\leqslant k\leqslant\varkappa(r)} \hat{u}t_{r,k}(A) \tag{54}$$

$$A^{\circ -1} \quad = \quad -A - \sum_{2\leqslant r} \check{u}t_{r,\varkappa(r)}(A) \tag{55}$$

$$A_* \quad = \quad A + \sum_{2\leqslant r}^{1\leqslant k\leqslant\varkappa(r)} c^{r,k}\,\check{u}t_{r,k}(A) \tag{56}$$

with coefficients $c^{r,k}$ that depend only on $ut_{r,k}$ as a non-rooted tree. In other words, $c^{r,k_1} = c^{r,k_2}$ when ut_{r,k_1} and ut_{r,k_2} coincide as graphs*, after erasure of the orientation on their edges.*

Thus, in (**??**) the 5-linear terms consist of three clusters

$$\begin{cases} +\frac{1}{30} & \big(\check{u}t_{5,1}(A) + \check{u}t_{5,5}(A) + \check{u}t_{5,7}(A)\big) \\ +\frac{1}{60} & \big(\check{u}t_{5,2}(A) + \check{u}t_{5,3}(A) + \check{u}t_{5,6}(A) + \check{u}t_{5,8}(A)\big) \\ -\frac{1}{30} & \big(\check{u}t_{5,4}(A) + \check{u}t_{5,9}(A)\big) \end{cases}$$

corresponding to the three non-rooted trees with 5 nodes.

Like the coefficients $\Gamma^{r,k}$ of (43)-(44), the new coefficients $c^{r,k}$ are given by a simple induction, but based this time on non-rooted trees. Thus we find:

$$c^{r,1} \quad = \quad \frac{r!}{[\frac{r-1}{2}]![\frac{r-1}{2}]!} \tag{57}$$

$$c^{r,2} \quad = \quad \begin{cases} \frac{r!}{(\frac{r-1}{2})!(\frac{r-3}{2})!} & \textit{if} \quad r \quad \textit{odd} \\ 0 & \textit{if} \quad r \quad \textit{even} \end{cases} \tag{58}$$

$$c^{r,\varkappa(r)} \quad = \quad c^{r,\varkappa(r-1)} \quad = \quad B_{r-1} \; (= \textit{Bernoulli number}) \; \textit{if} \; r \geqslant 3 \tag{59}$$

Note that the number $\bar{\varkappa}(r)$ of non-rooted, unordered trees with r nodes is much less than the number $\varkappa(r)$ of rooted, unordered trees, since $\bar{\varkappa}(r) \leqslant \varkappa(r-1)$. In fact, their respective generating series relate like

$$\bar{X}(t) \quad = \quad X(t) - \frac{1}{2}X^2(t) + \frac{1}{2}X(t^2) \quad \textit{with} \quad \begin{cases} X(t) := \sum_{1 \leqslant r} \varkappa(r)\, t^r \\ \bar{X}(t) := \sum_{1 \leqslant r} \bar{\varkappa}(r)\, t^r \end{cases} \tag{60}$$

The series $X(t)$ is the more basic of the two, being directly calculable from the relation:

$$X(t) \quad \equiv \quad t \; \exp\big(\sum_{1 \leqslant k} \frac{1}{k}\, X(t^k) \big) \tag{61}$$

3 Exotic pre-Lie products and exotic functional composition.

3.1 The pre-Lie products $\{a,b\}^{exo}_{\sigma}$ and $\{a,b\}^{exxo}$.

Consider the group $\mathbb{G}$ of identity-tangent local diffeomorphisms of $\mathbb{C}$ in its three main incarnations, corresponding to germs of type:

$$f: \quad t \mapsto t + \textstyle\sum_{1 \leqslant n} \alpha_n \, t^{1+n} \qquad\qquad t \quad \textit{small} \tag{62}$$

$$f: \quad t \mapsto t + \textstyle\sum_{1 \leqslant n} \beta_n \, t^{1-n} \qquad\qquad t \quad \textit{large} \tag{63}$$

$$f: \quad t \mapsto t + \textstyle\sum_{1 \leqslant n} \gamma_n \, e^{-n\,t} \qquad\qquad \Re(t) \; \textit{large} \tag{64}$$

The group law of $\mathbb{G}$ is ordinary composition $(f,g) \mapsto f \circ g$. The corresponding Lie algebra $\mathbb{L}$ has for bracket

$$[f_*, g_*] = f_*' \, g_* - f_* \, g_*' \tag{65}$$

with f_*, g_* standing for the infinitesimal generators of f, g.

Proposition 1. *The Lie bracket (65) is induced by the classical pre-Lie product (66) as well as two others 'exotic' pre-Lie products, namely:*

(i) the one-parameter family (67)
(ii) the isolated pre-Lie product (68)

$$\begin{aligned}
\langle f_*, g_* \rangle \quad &:= \quad f'_* \, g_* && (66)\\
\langle f_*, g_* \rangle^{exo}_{\sigma} \quad &:= \quad f'_* \, g_* - \sigma \int (f'_* \, g'_*) = \int (f''_* \, g_* + (1-\sigma) \, f'_* \, g'_*) && (67)\\
\langle f_*, g_* \rangle^{exxo} \quad &:= \quad f'_* \, g_* - \int (f'_* \, g'_*) - \iiint (f''_* \, g''_*) \;=\; \int (f''_* \, g_*) - \iiint (f''_* \, g''_*) && (68)
\end{aligned}$$

$$\text{with} \;\; (\textstyle\int f_*)(t) := \begin{cases} \int_0^t f_*(s)\, ds \;\; \text{for } f \text{ of type } (62) \\ \int_t^\infty f_*(s)\, ds \;\; \text{for } f \text{ of type } (63)\text{-}(64). \end{cases}$$

3.2 The exotic compositions $\circ^{exo}_{\sigma}$ and $\circ^{exxo}$.

There is a standard way – *two ways*, actually[14] – of expressing a group law in terms of the underlying pre-Lie product. In the present instance, however, we can do better than express $\circ^{exo}_{\sigma}$ or $\circ^{exxo}$ in terms of $\{.,.\}^{exo}_{\sigma}$ or $\{.,.\}^{exxo}$. We can directly relate the exotic composition to the standard one, via the detour through the common Lie algebra $\mathbb{L}$:

$$\begin{array}{ccccccc}
f = id + \underline{f} & \in & \mathbb{G} & & & & \\
 & & & \nwarrow\searrow & & & \\
 & & \uparrow\downarrow & & \mathbb{L} & \ni & f_* \\
 & & & \swarrow\nearrow & & & \\
f^{exo} = id + \underline{f}^{exo} & \in & \mathbb{G}^{exo} & & & &
\end{array}$$

and the explicit mappings:

$$\begin{cases} \underline{f} = f_* + \sum \frac{1}{n!} \, \langle \overrightarrow{f_*, ..., f_*} \rangle \\ \underline{f}^{exo} = f_* + \sum \frac{1}{n!} \, \langle \overrightarrow{f_*, ..., f_*} \rangle^{exo} \end{cases} \qquad \begin{cases} f_* = \underline{f} + \sum c_{\leftrightarrow} \, \langle \overleftrightarrow{\underline{f}, ..., \underline{f}} \rangle \\ f_* = \underline{f}^{exo} + \sum c_{\leftrightarrow} \, \langle \overleftrightarrow{\underline{f}^{exo}, ..., \underline{f}^{exo}} \rangle^{exo} \end{cases} \tag{69}$$

The algebra-to-group formulae (118-*left*) involve multiple pre-Lie brackets $\langle \overrightarrow{\ldots} \rangle$ or $\langle \overrightarrow{\ldots} \rangle^{exo}$ with forward parenthesising, so that at depth n we have just one term, with coefficient $1/n!$. The group-to-algebra formulae (118-*right*) also involve multiple pre-Lie brackets $\langle \overleftrightarrow{\ldots} \rangle$ or $\langle \overleftrightarrow{\ldots} \rangle^{exo}$, but with a mixed forward-backward parenthesising and suitably chosen rational coefficients $c_{\leftrightarrow}$. The expansion here is no longer unique, but uniqueness can be restored if we exclude (except in initial position) triplets of the form $\langle\langle \bullet, \bullet \rangle, \bullet \rangle$. We then get the so-called *pared-down formula*, which involves the least possible number of terms at depth n, namely 2^{n-1} n-uplets, each with a pleasantly explicit coefficient $c_{\leftrightarrow}$, whereas the brute force inversion of the algebra-to-group formula would yield a

[14] The better way, or *pared-down expansion*, involves, at each order, a minimal number of summands.

far larger number of n-uplets and uglier coefficients $c_{\leftrightarrow}$.

Proposition 2: the correspondence $\mathbb{G} \leftrightarrow \mathbb{G}^{exxo}$.
It is non-linear and fully determined by the formulae:

$$\partial_t^3 f^{exxo} \;=\; Sw\, f \quad with \;\; (f^{exxo} - f)(t) = \begin{cases} o(t^2) & for\ f\ of\ type\ (62) \\ o(t^{-1}) & for\ f\ of\ type\ (63) \\ o(e^{-t}) & for\ f\ of\ type\ (64) \end{cases} \tag{70}$$

and as usual $Sw(f) = (\frac{f''}{f'})' - \frac{1}{2}\,(\frac{f''}{f'})^2$.

To express the natural correspondence between $\mathbb{G} \leftrightarrow \mathbb{G}^{exo}_{\sigma}$, we require functions h_{σ_2,σ_1} along with their main properties:

$$\begin{aligned}
h_{\sigma_2,\sigma_1}(t) \;&:=\; t + \sum_{1\leqslant n} \frac{t^{n+1}}{(n+1)!} \prod_{1\leqslant k\leqslant n} \big(k\,(\sigma_1 - 1) - (\sigma_2 - 1)\big) && (71)\\
&=\; \frac{\big(1 + (1-\sigma_1)\,t\big)^{\frac{1-\sigma_2}{1-\sigma_1}} - 1}{1-\sigma_2} \qquad if\ \sigma_1 \neq 1, \sigma_2 \neq 1 && (72)\\
h_{\sigma,\sigma}(t) \;&=\; t && (73)\\
h_{1,0}(t) \;&=\; \log(1+t) && (74)\\
h_{0,1}(t) \;&=\; \exp(t) - 1 && (75)\\
h_{\sigma_3,\sigma_2} \circ h_{\sigma_2,\sigma_1} \;&=\; h_{\sigma_3,\sigma_2} && (76)\\
\partial_t\, h_{\sigma_2,\sigma_1}(t) \;&=\; \frac{1 - (\sigma_2 - 1)\, h_{\sigma_2,\sigma_1}(t)}{1 - (\sigma_1 - 1)\,t} && (77)\\
(Sw\, h_{\sigma_2,\sigma_1})(t) \;&=\; \frac{1}{2}\, \frac{(\sigma_1 - \sigma_2)(\sigma_1 + \sigma_2 - 2)}{\big(1 - (\sigma_1 - 1)\,t\big)^2} \qquad (Sw = Schwarzian) && (78)
\end{aligned}$$

Proposition 3: the correspondence $\mathbb{G} \leftrightarrow \mathbb{G}^{exo}_{\sigma}$.
The correspondence is non-linear and fully determined by either of the formulae:

$$h_{0,\sigma}\,(\partial_t\, f^{exo}_{\sigma} - 1) = (\partial_t\, f - 1) \quad , \quad h_{\sigma,0}\,(\partial_t\, f - 1) = (\partial_t\, f^{exo}_{\sigma} - 1) \tag{79}$$

More generally, the correspondence $\mathbb{G}^{exo}_{\sigma_1} \leftrightarrow \mathbb{G}^{exo}_{\sigma_2}$ *goes by the formula:.*

$$h_{\sigma_2,\sigma_1}\,(\partial_t\, f^{exo}_{\sigma_1} - 1) \;=\; (\partial_t\, f^{exo}_{\sigma_2} - 1) \tag{80}$$

Two exo-compositions stand out:

(i) One is the case $\sigma = 1$. It goes with the pre-Lie product $\{\underline{f}, \underline{g}\}^{exo}_1 = \int(\underline{f}''\, \underline{g})$ and the elementary connecting functions $h_{1,0}(t) = \log(1+t)$, $h_{0,1}(t) = \exp(t) - 1$.

(ii)The other is the case $\sigma = 2$. It corresponds to the pre-Lie product $\{\underline{f}, \underline{g}\}^{exo}_2 = \int(\underline{f}''\, \underline{g} - \underline{f}'\, \underline{g}')$, which has the property that $\{e^{-n\,t}, e^{-n\,t}\}^{exo}_2 \equiv 0$ for all n. This considerably simplifies calculations when exo-composing germs of type (64).

3.3 Left-linear expression of the exo-compositions.

Consider two elements f, g of $\mathbb{G}_\sigma^{exo}$ and their product $f \circ_\sigma^{exo} g$. Using the correspondence of Proposition 3, we can calculate their images f_{st}, g_{st} and $f_{st} \circ g_{st}$ in the standard group $\mathbb{G}$:

$$\begin{array}{lccccccc}
\textit{exo-composition} & \circ_\sigma^{exo} & : & (\mathbb{G}_\sigma^{exo} & , & \mathbb{G}_\sigma^{exo}) & \longrightarrow & \mathbb{G}_\sigma^{exo} \\
 & & & \downarrow & & \downarrow & & \uparrow \\
\textit{standard composition} & \circ & : & (\mathbb{G} & , & \mathbb{G}\ ) & \longrightarrow & \mathbb{G}
\end{array}$$

This gives us $f \circ_\sigma^{exo} g$ purely in terms of f, g and the standard composition $\circ$:

$$\begin{aligned}
(f \circ_\sigma^{exo} g)' - 1 &= h_{\sigma,0}\big((f_{st} \circ g_{st})' - 1\big) && (81)\\
&= h_{\sigma,0}\big(g'_{st} - 1 + (f_{st} - id) \circ g_{st})'\big) && (82)\\
&= h_{\sigma,0}\big(g'_{st} - 1 + (f_{st} - id)' \circ g_{st}.g'_{st}\big) && (83)\\
&= h_{\sigma,0}\big(g'_{st} - 1 + (h_{0,\sigma}(f' - 1)) \circ g_{st}.g'_{st}\big) && (84)\\
&= \sum_{0 \leqslant n} \frac{1}{n!}\, h_{\sigma,0}^{(n)}(g'_{st} - 1) \times (h_{0,\sigma}(f' - 1) \circ g_{st})^n.(g'_{st})^n && (85)
\end{aligned}$$

The above expression (85), on the face of it, is not linear in $f - id$. We know, however, that *it has to be.* So we are justified in neglecting the higher-order terms $\sum_{2 \leqslant n}$ and in replacing $h_{0,\sigma} = id + (\ldots)$ by id in $\sum_{1=n}$. As for the term $\sum_{0=n}$, it simply yields $g' - 1$. So, using $h_{\sigma,0} \circ h_{0,\sigma} = id$ and after a few elementary simplifications, we arrive at the final result:

Proposition 4: $\circ_\sigma^{exo}$ in terms of $\circ$. *Setting*

$$g_{st} := id + \int h_{0,\sigma}(g' - 1) = \int \Big(\sigma + (1 - \sigma)\, g'\Big)^{\frac{1}{1-\sigma}} \tag{86}$$

we get:

$$f \circ_\sigma^{exo} g = g + \int \Big((f \circ g_{st} - g_{st})'\ \frac{g''}{g''_{st}}\Big) \quad \textit{with} \quad \begin{cases} f(t) := t + \sum_{1 \leqslant n} \alpha_n\, t^{n+1} \\ g(t) := t + \sum_{1 \leqslant n} \beta_n\, t^{n+1} \\ g_{st} = \int h_{0,\sigma}(g') \end{cases} \tag{87}$$

$$a \circ_\sigma^{exo} g = \int \Big((a \circ g_{st})'\ \frac{g''}{g''_{st}}\Big) \quad \textit{with} \quad \begin{cases} a(t) := \sum_{1 \leqslant n} \alpha_n\, t^{n+1} \\ g(t) := t + \sum_{1 \leqslant n} \beta_n\, t^{n+1} \\ g_{st} = \int h_{0,\sigma}(g') \end{cases} \tag{88}$$

Or purely in terms of g_{st}:

$$\begin{aligned}
f \circ_\sigma^{exo} g &= g + \int \Big((f' \circ g_{st} - 1) \times (g'_{st})^{1-\sigma}\Big) && (89)\\
a \circ_\sigma^{exo} g &= \int \Big((a' \circ g_{st}) \times (g'_{st})^{1-\sigma}\Big) && (90)
\end{aligned}$$

Better proof:

$$(f_\sigma \circ_\sigma g_\sigma)' - 1 \;=\; h_{\sigma,0}((f\circ g)'-1) = \frac{(f\circ g)')^{1-\sigma}-1}{1-\sigma} \tag{91}$$

$$(f_\sigma \circ_\sigma g_\sigma)' - 1 \;=\; \frac{(g')^{1-\sigma}-1}{1-\sigma} + \big(\frac{(f')^{1-\sigma}-1}{1-\sigma}\big)\circ g\;\;(g')^{1-\sigma} \tag{92}$$

$$(f_\sigma \circ_\sigma g_\sigma)' - 1 \;=\; h_{\sigma,0}(g'-1) + \big(h_{\sigma,0}(f'-1)\big)\circ g\;\;(g')^{1-\sigma} \tag{93}$$

$$(f_\sigma \circ_\sigma g_\sigma)' - 1 \;=\; g'_\sigma - 1 + (f'_\sigma - 1)\circ g\,(g')^{1-\sigma} \tag{94}$$

Integrating the last indentity and adjusting the integration constants so as to make all germs vanish at 0, we arrive at (89).

Proposition 5: $\circ^{exxo}$ in terms of $\circ$. *Setting $g''' = Sw.g_{st}$, we get:*

$$f \circ^{exxo} g \;= g + \iiint \Big((f''' \circ g_{st}) \times (g'_{st})^2\Big) \qquad \mathit{if}\; \begin{cases} f(t) = t + o(t) \\ g(t) = t + o(t)\end{cases} \tag{95}$$

$$a \circ^{exxo} g \;\;=\;\; \iiint \Big((a''' \circ g_{st}) \times (g'_{st})^2\Big) \qquad \mathit{if}\; \begin{cases} a(t) = o(t) \\ g(t) = t + o(t)\end{cases} \tag{96}$$

3.4 Extending exotic composition to transseries: the three main steps.

We consider here germs f, g etc.. at $+\infty$ on the real axis.

(i) The formulae (118) and (79) extend exo-composition in a straightforward manner to the set of transserial mappings of the form $f := t \mapsto t + \sum m(t)$, with *elementary* transmonomial $m(t)$, i.e. transmonomials of type $\log m(t) = \mathcal{O}(t)$.

(ii) For general transserial mappings, the formulae (118) and (79) of Proposition 2 and 3 still hold, but when exo-composing large monomials (exponentials or towers of exponentials) one must carefully chose the integration constants implicit in the symbol $\int$.

(iii) When going beyond the range of transseries by allowing exponential iterates of *transfinite* order, one encounters the same *indeterminacy issues* as with ordinary composition

Remark: Except for $\sigma = 1$, there exist no bilinear *exotic* multiplications $\times^{exo}_\sigma$ that would ensure the Leibniz rule

$$\big\langle a \times^{exo}_\sigma b, g\big\rangle^{exo}_\sigma \quad\equiv\quad \big\langle a, g\big\rangle^{exo}_\sigma \times^{exo}_\sigma b \;\;+\;\; a \times^{exo}_\sigma \big\langle b, g\big\rangle^{exo}_\sigma \tag{97}$$

or, equivalently, the distributivity of post-composition:

$$(a \times^{exo}_\sigma b) \circ^{exo}_\sigma g \quad\equiv\quad (a \circ^{exo}_\sigma g) \times^{exo}_\sigma (b \circ^{exo}_\sigma g) \tag{98}$$

For $\sigma = 1$, there does exist an exotic multiplication $a \times^{exo}_1 b := \int(a'b')$, but it has unpleasant properties.

3.5 Exotic composition and resurgence.

We now turn to analytic germs at ∞ on the Riemann sphere. The above Propositions make it abundantly clear that exo-composition preserves the local analyticity of such germs. On the other hand, fractional exo-iteration (i.e. fractional iteration under exo-composition) turns local-analytic, identity-tangent germs into *resurgent ones*, just as in the case of standard fractional iteration, and that too with the same resurgence lattice, the same resurgence constants, and the same resurgence equations simply re-interpreted in the new context.[15]

Proofs. Let

$$
\begin{array}{llll}
g_w = g^{\circ w} & \in & \mathbb{G} & \searrow \\
 & & & \quad g_* \;\; \textit{common infinitesimal generator} \in \mathbb{L} \\
\gamma_w = g^{\circ^{exo}_{\sigma} w} & \in & \mathbb{G}^{exo}_{\sigma} & \nearrow
\end{array}
$$

$$
\begin{aligned}
\partial g_w &= g_*\, g'_w = \{g_w, g_*\} && (99)\\
\partial g'_w &= g'_*\, g'_w + g_*\, g''_w && (100)\\
\partial_w \gamma_w &= g_*\, \gamma'_w - \sigma \int (g'_* \gamma'_w) + \sigma g_* && (101)\\
\partial_w \gamma'_w &= (1-\sigma) g'_*\, \gamma'_w + g_* \gamma''_w) + \sigma g'_* && (102)
\end{aligned}
$$

Notice the difference: $\partial g_w = \langle g_w, g_* \rangle$ but $\partial \gamma_w = \langle g_w, g_* \rangle^{exo}_{\sigma} + \sigma\, g_*$. The parasitical term $\sigma\, g_*$ in the second identity comes from:

$$
\gamma_w = t + \sum_{1 \leqslant n} \frac{w^n}{n!} \langle g_*, \overrightarrow{\ldots}, g_* \rangle \tag{103}
$$

Now, let us check the identity

$$
\gamma'_w - 1 = h_{\sigma,0}(g'_w - 1) \tag{104}
$$

or rather the equation derived by applying ∂_w:

$$
\partial_w \gamma'_w = h'_{\sigma,0}(g'_w - 1)\, \partial_w g'_w \tag{105}
$$

After expressing $\partial_w \gamma'_w$ as in (102), $\partial_w g'_w$ as in (100) and replacing $h'_{\sigma,0}(g'_w - 1)$ by $(g'_w)^{-\sigma}$, we find that (105) turns into an identity. $\square$

4 The prototypal flexion algebra $Flex(\mathfrak{E})$.

To prepare the way for the proper object of this paper – the rapprochement between *tree* and *flexion* algebra – we must first recall some basic notions about the latter. (The reader already familiar with the subject may skip this section).

[15] Hint: for each resurgent germ, write down the standard *diplay* then re-phrase it in the exotic context.

We shall first (§4.1 through §4.4) present in rough terms the general setting, that is to say the polyalgebra *BIMU* of general bimoulds. We shall then (§4.5 through §4.7) zoom in on the monogenous polyalgebra $Flex(\mathfrak{E})$ generated by a so-called *flexion unit* $\mathfrak{E}$. Not only is $Flex(\mathfrak{E})$ the core part of *BIMU*; it is also the part that most naturally lends itself to an interpretation in terms of trees. Hence its relevance to the present investigation.

4.1 Flexion symbols. The space *BIMU* of bimoulds.

For now, let *BIMU* be simply the space of bimoulds, i.e. of moulds $M^\bullet$ indexed by double sequences $\boldsymbol{w} = \binom{\boldsymbol{u}}{\boldsymbol{v}} = \binom{u_1\ ,\dots,\ u_r}{v_1\ ,\dots,\ v_r}$, with the u_i's and v_i's ranging through some ring, generally $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$. Sequences (simple or double) are systematically noted in boldface, with their indices (when needed) in upper position. Their elements w_i, u_i, v_i are noted in ordinary print, with indices in lower position.

Elementary flexions.

In addition to ordinary, non-commutative mould multiplication *mu* (or $\times$):

$$A^\bullet = B^\bullet \times C^\bullet = \mathrm{mu}(B^\bullet, C^\bullet) \quad\Longleftrightarrow\quad A^{\boldsymbol{w}} = \sum_{\boldsymbol{w^1}.\boldsymbol{w^2}=\boldsymbol{w}}^{r(\boldsymbol{w^1}),r(\boldsymbol{w^2})\geqslant 0} B^{\boldsymbol{w^1}}\, C^{\boldsymbol{w^2}} \tag{106}$$

and its inverse *invmu*:

$$(\mathrm{invmu}.A)^{\boldsymbol{w}} = \sum_{1\leqslant s\leqslant r(\boldsymbol{w})} (-1)^s \sum_{\boldsymbol{w^1}\dots\boldsymbol{w^s}\,=\,\boldsymbol{w}} A^{\boldsymbol{w^1}} \dots A^{\boldsymbol{w^s}} \qquad (\boldsymbol{w^i} \neq \emptyset) \tag{107}$$

the bimoulds $A^\bullet$ in $BIMU = \oplus_{0\leqslant r} BIMU_r$ [16] can be subjected to a host of specific operations, all constructed from four elementary *flexions* $\lfloor, \rfloor, \lceil, \rceil$ that are always defined relative to a given factorisation of the total sequence $\boldsymbol{w}$. The way the flexions act is apparent from the following examples:

$$\begin{array}{llll}
\boldsymbol{w} = \boldsymbol{a}.\boldsymbol{b} & \boldsymbol{a} = \binom{u_1,\, u_2,\, u_3}{v_1,\, v_2,\, v_3} & \boldsymbol{b} = \binom{u_4,\, u_5,\, u_6}{v_4,\, v_5,\, v_6} & \\
\Longrightarrow & \boldsymbol{a}\rfloor = \binom{u_1,\ \ u_2,\ \ u_3}{v_{1:4},\, v_{2:4},\, v_{3:4}} & \lceil\boldsymbol{b} = \binom{u_{1234},\, u_5,\, u_6}{v_4,\ \ v_5,\, v_6} & \\
\boldsymbol{w} = \boldsymbol{b}.\boldsymbol{c} & \boldsymbol{b} = \binom{u_1,\, u_2,\, u_3}{v_1,\, v_2,\, v_3} & \boldsymbol{c} = \binom{u_4,\, u_5,\, u_6}{v_4,\, v_5,\, v_6} & \\
\Longrightarrow & \boldsymbol{b}\rceil = \binom{u_1,\, u_2,\, u_{3456}}{v_1,\, v_2,\ \ v_3} & \lfloor\boldsymbol{c} = \binom{u_4,\ \ u_5,\ \ u_6}{v_{4:3},\, v_{5:3},\, v_{6:3}} & \\
\boldsymbol{w} = \boldsymbol{a}.\boldsymbol{b}.\boldsymbol{c} & \boldsymbol{a} = \binom{u_1,\, u_2,\, u_3}{v_1,\, v_2,\, v_3} & \boldsymbol{b} = \binom{u_4,\, u_5,\, u_6}{v_4,\, v_5,\, v_6} & \boldsymbol{c} = \binom{u_7,\, u_8,\, u_9}{v_7,\, v_8,\, v_9} \\
\Longrightarrow & \boldsymbol{a}\rfloor = \binom{u_1,\ \ u_2,\ \ u_3}{v_{1:4},\, v_{2:4},\, v_{3:4}} & \lceil\boldsymbol{b}\rceil = \binom{u_{1234},\, u_5,\, u_{6789}}{v_4,\ \ v_5,\ \ v_6} & \lfloor\boldsymbol{c} = \binom{u_7,\ \ u_8,\ \ u_9}{v_{7:6},\, v_{8:6},\, v_{9:6}}
\end{array}$$

with the usual short-hand: $u_{i,\dots,j} := u_i + \dots + u_j$ and $v_{i:j} := v_i - v_j$. Here and throughout the sequel, we use boldface (with upper indexation) to denote sequences ($\boldsymbol{w}, \boldsymbol{w^i}, \boldsymbol{w^j}$ etc), and ordinary characters (with lower indexation) to denote single sequence elements (w_i, w_j etc), or sometimes sequences of length

[16] $BIMU_r$ of course regroups all bimoulds whose components of length other than r vanish. These are often dubbed "length-r bimoulds" (or "depth-r bimoulds") for short.

$r(\boldsymbol{w}) = 1$. Of course, the 'product' $\boldsymbol{w^1.w^2}$ denotes the concatenation of the two factor sequences.

Short and long indexations on bimoulds.
For bimoulds $M^\bullet \in BIMU_r$ it is sometimes convenient to switch from the usual *short indexation* (with r indices w_i's) to a more homogeneous *long indexation* (with a redundant initial w_0 which gets bracketted for distinctiveness). The correspondence goes like this:

$$M^{\binom{u_1\ ,\ldots,\ u_r}{v_1\ ,\ldots,\ v_r}} \cong M^{\binom{[u_0^*],\ u_1^*\ ,\ldots,\ u_r^*}{[v_0^*],\ v_1^*\ ,\ldots,\ v_r^*}} \tag{108}$$

with the dual conditions on upper and lower indices:

$$\begin{array}{llllll} u_0^* = -u_{1\ldots r} := -(u_1 + \ldots + u_r) & , & u_i^* & = & u_i & \forall i \geqslant 1 \\ v_0^* \quad \textit{arbitrary} & , & v_i^* - v_0^* & = & v_i & \forall i \geqslant 1 \end{array}$$

and of course $\sum_{1 \leqslant i \leqslant r} u_i v_i \equiv \sum_{0 \leqslant i \leqslant r} u_i^* v_i^*$.

Unary operations.
The following linear transformations on $BIMU$ are of constant use:[17]

$$\begin{array}{rcll}
B^\bullet = \text{minu}.A^\bullet & \Rightarrow & B^{w_1,\ldots,w_r} = -\,A^{w_1,\ldots,w_r} & (109) \\
B^\bullet = \text{pari}.A^\bullet & \Rightarrow & B^{w_1,\ldots,w_r} = (-1)^r\,A^{w_1,\ldots,w_r} & (110) \\
B^\bullet = \text{anti}.A^\bullet & \Rightarrow & B^{w_1,\ldots,w_r} = \ \ A^{w_r,\ldots,w_1} & (111) \\
B^\bullet = \text{mantar}.A^\bullet & \Rightarrow & B^{w_1,\ldots,w_r} = (-1)^{r-1}\,A^{w_r,\ldots,w_1} & (112) \\
B^\bullet = \text{neg}.A^\bullet & \Rightarrow & B^{w_1,\ldots,w_r} = \ \ A^{-w_1,\ldots,-w_r} & (113) \\
B^\bullet = \text{swap}.A^\bullet & \Rightarrow & B^{\binom{u_1\ ,\ldots,\ u_r}{v_1\ ,\ldots,\ v_r}} = A^{\binom{v_r\ \ ,\ldots,\ v_{3:4}\ ,\ v_{2:3},\ v_{1:2}}{u_{1..r}\ ,\ldots,\ u_{123},\ u_{12},\ u_1}} & (114) \\
B^\bullet = \text{pus}.A^\bullet & \Rightarrow & B^{\binom{u_1\ ,\ldots,\ u_r}{v_1\ ,\ldots,\ v_r}} = A^{\binom{u_r,\ u_1,\ u_2\ ,\ldots,\ u_{r-1}}{v_r,\ v_1,\ v_2\ ,\ldots,\ v_{r-1}}} & (115) \\
B^\bullet = \text{push}.A^\bullet & \Rightarrow & B^{\binom{u_1\ ,\ldots,\ u_r}{v_1\ ,\ldots,\ v_r}} = A^{\binom{-u_{1\ldots r},\ u_1\ \ ,\ u_2\ \ ,\ldots,\ u_{r-1}}{-v_r\ \ ,\ v_{1:r},\ v_{2:r}\ ,\ldots,\ v_{r-1:r}}} & (116)
\end{array}$$

All are involutions, save for *pus* and *push*, whose restrictions to each $BIMU_r$ reduce to circular permutations of order r resp. $r+1$:[18]

$$\begin{array}{rclll}
\text{push} & = & \text{neg.anti.swap.anti.swap} & & (117) \\
\text{leng}_r & = & \text{push}^{r+1}.\text{leng}_r & = \ \text{pus}^r.\text{leng}_r & (118)
\end{array}$$

with $leng_r$ standing for the natural projection of $BIMU$ onto $BIMU_r$.

[17] The reason for dignifying the humble sign change in (109) with the special name *minu* is that *minu* enters the definition of scores of operators acting on various algebras: the rule for forming the corresponding operators acting on the corresponding groups, is then simply to change the trivial, linear *minu*, which commutes with everybody, into the non-trivial, non-linear *invmu*, which commutes with practically nobody (see (107)). To keep the minus sign instead of *minu* (especially when it occurs twice and so cancels out) would be a sure recipe for getting the transposition wrong.

[18] *pus* resp. *push* is a circular permutation in the *short* resp. *long* indexation of bimoulds. Indeed: $(push.M)^{[w_0],w_1,\ldots,w_r} = M^{[w_r],w_0,\ldots,w_{r-1}}$.

4.2 Flexion operations. *BIMU* as polyalgebra.

Inflected derivations and automorphisms of *BIMU*.
Let $BIMU_*$ resp. $BIMU^*$ denote the subset of all bimoulds $M^\bullet$ such that $M^{\varnothing} = 0$ resp. $M^{\varnothing} = 1$. To each pair $\mathcal{A}^\bullet = (\mathcal{A}^\bullet_L, \mathcal{A}^\bullet_R) \in BIMU_* \times BIMU_*$ resp. $BIMU^* \times BIMU^*$ we attach two remarkable operators:

$$\mathrm{axit}(\mathcal{A}^\bullet) \in \mathrm{Der(BIMU)} \qquad \textit{resp.} \qquad \mathrm{gaxit}(\mathcal{A}^\bullet) \in \mathrm{Aut(BIMU)}$$

whose action on *BIMU* is given by:[19]

$$N^\bullet = \mathrm{axit}(\mathcal{A}^\bullet).M^\bullet \;\Leftrightarrow\; N^{\boldsymbol{w}} = \sum\nolimits^1 M^{\boldsymbol{a}\lceil\boldsymbol{c}} \mathcal{A}_L^{\boldsymbol{b}\rfloor} + \sum\nolimits^2 M^{\boldsymbol{a}\rfloor\boldsymbol{c}} \mathcal{A}_R^{\lfloor\boldsymbol{b}} \tag{119}$$

$$N^\bullet = \mathrm{gaxit}(\mathcal{A}^\bullet).M^\bullet \;\Leftrightarrow\; N^{\boldsymbol{w}} = \sum\nolimits^3 M^{\lceil\boldsymbol{b^1}\rceil\ldots\lceil\boldsymbol{b^s}\rceil} \mathcal{A}_L^{\boldsymbol{a^1}\rfloor} \ldots \mathcal{A}_L^{\boldsymbol{a^s}\rfloor} \mathcal{A}_R^{\lfloor\boldsymbol{c^1}} \ldots \mathcal{A}_R^{\lfloor\boldsymbol{c^s}} \tag{120}$$

and verifies the identities:

$$\mathrm{axit}(\mathcal{A}^\bullet).\mathrm{mu}(M_1^\bullet, M_2^\bullet) \equiv \mathrm{mu}(\mathrm{axit}(\mathcal{A}^\bullet).M_1^\bullet, M_2^\bullet) + \mathrm{mu}(M_1^\bullet, \mathrm{axit}(\mathcal{A}^\bullet).M_2^\bullet) \tag{121}$$

$$\mathrm{gaxit}(\mathcal{A}^\bullet).\mathrm{mu}(M_1^\bullet, M_2^\bullet) \equiv \mathrm{mu}(g\mathrm{axit}(\mathcal{A}^\bullet).M_1^\bullet, \mathrm{gaxit}(\mathcal{A}^\bullet).M_2^\bullet) \tag{122}$$

The *BIMU*-derivations *axit* are stable under the Lie bracket for operators. More precisely, the identity holds:

$$[\mathrm{axit}(\mathcal{B}^\bullet), \mathrm{axit}(\mathcal{A}^\bullet)] = \mathrm{axit}(C^\bullet) \quad \textit{with} \quad \mathcal{C}^\bullet = \mathrm{axi}(\mathcal{A}^\bullet, \mathcal{B}^\bullet) \tag{123}$$

relative to a Lie law *axi* on $BIMU_* \times BIMU_*$ given by:

$$\mathcal{C}_L^\bullet \;:=\; \mathrm{axit}(\mathcal{B}^\bullet).\mathcal{A}_L^\bullet - \mathrm{axit}(\mathcal{A}^\bullet).\mathcal{B}_L^\bullet + \mathrm{lu}(\mathcal{A}_L^\bullet, \mathcal{B}_L^\bullet) \tag{124}$$

$$\mathcal{C}_R^\bullet \;:=\; \mathrm{axit}(\mathcal{B}^\bullet).\mathcal{A}_R^\bullet - \mathrm{axit}(\mathcal{A}^\bullet).\mathcal{B}_R^\bullet - \mathrm{lu}(\mathcal{A}_R^\bullet, \mathcal{B}_R^\bullet) \tag{125}$$

Here, *lu* denotes the standard (non-inflected) Lie law on *BIMU*:

$$\mathrm{lu}(A^\bullet, B^\bullet) := \mathrm{mu}(A^\bullet, B^\bullet) - \mathrm{mu}(B^\bullet, A^\bullet) \tag{126}$$

Let *AXI* denote the Lie algebra consisting of all pairs $\mathcal{A}^\bullet \in BIMU_* \times BIMU_*$ under this law *axi*.

Likewise, the *BIMU*-automorphisms *gaxit* are stable under operator composition. More precisely:

$$\mathrm{gaxit}(\mathcal{B}^\bullet).\mathrm{gaxit}(\mathcal{A}^\bullet) = \mathrm{gaxit}(\mathcal{C}^\bullet) \quad \textit{with} \quad \mathcal{C}^\bullet = \mathrm{gaxi}(\mathcal{A}^\bullet, \mathcal{B}^\bullet) \tag{127}$$

relative to a law *gaxi* on $BIMU^* \times BIMU^*$ given by:

$$\mathcal{C}_L^\bullet \;:=\; \mathrm{mu}(\mathrm{gaxit}(\mathcal{B}^\bullet).\mathcal{A}_L^\bullet, \mathcal{B}_L^\bullet) \tag{128}$$

$$\mathcal{C}_R^\bullet \;:=\; \mathrm{mu}(\mathcal{B}_R^\bullet, \mathrm{gaxit}(\mathcal{B}^\bullet).\mathcal{A}_R^\bullet) \tag{129}$$

[19] The sum $\sum^1$ resp. $\sum^2$ extends to all sequence factorisations $\boldsymbol{w} = \boldsymbol{a.b.c}$ with $\boldsymbol{b} \neq \varnothing$, $\boldsymbol{c} \neq \varnothing$ resp. $\boldsymbol{a} \neq \varnothing$, $\boldsymbol{b} \neq \varnothing$. The sum $\sum^3$ extends to all factorisations $\boldsymbol{w} = \boldsymbol{a^1.b^1.c^1.a^2.b^2.c^2...a^s.b^s.c^s}$ such that $s \geqslant 1$, $\boldsymbol{b^i} \neq \varnothing$, $\boldsymbol{c^i.a^{i+1}} \neq \varnothing\ \forall i$. Note that the extreme factor sequences $\boldsymbol{a^1}$ and $\boldsymbol{c^s}$ may be $\varnothing$.

Let *GAXI* denote the Lie group consisting of all pairs $\mathcal{A}^\bullet \in BIMU^* \times BIMU^*$ under this law *gaxi*. This group *GAXI* clearly admits *AXI* as its Lie algebra.

The mixed operations *amnit = anmit***:**
For $\mathcal{A}^\bullet := (A^\bullet, 0^\bullet)$ and $\mathcal{B}^\bullet := (0^\bullet, B^\bullet)$ the operators $axit(\mathcal{A}^\bullet)$ and $axit(\mathcal{B}^\bullet)$ reduce to $amit(A^\bullet)$ and $anit(B^\bullet)$ respectively (see (137) and (138) *infra*) and the identity (462) becomes:

$$\mathrm{amnit}(A^\bullet, B^\bullet) \equiv \mathrm{anmit}(A^\bullet, B^\bullet) \qquad (\forall A^\bullet, B^\bullet \in \mathrm{BIMU}_*) \tag{130}$$

with

$$\mathrm{amnit}(A^\bullet, B^\bullet) \;:=\; \mathrm{amit}(A^\bullet).\mathrm{anit}(B^\bullet) - \mathrm{anit}(\mathrm{amit}(A^\bullet).B^\bullet) \tag{131}$$

$$\mathrm{anmit}(A^\bullet, B^\bullet) \;:=\; \mathrm{anit}(B^\bullet).\mathrm{amit}(A^\bullet) - \mathrm{amit}(\mathrm{anit}(B^\bullet).A^\bullet) \tag{132}$$

When one of the two arguments $(A^\bullet, B^\bullet)$ vanishes, the definitions reduce to:

$$\mathrm{amnit}(A^\bullet, 0^\bullet) = \mathrm{anmit}(A^\bullet, 0^\bullet) := \mathrm{amit}(A^\bullet) \tag{133}$$

$$\mathrm{amnit}(0^\bullet, B^\bullet) = \mathrm{anmit}(0^\bullet, B^\bullet) = \mathrm{anit}(B^\bullet) \tag{134}$$

Moreover, when *amnit* operates on a one-component bimould $M^\bullet \in BIMU_1$ (such as the *flexion units* $\mathfrak{E}^\bullet$, see §3.1 and §3.3 *infra*), its action drastically simplifies:

$$N^\bullet := \mathrm{amnit}(A^\bullet, B^\bullet).M^\bullet \equiv \mathrm{anmit}(A^\bullet, B^\bullet).M^\bullet \Leftrightarrow N^{\boldsymbol{w}} := \sum_{\boldsymbol{a}\, w_i \boldsymbol{b} = \boldsymbol{w}} A^{\boldsymbol{a}\rfloor} M^{\lceil w_i \rceil} B^{\lfloor \boldsymbol{b}} \tag{135}$$

Unary substructures.
We have two obvious subalgebras//subgroups of *AXI//GAXI*, answering to the conditions:

$$\begin{array}{llcll} \mathrm{AMI} \subset \mathrm{AXI}: & \mathcal{A}^\bullet_R = 0^\bullet & , & \mathrm{GAMI} \subset \mathrm{GAXI}: & \mathcal{A}^\bullet_R = 1^\bullet \\ \mathrm{ANI} \subset \mathrm{AXI}: & \mathcal{A}^\bullet_L = 0^\bullet & , & \mathrm{GANI} \subset \mathrm{GAXI}: & \mathcal{A}^\bullet_L = 1^\bullet \end{array}$$

but we are more interested in the *mixed* unary substructures, consisting of elements of the form:

$$\mathcal{A}^\bullet = (\mathcal{A}^\bullet_L, \mathcal{A}^\bullet_R) \quad \textit{with} \quad \mathcal{A}^\bullet_R \equiv \mathrm{h}(\mathcal{A}^\bullet_L) \quad \textit{and } \mathrm{h} \textit{ a fixed involution} \tag{136}$$

with everything expressible in terms of the left element $\mathcal{A}^\bullet_L$ of the pair $\mathcal{A}^\bullet$. There exist, up to isomorphism, exactly seven such mixed unary substructures:

algebra	h	*swap*	algebra	h
....	....	....	....	....
ARI	*minu*	↔	IRA	*minu.push*
ALI	*anti.pari*	↔	ILA	*anti.pari.neg*
ALA	$anti.pari.neg_u$	↔	ALA	$anti.pari.neg_u$
ILI	$anti.pari.neg_v$	↔	ILI	$anti.pari.neg_v$
AWI	*anti.neg*	↔	IWA	*anti*
AWA	$anti.neg_u$	↔	AWA	$anti.neg_u$
IWI	$anti.neg_v$	↔	IWI	$anti.neg_v$

group	h	*swap*	group	h
....	....	....	....	....
GARI	$invmu$	$\leftrightarrow$	GIRA	$push.swap.invmu.swap$
GALI	$anti.pari$	$\leftrightarrow$	GILA	$anti.pari.neg$
GALA	$anti.pari.neg_u$	$\leftrightarrow$	GALA	$anti.pari.neg_u$
GILI	$anti.pari.neg_v$	$\leftrightarrow$	GILI	$anti.pari.neg_v$
GAWI	$anti.neg$	$\leftrightarrow$	GIWA	$anti$
GAWA	$anti.neg_u$	$\leftrightarrow$	GAWA	$anti.neg_u$
GIWI	$anti.neg_v$	$\leftrightarrow$	GIWI	$anti.neg_v$

Each algebra in the first table (e.g. *ARI*) is of course *the* Lie algebra of the like-named group (e.g. *GARI*). Conversely, each Lie group in the second table is essentially determined by its eponymous Lie algebra *and* the condition of left-linearity.[20]

Dimorphic substructures.
Among all seven pairs of substructures, only two respect dimorphy, namely *ARI//GARI* and *ALI//GALI*. Moreover, when restricted to dimorphic objects, they actually coincide:

$$\begin{array}{rcll} \mathrm{ARI}^{\underline{\mathrm{al}}/\underline{\mathrm{al}}} &=& \mathrm{ALI}^{\underline{\mathrm{al}}/\underline{\mathrm{al}}} & \textit{with} \quad \{\underline{\mathrm{al}}/\underline{\mathrm{al}}\} = \{\text{alternal/alternal } \textit{and} \text{ even}\} \\ \mathrm{GARI}^{\underline{\mathrm{as}}/\underline{\mathrm{as}}} &=& \mathrm{GALI}^{\underline{\mathrm{as}}/\underline{\mathrm{as}}} & \textit{with} \quad \{\underline{\mathrm{as}}/\underline{\mathrm{as}}\} = \{\text{symmetral/symmetral } \textit{and} \text{ even}\} \end{array}$$

We shall henceforth work with the pair *ARI//GARI*, whose definition involves a simpler involution h (it dispenses with the sequence inversion *anti*: see above table).

4.3 Flexion polyalgebra.

Basic anti-actions.
The proper way to proceed is to define the anti-actions (on *BIMU*, with its uninflected product *mu* and bracket *lu*) first of the lateral pairs *AMI//GAMI*, *ANI//GANI* and then of the mixed pair *ARI//GARI*:

$$N^\bullet = \mathrm{amit}(A^\bullet).M^\bullet \quad\Leftrightarrow\quad N^{\boldsymbol{w}} = \sum\nolimits^1 M^{\boldsymbol{a}\lceil\boldsymbol{c}} A^{\boldsymbol{b}\rfloor} \tag{137}$$

$$N^\bullet = \mathrm{anit}(A^\bullet).M^\bullet \quad\Leftrightarrow\quad N^{\boldsymbol{w}} = \sum\nolimits^2 M^{\boldsymbol{a}\rfloor\boldsymbol{c}} A^{\lfloor\boldsymbol{b}} \tag{138}$$

$$N^\bullet = \mathrm{arit}(A^\bullet).M^\bullet \quad\Leftrightarrow\quad N^{\boldsymbol{w}} = \sum\nolimits^1 M^{\boldsymbol{a}\lceil\boldsymbol{c}} A^{\boldsymbol{b}\rfloor} - \sum\nolimits^2 M^{\boldsymbol{a}\rfloor\boldsymbol{c}} A^{\lfloor\boldsymbol{b}} \tag{139}$$

with sums $\sum^1$ (resp. $\sum^2$) ranging over all sequence factorisations $\boldsymbol{w} = \boldsymbol{abc}$ such

[20] meaning that the group operation (like $A^\bullet, B^\bullet \mapsto gari(A^\bullet, B^\bullet)$ in our example) is linear in $A^\bullet$ but highly non-linear in $B^\bullet$.

that $\boldsymbol{b} \neq \emptyset, \boldsymbol{c} \neq \emptyset$ (resp. $\boldsymbol{a} \neq \emptyset, \boldsymbol{b} \neq \emptyset$).

$$N^\bullet = \text{gamit}(A^\bullet).M^\bullet \;\Leftrightarrow\; N^{\boldsymbol{w}} = \sum\nolimits^1 M^{\lceil \boldsymbol{b^1} \cdots \lceil \boldsymbol{b^s}} A^{\boldsymbol{a^1}\rfloor} \ldots A^{\boldsymbol{a^s}\rfloor} \tag{140}$$

$$N^\bullet = \text{ganit}(A^\bullet).M^\bullet \;\Leftrightarrow\; N^{\boldsymbol{w}} = \sum\nolimits^2 M^{\boldsymbol{b^1}\rceil \cdots \boldsymbol{b^s}\rceil} A^{\lfloor \boldsymbol{c^1}} \ldots A^{\lfloor \boldsymbol{c^s}} \tag{141}$$

$$N^\bullet = \text{garit}(A^\bullet).M^\bullet \;\Leftrightarrow\; N^{\boldsymbol{w}} = \sum\nolimits^3 M^{\lceil \boldsymbol{b^1}\rceil \cdots \lceil \boldsymbol{b^s}\rceil} A^{\boldsymbol{a^1}\rfloor} \ldots A^{\boldsymbol{a^s}\rfloor} A_*^{\lfloor \boldsymbol{c^1}} \ldots A_*^{\lfloor \boldsymbol{c^s}} \tag{142}$$

with $A_*^\bullet := \text{invmu}(A^\bullet)$ and with sums $\sum^1, \sum^2, \sum^3$ ranging respectively over all sequence factorisations of the form :

$$\begin{array}{lcll} \boldsymbol{w} & = & \boldsymbol{a^1 b^1 \ldots a^s b^s} & (s \geqslant 1 \quad , \quad \textit{only } \boldsymbol{a^1} \textit{ may be } \emptyset) \\ \boldsymbol{w} & = & \boldsymbol{b^1 c^1 \ldots b^s c^s} & (s \geqslant 1 \quad , \quad \textit{only } \boldsymbol{c^s} \textit{ may be } \emptyset) \\ \boldsymbol{w} & = & \boldsymbol{a^1 b^1 c^1 \ldots a^s b^s c^s} & (s \geqslant 1 \quad , \quad \textit{with } \boldsymbol{b^i} \neq \emptyset \textit{ and } \boldsymbol{c^i a^{i+1}} \neq \emptyset) \end{array}$$

More precisely, in $\sum^3$ two *inner* neighbour factors $\boldsymbol{c^i}$ and $\boldsymbol{a^{i+1}}$ may vanish separately but not simultaneously, whereas the *outer* factors $\boldsymbol{a^1}$ and $\boldsymbol{c^s}$ may of course vanish separately or even simultaneously.

Lie brackets and group laws.
We can now concisely express the Lie algebra brackets *ami, ani, ari* and the group products *gami, gani, gari* :

$$\begin{aligned} \text{ami}(A^\bullet, B^\bullet) &:= \text{amit}(B^\bullet).A^\bullet - \text{amit}(A^\bullet).B^\bullet + \text{lu}(A^\bullet, B^\bullet) && (143) \\ \text{ani}(A^\bullet, B^\bullet) &:= \text{anit}(B^\bullet).A^\bullet - \text{anit}(A^\bullet).B^\bullet - \text{u}(A^\bullet, B^\bullet) && (144) \\ \text{ari}(A^\bullet, B^\bullet) &:= \text{arit}(B^\bullet).A^\bullet - \text{arit}(A^\bullet).B^\bullet + \text{lu}(A^\bullet, B^\bullet) && (145) \end{aligned}$$

$$\begin{aligned} \text{gami}(A^\bullet, B^\bullet) &:= \text{mu}(\text{gamit}(B^\bullet).A^\bullet), B^\bullet) && (146) \\ \text{gani}(A^\bullet, B^\bullet) &:= \text{mu}(B^\bullet, \text{ganit}(B^\bullet).A^\bullet)) && (147) \\ \text{gari}(A^\bullet, B^\bullet) &:= \text{mu}(\text{garit}(B^\bullet).A^\bullet), B^\bullet) && (148) \end{aligned}$$

Pre-Lie products ('pre-brackets').
Parallel with the three Lie brackets, we have three pre-Lie brackets:

$$\begin{aligned} \text{preami}(A^\bullet, B^\bullet) &:= \text{amit}(B^\bullet).A^\bullet + \text{mu}(A^\bullet, B^\bullet) && (149) \\ \text{preani}(A^\bullet, B^\bullet) &:= \text{anit}(B^\bullet).A^\bullet - \text{mu}(A^\bullet, B^\bullet) \quad (\textit{sign}!) && (150) \\ \text{preari}(A^\bullet, B^\bullet) &:= \text{arit}(B^\bullet).A^\bullet + \text{mu}(A^\bullet, B^\bullet) && (151) \end{aligned}$$

with the usual relations:

$$\begin{aligned} \text{ari}(A^\bullet, B^\bullet) &\equiv \text{preari}(A^\bullet, B^\bullet) - \text{preari}(B^\bullet, A^\bullet) && (152) \\ \text{assopreari}(A^\bullet, B^\bullet, C^\bullet) &\equiv \text{assopreari}(A^\bullet, C^\bullet, B^\bullet) && (153) \end{aligned}$$

with *assopreari* denoting the *associator*[21] of the pre-bracket *preari*. The same holds of course for *ami* and *ani*.

[21]Here, the associator *assobin* of a binary operation *bin* is straightforwardly defined as $assobin(a, b, c) := bin(bin(a, b), c) - bin(a, bin(b, c))$. Nothing to do with the Drinfeld associators of the sequel!

Exponentiation from *ARI* to *GARI*.
Provided we properly define the multiple pre-Lie brackets, i.e. from left to right:

$$\overrightarrow{\text{preari}}\,(A_1^\bullet,\ldots,A_s^\bullet)=\text{preari}((\overrightarrow{\text{preari}}\,(A_1^\bullet,\ldots,A_{s-1}^\bullet),A_s^\bullet) \tag{154}$$

we have a simple expression for the exponential mapping from a Lie algebra to its group. Thus, the exponential *expari* : *ARI* → *GARI* can be expressed as a series of pre-brackets:

$$\text{expari}(A^\bullet)=1^\bullet+\sum_{1\leqslant n}\frac{1}{n!}\;\overrightarrow{\text{preari}}\,(\overbrace{A^\bullet,\ldots,A^\bullet}^{n\ times}) \tag{155}$$

or, what amounts to the same, as a mixed *mu+arit*-expansion:

$$\text{expari}(A^\bullet)=1^\bullet+\sum_{1\leqslant r\,,\,1\leqslant n_i}\text{Ex}^{n_1,\ldots,n_r}\text{mu}(A_{n_1}^\bullet,\ldots,A_{n_r}^\bullet) \tag{156}$$

with $A_n^\bullet := \big(arit(A^\bullet)\big)^{n-1}.A^\bullet$ and with the symmetral mould $Ex^\bullet$:

$$\text{Ex}^{n_1,\ldots,n_r}:=\frac{1}{(n_1-1)!}\,\frac{1}{(n_2-1)!}\cdots\frac{1}{(n_r-1)!}\,\frac{1}{n_{1\ldots r}\,n_{2\ldots r}\ldots n_r} \tag{157}$$

The operation from *GARI* to *ARI* that inverses *expari* shall be denoted as *logari*. It, too, can be expressed as a series of multiple *pre-ari* products, but in a much less straightforward manner than (155).

For any *alternal* mould $L^\bullet$ we also have the identities:

$$\begin{aligned}&\sum_{\sigma\subset\mathfrak{S}(r)}L^{\omega_{\sigma(1)},\ldots,\omega_{\sigma(r)}}\text{preari}(A_{\sigma(1)}^\bullet,\ldots,A_{\sigma(r)}^\bullet)\quad\equiv\\&\frac{1}{r}\sum_{\sigma\subset\mathfrak{S}(r)}L^{\omega_{\sigma(1)},\ldots,\omega_{\sigma(r)}}\text{ari}(A_{\sigma(1)}^\bullet,\ldots,A_{\sigma(r)}^\bullet)\qquad(\forall A_1^\bullet,\ldots,A_r^\bullet)\end{aligned} \tag{158}$$

which actually characterise *preari*.

Adjoint actions.
We shall require the adjoint actions, *adgari* and *adari*, of *GARI* on *GARI* and *ARI* respectively. The definitions are straightforward:

$$\text{adgari}(A^\bullet).B^\bullet:=\text{gari}(A^\bullet,B^\bullet,\text{invgari}.A^\bullet)\quad(A^\bullet,B^\bullet\in\text{GARI}) \tag{159}$$

$$\text{adari}(A^\bullet).B^\bullet\;:=\;\text{logari}(\text{adgari}(A^\bullet).\text{expari}(B^\bullet)) \tag{160}$$

$$:=\;\text{fragari}(\text{preari}(A^\bullet,B^\bullet),A^\bullet)\quad(A^\bullet\in\text{GARI},B^\bullet\in\text{ARI}) \tag{161}$$

except for definition (161), which results from (160) and (148) and uses the *pre-ari* product [22] defined as in (151) *supra* and the *gari*-quotient[23] defined as in (**??**) *infra*.

Definition (161) has over the equivalent definition (160) the advantage of bringing out the $B^\bullet$-linearity of $adari(A^\bullet).B^\bullet$ and of leading to much simpler calculations.[24]

The centers of ARI **and** $GARI$**.**
The sets *Center(ARI)* resp. *Center(GARI)* consist of all bimoulds $M^\bullet$ that verify
(i) $M^\varnothing = 0$ resp. $M^\varnothing = 1$
(ii) $M^{\binom{u_1\,,\ldots,\,u_r}{0\,,\ldots,\,0}} = m_r \in \mathbb{C} \qquad \forall u_i$
(iii) $M^{\binom{u_1\,,\ldots,\,u_r}{v_1\,,\ldots,\,v_r}} = 0 \quad$ unless $\quad 0 = v_1 = \cdots = v_r$

Moreover, in view of (148), *gari*-multiplication by a central element $C^\bullet$ amounts to ordinary post-multiplication by that same $C^\bullet$:

$$\mathrm{gari}(C^\bullet, A^\bullet) \equiv \mathrm{gari}(A^\bullet, C^\bullet) \equiv \mathrm{mu}(A^\bullet, C^\bullet) \quad (C^\bullet \in \mathrm{Center(GARI)}) \tag{162}$$

4.4 Basic symmetries and symmetry conservations.

• **alternality and symmetrality**.
Like a mould, a bimould $A^\bullet$ is said to be *alternal* (resp. *symmetral*) if it verifies

$$\sum_{\boldsymbol{w}\in \mathrm{sha}(\boldsymbol{w}',\boldsymbol{w}'')} A^{\boldsymbol{w}} \equiv 0 \quad \big(resp. \;\equiv A^{\boldsymbol{w}'}A^{\boldsymbol{w}''}\big) \qquad \forall \boldsymbol{w}' \neq \varnothing\ ,\ \forall \boldsymbol{w}'' \neq \varnothing \tag{163}$$

with $\boldsymbol{w}$ running through the set $sha(\boldsymbol{w}',\boldsymbol{w}'')$ of all shufflings of $\boldsymbol{w}'$ and $\boldsymbol{w}''$.

• **{alternal}** $\Longrightarrow$**{mantar-invariant, pus-neutral}.**
Alternality implies *mantar*-invariance, with $mantar = minu.pari.anti$ defined as in (112).
It also implies *pus*-neutrality, which means this:

$$\Big(\sum_{1\leqslant l\leqslant r(\bullet)} \mathrm{pus}^l\Big).A^\bullet \equiv 0 \qquad i.e. \qquad \sum_{\boldsymbol{w}' \overset{\mathrm{circ}}{\sim} \boldsymbol{w}} A^{\boldsymbol{w}'} \equiv 0 \qquad (if\, r(\boldsymbol{w}) \geqslant 2) \tag{164}$$

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[22]Properly speaking, *preari* applies only to elements $M^\bullet$ of ARI, i.e. such that $M^\varnothing = 0$. Here, however, only $B^\bullet$ is in ARI, whilst $A^\bullet$ is in $GARI$ and therefore $A^\varnothing = 1$. But this is no obstacle to applying the rule (151).

[23]Properly speaking, *fragari* applies only to arguments $S_1^\bullet, S_2^\bullet$ in $GARI$, i.e. such that $S_i^\varnothing = 1$. Here, however, only $S_2^\bullet := A^\bullet$ is in $GARI$, whilst $S_1^\bullet := preari(A^\bullet, B^\bullet)$ is in ARI and therefore $S_1^\varnothing = 0$. But this is no obstacle to applying the rule:

$$\mathrm{fragari}(S_1^\bullet, S_2^\bullet) := \mathrm{mu}(\mathrm{garit}(S_2^\bullet)^{-1}.S_1^\bullet, \mathrm{invgari}.S_2^\bullet) = \mathrm{mu}(\mathrm{garit}(\mathrm{invgari}.S_2^\bullet).S_1^\bullet, \mathrm{invgari}.S_2^\bullet)$$

[24]Despite the spontaneous occurence of the *pre-ari* product in (161), it should be noted that $adari(A^\bullet)$ is an automorphisms of ARI but *not* of $PREARI$.

• **{symmetral} $\Longrightarrow$ {gantar-invariant, gus-neutral}.**
Symmetrality implies likewise *gantar*-invariance, with

$$\text{gantar} := \text{invmu.anti.pari} \tag{165}$$

as well as *gus*-neutrality, which means $\left(\sum_{1\leqslant l\leqslant r(\bullet)}\text{pus}^l\right).\text{logmu}.A^\bullet \equiv 0$ i.e.

$$\sum_{1\leqslant k\leqslant r(\boldsymbol{w})} (-1)^{k-1} \sum_{\boldsymbol{w^1}\ldots\boldsymbol{w^k} \overset{\text{circ}}{\sim} \boldsymbol{w}} A^{\boldsymbol{w^1}}\ldots A^{\boldsymbol{w^k}} \equiv 0 \qquad\qquad (\mathit{if}\, r(\boldsymbol{w}) \geqslant 2) \tag{166}$$

• **{bialternal} $\overset{\text{ess}^{\text{ly}}}{\Longrightarrow}$ {neg-invariant, push-invariant}.**
Bialternality implies not only invariance under *neg.push* but also separate *neg*-invariance and *push*-invariance for any $A^\bullet \in BIMU_r$ but the implication holds only if $r > 1$, since on $BIMU_1$ we have *neg=push*. So *neg.push=id*, meaning that there is no constraint at all on elements of $BIMU_1$. But we must nonetheless impose *neg*-invariance on $BIMU_1$ (or what amounts to the same, *push*-invariance) to ensure the stability of bialternals under the *ari*-bracket: see §2.7.

• **{bisymmetral} $\overset{\text{ess}^{\text{ly}}}{\Longrightarrow}$ {neg-invariant, gush-invariant}.**
Bisymmetrality implies not only invariance under *neg.gush*, with

$$\text{gush} := \text{neg.gantar.swap.gantar.swap} \tag{167}$$

but also separate *neg*-invariance and *gush*-invariance, but only if we assume *neg*-invariance for the component of length 1. If we do not make that assumption, every bisymmetral bimould in *GARI* splits into two bisymmetral factors: a regular right factor (invariant under *neg*) and an irregular left factor (invariant under *pari.neg*)

4.5 Flexion units.

As it happens, the most useful monogenous algebras $Flex(\mathfrak{E})$ are not those spawned by 'random' generators $\mathfrak{E}$ but on the contrary by very special ones - the so-called *flexion units.*

Exact flexion units. The tripartite relation.
A *flexion unit* is a bimould $\mathfrak{E}^\bullet \in BIMU_1$ that is *odd* in w_1 and verifies the *tripartite relation* below. More precisely:

$$\begin{aligned}
&\mathfrak{E}^{-w_1} \equiv -\mathfrak{E}^{w_1} \quad , \quad \mathfrak{E}^{w_1}\,\mathfrak{E}^{w_2} \equiv \mathfrak{E}^{w_1\rfloor}\,\mathfrak{E}^{\lceil w_2} + \mathfrak{E}^{w_1\rceil}\,\mathfrak{E}^{\lfloor w_2} \qquad i.e\\
&\mathfrak{E}^{\left(\begin{smallmatrix}-u_1\\ -v_1\end{smallmatrix}\right)} \equiv -\mathfrak{E}^{\left(\begin{smallmatrix}u_1\\ v_1\end{smallmatrix}\right)} \quad , \quad \mathfrak{E}^{\left(\begin{smallmatrix}u_1\\ v_1\end{smallmatrix}\right)}\mathfrak{E}^{\left(\begin{smallmatrix}u_2\\ v_2\end{smallmatrix}\right)} \equiv \mathfrak{E}^{\left(\begin{smallmatrix}u_1\\ v_{1:2}\end{smallmatrix}\right)}\mathfrak{E}^{\left(\begin{smallmatrix}u_{12}\\ v_2\end{smallmatrix}\right)} + \mathfrak{E}^{\left(\begin{smallmatrix}u_{12}\\ v_1\end{smallmatrix}\right)}\mathfrak{E}^{\left(\begin{smallmatrix}u_2\\ v_{2:1}\end{smallmatrix}\right)}
\end{aligned} \tag{168}$$

In view of the imparity of $\mathfrak{E}^\bullet$ the tripartite identity may also be written in more symmetric form:

$$\mathfrak{E}^{\left(\begin{smallmatrix}u_1\\ v_{1:0}\end{smallmatrix}\right)}\mathfrak{E}^{\left(\begin{smallmatrix}u_2\\ v_{2:0}\end{smallmatrix}\right)} + \mathfrak{E}^{\left(\begin{smallmatrix}u_2\\ v_{2:1}\end{smallmatrix}\right)}\mathfrak{E}^{\left(\begin{smallmatrix}u_0\\ v_{0:1}\end{smallmatrix}\right)} + \mathfrak{E}^{\left(\begin{smallmatrix}u_0\\ v_{0:2}\end{smallmatrix}\right)}\mathfrak{E}^{\left(\begin{smallmatrix}u_1\\ v_{1:2}\end{smallmatrix}\right)} \equiv 0 \quad \forall u_i, \forall v_i \ \mathit{with}\ \ u_0+u_1+u_2 = 0$$

Another way of characterising flexion units is via the *push*-neutrality of their powers $mu^n(\mathfrak{E}^\bullet)$. Indeed, if we set:

$$\mathrm{mu}^n(\mathfrak{E}^\bullet) = \mathrm{mu}(\overbrace{\mathfrak{E}^\bullet,\ldots,\mathfrak{E}^\bullet}^{n\ times}) \tag{169}$$

then $\mathfrak{E}$ is a flexion unit *iff* $mu^1(\mathfrak{E}^\bullet)$ and $mu^2(\mathfrak{E}^\bullet)$ are *push*-neutral, in which case it can be shown that *all* powers $mu^n(\mathfrak{E}^\bullet)$ are automatically *push*-neutral:

$$\Big\{\mathfrak{E}\ \textit{is a flexion unit}\Big\} \Leftrightarrow \Big\{\big(\sum_{0\leqslant k\leqslant n} \mathrm{push}^k\big).\mathrm{mu}^n(\mathfrak{E}^\bullet) = 0\ \ ,\ \ \forall n \in \mathbb{N}^*\Big\} \tag{170}$$

If two units $\mathfrak{E}^\bullet$ and $\mathfrak{O}^\bullet$ are *constant* respectively in v_1 and u_1 , then the sum $\mathfrak{E}^\bullet + \mathfrak{O}^\bullet$ is also a unit.

Lastly, if $\mathfrak{E}^\bullet$ is a unit, then for each $\alpha,\beta,\gamma,\delta \in \mathbb{C}$ the relation

$$\mathfrak{E}^{\binom{u_1}{v_1}}_{[\alpha,\beta,\gamma,\delta]} := \delta\, e^{\gamma\, u_1\, v_1}\, \mathfrak{E}^{\binom{u_1/\alpha}{v_1/\beta}} \tag{171}$$

defines a new unit $\mathfrak{E}^\bullet_{[\alpha,\beta,\gamma,\delta]}$.

Conjugate units:
If $\mathfrak{E}^\bullet$ is a unit, then the relation $\mathfrak{O}^{\binom{u_1}{v_1}} := \mathfrak{E}^{\binom{v_1}{u_1}}$ define another unit $\mathfrak{O}^\bullet$ – the so-called *conjugate* of $\mathfrak{E}^\bullet$. Indeed, setting $(u_1,u_2) := (v'_1, v'_2 - v'_1)$, $(v_1,v_2) := (u'_1 + u'_2, u'_2)$, then using the *imparity* of $\mathfrak{E}^\bullet$ and re-ordering the terms, we find that (168) becomes:

$$\mathfrak{O}^{\binom{u'_1}{v'_1}}\mathfrak{O}^{\binom{u'_2}{v'_2}} \equiv \mathfrak{O}^{\binom{u'_1}{v'_{1:2}}}\mathfrak{O}^{\binom{u'_{12}}{v'_2}} + \mathfrak{O}^{\binom{u'_{12}}{v'_1}}\mathfrak{O}^{\binom{u'_2}{v'_{2:1}}} \qquad \textit{with} \quad \mathfrak{O}^{\binom{u_1}{v_1}} := \mathfrak{E}^{\binom{v_1}{u_1}}$$

i.e. conserves its form.

Let us now mention the most useful flexion units, some *exact* and others only *approximate*. Throughout the sequel, we shall set:

$$P(t) := \frac{1}{t} \quad , \quad Q(t) := \frac{1}{\tan(t)} \quad , \quad Q_c(t) := \frac{c}{\tan(c\,t)} \tag{172}$$

Polar units:
They consist purely of poles at the origin:

$$\mathrm{Pa}^{w_1} = P(u_1) \tag{173}$$

$$\mathrm{Pi}^{w_1} = P(v_1) \tag{174}$$

$$\mathrm{Pai}^{w_1}_{\alpha,\beta} = P(\frac{u_1}{\alpha}) + P(\frac{v_1}{\beta}) = \frac{\alpha}{u_1} + \frac{\beta}{v_1} \tag{175}$$

$Pa^\bullet, Pi^\bullet, Pai^\bullet_{\alpha,\beta}$ are *exact* units.

Trigonometric units:
They are 'periodised' variants of the polar units:

$$\mathrm{Qa}_c^{w_1} \;=\; Q_c(u_1) \;=\; \frac{c}{\tan(c\,u_1)} \tag{176}$$

$$\mathrm{Qi}_c^{w_1} \;=\; Q_c(v_1) \;=\; \frac{c}{\tan(c\,v_1)} \tag{177}$$

$$\mathrm{Qai}_{c,\alpha,\beta}^{w_1} \;=\; Q_c(\frac{u_1}{\alpha}) + Q_c(\frac{v_1}{\beta}) \;=\; \frac{c}{\tan(\frac{c\,u_1}{\alpha})} + \frac{c}{\tan(\frac{c\,v_1}{\beta})} \tag{178}$$

$Qa_c^{\bullet}$, $Qi_c^{\bullet}$ are *approximate* units but $Qai_{c,\alpha,\beta}^{\bullet}$ is *exact*.

Elliptic units (after C. Brembilla):
Let $\sigma(z\,;\,g_2,g_3)$ be the classical Weierstrass sigma function:

$$\sigma(z\,;\,g_2,g_3) \;=\; z - \frac{g_2}{2^4.3.5}\,z^5 - \frac{g_3}{2^3.3.5.7}\,z^7 + \mathcal{O}(z^9) \qquad \textit{with}$$

$$\sigma(z\,;\,g_2,g_3) \;\equiv\; -\sigma(-z\,;\,g_2,g_3) \equiv t\,\sigma(z\,t^{-1}\,;\,g_2\,t^4, g_3\,t^6) \qquad (\forall t)$$

Then for all $g_2, g_3, \alpha, \beta, \gamma, \delta \in \mathbb{C}$ $(\alpha\beta \neq 0)$, the relation

$$\mathfrak{E}_{g_2,g_3}^{\binom{u_1}{v_1}} := \frac{\sigma(u_1+v_1\,;g_2,\,g_3)}{\sigma(u_1\,;g_2,\,g_3)\;\sigma(v_1\,;g_2,\,g_3)} \tag{179}$$

defines a two-parameter family of exact flexion units, which in turn, under the standard parameter saturation of (171), give rise to:

$$\mathfrak{E}_{g_2,g_3,\alpha,\beta,\gamma,\delta}^{\binom{u_1}{v_1}} \;:=\; \delta\;e^{\gamma\,u_1\,v_1}\;\;\mathfrak{E}_{g_2,g_3}^{\binom{u_1/\alpha}{v_1/\beta}} \tag{180}$$

$$\mathfrak{E}_{g_2,g_3,\alpha,\beta,\gamma,\delta}^{\bullet} \;\equiv\; \mathfrak{E}_{g_2\,t^4\,,g_3\,t^6\,,\alpha\,t\,,\beta\,t\,,\gamma\,,\delta\,t^{-1}}^{\bullet} \qquad (\forall t) \tag{181}$$

This six-parameter, five-dimensional complex variety of flexion units contains all previously listed *exact units* (polar or trigonometric) as limit cases. In fact, it would seem (the matter is still under investigation) that it exhausts *all* flexion units meromorphic in both u_1 and v_1.

We must now examine further units, exact or approximate, that fail to be meromorphic in one of these variables, or both.

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Bitrigonometric units:
$Qaa_c^{w_1}$ (resp. $Qii_c^{w_1}$) is defined for $u_1 \in \mathbb{C}$ and $v_1 \in \mathbb{Q}/\mathbb{Z}$ (resp. vice versa):

$$\mathrm{Qaa}_c^{\binom{u_1}{v_1}} \;:=\; \sum_{n_1\in\mathbb{Z}} \frac{c\,e^{-2\,\pi i n_1 v_1}}{\pi n_1 + c u_1} = \sum_{1\leqslant n_1\leqslant \mathrm{den}(v_1)} \frac{c\,e^{-2\pi i n_1 v_1}}{\mathrm{den}(v_1)}\;Q_c\Big(\frac{\pi\,n_1 + c\,u_1}{\mathrm{den}(v_1)}\Big) \tag{182}$$

$$\mathrm{Qii}_c^{\binom{u_1}{v_1}} \;:=\; \sum_{n_1\in\mathbb{Z}} \frac{c\,e^{-2\,\pi i n_1 u_1}}{\pi n_1 + c v_1} = \sum_{1\leqslant n_1\leqslant \mathrm{den}(u_1)} \frac{c\,e^{-2\pi i n_1 u_1}}{\mathrm{den}(u_1)}\;Q_c\Big(\frac{\pi\,n_1 + c\,v_1}{\mathrm{den}(u_1)}\Big) = \mathrm{Qaa}_c^{\binom{v_1}{u_1}}$$

with *den* denoting the denominator (of a rational number). $Qaa^\bullet_c$ and $Qii^\bullet_c$ are both *approximate* units (see (??),(??) below).

Flat units:
Let σ be the sign function on $\mathbb{R}$, i.e. $\sigma(\mathbb{R}^\pm) = \pm 1$ and $\sigma(0) = 0$. Then set:

$$\mathrm{Sa}^{w_1} = \sigma(u_1) \quad , \quad \mathrm{Si}^{w_1} = \sigma(v_1) \quad , \quad \mathrm{Sai}^{w_1} = \sigma(u_1) + \sigma(v_1) \tag{183}$$

$Sa^\bullet$, $Si^\bullet$ are *approximate* units but $Sai^\bullet$ is *exact*.[25]

Mixed units:

$$\mathrm{Qas}^{w_1}_{c,\pm} = \mathrm{Q}_c(u_1) \pm c\, i\, \sigma(v_1) \quad , \quad \mathrm{Qis}^{w_1}_{c,\pm} = \mathrm{Q}_c(v_1) \pm c\, i\, \sigma(u_1) \tag{184}$$

$Qas^\bullet_{c,\pm}$, $Qis^\bullet_{c,\pm}$ are *exact* units.

"False" units:

$$\mathrm{Qi}^{w_1}_{c,\pm} = Qi^{w_1}_c \pm c\, i = c\, Q(c\, v_1) \pm c\, i = \pm\, 2\, c\, i \frac{e^{\pm\, 2\, c\, i\, v_1}}{e^{\pm\, 2\, c\, i\, v_1} - 1} \tag{185}$$

$Qi^\bullet_{c,+}$ and $Qi^\bullet_{c,-}$ verify the exact *tripartite relation* but not the *imparity* condition.[26]

4.6 The prototypal polyalgebra $Flex(\mathfrak{E}^\bullet)$.

All polyalgebras generated by a *proper* flexion unit $\mathfrak{E}^\bullet$ are isomorphic, so that we are justified in referring to *the* polyalgebra $Flex(\mathfrak{E}^\bullet)$. Within the polyalgebra of general bimoulds, $Flex(\mathfrak{E}^\bullet)$ occupies a neuralgic, but somewhat paradoxical position.

Indeed, it appear to be the *most regular* part of *BIMU*, in the sense that it is on $Flex(\mathfrak{E}^\bullet)$, and on $Flex(\mathfrak{E}^\bullet)$ alone, that the involution *syap*,[27] which simply exchanges the u_i's and the v_i's, commutes with all flexion operations.

Yet at the same time, $Flex(\mathfrak{E}^\bullet)$ can be said to absorb, fixate, and concentrate on itself, all the *irregularity* inherent in *BIMU*, especially in the part of *BIMU* that is specially relevant to multizeta arithmetic and which consists of bimoulds $M^{\boldsymbol{w}}$ polynomial in either $\boldsymbol{u}$ or $\boldsymbol{v}$. It is also $Flex(\mathfrak{E}^\bullet)$, or rather its polar specialisations $Flex(Pa^\bullet)$ and $Flex(Pi^\bullet)$: cf next subsection) that contain the bisymmetral bimoulds $pal^\bullet/pil^\bullet$, whose importance can hardly be overstated, since they hold the key to an understanding of double symmetries, hence of arithmetical dimorphy.

[25] when viewed as a distribution or as an almost-everywhere defined function on $\mathbb{R}$. But when viewed as a function on $\mathbb{Z}$, it becomes an approximate unit.

[26] In terms of applications, the failure of imparity has more disruptive consequences than the failure to verify the exact *tripartite equation*, because it means that $\mathfrak{E}$ has no proper conjugate $\mathfrak{O}$, which in turn prevents it from serving as building block for dimorphic bimoulds such as $\mathfrak{ess}^\bullet$ etc.

[27] To be carefully distinguished from the involution *swap*, as defined in (114).

For our present purpose, however, $\mathit{Flex}(\mathfrak{E}^\bullet)$ has a more immediate distinction: of all substructures of the polyalgebra *BIMU*, it will turn out to be the one most readily, and most fully, describable in terms of *trees*.

4.7 $\mathit{Flex}(Pa^\bullet)$ and $\mathit{Flex}(Pi^\bullet)$: similarities/dissimilarities.

The most important incarnations of $\mathit{Flex}(\mathfrak{E}^\bullet)$, the ones we should constantly keep at the back of our minds, correspond to the *polar* flexion units $Pa^{w_1} := 1/u_1$ and $Pi^{w_1} := 1/v_1$. But although fully isomorphic as far as the flexion structure is concerned, $\mathit{Flex}(Pa^\bullet)$ and $\mathit{Flex}(Pi^\bullet)$ appear profoundly different when it comes to the shape of their elements: as algebraic functions (of the u_i's or v_i's), these will often differ markedly, in terms of degree, complexity, amenability to factorization, etc. We shall see striking illustrations of that fact in §5, when examining the semi-inflected operations on $\mathit{Flex}(\mathfrak{E}^\bullet)$ and their extendibility to *BIMU*, and again in §8, when describing the standard bases of $\mathit{Flex}(\mathfrak{E}^\bullet)$. So, when reasoning on the polyalgebra $\mathit{Flex}(\mathfrak{E}^\bullet)$, we should always harken back to its two polar specialisations, but also be prepared to constantly juggle them.

5 Alternators and counter-alternators.

5.1 Introduction.

Despite its rich array of binary operations, inflected or not, the flexion structure lacked so far a *single operation* capable of generating $\mathit{Flex}^{al}(\mathfrak{E}^\bullet)$ from $\mathfrak{E}^\bullet$ alone. It also lacked a *pair of operations* capable of generating the whole of $\mathit{Flex}(\mathfrak{E}^\bullet)$ from $\mathfrak{E}^\bullet$. To remedy this, we introduce in this section a weakly inflected derivation *de* on $\mathit{Flex}(\mathfrak{E}^\bullet)$ and from it we construct:

- a pre-Lie braket *dle* of *lu* that generates $\mathit{Flex}^{al}(\mathfrak{E}^\bullet)$ from $\mathfrak{E}^\bullet$.
- two operations *dme* and *mde*, pre-associative relative to *mu*, which jointly generates the whole of $\mathit{Flex}(\mathfrak{E}^\bullet)$ from $\mathfrak{E}^\bullet$. Either of them, in combination with *dle*, also generates $\mathit{Flex}(\mathfrak{E}^\bullet)$ from $\mathfrak{E}^\bullet$

Another benefit is this. While the alternality-preserving Lie brackets *ari* and *ali* already possessed pre-Lie brackets *preari* and *preali*, these did not preserve alternality. With *de*, however, we can slightly tweak their definitions to obtain alternality-preserving pre-Lie brakets *dari* and *dali*.

But the main dividends from the having the new operations *dle, dme, mde* will come with the introduction of the so-called *counter-alternators*. These are multivariate, multilinear applications of $\mathit{Flex}(\mathfrak{E}^\bullet)$ into itself that possess a counter-intuitive property: the more symmetrical (i.e. the less alternal) they are as functions of their arguments, the more alternal (i.e. the less symmetrical) they become as bimoulds, i.e. as functions of the sequence $\boldsymbol{w}$. This vague-sounding property actually lends itself to an exact description, and has two fortunate consequences:

- It leads to a neat decomposition $Flex_r(\mathfrak{E}^\bullet) = \oplus_{1\leqslant d\leqslant r} Flex_{r,d}(\mathfrak{E}^\bullet)$ of $Flex_r(\mathfrak{E}^\bullet)$ into subspaces consisting of bimoulds of co-alternality degree d, and yields neat formulae for $dim(Flex_{r,d}(\mathfrak{E}^\bullet))$

- It also permits a natural identification of the elements of $Flex(\mathfrak{E}^\bullet)$ (resp. $Flex^{al}(\mathfrak{E}^\bullet)$) with ordered (resp. unordered) trees, and, building on that, a far-going merger of *flexion* and *tree* algebra, with benefits in both directions.

5.2 Semi-inflected operations on $Flex(\mathfrak{E}^\bullet)$.

Any element of $Flex_r(\mathfrak{E})$ can be expressed as a sum

$$\mathfrak{M}^{\binom{u_1\ ,\dots,\ u_r}{v_1\ ,\dots,\ v_r}} = \sum_{1\leqslant i\leqslant r} \mathfrak{E}^{\binom{u_1+\dots+u_r}{v_i}} \mathfrak{M}_i^{\binom{u_1\ ,\dots,\ u_r}{v_1\ ,\dots,\ v_r}} \tag{186}$$

with components $\mathfrak{M}_i^\bullet$ that are themselves of the form

$$\mathfrak{M}_i^{\binom{u_1\ ,\dots,\ u_r}{v_1\ ,\dots,\ v_r}} = \sum_k m_{i,k} \prod_{k=1}^{r-1} \mathfrak{E}^{\binom{u'_{i,k}}{v'_{i,k}}} \quad \textit{and} \quad \begin{cases} m_{i,k} \in \mathbb{C} \\ \sum_{k=1}^{r-1} u'_{i,k} v'_{i,k} = \sum_{k=1}^{r} u_k(v_k - v_i) \end{cases} \tag{187}$$

The components $\mathfrak{M}_i^\bullet$ are uniquely defined, though their expansions (187) are not.[28] If we now set $de.\mathfrak{M}^\bullet := \sum \mathfrak{M}_i^\bullet$, we see at once that
(i) de is a linear bijection of $Flex_r(\mathfrak{E}^\bullet)$ onto $de.Flex_r(\mathfrak{E}^\bullet)$
(ii) both spaces $mu\big(Flex(\mathfrak{E}^\bullet), de.Flex(\mathfrak{E}^\bullet)\big)$ and $mu\big(de.Flex(\mathfrak{E}^\bullet), Flex(\mathfrak{E}^\bullet)\big)$ are subspaces of $de.Flex(\mathfrak{E}^\bullet)$
(iii) de is a derivation relative to the associative product mu, hence also to the Lie product lu.

We can therefore define on $Flex(\mathfrak{E}^\bullet)$ bilinear applications *dme, mde, dle* of $Flex(\mathfrak{E}^\bullet)$ into itself:

$$\begin{aligned} \mathrm{dme}(\mathfrak{A}^\bullet, \mathfrak{B}^\bullet) &:= \mathrm{de}^{-1}\,\mathrm{mu}(\mathrm{de}\,\mathfrak{A}^\bullet, \mathfrak{B}^\bullet) && (188)\\ \mathrm{mde}(\mathfrak{A}^\bullet, \mathfrak{B}^\bullet) &:= \mathrm{de}^{-1}\,\mathrm{mu}(\mathfrak{A}^\bullet, \mathrm{de}\,\mathfrak{B}^\bullet) && (189)\\ \mathrm{dle}(\mathfrak{A}^\bullet, \mathfrak{B}^\bullet) &:= \mathrm{de}^{-1}\,\mathrm{lu}(\mathrm{de}\,\mathfrak{A}^\bullet, \mathfrak{B}^\bullet) && (190) \end{aligned}$$

These operations are clearly pre-associative and pre-Lie relative to mu and lu, since they relate to the latter according to:

$$\begin{aligned} \mathrm{mu}(\mathfrak{A}^\bullet, \mathfrak{B}^\bullet) &\equiv \mathrm{dme}(\mathfrak{A}^\bullet, \mathfrak{B}^\bullet) + \mathrm{mde}(\mathfrak{A}^\bullet, \mathfrak{B}^\bullet) && (191)\\ \mathrm{lu}(\mathfrak{A}^\bullet, \mathfrak{B}^\bullet) &\equiv \mathrm{dle}(\mathfrak{A}^\bullet, \mathfrak{B}^\bullet) - \mathrm{dle}(\mathfrak{B}^\bullet, \mathfrak{A}^\bullet) && (192) \end{aligned}$$

while verifying the pre-associativity (resp. pre-Lie) identities:

$$\begin{aligned} \mathrm{anti.dme}(\mathfrak{A}^\bullet, \mathfrak{B}^\bullet) &\equiv \mathrm{mde}(\mathrm{anti}.\mathfrak{A}^\bullet, \mathrm{anti}.\mathfrak{B}^\bullet)\\ \mathrm{dme}(\mathrm{dme}(\mathfrak{A}^\bullet, \mathfrak{B}^\bullet), \mathfrak{C}^\bullet) &\equiv \mathrm{dme}(\mathfrak{A}^\bullet, \mathrm{dme}(\mathfrak{B}^\bullet, \mathfrak{C}^\bullet)) + \mathrm{dme}(\mathfrak{A}^\bullet, \mathrm{mde}(\mathfrak{B}^\bullet, \mathfrak{C}^\bullet))\\ \mathrm{mde}(\mathfrak{A}^\bullet, \mathrm{mde}(\mathfrak{B}^\bullet, \mathfrak{C}^\bullet)) &\equiv \mathrm{mde}(\mathrm{dme}(\mathfrak{A}^\bullet, \mathfrak{B}^\bullet), \mathfrak{C}^\bullet) + \mathrm{mde}(\mathrm{mde}(\mathfrak{A}^\bullet, \mathfrak{B}^\bullet), \mathfrak{C}^\bullet) \end{aligned}$$

[28]Nonetheless, they admit a preferred expansion in the 'binary tree' basis.

$$\mathrm{dle}(\mathrm{dle}(\mathfrak{A}^\bullet,\mathfrak{B}^\bullet),\mathfrak{C}^\bullet)-\mathrm{dle}(\mathfrak{A}^\bullet,\mathrm{dle}(\mathfrak{B}^\bullet,\mathfrak{C}^\bullet)) \equiv \mathrm{dle}(\mathrm{dle}(\mathfrak{A}^\bullet,\mathfrak{C}^\bullet),\mathfrak{B}^\bullet)-\mathrm{dle}(\mathfrak{A}^\bullet,\mathrm{dle}(\mathfrak{C}^\bullet,\mathfrak{B}^\bullet))$$

Though skin-deep, the above identities, when iterated and combined with the functional equation (168) of the flexion unit $\mathfrak{E}^\bullet$, produce far-reaching consequences, as we shall soon find out. But before proceeding, let us for clarity spell out the analytical expressions of *dme, mde, dle*:

$$\mathfrak{C}^\bullet = \mathrm{dme}(\mathfrak{A}^\bullet,\mathfrak{B}^\bullet) \;\Leftrightarrow\; \mathfrak{C}^{\boldsymbol{w}} = \sum_{i,j}^{\boldsymbol{w^1}\boldsymbol{w^2}=\boldsymbol{w}} \mathfrak{E}^{\binom{u_1+\ldots+u_r}{v_i}}\,\mathfrak{E}^{\binom{u_1+r_1+\ldots+u_r}{v_j-v_i}}\,\mathfrak{A}_i^{\boldsymbol{w^1}}\,\mathfrak{B}_j^{\boldsymbol{w^2}} \tag{193}$$

$$\mathfrak{C}^\bullet = \mathrm{mde}(\mathfrak{A}^\bullet,\mathfrak{B}^\bullet) \;\Leftrightarrow\; \mathfrak{C}^{\boldsymbol{w}} = \sum_{i,j}^{\boldsymbol{w^1}\boldsymbol{w^2}=\boldsymbol{w}} \mathfrak{E}^{\binom{u_1+\ldots+u_{r_1}}{v_i-v_j}}\,\mathfrak{E}^{\binom{u_1+\ldots+u_r}{v_j}}\,\mathfrak{A}_i^{\boldsymbol{w^1}}\,\mathfrak{B}_j^{\boldsymbol{w^2}} \tag{194}$$

$$\mathfrak{C}^\bullet = \mathrm{dle}(\mathfrak{A}^\bullet,\mathfrak{B}^\bullet) \;\Leftrightarrow\; \mathfrak{C}^{\boldsymbol{w}} = \sum_{i,j}^{\boldsymbol{w^1}\boldsymbol{w^2}=\boldsymbol{w}} \begin{cases} +\mathfrak{E}^{\binom{u_1+\ldots+u_r}{v_i}}\,\mathfrak{E}^{\binom{u_1+r_1+\ldots+u_r}{v_j-v_i}}\,\mathfrak{A}_i^{\boldsymbol{w^1}}\,\mathfrak{B}_j^{\boldsymbol{w^2}} \\ -\mathfrak{E}^{\binom{u_1+\ldots+u_{r_1}}{v_i-v_j}}\,\mathfrak{E}^{\binom{u_1+\ldots+u_r}{v_j}}\,\mathfrak{B}_i^{\boldsymbol{w^1}}\,\mathfrak{A}_j^{\boldsymbol{w^2}} \end{cases} \tag{195}$$

Here r, r_1, r_2 denote the lengths of the sequences $\boldsymbol{w}, \boldsymbol{w^1}, \boldsymbol{w^2}$, so that $r = r_1 + r_2$.

5.3 Semi-inflected operations on $Flex(Pa^\bullet)$ and $Flex(Pi^\bullet)$.

The most important flextion units are $Pa^\bullet$ and $Pi^\bullet$, followed by the approximate units $Qa^\bullet$ and $Qi^\bullet$. But whereas the derivation *de* reduces on $Flex(Pa^\bullet)$ to an elementary multiplication *da*

$$\mathrm{da}.M^{\boldsymbol{w}} \equiv (u_1 + \cdots + u_r)\, M^{\boldsymbol{w}} \tag{196}$$

its specialisation *di* on $Flex(Pi^\bullet)$ is less elementary. Indeed, although the derivative $di.M^\bullet = \sum M_i^\bullet$ and its components $M_i^\bullet$ may also be calculated from a general functional formula[29]

$$\mathrm{di}.M^{\boldsymbol{w}} \;\equiv\; \sum_{k=0}^{r-1} \frac{1}{k!}\,\sigma_{k+1}(v_1,...,v_r)\,(\partial_{v_1}+...+\partial_{v_r})^k M^{\boldsymbol{w}} \tag{197}$$

$$M_i^{\boldsymbol{w}} \;=\; v_i^r\,\Big(\sum_{k=0}^{r-1}\frac{1}{k!}\,\sigma_k(v_1,...,\widehat{v}_i,...,v_r)\,(\partial_{v_1}+...+\partial_{v_r})^k M^{\boldsymbol{w}}\Big) \prod_{1\leqslant j\leqslant r}^{j\neq i} \frac{1}{v_i-v_j}$$

the difference is not merely one of complexity. It also impacts the extension of *da*, *di* and their offspring to larger spaces. Thus, while the relation (196) immediately leads to operations *dma, mda* (pre-associative) and *dla* (pre-Lie) defined on the whole of *BIMU*

$$\begin{aligned} \mathrm{dma}(A^\bullet,B^\bullet) &:= \mathrm{da}^{-1}\,\mathrm{mu}(\mathrm{da}\,A^\bullet,B^\bullet) && (198)\\ \mathrm{mda}(A^\bullet,B^\bullet) &:= \mathrm{da}^{-1}\,\mathrm{mu}(A^\bullet,\mathrm{da}\,B^\bullet) && (199)\\ \mathrm{dla}(A^\bullet,B^\bullet) &:= \mathrm{da}^{-1}\,\mathrm{lu}(\mathrm{da}\,A^\bullet,B^\bullet) && (200) \end{aligned}$$

[29] where $\sigma_k(v_1,...,v_r)$ denotes the k^{th} symmetric function of the indices v_j, and $\hat{v}_i$ signals the removal of v_i: $\sum_{0\leqslant k\leqslant r}\sigma_k(v_1,...,v_r)\,t^k := \prod_{1\leqslant j\leqslant r}(1+t\,v_j)$

the relation (197) makes full sense only on the subspace $BIMU^{v\text{-}v}$ of $BIMU$ consisting of bimoulds of the form:

$$M^{\boldsymbol{w}} = \sum_{1\leqslant i\leqslant r(\boldsymbol{w})} P(v_i)\, M_i^{\boldsymbol{w}} \quad with \quad M^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ v_1-v_0 & ,\dots, & v_r-v_0 \end{smallmatrix}\right)} \equiv M^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ v_1 & ,\dots, & v_r \end{smallmatrix}\right)} \quad \forall v_0 \tag{201}$$

It leads there to operations *dmi, mdi* (pre-associative) and *dli* (pre-Lie):

$$C^\bullet = \mathrm{dmi}(A^\bullet, B^\bullet) \iff C^{\boldsymbol{w}} = \sum_{i,j}^{\boldsymbol{w^1}\boldsymbol{w^2}=\boldsymbol{w}} P(v_i)P(v_j-v_i)\, A_i^{\boldsymbol{w^1}}\, B_j^{\boldsymbol{w^2}} \tag{202}$$

$$C^\bullet = \mathrm{mdi}(A^\bullet, B^\bullet) \iff C^{\boldsymbol{w}} = \sum_{i,j}^{\boldsymbol{w^1}\boldsymbol{w^2}=\boldsymbol{w}} P(v_i-v_j)P(v_j)\, A_i^{\boldsymbol{w^1}}\, B_j^{\boldsymbol{w^2}} \tag{203}$$

$$C^\bullet = \mathrm{dli}(A^\bullet, B^\bullet) \iff C^{\boldsymbol{w}} = \sum_{i,j}^{\boldsymbol{w^1}\boldsymbol{w^2}=\boldsymbol{w}} \begin{cases} +P(v_i)P(v_j-v_i)\, A_i^{\boldsymbol{w^1}}\, B_j^{\boldsymbol{w^2}} \\ -P(v_i-v_j)P(v_j)\, B_i^{\boldsymbol{w^1}}\, A_j^{\boldsymbol{w^2}} \end{cases} \tag{204}$$

A further point deserves emphasizing, regarding the approximate units $Qa^\bullet$, $Qi^\bullet$. Although *dma, mda, dla* or *dmi, mdi, dli* may be made to act on $Flex(Qa^\bullet)$ or $Flex(Qi^\bullet)$, their action there is clearly not *internal*, nor can it me made so by tampering with the definitions.[30] This fact considerably complicates the investigation of the bimoulds $tal^\bullet/til^\bullet$ (– the trigonometric bisymmetrals, essentially equivalent to a Drinfeld associator–) in comparison with the simpler bimoulds $pal^\bullet/pil^\bullet$ (– the polar bisymmetrals, key to understanding flexion dimorphy–)

5.4 Alternality-preserving pre-Lie brackets.

Let us now revert to the general flexion algebra $Flex(\mathfrak{E}^\bullet)$. The pre-Lie brackets *preari*, *preali* hithertoo associated with the alternality-preserving Lie brackets *ari*,*ali*, themselves fail the preserve alternality.

$$\begin{aligned} \mathrm{preari}(A^\bullet, B^\bullet) &:= \mathrm{arit}(B^\bullet)A^\bullet + \mathrm{mu}(A^\bullet, B^\bullet) && (205)\\ \mathrm{preali}(A^\bullet, B^\bullet) &:= \mathrm{alit}(B^\bullet)A^\bullet + \mathrm{mu}(A^\bullet, B^\bullet) && (206) \end{aligned}$$

But the modified pre-Lie brackets *dari*, *dali* do preserve alternality:

$$\begin{aligned} \mathrm{dari}(A^\bullet, B^\bullet) &:= \mathrm{arit}(B^\bullet)A^\bullet + \mathrm{dle}(A^\bullet, B^\bullet) && (207)\\ \mathrm{dali}(A^\bullet, B^\bullet) &:= \mathrm{alit}(B^\bullet)A^\bullet + \mathrm{dle}(A^\bullet, B^\bullet) && (208) \end{aligned}$$

They have the further advantage (especially *dari*) of admitting a simple interpretation in terms of tree operations.

[30]That impossibility is easily proven. It is also worth noting that $Flex(Qa^\bullet)$ and $Flex(Qi^\bullet)$ are not isomorphic, unlike $Flex(Qa^\bullet)$ and $Flex(Qi^\bullet)$.

5.5 The counter-alternaltors *câlt*, *călt*, *cālt*.

We first require a trinary operation *mdme*, thus defined:

$$\begin{aligned} mdme\,(A^\bullet,B^\bullet,C^\bullet) &:= \mathrm{de}^{-1}.\mathrm{mu}(A^\bullet,\mathrm{de}.B^\bullet,C^\bullet) && (209)\\ &\equiv \mathrm{mde}(A^\bullet,\mathrm{dme}(B^\bullet,C^\bullet)) && (210)\\ &\equiv \mathrm{dme}(\mathrm{mde}(A^\bullet,B^\bullet),C^\bullet) && (211)\end{aligned}$$

We can now proceed with the definition of the counter-alternators *câlt*, *călt*, *cālt*. As usual with iterated binary operations, the arrows denote the direction of the multibrackets: rightward or leftward.

Definition 5.1 (Counter-alternators.)

$$\mathrm{c\hat{a}lt}_{H^\bullet}(A_1^\bullet,...,A_r^\bullet) := \sum_{0\leqslant i\leqslant r}(-1)^{r-i}\mathrm{mdme}\big(\overleftarrow{\mathrm{dme}}(A_1^\bullet,..,A_i^\bullet),H^\bullet,\overrightarrow{\mathrm{mde}}(A_{i+1}^\bullet,..,A_r^\bullet)\big) \quad (212)$$

$$\mathrm{c\check{a}lt}_{H^\bullet}(A_1^\bullet,...,A_r^\bullet) := \sum_{0\leqslant i\leqslant r}(-1)^{r-i}\mathrm{mdme}\big(\overrightarrow{\mathrm{mde}}(A_1^\bullet,..,A_i^\bullet),H^\bullet,\overleftarrow{\mathrm{dme}}(A_{i+1}^\bullet,..,A_r^\bullet)\big) \quad (213)$$

$$\mathrm{c\bar{a}lt}_{H^\bullet}(A_1^\bullet,...,A_r^\bullet) := \sum_{0\leqslant i\leqslant r}\frac{(-1)^{r-i}\,r!}{i!\,(r-i)!}\mathrm{mdme}\big(\mathrm{mu}(A_1^\bullet,..,A_i^\bullet),H^\bullet,\mathrm{mu}(A_{i+1}^\bullet,..,A_r^\bullet)\big) \quad (214)$$

etc. Of course, when the argument of a multibracket is the empty sequence, the result must be taken to be unit mould. Thus $\overleftarrow{\mathrm{dme}}(\varnothing)=\overrightarrow{\mathrm{mde}}(\varnothing)=1^\bullet$ and:

$$\mathrm{c\hat{a}lt}_{H^\bullet}(A_1^\bullet) := \begin{cases} +\mathrm{mdme}(\overleftarrow{\mathrm{dme}}(A_1^\bullet),H^\bullet,\mathit{1}^\bullet)\\ -\mathrm{mdme}(\mathit{1}^\bullet,H^\bullet,\overrightarrow{\mathrm{mde}}(A_1^\bullet))\end{cases} \quad (215)$$

$$\mathrm{c\hat{a}lt}_{H^\bullet}(A_1^\bullet,A_2^\bullet) := \begin{cases} +\mathrm{mdme}(\overleftarrow{\mathrm{dme}}(A_1^\bullet,A_2^\bullet),H^\bullet,\mathit{1}^\bullet)\\ -\mathrm{mdme}(\overleftarrow{\mathrm{dme}}(A_1^\bullet),H^\bullet,\overrightarrow{\mathrm{mde}}(A_2^\bullet))\\ +\mathrm{mdme}(\mathit{1}^\bullet,H^\bullet,\overrightarrow{\mathrm{mde}}(A_1^\bullet,A_2^\bullet))\end{cases} \quad (216)$$

Note that since $\overleftarrow{\mathrm{dme}}(A_1^\bullet)=\overrightarrow{\mathrm{mde}}(A_1^\bullet)=A_1^\bullet$, for $r=1$ the three counter-alternators $\mathit{c\hat{a}lt}_{H^\bullet}(A_1^\bullet)$, $\mathit{c\check{a}lt}_{H^\bullet}(A_1^\bullet)$, $\mathit{c\bar{a}lt}_{H^\bullet}(A_1^\bullet)$ all reduce to $-\mathit{dle}(H^\bullet,A_1^\bullet)$. But as soon as $r\geqslant 2$ they cease to be expressible in terms of the sole pre-Lie product *dle* and requires *dme* and *mde*.

It is also worth noting that there is complete rigidity in the definition of the counter-alternators. In particular no formula involving $\overrightarrow{\mathrm{dme}},\overleftarrow{\mathrm{mde}}$ instead of $\overleftarrow{\mathrm{dme}},\overrightarrow{\mathrm{mde}}$ would yield interesting results.

Remark: The counter-alternators are also capable of an inductive construction by depth r. Thus for *câlt* we have:

$$\mathrm{câlt}_{H^\bullet}(A_1^\bullet,..,A_r^\bullet) := \begin{cases} -\mathrm{dle}(H^\bullet, \overleftarrow{\mathrm{dme}}(A_1^\bullet,..,A_r^\bullet)) \\ -\sum_{1\leqslant j<r} \overleftarrow{\mathrm{dme}}\big(\mathrm{câlt}_{H^\bullet}(A_1^\bullet,..,A_j^\bullet), A_{j+1}^\bullet,..,A_r^\bullet)\big) \end{cases} \tag{217}$$

$$\equiv \begin{cases} (-1)^r\,\mathrm{dle}(H^\bullet, \overrightarrow{\mathrm{mde}}(A_1^\bullet,..,A_r^\bullet)) \\ -\sum_{1\leqslant j<r}(-1)^j\overrightarrow{\mathrm{mde}}\big(A_1^\bullet,..,A_j^\bullet), \mathrm{câlt}_{H^\bullet}(A_{j+1}^\bullet,..,A_r^\bullet)\big) \end{cases} \tag{218}$$

5.6 The symmetrical alternaltors *sâlt*, *sǎlt*, *sālt*.

The symmetrical alternators can be defined directly, by full symmetrization of the arguments in the counter-alternators:

Definition 5.2 (Alternators.)

$$\mathrm{sâlt}_{H^\bullet}(A_1^\bullet,..,A_r^\bullet) := \sum_{\sigma\in\mathfrak{S}_r} \mathrm{câlt}_{H^\bullet}(A_{\sigma(1)}^\bullet,..,A_{\sigma(r)}^\bullet) \tag{219}$$

$$\mathrm{sǎlt}_{H^\bullet}(A_1^\bullet,..,A_r^\bullet) := \sum_{\sigma\in\mathfrak{S}_r} \mathrm{cǎlt}_{H^\bullet}(A_{\sigma(1)}^\bullet,..,A_{\sigma(r)}^\bullet) \tag{220}$$

$$\mathrm{sālt}_{H^\bullet}(A_1^\bullet,..,A_r^\bullet) := \sum_{\sigma\in\mathfrak{S}_r} \mathrm{cālt}_{H^\bullet}(A_{\sigma(1)}^\bullet,..,A_{\sigma(r)}^\bullet) \tag{221}$$

Two of them, *sâlt* and *sālt*, also admit an inductive construction:

$$\mathrm{sâlt}_{H^\bullet}(A_1^\bullet,..,A_r^\bullet) := \begin{cases} -\mathrm{dle}\big(\mathrm{sâlt}_{H^\bullet}(A_1^\bullet,..,A_{r-1}^\bullet), A_r^\bullet\big) \\ +\sum_{1\leqslant j<r} \mathrm{sâlt}_{H^\bullet}\big(A_1^\bullet,..,\widehat{A}_j^\bullet,..,A_{r-1}^\bullet, \mathrm{dle}(A_j^\bullet, A_r^\bullet)\big) \end{cases} \tag{222}$$

$$\mathrm{sālt}_{H^\bullet}(A_1^\bullet,..,A_r^\bullet) := -\mathrm{dle}\big(\mathrm{sālt}_{H^\bullet}(A_1^\bullet,..,\widehat{A}_j^\bullet,..,A_r^\bullet), A_j^\bullet\big) \tag{223}$$

Remarkably, no such induction exists for *sǎlt*.

The symmetrical alternator purely in terms of *dle*:
To state the result, we once again require the specific combinations of leftward and rightward multiple pre-Lie brackets already encountered in the 'optimal formulae' of §2.2, but this time with distinct arguments $A_i^\bullet$. So we set:

$$\mathrm{le}_{r_1,\ldots,r_s}(A_1^\bullet,\ldots,A_r^\bullet) \quad := \quad \overrightarrow{\mathrm{dle}}\big(\overleftarrow{\mathrm{dle}}(\boldsymbol{A}^1),\ldots,\overleftarrow{\mathrm{dle}}(\boldsymbol{A}^s)\big) \tag{224}$$

$$\textit{with} \qquad \begin{cases} \boldsymbol{A} = \boldsymbol{A}^1\ldots\boldsymbol{A}^s = (A_1^\bullet,\ldots,A_r^\bullet) \\ \textit{length}(\boldsymbol{A}^i) = r_i \\ r_1+\ldots+r_s = r \end{cases}$$

We alse require, to express the coefficients of our expansions, two symmetral moulds $s\hat{a}^\bullet$ and $s\check{a}^\bullet$:

$$\mathrm{s\hat{a}}^{s_1,\ldots,s_r} \quad = \quad (-1)^r \prod_{1\leqslant i\leqslant r} \frac{1}{s_1+\ldots+s_i} \tag{225}$$

$$\mathrm{s\check{a}}^{s_1,\ldots,s_r} \quad = \quad (-1)^{s_1+\ldots+s_r} \prod_{1\leqslant i\leqslant r} \frac{1}{s_i+\ldots+s_r} \tag{226}$$

Proposition 5.1 (*dle*-expansions of the alternators.)

$$\text{sâlt}_{A_0^\bullet}(A_1^\bullet,..,A_r^\bullet) \;:=\; \sum_{\sigma\in\mathfrak{S}(r)} \sum_{1\leqslant s\leqslant r}^{\sum r_i=r} \text{sâ}^{r_1,\dots,r_s}\,\text{le}_{1+r_1,\dots,r_s}(A_0^\bullet, A_{\sigma(1)}^\bullet,\dots,A_{\sigma(r)}^\bullet) \quad (227)$$

$$\text{sălt}_{A_0^\bullet}(A_1^\bullet,..,A_r^\bullet) \;:=\; \sum_{\sigma\in\mathfrak{S}(r)} \sum_{1\leqslant s\leqslant r}^{\sum r_i=r} \text{să}^{r_1,\dots,r_s}\,\text{le}_{1+r_1,\dots,r_s}(A_0^\bullet, A_{\sigma(1)}^\bullet,\dots,A_{\sigma(r)}^\bullet) \quad (228)$$

$$\text{sālt}_{A_0^\bullet}(A_1^\bullet,..,A_r^\bullet) \;:=\; \sum_{\sigma\in\mathfrak{S}(r)} \sum_{1\leqslant s\leqslant r}^{\sum r_i=r} (-1)^r \;\overrightarrow{\text{le}}(A_0^\bullet, A_{\sigma(1)}^\bullet,\dots,A_{\sigma(r)}^\bullet) \quad (229)$$

5.7 Main properties of the alternators:

Recall that a mould or bimould $A^\bullet$ is said to have alternality codegree δ if it meets either of the two equivalent conditions:

$$\sum_{\boldsymbol{w}\in\text{sha}(\boldsymbol{w^1},\dots,\boldsymbol{w^{\delta+1}})} A^{\boldsymbol{w}} \;\equiv\; 0 \qquad (\forall\ \boldsymbol{w^1},\dots,\boldsymbol{w^{\delta+1}}) \quad (230)$$

$$\deg\Big(\sum_{\sigma\in\mathfrak{S}_r} A^{w_{\sigma(1)},\dots,w_{\sigma(r)}} D_1\dots D_r\Big) \;\equiv\; \delta \qquad (\forall\ D_1,\dots,D_r) \quad (231)$$

Here, $\boldsymbol{w}$ runs through all shuffle products of $\delta{+}1$ non-empty sequences $\boldsymbol{w^j}$, and the D_j denote independent abstract derivations.

Proposition 5.2 (Properties of the strict counter-alternators *câlt*, *călt***)** *Let here* calt, salt *stand for either* câlt, sâlt *or* călt, sălt*, and consider a superposition* $B^{w_1,\dots,w_d}$ *of counter-alternators involving the same arguments* $A_1^\bullet,..,A_r$ *in different arrangements:*

$$B^\bullet := \sum_{\sigma\in\mathfrak{S}_r} c_\sigma\,\text{calt}_{H^\bullet}(A_{\sigma(1)},\dots,A_{\sigma(r)}) \;\; \textit{with} \;\; \begin{cases} c_\sigma\in\mathbb{C} \\ \textit{depth}(H^\bullet)=d_0 \\ \textit{depth}(A_i^\bullet)=d_i \\ d=d_0+d_1+\dots+d_r \end{cases} \quad (232)$$

(i) The **less** *alternal* $B^{w_1,\dots,w_d}$ *is as a function of* $(A_1^\bullet,\dots,A_r^\bullet)$*, the* **more** *alternal it is as a function of* $(w_1,\dots,w_d)$ *- which of course is why* calt *is called a* counter-alternator.
(ii) More precisely, let the bimoulds $H^\bullet$ *and* $A_i^\bullet$ *have alternality co-degrees* δ_0 *and* δ_i *(as functions of their indices* $\boldsymbol{w}$*) and let* $B^\bullet$ *have alternality co-degree* δ_* *as a function of its bimould arguments* $(A_1^\bullet,\dots,A_r^\bullet)$*, then that same* $B^\bullet$*, as a function of its indices* $\boldsymbol{w}=(w_1,\dots,w_d)$*, has alternality co-degree*

$$\delta=\delta_0+\delta_1+\dots+\delta_r-\delta_* \quad (233)$$

(iii) In particular, if $c_\sigma\equiv 1\ \forall\sigma$*, then* $\delta_*=r$ *and* $B^\bullet$ *reduces to the symmetrical*

alternator $\mathrm{salt}_{H^\bullet}(A_1^\bullet,...,A_r^\bullet)$. *As a function of its indices* $\boldsymbol{w}=(w_1,...,w_d)$ *it has therefore co-degree* $\delta=\delta_0+\delta_1+...+\delta_r-r$. *If moreover, the bimoulds* $H^\bullet$ *and* $A_i^\bullet$ *are all alternal, then* $\delta_0=\delta_i=1$ *so that* $\delta=1$, *meaning that* $B^\bullet$ *also is alternal. In that sense, it may be said that* salt *preserves alternality.*

Proposition 5.3 (Properties of the loose counter-alternator *cālt***)** .
The same holds for cālt, *but with (233) giving way to a simple inequality*

$$\delta \leqslant \delta_0+\delta_1+...+\delta_r-\delta_* \tag{234}$$

except of course when the right-hand side is 1, in which case δ *has to be 1 also. This is exactly what happens when* $c_\sigma\equiv 1$ *and all bimould inputs* $H^\bullet$, $A_i^\bullet$ *are alternal. Thus, although* cālt *may be dismissed as a loose counter-alternator*[31], sālt *is very much a strict alternator.*[32]

5.8 Alternality projectors:

For a better grasp of the phenomenon of 'counter-alternativity', let us decompose the algebra $\mathbb{M}$ of all moulds into subspaces $\mathbb{M}_{:d}$ consisting of d-alternal bimoulds expressible as fully symmetrized sums of *mu*-products of alternal moulds. Similarly, let us decompose the algebra $\mathbb{E}$ freely generated by abstract degree-1 derivations $e_1,e_2...$ into subspaces $\mathbb{E}^{:d}$ spanned by degree-d derivations expressible as fully symmetrized products of d degree-1 derivations (those in turn being expressible as multiple Lie-brackets of the generators $e_1,e_2....$

$$\mathbb{M}=\oplus_{1\leqslant d}\ \mathbb{M}_{:d} \quad , \quad \mathrm{pr}_{:d}:\ \mathbb{M}\overset{projection}{\longrightarrow}\mathbb{M}_{:d} \tag{235}$$

$$\mathbb{E}\ =\oplus_{1\leqslant d}\ \mathbb{E}^{:d} \quad , \quad \mathrm{pr}^{:d}:\ \mathbb{E}\overset{projection}{\longrightarrow}\mathbb{E}^{:d} \tag{236}$$

Due to (231), the projectors $pr_{:d}$ and $pr^{:d}$ are mutually transposed operators, and necessarily of the form:

$$\mathrm{pr}_{:d}(M^\bullet)=M^\bullet_{:d} \quad \textit{with} \quad M_{:d}^{w_1,...,w_r}=\sum_{\sigma\in\mathfrak{S}_r} h_{r,d}(\sigma)\, M^{w_{\sigma(1)},...,w_{\sigma(1)}} \tag{237}$$

$$\mathrm{pr}^{:d}(e_\bullet)\ =e_\bullet^{:d} \quad \textit{with} \quad (e_1...e_r)^{:d}=\sum_{\sigma\in\mathfrak{S}_r} h_{r,d}(\sigma^{-1})\, e_{\sigma(1)}...e_{\sigma(r)} \tag{238}$$

The simplest way to calculate the coefficients $h_{r,d}(\sigma)$ is to consider symmetral moulds $S^\bullet$ and form their t^{th} powers (respective to mould mutiplication:

$$(S^\bullet)^{\times t}:=1^\bullet+\sum_{1\leqslant n}\frac{t(t-1)...(t-n+1)}{n!}\ \overbrace{(S^\bullet-1^\bullet)\times\cdots\times(S^\bullet-1^\bullet)}^{n\ times} \tag{239}$$

[31]This is essentially a consequence if its definition involving the uninflected product *mu* instead of the inflected *midme*,*mde*.

[32]This difference explains why, whereas *cālt* cannot replace *căl t* or *câlt* as counter-alternator, *sālt* often proves – due to the greater simplicity of its definition – a more convenient alternator than *sâlt* or *sălt* .

We then take advantage of the symmetrality relations to linearize all terms $S^{\boldsymbol{w^1}}...S^{\boldsymbol{w^n}}$ present on the right-hand side of (239). Next, we collect all terms in front of the powers t^d: their sum is garanteed to be d-alternal. This gives us the action of the projector $pr_{:d}$ not just on *symmetral moulds*, but also, due to the universality and uniqueness of the expansion (239), on *all moulds*.[33]

We can now re-interpret and sharpen Proposition 5.1 and 5.2.

Proposition 5.4 (Measuring counter-alternality) .
Let calt denote any of the three counter-alternators. Then, for any mould $B^\bullet$ of the form

$$B^\bullet := \sum_{\sigma\in\mathfrak{S}_r} h_{r,\delta_*}(\sigma)\,\mathrm{calt}_{H^\bullet}(A^\bullet_{\sigma(1)},...,A^\bullet_{\sigma(r)}) \quad \mathit{with} \quad \begin{cases} H^\bullet \in \mathbb{M}_{:\delta_0} \\ A^\bullet_i \in \mathbb{M}_{:\delta_i} \end{cases} \tag{240}$$

the inclusion holds:

$$B^\bullet \in \mathbb{M}_{:\delta} \oplus \mathbb{M}_{:\delta-2} \oplus \mathbb{M}_{:\delta-4} \dots \quad \mathit{with} \quad \delta := \delta_0+\delta_1+...+\delta_r - \delta_* \tag{241}$$

The minus sign in front of δ_* is precisely the algebraic expression of 'counter-alternality'. As for the absence of components $\mathbb{M}_{:\delta-1}$, $\mathbb{M}_{:\delta-3}$ etc, it simply follows from the fact that all elements $M^\bullet$ of $\mathbb{M}_{:d}$ verify $anti M^\bullet \equiv (-1)^{d+1} M^\bullet$.[34]

Here are two elementary but useful lemmas about the stability properties of d-alternality:

Lemma 5.1 *We have the following inclusions:*

$$\begin{aligned}
\mathrm{mu}(\mathbb{M}_{:d_1}, \mathbb{M}_{:d_2}) &\subset \oplus_{0\leqslant\delta}\mathbb{M}_{:d_1+d_2-\delta} \\
\mathrm{lu}(\mathbb{M}_{:d_1}, \mathbb{M}_{:d_2}) &\subset \oplus_{0\leqslant\delta}\mathbb{M}_{:d_1+d_2-2\,\delta} \\
\mathrm{dme}\big(\mathrm{Flex}_{:d_1}(\mathfrak{E}), \mathrm{Flex}_{:d_2}(\mathfrak{E})\big) &\subset \oplus_{0\leqslant\delta}\mathrm{Flex}_{:d_1+d_2-\delta}(\mathfrak{E}) \\
\mathrm{mde}\big(\mathrm{Flex}_{:d_1}(\mathfrak{E}), \mathrm{Flex}_{:d_2}(\mathfrak{E})\big) &\subset \oplus_{0\leqslant\delta}\mathrm{Flex}_{:d_1+d_2-\delta}(\mathfrak{E}) \\
\mathrm{dle}\big(\mathrm{Flex}_{:d_1}(\mathfrak{E}), \mathrm{Flex}_{:d_2}(\mathfrak{E})\big) &\subset \oplus_{0\leqslant\delta}\mathrm{Flex}_{:d_1+d_2-1-\delta}(\mathfrak{E})
\end{aligned}$$

Lemma 5.2 *Let $H^{\boldsymbol{w}} := h(u_1v_1+...+u_rv_r)$ for some function h. Let $r_1+r_2 = r$ and denote by $H^{\boldsymbol{w}}_{r_1,r_2}$ the sum of all terms resulting from shuffling the r_1 first u_i's with the r_2 last u_i's while leaving the v_i's in place. Next, take $d_1 \leqslant r_1$, $d_2 \leqslant r_2$ and let the projectors $\mathrm{pr}_{:d_1}$ and $\mathrm{pr}_{:d_2}$ act respectively on the variables $u_1,...,u_{r_1}$ and $u_{r_1+1},...,u_r$ of $H^{\boldsymbol{w}}_{r_1,r_2}$. Then the result, viewed as a function of the sole v_i's, is in $\mathrm{BIMU}^{:d_1+d_2}$. In other words:*

$$\mathrm{pr}^{:d}_{\boldsymbol{v}}.\mathrm{pr}^{\boldsymbol{u^1}}_{:d_1}.\mathrm{pr}^{\boldsymbol{u^2}}_{:d_2}.\mathrm{sha}^{\boldsymbol{u}}_{r_1,r_2}.H^{\boldsymbol{w}} = \begin{cases} H^{\boldsymbol{w}} & if\;\; d = d_1+d_2 \\ 0 & otherwise \end{cases}$$

[33]For details, see "Combinatorial tidbits" on our homepage.
[34]Recall that *anti* denotes the reversal of the sequence $\bullet$.

Remark 1: If we were to replace $h_{r,\delta_*}(\sigma)$ by $h_{r,\delta_*}(\sigma^{-1})$ in (240), we couldn't draw any definite conclusions about the alternality co-degree of $B^\bullet$. This underscores the (non-obvious) fact that the arguments $A_j^\bullet$ of a counter-alternator must be treated like the indices w_j of a mould rather than like the factors D_j of a product of derivations.

Remark 2: The elementary projectors

$$\mathrm{pr}_{:d}: \quad M^{w_1,\dots,w_r} \mapsto M_{:d}^{w_1,\dots,w_r} = \sum_{\sigma\in\mathfrak{S}_r} h_{r,d}(\sigma)\, M^{w_{\sigma(1)},\dots,w_{\sigma(1)}}$$

of $BIMU_r$ onto $BIMU_{r,d}$ cease to act internally on $Flex_r(\mathfrak{E}^\bullet)$ from $r=4$ onward. For $r=4$, we still have elementary projectors of $Flex_r(\mathfrak{E}^\bullet)$ onto $Flex_{r,d}(\mathfrak{E}^\bullet)$, of the form:

$$\mathrm{pr}^*_{:d}: \quad M^{w_1,\dots,w_r} \mapsto M_{:d}^{w_1,\dots,w_4} = \sum_{\sigma\in\mathfrak{S}_4} h^*_{4,d}(\sigma)\, M^{w_{\sigma(1)},\dots,w_{\sigma(4)}}$$

but with $pr_{:1} \neq pr^*_{:1}$ and $pr_{:3} \neq pr^*_{:3}$. For comparison, here are the coefficients $h_{4,d}(\sigma)$ and $h^*_{4,d}(\sigma)$ in vis-a-vis:

$\sigma(\boldsymbol{w})$	$h_{4,1}$	$h^*_{4,1}$	$h_{4,2}$	$h^*_{4,2}$	$h_{4,3}$	$h^*_{4,3}$	$h_{4,4}$	$h^*_{4,4}$
w_1,w_2,w_3,w_4	6	6	11	11	6	6	1	1
w_1,w_2,w_4,w_3	−2	−6	−1	−1	2	6	1	1
w_1,w_3,w_2,w_4	−2	6	−1	−1	2	−6	1	1
w_1,w_3,w_4,w_2	−2	6	−1	−1	2	−6	1	1
w_1,w_4,w_2,w_3	−2	−6	−1	−1	2	6	1	1
w_1,w_4,w_3,w_2	2	6	−1	−1	−2	−6	1	1
w_2,w_1,w_3,w_4	−2	−6	−1	−1	2	6	1	1
w_2,w_1,w_4,w_3	2	−6	−1	−1	−2	6	1	1
w_2,w_3,w_1,w_4	−2	−6	−1	−1	2	6	1	1
w_2,w_3,w_4,w_1	−2	−6	−1	−1	2	6	1	1
w_2,w_4,w_1,w_3	2	−6	−1	−1	−2	6	1	1
w_2,w_4,w_3,w_1	2	−6	−1	−1	−2	6	1	1
w_3,w_1,w_2,w_4	−2	6	−1	−1	2	−6	1	1
w_3,w_1,w_4,w_2	−2	6	−1	−1	2	−6	1	1
w_3,w_2,w_1,w_4	2	6	−1	−1	−2	−6	1	1
w_3,w_2,w_4,w_1	2	6	−1	−1	−2	−6	1	1
w_3,w_4,w_1,w_2	−2	6	−1	−1	2	−6	1	1
w_3,w_4,w_2,w_1	2	6	−1	−1	−2	−6	1	1
w_4,w_1,w_2,w_3	−2	−6	−1	−1	2	6	1	1
w_4,w_1,w_3,w_2	2	6	−1	−1	−2	−6	1	1
w_4,w_2,w_1,w_3	2	−6	−1	−1	−2	6	1	1
w_4,w_2,w_3,w_1	2	−6	−1	−1	−2	6	1	1
w_4,w_3,w_1,w_2	2	6	−1	−1	−2	−6	1	1
w_4,w_3,w_2,w_1	−6	−6	11	11	−6	−6	1	1

For $r \geqslant 5$ we no longer have projectors of $Flex_r(\mathfrak{E}^\bullet)$ onto $Flex_{r,d}(\mathfrak{E}^\bullet)$ expressible purely via index substitutions. In place of these elementary projectors, we must resort to the *partially symmetrized alternators* of the next subsection.[35]

5.9 Partially symmetrized alternators:

Definition 5.3 (d-alternators.) *For $1 \leqslant d \leqslant r$ we set:*

$$\begin{aligned}
\mathrm{c\hat{a}lt}^{r,d}_{H^\bullet}(A_1^\bullet,...,A_r^\bullet) &:= \sum_{\sigma\in\mathfrak{S}_r} h_{r,1+r-d}(\sigma)\,\mathrm{c\hat{a}lt}_{H^\bullet}(A^\bullet_{\sigma(1)},...,A^\bullet_{\sigma(r)}) && (242)\\
\mathrm{c\check{a}lt}^{r,d}_{H^\bullet}(A_1^\bullet,...,A_r^\bullet) &:= \sum_{\sigma\in\mathfrak{S}_r} h_{r,1+r-d}(\sigma)\,\mathrm{c\check{a}lt}_{H^\bullet}(A^\bullet_{\sigma(1)},...,A^\bullet_{\sigma(r)}) && (243)\\
\mathrm{c\bar{a}lt}^{r,d}_{H^\bullet}(A_1^\bullet,...,A_r^\bullet) &:= \sum_{\sigma\in\mathfrak{S}_r} h_{r,1+r-d}(\sigma)\,\mathrm{c\bar{a}lt}_{H^\bullet}(A^\bullet_{\sigma(1)},...,A^\bullet_{\sigma(r)}) && (244)
\end{aligned}$$

with the coefficients $\mathrm{h}_{r,\delta}(\sigma)$ associated with the projectors $pr_{:d}$: see (237).

Proposition 5.4 tells us, inter alia, that when we insert alternal arguments $H^\bullet$ and $A_i^\bullet$ into our d-alternators, the result is going to be a d-alternal bimould $M^\bullet$. It is therefore tempting to try to rephrase the definition of the d-alternators in a way that would make their signature property manifest. This is indeed possible, due to:

Proposition 5.5 (d-alternality made manifest) .
The d-*alternators are capable of an equivalent definition, of type:*

$$\begin{aligned}
\mathrm{c\hat{a}lt}^{r,d}_{H^\bullet}(A_1^\bullet,...,A_r^\bullet) &\equiv \sum_{\sigma\in\mathfrak{S}_r} h_{r,1+r-d}(\sigma)\,\mathrm{k\hat{a}lt}^{r,d}_{H^\bullet}(A^\bullet_{\sigma(1)},...,A^\bullet_{\sigma(r)}) && (245)\\
\mathrm{c\check{a}lt}^{r,d}_{H^\bullet}(A_1^\bullet,...,A_r^\bullet) &\equiv \sum_{\sigma\in\mathfrak{S}_r} h_{r,1+r-d}(\sigma)\,\mathrm{k\check{a}lt}^{r,d}_{H^\bullet}(A^\bullet_{\sigma(1)},...,A^\bullet_{\sigma(r)}) && (246)\\
\mathrm{c\bar{a}lt}^{r,d}_{H^\bullet}(A_1^\bullet,...,A_r^\bullet) &\equiv \sum_{\sigma\in\mathfrak{S}_r} h_{r,1+r-d}(\sigma)\,\mathrm{k\bar{a}lt}^{r,d}_{H^\bullet}(A^\bullet_{\sigma(1)},...,A^\bullet_{\sigma(r)}) && (247)
\end{aligned}$$

where each $(r+1)$-linear term $\mathrm{kalt}^{r,d}_{H^\bullet}(A^\bullet_{\sigma(1)},...,A^\bullet_{\sigma(r)})$ can be written as a finite sum of elementary summands involving the operations dle, dme, mde *respectively r_0, r_1, r_2 times, with $r_0 = r+1-d$ and $r_1+r_2 = d-1$.*

Unfortunately, there exist scores of equivalent expressions for $kalt^{r,d}_{H^\bullet}(...)$, and so far we failed to come up with a clearly privileged choice, except in the two extreme cases, namely when $d = 1$ or $d = r$. Indeed:

Proposition 5.6 (1-alternality and r-alternality made manifest) .
The 1-*alternators* $\mathrm{calt}^{r,d}_{H^\bullet}$ *coincide with the alternators* $\mathrm{salt}^{r,d}_{H^\bullet}$ *of Definition 5.2, and for these the formulae (227)-(229) clearly amount to expansions of type*

[35] usually with the index $H^\bullet := \mathfrak{E}^\bullet$.

(245)-(247). As for the r-alternators, they too admit expansions of such type, to wit:

$$\text{kâlt}^{r,r}_{H^\bullet}(A^\bullet_1,..,A^\bullet_r) := \sum_{1\leqslant s\leqslant r}(-1)^{s-1}\overrightarrow{\text{dme}}\Big(\text{dle}\big(H^\bullet,\overleftarrow{\text{dme}}(\boldsymbol{A^1})\big),\overleftarrow{\text{dme}}(\boldsymbol{A^2}),...,\overleftarrow{\text{dme}}(\boldsymbol{A^s})\Big)$$

$$\text{kălt}^{r,r}_{H^\bullet}(A^\bullet_1,..,A^\bullet_r) := \sum_{1\leqslant s\leqslant r}(-1)^{s-1}\overrightarrow{\text{dme}}\Big(\text{dle}\big(H^\bullet,\overrightarrow{\text{mde}}(\boldsymbol{A^1})\big),\overrightarrow{\text{mde}}(\boldsymbol{A^2}),...,\overrightarrow{\text{mde}}(\boldsymbol{A^s})\Big)$$

$$\text{kālt}^{r,r}_{H^\bullet}(A^\bullet_1,..,A^\bullet_r) := \sum_{1\leqslant s\leqslant r}(-1)^{s-1}\text{dme}\Big(\text{dle}\big(H^\bullet,\text{mu}(A^\bullet_1,..,A^\bullet_s)\big),\text{mu}(A^\bullet_{s+1},..,A^\bullet_r)\Big)$$

5.10 Complement: dimensions of the component spaces.

Let us illustrate the dual gradations by degree and codegree, while focusing on the former, for greater ease of notations. Let $\mathbb{E}$ be the associative algebra freely generated by non-commutative variables $e_1, e_2...$ and for any sequence $\boldsymbol{s} = (s_1,...,s_n)$ let $\mathbb{E}^{\boldsymbol{s}}$ be the subspace of $\mathbb{E}$ spanned by elements of degree s_1 in e_1, s_2 in e_2 etc. $\mathbb{E}^{\boldsymbol{s}}$ admits a natural decomposition $\mathbb{E} = \oplus_{1\leqslant\delta\leqslant|\boldsymbol{s}|}\mathbb{E}^{\boldsymbol{s}}_\delta$ spanned by elements of 'differential degree' δ [36]. $\mathbb{E}^{\boldsymbol{s}}_1$ is simply the subspace of Lie elements, with dimension

$$\dim(\mathbb{E}^{\boldsymbol{s}}_1) = \frac{1}{s}\sum_{d|s_i}\mu(d)\frac{(s/d)!}{(s_1/d)!...(s_n/d)!} \qquad (s := |\boldsymbol{s}|, \mu = \textit{mobius function})$$

For $\delta \geqslant 2$, $\mathbb{E}^{\boldsymbol{s}}_\delta$ is defined as follows.We consider all δ-partitions $\mathcal{S}_\delta$ of $\boldsymbol{s}$:

$$\mathcal{S}_\delta \quad : \quad \boldsymbol{s} = \boldsymbol{s^1} + \cdots + \boldsymbol{s^\delta} \quad \textit{with} \quad \boldsymbol{s^i} = (s^i_1,...,s^i_n) \quad \textit{and} \quad s^i_k \geqslant 0 \ \ \textit{but} \ \ |\boldsymbol{s^i}| \geqslant 1$$

We then define $\mathbb{E}^{\boldsymbol{s}}_\delta$ as the sum, for all δ-partitions of $\boldsymbol{s}$, of the symmetrized products of the Lie spaces $\mathbb{E}^{\boldsymbol{s^i}}_1$:

$$\mathbb{E}^{\boldsymbol{s}}_\delta := \oplus_{\mathcal{S}_\delta}\,\textit{symmetrize}(\mathbb{E}^{\boldsymbol{s^1}}_1\mathbb{E}^{\boldsymbol{s^2}}_1\ldots\mathbb{E}^{\boldsymbol{s^\delta}}_1) \tag{248}$$

Here, $\textit{symmetrize}(\mathbb{E}^{\boldsymbol{s^1}}_1\mathbb{E}^{\boldsymbol{s^2}}_1\ldots\mathbb{E}^{\boldsymbol{s^\delta}}_1)$ obviously denotes the space spanned by all the symmetrized products of Lie elements $\boldsymbol{e^1}, \boldsymbol{e^2}, ..., \boldsymbol{e^\delta}$, with each $\boldsymbol{e^i}$ running through some basis of $\mathbb{E}^{\boldsymbol{s^i}}_1$. The resulting space $\mathbb{E}^{\boldsymbol{s}}_\delta$ does not depend on the choice of those bases. Moreover:

$$\dim(\mathbb{E}^s_\delta) = \sum_{\mathcal{S}_\delta}\dim(\mathbb{E}^{\boldsymbol{s^i}}_1)\ldots\dim(\mathbb{E}^{\boldsymbol{s^\delta}}_1) \quad ; \quad \sum_\delta \dim(\mathbb{E}^s_\delta) = \dim(\mathbb{E}^s) = \frac{s!}{s_1!\ldots s_n!}$$

In the case of distinct generators, i.e. for $\boldsymbol{s} = (1,...,1)$ (r *times*), we get:

$$\sum_{1\leqslant d\leqslant r}\dim(\mathbb{E}^{1,...,1}_d\, x^d = x\,(x+1)\ldots(x+d-1)$$

[36]

To illustrate the general case, let us consider the sequence $\boldsymbol{s} = (1,1,2)$. We then get the decomposition $\mathbb{E}^{\boldsymbol{s}} = \oplus_\delta \mathbb{E}_1^{\boldsymbol{s}}$ with

$$\dim(\mathbb{E}_1^{\boldsymbol{s}}) = 3 \;\;,\;\; \dim(\mathbb{E}_2^{\boldsymbol{s}}) = 5 \;\;,\;\; \dim(\mathbb{E}_3^{\boldsymbol{s}}) = 3 \;\;,\;\; \dim(\mathbb{E}_1^{\boldsymbol{s}}) = 1$$

and here is a possible choice of basis elements $\boldsymbol{\epsilon}_{\delta,k}$:

$$basis(\mathbb{E}_1^{\boldsymbol{s}}) \;:\; \begin{cases} \boldsymbol{\epsilon}_{1,1} = [[[e_1,e_3],e_3],e_2], \\ \boldsymbol{\epsilon}_{1,2} = [[e_1,e_3],[e_2,e_3]], \\ \boldsymbol{\epsilon}_{1,3} = [e_1,[[e_2,e_3],e_3]] \end{cases}$$

$$basis(\mathbb{E}_2^{\boldsymbol{s}}) \;:\; \begin{cases} \boldsymbol{\epsilon}_{2,1} = [[e_3,e_1],e_2].e_3 + e_3.[[e_3,e_1],e_2], \\ \boldsymbol{\epsilon}_{2,2} = [[e_3,e_2],e_1].e_3 + e_3.[[e_3,e_2],e_1], \\ \boldsymbol{\epsilon}_{2,3} = [[e_1,e_3],e_3].e_2 + e_2.[[e_1,e_3],e_3], \\ \boldsymbol{\epsilon}_{2,4} = [[e_2,e_3],e_3].e_1 + e_1.[[e_2,e_3],e_3], \\ \boldsymbol{\epsilon}_{2,5} = [e_1,e_3].[e_2,e_3] + [e_2,e_3].[e_1,e_3] \end{cases}$$

$$basis(\mathbb{E}_3^{\boldsymbol{s}}) \;:\; \begin{cases} \boldsymbol{\epsilon}_{3,1} = \quad [e_1,e_2].e_3.e_3 + e_3.[e_1,e_2].e_3 + e_3.e_3.[e_1,e_2], \\ \boldsymbol{\epsilon}_{3,2} = \begin{cases} \;\;[e_1,e_3].e_2.e_3 + e_2.[e_1,e_3].e_3 + e_2.e_3.[e_1,e_3] \\ +[e_1,e_3].e_3.e_2 + e_3.[e_1,e_3].e_2 + e_3.e_2.[e_1,e_3], \end{cases} \\ \boldsymbol{\epsilon}_{3,3} = \begin{cases} \;\;[e_2,e_3].e_1.e_3 + e_1.[e_2,e_3].e_3 + e_1.e_3.[e_2,e_3] \\ +[e_2,e_3].e_3.e_1 + e_3.[e_2,e_3].e_1 + e_3.e_1.[e_2,e_3] \end{cases} \end{cases}$$

$$basis(\mathbb{E}_4^{\boldsymbol{s}}) \;:\; \boldsymbol{\epsilon}_{4,1} = \begin{cases} \;\;e_1.e_2.e_3.e_3 + e_1.e_3.e_2.e_3 + e_1.e_3.e_3.e_2 + e_2.e_1.e_3.e_3 \\ +e_2.e_3.e_1.e_3 + e_2.e_3.e_3.e_1 + e_3.e_1.e_2.e_3 + e_3.e_1.e_3.e_2 \\ +e_3.e_2.e_1.e_3 + e_3.e_2.e_3.e_1 + e_3.e_3.e_1.e_2 + e_3.e_3.e_2.e_1 \end{cases}$$

The bottomline is this:
(i) choosing a basis for all $\mathbb{E}_d^{\boldsymbol{s}}$ entirely reduces to choosing a basis for all $\mathbb{E}_1^{\boldsymbol{s}}$
(ii) but picking a basis for any $\mathbb{E}_1^{\boldsymbol{s}}$ necessarily involves some arbitrariness.

6 Tree-indexed bases: binary, ordered, stacked.

6.1 Binary and ordered trees.

Let $\mathbb{BT}_r$ denote the set of binary trees with r nodes and *one* root, and let $\mathbb{OT}_r$ be the set of all ordered trees with r nodes and *one* or *more than one* root. It is well-known that both have the same number of elements:

$$\#(\mathbb{BT}_r) = \#(\mathbb{OT}_r) = \kappa_r := \frac{2\,r!}{r!\,(r+1)!} \tag{249}$$

while the subset of $\mathbb{OT}_r$ consisting of one-rooted trees has κ_{r-1} elements.

For convenient enumeration, we introduce on the elements $bt_{r,k}$ of $\mathbb{BT}_r$ and

$ot_{r,k}$ of $\mathbb{OT}_r$ various orderings[37] that reflect the trees' inductive construction:

$$\mathrm{bt}_{r,k} \ = \ \mathfrak{h}(\mathrm{bt}_{r_1,k_1},\mathrm{bt}_{r_2,k_2}) \quad \textit{with} \qquad \begin{cases} r_1+r_2=r-1 \\ 1\leqslant k_i \leqslant \kappa_{r_i} \end{cases} \tag{250}$$

$$\mathrm{ot}_{r,k} \ = \ \begin{cases} \textit{either} \quad \mathfrak{h}(\mathrm{ot}_{r_1,k_1},...,\mathrm{ot}_{r_s,k_s}) \quad \textit{with} \ \begin{cases} r_1+\dots+r_s=r-1 \\ 1\leqslant s \ \ , \ \ 1\leqslant k_i\leqslant \kappa_{r_i-1} \end{cases} \\ \\ \quad \textit{or} \qquad \mathfrak{j}(\mathrm{ot}_{r_1,k_1},...,\mathrm{ot}_{r_s,k_s}) \quad \textit{with} \ \begin{cases} r_1+\dots+r_s=r \\ 2\leqslant s \ \ , \ \ 1\leqslant k_i\leqslant \kappa_{r_i-1} \end{cases} \end{cases} \tag{251}$$

The gothic $\mathfrak{h}$ (for *heave*) in (250) signals that bt_{r_1,k_1} and bt_{r_2,k_2} get attached to a root as left- and right-leaning branches, to produce a new binary tree. That same $\mathfrak{h}$ in (251) means that the various bt_{r_i,k_i} get attached to a root, as separate branches ordered from left to right, to produce a new, one-rooted ordered tree. Lastly, the gothic $\mathfrak{j}$ (for *juxtapose*) means that the bt_{r_i,k_i} simply get juxtaposed, resulting in an s-rooted, ordered tree.

Although we impose $1 \leqslant k_i \leqslant \kappa_{r_i}$ in (250) and $1 \leqslant k_i \leqslant \kappa_{r_i-1}$ in (**??**), the effect is the same: it allows only one-rooted trees inside $\mathfrak{h}$ or $\mathfrak{j}$.

We can now return to the task of tree indexation. A quick, if artificial-looking way of going about this is by associating with each tree a function germ at $+\infty$. Here are the most useful choices, with the monomials given in order of decreasing dominance:

$$\mathrm{bt}_{r,k} \mapsto \mathrm{gbt}_{r,k}(x) \ := \ -\big(\frac{r_1}{x}+\frac{r_2}{x^2}\big)+e^{-x}\,\big(\frac{k_1}{x}+\frac{k_2}{x^2}\big) \tag{252}$$

$$\mathrm{bt}_{r,k} \mapsto \mathrm{gbt}_{r,k}(x) \ := \ -\big(\frac{r_1}{x}+\frac{r_2}{x^2}\big)+e^{-x}\,\big(k_2\,x^2+k_1x\big) \tag{253}$$

$$\mathrm{ot}_{r,k} \mapsto \mathrm{got}_{r,k}(x) \ := \ \epsilon\,e^{x}-\big(\frac{r_1}{x}+...+\frac{r_s}{x^s}\big)+e^{-x}\,\big(\frac{k_1}{x}+...+\frac{k_s}{x^s}\big) \tag{254}$$

$$\mathrm{ot}_{r,k} \mapsto \mathrm{got}_{r,k}(x) \ := \ \epsilon\,e^{x}-\big(\frac{r_1}{x}+...+\frac{r_s}{x^s}\big)+e^{-x}\,\big(k_s\,x^s+...+k_1x\big) \tag{255}$$

$$\mathrm{ot}_{r,k} \mapsto \mathrm{got}_{r,k}(x) \ := \ \epsilon\,e^{x}+x^s-\big(\frac{r_1}{x}+...+\frac{r_s}{x^s}\big)+e^{-x}\,\big(\frac{k_1}{x}+...+\frac{k_s}{x^s}\big) \tag{256}$$

$$\mathrm{ot}_{r,k} \mapsto \mathrm{got}_{r,k}(x) \ := \ \epsilon\,e^{x}+x^s-\big(\frac{r_1}{x}+...+\frac{r_s}{x^s}\big)+e^{-x}\,\big(k_s\,x^s+...+k_1x\big) \tag{257}$$

$$\textit{with} \quad \epsilon \ = \ \begin{cases} 0 \quad \textit{if} \quad ot_{r,k} \ \textit{is one-rooted}, \ \textit{i.e. of the form} \ \mathfrak{h}(\dots) \\ 1 \quad \textit{if} \quad ot_{r,k} \ \textit{is many-rooted}, \ \textit{i.e. of the form} \ \mathfrak{j}(\dots) \end{cases}$$

We can then endow our trees with the k-indexation ($1 \leqslant k \leqslant \kappa_r$) that mirrors the germs' behaviour at infinity:

$$\textit{on} \ \ \mathbb{BT}_r: \quad \{k' < k''\} \Longleftrightarrow \mathrm{gbt}_{r,k'}(x) < \mathrm{gbt}_{r,k''}(x) \ \textit{for} \ 1 << x \tag{258}$$

$$\textit{on} \ \ \mathbb{OT}_r: \quad \{k' < k''\} \Longleftrightarrow \mathrm{got}_{r,k'}(x) < \mathrm{got}_{r,k''}(x) \ \textit{for} \ 1 << x \tag{259}$$

Different applications require different orderings. Whenever necessary, we shall specify which ordering we are working with.

[37] For clarity, let us speak of *orderings* when comparing tres, and of *orders* when comparing edges issuing from the same node in a given tree.

6.2 Binary trees for basis indexation.

To any basis $\{bt_{r,k}, 1 \leqslant k \leqslant \kappa_r\}$ of $\mathbb{BT}_r$ we associate a basis $\{\mathfrak{bte}^\bullet_{r,k}, 1 \leqslant k \leqslant \kappa_r\}$ of $Flex_r(\mathfrak{E}^\bullet)$ by means of inductions that run parallel to **??**:

$$\mathfrak{bte}^\bullet_{1,1} = \mathfrak{E}^\bullet \quad ; \quad \mathfrak{bte}^\bullet_{r,k} = \mathrm{mdme}(\mathfrak{bte}^\bullet_{r_1,k_1}, \mathfrak{E}^\bullet, \mathfrak{bte}^\bullet_{r_2,k_2}) \tag{260}$$

with the familiar semi-inflected ternary operation:

$$\mathrm{mdme}\,(A^\bullet, B^\bullet, C^\bullet) \;:=\; \mathrm{de}^{-1}.\mathrm{mu}(A^\bullet, \mathrm{de}.B^\bullet, C^\bullet) \tag{261}$$

We denote by $\{bta^\bullet_{r,k}, 1 \leqslant k \leqslant \kappa_r\}$ and $\{bti^\bullet_{r,k}, 1 \leqslant k \leqslant \kappa_r\}$ the corresponding bases of $Flex_r(Pa^\bullet)$ and $Flex_r(Pi^\bullet)$.

6.3 Ordered trees for basis indexation.

To any basis $\{ot_{r,k}, 1 \leqslant k \leqslant \kappa_r\}$ of $\mathbb{OT}_r$ we associate three variously accentuated sets $\{\mathfrak{ote}^\bullet_{r,k}, 1 \leqslant k \leqslant \kappa_r\}$ of $Flex_r(\mathfrak{E}^\bullet)$ by inductions that run parallel to (251). Only the first two, $\{\hat{\mathfrak{o}}\mathfrak{te}^\bullet_{r,k}, 1 \leqslant k \leqslant \kappa_r\}$ and $\{\check{\mathfrak{o}}\mathfrak{te}^\bullet_{r,k}, 1 \leqslant k \leqslant \kappa_r\}$, will turn out to be proper bases of $Flex_r(\mathfrak{E}^\bullet)$.

$$\hat{\mathfrak{o}}\mathfrak{te}^\bullet_{1,1} = \mathfrak{E}^\bullet \quad , \quad \hat{\mathfrak{o}}\mathfrak{te}^\bullet_{r,k} = \begin{cases} \mathrm{c\hat{a}lt}_{\mathfrak{E}^\bullet}(\hat{\mathfrak{o}}\mathfrak{te}^\bullet_{r_1,k_1}, ..., \hat{\mathfrak{o}}\mathfrak{te}^\bullet_{r_s,k_s}) & \text{if} \;\; k \leqslant \kappa_{r-1} \\ \overleftarrow{\mathrm{dme}}\,(\hat{\mathfrak{o}}\mathfrak{te}^\bullet_{r_1,k_1}, ..., \hat{\mathfrak{o}}\mathfrak{te}^\bullet_{r_s,k_s}) & \text{if} \;\; k > \kappa_{r-1} \end{cases} \tag{262}$$

$$\check{\mathfrak{o}}\mathfrak{te}^\bullet_{1,1} = \mathfrak{E}^\bullet \quad , \quad \check{\mathfrak{o}}\mathfrak{te}^\bullet_{r,k} = \begin{cases} \mathrm{c\check{a}lt}_{\mathfrak{E}^\bullet}(\check{\mathfrak{o}}\mathfrak{te}^\bullet_{r_1,k_1}, ..., \check{\mathfrak{o}}\mathfrak{te}^\bullet_{r_s,k_s}) & \text{if} \;\; k \leqslant \kappa_{r-1} \\ \overleftarrow{\mathrm{dme}}\,(\check{\mathfrak{o}}\mathfrak{te}^\bullet_{r_1,k_1}, ..., \check{\mathfrak{o}}\mathfrak{te}^\bullet_{r_s,k_s}) & \text{if} \;\; k > \kappa_{r-1} \end{cases} \tag{263}$$

$$\bar{\mathfrak{o}}\mathfrak{te}^\bullet_{1,1} = \mathfrak{E}^\bullet \quad , \quad \bar{\mathfrak{o}}\mathfrak{te}^\bullet_{r,k} = \begin{cases} \mathrm{c\bar{a}lt}_{\mathfrak{E}^\bullet}(\bar{\mathfrak{o}}\mathfrak{te}^\bullet_{r_1,k_1}, ..., \bar{\mathfrak{o}}\mathfrak{te}^\bullet_{r_s,k_s}) & \text{if} \;\; k \leqslant \kappa_{r-1} \\ \overleftarrow{\mathrm{dme}}\,(\bar{\mathfrak{o}}\mathfrak{te}^\bullet_{r_1,k_1}, ..., \bar{\mathfrak{o}}\mathfrak{te}^\bullet_{r_s,k_s}) & \text{if} \;\; k > \kappa_{r-1} \end{cases} \tag{264}$$

Of course, when dealing with $Flex_r(Pa^\bullet)$ or $Flex_r(Pi^\bullet)$, we revert from gothic to roman, and change $\mathfrak{e}$ to a or i.

Remark: In view of the systematic exchange $\overleftarrow{dme} \leftrightarrow \overrightarrow{mde}$ in the definitions of the first two alternators (see §5.5), one might expect $\overleftarrow{dme}$ and $\overrightarrow{mde}$ on the lower right-hand sides of (262) and (262) respectively — or the reverse! In fact, these choices would work fine, and still lead to proper bases, but with them the formulae for basis change would be slightly more awkward.

6.4 Stacked trees for basis indexation.

Stacked trees aren't trees proper, but rather linear combinations of ordered trees subject to definite alternation conditions on their branches. Before defining them, we require an auxiliary construction. Let $\mathbb{E}$ be the associative algebra freely generated by non-commutative variables $e_1, e_2 ...$, as in §5.10, and for any sequence $\boldsymbol{s} = (s_1, ..., s_n)$ let $\mathbb{E}^{\boldsymbol{s}}$ be the subspace of $\mathbb{E}$ spanned by elements of degree s_1 in e_1, s_2 in e_2 etc. $\mathbb{E}^{\boldsymbol{s}}$ admits a natural decomposition $\mathbb{E} = \oplus_{1 \leqslant \delta \leqslant |\boldsymbol{s}|} \mathbb{E}^{\boldsymbol{s}}_\delta$ spanned by elements of 'differential degree' δ.

Construct, as in §5.10, a basis of $\mathbb{E}_\delta^{\boldsymbol{s}}$, consisting of elements $\boldsymbol{\epsilon}_{\delta,i}$

$$\boldsymbol{\epsilon}_{\delta,i} = \sum_\sigma n_{i,\sigma}\, e_{\sigma(1)}.e_{\sigma(2)} \dots e_{\sigma(s)} \quad \textit{with} \quad \begin{cases} n_{i,\sigma} \in \mathbb{Z} \\ \sigma : \{1,...,s\} \to \{1,...,n\} \end{cases} \tag{265}$$

and consider the dual expansion in $\mathbb{M}$:

$$\boldsymbol{m}_{\delta,i} = \sum_\sigma n^*_{i,\sigma}\, m_{\sigma(1)}.m_{\sigma(2)} \dots m_{\sigma(s)} \quad \textit{with} \quad \begin{cases} n^*_{i,\sigma} \in \mathbb{Z} \\ \sigma : \{1,...,s\} \to \{1,...,n\} \end{cases} \tag{266}$$

We are now in a position to inductively construct a complete system $\mathbb{ST}_r$ of stacked trees and, parallet with it, convenient bases of $Flex_r(\mathfrak{E}^\bullet)$. However, we must now sharply distinguish the *one-rooted* stacked trees, to which there will correspond *dme*-generators of the $Flex(\mathfrak{E}^\bullet)$, and the *many-rooted* stacked trees, to which there will correspond *dme*-products of generators, which together shall provided a *graded* basis of $Flex(\mathfrak{E}^\bullet)$.

One-rooted stacked trees with r nodes and co-degree d are inductively defined as the superposition: [38]

$$\mathrm{st}_{r,d,k} = \sum_\sigma n^*_{i,\sigma}\, \mathfrak{h}(\mathrm{st}_{r_{\sigma(1)},d_{\sigma(1)},k_{\sigma(1)}}, ..., \mathrm{st}_{r_{\sigma(s)},d_{\sigma(s)},k_{\sigma(s)}}) \tag{267}$$

$$\textit{with} = r = 1 + r_1 + \dots + r_s \qquad \textit{and} \qquad d = 1 - d_0 + d_1 + ... + d_s \tag{268}$$

We have here the same coefficients $n^*_{i,\sigma}$ as in (266), and $(\mathrm{st}_{r_1,d_1,k_1}, ..., \mathrm{st}_{r_s,d_s,k_s})$ is still a sequence of multiplicity type, i.e. with a first element repeated s_1 times, a second element s_2 times, etc. The corresponding elements of $Flex(\mathfrak{E}^\bullet)$ are defined by:

$$\hat{\mathfrak{ste}}^\bullet_{r,d,k} = \sum_\sigma n^*_{i,\sigma}\, \hat{\mathrm{calt}}_{\mathfrak{E}^\bullet}(\hat{\mathfrak{ste}}^\bullet_{r_{\sigma(1)},d_{\sigma(1)},k_{\sigma(1)}}, ..., \hat{\mathfrak{ste}}^\bullet_{r_{\sigma(s)},d_{\sigma(s)},k_{\sigma(s)}}) \tag{269}$$

$$\check{\mathfrak{ste}}^\bullet_{r,d,k} = \sum_\sigma n^*_{i,\sigma}\, \check{\mathrm{calt}}_{\mathfrak{E}^\bullet}(\check{\mathfrak{ste}}^\bullet_{r_{\sigma(1)},d_{\sigma(1)},k_{\sigma(1)}}, ..., \check{\mathfrak{ste}}^\bullet_{r_{\sigma(s)},d_{\sigma(s)},k_{\sigma(s)}}) \tag{270}$$

$$\bar{\mathfrak{ste}}^\bullet_{r,d,k} = \sum_\sigma n^*_{i,\sigma}\, \bar{\mathrm{calt}}_{\mathfrak{E}^\bullet}(\bar{\mathfrak{ste}}^\bullet_{r_{\sigma(1)},d_{\sigma(1)},k_{\sigma(1)}}, ..., \bar{\mathfrak{ste}}^\bullet_{r_{\sigma(s)},d_{\sigma(s)},k_{\sigma(s)}}) \tag{271}$$

The main point is of course the occurence of d_0 with a *minus* sign in the second sum (268). That minus sign reflects the main property of the counter-alternators: the *more* alternating they are as functions of their arguments $(A_1^\bullet, ..., A_s^\bullet)$, the *less* alternating they are as bimoulds, i.e. as functions of $\bullet$. In fact, the indexation by stacked trees is what shall enable us, in the next section (§8) to decompose $Flex_r(\mathfrak{E}^\bullet)$ as a sum $Flex_{r,d}(\mathfrak{E}^\bullet)$ of subspaces with elements of co-degree d and to calculate their dimensions $\dim(Flex_{r,d}(\mathfrak{E}^\bullet))$ by means of interesting generating series.

[38] since the st_{r_i,d_i,k_i} inside $\mathfrak{h}$ are tree superpositions ('stacked trees'), $\mathfrak{h}$ must be viewed as a multilinear function of its arguments.

But before proceeding, we must dispose of the case of *many-rooted* stacked trees. Here, the definition is more straightforward: [39]

$$\mathrm{st}_{r,d,k} = \mathfrak{j}(\mathrm{st}_{r_1,d_1,k_1},\ldots,\mathrm{st}_{r_s,d_s,k_s}) \quad \textit{with} \quad \begin{cases} s \geqslant 2 \\ r = r_1 + \cdots + r_s \\ d = d_1 + \cdots + d_s \end{cases} \tag{272}$$

and gives rise to the parallel construction on $Flex(\mathfrak{E}^\bullet)$:

$$\begin{cases} \hat{\mathfrak{ste}}^\bullet_{r,d,k} = \overleftarrow{\mathrm{dme}}\,(\hat{\mathfrak{ste}}^\bullet_{r_1,d_1,k_1},\ldots,\hat{\mathfrak{ste}}^\bullet_{r_s,d_s,k_s}) \\ \check{\mathfrak{ste}}^\bullet_{r,d,k} = \overleftarrow{\mathrm{dme}}\,(\check{\mathfrak{ste}}^\bullet_{r_1,d_1,k_1},\ldots,\check{\mathfrak{ste}}^\bullet_{r_s,d_s,k_s}) \\ \bar{\mathfrak{ste}}^\bullet_{r,d,k} = \overleftarrow{\mathrm{dme}}\,(\bar{\mathfrak{ste}}^\bullet_{r_1,d_1,k_1},\ldots,\bar{\mathfrak{ste}}^\bullet_{r_s,d_s,k_s}) \end{cases} \quad \textit{with} \quad \begin{cases} s \geqslant 2 \\ r = r_1 + \ldots + r_s \\ d = d_1 + \ldots + d_s \end{cases} \tag{273}$$

6.5 Ordered trees and the pre-Lie products.

Anticipating on the systematic investigation of the bases $\{\hat{\mathfrak{ote}}^\bullet_{r,k}\}$ and $\{\check{\mathfrak{ote}}^\bullet_{r,k}\}$ in §8, let us mention here two statements that justify after the event their seemingly artificial construction. For simplicity, we limit ourselves to $\{\hat{\mathfrak{ote}}^\bullet_{r,k}\}$ and look at the action of the alternality-preserving pre-Lie brackets *dle* and *dari* as defined in §5.5.

Proposition 6.1 (The pre-Lie product ***dle*** **as tree attachment)** .
For any pair of rooted, ordered trees $\mathrm{ot}_{r_1,k_1}, \mathrm{ot}_{r_2,k_2}$*, the identity holds:*

$$\mathrm{dle}(\hat{\mathfrak{ote}}^\bullet_{r_1,k_1},\hat{\mathfrak{ote}}^\bullet_{r_2,k_2}) = -\sum_{1\leqslant k\leqslant \kappa_{r-1}} l_{r,k}\;\hat{\mathfrak{ote}}^\bullet_{r,k} \quad \textit{with} \quad \begin{cases} 1 \leqslant k_i \leqslant \kappa_{r_i-1} \\ l_{r,k} \in \mathbb{N} \end{cases} \tag{274}$$

The ordered trees $\mathrm{ot}_{r,k}$ *on the right-hand side are exactly the trees (counted with their multiplicities) that can be obtained by attaching the tree* ot_{r_2,k_2} *to the tree* ot_{r_1,k_1}*, i.e. by attaching* ot_{r_2,k_2} *successively to all the nodes of* ot_{r_1,k_1}*.*

Proposition 6.2 (The pre-Lie product ***dari*** **as tree insertion)** .
For any pair of rooted, ordered trees $\mathrm{ot}_{r_1,k_1}, \mathrm{ot}_{r_2,k_2}$*, the identity holds:*

$$\mathrm{dari}(\hat{\mathfrak{ote}}^\bullet_{r_1,k_1},\hat{\mathfrak{ote}}^\bullet_{r_2,k_2}) = +\sum_{1\leqslant k\leqslant \kappa_{r-1}} d_{r,k}\;\hat{\mathfrak{ote}}^\bullet_{r,k} \;\textit{with}\; \begin{cases} 1 \leqslant k_i \leqslant \kappa_{r_i-1} \\ d_{r,k} \in \mathbb{N} \end{cases} \tag{275}$$

The ordered trees $\mathrm{ot}_{r,k}$ *on the right-hand side are exactly the trees (counted with their multiplicities) that can be obtained by inserting the tree* ot_{r_2,k_2} *into the tree* ot_{r_1,k_1}*, i.e. by inserting some rooted branch* $\mathrm{branch}(b_*)$ *of* ot_{r_2,k_2} *into some edge* (a_1,a_2) *of* ot_{r_1,k_1}*.*

[39] once again, since the st_{r_i,d_i,k_i} inside $\mathfrak{j}$ are tree superpositions ('stacked trees'), $\mathfrak{j}$ must be viewed as a multilinear function of its arguments.

Explanation: An edge (x_1, x_2) of a tree ot links two consecutive nodes. A branch $branch(x_*)$ of ot consists of all nodes x anterior to x_*. It therefore contains the root x_0 of ot. Note that x_* neednot be an end-point of the tree. It may even be the root of the tree, in which case $branch(x_0) = \{x_0\}$.

Remark 1: Since in Proposition 6.2 $a_1 \neq a_2$, it follows that no tree can be inserted into a tree that reduces to a root. Therefore:

$$\mathrm{dari}(\hat{\mathfrak{ote}}^\bullet_{1,1}, \hat{\mathfrak{ote}}^\bullet_{r_2,k_2}) \equiv 0 \tag{276}$$

On the other hand, since a rooted branch may reduce to the root, it follows that:

$$\mathrm{dari}(\hat{\mathfrak{ote}}^\bullet_{r_1,k_1}, \hat{\mathfrak{ote}}^\bullet_{1,1}) \neq 0 \quad \textit{iff} \quad r_1 > 1 \tag{277}$$

This is in sharp contrast to Proposition 6.1, where the right-hand side of (274) never vanishes.

In fact, as a *dle* pre-Lie algebra, $Flex^{al}(\mathfrak{E}^\bullet)$ is freely generated by the single element $\mathfrak{E}^\bullet_{al}$, whereas as a *dari* pre-Lie algebra, all non-trivial identities[40] in $Flex(\mathfrak{E}^\bullet)$ are generated by the identities (276).

Remark 2: Propositions 6.1 and 6.2 show that the subspace $Flex_{root}(\mathfrak{E}^\bullet)$ spanned by the basis elements $\hat{\mathfrak{ote}}^\bullet_{r,k}$ corresponding to rooted trees $\hat{\mathfrak{ote}}_{r,k}$ $(k \leqslant \kappa_{r-1})$, is stable under the pre-Lie products *dle* and *dari*, and therefore under the Lie brackets *lu* and *ari*. This is no longer the case with the basis $\{\check{\mathfrak{ote}}^\bullet_{r,k}\}$ or the system $\{\bar{\mathfrak{ote}}^\bullet_{r,k}\}$(not a basis).

Remark 3: There is no way of modifying our definitions so as to get the same signs in front of $\sum$, on the right-hand sides of both (274) and (275). Here, we chose to have the plus sign for *dari* and the minus sign for *dle* because, on balance, it simplifies a larger number of formulae (even though it clashes with the convention usually adopted when defining the standard pre-Lie product on non-ordered trees).

Remark 4: Assume we have already constructed a system $\{\hat{\mathfrak{ote}}^\bullet_{r,k}\}$ that verifies (274) for all $r < r_0$. Then the same identities (274) fully determine – in fact, overdetermine – the system $\{\hat{\mathfrak{ote}}^\bullet_{r,k}\}$ for $r = r_0$. In that sense, there is no latitude in the choice of system. Actually, solving (274) is how the system $\{\hat{\mathfrak{ote}}^\bullet_{r,k}\}$ was found and how, once found, it led to the counter-alternator *câlt*. The other counter-alternators followed by analogy.

6.6 From *câlt* to *Câlt* : an illusory, yet useful extension.

We are now going to construct counter-alternators *Câlt* indexed by ordered trees $ot_{r,k}$, or simply by (r, k). These capitalised $Câlt_{r,k}(A^\bullet_1, ..., A^\bullet_r)$

[40] i.e. all identities that do not result from the universal pre-Lie identities

$$\mathrm{dari}(\mathrm{dari}(A^\bullet, B^\bullet), C^\bullet) - \mathrm{dari}(A^\bullet, \mathrm{dari}(B^\bullet, C^\bullet)) = \mathrm{dari}(\mathrm{dari}(A^\bullet, C^\bullet), B^\bullet) - \mathrm{dari}(A^\bullet, \mathrm{dari}(C^\bullet, B^\bullet))$$

- appear to be more general than the ordinary $\hat{calt}_{H^\bullet}(A_1^\bullet, ..., A_r^\bullet)$
- unexpectedly turn out to reduce to linear combinations of these ordinary counter-alternators
- nonetheless prove very useful to relate the *dme* and *mde* generation
- have the merit of highlighting the advantages of *câlt* over *călt* and *cālt*.

Definition 6.1 (The tree-indexed counter-alternators $\mathrm{Câlt}_{r,k}(\boldsymbol{A})$) .
For any sequence $\boldsymbol{A} := (A_1^\bullet, ..., A_r^\bullet)$ *with* $A_i^\bullet \in Flex(\mathfrak{E}^\bullet)$ *we set:*

$$\begin{aligned}
\textbf{\textit{one-rooted case}} &\ : \ 1 \leqslant k \leqslant \kappa_{r-1} \textit{ and } \mathrm{ot}_{r,k} = \mathfrak{h}(\mathrm{ot}_{r_1,k_1}, \ldots, \mathrm{ot}_{r_s,k_s}) \\
\mathrm{Câlt}_{r,k}(\boldsymbol{A}) &:= \mathrm{câlt}_{A_1^\bullet}(\mathrm{Câlt}_{r_1,k_1}(\boldsymbol{A^1}), ..., \mathrm{Câlt}_{r_s,k_s}(\boldsymbol{A^s})) \qquad (278)\\
\textit{with} &\quad \begin{cases} 1 \leqslant k_i \leqslant \kappa_{r_i-1} & \sum r_i = r \\ \boldsymbol{A^1}...\boldsymbol{A^s} = (A_2^\bullet, ..., A_s^\bullet) & \mathrm{length}(\boldsymbol{A^i}) = r_i \end{cases} \qquad (279)\\
\textbf{\textit{many-rooted case}} &\ : \ \kappa_{r-1} < k \leqslant \kappa_r \textit{ and } \mathrm{ot}_{r,k} = \mathfrak{j}(\mathrm{ot}_{r_1,k_1}, \ldots, \mathrm{ot}_{r_s,k_s}) \\
\mathrm{Câlt}_{r,k}(\boldsymbol{A}) &:= \overleftarrow{\mathrm{dme}}\,(\mathrm{Câlt}_{r_1,k_1}(\boldsymbol{A^1}), ..., \mathrm{Câlt}_{r_s,k_s}(\boldsymbol{A^s})) \qquad (280)\\
\textit{with} &\quad \begin{cases} 1 < k_i \leqslant \kappa_{r_i-1} & \sum r_i = r \\ \boldsymbol{A^1}...\boldsymbol{A^s} = (A_2^\bullet, ..., A_s^\bullet) & \mathrm{length}(\boldsymbol{A^i}) = r_i \end{cases} \qquad (281)
\end{aligned}$$

Pay attention: in all cases, whether one- or many-rooted, $\hat{Calt}_{r,k}$ is recursively defined in terms of one-rooted predecessors $\hat{Calt}_{r_i,k_i}$, with $1 \leqslant k_i \leqslant \kappa_{r_i}$. On the other hand, in the one-rooted tree, it is the sequence $^*\boldsymbol{A} := (A_2^\bullet, ..., A_r^\bullet)$ that gets factored into subsequences $\boldsymbol{A^i}$, whereas in the many-rooted case, it is the full $\boldsymbol{A} := (A_1^\bullet, ..., A_r^\bullet)$. Lastly, note that, due to the inclusion of $A_1^\bullet$, we have $\sum r_i = r$ in (279)-(281) instead of $r-1$ in (251) .

Clearly, the new $\hat{Calt}(\ldots)$ are closely related to the basis elements $\{\hat{ote}_{r,k}\}$. Indeed:

$$\mathrm{ôte}_{r,k}^\bullet = \mathrm{Câlt}_{r,k}(\overbrace{\mathfrak{E}^\bullet, \ldots, \mathfrak{E}}^{r \ times}) \qquad (282)$$

But it would seem that by allowing arbitary bimoulds $A_1^\bullet$ in index position $\hat{calt}_{A_1^\bullet}(...)$ in (278), we are going to generate new elements not covered by the $\{\hat{ote}_{r,k}\}$ system. This, however, is not the case:

Proposition 6.3 (From *Câlt* to *câlt*) .
For any sequence $\{A_i^\bullet := \mathrm{ôte}_{r_i,k_i}, 1 \leqslant k_i \leqslant \kappa_{r_i-1}\}$ *of one-rooted basis element, the identity holds:*

$$\mathrm{câlt}_{A_1^\bullet}(A_2^\bullet, ..., A_s^\bullet) = \sum_{1 \leqslant p \leqslant \kappa_{r-1}} m_{r,p}\ \mathrm{ôte}_{r,p}^\bullet \quad \textit{with} \quad \begin{cases} r = r_1 + ... + r_s \\ m_{r,p} \in \mathbb{N} \end{cases} \qquad (283)$$

with the sum extending to all one-rooted ordered trees $\mathrm{ot}_{r,p}$ *(counted with their multiplicities) that can be obtained by* attaching*, in an order compatible manner, the one-rooted trees* $\mathrm{ot}_{r_2,k_2}, ..., \mathrm{ot}_{r_s,k_s}$ *to the one-rooted tree* ot_{r_1,k_1} *featuring in index position.*

"In an order compatible manner" obviously means than two trees $\mathrm{ot}_{r_i,k_i}, \mathrm{ot}_{r_j,k_j}$ should never be attached to two nodes of ot_{r_1,k_1} located in reverse order, and that, if attached to the same node, the order $i < j$ should be respected. Of course, some trees $\mathrm{ot}_{r,p}$ may occur several times ($m_{r,p} > 1$) or none at all ($m_{r,p} = 0$). In the special case of just two trees, we fall back, signs aside, to the formula:

$$\mathrm{calt}_{A_1^\bullet}(A_2^\bullet) = -\mathrm{dle}(A_1^\bullet, A_2^\bullet)$$

.

So, if the 'generalised' counter-alternators $C\hat{a}lt_{r,k}$ do not produce anything really new, why bother with them? The answer is that they are needed to bridge the gap between the *dme*- and *mde*-products of one-rooted basis elements:

Proposition 6.4 (From $\overleftarrow{\mathrm{dme}}$ to $\overrightarrow{\mathrm{mde}}$) .
For any number of one-rooted trees ot_{r_i,k_i} $(1 \leqslant k_i \leqslant \kappa_{r_i-1})$*, the identity holds:*

$$\overrightarrow{\mathrm{mde}}\,(\hat{\mathrm{o}}\mathrm{te}^\bullet_{r_1,k_1}, ..., \hat{\mathrm{o}}\mathrm{te}^\bullet_{r_s,k_s}) = \sum_{1 \leqslant k \leqslant \kappa_r}^{r=r_1+...+r_s} \mathrm{C\hat{a}lt}_{r,k}(\hat{\mathrm{o}}\mathrm{te}^\bullet_{r_s,k_s}, ..., \hat{\mathrm{o}}\mathrm{te}^\bullet_{r_1,k_1}) \tag{284}$$

Note the order reversion on both sides of (284). Note further that, although all arguments are one-rooted ($k_i \leqslant \kappa_{r_i-1}$), the sum in (284) extends to all trees $\mathrm{ot}_{r,k}$, one- or many-rooted: all are required. Note lastly that, due to the preceding Proposition 6.3, each term on the right-hand side of (284) reduces to a sum of basis elements $\hat{\mathrm{o}}\mathrm{te}^\bullet_{r,k}$. The involutive identity (284) therefore does exactly what was requested of it: expressing $\overrightarrow{mde}$ products in terms of $\overleftarrow{dme}$ ones.

Remark 5: Neither Proposition 6.3 nor Proposition 6.4 would hold if we were to replace the pair $(c\hat{a}lt, C\hat{a}lt)$ by a similarly constructed pair $(c\breve{a}lt, C\breve{a}lt)$ or $(c\bar{a}lt, C\bar{a}lt)$. This again goes to show the privileged status of the first counter-alternator.

6.7 Stacked trees and co-degree stratification.

Gradation of $Flex_{root}(\mathfrak{E}^\bullet)$ by codegree.

In view of what precedes, the subspaces $\widetilde{Flex^{(d)}_{root}}(\mathfrak{E}^\bullet)$ of $Flex_{root}(\mathfrak{E}^\bullet)$ spanned by the basis elements $\hat{s}te^\bullet_{r,d',k}$ corresponding to one-rooted stacked trees $\hat{s}te^\bullet_{r,d',k}$ with $d' \leqslant d$ clearly constitute a filtration by codegree of $Flex_{root}(\mathfrak{E}^\bullet)$. In fact, the smaller $Flex^{(d)}_{root}(\mathfrak{E}^\bullet)$ spanned by the sole elements $ste^\bullet_{r,d',k}$ constitute a genuine *polyalgebra gradation* of $Flex_{root}(\mathfrak{E}^\bullet)$, in view of numerous inclusions of type:

$$\mathrm{lu}(Flex^{(d_1)}_{root}(\mathfrak{E}^\bullet), Flex^{(d_2)}_{root}(\mathfrak{E}^\bullet)) \quad \subset \quad Flex^{(d_1+d_2-1)}_{root}(\mathfrak{E}^\bullet) \tag{285}$$

$$\mathrm{ari}(Flex^{(d_1)}_{root}(\mathfrak{E}^\bullet), Flex^{(d_2)}_{root}(\mathfrak{E}^\bullet)) \quad \subset \quad Flex^{(d_1+d_2-1)}_{root}(\mathfrak{E}^\bullet) \tag{286}$$

Free generation of $Flex(\mathfrak{E}^\bullet)$ from $Flex_{root}(\mathfrak{E}^\bullet)$.

The whole of $Flex(\mathfrak{E}^\bullet)$ can be *freely* generated from $Flex_{root}(\mathfrak{E}^\bullet)$ under either of the pre-associative products *dme* and *mde*, in the sense that each many-rooted $\hat{ste}^\bullet_{r,d,k}$ admits these two exressions, each of them unique:

$$\hat{\mathrm{ste}}^\bullet_{r,d,k} = \overleftarrow{\mathrm{dme}}\,(\hat{\mathrm{ste}}^\bullet_{r_1,d_1,k_1},\ldots,\hat{\mathrm{ste}}^\bullet_{r_s,d_s,k_s}) \tag{287}$$

$$\hat{\mathrm{ste}}^\bullet_{r,d,k} = \sum \beta_{r_1,d_1,k_1;\ldots,r_s,d_s,k_s}\ \overrightarrow{\mathrm{mde}}\,(\hat{\mathrm{ste}}^\bullet_{r_1,d_1,k_1},\ldots,\hat{\mathrm{ste}}^\bullet_{r_s,d_s,k_s}) \tag{288}$$

with $2 \leqslant s$, $\sum r_i = r$, $\sum d_i = d$, $\hat{\mathrm{ste}}^\bullet_{r_i,d_i,k_i} \in \mathrm{Flex}_{\mathrm{root}}(\mathfrak{E}^\bullet)$ and with integer coefficients $\beta_{r_1,d_1,k_1;\ldots,r_s,d_s,k_s} \in \mathbb{Z}$ calculable by (283) and (284)

Gradation of $Flex(\mathfrak{E}^\bullet)$ by codegree.

Here again, rather than a mere filtration of $Flex(\mathfrak{E}^\bullet)$ by codegree, we have a genuine gradation by the subspaces $Flex^{(d)}(\mathfrak{E}^\bullet)$ spanned all the (one- or many-rooted) basis elements $ste^\bullet_{r,d,k}$.

6.8 Improbable explicitness of the basis changes.

From binary to ordered trees, and back.

Set as usual $\kappa_r := \frac{(2\,r)!}{r!(r+1)!}$ and consider on $Flex_r(\mathfrak{E}^\bullet)$ the basis changes

$$\hat{\mathrm{ote}}^\bullet_{r,p} = \sum_{1\leqslant q\leqslant\kappa_r} \mathrm{mob}_r^{p,q}\ \mathrm{bte}^\bullet_{r,q} \qquad (\textit{binary to ordered}) \tag{289}$$

$$\mathrm{bte}^\bullet_{r,p} = \sum_{1\leqslant q\leqslant\kappa_r} \mathrm{mbo}_r^{p,q}\ \hat{\mathrm{ote}}^\bullet_{r,q} \qquad (\textit{ordered to binary}) \tag{290}$$

and the corresponding matrices $mob_r := [mob_r^{p,q}]$, $mbo_r := [mbo_r^{p,q}]$.

Proposition 6.5 (Properties of the matrices mob_r and mbo_r) .

$(\mathcal{P}_0)$ $\det(\mathrm{mob}_r) = \det(\mathrm{mbo}_r) \in \{1,-1\}$

$(\mathcal{P}_1)$ mob_r *has all its coefficients* $\mathrm{mob}_r^{p,q}$ *in* $\{0,1,-1\}$

$(\mathcal{P}_2)$ mbo_r *has only integer, non-negative coefficients* $\mathrm{mbo}_r^{p,q}$

$(\mathcal{P}_3)$ $\sum_{1\leqslant p,q\leqslant\kappa_r} \mathrm{mob}_r^{p,q} \equiv 1$

$(\mathcal{P}'_3)$ $\sum_{1\leqslant p,q\leqslant\kappa_r} |\mathrm{mob}_r^{p,q}| \equiv \sum_{0\leqslant k\leqslant r-1} \frac{(r+k-1)!\,(r-k)}{r!\ \,k!}\ 2^k$

$(\mathcal{P}_4)$ $\sum_{1\leqslant p,q\leqslant\kappa_r} \mathrm{mbo}_r^{p,q} \equiv 1\times 3\times 5\times\cdots\times(2\,r-1)$

$(\mathcal{P}_5)$ $\sum_{1\leqslant p\leqslant\kappa_r} \mathrm{mob}_r^{p,q} = \begin{cases} 1 & \textit{if}\ \ q=1 \\ 0 & \textit{if}\ \ q>1 \end{cases}$

$(\mathcal{P}'_5)$ $\sum_{1\leqslant p\leqslant\kappa_r} |\mathrm{mob}_r^{p,q}| = 2^{r-\#(\textit{leftmost branch of}\ \mathrm{bt}_{r,q})}$

$(\mathcal{P}_6)$ $\sum_{1\leqslant p\leqslant\kappa_r} \mathrm{mbo}_r^{p,q} = \frac{r!}{(\mathrm{ot}_{r,q})!} \in \mathbb{N}$

Remarkably, it is not the q-sums (rows) but the p-sums of the matrix elements that admit simple expressions.

In $(\mathcal{P}'_5)$, the p-sum is equal to 1 if all the binary tree $bt_{r,q}$ consists of a single left-leaning branch, and in general it is equal to 2 to the number of nodes *off* the left-most branch of $bt_{r,q}$.

In $(\mathcal{P}_6)$, the expression of the p-sum involves the factorial of the ordered tree $ot_{r,q}$, defined in the usual way:

$$(\mathrm{ot})! := \prod_{i \in Nodes(\mathrm{ot})} \Big(\sum_{i \leqslant^{\mathrm{ot}} j} 1 \Big) \qquad\qquad (\mathrm{ot} \in \mathbb{OT}) \tag{291}$$

with $\leqslant^{\mathrm{ot}}$ denoting the partial order on $Nodes(ot)$ induced by the *vertical* [41] tree structure of ot. Note that the ratio $r!/(ot)!$ is also equal to the number of *total orders* on $Nodes(ot)$ that are compatible with the (vertical) partial order on ot.

Proposition 6.6 (Making the matrices ob_r and bo_r triangular) .
If we adopt on the binary trees the k*-indexation induced by (252) and on the ordered trees the* k*-indexation induced by (254), then the matrices* mob_r *and* mbo_r *become upper-triangular, with all diagonal entries equal to* 1*.*

Proposition 6.7 (Direct calculation of the matrix entries in mob_r) .
Here is an algorithm for turning binary trees $\mathrm{bt}_{r,q}$ *into linear combinations*

$$\mathrm{fold}^{\infty}(\mathrm{bt}_{r,q}) = \sum_p \mathrm{mob}_r^{p,q}\ \mathrm{ot}_{r,p} \qquad \begin{cases} \mathrm{ob}_{r,q}\mathbb{BT}\ ;\ \mathrm{ot}_{r,q}\mathbb{OT}\ ; \\ \mathrm{mob}_r^{p,q} \in \{0,1,-1\} \end{cases} \tag{292}$$

of ordered trees $\mathrm{ot}_{r,q}$ *that carries all the information about the matrix entries* $\mathrm{mob}_r^{p,q}$*. The procedure applies recursively a linear operator* fold *to produce, at each step s, linear combinations*

$$\mathrm{fold}^{s}(\mathrm{bt}_{r,q}) = \sum_{1 \leqslant k \leqslant 2^s n_1..n_s} \epsilon_{r,s}^{q,k}\ \mathrm{obt}_{r,k} \qquad \textit{with} \qquad \epsilon_{r,s}^{q,k} \in \{0,1,-1\} \tag{293}$$

of hybrid trees $\mathrm{obt}_{r,k}$ *– part binary, part ordered – with the 'binary tree' aspect decreasing at each step* s*, and the 'ordered tree' aspect becoming dominant towards the end. We also require the notion of* anchor*. For any node x of a binary tree, let x', x'' etc denote the successive antecedent nodes. The anchor x^* of x is the first node $x^{(l)}$ not aligned with the earlier antecedents. Each node that lies outside the leftmost and rightmost branches possesses a well-defined anchor.* [42]

- **Step 0:** *We mark all the n_1 nodes x that lie ouside the leftmost branch of* $\mathrm{bt}_{r,q}$ *as* movable *by writing them in boldface: $x \to \boldsymbol{x}$. If* $\mathrm{bt}_{r,q}$ *has a non-void right branch issuing from its root x_0, we attach x_0 (and with it the whole of* $\mathrm{bt}_{r,q}$*) to a new root x_{00} so as to get a new tree* $\mathrm{bt}_{r+1,q}$*, all movable nodes of which possess an anchor.*

[41] without regard for the order on the branches issuing from a given node.

[42] i.e. the first $x^{(l)}$ such that the twig $(x^{(l)}, x^{(l-1)})$ has not the same slant as the twig (x', x).

- **Step 1:** *We take each one of the movable nodes x located at end points of* $\mathrm{ob}_{r,q}$ *and*
(i) either leave it in place after unmarking *and overlining it:* $\boldsymbol{x} \to \overline{x}$
(ii) or we detach it from it direct antecedent x' and attach it properly [43] *to its anchor, after* unmarking *and underlining it:* $\boldsymbol{x} \to \underline{x}$.
This results in an expansion of type:

$$\mathrm{fold}(\mathrm{bt}_{r,q}) = \sum_{1 \leqslant k \leqslant 2n_1} \epsilon_{r,1}^{q,k}\, \mathrm{obt}_{r,k} \qquad \textit{with} \qquad \epsilon_{r,1}^{q,k} \in \{1, -1\} \tag{294}$$

with coefficients defined in this way:

$$\tag{295}$$

- **Step s:** *We repeat the procedure of Step 1 for each of the hybrid trees obt$_{r,k}$ featuring in the expansion of type (294) obtained at Step $s-1$. Each such obt$_{r,k}$ possesses the same number n_s of (still untouched) movable nodes $\boldsymbol{x}$ situated in* extreme position, *i.e. at end points or with succesors that are all of type $\overline{y}$ (movable points previously unmarked).These new $\boldsymbol{x}$ are the ones that get unmarked at step s. They are either kept in place and overlined ($\boldsymbol{x} \to \overline{x}$) or* properly *attached to their anchors x^* and underlined ($\boldsymbol{x} \to \underline{x}$). In this way, the various hybrid obt$_{r,k}$ from the preceding step producs n_s new hybrid trees, each preceded by a $\pm$ sign calculated according to the rules (295).*

- **Final step:** *When we reach the stage when all the movable points have been unmarked, we are left with an expansion (294) where all the hybrid trees* $\mathrm{obt}_{r,k}$ *have completely shed their 'binary' nature. In other words, we have exactly the expansion (292) which we had set out to construct. However, if we had to introduce an additional root x_{00} at Step 0, we must now remove it and replace each one-rooted tree* $\mathrm{ot} = \mathfrak{h}(\mathrm{ot}_{r_1,q_1}, ..., \mathrm{ot}_{r_s,q_s})$ *by the juxtaposition* $\mathfrak{j}(\mathrm{ot}_{r_1,q_1}, ..., \mathrm{ot}_{r_s,q_s})$ *of the various branches* ot_{r_i,q_i} *which may, in the course of the successive 'foldings', have attached themselves to x_{00} as their anchor.*

Remark 1: 'Properly' attaching a movable node $\boldsymbol{x}$ to its anchor $x^* = x^{(l)}$ means two things:
(i) If the original twig $(x', \boldsymbol{x})$ is rigth-leaning (resp. leftleaning), then after detaching $\boldsymbol{x}$ from its immediate antecedent x' and attaching it to its anchor $x^{(l)}$, the new twig $(x^{(l)}, \underline{x})$ must be squeezed between the twig $(x^{(l)}, x^{(l-1)})$ and the various twigs which may already originate from $x^{(l)}$ to the right (resp. to the left) of $(x^{(l)}, x^{(l-1)})$ as a result of previous relocations.
(ii) The movable node $\boldsymbol{x}$ doesn?t migrate alone to its anchor $x^* = x^{(l)}$ but it carries with it the whole branch of previously unmarked and left-in-place nodes

[43]See details *infra*.

$\overline{x}_1, \overline{x}_2 \ldots$ which may be attached to it. That branch, though originally endowed with a binary tree structure, should henceforth be viewed as an ordered tree.

Provided we take these precautions, it doesn't matter in which order we detach and re-attach the various movable nodes at step s. But, at any given step, we must re-locate only the movable nodes situated in extreme position. Re-locating other, non-extremal nodes 'before time' would lead to completely wrong results.

Remark 2: The expansion (292) always contain on the right-hand side one ordered tree $\mathrm{ot}_{r,p}$ that coincides with the binary tree $\mathrm{bt}_{r,q}$ of the left-hand side, but stripped of its 'binary' structure, and preceded by a + (resp. -) sign if the original $\mathrm{bt}_{r,q}$ carries an even (resp. odd) number of right-leaning unit twigs.

Let us now give a series of examples to cover all the intricacies of the 'folding' procedure[44] of Proposition 6.7. Trees (whether binary, ordered, or mixed) are represented by (clumsy but unambiguous) parenthesisings, of type (...) for one-rooted and (...)...(...) for many-rooted trees. Each node is assigned a number[45]. The over- and underlinings are there simply to keep track of the folding history.

Here is an example with just two movable nodes and no need for an auxiliary root x_{00}.

$$
\begin{aligned}
f : ((1,(\mathbf{2},(\mathbf{3}))),4) \quad &\mapsto \quad \pm((1,(\mathbf{2},(\overline{3}))),4) \pm ((1,(\mathbf{2})),4,(\underline{3})) \\
f^2 : ((1,(\mathbf{2},(\mathbf{3}))),4) \quad &\mapsto \quad \begin{cases} +((1,(\overline{2},(\overline{3}))),4) - ((1),4,(\underline{2},(\overline{3}))) \\ -((1,(\overline{2})),4,(\underline{3})) + ((1),(\underline{2}),4,(\underline{3})) \end{cases}
\end{aligned}
$$

Here we have three movable nodes and require an auxiliary root.

$$
\begin{aligned}
f : (1,(((\mathbf{2}),\mathbf{3}),\mathbf{4})) \quad &\mapsto \quad \pm(1,(((\overline{2}),\mathbf{3}),\mathbf{4})) \pm ((\underline{2}),1,((\mathbf{3}),\mathbf{4})) \\
f^2 : (1,(((\mathbf{2}),\mathbf{3}),\mathbf{4})) \quad &\mapsto \quad \begin{cases} \pm(1,(((\overline{2}),\overline{3}),\mathbf{4})) \pm (((\overline{2}),\underline{3}),1,(\mathbf{4})) \\ \pm((\underline{2}),1,((\overline{3}),\mathbf{4})) \pm ((\underline{2}),(\underline{3}),1,(\mathbf{4})) \end{cases} \\
f^3 : (1,(((\mathbf{2}),\mathbf{3}),\mathbf{4})) \quad &\mapsto \quad \begin{cases} -(1,(((\overline{2}),\overline{3}),\overline{4})) + \{(1);(((\overline{2}),\overline{3}),\underline{4})\} \\ +(((\overline{2}),\underline{3}),1,(\overline{4})) - \{(((\overline{2}),\underline{3}),1);(\underline{4})\} \\ +((\underline{2}),1,((\overline{3}),\overline{4})) - \{((\underline{2}),1);((\overline{3}),\underline{4})\} \\ -((\underline{2}),(\underline{3}),1,(\overline{4})) + \{((\underline{2}),(\underline{3}),1);(\underline{4})\} \end{cases}
\end{aligned}
$$

Here, we have again three movable nodes, but they are disposed of in just two

[44] the procedure is actually simpler to program than to expound!

[45] Since the procedure starts with a binary tree bt, we resort to the natural enumeration of its nodes, that is to say, the enumeration for which the image $bta^{\bullet}$ of bt in $Flex(Pa)$ is of the form $\mathrm{bta}^{\boldsymbol{w}} = \prod_i P(\sum_{i \leqslant^{bt} j} u_j)$

steps.

$$f:(1,((\mathbf{2}),\mathbf{3},(\mathbf{4}))) \quad \mapsto \quad \begin{cases} \pm(1,((\overline{2}),\mathbf{3},(\overline{4}))) \pm \{(1,((\overline{2}),\mathbf{3}));(\underline{4})\} \\ \pm((\underline{2}),1,(\mathbf{3},(\overline{4}))) \pm \{((\underline{2}),1,(\mathbf{3}));(\underline{4})\} \end{cases}$$

$$f^2:(1,((\mathbf{2}),\mathbf{3},(\mathbf{4}))) \quad \mapsto \quad \begin{cases} +(1,((\overline{2}),\overline{3},(\overline{4}))) - \{(1);((\overline{2}),\underline{3},(\overline{4}))\} \\ -\{(1,((\overline{2}),\overline{3}));(\underline{4})\} + \{(1);((\overline{2}),\underline{3});(\underline{4})\} \\ -((\underline{2}),1,(\overline{3},(\overline{4}))) + \{((\underline{2}),1);(\underline{3},(\overline{4}))\} \\ +\{((\underline{2}),1,(\overline{3}));(\underline{4})\} - \{((\underline{2}),1);(\underline{3});(\underline{4})\} \end{cases}$$

The next case, with four movable nodes, it the simplest example to illustrate the caveat of Remark 1 on how to 'properly' attach movable nodes to their anchors. Disregarding it would yield wrong results.

$$f:(((1,(\mathbf{2})),3,(((\mathbf{4}),\mathbf{5}),\mathbf{6})),7) \mapsto$$

$$\begin{cases} \pm(((1,(\overline{2})),3,(((\overline{4}),\mathbf{5}),\mathbf{6})),7) \pm (((1),3,(\underline{2}),(((\overline{4}),\mathbf{5}),\mathbf{6})),7) \\ \pm(((1,(\overline{2})),3,(\underline{4}),((\mathbf{5}),\mathbf{6})),7) \pm (((1),3,(\underline{2}),(\underline{4}),((\mathbf{5}),\mathbf{6})),7) \end{cases}$$

$$f^2:(((1,(\mathbf{2})),3,(((\mathbf{4}),\mathbf{5}),\mathbf{6})),7) \mapsto$$

$$\begin{cases} +(((1,(\overline{2})),3,(((\overline{4}),\overline{5}),\mathbf{6})),7) - (((1,(\overline{2})),3,((\overline{4}),\underline{5}),(\mathbf{6})),7) \\ -(((1),3,(\underline{2}),(((\overline{4}),\overline{5}),\mathbf{6})),7) + (((1),3,(\underline{2}),((\overline{4}),\underline{5}),(\mathbf{6})),7) \\ -(((1,(\overline{2})),3,(\underline{4}),((\overline{5}),\mathbf{6})),7) + (((1,(\overline{2})),3,(\underline{4}),(\underline{5}),(\mathbf{6})),7) \\ +(((1),3,(\underline{2}),(\underline{4}),((\overline{5}),\mathbf{6})),7) - (((1),3,(\underline{2}),(\underline{4}),(\underline{5}),(\mathbf{6})),7) \end{cases}$$

$$f^3:(((1,(\mathbf{2})),3,(((\mathbf{4}),\mathbf{5}),\mathbf{6})),7) \mapsto$$

$$\begin{cases} +(((1,(\overline{2})),3,(((\overline{4}),\overline{5}),\overline{6})),7) - (((1,(\overline{2})),3),(((\overline{4}),\overline{5}),\underline{6}),7) \\ -(((1,(\overline{2})),3,((\overline{4}),\underline{5}),(\overline{6})),7) + (((1,(\overline{2})),3,((\overline{4}),\underline{5})),(\underline{6}),7) \\ -(((1),3,(\underline{2}),(((\overline{4}),\overline{5}),\overline{6})),7) + (((1),3,(\underline{2})),(((\overline{4}),\overline{5}),\underline{6}),7) \\ +(((1),3,(\underline{2}),((\overline{4}),\underline{5}),(\overline{6})),7) - (((1),3,(\underline{2}),((\overline{4}),\underline{5})),(\underline{6}),7) \\ -(((1,(\overline{2})),3,(\underline{4}),((\overline{5}),\overline{6})),7) + (((1,(\overline{2})),3,(\underline{4})),((\overline{5}),\underline{6}),7) \\ +(((1,(\overline{2})),3,(\underline{4}),(\underline{5}),(\overline{6})),7) - (((1,(\overline{2})),3,(\underline{4}),(\underline{5})),(\underline{6}),7) \\ +(((1),3,(\underline{2}),(\underline{4}),((\overline{5}),\overline{6})),7) - (((1),3,(\underline{2}),(\underline{4})),((\overline{5}),\underline{6}),7) \\ -(((1),3,(\underline{2}),(\underline{4}),(\underline{5}),(\overline{6})),7) + (((1),3,(\underline{2}),(\underline{4}),(\underline{5})),(\underline{6}),7) \end{cases}$$

This last example, after the introduction of an auxiliary root $x_{00} = x_7$, takes us back to the preceding case and shows how, after removal of x_7, we are left with eight one-rooted and eight multi-rooted trees.

$$
\begin{aligned}
& f^3 : ((1,(\mathbf{2})),3,(((\mathbf{4}),\mathbf{5}),\mathbf{6})) \mapsto \\
& \qquad \begin{cases}
+((1,(\overline{2})),3,(((\overline{4}),\overline{5}),\overline{6})) - \{((1,(\overline{2})),3);(((\overline{4}),\overline{5}),\underline{6})\} \\
-((1,(\overline{2})),3,((\overline{4}),\underline{5}),(\overline{6})) + \{((1,(\overline{2})),3,((\overline{4}),\underline{5}));(\underline{6})\} \\
-((1),3,(\underline{2}),(((\overline{4}),\overline{5}),\overline{6})) + \{((1),3,(\underline{2}));(((\overline{4}),\overline{5}),\underline{6})\} \\
+((1),3,(\underline{2}),((\overline{4}),\underline{5}),(\overline{6})) - \{((1),3,(\underline{2}),((\overline{4}),\underline{5}));(\underline{6})\} \\
-((1,(\overline{2})),3,(\underline{4}),((\overline{5}),\overline{6})) + \{((1,(\overline{2})),3,(\underline{4}));((\overline{5}),\underline{6})\} \\
+((1,(\overline{2})),3,(\underline{4}),(\underline{5}),(\overline{6})) - \{((1,(\overline{2})),3,(\underline{4}),(\underline{5}));(\underline{6})\} \\
+((1),3,(\underline{2}),(\underline{4}),((\overline{5}),\overline{6})) - \{((1),3,(\underline{2}),(\underline{4}));((\overline{5}),\underline{6})\} \\
-((1),3,(\underline{2}),(\underline{4}),(\underline{5}),(\overline{6})) + \{((1),3,(\underline{2}),(\underline{4}),(\underline{5}));(\underline{6})\}
\end{cases}
\end{aligned}
$$

Remark 3: There exists a similar algorithm, also based on tree surgery, for calculating the matrix entries $mbo_r^{p,q}$.

From ordered to stacked trees.

The equivalence, for each r, of the systems $\{ot_{r,k}\}$ (for ordered trees) and $\{st_{r,d,k}\}$ (for stacked trees) isn't far-fetched. Indeed:

- They both have the same dimension $\kappa_r := \frac{(2\,r)!}{r!\,(r+1)!}$.
- We have shown that the $ot_{r,k}$, being independent, form a basis of $\mathbb{OT}_r$.
- If, in the recursive construction of $\{st_{r,d,k}\}$, we *divide* each symmetrised product by the number of summands, we get an equivalent system $\{st^*_{r,d,k}\}$, but the matrix that takes us from $\{st^*_{r,d,k}\}$ to $\{ot_{r,k}\}$ is now guaranteed to have its determinant equal to ± 1.

6.9 Scalar product on trees.

The correspondence between $Flex_r(\mathfrak{E}^\bullet)$ and $\mathbb{BT}_t$ (binary trees; see §6.2) or $\mathbb{OT}_r$ (ordered trees; see §6.3) makes it possible to define on trees extremely useful scalar products. Let us give here just one example.

Consider the 'flat' flexion unit $Flat^\bullet$:

$$
\mathrm{Flat}^{\binom{u_1}{v_1}} := \frac{1}{2}\big(sign(u_1) + sign(v_1)\big) \qquad (u_1, v_1 \in \mathbb{R}) \tag{296}
$$

Viewed as an almost everywhere defined function, it does indeed verify a flexion unit's functional identities (see §4.5). Let us set, with self-explanatory notations:

$$
\mathrm{mbt}_r^{p,q} \;:=\; \frac{1}{2^r}\int_{|u_i|<1,|v_i|<1} \mathrm{btflat}^{\boldsymbol{w}}_{r,p}\ \mathrm{btflat}^{\boldsymbol{w}}_{r,q}\ du_1...du_r\, dv_1...dv_r \tag{297}
$$

$$
\mathrm{mot}_r^{p,q} \;:=\; \frac{1}{2^r}\int_{|u_i|<1,|v_i|<1} \hat{\mathrm{o}}\mathrm{tflat}^{\boldsymbol{w}}_{r,p}\ \hat{\mathrm{o}}\mathrm{tflat}^{\boldsymbol{w}}_{r,q}\ du_1...du_r\, dv_1...dv_r \tag{298}
$$

These elementary integrals induce scalar products on $\mathbb{BT}_r$ and $\mathbb{OT}_r$:

$$\big\langle \mathrm{bt}_{r,p}, \mathrm{bt}_{r,p} \big\rangle := \mathrm{mbt}_r^{p,q} \qquad (\mathrm{bt}_{r,p}, \mathrm{bt}_{r,q} \in \mathbb{BT}_r) \tag{299}$$

$$\big\langle \mathrm{ot}_{r,p}, \mathrm{ot}_{r,p} \big\rangle := \mathrm{m\hat{o}t}_r^{p,q} \qquad (\mathrm{ot}_{r,p}, \mathrm{ot}_{r,q} \in \mathbb{OT}_r) \tag{300}$$

Here are the matrices $mbt_r := [mbt_r^{p,q}]$ and $m\hat{o}t_r := [m\hat{o}t_r^{p,q}]$ up to $r = 3$:

$$\mathrm{mbt}_2 = \frac{1}{4}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \quad , \quad \mathrm{m\hat{o}t}_2 = \frac{1}{4}\begin{bmatrix} 8 & -4 \\ -4 & 3 \end{bmatrix}$$

$$\mathrm{mbt}_3 \quad = \quad \frac{1}{8} \begin{bmatrix} \frac{13}{3} & -2 & -\frac{3}{2} & -\frac{7}{18} & \frac{8}{9} \\ -2 & \frac{13}{3} & \frac{1}{2} & -\frac{10}{9} & -\frac{7}{18} \\ -\frac{3}{2} & \frac{1}{2} & \frac{14}{3} & \frac{1}{2} & -\frac{3}{2} \\ -\frac{7}{18} & -\frac{10}{9} & \frac{1}{2} & \frac{13}{3} & -2 \\ \frac{8}{9} & -\frac{7}{18} & -\frac{3}{2} & -2 & \frac{13}{3} \end{bmatrix}$$

$$\mathrm{m\hat{o}t}_3 \quad = \quad \frac{1}{8} \begin{bmatrix} \frac{238}{9} & -\frac{65}{9} & -\frac{119}{9} & -6 & \frac{137}{18} \\ -\frac{65}{9} & \frac{82}{9} & \frac{65}{18} & -\frac{11}{2} & -\frac{8}{9} \\ -\frac{119}{9} & \frac{65}{18} & \frac{38}{3} & 2 & -\frac{19}{3} \\ -6 & -\frac{11}{2} & 2 & \frac{34}{3} & -\frac{23}{6} \\ \frac{137}{18} & -\frac{8}{9} & -\frac{19}{3} & -\frac{23}{6} & \frac{13}{3} \end{bmatrix}$$

7 Filtration by co-degree. Dimensions.

7.1 Filtrations, gradations, dimensions: a road map.

We have already encountered the filtration/gradation of the polygebras $Flex(\mathfrak{E}^\bullet)$ and $Flex^{root}(\mathfrak{E}^\bullet)$ by depth r and alternality co-degree d:

$$\mathrm{Flex}(\mathfrak{E}^\bullet) \;\; = \oplus_{r,d}\mathrm{Flex}_{r,d}(\mathfrak{E}^\bullet) \;\; \textit{with} \;\; \mathrm{Flex}_{r,d}(\mathfrak{E}^\bullet) \sim \mathrm{Flex}_{r,(d)}(\mathfrak{E}^\bullet)/\mathrm{Flex}_{r,(d-1)}(\mathfrak{E}^\bullet)$$

$$\mathrm{Flex}^{\mathrm{root}}(\mathfrak{E}^\bullet) = \oplus_{r,d}\mathrm{Flex}^{\mathrm{root}}_{r,d}(\mathfrak{E}^\bullet) \;\; \textit{with} \;\; \mathrm{Flex}^{\mathrm{root}}_{r,d}(\mathfrak{E}^\bullet) \sim \mathrm{Flex}^{\mathrm{root}}_{r,(d)}(\mathfrak{E}^\bullet)/\mathrm{Flex}^{\mathrm{root}}_{r,(d-1)}(\mathfrak{E}^\bullet)$$

Recall that $Flex_{r,(d)}(\mathfrak{E}^\bullet)$ and $Flex^{root}_{r,(d)}(\mathfrak{E}^\bullet)$ denote the subspaces whose elements have depth r and co-degree at most d (filtration), while $Flex_{r,d}(\mathfrak{E}^\bullet)$ and $Flex^{root}_{r,d}(\mathfrak{E}^\bullet)$ denote the natural incarnation[46] of the corresponding quotients (gradation).

Our aim in this section is to calculate the dimensions of the above subspaces. These dimensions matter on many counts – as important features of $Flex(\mathfrak{E}^\bullet)$; as key to the structure of the *pre-associative algebras* in general (cf §10); and because of the rich combinatorics involved in their calculation.

Now, it will turn out that:

[46] given by the basis $\{ste^\bullet_{r,d,k}\}$ indexed by stacked trees.

- The proper numbers to consider are the dimensions corresponding to the gradation rather than the filtration.
- The hard part is to find the dimensions on $Flex^{root}(\mathfrak{E}^\bullet)$; those on $Flex(\mathfrak{E}^\bullet)$ easily follow.
- The pertinent indexation for the dimensions on $Flex(\mathfrak{E}^\bullet)$ is by the co-degree d and on $Flex^{root}(\mathfrak{E}^\bullet)$ by the shifted co-degree $\delta := d-1$.
- While there exist no manageable, closed expressions for the dimensions themselves, such formulae, fairly complex yet elegant, do exist for their simple generating series (in r) and even more so for their double generating series (in r and d),

So we are justified in setting:

$$\gamma_{r,\delta} := \dim\left(\mathrm{Flex}^{\mathrm{root}}_{r,\delta+1}(\mathfrak{E}^\bullet)\right) \quad \| \quad \gamma_\delta(t) := \sum \gamma_{r,\delta}\, t^r \quad \| \quad \Gamma(t,x) := \sum \gamma_\delta(t)\, x^\delta$$
$$\xi_{r,\delta} := \dim\left(\mathrm{Flex}_{r,d}(\mathfrak{E}^\bullet)\right) \quad \| \quad \xi_d(t) := \sum \xi_{r,d}\, t^d \quad \| \quad \Xi(t,x) := \sum \xi_d(t)\, x^d$$

We shall proceed as follows:

- Introduce the main tools – the *framing function* and *pilot polynomials* – needed for the calculations.
- State the main results, with some illustrations.
- Sketch two proofs, one indirect but natural, the other more direct but with a whiff of artificiality about it.
- Provide summary tables at the end of this section, and more extensive ones towards the end of the paper.

7.2 Framing function and pilot polynomials.

Proposition 7.1 (Framing function) .
The framing function, defined by the infinite product[47]

$$\mathcal{F}(x;\, y_1, y_2, y_3, ...) := \prod_{1\leqslant d} \Big(1 - x^d\; y_d\Big)^{-\frac{1}{d}\sum_{d_1|d}\mu(\frac{d}{d_1})\,x^{-d_1}} \tag{301}$$

may be viewed as a power series of $x, y_1, y_2, y_3 \ldots$. It admits a factorisation:

$$\mathcal{F}(x;\, y_1, y_2, y_3, ...) \quad = \quad \mathcal{P}(x;\, y_1, y_2, y_3, ...)\, \exp\Big(\sum_{1\leqslant n} \frac{1}{n}\, y_n\Big) \tag{302}$$

[47] where μ denotes the Möbius function.

into an elementary exponential and a factor $\mathcal{P}$ which itself possesses a rather elementary logarithm:

$$\mathcal{P}(x;\,y_1,y_2,y_3,...) \;=\; 1+\sum_{1\leqslant n} x^n\,P_n(y_1,\dots,y_{2n}) \qquad (P_n\ \ \textit{polynomial}) \tag{303}$$

$$\log\mathcal{P}(x;\,y_1,y_2,y_3,...) \;=\; \sum_{1\leqslant n} x^n\,L_n(y_1,\dots,y_{2n}) \qquad (L_n\ \ \textit{polynomial}) \tag{304}$$

Each 'pilot polynomial' P_n *and* L_n *splits into 'homogeneous' and highly lacunary*[48] *components* $\mathrm{L}_{r,d}$ *and* $\mathrm{P}_{r,d}$*:*

$$P_n(\boldsymbol{y}) \;=\; \sum_{n<d\leqslant 2\,n} P_{n,d}(\boldsymbol{y}) \tag{305}$$

$$L_n(\boldsymbol{y}) \;=\; \sum_{\delta|n} L_{n,n+\delta}(\boldsymbol{y}) \quad \textit{with} \quad L_{n,n+\delta}(\boldsymbol{y}) = \frac{1}{n+\delta}\sum_{\delta_1|\frac{n+\delta}{\delta}}\mu(\delta_1)\,(y_{\delta_1\delta})^{\frac{n+\delta}{\delta_1\delta}} \tag{306}$$

$$\textit{with}\quad \begin{cases} P_{n,d}(y_1\,y,y_2\,y^2,\dots,y_d\,y^d)\equiv y^d\,P_{n,d}(y_1,y_2,\dots,y_d)\\ L_{n,d}(y_1\,y,y_2\,y^2,\dots,y_d\,y^d)\equiv y^d\,L_{n,d}(y_1,y_2,\dots,y_d)\end{cases}$$

Pilot polynomials.

Though much more complex than the L_n, and lacking in closed expressions of type (306), the pilot polynomials P_n also matter, by reason of their close relation with the 'copilot polynomials' Q_n (see *infra*) which count the dimensions $\dim(\mathbb{E}_n^{n_1,\dots,n_s})$.

Here are the first four pilot polynomials P_n with their n homogeneous parts. For more extensive tables, see §11.

$$P_1(y_1,y_2) := +\frac{1}{2}\,(y_1^2-y_2)$$

$$P_2(y_1,..,y_4) := \begin{cases} +\frac{1}{3}\ \ (y_1^3-y_3)\\ +\frac{1}{8}\ \ (y_1^4-2\,y_1^2y_2+3\,y_2^2-2\,y_4)\end{cases}$$

$$P_3(y_1,..,y_6) := \begin{cases} +\frac{1}{4}\ \ (y_1^4-y_2^2)\\ +\frac{1}{6}\ \ (y_1^6-y_3^2)\\ +\frac{1}{48}\begin{cases} +y_1^6-3\,y_1^4y_2+9\,y_1^2y_2^2-6\,y_1^2y_4\\ -7\,y_2^3+6\,y_2y_4+8\,y_3^2-8\,y_6\end{cases}\end{cases}$$

$$P_4(y_1,..,y_8) := \begin{cases} +\frac{1}{5}\ \ (y_1^5-y_5)\\ +\frac{1}{72}\begin{cases} +13\,y_1^6-9\,y_1^4y_2-8\,y_1^3y_3\\ -9\,y_1^2y_2^2+21\,y_2^3+4\,y_3^2-12\,y_6\end{cases}\\ +\frac{1}{24}\ \ (y_1^4-2\,y_1^2y_2+3\,y_2^2-2\,y_4)\,(y_1^3-y_3)\\ +\frac{1}{384}\begin{cases} +y_1^8-4\,y_1^6y_2+18\,y_1^4y_2^2-12\,y_1^4y_4-28\,y_1^2y_2^3\\ +24\,y_1^2y_2y_4+32\,y_1^2y_3^2+25\,y_2^4-32\,y_1^2y_6\\ -36\,y_2^2y_4-32\,y_2y_3^2+32\,y_2y_6+60\,y_4^2-48\,y_8\end{cases}\end{cases}$$

[48] That applies above all to the components $L_{n,d}$.

Observe that the homogeneous parts $P_{n,d}(y_1,...,y_d)$ may be constant in some of their variables y_i.

The pilot polynomials possess many remarkable properties, but since these aren't directly relevant to our purpose, we shall mention but a few.

The lowest homogeneous components are $P_{r,r+1}$ are particularly simple, given that they coincide with $L_{r,r+1}$:

$$P_{r,r+1}(y_1,...,y_{r+1}) \quad = \quad \frac{1}{r+1}\sum_{d|r+1} \mu(d)\,(y_d)^{\frac{r+1}{d}} \quad = \quad L_{r,r+1}(y_1,...,y_{r+1}) \tag{307}$$

The highest homogeneous components, on the other hand, verify:

$$P_{r,2r}(y,...,y) \quad = \quad \frac{1}{2^r\,r!}\prod_{0\leqslant d\leqslant r-1}(y^2-y+2\,d) \tag{308}$$

When $y_1,...,y_r$ assume distinct integer values picked at random, the fractions $P_{r,d}(y_1,...,y_d)$ tend to have large denominators, but the situation changes completely when all y_i coincide: for any integer y, $P_{r,d}(y,...,y)$ is itself an integer[49] That case gives rise to many special identities such as:

$$\begin{aligned}
P_{r,d}(1,...,1) \quad &= \quad 0 \quad \forall r,d \\
P_{r,d}(-1,....,-1) \quad &= \quad \begin{cases} 1 & \mathit{if} \;\; d=2\,r \\ 0 & \mathit{otherwise} \end{cases} \\
P_{r,d}(2,....,2) \quad & \mathit{is\;\;of\;\;the\;\;form} \quad \theta(2\,r-d) \;\; (\in \mathbb{N}) \;\; \mathit{if} \;\; \frac{3\,r}{2} \leqslant d
\end{aligned}$$

7.3 Dimensions. Main statements.

Recall the definitions of the generating series

$$\gamma_\delta(t) := \sum_{1\leqslant r}\gamma_{r,\delta}\,t^r \qquad \mathit{with} \qquad \gamma_{r,\delta} := \dim\Big(\mathrm{Flex}^{\mathrm{root}}_{r,\delta+1}(\mathfrak{E}^\bullet)\Big) \tag{309}$$

$$\xi_d(t) := \sum_{1\leqslant r}\xi_{r,d}\,t^r \qquad \mathit{with} \qquad \xi_{r,d} := \dim\Big(\mathrm{Flex}_{r,d}(\mathfrak{E}^\bullet)\Big) \tag{310}$$

and of their own generating series:

$$\Gamma(t,x) \quad := \quad \gamma_0(t)+\gamma_1(t)\,x+\gamma_2(t)\,x^2+\ldots \tag{311}$$

$$\Xi(t,x) \quad := \quad 1+\xi_1(t)\,x+\xi_2(t)\,x^2+\ldots \tag{312}$$

Proposition 7.2 (The series $\xi_d(t)$ from the series $\gamma_\delta(t)$) .
The series $\xi_d(t)$ and $\Xi(t,x)$ readily follow from the $\gamma_d(t)$ and $\Gamma(t,x)$

$$\Xi(t,x) \equiv \frac{1}{1-x\,\Gamma(t,x)} \tag{313}$$

with the x in front of $\Gamma(t,x)$ accounting for the shift $\delta=d-1$.

[49]For $P_{r,2r}$ this can be checked directly with the help of formula (**??**). For the other homogeneous components $P_{r,d}$, this is not a trivial consequence of the way the pilot polynomials are defined.

We find in particular

$$\xi_1 = \gamma_0 \quad ; \quad \xi_2 = \gamma_1 + \gamma_0^2 \quad ; \quad \xi_3 = \gamma_2 + 2\,\gamma_0\,\gamma_1 + \gamma_0^3 \quad ; \quad \dots$$

Proposition 7.3 (Enumerating the alternals) .
All alternals automatically belong to $\mathrm{Flex}^{\mathrm{root}}(\mathfrak{E}^\bullet)$ *and the corresponding dimensions* $\gamma_{r,0} \equiv \xi_{r,1}$ *are inductively calculable from a functional equation verified by the generating series* $\gamma_0(t)$*:*

$$\gamma_0(t) \quad \equiv \quad t\ \exp\big(\gamma_0(t) + \frac{1}{2}\gamma_0(t^2) + \frac{1}{3}\,\gamma_0(t^3) + \dots\big) \tag{314}$$

The proof here is straightforward. Since in the basis $\{ste^\bullet_{r,d,k}\}$ of $Flex(\mathfrak{E}^\bullet)$, the alternal elements $\{ste^\bullet_{r,d,k}\}$ naturally indexed by *unordered rooted trees* (elements of $\mathbb{UT}$), we fall back on the induction for generating such trees, and that induction readily translates into the functional equation (313). In any case, (314) is a well-known formula for enumerating the unordered rooted trees.

Proposition 7.4 (General formula: theoretical version) .
The general dimensions $\gamma_{r,\delta}$ *also are inductively calculable from a functional equation bearing on the double generating series* $\Gamma(t,x)$*:*

$$\Gamma(t,x) \quad \equiv \quad t\ \mathcal{F}\big(x\,;\Gamma(t,x),\Gamma(t^2,x^2),\Gamma(t^3,x^3),\dots\big) \tag{315}$$

Apart from involving the highly complex *framing function* $\mathcal{F}$, the functional equation (315) closely resembles (314). Indeed, for $x=0$, (315) reduces to (314). But for all its indisputable elegance, (315) has a drawback: when expanded in powers of x, it expresses any given $\gamma_r(t)$ as a polynomial in $\gamma_1(t),\dots,\gamma_{r-1}(t)$ and their first 'dilatees' $\gamma_i(t^k)$, but as a full-blown *entire function* of γ_0 and its dilatees.[50]

To remedy this, we must replace (315) by the more practical, if less shapely, formula (316) below, which is obtained by factoring away from (315) the equation (314), which is itself but a special case of (315) for $x=0$.

Proposition 7.5 (General formula: practical version) .
The double generating function $\Gamma(t,x)$ *verifies the functional equation*

$$\Gamma(t,x) \ = \ \Gamma_0(t,0) \times \begin{cases} \exp\big(\sum_{1\leqslant n} \frac{1}{n}\,(\Gamma(t^n,x^n) - \Gamma_0(t^n,0))\,\big)\times \\ \mathcal{P}\big(x\,;\,\Gamma(t,x),\Gamma(t^2,x^2),\Gamma(t^3,x^3),\dots\big) \end{cases} \tag{316}$$

which may also be written as

$$\Gamma(t,x) \ = \ \gamma_0(t) \times \begin{cases} \exp\big(\sum_{1\leqslant n} \frac{1}{n}\,(\Gamma(t^n,x^n) - \gamma_0(t^n))\,\big)\times \\ \mathcal{P}\big(x\,;\,\Gamma(t,x),\Gamma(t^2,x^2),\Gamma(t^3,x^3),\dots\big) \end{cases} \tag{317}$$

[50] Indeed, since in this context the component γ_0 behaves as an object of homogeneous degree 0, there is nothing to prevent all powers γ_0^n from occurring on the right-hand side of (315). Not so the other components γ_i, which have positive homogeneous degree.

or better still:

$$\log \frac{\Gamma(t,x)}{\gamma_0(t)} \;=\; \times \begin{cases} \big(\sum_{1\leqslant n} \frac{1}{n}\,(\Gamma(t^n,x^n)-\gamma_0(t^n))\,\big)\times \\ +\log \mathcal{P}\big(x\,;\,\Gamma(t,x),\Gamma(t^2,x^2),\Gamma(t^3,x^3),\dots\big) \end{cases} \tag{318}$$

When expanded in powers of x, this equation expresses any given $\gamma_r(t)$ as a polynomial in $\gamma_0(t),\gamma_1(t),\dots,\gamma_{r-1}(t)$ and their first 'dilatees'.

Before sketching, in §7.6 and §7.7, two proofs of the central Proposition 7.4 (of which Proposition 7.5 is a mere corollary), let us make a few comments and give some illustrations.

Relations between the generating series.

Since $\dim\big(\mathit{Flex}_r(\mathfrak{E}^\bullet)\big)=\kappa_r$ and $\dim\big(\mathit{Flex}_r^{root}(\mathfrak{E}^\bullet)\big)=\kappa_{r-1}$ and since the Catalan numbers $\kappa_r := \frac{2\,r)!}{r!\,(r+1)!}$ admit the generating function

$$\kappa(t) := \sum_{0\leqslant r} \kappa_r\, t^r = \frac{1}{2\,t}\big(1-(1-4\,t)^{\frac{1}{2}}\big) \quad (\textit{solution of } \kappa^2 - t\,\kappa + 1 = 0) \tag{319}$$

our generating functions $\xi_d(t)$ and $\gamma_\delta(t)$ must clearly add up to $\kappa(t)$ and $t\,\kappa(t)$ respectively:

$$\sum_{0\leqslant r} \xi_r(t) \equiv \kappa(t) \qquad \textit{i.e.} \qquad \Xi(t,1)\equiv\kappa(t) \tag{320}$$

$$\sum_{1\leqslant r} \gamma_r(t) \equiv t\,\kappa(t) \qquad \textit{i.e.} \qquad \Gamma(t,1)\equiv t\,\kappa(t) \tag{321}$$

The relation (321) immediately follows from (313) due to (319) and κ's functional equation $\kappa(t)=(1-t\,\kappa(t))^{-1}$.

Practical calculations.

When taking the coefficient of x^n on both sides of (317), or preferably (318), we find respectively:[51]

$$\gamma_n(t) = \sum_{0\leqslant p\leqslant n} \left[\begin{cases} \gamma_0(t)\,\exp\Big(\sum_{1\leqslant r}^{1\leqslant k} \gamma_r(t^k)\frac{x^{k\,r}}{k}\Big)\times \\ P_p\big(\sum_{0\leqslant r}\gamma_r(t)x^{1.r},...,\sum_{0\leqslant r}\gamma_r(t^{2pr})x^{2pr}\big) \end{cases} \right]_{x^{n-p}} \tag{322}$$

$$\left[\log\big(1+\sum_{r\leqslant n}\frac{\gamma_r(t)}{\gamma_0(t)}x^r\big)\right]_{x^n} =$$

$$\sum_{kr=n}\frac{1}{k}\gamma_r(t^k) + \sum_{0\leqslant n\leqslant p}\left[L_p\big(\sum_{0\leqslant r}\gamma_r(t)x^{1.r},...,\sum_{0\leqslant r}\gamma_r(t^{2pr})x^{2pr}\big)\right]_{x^{n-p}} \tag{323}$$

[51] For consistency, we must set $P_0\equiv 1$.

with, in the second case, the elementary polynomial L_n of (304) and (306).
As already pointed out, the 'practical formulae' have the advantage of producing, on either side, polynomials in finitely many terms of the form $\gamma_i(t^k)$ with $0 \leqslant i \leqslant n-1$ and with products $i\,k$ bounded by simple homogeneousness conditions. In fact, when expanding (323), we get:

$$\frac{\gamma_n(t)}{\gamma_0(t)} + \sum_{n_j<n}^{s\leqslant n} a^{n_1,\dots,n_s}\, \frac{\gamma_{n_1}(t)}{\gamma_0(t)} \dots \frac{\gamma_{n_s}(t)}{\gamma_0(t)} = \gamma_n(t) + \sum_{n_j<n}^{s\leqslant n} b^{n_1,\dots,n_s}_{k_1,\dots,k_s}\, \gamma_{n_1}(t^{k_1})\dots\gamma_{n_s}(t^{k_s})$$

with $a^{n_1,\dots,n_s}$ and $b^{n_1,\dots,n_s}_{k_1,\dots,k_s} \in \mathbb{Q}$ and $\sum k_j\, n_j \leqslant n$. Since γ_0 is known, (323) amounts, for the generating functions γ_n, to a recursion of the form:

$$\gamma_n(t) = \frac{\gamma_0(t)}{1-\gamma_0(t)} \sum_{n_j<n}^{s\leqslant n} c^{n_1,\dots,n_s}_{k_1,\dots,k_s}\, \gamma_{n_1}(t^{n_1 k_1})\dots\gamma_{n_s}(t^{n_s k_s}) \qquad \begin{cases} c^{n_1,\dots,n_s}_{k_1,\dots,k_s} \in \mathbb{Q} \\ \sum k_j\, n_j \leqslant n \end{cases} \tag{324}$$

The first five generating series $\gamma_r(t)$ are given in §11.5. With the pilot polynomials, tabulated §11.4, one can easily calculate the next $\gamma_r(t)$ which, however, become too unwieldy to write down.

In fact, by repeatedly re-injecting earlier versions of (324) (i.e. versions relative to smaller values of n) into itself, we can rid the right-hand side of (324) of all terms $\gamma_{n_i}(t^{n_i\,k_i})$ with $1 \leqslant n_i$. Eventually, we get $\gamma_n(t)$ expressed as a rational function of the already known series $\gamma_0(t)$ and its dilatees $\gamma_0(t^2), \dots, \gamma_0(t^{2n})$. These successive eliminations, however, tend to complicate rather than simplify the expression of $\gamma_n(t)$. This is why we opted, in the tables of §11.15, for the 'raw' form (324).

7.4 Codimensions. Main statements.

$$\gamma^{\text{co}}_{r,d} := \dim\left(\text{Flex}^{\text{root}}_{r,r-d}(\mathfrak{E}^\bullet)\right) \;\;\|\;\; \gamma^{\text{co}}_d(t) := \sum \gamma^{\text{co}}_{r,d}\, t^r \;\;\|\;\; \Gamma^{\text{co}}(t,x) := \textstyle\sum_{0\leqslant d} \gamma_\delta(t)\, x^\delta$$

$$\xi^{\text{co}}_{r,d} := \dim\left(\text{Flex}_{r,r-d}(\mathfrak{E}^\bullet)\right) \;\;\|\;\; \xi^{\text{co}}_d(t) := \sum \xi^{\text{co}}_{r,d}\, t^d \;\;\|\;\; \Xi^{\text{co}}(t,x) := \textstyle\sum_{0\leqslant d} \xi^{\text{co}}_d(t)\, x^d$$

Here is how new generating series relate to the old, and to each other:[52]

$$\Gamma^{\text{co}}(t,x) = x^{-1}\, \Gamma(t\,x, x^{-1}) \tag{325}$$

$$\Xi^{\text{co}}(t,x) = \Xi(t\,x, x^{-1}) = \Big(1-\Gamma^{\text{co}}(t,x)\Big)^{-1} \tag{326}$$

The functional equation for $\Gamma^{co}(t,x)$.

Clearly, $\Gamma^{\text{co}}(t,x) = t + x\,t^2 + \sum_{2\leqslant n} x^n\, \gamma^{\text{co}}_n(t)$, and due to (315) the functional equation (325) becomes:

$$\log\Big(\frac{\Gamma^{\text{co}}(t,x)}{t}\Big) = -\sum_{1\leqslant n} \log\left(1-\Gamma^{\text{co}}(t^n,x^n)\right) \times \Big(\frac{1}{n}\sum_{d|n} \mu\big(\frac{n}{d}\big)\, x^n\Big) \tag{327}$$

[52]The presence of a factor x^{-1} in front of $\Gamma(t\,x, x^{-1})$ but not $\Xi(t\,x, x^{-1})$ comes from the shift between the δ and d-indexation for Γ and Ξ, which has no parallel for Γ^{co} and Ξ^{co}.

Written in this way, it is inconvenient, because comparing the coefficient of x^n on both sides doesn't give $\gamma_n^{\mathrm{co}}(t)$ as a finite expression of x and the earlier $\gamma_{n'}^{\mathrm{co}}(t)$. However, by 'Möbius inversion', the identity $\sum_{1\leqslant n} \frac{t^n}{n} = -\log(1-t)$ yields successively

$$t \equiv -\sum_{1\leqslant n} \log(1-t^n)\,\frac{\mu(n)}{n} \tag{328}$$

$$-\log(1-t\,x) \equiv -\sum_{1\leqslant n} \log(1-t^n)\,\frac{1}{n}\times\Big(\sum_{d|n}\mu(\frac{n}{d})\,x^n\Big) \tag{329}$$

By subtracting (329) from (327), we can re-write the functional equation as

$$\log(\frac{\Gamma^{\mathrm{co}}(t,x)}{t})+\log(1-t\,x)= \\ -\sum_{1\leqslant n}\log\Big(1-\frac{\Gamma^{\mathrm{co}}(t^n,x^n)-t^n}{1-t^n}\Big)\times\Big(\frac{1}{n}\sum_{d|n}\mu(\frac{n}{d})\,x^n\Big) \tag{330}$$

which yields a proper induction, since by equating the coefficients of x^n on both sides, we now find $\gamma_n(t)$ expressed in closed form in terms of the earlier $\gamma_{n'}(t)$.

The generating series $\gamma_d^{co}(t)$.

Proposition 7.6 (Rationalness of $\gamma_d^{co}(t)$) .

Unlike the $\gamma_d(t)$, the $\gamma_d^{co}(t)$ are rational functions of t.
Thus, $\gamma_0^{\mathrm{co}}(t)=t$, $\gamma_1^{\mathrm{co}}(t)=t^2$ and for $n\geqslant 2$:

$$\gamma_n^{\mathrm{co}}(t) = t^{n+1}\,\frac{\big(\gamma_{0,n+1}+\cdots+(-1)^{n-1}\,t^{\frac{n(n-1)}{2}}\big)}{\prod_{k=1}^{n-1}(1-t^k)} = \frac{t^{n+1}\,\widehat{\gamma}_n^{\mathrm{co}}(t)}{\prod_{k=1}^{n-1}(1-t^k)} \tag{331}$$

The numerator $\widehat{\gamma}_n^{\mathrm{co}}(t)$ in (331) is a polynomial with integer coefficients of mixed signs. Its first coefficient $\gamma_{0,n+1}$ is the number of rooted, non-ordered trees with $n+1$ nodes.

The generating series $\xi_d^{co}(t)$.

Proposition 7.7 (Rationalness and positiveness of $\xi_d^{co}(t)$) .

The $\xi_d^{co}(t)$'s are also rational functions of t, but of a more regular type than the $\gamma_d^{co}(t)$'s. Thus, $\xi_0(t)=(1-t)^{-1}$, $\xi_1(t)=t^2\,(1-t)^{-2}$ and for $n\geqslant 2$:

$$\xi_n^{\mathrm{co}}(t) = t^{n+1}\,\frac{\big(\gamma_{0,n+1}+\cdots+c_n\,t^{[\frac{(n-1)}{1}]}\big)}{(1-t)^2\,\prod_{k=1}^{n-1}(1-t^k)} = \frac{t^{n+1}\,\widehat{\xi}_n^{\mathrm{co}}(t)}{(1-t)^2\,\prod_{k=1}^{n-1}(1-t^k)} \tag{332}$$

with $[\frac{(n-1)}{2}]$ denoting the integer part of $\frac{(n-1)}{2}$ and $c_n=\begin{cases} n/2+2 & \text{if } n \text{ even} \\ 1 & \text{if } n \text{ odd}\end{cases}$

The numerator $\widehat{\xi}_n^{\mathrm{co}}(t)$ in (332) is a polynomial with positive integer coefficients.

We may get Ξ^{co} from Γ^{co} or directly from the Möbius inverse[53] of (333):

$$\log\Big(\Xi^{\rm co}(t,x)\Big) \;=\; \sum_{1\leqslant n} \log\Big(\frac{\Xi^{\rm co}(t^n,x^n)-1}{t^n\,\Xi^{\rm co}(t^n,x^n)}\Big)\,\frac{1}{n}\sum_{d|n}\mu(\frac{n}{d})\,\frac{1}{x^d} \tag{333}$$

Here again, to get a finite induction, we must subtract from (326) the following identity

$$\log\Big(\frac{1-t^2\,x}{1-t}\Big) \;=\; \sum_{1\leqslant n}\log\Big(1+t^n\,x^n\Big)\,\frac{1}{n}\sum_{d|n}\mu(\frac{n}{d}) \tag{334}$$

and write

$$\log\big((1-t)\,\Xi^{\rm co}(t,x)\big) = \begin{cases} +\log(1-t^2\,x) \\ +\sum_{1\leqslant n}\log\big(\frac{\Xi^{\rm co}(t^n,x^n)-1}{t^n\,(1+t^n\,x^n)\,\Xi^{\rm co}(t^n,x^n)}\big)\,\frac{1}{n}\sum_{d|n}\mu(\frac{n}{d})\,\frac{1}{x^d}\end{cases} \tag{335}$$

The fact, not immediately apparent from the induction (335), that $\widehat{\xi}^{\rm co}_n(t)$ has only positive integer coefficients is quite significant. It implies – or should we say, suggests – that we can produce, from a *finite* number of elements of $Flex(\mathfrak{E}^\bullet)$ a basis for the infinite dimensional space $\oplus_r Flex_{r,r-d}(\mathfrak{E}^\bullet)$.

Remark: As should be expected (see below §7.5), the polynomials $\widehat{\gamma}^{\rm co}_n(t)$ and $\widehat{\xi}^{\rm co}_n(t)$ often assume remarkable values when t is a unit root. Leaving aside the relations that trivially follow from

$$\widehat{\gamma}^{\rm co}_n(t) \equiv \widehat{\xi}^{\rm co}_n(t) \qquad \text{mod } (1+t+t^2+\cdots+t^{n-2})$$

we have numerous identities of type

$$\left|\frac{\widehat{\gamma}^{\rm co}_{4n+1}(i)}{\widehat{\gamma}^{\rm co}_{4n}(i)}\right|^2 \equiv 9 \qquad , \qquad \left|\frac{\widehat{\xi}^{\rm co}_{4n+1}(i)}{\widehat{\xi}^{\rm co}_{4n}(i)}\right|^2 \equiv 5$$

7.5 Analytic properties of $\Gamma(t,x)$ and $\Xi(t,x)$.

Special Möbius inversion.

Lemma 7.1 (Special Möbius inversion) .
Setting $\chi_{s,d}(x) := d^{-s}\sum_{d_1|d}\mu(\frac{d}{d_1})\,x^{d_1}$, we have the formal equivalence

$$\Big\{A(t,x) = \prod_{1\leqslant n}\chi_{s,d}(x)\,B(t^n,x^n)\Big\} \Longleftrightarrow \Big\{B(t,x) = \prod_{1\leqslant n}\chi_{s,d}(\frac{1}{x})\,A(t^n,x^n)\Big\} \tag{336}$$

resting on the elementary identity

$$\sum_{d_1\,d_2=d}\chi_{s,d_1}(x)\,\chi_{s,d_2}(x^{-d_1}) \equiv \begin{cases} 1 & if\;\; d=1 \\ 0 & otherwise\end{cases} \tag{337}$$

[53] See (**??**) below.

For a clearer overview, let us again write the two pairs of mutually inverse identities:

$$\frac{\Gamma(t,x)}{t} = \prod_{1\leqslant n}\Big(\Xi(t^n,x^n)\Big)^{\chi_{1,n}(\frac{1}{x})} \quad \textit{with} \quad \Xi(t,x) = \frac{1}{1-x\,\Gamma(t,x)} \tag{338}$$

$$\Xi(t,x) = \prod_{1\leqslant n}\Big(\frac{\Gamma(t^n,x^n)}{t^n}\Big)^{\chi_{1,n}(x)} \quad \textit{with} \quad \Gamma(t,x) = \frac{\Xi(t,x)-1}{x\,\Xi(t,x)} \tag{339}$$

$$\frac{\Gamma^{\mathrm{co}}(t,x)}{t} = \prod_{1\leqslant n}\Big(\Xi^{\mathrm{co}}(t^n,x^n)\Big)^{\chi_{1,n}(x)} \quad \textit{with} \quad \Xi^{\mathrm{co}}(t,x) = \frac{1}{1-\Gamma^{\mathrm{co}}(t,x)} \tag{340}$$

$$\Xi^{\mathrm{co}}(t,x) = \prod_{1\leqslant n}\Big(\frac{\Gamma^{\mathrm{co}}(t^n,x^n)}{t^n}\Big)^{\chi_{1,n}(\frac{1}{x})} \quad \textit{with} \quad \Gamma^{\mathrm{co}}(t,x) = \frac{\Xi^{\mathrm{co}}(t,x)-1}{\Xi^{\mathrm{co}}(t,x)} \tag{341}$$

Neither correspondence $\Gamma \leftrightarrow \Xi$ or $\Gamma^{co} \leftrightarrow \Xi^{co}$ being involutive, the above identities do not in any way relate our four functions for inverse values x and $1/x$. It is not even clear whether they admit (ramified) analytic continuations at infinity in x, or in t, for that matter. Nevertheless, these two pairs of identities raise the question as to the analytic nature of Γ, Γ^{co} and Ξ, Ξ^{co}. Let us limit ourselves to a few sketchy indications.

Analyticity of $\Gamma(t,x)$. The case $x = \pm 1$.

$$\Gamma(t,1) = \frac{1}{2}\Big(1-(1-4\,t)^{\frac{1}{2}}\Big) \quad ; \quad \Xi(t,1) = \frac{1}{2\,t}\Big(1-(1-4\,t)^{\frac{1}{2}}\Big) \tag{342}$$

$$\Gamma(t,-1) = \frac{1}{2}\Big(1-(\frac{1+2\,t}{1-2\,t})^{\frac{1}{2}}\Big) \quad ; \quad \Xi(t,-1) = \frac{1}{2\,t}\Big(-1+2\,t+(1-4\,t^2)^{\frac{1}{2}}\Big) \tag{343}$$

$$\mathcal{P}_4 := \Big\{ n\ ;\ n = p_1^{\pi_1}\dots p_s^{\pi_s}\ \textit{with}\ s \geqslant 1\ \textit{and}\ \begin{cases} p_i\ \textit{prime} \geqslant 3 \\ p_i \equiv -1 \mod 4 \end{cases} \Big\} \tag{344}$$

It follows that, for any fixed x, $\Gamma(t,x)$ and $\Xi(t,x)$ as series of t, have positive convergence radii.

The case when $x^3 = 1$ or $x^4 = 1$.

Consider the integer sets

$$\mathcal{P}_3 := \Big\{ n\ ;\ n = p_1^{\pi_1}\dots p_s^{\pi_s}\ \ \textit{with}\ \ p_i\ \textit{prime}\ \ \textit{and}\ \ p_i \equiv -1 \mod 3 \Big\} \tag{345}$$

$$\mathcal{P}_4 := \Big\{ n\ ;\ n = p_1^{\pi_1}\dots p_s^{\pi_s}\ \ \textit{with}\ \ p_i\ \textit{prime}\ \ \textit{and}\ \ p_i \equiv -1 \mod 4 \Big\} \tag{346}$$

and for $n \in \mathcal{P}_3$ or $\mathcal{P}_4$ set $\epsilon(n) := (-1)^{\pi_1+\dots+\pi_s}$ and $\sigma(n) := 2^s$. Then

$$\Gamma(t,j) \;=\; \begin{cases} t\left(1-j\,\Gamma(t,j)\right)^{-j}\left(1-\Gamma(t^3,1)\right)^{-\frac{1}{2}-\frac{1}{6}(j-\bar{j})} & \times \\ \prod_{\epsilon(n)=+1}^{n\in\mathcal{P}_3}\left(1-j\,\Gamma(t^n,j)\right)^{\frac{1}{2n}\,\epsilon(n)\,\sigma(n)\,(j-\bar{j})} & \times \\ \prod_{\epsilon(n)=-1}^{n\in\mathcal{P}_3}\left(1+\bar{j}\,\Gamma(t^n,\bar{j})\right)^{\frac{1}{2n}\,\epsilon(n)\,\sigma(n)\,(j-\bar{j})} & \times \\ \prod_{\epsilon(n)=\pm 1}^{n\in\mathcal{P}_3}\left(1-\;\Gamma(t^{3n},1)\right)^{-\frac{1}{6n}\,\epsilon(n)\,\sigma(n)\,(j-\bar{j})} \end{cases} \tag{347}$$

with $j = e^{2\pi\, i/3}$.

$$\Gamma(t,i) \;=\; \begin{cases} t\left(1-i\,\Gamma(t,i)\right)^{i}\left(1+\Gamma(t^2,-1)\right)^{1-i}\left(1-\Gamma(t^4,1)\right)^{-\frac{1}{2}} & \times \\ \prod_{\epsilon(n)=+1}^{n\in\mathcal{P}_4}\left(1-i\,\Gamma(t^n,i)\right)^{\frac{1}{n}\,\epsilon(n)\,\sigma(n)\,i} & \times \\ \prod_{\epsilon(n)=-1}^{n\in\mathcal{P}_4}\left(1+i\,\Gamma(t^n,-i)\right)^{\frac{1}{n}\,\epsilon(n)\,\sigma(n)\,i} & \times \\ \prod_{\epsilon(n)=\pm 1}^{n\in\mathcal{P}_4}\left(1+\;\Gamma(t^{2n},-1)\right)^{-\frac{1}{2\,n}\,\epsilon(n)\,\sigma(n)\,i} \end{cases} \tag{348}$$

The relations (347) and (348), together with the conjugate relations, completely determine $\Gamma(t,j)$ and $\Gamma(t,i)$.

The case when x is a general unit root.

Similar systems obtain when $x=\epsilon$, for ϵ a prime unit root of order $q := \prod q_i^{m_i}$ (q_i prime), but with this difference that we no longer have simple formulae for expressing the exponents $h(n,\epsilon) := \sum_{d|n}\mu(n/d)\,\epsilon^{-d}$. Still, the set $\mathcal{P}_q$ of relevant integers n, i.e. of all n for which $h(n,\epsilon) \neq 0$, remains fairly lacunary, as it excludes (in particular) all n that are

- divisible by $\prod q_i^{m_i+1}$
- or divisible by some q_0 with $q_0 \equiv 1 \mod q$
- or divisible by some $q_0 \prod q_i^{m_i'}$ with $q_0 \equiv 1 \mod \prod q_i^{m_i''}$ and $m_i'+m_i'' > m_i$.

If ϵ is a prime root of order q, then $\Gamma(t,\epsilon)$ and $\Xi(t,\epsilon)$ have radius of convergence $4^{-1/q}$, with (unless $\epsilon = \pm 1$) an analytic boundary at the unit circle, on all their Riemann sheets.

If $\epsilon = e^{2\pi i\theta}$ is not a unit root, it would seem that $\Gamma(t,\epsilon)$ and $\Xi(t,\epsilon)$ have 1 as radius of convergence, unless perhaps for strongly Liouvillian θ.

No special significance attaches to the case t unit root, x small.

Lastly, the case when t and x are both unit roots (but $x \neq \pm 1$) falls outside the domain of definition of $\Gamma(t,x)$ and $\Xi(t,x)$.

7.6 Sketch of proof: first approach.

Roadmap.

We consider the following generating series $\gamma_0(t), \gamma_\delta(t), \gamma^*_\delta(t)$:

$$\begin{aligned}
\gamma_0(t) &:= \sum \gamma_{r,1}\, t^r \quad \textit{with} \quad \gamma_{r,1} := \dim\left(\mathrm{Flex}^{\mathrm{root}}_{r,1}(\mathfrak{E}^\bullet)\right) = \dim\left(\mathrm{Flex}^{\mathrm{al}}_{r}(\mathfrak{E}^\bullet)\right)\\
\gamma_\delta(t) &:= \sum \gamma_{r,1+\delta}\, t^r \quad \textit{with} \quad \gamma_{r,1+\delta} := \dim\left(\mathrm{Flex}^{\mathrm{root}}_{r,1+\delta}(\mathfrak{E}^\bullet)\right)\\
\gamma^*_\delta(t) &:= \sum \gamma^*_{r,1+\delta}\, t^r \quad \textit{with} \quad \gamma^*_{r,1+\delta} := \sum \dim\left(\mathrm{sym.calt}_\delta(\mathrm{Flex}^{\mathrm{al}}_{r_1}(\mathfrak{E}^\bullet), .., \mathrm{Flex}^{\mathrm{al}}_{r_s}(\mathfrak{E}^\bullet)\right)
\end{aligned}$$

where the 'restricted dimension' $\gamma^*_{r,\delta+1}$ denotes the dimension of the space spanned by all bimoulds $B^\bullet$ of depth r and co-alternality $\delta+1$ that can be formed *directly* from alternal bimoulds $A^\bullet_i$ by forming partially symmetrised counter-alternators according to the procedure of §6.4; in other words, by all $B^\bullet$ of the form:

$$B^\bullet := \sum_\sigma n^*_{i,\sigma}\ \mathrm{calt}_{\mathfrak{E}^\bullet}(A^\bullet_{\sigma(1)}, \ldots, A^\bullet_{\sigma(s)}) \tag{349}$$

We then proceed as follows:

- We provisionally assume $\gamma_0(t)$ to be known.
- We directly calculate the 'restricted dimensions' $\gamma^*_{r,\delta+1}$ by using the 'copilot polynomials'.
- We form the corresponding generating series $\gamma^*_\delta(t)$ by using the conversion matrices $\boldsymbol{moe}_n$.
- Lastly, we easily go from the 'restricted' series $\gamma^*_\delta(t)$ to the full series $\gamma_\delta(t)$.

The conversion matrices.

The first ingredient in this approach are the *conversion matrices* meo_n and their inverses moe_n, whose entries are indexed by $\mathfrak{p}, \mathfrak{q}$ ranging through the set of all partitions of the integer n.

Proposition 7.8 (The matrices $\boldsymbol{meo}_n$ and $\boldsymbol{meo}_n$) .
For any partition $\mathfrak{n} = [n_1, ..., n_s]$ of n, we set

$$\boldsymbol{y}^{\mathfrak{n}} := \prod_{i=1}^{i=n} y_i^{n_i} \tag{350}$$

$$\boldsymbol{y}_{\mathfrak{n}} := \prod_{i=1}^{i=n} \sum_{j=1}^{j=n} y_j^{n_i} \tag{351}$$

$$< \prod y_i^{p_i}, \prod y_i^{q_i} > \quad := \quad \begin{cases} 1 & \textit{if} \ \ p_i \equiv q_i \\ 0 & \textit{otherwise} \end{cases} \tag{352}$$

For a suitable ordering[54] *of the set* Par_n *of all partitions of* n*, the matrix* meo_n *with positive integer entries*

$$\mathrm{meo}_n[\mathfrak{p},\mathfrak{q}] \quad := \quad < \boldsymbol{y}^{\mathfrak{p}}, \boldsymbol{y}_{\mathfrak{q}} > \qquad \big(\mathfrak{p} \in \mathrm{Par}_n, \mathfrak{q} \in \mathrm{Par}_n\big) \tag{353}$$

is inferior triangular, with non-vanishing diagonal elements. The inverse matrix $\mathrm{moe}_n = \mathrm{meo}_n^{-1}$ *has rational entries of mixted signs. Moreover:*

$$\sum_{\mathfrak{p},\mathfrak{q}\in\mathrm{Par}_n} \mathrm{moe}_n[\mathfrak{p},\mathfrak{q}] \quad \equiv \quad 1 \qquad \forall n \tag{354}$$

$$\sum_{\mathfrak{p},\mathfrak{q}\in\mathrm{Par}_n} |\,\mathrm{moe}_n[\mathfrak{p},\mathfrak{q}]\,| \quad \equiv \quad 2^{n-1} \qquad \forall n \tag{355}$$

$\mathrm{Par}_1 = [[1]]$, $\mathrm{Par}_2 = [[1,1],[2]]$, $\mathrm{Par}_3 = [[1,1,1],[2,1],[3]]$,
$\mathrm{Par}_4 = [[1,1,1,1],[2,1,1],[2,2],[3,1],[4]]$,
$\mathrm{Par}_5 = [[1,1,1,1,1],[2,1,1,1],[2,2,1],[3,1,1],[3,2],[4,1],[5]]$,
$\mathrm{Par}_6 = [[1,1,1,1,1,1],[2,1,1,1,1],[2,2,1,1],[3,1,1,1],[2,2,2],[3,2,1],[4,1,1],[3,3],[4,2],[5,1],[6]]$

Here are the first six matrices meo_n, duly diagonal:

$$\begin{bmatrix}1\end{bmatrix},\ \begin{bmatrix}2&0\\1&1\end{bmatrix},\ \begin{bmatrix}6&0&0\\3&1&0\\1&1&1\end{bmatrix},\ \begin{bmatrix}24&0&0&0&0\\12&2&0&0&0\\6&2&2&0&0\\4&2&0&1&0\\1&1&1&1&1\end{bmatrix},\ \begin{bmatrix}120&0&0&0&0&0&0\\60&6&0&0&0&0&0\\30&6&2&0&0&0&0\\20&6&0&2&0&0&0\\10&4&2&1&1&0&0\\5&3&1&2&0&1&0\\1&1&1&1&1&1&1\end{bmatrix}$$

$$\mathrm{meo}_6 := \begin{bmatrix}720&0&0&0&0&0&0&0&0&0&0\\360&24&0&0&0&0&0&0&0&0&0\\180&24&4&0&0&0&0&0&0&0&0\\120&24&0&6&0&0&0&0&0&0&0\\90&18&6&0&6&0&0&0&0&0&0\\60&16&4&3&0&1&0&0&0&0&0\\30&12&2&6&0&0&2&0&0&0&0\\20&8&4&2&0&2&0&2&0&0&0\\15&7&3&3&3&1&1&0&1&0&0\\6&4&2&3&0&1&2&0&0&1&0\\1&1&1&1&1&1&1&1&1&1&1\end{bmatrix}$$

[54] Roughly, from [1,...,1] to [n]. See the examples below (355).

And here are the first six matrices moe_n:

$$
[1]\,,\begin{bmatrix} \frac{1}{2} & 0 \\ \frac{\overline{1}}{2} & 1 \end{bmatrix},\begin{bmatrix} \frac{1}{6} & 0 & 0 \\ \frac{\overline{1}}{2} & 1 & 0 \\ \frac{1}{3} & \overline{1} & 1 \end{bmatrix},\begin{bmatrix} \frac{1}{24} & 0 & 0 & 0 & 0 \\ \frac{\overline{1}}{4} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{8} & \frac{\overline{1}}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & \overline{1} & 0 & 1 & 0 \\ \frac{\overline{1}}{4} & 1 & \frac{\overline{1}}{2} & \overline{1} & 1 \end{bmatrix},\begin{bmatrix} \frac{1}{120} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\overline{1}}{12} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{8} & \frac{\overline{1}}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{6} & \frac{\overline{1}}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{\overline{1}}{6} & \frac{5}{6} & \overline{1} & \frac{\overline{1}}{2} & 1 & 0 & 0 \\ \frac{\overline{1}}{4} & 1 & \frac{\overline{1}}{2} & \overline{1} & 0 & 1 & 0 \\ \frac{1}{5} & \overline{1} & 1 & 1 & \overline{1} & \overline{1} & 1 \end{bmatrix}
$$

$$
\mathrm{moe}_n := \begin{bmatrix} \frac{1}{720} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\overline{1}}{48} & \frac{1}{24} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{16} & \frac{\overline{1}}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \dots & & & & & & & & & & \\ \frac{\overline{1}}{48} & \frac{1}{8} & \frac{\overline{1}}{4} & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\overline{1}}{6} & \frac{5}{6} & \overline{1} & \frac{\overline{1}}{2} & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{\overline{1}}{8} & \frac{1}{2} & \frac{\overline{1}}{4} & \frac{\overline{1}}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{18} & \frac{\overline{1}}{3} & \frac{1}{2} & \frac{1}{3} & 0 & \overline{1} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{8} & \frac{\overline{3}}{4} & \frac{5}{4} & \frac{1}{2} & \frac{\overline{1}}{2} & \overline{1} & \frac{\overline{1}}{2} & 0 & 1 & 0 & 0 \\ \frac{1}{5} & \overline{1} & 1 & 1 & 0 & \overline{1} & \overline{1} & 0 & 0 & 1 & 0 \\ \frac{\overline{1}}{6} & 1 & \frac{\overline{3}}{2} & \overline{1} & \frac{1}{3} & 2 & 1 & \frac{\overline{1}}{2} & \overline{1} & \overline{1} & 1 \end{bmatrix}
$$

Proposition 7.9 (Properties of the matrices $\boldsymbol{moe}_6$) .
Despite having rational rather than integer entries, the matrices $\boldsymbol{moe}_n$ are in some respects more regular than the matrices $\boldsymbol{meo}_n$. In particular:

$$
\sum_{\mathfrak{p}} \mathrm{moe}_n[\mathfrak{p},\mathfrak{q}] \;\equiv\; 1 \quad \textit{if} \quad \mathfrak{q} = [n] \quad \textit{and} \quad 0 \quad \textit{otherwise} \tag{356}
$$

$$
\sum_{\mathfrak{p}} |\mathrm{moe}_n[\mathfrak{p},\mathfrak{q}]| \;\equiv\; \frac{(d_1 + ... + d_s)!}{d_1! \dots d_s!} \quad \textit{if} \;\; \mathfrak{q} = [q_1^{(d_1)}, \dots, q_s^{(d_s)}] \tag{357}
$$

$$
\sum_{\mathfrak{q}} \mathrm{moe}_n[\mathfrak{p},\mathfrak{q}] \;\equiv\; \frac{1}{d_1!\, p_1^{d_1} ...\, d_s!\, p_s^{d_s}} \quad \textit{if} \;\; \mathfrak{p} = [p_1^{(d_1)}, \dots, p_s^{(d_s)}] \tag{358}
$$

The rows and columns of $\boldsymbol{meo}_n$, on the other hand, have completely unremarkable sums.

The second ingredient in this approach are the dual *pilot* and *copilot* polynomials. They derive from the conversion matrices. Actually, the copilot polynomials come first.

The copilot polynomials.

$Q_{1,2}(x_1) =$	$[\delta_1]$	$[\sigma_1, \sigma_2 ...]$
$+\frac{1}{2}\ \hat{x}_1\,(\hat{x}_1-1)$	$[1]$	$[1,1]$
$Q_{2,3}(x_1,x_2) =$	$[\delta_1]$	$[\sigma_1, \sigma_2 ...]$
$+\frac{1}{3}\ \hat{x}_1\,(\hat{x}_1-1)\,(\hat{x}_1-2)$	$[2]$	$[1,1,1]$
$+1\ \hat{x}_2\,(\hat{x}_1-1)$	$[2]$	$[2,1]$
$Q_{2,4}(x_1,x_2) =$	$[\delta_1, \delta_2]$	$[\sigma_1, \sigma_2 ...]$
$+\frac{1}{8}\ \hat{x}_1\,(\hat{x}_1-1)\,(\hat{x}_1-2)\,(\hat{x}_1-3)$	$[1,1]$	$[1,1,1,1]$
$+\frac{1}{2}\ \hat{x}_2\,(\hat{x}_1-1)\,(\hat{x}_1-2)$	$[1,1]$	$[2,1,1]$
$+\frac{1}{2}\ \hat{x}_2\,(\hat{x}_2-1)$	$[1,1]$	$[2,2]$
$Q_{3,4}(x_1,x_2,x_3) =$	$[\delta_1]$	$[\sigma_1, \sigma_2 ...]$
$+\frac{1}{4}\ \hat{x}_1\,(\hat{x}_1-1)\,(\hat{x}_1-2)\,(\hat{x}_1-3)$	$[3]$	$[1,1,1,1]$
$+\frac{3}{2}\ \hat{x}_2\,(\hat{x}_1-1)\,(\hat{x}_1-2)$	$[3]$	$[2,1,1]$
$+\frac{1}{2}\ \hat{x}_2\,(\hat{x}_2-1)$	$[3]$	$[2,2]$
$+1\ \hat{x}_3\,(\hat{x}_1-1)$	$[3]$	$[3,1]$
$Q_{3,5}(x_1,x_2,x_3) =$	$[\delta_1, \delta_2]$	$[\sigma_1, \sigma_2 ...]$
$+\frac{1}{6}\ \hat{x}_1\,(\hat{x}_1-1)\,(\hat{x}_1-2)\,(\hat{x}_1-3)\,(\hat{x}_1-4)$	$[2,1]$	$[1,1,1,1,1]$
$+\frac{3}{2}\ \hat{x}_2\,(\hat{x}_1-1)\,(\hat{x}_1-2)\,(\hat{x}_1-3)$	$[2,1]$	$[2,1,1,1]$
$+2\ \hat{x}_2\,(\hat{x}_2-1)\,(\hat{x}_1-2)$	$[2,1]$	$[2,2,1]$
$+1\ \hat{x}_3\,(\hat{x}_1-1)\,(\hat{x}_1-2)$	$[2,1]$	$[3,1,1]$
$+1\ \hat{x}_3\,(\hat{x}_2-1)$	$[2,1]$	$[3,2]$
$Q_{3,6}(x_1,x_2,x_3) =$	$[\delta_1, \delta_2, \delta_3]$	$[\sigma_1, \sigma_2 ...]$
$+\frac{1}{48}\ \hat{x}_1\,(\hat{x}_1-1)\,(\hat{x}_1-2)\,(\hat{x}_1-3)\,(\hat{x}_1-4), (\hat{x}_1-5)$	$[1,1,1]$	$[1,1,1,1,1,1]$
$+\frac{1}{4}\ \hat{x}_2\,(\hat{x}_1-1)\,(\hat{x}_1-2)\,(\hat{x}_1-3)\,(\hat{x}_1-4)$	$[1,1,1]$	$[2,1,1,1,1]$
$+\frac{1}{6}\ \hat{x}_3\,(\hat{x}_1-1)\,(\hat{x}_1-2)\,(\hat{x}_1-3)$	$[1,1,1]$	$[3,1,1,1]$
$+\frac{3}{4}\ \hat{x}_2\,(\hat{x}_2-1)\,(\hat{x}_1-2)\,(\hat{x}_1-3)$	$[1,1,1]$	$[2,2,1,1]$
$+\frac{1}{6}\ \hat{x}_2\,(\hat{x}_2-1)\,(\hat{x}_2-2)$	$[1,1,1]$	$[2,2,2]$
$+1\ \hat{x}_3\,(\hat{x}_2-1)\,(\hat{x}_1-2)$	$[1,1,1]$	$[3,2,1]$
$+\frac{1}{2}\ \hat{x}_3\,(\hat{x}_3-1)$	$[1,1,1]$	$[3,3]$

The pilot-copilot correspondence.

Proposition 7.10 (The $P_r \leftrightarrow Q_r$ and $P_{r,d} \leftrightarrow Q_{r,d}$ correspondence) .
For any n, let Par_n be set of all partitions $\mathfrak{n} := [n_1, \ldots, n_s]$ of n, with the n_i arranged in non-increasing order. To each n we also attach a matrix $\boldsymbol{moe}_n$ and two vectors $\boldsymbol{vo}_n(\boldsymbol{y})$, $\boldsymbol{ve}(Q_r)$:
(i) $\boldsymbol{moe}_n$ is the above defined square matrix, of entries $\boldsymbol{moe}_n[\mathfrak{p}, \mathfrak{q}]$, with $\mathfrak{p}$ and $\mathfrak{q}$ running through Par_n.
(ii) $\boldsymbol{vo}_n(\boldsymbol{y})$ is the vector of entries $\boldsymbol{y}^{\mathfrak{p}} = \prod y_i^{p_i}$ with $\mathfrak{p} = [p_1, p_2, ...]$ also running

through Par_n.
(iii) $\boldsymbol{ve}(Q_r)$ is the vector of entries $Q_r(\boldsymbol{q}^*) = Q_r(q_1^*, q_2^*, ...)$ with $\mathfrak{q} = [q_1, q_2, ...]$ again running through Par_n and $\mathfrak{q}^*$ denoting the conjugate partition:

$$\mathfrak{q}^* = [q_1^*, q_2^*, \dots] \quad with \quad q_i^* = \sum_{i \leqslant q_j} 1 \qquad \big(\sum q_i^* \equiv \sum q_j\big) \tag{359}$$

Then the identity holds:

$$P_r(\boldsymbol{y}) \exp(\sum \frac{1}{n} y_n) \quad \equiv \quad \sum_{r \leqslant n} <\boldsymbol{vo}_n(\boldsymbol{y}), \boldsymbol{moe}_n, \boldsymbol{ve}_n(Q_r)> \tag{360}$$

with both sides viewed as power series of $y_1, y_2, \dots$. If we view y_i as having homogeneous degree i and introduce a variable y to order the terms according to their global homogeneous degree, (360) becomes:

$$\big(\sum_{d=r+1}^{d=2\,r} y^d\, P_{r,d}(\boldsymbol{y})\big) \exp(\sum \frac{y^n}{n} y_n) \equiv \sum_{r \leqslant n} y^n <\boldsymbol{vo}_n(\boldsymbol{y}), \boldsymbol{moe}_n, \boldsymbol{ve}_n(Q_r)> \tag{361}$$

The identity (361) actually holds true for each pair $(P_{r,d}, Q_{r,d})$ separately:

$$y^d\, P_{r,d}(\boldsymbol{y}) \exp(\sum \frac{y^n}{n} y_n) \quad \equiv \quad \sum_{r \leqslant n} y^n <\boldsymbol{vo}_n(\boldsymbol{y}), \boldsymbol{moe}_n, \boldsymbol{ve}_n(Q_{r,d})> \tag{362}$$

$$\equiv \quad \sum_{d \leqslant n} y^n <\boldsymbol{vo}_n(\boldsymbol{y}), \boldsymbol{moe}_n, \boldsymbol{ve}_n(Q_{r,d})> \tag{363}$$

The elementary identities $\boldsymbol{ve}_n(Q_{r,d}) \equiv 0$ for n in the interval $[r, d[$ ensure the equivalence of (362) and (363).

The pilot-copilot correspondence at the most basic level.

Proposition 7.11 (The $P^{[r_1,\dots,r_s]} \leftrightarrow Q^{[r_1,\dots,r_s]}$ and correspondence) .
If we define the atomic pilot and copilot polynomials as follows

$$Q^{[r_1,\dots,r_s]}(x_1, x_2, ...) := \prod_{1 \leqslant i \leqslant r} (x_{r_i} - i + 1) \qquad (r_1 \geqslant r_2 \geqslant ...r_s) \tag{364}$$

$$P^{[r_1,\dots,r_s]}(y_1, y_2, ...) := \sum_{\mathcal{J}_1 \cup \dots \mathcal{J}_t = \{1,\dots,s\}}^{1 \leqslant t \leqslant s} (-1)^{s-t} \prod_{1 \leqslant i \leqslant s} \Gamma(\mathcal{J}_i)\; y_{r(\mathcal{J}_i)} \tag{365}$$

$$with \; := \begin{cases} \Gamma(\mathcal{J}_i) & := (\#(\mathcal{J}_i) - 1)! \\ r(\mathcal{J}_i) & := \sum_{k \in \mathcal{J}_i} \; r_k \end{cases} \tag{366}$$

the early correspondence still holds

$$y^{|\boldsymbol{r}|} P^{[r_1,..,r_s]}(\boldsymbol{y}) \exp(\sum \frac{y^n}{n} y_n) \equiv \sum_{r_1 \leqslant n} y^n <\boldsymbol{vo}_n(\boldsymbol{y}), \boldsymbol{moe}_n, \boldsymbol{ve}_n(Q^{[r_1,..,r_s]})> \tag{367}$$

$$\equiv \sum_{|\boldsymbol{r}| \leqslant n} y^n <\boldsymbol{vo}_n(\boldsymbol{y}), \boldsymbol{moe}_n, \boldsymbol{ve}_n(Q^{[r_1,..,r_s]})> \tag{368}$$

with $|\boldsymbol{r}| := r_1 + \cdots + r_s$ *and with the elementary identities*

$$\boldsymbol{ve}_n(Q^{[r_1,\ldots,r_s]}) \equiv 0 \qquad \forall n \in [\, r_1, |\boldsymbol{r}|\, [$$

ensuring the equivalence of (367) and (368).

Except in the trivial case $P^{[n_1]}(\boldsymbol{y}) = y_{n_1}, Q^{[n_1]}(\boldsymbol{x}) = x_{n_1}$, the construction (367)-(368) implies

$$\begin{aligned} P^{[n_1,\ldots,n_s]}(y, y^2, y^3, \ldots) &\equiv 0 && (369)\\ P^{[n_1,\ldots,n_s]}(y, y, y, \ldots) &\equiv y\,(y-1)\,(y-2)\ldots(y-s+1) && (370) \end{aligned}$$

Here are two elementary instances of pilot-copilot pairs:

$$\begin{cases} P^{[n_1,n_2]}(\boldsymbol{y}) = y_{n_1} y_{n_2} - y_{n_1+n_2} \\ Q^{[n_1,n_2]}(\boldsymbol{x}) = x_{n_1}(x_{n_2}-1) \qquad (n_1 \geqslant n_2) \end{cases}$$

$$\begin{cases} P^{[n_1,n_2,n_3]}(\boldsymbol{y}) = y_{n_1} y_{n_2} y_{n_3} - y_{n_1+n_2} y_{n_3} - y_{n_1+n_3} y_{n_2} - y_{n_2+n_3} y_{n_1} + 2\, y_{n_1+n_2+n_3} \\ Q^{[n_1,n_2,n_3]}(\boldsymbol{x}) = x_{n_1}(x_{n_2}-1)\,(x_{n_3}-2) \qquad (n_1 \geqslant n_2 \geqslant n_3) \end{cases}$$

Here is yet another example, with all n_i's equal to distinct powers of 2 to preempt repetitions in the sums of n_i's:

$$\begin{cases} P^{[8,4,2,1]}(\boldsymbol{y}) = \begin{cases} +0!\,0!\,0!\,0!\, y_1\, y_2\, y_4\, y_8 \\ -0!\,0!\,1!\,\big(y_1 y_4 y_{10} + y_1 y_2 y_{12} + y_1 y_8 y_6 + y_2 y_4 y_9 + y_2 y_8 y_5 + y_4 y_8 y_3\big) \\ +0!\,2!\,\big(y_1 y_{14} + y_4 y_{11} + y_2 y_{13} + y_8 y_7\big) \\ +1!\,1!\,\big(y_6 y_9 + y_3 y_{12} + y_5 y_{10}\big) \\ -3!\, y_{15}; \end{cases} \\ Q^{[8,4,2,1]}(\boldsymbol{x}) = x_8\,(x_4-1)\,(x_2-2)\,(x_1-3) \end{cases}$$

Remark: Let $Par_r(\boldsymbol{y})$ be the linear space spanned by the monomials $\boldsymbol{y}^{\mathfrak{r}} := \prod y_{r_i}$, with $\mathfrak{r} = [r_1, ..., r_s]$ running through Par_r. Then the endomorphism f of $Par_r(\boldsymbol{y})$

$$f: \qquad y_{r_1} \ldots y_{r_s} \mapsto \sum_{\mathcal{J}_1 \cup \ldots \mathcal{J}_t = \{1,\ldots,s\}}^{1 \leqslant t \leqslant s} (-1)^{s-t} \prod_{1 \leqslant i \leqslant s} \Gamma(\mathcal{J}_i)\;\, y_{r(\mathcal{J}_i)} \tag{371}$$

which according to Proposition 8.9 encodes the correspondence $Q^{\boldsymbol{r}} \mapsto P^{\boldsymbol{r}}$, admits an even simpler inverse

$$f^{-1}: \qquad y_{r_1} \ldots y_{r_s} \mapsto \sum_{\mathcal{J}_1 \cup \ldots \mathcal{J}_t = \{1,\ldots,s\}}^{1 \leqslant t \leqslant s} \;\prod_{1 \leqslant i \leqslant s} y_{r(\mathcal{J}_i)} \tag{372}$$

which encodes the correspondence $P^{\boldsymbol{r}} \mapsto Q^{\boldsymbol{r}}$

Higher order contributions: going from $\gamma^*_\delta(t)$ to $\gamma_\delta(t)$.

Piecing together the results of this section, we can state that the restricted, double generating function:

$$\Gamma^*(t,x) := \gamma^*_0(t) + \gamma^*_1(t)\,x + \gamma^*_2(t)\,x^2 + \dots \quad \begin{cases} \gamma^*_0 = \gamma_0 \\ \gamma^*_\delta \neq \gamma_\delta \;\; \textit{if} \;\; 1 \leqslant \delta \end{cases} \tag{373}$$

verifies the equation

$$\Gamma^*(t,x) \;=\; t\,\mathcal{F}\big(x, \Gamma^*(t,0), \Gamma^*(t^2,0), \Gamma^*(t^3,0), \dots\big) \tag{374}$$

$$\;=\; t\,\mathcal{F}\big(x, \gamma(t), \gamma(t^2), \gamma(t^3), \dots\big) \tag{375}$$

though with a framing function $\mathcal{F}$ defined, not directly by (302) but rather by (302) plus (303) relative to the pilot polynomials $P_{r,d}$ determined in Proposition 7.11.

Finally, to 'unrestrict' the series $\Gamma^*(t,x)$ to the full series $\Gamma(t,x)$ and establish the functional equation

$$\Gamma(t,x) = t\,\mathcal{F}\big(x, \Gamma(t,x), \Gamma(t^2,x^2), \Gamma(t^3,x^3), \dots\big)$$

we use the following lemma:

Lemma 7.2 ....

In conclusion, for all its detours and meanderings, the approach followed in this section has the merit of drawing attention to the remarkable conversion matrices $\boldsymbol{meo}_n.\boldsymbol{moe}_n$ and of breaking down the pilot polynomials $P_{r,d}$ into completely explicit 'atoms':

$$P_{r,d}(t) = \sum \alpha_{d_1,\dots,d_s}\, P^{[d_1,\dots,d_s]}(t) \quad \textit{with} \quad \begin{cases} d_1 + \dots + d_s = d \\ \alpha_{d_1,\dots,d_s} \in \mathbb{Q}^+ \end{cases}$$

7.7 Sketch of proof: second approach.

As in §5.10 let us decompose the associative algebra $\mathbb{E}$ freely generated by elements $e_1, e_2 \dots$ into spaces $\mathbb{E}^{s_1,\dots,s_r}$ of degree s_i in e_i, and let us split these spaces into sums

$$\mathbb{E}^{s_1,\dots,s_r} = \oplus_{1 \delta \leqslant s} \mathbb{E}^{s_1,\dots,s_r}_d \quad (s = \sum s_i)$$

of subspacesc $\mathbb{E}^{\boldsymbol{s}}_d$ consisting of distinguished elements of 'differential' degree d. The immediate aim is to calculate $\dim(\mathbb{E}^{s_1,\dots,s_r}_d)$. For $d=1$ we have classically:

$$\dim(\mathbb{E}^{s_1,\dots,s_r}) = \frac{s!}{s_1! \dots s_r!} \quad ; \quad \dim(\mathbb{E}^{s_1,\dots,s_r}_1) = \frac{1}{s} \sum_{\delta | s_i} \mu(d) \frac{(s/d)!}{(s_1/d)! \dots (s_r/d)!}$$

Next, consider the three cgenerating series:

$$S(\boldsymbol{x}) \;=\; 1+\sum_{1\leqslant r}\sum_{s_1,s_2\ldots}^{i_1<i_2\ldots} x_{i_1}^{s_1}...x_{i_r}^{s_r}\,\dim(\mathbb{E}_1^{s_1,\ldots,s_r}) \;=\; \big(1-\sum x_i\big)^{-1} \tag{376}$$

$$M(\boldsymbol{x}) \;=\; \sum_{1\leqslant r}\sum_{s_1,s_2\ldots}^{i_1<i_2\ldots} x_{i_1}^{s_1}...x_{i_r}^{s_r}\,\dim(\mathbb{E}_1^{s_1,\ldots,s_r}) \tag{377}$$

$$S_w(\boldsymbol{x}) \;=\; 1+\sum_{1\leqslant r}^{1\leqslant d}\sum_{s_1,s_2\ldots}^{i_1<i_2\ldots} x_{i_1}^{s_1}...x_{i_r}^{s_r}\,\dim(\mathbb{E}_d^{s_1,\ldots,s_r}) \tag{378}$$

Clearly

$$S_w(\boldsymbol{x}) \;=\; \exp\Big(\sum_{1\leqslant k}\frac{w^k}{k}\,M(x_1^k,x_2^k,\ldots)\Big) \qquad \textit{with} \qquad S_1(\boldsymbol{x})\equiv S(\boldsymbol{x}) \tag{379}$$

and in the special case $w=1$:

$$S_1(\boldsymbol{x}) \;=\; \exp\Big(\sum_{1\leqslant k}\frac{w^k}{k}\,M(x_1^k,x_2^k,\ldots)\Big) \equiv S(\boldsymbol{x}) \tag{380}$$

By Möbius inversion, (380) leads to

$$M(\boldsymbol{x}) \;=\; \sum_{1\leqslant k}\frac{1}{k}\,\log\big(S(x_1^k,x_2^k,\ldots)\big) = -\sum_{1\leqslant k}\frac{1}{k}\,\log\big((1-\sum x_i^k)\big)\big) \tag{381}$$

and then to

$$S_w(x) \;=\; \prod\Big(1-\sum_i x_i^k\Big)^{-\frac{1}{k}\sum_{d|k}\mu(\frac{k}{d})\,w^d} \tag{382}$$

So we can now calculate $\dim(\mathbb{E}_d^{\boldsymbol{s}})$ from (382). However, as noted in §7, the elements of $Flex_r^{root}(\mathfrak{E})$ of alternality co-degree $\delta=d-1$ that can be obtained as counter-alternaltors of elements alternal elements correspond do not correspond to elements of $\mathbb{E}^{\boldsymbol{s}}$ of degree d, but of supplementory degree $s-d$. This leads us to replace the series S_w by $\mathcal{G}$ with:

$$\mathcal{G}(z;x_1,x_2,\ldots) \;=\; S_{z^{-1}}(x_1\,z,x_2\,z,\ldots) \tag{383}$$

$$=\; \prod\Big(1-\sum x_i^k\,z^k\Big)^{-\imath k\sum_{d|k}\mu(kd)\,z^{-d}} \tag{384}$$

But going from $\mathcal{G}(z;\boldsymbol{x})$ as just defined to $\mathcal{F}(x;\boldsymbol{y})$ as defined in (302) exactly corresponds to applying the transform (367) with the matrices moe_r. So, the restricted generating series $\Gamma^*(t,x)$ verifies the functional equation (374). We can then go to the full generation series $\Gamma(t,x)$ and its functional equation (315) by using the same trick than at the end of §7.6.

7.8 Dimensions.

Here are the first dimensions $\dim\big(Flex^{root}_{r,d}\big)$ and $\dim\big(Flex_{r,d}\big)$. All calculations are based on the functional equation (315) in the 'practical' form (318). The columns are indexed by the depth r and the row by the co-degree d. For consistency, we used d rather than the shifted co-degree $\delta = d-1$ even in the first table. The reader may check that, in both tables, the entries in each column sum up to a Catalan number; and also that the first lines of either table (corresponding to the alternals) do coincide, as implied by (320)-(321).

Table for $\dim\big(Flex^{root}_{r,d}\big)$.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	1	2	4	9	20	48	115	286	719	1842	4766	12486	32973	87811
2	0	0	0	1	4	15	49	156	479	1452	4343	12908	38146	112358	330064
3	0	0	0	0	1	6	27	108	405	1446	5013	16953	56321	184385	596741
4	0	0	0	0	0	1	7	40	191	839	3440	13475	50889	186888	670807
5	0	0	0	0	0	0	1	8	58	317	1568	7197	31258	129898	521166
6	0	0	0	0	0	0	0	1	10	76	476	2654	13539	64729	293759
7	0	0	0	0	0	0	0	0	1	12	100	693	4249	23749	123608
8	0	0	0	0	0	0	0	0	0	1	13	124	954	6433	39183
9	0	0	0	0	0	0	0	0	0	0	1	15	153	1285	9391
10	0	0	0	0	0	0	0	0	0	0	0	1	16	183	1672
11	0	0	0	0	0	0	0	0	0	0	0	0	1	18	218
12	0	0	0	0	0	0	0	0	0	0	0	0	0	1	19
13	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

Table for $\dim\big(Flex_{r,d}\big)$.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	1	2	4	9	20	48	115	286	719	1842	4766	12486	32973	87811
2	0	1	2	6	16	45	123	344	957	2687	7557	21358	60516	172034	490204
3	0	0	1	3	12	41	138	446	1428	4497	14068	43668	134911	414952	1272130
4	0	0	0	1	4	20	83	328	1222	4422	15554	53702	182423	611986	2031248
5	0	0	0	0	1	5	30	147	667	2815	11364	44164	166881	615935	2230554
6	0	0	0	0	0	1	6	42	237	1216	5737	25586	108917	447319	1783137
7	0	0	0	0	0	0	1	7	56	358	2049	10687	52194	241591	1071839
8	0	0	0	0	0	0	0	1	8	72	514	3249	18566	98584	493086
9	0	0	0	0	0	0	0	0	1	9	90	710	4911	30517	175067
10	0	0	0	0	0	0	0	0	0	1	10	110	950	7140	47938
11	0	0	0	0	0	0	0	0	0	0	1	11	132	1239	10053
12	0	0	0	0	0	0	0	0	0	0	0	1	12	156	1581
13	0	0	0	0	0	0	0	0	0	0	0	0	1	13	182
14	0	0	0	0	0	0	0	0	0	0	0	0	0	1	14
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

8 Main bases of $Flex(\mathfrak{E}^\bullet)$ and $Flex^{al}(\mathfrak{E}^\bullet)$.

8.1 mu-generation versus dmu-generation.

Instead of the counter-alternators, consider the simpler operation $malt$:

$$\begin{array}{rcl} A_0^\bullet,\dots,A_s^\bullet & \mapsto & \mathrm{malt}_{A_0^\bullet}(A_1^\bullet,...,A_s^\bullet) := \mathrm{dmu}\big(A_0^\bullet, \mathrm{mu}(A_1^\bullet,...,A_s)\big) \\ \otimes_i Flex_{r_i}(\mathfrak{E}^\bullet) & \rightarrow & Flex_{r_0+...+r_s}(\mathfrak{E}^\bullet) \end{array} \qquad (385)$$

and the system $\{\mathfrak{mote}^\bullet_{r,k}\}$ constructed parallel to the system $\{\hat{\mathfrak{o}}\mathfrak{te}^\bullet_{r,k}\}$, but with the operations *malt, mu* now playing the part of *cǎlt*, *dmu*, as follows:

$$
\begin{array}{lcll}
\mathfrak{mote}^\bullet_{1,1} & := & \mathfrak{E}^\bullet & (\textit{prime element}) \\[6pt]
\mathfrak{mote}^\bullet_{2,1} & := & \mathrm{malt}_{\mathfrak{E}^\bullet}(\mathfrak{mote}^\bullet_{1,1}) & (\textit{prime element}) \\
\mathfrak{mote}^\bullet_{2,2} & := & \mathrm{mu}(\mathfrak{mote}^\bullet_{1,1},\mathfrak{mote}^\bullet_{1,1}) & (\textit{composite element}) \\[6pt]
\mathfrak{mote}^\bullet_{3,1} & := & \mathrm{malt}_{\mathfrak{E}^\bullet}(\mathfrak{mote}^\bullet_{2,1}) & (\textit{prime element}) \\
\mathfrak{mote}^\bullet_{3,2} & := & \mathrm{malt}_{\mathfrak{E}^\bullet}(\mathfrak{mote}^\bullet_{1,1},\mathfrak{mote}^\bullet_{1,1}) & (\textit{prime element}) \\
\mathfrak{mote}^\bullet_{3,3} & := & \mathrm{mu}(\mathfrak{mote}^\bullet_{1,1},\mathfrak{mote}^\bullet_{2,1}) & (\textit{composite element}) \\
\mathfrak{mote}^\bullet_{3,4} & := & \mathrm{mu}(\mathfrak{mote}^\bullet_{2,1},\mathfrak{mote}^\bullet_{1,1}) & (\textit{composite element}) \\
\mathfrak{mote}^\bullet_{3,5} & := & \mathrm{mu}(\mathfrak{mote}^\bullet_{1,1},\mathfrak{mote}^\bullet_{1,1},\mathfrak{mote}^\bullet_{1,1}) & (\textit{composite element}) \\[6pt]
\textit{etc}\ldots & & &
\end{array}
$$

Lemma 8.1 (*mu*-generated basis) .
The 'prime elements' or 'generators' $\{\mathfrak{mote}^\bullet_{r,k}; k \leqslant \kappa_{r-1}\}$*, together with their* mu-*products* $\{\mathfrak{mote}^\bullet_{r,k}; \kappa_{r-1} < k \leqslant \kappa_r\}$*, constitute a basis of* $\mathrm{Flex}(\mathfrak{E}^\bullet)$ *naturally indexed by ordered trees.*

The shortest proof again lies in the fact that, for a suitable ordering on $\mathbb{OT}$ and $\mathbb{BT}$, the $\mathbb{OT}$-indexed $\mathfrak{mote}^\bullet_{r,k}$ relate to the $\mathbb{BT}$-indexed $\mathfrak{bte}^\bullet_{r,k}$ according to

$$
\begin{array}{lcllll}
\mathfrak{mote}^\bullet_{r,k_1} & = & \sum_{k_2\leqslant\kappa_r} a^{r,k_2}_{r,k_1}\ \mathfrak{bte}^\bullet_{r,k_2} & \textit{with} & a^{r,k_2}_{r,k_1} \in \{0,1\} \\
\mathfrak{bte}^\bullet_{r,k_1} & = & \sum_{k_2\leqslant\kappa_r} b^{r,k_2}_{r,k_1}\ \mathfrak{mote}^\bullet_{r,k_2} & \textit{with} & b^{r,k_2}_{r,k_1} \in \{0,1,-1\}
\end{array}
$$

with triangular matrices $[a^\bullet_\bullet]$ and $[b^\bullet_\bullet]$ whose diagonal elements are all $\equiv 1$.

The system $\{\mathfrak{mote}^\bullet_{r,k}\}$ is essentially the simplest[55] basis of $Flex(\mathfrak{E}^\bullet)$ that comes with a natural $\mathbb{OT}$-indexation, but it does not lead to simple expressions for the basic mould operations. Worse still, it does not, any more than $\{\mathfrak{bte}^\bullet_{r,k}\}$, reflect the alternality stratification of $Flex(\mathfrak{E}^\bullet)$. Actually, no system relying on *mu*-generation can reflect that stratification. The reason is that, if the space spanned by the 'prime elements' or 'generators' is to contain the whole of $Flex^{al}(\mathfrak{E}^\bullet)$, then the 'composites elements' will automatically reproduce some of the alternal elements already constructed (since superpositions of *mu*-products cover all *lu*-bracketings) and cannot therefore be truly independent of the 'primes'. So, to get alternality-friendly bases, we must now return to the proper counter-alternators.

8.2 Main bases of $Flex(\mathfrak{E}^\bullet)$.

Proposition 8.1 (Counter-alternators and bases of $Flex(\mathfrak{E}^\bullet)$) .
(i) The elements $\hat{\mathfrak{o}}\mathfrak{te}^\bullet_{r,k}$ *indexed by ordered, one-rooted trees* $\mathrm{ot}_{r,k}$ *($k \leqslant \kappa_{r-1}$) and built from the main counter-alternator* calt *according to the procedure of*

[55]in the sense of making minimal use of inflected operations.

§.. condtitute a full system of dmu- *and* mdu-*generators, in the sense that any element* $\mathfrak{e}^\bullet \in Flex(\mathfrak{E}^\bullet)$ *admits a unique expansion of the form*

$$\mathfrak{e}^\bullet \;=\; \sum_{1\leqslant s\leqslant r}^{r_1+\ldots+r_s=r} \;\sum_{k_i\leqslant \kappa_{r_i-1}} \hat{a}^{\left(\begin{smallmatrix} r_1 & \ldots & r_s \\ k_1 & \ldots & k_s\end{smallmatrix}\right)}\, \overleftarrow{\mathrm{dmu}}\big(\hat{\mathfrak{ote}}^\bullet_{r_1,k_1},\ldots,\hat{\mathfrak{ote}}^\bullet_{r_s,k_s}\big) \qquad (386)$$

and another of the form

$$\mathfrak{e}^\bullet \;=\; \sum_{1\leqslant s\leqslant r}^{r_1+\ldots+r_s=r} \;\sum_{k_i\leqslant \kappa_{r_i-1}} \hat{b}^{\left(\begin{smallmatrix} r_1 & \ldots & r_s \\ k_1 & \ldots & k_s\end{smallmatrix}\right)}\, \overrightarrow{\mathrm{mdu}}\big(\hat{\mathfrak{ote}}^\bullet_{r_1,k_1},\ldots,\hat{\mathfrak{ote}}^\bullet_{r_s,k_s}\big) \qquad (387)$$

with a conversion formula $\hat{a}^\bullet \leftrightarrow \hat{b}^\bullet$ *given by the identity (??).*
The closely related system $\{\mathrm{ate}^\bullet_{r,d,r}; k \leqslant \gamma_{r,d}\}$ *indexed by aggregated trees* $\mathrm{at}_{r,d,k}$ *faithfully reflects the alternality gradation of* $\mathrm{Flex}(\mathfrak{E}^\bullet)$*.*

(ii) The same holds for the elements $\check{\mathfrak{ote}}^\bullet_{r,k}$ *built from the second counter-alternator* $\check{\mathrm{calt}}$*, but with a less simple concersion rule for the expansions*

$$\mathfrak{e}^\bullet \;=\; \sum_{1\leqslant s\leqslant r}^{r_1+\ldots+r_s=r} \;\sum_{k_i\leqslant \kappa_{r_i-1}} \check{a}^{\left(\begin{smallmatrix} r_1 & \ldots & r_s \\ k_1 & \ldots & k_s\end{smallmatrix}\right)}\, \overleftarrow{\mathrm{dmu}}\big(\check{\mathfrak{ote}}^\bullet_{r_1,k_1},\ldots,\check{\mathfrak{ote}}^\bullet_{r_s,k_s}\big) \qquad (388)$$

$$\mathfrak{e}^\bullet \;=\; \sum_{1\leqslant s\leqslant r}^{r_1+\ldots+r_s=r} \;\sum_{k_i\leqslant \kappa_{r_i-1}} \check{b}^{\left(\begin{smallmatrix} r_1 & \ldots & r_s \\ k_1 & \ldots & k_s\end{smallmatrix}\right)}\, \overrightarrow{\mathrm{mdu}}\big(\check{\mathfrak{ote}}^\bullet_{r_1,k_1},\ldots,\check{\mathfrak{ote}}^\bullet_{r_s,k_s}\big) \qquad (389)$$

and with further differences pertaining to the behaviour of the generators $\hat{\mathrm{s}}\mathrm{te}^\bullet_{r,k}$ *under the brackets lu and ari.*

(iii) The elements $\bar{\mathfrak{ote}}^\bullet_{r,k}$ *constructed in the same way from the weakly inflected counter-alternator* $\bar{\mathrm{calt}}$ *do not constitute a full system of* dmu *or* mdu *generators. Indeed, from* $r = 7$ *onwards the generators* $\bar{\mathfrak{ote}}^\bullet_{r,k}$ $(k \leqslant \kappa_{r-1})$ *cease to be independent; and the same holds even earlier, starting from* $r = 4$*, for the would-be basis elements* $\bar{\mathfrak{ote}}^\bullet_{r,k}$ $(k \leqslant \kappa_r)$*.*

However, the elements $\bar{\mathfrak{ute}}^\bullet_{r,1,k}$ $(k \leqslant \gamma_{1,r})$ *constructed from* $\bar{\mathrm{calt}}$ *and indexed by unordered trees in* $\mathbb{UT}$[56] *do constitute a basis of* $\mathrm{Flex}^{\mathrm{al}}_r(\mathfrak{E}^\bullet)$ *(alternal elements) which in some contexts compares advantageously with the analogous bases derived from* calt *or* $\hat{\mathrm{calt}}$*. See §9... and §9 infra.*

Sketch of proof:

(i) The independence of the generator set $\hat{\mathfrak{ote}}^\bullet_{r,k}$ $(k \leqslant \kappa_{r-1})$ follows from that of the full set $\hat{\mathfrak{ote}}^\bullet_{r,k}$ $(k \leqslant \kappa_r)$ and the latter follows from the form (lower triangular, with non-zero diagonal entries) of the conversion matrices exchanging the $\mathbb{OT}$-indexed systems $\{\hat{\mathfrak{ote}}^\bullet_{r,k}\,;\, k \leqslant \kappa_r\}$ and the $\mathbb{BT}$-indexed systems $\{\hat{\mathfrak{bte}}^\bullet_{r,k}\,;\, k \leqslant \kappa_r\}$, as explained in §...

[56] or, what amounts to the same, by aggregated trees in $\mathbb{AT}^{(1)}$.

(ii) Here again, the proofs relies on the triangular nature of the conversion matrices. Due to the lesser importance of that case, we haven't described these matrices, but some instances may be found in the tables of §13.3.

(iii) Even for small depths r, the dimensions $\bar{\kappa}_{r-1}$ of the spaces spanned by the $c\bar{a}lt$-based generators $\{\bar{\mathfrak{o}}\mathfrak{te}_{r,k}\,;\, k \leqslant \kappa_{r-1}\}$ start lagging behind the required number κ_{r-1}, and the dimensions $\bar{\bar{\kappa}}_r$ of the full systems $\{\bar{\mathfrak{o}}\mathfrak{te}_{r,k}\,;\, k \leqslant \kappa_r\}$ start diverging from κ_r even sooner and faster, as shown by this table:

r	1	2	3	4	5	6	7	8
$(\bar{\kappa}_{r-1},\kappa_{r-1})$	$(1,1)$	$(1,1)$	$(2,2)$	$(5,5)$	$(14,14)$	$(42,42)$	$(131,132)$	$(423,429)$
$(\bar{\bar{\kappa}}_r,\kappa_r)$	$(1,1)$	$(2,2)$	$(5,5)$	$(13,14)$	$(36,42)$	$(104,132)$	$(316,429)$	$(...,1430)$

To illustrate the mechanism behind these discrepancies, here are the first dependence relations between general elements or between generators, at depth $r = 4$ and $r = 7$ respectively:

$$0 \equiv +\bar{\mathfrak{o}}\mathfrak{te}^\bullet_{4,1} - \bar{\mathfrak{o}}\mathfrak{te}^\bullet_{4,2} + \bar{\mathfrak{o}}\mathfrak{te}^\bullet_{4,3} - \bar{\mathfrak{o}}\mathfrak{te}^\bullet_{4,4} + 2\,\bar{\mathfrak{o}}\mathfrak{te}^\bullet_{4,8} - 2\,\bar{\mathfrak{o}}\mathfrak{te}^\bullet_{4,9} \tag{390}$$

$$0 \equiv +3\,\mathfrak{e}^\bullet_1 + 4\,\mathfrak{e}^\bullet_2 + 3\,\mathfrak{e}^\bullet_3 \tag{391}$$

with

$$\begin{cases} \mathfrak{e}^\bullet_1 = +\bar{\mathfrak{o}}\mathfrak{te}^\bullet_{7,62} - \bar{\mathfrak{o}}\mathfrak{te}^\bullet_{7,63} - \bar{\mathfrak{o}}\mathfrak{te}^\bullet_{7,64} + \bar{\mathfrak{o}}\mathfrak{te}^\bullet_{7,65} \\ \mathfrak{e}^\bullet_2 = +\bar{\mathfrak{o}}\mathfrak{te}^\bullet_{7,90} - \bar{\mathfrak{o}}\mathfrak{te}^\bullet_{7,91} - \bar{\mathfrak{o}}\mathfrak{te}^\bullet_{7,99} + \bar{\mathfrak{o}}\mathfrak{te}^\bullet_{7,100} - \bar{\mathfrak{o}}\mathfrak{te}^\bullet_{7,104} + \bar{\mathfrak{o}}\mathfrak{te}^\bullet_{7,105} + \bar{\mathfrak{o}}\mathfrak{te}^\bullet_{7,106} - \bar{\mathfrak{o}}\mathfrak{te}^\bullet_{7,107} \\ \mathfrak{e}^\bullet_2 = -\bar{\mathfrak{o}}\mathfrak{te}^\bullet_{7,118} + \bar{\mathfrak{o}}\mathfrak{te}^\bullet_{7,119} + \bar{\mathfrak{o}}\mathfrak{te}^\bullet_{7,122} - -\bar{\mathfrak{o}}\mathfrak{te}^\bullet_{7,123} \end{cases}$$

Let us parse (**??**) first:

$$\begin{aligned} \bar{\mathfrak{o}}\mathfrak{te}^\bullet_{4,1}\ ,\ \bar{\mathfrak{o}}\mathfrak{te}^\bullet_{4,1} &\in \mathrm{Flex}_{4,1}(\mathfrak{E}^\bullet) \\ \bar{\mathfrak{o}}\mathfrak{te}^\bullet_{4,3} - \bar{\mathfrak{o}}\mathfrak{te}^\bullet_{4,4} &\in \mathrm{Flex}_{4,2}(\mathfrak{E}^\bullet) \\ \bar{\mathfrak{o}}\mathfrak{te}^\bullet_{4,8} - \bar{\mathfrak{o}}\mathfrak{te}^\bullet_{4,9} &= \mathrm{dmu}(\bar{\mathfrak{o}}\mathfrak{te}^\bullet_{2,1}, \bar{\mathfrak{o}}\mathfrak{te}^\bullet_{2,1}) - \mathrm{dmu}(\bar{\mathfrak{o}}\mathfrak{te}^\bullet_{3,1}, \bar{\mathfrak{o}}\mathfrak{te}^\bullet_{1,1}) \in \mathrm{Flex}_{4,2}(\mathfrak{E}^\bullet) \end{aligned}$$

Now, to (**??**). Here are the three elements of $Flex(\mathfrak{E}^\bullet)$

$$\begin{aligned} \mathfrak{a}^\bullet_1 &:= \bar{\mathfrak{o}}\mathfrak{te}^\bullet_{1,1} \in \mathrm{Flex}_{1,1}(\mathfrak{E}^\bullet) \\ \mathfrak{a}^\bullet_2 &:= \bar{\mathfrak{o}}\mathfrak{te}^\bullet_{2,1} \in \mathrm{Flex}_{2,1}(\mathfrak{E}^\bullet) \\ \mathfrak{a}^\bullet_3 &:= \bar{\mathfrak{o}}\mathfrak{te}^\bullet_{3,1} - \bar{\mathfrak{o}}\mathfrak{te}^\bullet_{3,2} \in \mathrm{Flex}_{3,1}(\mathfrak{E}^\bullet) \end{aligned}$$

that go into the making of $\mathfrak{e}^\bullet_1, \mathfrak{e}^\bullet_2, \mathfrak{e}^\bullet_3$:

$$\begin{aligned} \mathfrak{e}^\bullet_1 &= \mathrm{c\bar{a}lt}_{\mathfrak{E}}(\mathfrak{a}^\bullet_3, \mathfrak{a}^\bullet_3) && \in \mathrm{Flex}_{7,2}(\mathfrak{E}^\bullet) \\ \mathfrak{e}^\bullet_2 &= \begin{cases} +\mathrm{c\bar{a}lt}_{\mathfrak{E}^\bullet}(\mathfrak{a}^\bullet_1, \mathfrak{a}^\bullet_2, \mathfrak{a}^\bullet_3) + \mathrm{c\bar{a}lt}_{\mathfrak{E}^\bullet}(\mathfrak{a}^\bullet_3, \mathfrak{a}^\bullet_1, \mathfrak{a}^\bullet_2) \\ -\mathrm{c\bar{a}lt}_{\mathfrak{E}^\bullet}(\mathfrak{a}^\bullet_2, \mathfrak{a}^\bullet_1, \mathfrak{a}^\bullet_3) - \mathrm{c\bar{a}lt}_{\mathfrak{E}^\bullet}(\mathfrak{a}^\bullet_3, \mathfrak{a}^\bullet_2, \mathfrak{a}^\bullet_1) \end{cases} && \in \mathrm{Flex}_{7,3}(\mathfrak{E}^\bullet) \\ \mathfrak{e}^\bullet_3 &= \begin{cases} +\mathrm{c\bar{a}lt}_{\mathfrak{E}^\bullet}(\mathfrak{a}^\bullet_1, \mathfrak{a}^\bullet_2, \mathfrak{a}^\bullet_1, \mathfrak{a}^\bullet_2) + \mathrm{c\bar{a}lt}_{\mathfrak{E}^\bullet}(\mathfrak{a}^\bullet_2, \mathfrak{a}^\bullet_1, \mathfrak{a}^\bullet_2, \mathfrak{a}^\bullet_1) \\ -\mathrm{c\bar{a}lt}_{\mathfrak{E}^\bullet}(\mathfrak{a}^\bullet_1, \mathfrak{a}^\bullet_2, \mathfrak{a}^\bullet_2, \mathfrak{a}^\bullet_1) - \mathrm{c\bar{a}lt}_{\mathfrak{E}^\bullet}(\mathfrak{a}^\bullet_2, \mathfrak{a}^\bullet_1, \mathfrak{a}^\bullet_1, \mathfrak{a}^\bullet_2) \end{cases} && \in \mathrm{Flex}_{7,3}(\mathfrak{E}^\bullet) \end{aligned}$$

8.3 Main bases of $Flex^{al}(\mathfrak{E}^{\bullet})$.

8.4 Lie and pre-Lie brackets in the main bases.

Lemma 8.2 (Tree operations: attachment and insertion) .
Let $\mathrm{T}_1, \mathrm{T}_2, \mathrm{T}_3$ *denote ordered, planar, one-rooted trees, i.e. elements of* $\mathbb{OT}^{(1)}$.

(i) Attaching T_2 *to* T_1 *means connecting the root node* j_* *of* T_2 *to some node* i *of* T_1 *via a new vertex* v *from* i *to* j_**, and assigning* v *a definite position among the various edges of* T_1 *issuing from* i*, so as to unambiguously define a new ordered tree* T_3*. If* T_1 *has* r_1 *nodes, there are exactly* $2\,r_1-1$ *ways of attaching to it any given* T_2.

(ii) Inserting T_2 *into* T_1 *means producing a tree* T_3 *whose nodes can be assigned colours 1 or 2 in such a way that:*
(1) The 2-coloured nodes of T_3 *constitute a connected sub-tree* T'_2 *of* T_3 *isomorphic to* T_2.
(2) There is at least one 2-coloured node immediately preceding one 1-coloured node.[57]
(3) The root node of T'_2 *is distinct from the root node of* T_3 *and has for immediate antecedent some 1-coloured node* i *of* T_3.
(4) By retaining only the 1-coloured nodes of T_3*, i.e. by collapsing the whole of* T'_2 *to the 1-coloured node* i *of* T_3*, one gets a tree* T'_1 *isomorphic to* T_1.

Let r_2 *be the node number of* T_2 *and for each node* i *of* T_1*, let* p_i *be the number of vertices issuing from* i*. Then there exist exactly* $\sum_i h(p_i, r_2)$ *distinct ways of inserting* T_2 *into* T_1*, with*

$$h(p,q) := \frac{(p+2\,q)!}{p!\,(2\,q)!} - 1 - p \tag{392}$$

The only point to check is formula (**??**). Assume first that T_2 is linear, with each node (other than the end node) having just one successor. Let i be the node of T'_1 immediately preceding the root node j_* of T'_2. In the insertion process, each of the p_i nodes of T_1 immediately following i either remains attached to i or moves upward to attach itself to some of the r_2 nodes of T_2, with the only proviso that not *all* p_i successor nodes should remain attached to i (for that would contradict clause (2); it would in effect mean that T_2 is getting *attached* to T_1 rather than inserted into it). Now, the number of distinct ways to achieve this is exactly

$$-1 - p_i + \sum_{1 \leqslant k \leqslant p_i} \frac{(2\,r_2+1)!}{k!\,(2\,r_2+1-k)!}\;\frac{(p_i-1)!}{(k-1)!(p_i-k)!} \equiv h(p_i, r_2) \tag{393}$$

The k^{th} summand counts all possible ways of attaching the p_i succesor nodes of i to k distinct nodes of T_2 and the corrective term $-1-p_i$ accounts for the

[57] This clause ensures that one and the same T_3 cannot simultaneously be the *attachment* of T_2 onto T_1 and the *insertion* of T_2 into T_1.

impossibility of keeping *all* successor nodes attached to i. To complete the proof, it suffices to check that nothing changes if we gradually modify the structure of T_2, from linear to arbitrary, while keeping the total number of nodes equal to r_2.

Just three more remarks:
(i) The end nodes i of T_1 play no part in the insertion process, since for them $p_i = 0$ and so $h(p_i, r_2) \equiv 0$.
(ii) For $r_1 = 1$, i.e. when T_1 reduces to a root node, the definition makes it clear that T_1 cannot have anything *inserted* into it.
(iii) Conversely, when $r_2 = 1$, inserting T_2 into T_1 reduces to grafting the single node of T_2 onto any one of T_1's edges.

Proposition 8.2 (Generators $\mathfrak{ote}^{\bullet}_{r,k}$ and pre-Lie products) .
(i) The generators $\hat{\mathfrak{ote}}^{\bullet}_{r,k}$ $(k \leqslant \kappa_{r-1})$ *constructed from the main counter-alternator* câlt *and indexed by ordered trees* $\mathrm{ot}_{r,k}$ *behave optimally simply under the pre-Lie products* dlu *and* dari*:*

$$\mathrm{dlu}(\hat{\mathfrak{ote}}^{\bullet}_{r_1,k_1}, \hat{\mathfrak{ote}}^{\bullet}_{r_2,k_2}) \;=\; -\sum_{k\leqslant\kappa_{r-1}}^{r=r_1+r_2} \hat{\alpha}^{\,r,k}_{r_1,k_1;r_2,k_2}\;\; \hat{\mathfrak{ote}}^{\bullet}_{r,k} \quad \textit{with} \quad \alpha^{\bullet}_{\bullet,\bullet} \in \mathbb{N} \tag{394}$$

$$\mathrm{dari}(\hat{\mathfrak{ote}}^{\bullet}_{r_1,k_1}, \hat{\mathfrak{ote}}^{\bullet}_{r_2,k_2}) \;=\; +\sum_{k\leqslant\kappa_{r-1}}^{r=r_1+r_2} \hat{\beta}^{\,r,k}_{r_1,k_1;r_2,k_2}\;\; \hat{\mathfrak{ote}}^{\bullet}_{r,k} \quad \textit{with} \quad \beta^{\bullet}_{\bullet,\bullet} \in \mathbb{N} \tag{395}$$

In terms of the indexing trees $\mathrm{ot}_{r,k}$ *, the operation* dlu *corresponds to* attaching *the second tree* ot_{r_2,k_2} *successively to each node of the first tree* ot_{r_1,k_1}*, in all possible arrangements, and the operation* dari *corresponds to* inserting *the second tree* ot_{r_2,k_2} *into the first tree* ot_{r_1,k_1}*, again in all possible ways (see details below). As for the integer coefficients involved in these identities, they verify*

$$\sum_k \hat{\alpha}^{r,k}_{r_1,k_1;r_2,k_2} \;=\; 2\,r_1 - 1 \tag{396}$$

$$\sum_k \hat{\beta}^{r,k}_{r_1,k_1;r_2,k_2} \;=\; \sum_{p_i=\mathrm{ramif}(i)}^{i\in\mathrm{nodes}(\mathrm{ot}_{r_1,k_1})} \Big(\frac{(p_i+2\,r_2)!}{p_i!\,(2\,r_2)!} - 1 - p_i\Big) \tag{397}$$

*In (**??**), the $\hat{\alpha}$-sum depends only on the node number r_1 of the first tree* ot_{r_1,k_1}*, and not at all on the second tree. In (**??**), the $\hat{\beta}$-sum depends on the ramification numbers p_i at each node i of the first tree* ot_{r_1,k_1} *(i.e. on the number of branches issuing from i) but only on the node number r_2 of the second tree.*

(ii) The generators $\check{\mathfrak{ote}}^{\bullet}_{r,k}$ $(k \leqslant \kappa_{r-1})$ *constructed from the second counter-alternator* călt *and indexed by ordered trees* $\mathrm{ot}_{r,k}$ *behave under the pre-Lie products* dlu *and* dari *under (slightly less) simple rules. The formulae now involve*

coefficient $\check{\alpha}, \check{\beta}$ of mixed signs:

$$\mathrm{dlu}(\check{\mathfrak{ote}}^\bullet_{r_1,k_1}, \check{\mathfrak{ote}}^\bullet_{r_2,k_2}) \;=\; - \sum_{k\leqslant \kappa_r}^{r=r_1+r_2} \check{\alpha}^{\,r,k}_{r_1,k_1;r_2,k_2} \;\; \check{\mathfrak{ote}}^\bullet_{r,k} \qquad \textit{with} \quad \check{\alpha}^\bullet_{\bullet,\bullet} \in \mathbb{Z} \tag{398}$$

$$\mathrm{dari}(\check{\mathfrak{ote}}^\bullet_{r_1,k_1}, \check{\mathfrak{ote}}^\bullet_{r_2,k_2}) \;=\; + \sum_{k\leqslant \kappa_r}^{r=r_1+r_2} \check{\beta}^{\,r,k}_{r_1,k_1;r_2,k_2} \;\; \check{\mathfrak{ote}}^\bullet_{r,k} \qquad \textit{with} \quad \check{\beta}^\bullet_{\bullet,\bullet} \in \mathbb{Z} \tag{399}$$

The integer coefficients involved in these identities verify

$$\sum_k \check{\alpha}^{r,k}_{r_1,k_1;r_2,k_2} \;=\; 1 \tag{400}$$

$$\sum_k \check{\beta}^{r,k}_{r_1,k_1;r_2,k_2} \;=\; r_1 - 1 \tag{401}$$

But the main difference is that in the new expansions k may run up to κ_r instead of κ_{r-1} in (i), meaning that, beside simple generators, we may also get dmu *products of these. The remark applies not just to* dlu *and* dari *but also to* lu *and* ari.

(iii) Turning now to the weak counter-alternator $\bar{\mathrm{c}}\mathrm{alt}$, *the system $\bar{\mathfrak{ote}}^\bullet_{r,k}$ $(k \leqslant \kappa_r)$ indexed by ordered trees is not complete in* $\mathrm{Flex}_r(\mathfrak{E}^\bullet)$, *but the system $\bar{\mathfrak{ute}}^\bullet_{r,k}$ $(k \leqslant \varkappa_r)$ indexed by unordered trees does span* $\mathrm{Flex}^{\mathrm{al}}_r(\mathfrak{E}^\bullet)$ *and gives rise there to expansions of the form:*

$$\mathrm{dlu}(\bar{\mathfrak{ute}}^\bullet_{r_1,k_1}, \bar{\mathfrak{ute}}^\bullet_{r_2,k_2}) \;=\; - \sum_{k\leqslant \varkappa_r}^{r=r_1+r_2} \bar{\alpha}^{\,r,k}_{r_1,k_1;r_2,k_2} \;\; \bar{\mathfrak{ute}}^\bullet_{r,k} \qquad \textit{with} \quad \bar{\alpha}^\bullet_{\bullet,\bullet} \in \mathbb{Q} \tag{402}$$

$$\mathrm{dari}(\bar{\mathfrak{ute}}^\bullet_{r_1,k_1}, \bar{\mathfrak{ute}}^\bullet_{r_2,k_2}) \;=\; + \sum_{k\leqslant \varkappa_r}^{r=r_1+r_2} \bar{\beta}^{\,r,k}_{r_1,k_1;r_2,k_2} \;\; \bar{\mathfrak{ute}}^\bullet_{r,k} \qquad \textit{with} \quad \bar{\beta}^\bullet_{\bullet,\bullet} \in \mathbb{Q} \tag{403}$$

with rational coefficients that verify

$$\sum_k \bar{\alpha}^{r,k}_{r_1,k_1;r_2,k_2} \;=\; 1 \tag{404}$$

$$\sum_k \bar{\beta}^{r,k}_{r_1,k_1;r_2,k_2} \;=\; r_1 - 1 \tag{405}$$

Sketch of proof:
(i) The counter-alternator *ĉalt* was precisely devised to ensure that calculating $\mathrm{dlu}(\hat{\mathfrak{ote}}^\bullet_{r_1,k_1}, \hat{\mathfrak{ote}}^\bullet_{r_2,k_2})$ should reduce to *attaching* the tree ot_{r_2,k_2} to the tree ot_{r_1,k_1}, in all possible ways. To infer from that the more complicated *insertion* rule for the *dari* product, one starts from the trivial identity $\mathrm{dari}(\hat{\mathfrak{ote}}^\bullet_{r_1,k_1}, \hat{\mathfrak{ote}}^\bullet_{r_2,k_2}) \equiv 0$ when $r_1 = k_1 = 1$ (i.e. $\hat{\mathfrak{ote}}^\bullet_{r_1,k_1} = \mathfrak{E}^\bullet$). One then observes

- that any $\hat{\mathfrak{ote}}^\bullet_{r_1,k_1}$ can be constructed from units $\mathfrak{E}^\bullet$ through a succession of operations *dlu, dmu, mdu*;

- that $dari(A^\bullet, B^\bullet) = arit(B^\bullet)A^\bullet + dlu(A^\bullet, B^\bullet)$;
- that the operator $arit(B^\bullet)$???is a *dlu*-, *dmu*-, *mdu*-automorphism;
- that at each stage of $\hat{\mathfrak{ote}}^\bullet_{r_1,k_1}$'s reconstruction the *insertion* rule applies.

(ii) and (iii). Since the conversion matrices from $\{\hat{\mathfrak{ote}}^\bullet_{r,k}\}$ to $\{\check{\mathfrak{ote}}^\bullet_{r,k}\}$, and back, have only integer entries, the coefficients $\check{\alpha}, \check{\beta}$ will be integers, just like $\hat{\alpha}, \hat{\beta}$, though not necessarily positive. The coefficients $\bar{\alpha}, \bar{\beta}$, on the other hand, are merely rational. As for the curious identities (**??**)-(**??**) and (**??**)-(**??**), they are mentioned here only for the record, but can be checked (painstakingly) by induction on r_1.

8.5 Main sub-algebras.

$$\mathrm{he}A^\bullet := \mathrm{arit}(A^\bullet).\mathfrak{E}^\bullet = \mathrm{c\hat{a}lt}_{\mathfrak{E}}(A^\bullet) = \mathrm{c\check{a}lt}_{\mathfrak{E}}(A^\bullet) = \mathrm{c\bar{a}lt}_{\mathfrak{E}}(A^\bullet) \tag{406}$$

$\forall A^\bullet \in \mathrm{Flex}(\mathfrak{E})$

Proposition 8.3 ($\mathrm{Flex}(\mathfrak{E})$ **and its subalgebras)** .

These are the main ari*-subalgebras of* $\mathrm{Flex}(\mathfrak{E})$ *or* Flex *for short :*

$$\mathrm{he}^\infty\mathrm{Flex}^{\mathrm{al}} \subset ...\mathrm{he}^2\mathrm{Flex}^{\mathrm{al}} \subset \mathrm{he}^1\mathrm{Flex}^{\mathrm{al}} \subset \mathrm{Flex}^{\mathrm{al}} = \oplus_{1\leqslant n}\mathrm{be}^n\mathrm{Flex}^{\mathrm{al}} \subset \mathrm{Flex}^{\mathrm{root}} \tag{407}$$

Each $\mathrm{he}^n\mathrm{Flex}^{\mathrm{al}}$ $(0 \leqslant n \leqslant \infty)$ *and each* $\mathrm{be}^n\mathrm{Flex}^{\mathrm{al}}$ $(1 \leqslant n < \infty)$ *is stable under* ari*, while* $\mathrm{Flex}^{\mathrm{al}}$ *and* $\mathrm{Flex}^{\mathrm{root}}$ *are stable under both* ari *and* lu*. Moreover:*

$$\mathrm{dari}(\mathrm{be}^{\mathrm{n}_1}\,\mathrm{Flex}^{\mathrm{al}}, \mathrm{be}^{\mathrm{n}_2}\,\mathrm{Flex}^{\mathrm{al}}) \in \mathrm{be}^{\mathrm{n}_1}\,\mathrm{Flex}^{\mathrm{al}} \tag{408}$$

$$\mathrm{ari}(\mathrm{be}^{\mathrm{n}_1}\,\mathrm{Flex}^{\mathrm{al}}, \mathrm{be}^{\mathrm{n}_2}\,\mathrm{Flex}^{\mathrm{al}}) \in \mathrm{be}^{\mathrm{n}_1}\,\mathrm{Flex}^{\mathrm{al}} \oplus \mathrm{be}^{\mathrm{n}_2}\,\mathrm{Flex}^{\mathrm{al}} \tag{409}$$

Here are the identities responsible for the ari*-stability of* $\mathrm{he}^n\mathrm{Flex}^{\mathrm{al}}$*:*

$$\begin{array}{lcccc} \mathrm{ari} & : & \Big(\mathrm{he}^n\mathrm{Flex}_{r_1}(\mathfrak{E})\,,\,\mathrm{he}^n\mathrm{Flex}_{r_2}(\mathfrak{E})\Big) & \to & \mathrm{he}^n\mathrm{Flex}_{r_1+r_2+n}(\mathfrak{E}) \\ \mathrm{arihe_n} & : & \Big(\mathrm{Flex}_{r_1}(\mathfrak{E})\,,\,\mathrm{Flex}_{r_2}(\mathfrak{E})\Big) & \to & \mathrm{Flex}_{r_1+r_2+n}(\mathfrak{E}) \end{array}$$

$$\mathrm{ari}(\mathrm{he}^n\,A^\bullet, \mathrm{he}^n\,B^\bullet) \equiv \mathrm{he}^n\,\mathrm{arihe}_n(A^\bullet, B^\bullet) \tag{410}$$

$$\textit{with} \qquad \mathrm{arihe}_n(A^\bullet, B^\bullet) := \begin{cases} -\mathrm{arit}(\mathrm{he}^n A^\bullet)\,B^\bullet + \mathrm{arit}(\mathrm{he}^n B^\bullet)\,A^\bullet \\ + \sum_{n_1+n_2=n}^{0\leqslant n_i} \mathrm{lu}(\mathrm{he}^{\mathrm{n}_1} A^\bullet, \mathrm{he}^{\mathrm{n}_2} B^\bullet) \end{cases} \tag{411}$$

And here is the identity responsible for the ari*-stability of* $\mathrm{be}^n\mathrm{Flex}^{\mathrm{al}}$*:*

$$\mathrm{dari}\Big(\mathrm{c\bar{a}lt}_{\mathfrak{E}}(A_1^\bullet, \ldots, A_s^\bullet), B^\bullet\Big) \equiv \sum_j \mathrm{c\bar{a}lt}_{\mathfrak{E}}\Big(A_1^\bullet, \ldots, \mathrm{tari}(A_j^\bullet, B^\bullet), \ldots, A_s^\bullet\Big) \tag{412}$$

$$\textit{with} \begin{cases} dari(A^\bullet, B^\bullet) & := \mathrm{arit}(B^\bullet).A^\bullet + \mathrm{dlu}(A^\bullet, B^\bullet) \\ tari(A^\bullet, B^\bullet) & := \mathrm{arit}(B^\bullet).A^\bullet + \mathrm{lu}(A^\bullet, B^\bullet) \end{cases} \tag{413}$$

8.6 *Push*-invariance.

Proposition 8.4 (câlt **,** călt **,** cālt **and** ***push*****-invariance)** .
(i) In the $\{\hat{\mathfrak{o}}\mathfrak{te}^\bullet_{r,k}\}$*-basis, we have expansions of type*

$$\sum_{0\leqslant n\leqslant r} \mathrm{push}^n.\, \hat{\mathfrak{o}}\mathfrak{te}^\bullet_{r,p} = \sum_q \hat{a}_{p,q}\, \hat{\mathfrak{o}}\mathfrak{te}^\bullet_{r,q} \qquad (p,q \leqslant \kappa_r\,;\, \hat{a}_{p,q} \in \mathbb{Z}) \tag{414}$$

with integers $\hat{a}_{p,q}$ *whose sum* $\sum_q \hat{a}_{p,q}$ *is divisible by* $2\,r+1$ *whenever* $2\,r+1$ *is prime.*

(ii) In the $\{\check{\mathfrak{o}}\mathfrak{te}^\bullet_{r,k}\}$*-basis, we have expansions of type*

$$\sum_{0\leqslant n\leqslant r} \mathrm{push}^n.\, \check{\mathfrak{o}}\mathfrak{te}^\bullet_{r,p} = \sum_q \check{a}_{p,q}\, \check{\mathfrak{o}}\mathfrak{te}^\bullet_{r,q} \qquad (p,q \leqslant \kappa_r\,;\, \check{a}_{p,q} \in \mathbb{Z}) \tag{415}$$

with integers $\check{a}_{p,q}$ *whose sum* $\sum_q \check{a}_{p,q}$ *is always* $\equiv 0$*.*

(iii) The system $\{\bar{\mathfrak{o}}\mathfrak{te}^\bullet_{r,k}\}$ *being no basis of* $\mathrm{Flex}_r(\mathfrak{E}^\bullet)$*, the above relations have no exact counterpart here. However, the system* $\{\bar{\mathfrak{u}}\mathfrak{te}^\bullet_{r,k}\}$ *indexed by unordered trees of* $\mathbb{UT}_r$ *is a basis of the alternal subspace* $\mathrm{Flex}^{\mathrm{al}}_r(\mathfrak{E}^\bullet)$*, and gives rise to remarkable expansions of type:*

$$\sum_{0\leqslant n\leqslant r} \mathrm{push}^n.\, \bar{\mathfrak{u}}\mathfrak{te}^\bullet_{r,p} = \sum_q \bar{a}_{p,q}\, \bar{\mathfrak{u}}\mathfrak{te}^\bullet_{r,q} \qquad (p,q \leqslant \varkappa_r\,;\, \bar{a}_{p,q} \in \mathbb{Q}) \tag{416}$$

that involve only basis elements $\bar{\mathfrak{u}}\mathfrak{te}^\bullet_{r,q}$ *indexed by unordered trees with* odd-branching *roots, i.e. with roots having a odd number of edges issuing from them. Thus, in (??), about half the basis elements are automatically ruled out. Moreover, if we normalise the basis* $\{\bar{\mathfrak{u}}\mathfrak{te}^\bullet_{r,k}\}$ *(see (??)), the remaining coefficients sum up to zero:* $\sum_q \bar{a}_{p,q} \equiv 0$

9 Expanding the dimorphic bimoulds.

9.1 The main bases and the universal alternals.

We shall construct in *Flex*$(\mathfrak{E})$ two elementary and three semi-elementary series of alternals, namely $\{\mathfrak{re}^\bullet_r\}, \{\mathfrak{le}^\bullet_r\}$ and $\{\mathfrak{he}^\bullet_r\}, \{\mathfrak{ke}^\bullet_{2r}\}, \{\mathfrak{ge}^\bullet_r\}$, by giving in each case a direct description side by side with an inductive definition.

The first alternal series $\{\mathfrak{re}^\bullet_r\}$ **.**

The inductive definition, which immediately implies alternality, reads:

$$\mathfrak{re}^\bullet_1 := \mathfrak{E}^\bullet \quad ; \quad \mathfrak{re}_{\mathfrak{r}}{}^\bullet := \mathrm{arit}(\mathfrak{re}^\bullet_{r-1})\, \mathfrak{E}^\bullet \qquad (\forall r \geqslant 2) \tag{417}$$

The most outstanding property of the alternals $\mathfrak{re}^\bullet_r$ is their self-reproduction à la Witt under the *ari* bracket:

$$\mathrm{ari}(\mathfrak{re}^\bullet_{r_1}, \mathfrak{re}^\bullet_{r_2}) = (r_1 - r_2)\, \mathfrak{re}^\bullet_{r_1+r_2} \tag{418}$$

The second alternal series $\{\mathfrak{le}^\bullet_r\}$.

Here the direct definition reads:

$$\mathfrak{le}_r^{w_1,\dots,w_r} \quad := \quad \sum_{1\leqslant i\leqslant r} (-1)^{i-1}\frac{(r-1)!}{(i-1)!(r-i)!} \ \mathfrak{E}^{\binom{u_1+\dots+u_r}{v_i}} \prod_{j\neq i} \mathfrak{E}^{\binom{u_j}{v_j-v_i}} \tag{419}$$

The third alternal series $\{\mathfrak{he}^\bullet_r\}$.

The series $\{\mathfrak{he}^\bullet_r\}$ is best defined in the binary basis $\{\mathfrak{e}^\bullet_{\boldsymbol{t}}\}$:

$$\mathfrak{he}^\bullet_r \ = \sum_{\boldsymbol{t}\in\mathbb{BT}_r} \mathrm{slant}(\boldsymbol{t})\,\mathfrak{e}^\bullet_{\boldsymbol{t}} \ \ \textit{with} \ \ \begin{cases} \boldsymbol{t} = h(\boldsymbol{t_1},\boldsymbol{t_2}) \\ \mathfrak{e}^\bullet_{\boldsymbol{t}} = \mathit{amnit}(\mathfrak{e}^\bullet_{\boldsymbol{t_1}},\mathfrak{e}^\bullet_{\boldsymbol{t_2}})\mathfrak{E}^\bullet \\ \mathrm{slant}(\boldsymbol{t}) = \mathrm{slant}\binom{p_1}{q_1}\!\Big|\!\binom{p_2}{q_2} \end{cases} \tag{420}$$

with *slant* coefficients

$$\mathrm{slant}\begin{pmatrix} p_1 \,|\, p_2 \\ q_1 \,|\, q_2\end{pmatrix} \ = \ (-1)^{q_{12}-1}\frac{(p_1+p_2)!(q_1+q_2)!}{(p_1+p_2+q_1+q_2)!} \ \det\begin{bmatrix} p_1 & 1+p_2 \\ 1+q_1 & q_2\end{bmatrix} \tag{421}$$

that depend only on the number p_1, p_2 (resp. q_1, q_2) of left-leaning (resp. right-leaning) edges in the subtrees $\boldsymbol{t_1}, \boldsymbol{t_2}$ attached to the root-node of $\boldsymbol{t}$. One of these subtrees may be void, in which case the corresponding pair (p_1, q_1) or (p_2, q_2) is taken to be $(0, 0)$.

The fourth alternal series $\{\mathfrak{ke}^\bullet_{2r_*}\}$.

The series $\{\mathfrak{ke}^\bullet_{2r}\}$ is defined only at even depths $2r$:

$$\mathfrak{ke}^\bullet_{2r} \ = \sum_{\boldsymbol{t}\in\mathbb{BT}_{2r}} \mathrm{stack}(\boldsymbol{t})\,\mathfrak{e}^\bullet_{\boldsymbol{t}} \ \ \textit{with} \ \ \begin{cases} \boldsymbol{t} = h(\boldsymbol{t_1},\boldsymbol{t_2}) \\ \mathfrak{e}^\bullet_{\boldsymbol{t}} = \mathit{amnit}(\mathfrak{e}^\bullet_{\boldsymbol{t_1}},\mathfrak{e}^\bullet_{\boldsymbol{t_2}})\mathfrak{E}^\bullet \\ \mathrm{stack}(\boldsymbol{t}) = \mathrm{stack}\binom{m_1}{n_1}\!\Big|\!\binom{m_2}{n_2} \end{cases} \tag{422}$$

with *stack* coefficients

$$\mathrm{stack}\begin{pmatrix} p_1 \,|\, p_2 \\ q_1 \,|\, q_2\end{pmatrix} \ = \ (-2)^{m_{12}-1}(m_1+m_2-1)!\frac{(n_1+n_2-m_1-m_2)!!}{(n_1+n_2+m_1+m_2-2)!!} \ \det\begin{bmatrix} m_1 & m_2 \\ 1+n_1 & 1+n_2\end{bmatrix} \tag{423}$$

that depend only on the number m_1, m_2 (resp. n_1, n_2) of end-nodes (resp. non end-nodes) in the subtrees $\boldsymbol{t_1}, \boldsymbol{t_2}$ attached to the root-node of $\boldsymbol{t}$.

The fifth alternal series $\{\mathfrak{ge}^\bullet_r\}$.

$$\mathrm{gi}_r^{w_1,\dots,w_r} := \Big(\sum_{1\leqslant i\leqslant r} P(v_i)\Big) \prod_{1\leqslant i<r} P(v_i-v_{i+1}) \tag{424}$$

$$\mathrm{gic}_r^{w_1,\dots,w_r} := \Big(\sum_{1\leqslant i\leqslant r} Q(v_i)\Big)\Big[\big(1+\sum\frac{(i\,c)^{2n}}{1+2n}\partial_t^{2n}\big)\prod_{1\leqslant i<r}\big(Q(v_i-v_{i+1})+t\big)\Big]_{t=0} \tag{425}$$

with as usual $P(v_i) := \frac{1}{v_i}$ and $Q(v_i) := \frac{c}{\tan(c\,v_i)}$

9.2 The main bases and the bisymmetrals.

$$
\begin{array}{rcrcrcl}
\mathrm{pal}^\bullet &=& \mathrm{mdu}(\mathrm{pal}^\bullet, \mathrm{dupal}^\bullet) & \textit{with} & \mathrm{dupal}^\bullet &=& -\sum_{1\leqslant r} \alpha_r \, \mathrm{la}_r^\bullet \\
\mathrm{pal}_{\mathrm{ev}}^\bullet &=& \mathrm{mdu}(\mathrm{pal}_{\mathrm{ev}}^\bullet, \mathrm{dupal}_{\mathrm{ev}}^\bullet) & \textit{with} & \mathrm{dupal}_{\mathrm{ev}}^\bullet &=& -\sum_{1\leqslant r} \alpha_{2r} \, \mathrm{la}_{2r}^\bullet \\
\mathrm{d.pil}^\bullet &=& \mathrm{preari}(\mathrm{pil}^\bullet, \mathrm{dipil}^\bullet) & \textit{with} & \mathrm{dipil}^\bullet &=& -\sum_{1\leqslant r} \frac{1}{(r+1)!} \, \mathrm{ri}_r^\bullet \\
\mathrm{d.pil}_{\mathrm{ev}}^\bullet &=& \mathrm{preari}(\mathrm{pil}_{\mathrm{ev}}^\bullet, \mathrm{dipil}_{\mathrm{ev}}^\bullet) & \textit{with} & \mathrm{dipil}_{\mathrm{ev}}^\bullet &=& -\sum_{1\leqslant r} \frac{1}{(r+1)!} \, \mathrm{ri}_{2r}^\bullet \\
\mathrm{d.ripal}^\bullet &=& \mathrm{preari}(\mathrm{ripal}^\bullet, \mathrm{diripal}^\bullet) & \textit{with} & \mathrm{diripal}^\bullet &=& \sum_{1\leqslant r} \frac{1}{r(r+1)} \, \mathrm{ha}_r^\bullet \\
\mathrm{d.ripal}_{\mathrm{ev}}^\bullet &=& \mathrm{preari}(\mathrm{ripal}_{\mathrm{ev}}^\bullet, \mathrm{diripal}_{\mathrm{ev}}^\bullet) & \textit{with} & \mathrm{diripal}_{\mathrm{ev}}^\bullet &=& \sum_{1\leqslant r} \frac{2^{1-2r}}{4r^2-1} \, \mathrm{ka}_{2r}^\bullet \\
\mathrm{d.ripil}^\bullet &=& \mathrm{preari}(\mathrm{ripil}^\bullet, \mathrm{diripil}^\bullet) & \textit{with} & \mathrm{diripil}^\bullet &=& \sum_{1\leqslant r} \frac{1}{r(r+1)} \, \mathrm{ri}_r^\bullet \\
\mathrm{d.ripal}_{\mathrm{ev}}^\bullet &=& \mathrm{preari}(\mathrm{ripil}_{\mathrm{ev}}^\bullet, \mathrm{diripil}_{\mathrm{ev}}^\bullet) & \textit{with} & \mathrm{diripil}_{\mathrm{ev}}^\bullet &=& \sum_{1\leqslant r} \frac{2^{1-2r}}{4r^2-1} \, \mathrm{ri}_{2r}^\bullet
\end{array}
$$

$$
\begin{array}{l}
[\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0\,0\,0\,0 \;\; 0\,0\,0 \;\; 0 \;\; 0 \;\; \frac{1}{2} \;\; 1 \;\; \frac{3}{2} \;\; \frac{3}{2} \;\; 3 \;\; 0 \;\; 0 \ldots \\
[\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0\,0\,0\,0 \;\; 0\,0\,0 \;\; 0 \;\; 0 \; -\frac{1}{2} \;\; 1 \; -\frac{3}{2} \; -\frac{3}{2} \;\; 3 \;\; 0 \;\; 0 \ldots \\
[\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0\,0\,0\,0 \;\; 0\,0\,0 \;\; 0 \;\; 0 \;\; \frac{1}{2} \;\; \frac{5}{2} \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \ldots \\
[\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0\,0\,0\,0 \;\; 0\,0\,0 \;\; 0 \;\; 0 \; -2 \;\; \frac{5}{2} \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \ldots
\end{array}
$$

$$
\begin{array}{l}
\ldots 0 \;\; 0 \;\; \frac{1}{2} \;\; 1 \;\; \frac{3}{2} \;\; \frac{3}{2} \;\; 3\,0\,0 \; -\frac{5}{2} \;\; 0\,0 \; -\frac{5}{2} \; -\frac{5}{2} \;\; 0 \;\; 0 \; -\frac{15}{2} \; -\frac{15}{2} \; -\frac{15}{2} \; -\frac{15}{2} \; -\frac{45}{2} \;\;] \\
\ldots 0 \;\; 0 \; -\frac{1}{2} \;\; 1 \; -\frac{3}{2} \; -\frac{3}{2} \;\; 3\,0\,0 \; -\frac{5}{2} \;\; 0\,0 \; -\frac{5}{2} \; -\frac{5}{2} \;\; 0 \;\; 0 \; -\frac{15}{2} \; -\frac{15}{2} \; -\frac{15}{2} \; -\frac{15}{2} \; -\frac{45}{2} \;\;] \\
\ldots 0 \;\; 0 \;\; \frac{1}{2} \;\; \frac{5}{2} \;\; 0 \;\; 0\,0\,0\,0 \; -\frac{5}{2} \;\; 0\,0 \; -\frac{5}{2} \; -\frac{5}{2} \;\; 0 \;\; 0 \; -\frac{15}{2} \; -\frac{15}{2} \; -\frac{15}{2} \; -\frac{15}{2} \; -\frac{45}{2} \;\;] \\
\ldots 0 \;\; 0 \; -2 \;\; \frac{5}{2} \;\; 0 \;\; 0\,0\,0\,0 \; -\frac{5}{2} \;\; 0\,0 \; -\frac{5}{2} \; -\frac{5}{2} \;\; 0 \;\; 0 \; -\frac{15}{2} \; -\frac{15}{2} \; -\frac{15}{2} \; -\frac{15}{2} \; -\frac{45}{2} \;\;]
\end{array}
$$

9.3 The main bases and the alternal dilators.

9.4 The main bases and the bialternal dilators.

$$
\mathrm{pal}^\bullet = \mathrm{gari}(\mathrm{par}^\bullet, \mathrm{eral}^\bullet) \quad \textit{with} \quad \begin{cases} \mathrm{pal}^\bullet, \mathrm{pal}^\bullet & \in \mathrm{GARI}^{\mathrm{as/as}} \\ \quad \mathrm{eral}^\bullet & \in \mathrm{GARI}^{\underline{\mathrm{as}}/\underline{\mathrm{as}}} \end{cases} \tag{426}
$$

$$
\mathrm{pil}^\bullet = \mathrm{gari}(\mathrm{pir}^\bullet, \mathrm{eril}^\bullet) \quad \textit{with} \quad \begin{cases} \mathrm{pil}^\bullet, \mathrm{pil}^\bullet & \in \mathrm{GARI}^{\mathrm{as/as}} \\ \quad \mathrm{eril}^\bullet & \in \mathrm{GARI}^{\underline{\mathrm{as}}/\underline{\mathrm{as}}} \end{cases} \tag{427}
$$

$$
\mathrm{ral}^\bullet = \mathrm{logari}(\mathrm{eral}^\bullet) \qquad \textit{with} \qquad \mathrm{ral}^\bullet \in \mathrm{ARI}^{\underline{\mathrm{al}}/\underline{\mathrm{al}}} \tag{428}
$$

$$
\mathrm{ril}^\bullet = \mathrm{logari}(\mathrm{eril}^\bullet) \qquad \textit{with} \qquad \mathrm{ril}^\bullet \in \mathrm{ARI}^{\underline{\mathrm{al}}/\underline{\mathrm{al}}} \tag{429}
$$

$$
\{\mathrm{pal}^\bullet \,,\, \mathrm{par}^\bullet \,,\, \mathrm{ral}^\bullet\} \quad \stackrel{\textit{swap}}{\longleftrightarrow} \quad \{\mathrm{pil}^\bullet \,,\, \mathrm{pir}^\bullet \,,\, \mathrm{ril}^\bullet\} \tag{430}
$$

$$
\{\mathrm{pal}^\bullet \,,\, \mathrm{par}^\bullet \,,\, \mathrm{ral}^\bullet\} \quad \stackrel{\textit{slap}}{\longleftrightarrow} \quad \{\mathrm{pir}^\bullet \,,\, \mathrm{pil}^\bullet \,,\, \mathrm{lir}^\bullet = -\mathrm{ril}^\bullet\} \tag{431}
$$

$$
\begin{array}{rlcl}
5!\,\mathrm{ral}_4^{\bullet} & \textit{in the } \mathrm{c\bar{a}lt}\textit{-basis} & : & [\tfrac{1}{4}\ \ -\tfrac{5}{12}\ \ \ 0\ \ \tfrac{1}{6}] \\
5!\,\mathrm{ral}_4^{\bullet} & \textit{in the } \mathrm{c\hat{a}lt}\textit{-basis} & : & [0\ \ -\tfrac{1}{2}\ \ \ 1\ \ 1] \\
5!\,\mathrm{ral}_4^{\bullet} & \textit{in the } \mathrm{c\check{a}lt}\textit{-basis} & : & [\tfrac{1}{2}\ \ -\tfrac{1}{2}\ -1\ \ 1]
\end{array}
$$

Here is $7!\, ral_6^{\bullet}$ successively in the $c\bar{a}lt$-, $c\hat{a}lt$-, $c\check{a}lt$-bases:

$$
\begin{array}{c}
[-\tfrac{5}{12}\ \ \tfrac{35}{36}\ \ 0\ -\tfrac{35}{36}\ \ 0\ \ 0\ \ 0\ \ 0\ \ \tfrac{7}{12}\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ -\tfrac{1}{6}] \\
[\ 0\ \ \tfrac{5}{6}\ -\tfrac{10}{3}\ -\tfrac{10}{3}\ \ \tfrac{10}{3}\ \ \tfrac{20}{3}\ \ \tfrac{5}{2}\ \ 15\ \ 10\ -\tfrac{5}{3}\ -\tfrac{10}{3}\ -10\ -10\ -\tfrac{10}{3}\ -\tfrac{20}{3}\ -10\ -20\ -15\ -40\ -20] \\
[\ 0\ -\tfrac{5}{6}\ \ \tfrac{25}{3}\ -\tfrac{25}{3}\ -\tfrac{10}{3}\ \ \tfrac{20}{3}\ \ \tfrac{5}{2}\ -15\ \ 10\ -\tfrac{5}{3}\ \ \tfrac{10}{3}\ -10\ \ 10\ -\tfrac{10}{3}\ \ \tfrac{20}{3}\ \ 10\ -20\ -15\ \ 40\ -20]
\end{array}
$$

Lastly, here is $9!\, ral_8^{\bullet}$ successively in the $c\bar{a}lt$-basis:

$$
\left[\begin{array}{ccccccccccccccccc}
\frac{7}{4} & -\frac{133}{32} & \frac{7}{12} & \frac{427}{96} & -\frac{7}{8} & \frac{35}{48} & 0 & -\frac{49}{32} & -\frac{1463}{480} & -\frac{63}{40} & \frac{49}{20} & \frac{21}{20} & -\frac{77}{40} & 0 & 0 & -\frac{21}{40} & \frac{7}{5} \\
0 & \frac{21}{40} & \frac{7}{4} & -\frac{21}{16} & \frac{77}{40} & \frac{7}{8} & -\frac{119}{80} & \frac{21}{10} & -\frac{147}{40} & 0 & \frac{21}{80} & \frac{21}{16} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{189}{160} & -\frac{273}{160} & -\frac{63}{80} & \frac{21}{16} & 0 & 0 & 0 & \frac{21}{40} & -\frac{7}{8} & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{189}{160} & -\frac{273}{160} & -\frac{63}{80} & \frac{21}{16} & -\frac{63}{40} & \frac{21}{8} & 0 & 0 & -\frac{21}{20} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{10} & & &
\end{array}\right]
$$

9.5 Survey.

- On the flexion algebra $Flex(\mathfrak{E}^{\bullet})$ there exist natural systems of mu-generators, but none can adequately reflect the alternality filtration. Only systems of dmu-generators built from the counter-alternators $c\hat{a}lt$ or $c\check{a}lt$ can do that. Systems of dmu-generators built from the weakly inflected counter-alternator $c\bar{a}lt$ do reflect the alternality gradation, but are not complete.

- The system $\{\hat{\mathfrak{ote}}^{\bullet}_{r,k}\,;\ ot_{r,k} \in \mathbb{OT}_r, k \leqslant \kappa_{r-1}\}$ of dmu-generators built from $c\hat{a}lt$ (and indexed by one-rooted ordered trees) is stable not only under the Lie brackets lu and ari, but also under the pre-Lie products dlu and $dari$, and that too in a transparent manner: dlu corresponds to the operation of *tree attachment* and $dari$ to that of *tree insertion*.

- The system $\{\check{\mathfrak{ote}}^{\bullet}_{r,k}\,;\ ot_{r,k} \in \mathbb{OT}_r, k \leqslant \kappa_{r-1}\}$ of dmu-generators built from $c\check{a}lt$ is stable under neither lu nor ari, nor even under dlu and $dari$. There still exist rather simple formulae for all four operations, but they are not 'closed', in the sense that they also involve dmu-products, i.e. 'many-rooted' elements $\hat{ote}^{\bullet}_{r,k}$ with $k > \kappa_{r-1}$,

- The full systems $\{\hat{\mathfrak{ote}}^{\bullet}_{r,k}\,;\ ot_{r,k} \in \mathbb{OT}_r, k \leqslant \kappa_r\}$ and $\{\check{\mathfrak{ote}}^{\bullet}_{r,k}\,;\ ot_{r,k} \in \mathbb{OT}_r, k \leqslant \kappa_r\}$ indexed by one- or many rooted oredered trees (hence $k \leqslant \kappa_r$ instead of $k \leqslant \kappa_{r-1}$) constitute each a basis of $Flex_r(\mathfrak{E}^{\bullet})$. The conversion matrices from either basis to the basis $\{bte^{\bullet}_{r,t}\,;\ bt_{r,k} \in \mathbb{BT}_r, k \leqslant \kappa_r\}$ are lower-triangular (for a suitable ordering on $\mathbb{OT}_r$ and $\mathbb{BT}_r$) with integer entries given by rather explicit formulae, and of course non-zero diagonal elements.

- The closely related systems $\{ste^{\bullet}_{r,k}\ ;\ st_{r,k} \in \mathbb{ST}_r, k \leqslant \kappa_r\}$ and $\{\hat{s}te^{\bullet}_{r,k}\ ;\ st_{r,k} \in \mathbb{ST}_r, k \leqslant \kappa_r\}$ accurately reflect the alternality gradation, and permit the calculation of the dimensions $\dim\big(Flex^{root}_{r,d}(\mathfrak{E}^{\bullet})\big)$ and $\dim\big(Flex_{r,d}(\mathfrak{E}^{\bullet})\big)$ via the generating series $\Gamma(t,x)$ and $\Xi(t,x)$ and the functional equations that these verify.

- The system $\{\bar{u}te^{\bullet}_{r,k}\ ;\ ut_{r,k} \in \mathbb{UT}_r, k \leqslant \kappa_{r-1}\}$ built from the weakly inflected counter-alternator $c\bar{a}lt$ and indexed by one-rooted, unordered trees,[58] constitutes a basis of the alternal sub-algebra $Flex^{al}_r(\mathfrak{E}^{\bullet})$. Even though it does not extend to a full basis of $Flex_r(\mathfrak{E}^{\bullet})$ as the *calt*- or $c\hat{a}lt$-based systems do, it is a valuable tool for expanding the elements of $Flex^{al}_r(\mathfrak{E}^{\bullet})$ (alternals) or $Flex^{al/push}_r(\mathfrak{E}^{\bullet})$ (*push*-invariant alternals) or $Flex^{\underline{al}/\underline{al}}_r(\mathfrak{E}^{\bullet})$ (bialternals), due mainly to the fact that, in this system, the *ari*-bracket respects the number of edges issuing from the root of the indexing tree, and also to the fact that *push*-invariant alternals automatically project onto basis elements $\bar{u}te^{\bullet}_{r,k}$ indexed by trees with odd-branching roots.[59]

10 Pre-Lie and pre-associative algebras.

This informal and avowedly tentative section attempts to detach the main results of the present paper from their origin in flexion theory, especially in $Flex(\mathfrak{E}^{\bullet})$ and its two 'polar' models, and to put them on a neat axiomatic foundation. The proper framework appears to be that of *pre-associative algebras*[60], taken in their natural context:

$$\begin{array}{ccc} \textit{pre-associative} & \rightarrow & \textit{associative} \\ \downarrow & & \downarrow \\ \textit{pre-Lie} & \rightarrow & \textit{Lie} \end{array}$$

Although it is way too early to say if these algebras will justify the hopes reposited in them (– of rendering roughly the same services as pre-Lie algebras do –), they have at least one immediate use: shedding an oblique light on the counter-alternators and providing a quick proof of their main property.

[58] Unordered trees being a subset of stacked trees, the above (free) system is a sub-system of the (non-free, as already ponted out) system $\{\bar{s}te^{\bullet}_{r,k}\ ;\ not_{r,k} \in \mathbb{ST}_r, k \leqslant \kappa_r\}$.

[59] i.e. with roots from which there issue an odd number of edges.

[60] though not in the sense usually given to 'pre-associative'.

10.1 Pre-associativity: definitions and first properties.

A pre-associative algebra is a vector space endowed with a linear involution $x \mapsto x_*$ and two bilinear binary operations μ and $\overline{\mu}$ that verify:

$$
\begin{array}{llcl}
\mathcal{R}_0 : & \mu_*(x,y) & \equiv & (\mu(y_*,x_*))_* \\
\mathcal{R}_1 : & \mu(\mu_*(x,y),z) & \equiv & \mu_*(x,\mu(y,z)) \\
\mathcal{R}_2 : & \mu(\mu(x,y),z) & \equiv & \mu(x,\mu(y,z)) + \mu(x,\mu_*(y,z)) \\
\mathcal{R}_3 : & \mu_*(x,\mu_*(y,z)) & \equiv & \mu_*(\mu(x,y),z) + \mu_*(\mu_*(x,y),z)
\end{array}
$$

Since μ_* and $\mathcal{R}_3$ follow from μ and $\mathcal{R}_2$ under the involution, the whole structure is fixed as soon as we define an operation μ compatible with the axioms $\mathcal{R}_1, \mathcal{R}_2$.

Proposition 10.1 (The offspring of a pre-associative product) .
If we set:

$$m(x,y) := \mu(x,y) + \mu_*(x,y) \tag{432}$$

$$\lambda(x,y) := \mu(x,y) - \mu_*(y,x) \tag{433}$$

$$\lambda_*(x,y) := \mu_*(x,y) - \mu(y,x) \ \equiv \ (\lambda(x_*,y_*))_* \tag{434}$$

$$l(x,y) := m(x,y) - m(y,x) \ \equiv \ \lambda(x,y) - \lambda(y,x) \ \equiv \ \lambda_*(x,y) - \lambda_*(y,x) \tag{435}$$

and if μ, μ_* verify the pre-associativity axioms, then
(i) $\boldsymbol{m}$ is an associative product, with e as unit element.
(ii) $\boldsymbol{l}$ is the Lie bracket associated with $\boldsymbol{m}$
(iii) $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}_*$ define a right resp. left pre-Lie product for $\boldsymbol{l}$

Indeed, if we set

$$
\begin{array}{lcll}
R_1(x,y,z) & := & \mu(\mu_*(x,y),z) - \mu_*(x,\mu(y,z)) & (\equiv 0) \\
R_2(x,y,z) & := & \mu(\mu(x,y),z) - \mu(x,\mu(y,z)) - \mu(x,\mu_*(y,z)) & (\equiv 0) \\
R_3(x,y,z) & := & \mu_*(x,\mu_*(y,z)) - \mu_*(\mu(x,y),z) - \mu_*(\mu_*(x,y),z) & (\equiv 0)
\end{array}
$$

we find

$$m(m(x,y),z) - m(x,m(y,z)) \equiv R_1(x,y,z) + R_2(x,y,z) - R_3(x,y,z) \equiv 0$$

This confirms the associativity of $\boldsymbol{m}$. Next we find

$$
\begin{cases}
+\lambda(\lambda(x,y),z) - \lambda(x,\lambda(y,z)) \\
-\lambda(\lambda(x,z),y) + \lambda(x,\lambda(z,y))
\end{cases}
\equiv
\begin{cases}
+R_1(z,x,y) - R_1(y,x,z) \\
+R_2(x,y,z) - R_2(x,z,y) \\
+R_3(z,y,x) - R_3(y,z,x)
\end{cases}
\equiv \ 0
$$

This confirms the pre-Lie nature of $\boldsymbol{\lambda}$ and, by symmetry, of $\boldsymbol{\lambda}_*$ as well.

Remark: It would be tempting to introduce a 'unit element' [61] e such that:

$$
\begin{cases}
\mu(x,e) \equiv x \quad ; \quad \mu_*(e,y) \equiv y & (\forall x, y \neq e) \\
\mu(e,y) \equiv 0 \quad ; \quad \mu_*(x,e) \equiv 0 & (\forall x, y \neq e) \\
\mu(e,e) = \mu_*(e,e) = \frac{1}{2}\, e &
\end{cases}
\tag{436}
$$

but in formulae involving several consecutive units we would run into unsurmountable contradictions.

[61] such an element would indeed be a unit for the associative product m associated with μ.

10.2 Counter-alternality defined.

A *co-degree* on a pre-associative algebra $\mathcal{PA}$ is function $d : \mathcal{PA} \to \mathbb{N}^*$ giving rise to a filtration and a gradation:

$$\mathcal{PA}_{(d)} \subset \mathcal{PA}_{(d+1)} \quad ; \quad \mathcal{PA}_d = \mathcal{PA}_{(d)}/\mathcal{PA}_{(d-1)} \tag{437}$$

such that:

$$\begin{cases} \mu\big(\mathcal{PA}_{(d_1)}, \mathcal{PA}_{(d_1)}\big) & \subset & \mathcal{PA}_{(d_1+d_2)} \\ \mu_*\big(\mathcal{PA}_{(d_1)}, \mathcal{PA}_{(d_1)}\big) & \subset & \mathcal{PA}_{(d_1+d_2)} \\ \lambda\big(\mathcal{PA}_{(d_1)}, \mathcal{PA}_{(d_1)}\big) & \subset & \mathcal{PA}_{(d_1+d_2-1)} \end{cases} \tag{438}$$

The the natural tools for investigating a co-degree gradation are the three *counter-alternators*,[62] which are defined as follows. In view of $\mathcal{R}_1$ we may set:

$$\widetilde{\mu}(x, y, z) := \mu(\mu_*(x, y), z) = \mu_*(x, \mu(y, z)) \tag{439}$$

We then transpose the construction of §5.5:

Definition 10.1 (Strict and loose counter-alternators) .

$$\text{câlt}_{x_0}(x_1, .., x_s) \ := \sum_{0 \leqslant i \leqslant s} (-1)^{s-i}\, \widetilde{\mu}\big(\overleftarrow{\mu}\,(x_1, .., x_i), x_0, \vec{\mu}_*(x_{i+1}, .., x_s)\big) \tag{440}$$

$$\text{cǎlt}_{x_0}(x_1, .., x_s) \ := \sum_{0 \leqslant i \leqslant s} (-1)^{s-i}\, \widetilde{\mu}\big(\vec{\mu}_*(x_1, .., x_i), x_0, \overleftarrow{\mu}\,(x_{i+1}, .., x_s)\big) \tag{441}$$

$$\text{cālt}_{x_0}(x_1, .., x_s) \ := \sum_{0 \leqslant i \leqslant s} \frac{(-1)^{s-i} s!}{i!\,(s-i)!}\, \widetilde{\mu}\big(m(x_1, .., x_i), x_0, m(x_{i+1}, .., x_s)\big) \tag{442}$$

Next, we construct the partially symmetrized alternators exactly as we did on $Flex(\mathfrak{E}^\bullet)$:

Definition 10.2 (d-alternators.) *For $1 \leqslant d \leqslant r$ we set:*

$$\text{câlt}^{r,d}_{x_0}(x_1, ..., x_r) \quad := \quad \sum_{\sigma \in \mathfrak{S}_r} h_{r,1+r-d}(\sigma)\, \text{câlt}_{x_0}(x_{\sigma(1)}, ..., x_{\sigma(r)}) \tag{443}$$

$$\text{cǎlt}^{r,d}_{x_0}(x_1, ..., x_r) \quad := \quad \sum_{\sigma \in \mathfrak{S}_r} h_{r,1+r-d}(\sigma)\, \text{cǎlt}_{x_0}(x_{\sigma(1)}, ..., x_{\sigma(r)}) \tag{444}$$

$$\text{cālt}^{r,d}_{x_0}(x_1, ..., x_r) \quad := \quad \sum_{\sigma \in \mathfrak{S}_r} h_{r,1+r-d}(\sigma)\, \text{cālt}_{x_0}(x_{\sigma(1)}, ..., x_{\sigma(r)}) \tag{445}$$

with the coefficients $\mathrm{h}_{r,\delta}(\sigma)$ *associated with the projectors* $pr_{:d}$ *as in §5.8.*

[62]For reasons we have already encountered in the flexion context, the first two (*câlt*, *cǎlt*) are declared *strict*, the last one *loose*.

10.3 Counter-alternality modelized and proved.

Let $\mathbb{E}$ be the graded, non-associative algebra freely generated by the symbols $e_0, e_1, e_2 ...$, with coefficients in the commutative algebra $\mathbb{Q}\{\alpha_0, \alpha_1, \alpha_2 ...\}$, whereby the α_i's are assumed to be positive, transcendental, and algebraically independent. We assign the grade α_i to each generator e_i and consider on $\mathbb{D}$ the derivation ∂ (with its antiderivation ∂^{-1}) acting as follows:

$$\partial\, e_i := \alpha_i e_i \quad ; \quad \partial^{-1} e_i := \alpha_i^{-1} e_i \quad ; \quad \partial \alpha_i := 0 \tag{446}$$

Finally, we notice that the operations μ, μ_* thus defined:

$$\mu\,(E_1, E_2) \quad := \quad \partial^{-1}\Big((\partial E_1) E_2\Big) \qquad (\forall E_1, D_2 \in \mathbb{D}) \tag{447}$$

$$\mu_*(E_1, E_2) \quad := \quad \partial^{-1}\Big(D_1 (\partial D_2)\Big) \qquad (\forall E_1, E_2 \in \mathbb{D}) \tag{448}$$

endow $\mathbb{E}$ with a pre-associative structure, which we shall call $\mathcal{E}$ for distinction. By assigning co-degree 1 to all generators e_i, we also endow $\mathcal{E}$ with a co-degree filtration $\mathcal{E} = \cup^{\uparrow} \mathcal{E}_d$.

Though free as an associative algebra, $\mathcal{E}$ is far from free with respect to its ∂-induced pre-associative structure.[63] Nonetheless, the subspace of elements separately linear in $e_0, e_1, ..., e_r$ is free in the sense that it faithfully reflects the structure of the corresponding subspace in the *free* pre-associative algebra generated by $e_0, e_1, e_2 ..$ – a fortunate circumstance that leads, first to an interpretation of the counter-alternators, then to a proof of their signature properties, namely

$$\mathrm{codeg}\big(\mathrm{calt}^{r,d}_{x_0}(x_1, ..., x_r)\big) \leqslant d_0{+}d_1{+}...{+}d_r{+}d{-}r{-}1 \quad \textit{if} \quad \mathrm{codeg}(x_i) = d_i \tag{449}$$

with exact equality for the strict counter-alternators $c\hat{a}lt$ or $c\check{a}lt$.

Consider now these particular identities in $\mathcal{E}$:

$$\mathrm{c\hat{a}lt}_{e_0}(e_1, ..., e_r) \quad \equiv \quad \hat{c}_r \sum_{0 \leqslant i \leqslant r} \mathrm{t\hat{e}}^{1,...,i,0,i+1,...,r} e_1 ... e_i\, e_0\, e_{i+1} ... e_r \tag{450}$$

$$\mathrm{c\check{a}lt}_{e_0}(e_1, ..., e_r) \quad \equiv \quad \hat{c}_r \sum_{0 \leqslant i \leqslant r} \mathrm{t\check{e}}^{1,...,i,0,i+1,...,r} e_1 ... e_i\, e_0\, e_{i+1} ... e_r \tag{451}$$

with pre-sum factors:

$$\hat{c}_r = \frac{\alpha_0\, \alpha_1 \ldots \alpha_r}{\alpha_0 + \alpha_1 + \ldots + \alpha_r} \qquad ; \qquad \hat{c}_r = \frac{\alpha_0\, \alpha_1 \ldots \alpha_r}{\alpha_0 + \alpha_1 + \ldots + \alpha_r} \tag{452}$$

and alternal moulds $t\hat{e}^{\bullet}, t\check{e}^{\bullet}$:

$$\mathrm{t\hat{e}}^{1,...,i,0,i+1,...,r} \quad := \quad \mathrm{se}^{1,...,i} \mathrm{ze}^{i+1,...r} \tag{453}$$

$$\mathrm{t\check{e}}^{1,...,i,0,i+1,...,r} \quad := \quad \mathrm{ze}^{1,...,i} \mathrm{se}^{i+1,...r} \tag{454}$$

[63] Think of the sub-algebras generated from one single e_i under repeated action of μ and μ_*.

themselves built from two symmetral, mutually inverse moulds $se^\bullet, ze^\bullet$:

$$\mathrm{se}^{1,\dots,r} := \prod_{i=1}^{r} \frac{1}{\alpha_i + \dots + \alpha_r} \quad ; \quad \mathrm{ze}^{1,\dots,r} := \prod_{i=1}^{r} \frac{(-1)^r}{\alpha_1 + \dots + \alpha_i} \tag{455}$$

Any mould $t^\bullet$ of type $t^\bullet = s^\bullet \times id^\bullet \times z^\bullet$ with $s^\bullet, z^\bullet$ symmetral and mutually inverse, is obviously alternal, but here $t\hat{e}^\bullet$ and $t\check{e}^\bullet$ are of a distinct type, and their alternality results from the identities:

$$\sum_{\boldsymbol{\gamma}\in \mathrm{sha}(\boldsymbol{\alpha},\boldsymbol{\beta})} \mathrm{t\hat{e}}^{\boldsymbol{\gamma}} \equiv \begin{cases} \mathrm{se}^{\boldsymbol{\alpha}'}\mathrm{ze}^{\boldsymbol{\alpha}''}\big(\sum_{\boldsymbol{\beta}'\,\boldsymbol{\beta}''=\boldsymbol{\beta}} \mathrm{se}^{\boldsymbol{\beta}'}\mathrm{ze}^{\boldsymbol{\beta}''}\big) \equiv 0 & \textit{if} \ \ \boldsymbol{\alpha} = (\boldsymbol{\alpha}', 0, \boldsymbol{\alpha}'') \\ \big(\sum_{\boldsymbol{\alpha}'\,\boldsymbol{\alpha}''=\boldsymbol{\alpha}} \mathrm{se}^{\boldsymbol{\alpha}'}\mathrm{ze}^{\boldsymbol{\alpha}''}\big)\,\mathrm{se}^{\boldsymbol{\beta}'}\mathrm{ze}^{\boldsymbol{\beta}''} \equiv 0 & \textit{if} \ \ \boldsymbol{\beta} = (\boldsymbol{\beta}', 0, \boldsymbol{\beta}'') \end{cases} \tag{456}$$

$$\sum_{\boldsymbol{\gamma}\in \mathrm{sha}(\boldsymbol{\alpha},\boldsymbol{\beta})} \mathrm{t\check{e}}^{\boldsymbol{\gamma}} \equiv \begin{cases} \mathrm{ze}^{\boldsymbol{\alpha}'}\mathrm{se}^{\boldsymbol{\alpha}''}\big(\sum_{\boldsymbol{\beta}'\,\boldsymbol{\beta}''=\boldsymbol{\beta}} \mathrm{ze}^{\boldsymbol{\beta}'}\mathrm{se}^{\boldsymbol{\beta}''}\big) \equiv 0 & \textit{if} \ \ \boldsymbol{\alpha} = (\boldsymbol{\alpha}', 0, \boldsymbol{\alpha}'') \\ \big(\sum_{\boldsymbol{\alpha}'\,\boldsymbol{\alpha}''=\boldsymbol{\alpha}} \mathrm{ze}^{\boldsymbol{\alpha}'}\mathrm{se}^{\boldsymbol{\alpha}''}\big)\,\mathrm{ze}^{\boldsymbol{\beta}'}\mathrm{se}^{\boldsymbol{\beta}''} \equiv 0 & \textit{if} \ \ \boldsymbol{\beta} = (\boldsymbol{\beta}', 0, \boldsymbol{\beta}'') \end{cases} \tag{457}$$

As usual, $sha(\boldsymbol{\alpha}, \boldsymbol{\beta})$ denotes the set of all shuffle products of the sequences $\boldsymbol{\alpha}, \boldsymbol{\beta}$, and the right-hand sides in (456) or (457) vanish due to the factors $\sum \mathrm{se}^{\boldsymbol{\alpha}'}\mathrm{ze}^{\boldsymbol{\alpha}''} \equiv 0$ or $\sum \mathrm{se}^{\boldsymbol{\beta}'}\mathrm{ze}^{\boldsymbol{\beta}''} \equiv 0$ corresponding to the sequence ($\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$) that *does not* contain the exceptional term 0.

Let us return to the proof. To this end, we fix $d \in [1, r]$; replace $(c\hat{a}lt, c\check{a}lt)$ by $(c\hat{a}lt^{r,d}, c\check{a}lt^{r,d})$ in (450)-(451); and call $\hat{X}^{r,d}, \check{X}^{r,d}$ the new values assumed by the right-hand sides of (450)-(451). To prove that $\hat{X}^{r,d}, \check{X}^{r,d}$ have 'differential degree' d as elements of $\mathbb{D}$ or, equivalently, co-degree d as elements of $\mathcal{D}$, we use the earlier Lemma 5.2 in the special case $(d_1, d_2) = (1, r)$, and the following Lemma 10.1 in the special case $\delta_0 = 1 + r - d$, $\delta = 1$ and $T^\bullet = t\hat{e}^\bullet$ or $t\check{e}^\bullet$:

Lemma 10.1 (Mould-comould contractions) .
Let $(\epsilon_1, \dots, \epsilon_r)$ run through all permutations of $(1, \dots, r)$ and consider the mould $T^\bullet$ defined by

$$H^{\epsilon_1,\dots,\epsilon_r} := \sum_{\sigma\in\mathfrak{S}_r} h_{r,\delta_0}(\sigma)\; A^{\sigma(\epsilon_1),\dots,\sigma(\epsilon_r)}\; e_{\sigma(\epsilon_1)}\dots e_{\sigma(\epsilon_r)} \tag{458}$$

with the coefficients $h_{r,d}(\sigma)$ of ... and with some mould $T^\bullet$ of alternality co-degree δ_1 and with values in $\mathbb{Q}[\cup_i \alpha_i]/\mathbb{Q}[\cup_i \alpha_i]$. Then, as an element of $\mathbb{E}$, $H^\bullet$ has degree $\delta_2 = \delta_1 - \delta_0 + r$. In particular, if $\delta_0 = 1 + r - d$ as in (443)-(444) and $\delta_1 = 1$ as with $t\hat{e}^\bullet$ and $t\check{e}^\bullet$, we get $\delta_2 = d$.

Writing (458) compactly as $H^\bullet = \sum_\sigma h_{r,\delta_0}(\sigma) T^{\sigma(\bullet)}\, e_{\sigma(\bullet)}$; then invoking the δ_1-alternality of the mould $T^\bullet$ to express it as $T^\bullet = \sum_{\sigma_1} h_{r,\delta_1}(\sigma_1) A^{\sigma_1(\bullet)}$ for some arbitrary mould $A^\bullet$; and lastly expressing that the projector $pr_{:\delta_2}$ acting solely on the bimould part annihilates $H^\bullet$ if $\delta_2 - \delta_1 + \delta_0 > r$, we find that Lemma 10.1 amouts to the identity:

$$\sum_{\tau\in\mathfrak{S}_r} h_{\delta_0}(\tau)\, h_{\delta_1}(\tau_1 \circ \tau)\, h_{\delta_2}(\tau^{-1} \circ \tau_2) \equiv 0 \quad \textit{if} \quad \delta_2 - \delta_1 + \delta_0 > r \ \ (\forall \tau_1, \tau_2) \tag{459}$$

This applies in fact even when $\delta_2 - \delta_1 + \delta_0 - r$ is negative but *odd*, for trivial reasons of invariance under $\pm anti$.

10.4 Counter-alternality made manifest.

The *partially symmetrized* counter-alternators $calt^{r,d}$ can be written as *partially symmetrized* expressions $kalt^{r,d}$ whose co-degree d is immediately apparent – instead of hidden, as in the definitions (440)-(442) and (443)-(445). Indeed:

Proposition 10.2 (d-alternality made manifest) .
The d*-alternators are capable of an equivalent definition, of type:*

$$\hat{\mathrm{calt}}^{r,d}_{x_0}(x_1,...,x_r) \equiv \sum_{\sigma\in\mathfrak{S}_r} h_{r,1+r-d}(\sigma)\,\hat{\mathrm{kalt}}^{r,d}_{x_0}(x_{\sigma(1)},...,x_{\sigma(r)}) \tag{460}$$

$$\check{\mathrm{calt}}^{r,d}_{x_0}(x_1,...,x_r) \equiv \sum_{\sigma\in\mathfrak{S}_r} h_{r,1+r-d}(\sigma)\,\check{\mathrm{kalt}}^{r,d}_{x_0}(x_{\sigma(1)},...,x_{\sigma(r)}) \tag{461}$$

$$\bar{\mathrm{calt}}^{r,d}_{x_0}(x_1,...,x_r) \equiv \sum_{\sigma\in\mathfrak{S}_r} h_{r,1+r-d}(\sigma)\,\bar{\mathrm{kalt}}^{r,d}_{x_0}(x_{\sigma(1)},...,x_{\sigma(r)}) \tag{462}$$

where each $(r+1)$-linear term $\mathrm{kalt}^{r,d}_{x_0}(x_{\sigma(1)},...,x_{\sigma(r)})$ can be written as a finite sum of elementary summands involving the operations λ,μ,μ_* respectively r_0,r_1,r_2 times, with $r_0=r+1-d$ and $r_1+r_2=d-1$.

Unfortunately, there exist scores of possible expressions for $kalt^{r,d}$, and so far we found universal (for all r) and compellingly natural expressions only in the extreme cases $d=1$ or $d=r$, corresponding to *maximal* and *minimal* symmetrization. Here are these natural expressions:

Proposition 10.3 (Maximally symmetrized counter-alternators)

$$\hat{\mathrm{kalt}}^{r,1}_{x_0}(x_1,..,x_r) := \sum_{1\leqslant s\leqslant r} \hat{\mathrm{sa}}^{s_1,...,s_r}\ \vec{\lambda}\Big(\overleftarrow{\lambda}(x_0,\boldsymbol{x^1}),\overleftarrow{\lambda}(\boldsymbol{x^2}),...,\overleftarrow{\lambda}(\boldsymbol{x^s})\Big) \tag{463}$$

$$\check{\mathrm{kalt}}^{r,1}_{x_0}(x_1,..,x_r) := \sum_{1\leqslant s\leqslant r} \check{\mathrm{sa}}^{s_1,...,s_r}\ \vec{\lambda}\Big(\overleftarrow{\lambda}(x_0,\boldsymbol{x^1}),\overleftarrow{\lambda}(\boldsymbol{x^2}),...,\overleftarrow{\lambda}(\boldsymbol{x^s})\Big) \tag{464}$$

$$\bar{\mathrm{kalt}}^{r,1}_{x_0}(x_1,..,x_r) := \sum_{1\leqslant r\leqslant s}^{\sum s_i=s} (-1)^r\ \vec{\lambda}\,(x_0,x_{\sigma(1)},...,x_{\sigma(s)}) \tag{465}$$

with the coefficients:

$$\hat{\mathrm{sa}}^{s_1,...,s_r} = (-1)^r \prod_{1\leqslant i\leqslant r}\frac{1}{s_1+...+s_i} \tag{466}$$

$$\check{\mathrm{sa}}^{s_1,...,s_r} = (-1)^{s_1+...+s_r} \prod_{1\leqslant i\leqslant r}\frac{1}{s_i+...+s_r} \tag{467}$$

Proposition 10.4 *(Minimally symmetrized counter-alternators)*

$$\hat{\mathrm{kalt}}^{r,r}_{x_0}(x_1,..,x_r) := \sum_{1\leqslant s\leqslant r}(-1)^{s-1}\ \vec{\mu}\Big(\lambda\big(x_0,\overleftarrow{\mu}(\boldsymbol{x^1})\big),\overleftarrow{\mu}(\boldsymbol{x^2}),...,\overleftarrow{\mu}(\boldsymbol{x^s})\Big) \tag{468}$$

$$\check{\mathrm{kalt}}^{r,r}_{x_0}(x_1,..,x_r) := \sum_{1\leqslant s\leqslant r}(-1)^{s-1}\ \vec{\mu}\Big(\lambda\big(x_0,\vec{\mu}_*(\boldsymbol{x^1})\big),\vec{\mu}_*(\boldsymbol{x^2}),...,\vec{\mu}_*(\boldsymbol{x^s})\Big) \tag{469}$$

$$\bar{\mathrm{kalt}}^{r,r}_{x_0}(x_1,..,x_r) := \sum_{1\leqslant s\leqslant r}(-1)^{r-s+1}\mu\Big(\lambda\big(x_0,\mathrm{m}(x_1,..,x_s)\big),\mathrm{m}(x_{s+1},..,x_r)\Big) \tag{470}$$

10.5 Structure of free pre-associative algebras.

A pre-associative algebra generated by *one* element is automatically isomorphic to $Flex(\mathfrak{E}^\bullet)$. It also admits a tree theoretical model, namely the space $\mathbb{C}[\mathbb{OT}]$ of linear combinations of orderer trees $ot_{r,k}$ equipped with the operations λ, μ, μ_* 'lifted' from dle, dme, mde via the correspondence $\hat{\mathfrak{ote}}^\bullet_{r,k} \mapsto ot_{r,k}$.

With pre-associative algebras freely generated by n elements, the situation is slightly different: while such an algebra still admits a privileged tree theoretical model, it admits several mould theoretical ones. Chief amongst the latter are the sub-algebras of $BIMU$ generated under the operations dla, dma, mda [64] by n depth-1 bimoulds $A_1^{w_1} := a_1(u_1), \ldots, A_n^{w_1} := a_n(u_1)$, for any choice of transcendental functions $a_i(u_1)$, algebraically independent and verifying no functional equations. As for the tree theoretical model, it is the same as in the case of one generators, but with n-decorated trees, i.e. trees whose nodes are assigned various colours, from a set of n.

10.6 Filtration by co-degree. Dimensions.

For free pre-associative algebras with one generator, he co-degree filtration and gradation, along with the corresponding dimensions, are exactly those of $Flex(\mathfrak{E}^\bullet)$: see §7.3. The case of several generators immediately follows, modulo the introduction of colours.

10.7 Examples of non-free pre-associative algebras.

Any subalgebra of the bimould algebra $BIMU^*$[65] generated by the operations dla, dma, mda (see (198)-(200) in §5.3.) as stand-ins for λ, μ, μ_*, from any set of bimoulds $A_i^\bullet \in BIMU^\bullet$ patently constitutes a pre-associative algebra, generally non-free, and with a structure that entirely depends on the mutual relations that the generators $A_i^\bullet$ may entertain.

The same applies with $BIMU^{v\text{-}v}$ (see (201)) in place of $BIMU^*$ and dli, dmi, mdi (see (202)-(204)) as basic operations in place of dla, dma, mda.

But there also exist numerous pre-associative algebras with no apparent link to $BIMU$. For instance, given any associative algebra $\mathbb{E}$, free or not, and any uniquely invertible derivation ∂ on $\mathbb{E}$, the operations μ, μ_* defined as in (447)-(448) turn $\mathbb{E}$ into a pre-associative algebra $\mathcal{E}$, non-free even when, as an associative algebra, $\mathbb{E}$ is free. An important sub-instance is that of a derivation ∂ induced as in (446) by a scalar gradation on $\mathbb{E}$ with values in $\mathbb{R}^+$.

[64] See (198)-(200) in §5.3.

[65] $BIMU^*$ is $BIMU$ minus the unit mould $1^\bullet$ and its multiples. Regarding the necessary absence of *units* in pre-associative algebras, see the remark at the end of §10.1.

10.8 Enveloping and enfolding algebras.

To sum up, we have a pleasant four-fold scheme:

$$\begin{array}{ccccccccccc}
\left\{\begin{array}{l}\textit{enfolding algebra}\\ (\textit{pre-associative})\end{array}\right. & \begin{array}{c}\longleftarrow\\ \Longrightarrow\end{array} & \left\{\begin{array}{c}\textit{enveloping algebra}\\ (\textit{associative})\end{array}\right. & \| & \mathfrak{M} & \supset & M & \| & \boldsymbol{\mu} & \Rightarrow & \boldsymbol{m}\\
\Updownarrow & & \Updownarrow & \| & \cup & & \cup & \| & \Downarrow & & \Downarrow\\
\textit{pre-Lie algebra} & \begin{array}{c}\longleftarrow\\ \Longrightarrow\end{array} & \textit{Lie algebra} & \| & \mathfrak{L} & \supset & L & \| & \boldsymbol{\lambda} & \Rightarrow & \boldsymbol{l}
\end{array}$$

And coursing through that four-fold scheme, we have a double movement of *restriction* (right- and downward arrows) and *unfolding* (left- and upward arrows), with uniqueness of construction in the case of double-barred arrows, but not in the case of simple arrows.

10.9 Dynkin-like projectors.

Known projectors:

$$\begin{array}{ccc} & & \textbf{Associative}\\ & & \downarrow\\ \textbf{Pre-Lie} & \rightarrow & \textbf{Lie}\end{array}$$

If a sum $\sum a_i\, m(x_{i,1},...,x_{i,n})$ of n-linear associative summands is known to be in the Lie algebra, we have the well-known Dynkin projectors

$$\sum_i a_i\, m(x_{i,1},...,x_{i,n}) = \frac{1}{n}\sum_i a_i \overrightarrow{l}\,(x_{i,1},...,x_{i,n}) = \frac{1}{n}\sum_i a_i \overleftarrow{l}\,(x_{i,1},...,x_{i,n}) \qquad (471)$$

to make that Lie nature manifest.

We have similar projectors for sums of n-linear pre-Lie brackets that are known to be in the Lie algebra, especially when the bracketing is uniformly forward or backward, i.e. of the form $\sum_i a_i \overrightarrow{\lambda}\,(x_{i,1},...,x_{i,n})$ or $\sum_i a_i \overleftarrow{\lambda}\,(x_{i,1},...,x_{i,n})$.

Wanted projectors: But *quid* of the following projectors?

$$\begin{array}{ccc} \textbf{Pre-associative} & \stackrel{?}{\rightarrow} & \textbf{Associative}\\ \downarrow ? & & \\ \textbf{Pre-Lie} & & \end{array}$$

And *quid* of the projectors of a pre-associative algebra $\mathcal{PA}$ on its components $\mathcal{PA}_{(d)}$ of co-degree d? Such projectors ought to exist in *explicit* presentation[66] and manageable form, and would come quite handy (they would in particular automatically determine *privileged* choices for the $kalt^{r,d}$ of Proposition 10.2, but finding them seems to be no trivial matter.[67]

[66] as opposed to the *implicit* presentation of the counter-alternators $calt^{r,d}$, which work just fine as projectors, but not *transparently* so.

[67] due to the enormous number of *a priori* relations that connect multiple superpositions of the three basic operations λ, μ, μ_*.

10.10 Notion of Janus algebra.

On $Flex(\mathfrak{E}^\bullet)$ the identities

$$\text{ari}(A^\bullet, B^\bullet) = \text{dari}(A^\bullet, B^\bullet) - \text{dari}(B^\bullet, A^\bullet) \tag{472}$$
$$\text{dari}(A^\bullet, B^\bullet) = \text{arit}(B^\bullet).A^\bullet + dle(A^\bullet, B^\bullet) \tag{473}$$

reduce the calculation of *ari* to that of *arit* and *dle*. But the operator $\text{arit}(B^\bullet)$ is a derivation not only with respect to the non-inflected products *lu* and *mu*, but also with respect to the semi-inflected ones *dle, dme, mde*:

$$\text{arit}(B^\bullet)\,\gamma(M_1^\bullet, M_2^\bullet) \equiv \gamma(\text{arit}(B^\bullet)\,M_1^\bullet, M_2^\bullet) + \gamma(M_1^\bullet, \text{arit}(B^\bullet)\,M_2^\bullet) \tag{474}$$
$$\forall\gamma \in \{\text{dle}, \text{dme}, \text{mde}, \text{lu}, \text{mu}\}$$

Therefore, for any element $A^\bullet$ in $Flex(\mathfrak{E}^\bullet)$ and any expression of $A^\bullet$ as a sum of r-linear monomials h_i involving the operations $\text{dle}, \text{dme}, \text{mde}$ respectively r_0, r_1, r_2 times, with $r_0 + r_1 + r_2 = r - 1$:

$$A^\bullet = \sum_i h_i(\overbrace{\mathfrak{E}^\bullet, \ldots, \mathfrak{E}^\bullet}^{r\ times}) \tag{475}$$

we can write:

$$\text{arit}(B^\bullet)\,A^\bullet = \sum_i \sum_j h_i(\mathfrak{E}^\bullet, \ldots, \overbrace{\text{arit}(B^\bullet)\,\mathfrak{E}^\bullet}^{j^{th}\ term}, \ldots, \mathfrak{E}^\bullet) \tag{476}$$

Moreover, setting $A^\bullet = \mathfrak{E}^\bullet$ in (473) and using $dari(\mathfrak{E}^\bullet, B^\bullet) \equiv 0$ (see §8.4), we find:

$$\text{arit}(B^\bullet)\,\mathfrak{E}^\bullet \equiv -\text{dle}(\mathfrak{E}^\bullet, B^\bullet) \tag{477}$$

Combining (476)-(477) with the earlier identities (472)-(473), we find that, on the one-generator, free pre-associative algebra $Flex(\mathfrak{E}^\bullet)$, the operations *dari* and *ari* can be expressed entirely in terms of *dle, dme, mde*. In view of §10.5, we can duplicate that in any free pre-associative algebra, no matter with how many generators, simply by reasoning on the tree theoretical model $\mathbb{C}[\mathbb{OT}; n\ colours]$ and by transposing the preceding operations in terms of λ, μ, μ_*. The bottomline is this:

Proposition 10.5 (Janus algebra) .
Any free pre-associative algebra, side by side with the 'outer' operations

$$\boldsymbol{\lambda}\ \ (\textit{pre-Lie}) \quad \textit{and} \quad \boldsymbol{l}\ \ (\textit{Lie})$$

automatically possesses two 'inner' operations

$$\boldsymbol{\rho}\ \ (\textit{pre-Lie}) \quad \textit{and} \quad \boldsymbol{r}\ \ (\textit{Lie})$$

with (ρ, r) relating to (λ, l) exactly as $(dari, ari)$ to (dle, lu).

Here, *inner* and *outer* allude to the shape that these operations assume in the tree theoretical model: simple *tree insertion* for ρ and r; subtle *tree attachment* for λ and l. It is this inner/outer dichotomy, corresponding to the inflected/uninflected dichotomy familiar from the flexion model, that (arguably) justifies attaching the name of Janus (the god of thresholds; he of the two faces, inward- and outward-looking) to these algebras. But we are treading on thin ice here: for the name to stick, and the notion to prove its worth, it would take the discovery of interesting examples of *non-free* Janus algebras. And the difficulty in the non-free context is clearly that the procedure (475)-(476) no longer applies: different expansions (476) of a given $A^\bullet$ may no longer lead to an unambiguous action (476) of $arit(B^\bullet)$ on $A^\bullet$.

11 Tables.

11.1 From binary to ordered trees.

Here are the first matrices mob_r corresponding to the basis change of $Flex_r(\mathfrak{E}^\bullet)$, relative to the standard k-indexation:

$$\begin{array}{rcll} \{\mathfrak{bte}^\bullet_{r,k}\} & \mapsto & \{\hat{\mathfrak{o}}\mathfrak{te}^\bullet_{r,k}\} & \quad (\textit{binary to ordered}) \\ \hat{\mathrm{o}}\mathrm{te}^\bullet_{r,p} & = & \displaystyle\sum_{1\leqslant q\leqslant \kappa_r} \mathrm{mob}_r^{p,q}\ \mathrm{bte}^\bullet_{r,q} & \end{array}$$

Their coefficients $mob_r^{p,q}$ are all of the form 0, 1, or $\bar{1} := -1$ and verify the properties listed in §6.8.

$$\mathrm{mob}_1 := [1] \qquad \mathrm{mob}_2 := \begin{bmatrix} 1 & \bar{1} \\ 0 & 1 \end{bmatrix} \qquad \mathrm{mob}_3 := \begin{bmatrix} 1 & \bar{1} & 0 & \bar{1} & 1 \\ 0 & 1 & \bar{1} & 1 & 0 \\ 0 & 0 & 1 & \bar{1} & \bar{1} \\ 0 & 0 & 0 & 1 & \bar{1} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{mob}_4 = \begin{bmatrix}
1 & \bar{1} & 0 & \bar{1} & 1 & 0 & 0 & 0 & 0 & \bar{1} & 1 & 0 & 1 & \bar{1} \\
0 & 1 & \bar{1} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{1} & 1 & \bar{1} & 0 \\
0 & 0 & 1 & \bar{1} & \bar{1} & \bar{1} & 1 & 0 & 0 & 1 & \bar{1} & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & \bar{1} & 0 & 0 & \bar{1} & 1 & 1 & 1 & \bar{1} & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & \bar{1} & 1 & 0 & \bar{1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & \bar{1} & 0 & 0 & \bar{1} & 1 & \bar{1} & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & \bar{1} & \bar{1} & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \bar{1} & \bar{1} & \bar{1} & 1 & \bar{1} & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \bar{1} & 0 & \bar{1} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \bar{1} & 0 & \bar{1} & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \bar{1} & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \bar{1} & \bar{1} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \bar{1} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

$$\mathrm{mob}_5 = \begin{bmatrix} \mathrm{mob}_5^{1,1} & \mathrm{mob}_5^{1,2} & \mathrm{mob}_5^{1,3} \\ [0] & \mathrm{mob}_5^{2,2} & \mathrm{mob}_5^{2,3} \\ [0] & [0] & \mathrm{mob}_5^{3,3} \end{bmatrix} \quad \textit{with} \quad \begin{cases} \mathrm{mob}_5^{1,1} = \mathrm{mob}_5^{3,3} = \mathrm{mob}_4 \\ \textit{and the other } \mathrm{mob}_5^{i,j} \textit{ as follows} \end{cases}$$

$$\mathrm{mob}_5^{1,2} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\bar{1} & 1 & 0 & 1 & \bar{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \bar{1} & 1 & \bar{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \bar{1} & 1 & 1 & \bar{1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \bar{1} & 1 & 1 & 1 & \bar{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{1} & 1 & 0 & 1 & \bar{1} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{1} & 1 & \bar{1} & 0 \\
0 & 0 & 0 & \bar{1} & 1 & 0 & 0 & 0 & 0 & 1 & \bar{1} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \bar{1} & 0 & 1 & 1 & 1 & \bar{1} & 0 & 0 \\
0 & 0 & 0 & 0 & \bar{1} & 0 & 1 & 0 & 0 & \bar{1} & 0 & 0 & 0 & 0
\end{bmatrix}$$

$$\mathrm{mob}_5^{1,3} = \begin{bmatrix}
\bar{1} & 1 & 0 & 1 & \bar{1} & 0 & 0 & 0 & 0 & 1 & \bar{1} & 0 & \bar{1} & 1 \\
0 & \bar{1} & 1 & \bar{1} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \bar{1} & 1 & 0 \\
0 & 0 & \bar{1} & 1 & 1 & 1 & \bar{1} & 0 & 0 & \bar{1} & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & \bar{1} & 1 & 0 & 0 & 1 & \bar{1} & \bar{1} & \bar{1} & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & \bar{1} & 0 & 1 & \bar{1} & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & \bar{1} & 0 & \bar{1} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & \bar{1} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & \bar{1} & 1 & \bar{1} & \bar{1} & \bar{1} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\bar{1} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & \bar{1} & 1 & \bar{1} & 0 & 0 & \bar{1} & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & \bar{1} & \bar{1} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\bar{1} & \bar{1} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\bar{1} & 0 & \bar{1} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

$$\mathrm{mob}_5^{2,2} = \begin{bmatrix}
1 & \bar{1} & 0 & \bar{1} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & \bar{1} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & \bar{1} & \bar{1} & \bar{1} & 1 & \bar{1} & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & \bar{1} & 0 & 0 & 0 & 0 & \bar{1} & 1 & \bar{1} & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & \bar{1} & 0 & \bar{1} & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & \bar{1} & \bar{1} & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \bar{1} & \bar{1} & \bar{1} & 1 & \bar{1} & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \bar{1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \bar{1} & 0 & \bar{1} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \bar{1} & 0 & \bar{1} & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \bar{1} & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \bar{1} & \bar{1} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \bar{1} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

$$\mathrm{mob}_5^{2,3} = \begin{bmatrix}
\bar{1} & 1 & 0 & 1 & \bar{1} & \bar{1} & 1 & 0 & 0 & 1 & \bar{1} & 1 & \bar{1} & \bar{1} \\
0 & \bar{1} & 1 & \bar{1} & 0 & 0 & \bar{1} & 1 & 1 & \bar{1} & 0 & \bar{1} & 0 & 0 \\
1 & \bar{1} & 0 & 0 & 0 & 1 & \bar{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & \bar{1} & 0 & 0 & 1 & 1 & \bar{1} & \bar{1} & 0 & 0 & 0 & 0 & 0 \\
\bar{1} & 0 & 0 & 0 & 0 & \bar{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\bar{1} & 1 & \bar{1} & 1 & 1 & 1 & \bar{1} & \bar{1} & 1 & 1 & 1 & \bar{1} & 1 & \bar{1} \\
1 & 0 & 1 & 0 & 0 & \bar{1} & 0 & 1 & \bar{1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \bar{1} & 1 & 0 & \bar{1} & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\bar{1} & \bar{1} & 1 & \bar{1} & 1 & 0 & 0 & 1 & \bar{1} & \bar{1} & 1 & 0 & 1 & \bar{1} \\
0 & 0 & \bar{1} & 0 & \bar{1} & 1 & 1 & \bar{1} & 0 & 0 & \bar{1} & 1 & \bar{1} & 0 \\
0 & 0 & 0 & 0 & 0 & \bar{1} & \bar{1} & 1 & 1 & 0 & 0 & \bar{1} & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{1} & 1 & 0 & 0 & 0 & \bar{1} & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{1} & 0 & 0 & 0 & 0 & \bar{1}
\end{bmatrix}$$

11.2 From ordered to binary trees.

Here are the first matrices mbo_r corresponding to the basis change of $Flex_r(\mathfrak{E}^\bullet)$, relative to the standard k-indexation:

$$\begin{array}{rcl} \{\hat{\mathfrak{ote}}^\bullet_{r,k}\} & \mapsto & \{\mathfrak{bte}^\bullet_{r,k}\} \qquad (\textit{ordered to binary}) \\ \mathrm{bte}^\bullet_{r,p} & = & \displaystyle\sum_{1\leqslant q\leqslant \kappa_r} \mathrm{mbo}_r^{p,q}\ \hat{\mathrm{ote}}^\bullet_{r,q} \end{array}$$

Their coefficients $mob_r^{p,q}$ are always non-negative integers and verify the properties listed in §6.8.

$$\mathrm{mbo}_1 = [1] \qquad \mathrm{mbo}_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad \mathrm{mbo}_3 = \begin{bmatrix} 1&1&1&1&1 \\ 0&1&1&0&1 \\ 0&0&1&1&2 \\ 0&0&0&1&1 \\ 0&0&0&0&1 \end{bmatrix}$$

$$\mathrm{mbo}_4 = \begin{bmatrix}
1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\
0&1&1&0&1&1&1&0&1&0&1&1&0&1 \\
0&0&1&1&2&1&2&1&2&0&0&1&1&2 \\
0&0&0&1&1&0&1&1&1&0&0&0&1&1 \\
0&0&0&0&1&0&1&0&1&0&0&0&0&1 \\
0&0&0&0&0&1&1&1&2&1&1&2&2&3 \\
0&0&0&0&0&0&1&1&2&0&1&1&1&3 \\
0&0&0&0&0&0&0&1&1&1&2&2&2&3 \\
0&0&0&0&0&0&0&0&1&0&0&1&1&3 \\
0&0&0&0&0&0&0&0&0&1&1&1&1&1 \\
0&0&0&0&0&0&0&0&0&0&1&1&0&1 \\
0&0&0&0&0&0&0&0&0&0&0&1&1&2 \\
0&0&0&0&0&0&0&0&0&0&0&0&1&1 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{bmatrix}$$

$$\mathrm{mbo}_5 = \begin{bmatrix} \mathrm{mbo}_5^{1,1} & \mathrm{mbo}_5^{1,2} & \mathrm{mbo}_5^{1,3} \\ [0] & \mathrm{mbo}_5^{2,2} & \mathrm{mbo}_5^{2,3} \\ [0] & [0] & \mathrm{mbo}_5^{3,3} \end{bmatrix} \quad \textit{with} \quad \begin{cases} \mathrm{mbo}_5^{1,1} = \mathrm{mbo}_5^{1,3} = \mathrm{mbo}_5^{3,3} = \mathrm{mbo}_4 \\ \textit{and the other } \ \mathrm{mbo}_5^{i,j} \ \textit{ as follows} \end{cases}$$

$$\mathrm{mbo}_5^{1,2} = \begin{bmatrix}
1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\
1&1&1&1&1&0&0&1&1&0&1&1&0&1 \\
1&2&2&1&2&1&1&2&2&0&0&1&1&2 \\
0&1&1&0&1&1&1&1&1&0&0&0&1&1 \\
0&1&1&0&1&0&0&1&1&0&0&0&0&1 \\
1&1&2&2&3&1&2&2&3&1&1&2&2&3 \\
0&1&2&1&3&1&1&2&3&0&1&1&1&3 \\
0&0&1&2&3&1&2&1&3&1&2&2&2&3 \\
0&0&1&1&3&0&1&1&3&0&0&1&1&3 \\
0&0&0&1&1&0&1&0&1&1&1&1&1&1 \\
0&0&0&1&1&0&0&0&1&0&1&1&0&1 \\
0&0&0&1&2&0&1&0&2&0&0&1&1&2 \\
0&0&0&0&1&0&1&0&1&0&0&0&1&1 \\
0&0&0&0&1&0&0&0&1&0&0&0&0&1
\end{bmatrix}$$

$$
\mathrm{mbo}_5^{2,2} = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 3 \\
0 & 1 & 1 & 0 & 1 & 1 & 1 & 2 & 2 & 0 & 1 & 1 & 1 & 3 \\
0 & 0 & 1 & 1 & 2 & 1 & 2 & 2 & 4 & 1 & 2 & 3 & 3 & 6 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 2 & 1 & 1 & 2 & 2 & 3 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 2 & 0 & 1 & 1 & 1 & 3 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 2 & 2 & 2 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

$$
\mathrm{mbo}_5^{2,3} = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 2 & 2 & 3 & 3 & 4 \\
0 & 1 & 1 & 0 & 1 & 1 & 2 & 1 & 3 & 0 & 2 & 2 & 1 & 4 \\
0 & 0 & 1 & 1 & 2 & 1 & 2 & 2 & 4 & 1 & 2 & 4 & 4 & 8 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 2 & 3 & 4 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 4 \\
1 & 1 & 2 & 2 & 3 & 2 & 3 & 3 & 4 & 2 & 3 & 4 & 4 & 6 \\
0 & 1 & 1 & 1 & 3 & 1 & 3 & 1 & 3 & 1 & 3 & 3 & 3 & 6 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 3 & 1 & 1 & 3 & 3 & 6 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 0 & 1 & 2 & 2 & 6 \\
1 & 2 & 2 & 2 & 3 & 2 & 3 & 2 & 3 & 2 & 3 & 3 & 3 & 4 \\
0 & 0 & 1 & 1 & 3 & 1 & 3 & 1 & 3 & 0 & 1 & 2 & 1 & 4 \\
0 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 4 & 1 & 2 & 4 & 4 & 8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 3 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 4
\end{bmatrix}
$$

The block decompositions

$$
\mathrm{mob}_5 = \begin{bmatrix} \mathrm{mob}_r^{1,1} & \mathrm{mob}_r^{1,2} & \mathrm{mob}_r^{1,3} \\ [0] & \mathrm{mob}_r^{2,2} & \mathrm{mob}_r^{2,3} \\ [0] & [0] & \mathrm{mob}_r^{3,3} \end{bmatrix} , \ \mathrm{mbo}_5 = \begin{bmatrix} \mathrm{mbo}_r^{1,1} & \mathrm{bo}_r^{1,2} & \mathrm{mbo}_r^{1,3} \\ [0] & \mathrm{mbo}_r^{2,2} & \mathrm{mbo}_r^{2,3} \\ [0] & [0] & \mathrm{mbo}_r^{3,3} \end{bmatrix} \tag{478}
$$

with the identities $\begin{cases} \mathrm{mob}_r^{1,1} = \mathrm{mob}_r^{3,3} = \mathrm{mob}_{r-1} \\ \mathrm{mbo}_r^{1,1} = \mathrm{mbo}_r^{1,3} = \mathrm{mbo}_r^{3,3} = \mathrm{mbo}_{r-1} \end{cases}$ hold for all values of r, but only for $r = 5$ do we get square blocks $\mathrm{mob}_r^{1,2}, \mathrm{mob}_r^{2,3}, \mathrm{mbo}_r^{1,2}, \mathrm{mbo}_r^{2,3}$.

11.3 The căłt- and cālt-based conversion matrices.

11.4 Pilot polynomials.

The pilot polynomials $P_{r,d}, Q_{r,d}, L_{r,d}$ enter, as their main ingredient, the formulae for $dim\big(Flex_{r,d}(\mathfrak{E}^\bullet)\big)$ — the number of independent elements in $Flex_{r,d}(\mathfrak{E}^\bullet)$ with depth r and co-alternality degree d. We tabulate here only the first $P_{r,d}$

which, unlike the other two series, don't admit explicit expansions.

$$
\begin{aligned}
P_0 &= 1 \\
P_{1,2} &= \frac{1}{2}(y_1^2 - y_2) \\
P_{2,3} &= \frac{1}{3}(y_1^3 - y_3) \\
P_{2,4} &= \frac{1}{8}(y_1^4 - 2\,y_1^2\,y_2 + 3\,y_2^2 - 2\,y_4) \\
P_{3,4} &= \frac{1}{4}(y_1^2 - y_2)\,(y_1^2 + y_2) \\
P_{3,5} &= \frac{1}{6}(y_1^3 - y_3)\,(y_1^2 - y_2) \\
P_{3,6} &= \frac{1}{48}\Big\{(y_1^6 - 3\,y_1^4\,y_2 + 9\,y_1^2\,y_2^2 - 6\,y_1^2\,y_4 - 7\,y_2^3 + 6\,y_2\,y_4 + 8\,y_3^2 - 8\,y_6)
\end{aligned}
$$

$$
\begin{aligned}
P_{4,5} &= \frac{1}{5}(y_1^5 - y_5) \\
P_{4,6} &= \frac{1}{72}(13\,y_1^6 - 9\,y_1^4 y_2 - 8\,y_1^3 y_3 - 9\,y_1^2 y_2^2 + 21\,y_2^3 + 4\,y_3^2 - 12\,y_6) \\
P_{4,7} &= \frac{1}{24}(y_1^3 - y_3)\,(y_1^4 - 2\,y_1^2 y_2 + 3\,y_2^2 - 2\,y_4) \\
P_{4,8} &= \frac{1}{384}\begin{cases} +y_1^8 - 4\,y_1^6 y_2 + 18\,y_1^4 y_2^2 - 12\,y_1^4 y_4 - 28\,y_1^2 y_2^3 + 24\,y_1^2 y_2 y_4 + 32\,y_1^2 y_3^2 \\ +25\,y_2^4 - 32\,y_1^2 y_6 - 36\,y_2^2 y_4 - 32\,y_2 y_3^2 + 32\,y_2 * y_6 + 60\,y_4^2 - 48\,y_8 \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
P_{5,6} &= \frac{1}{6}(y_1^6 - y_2^3 - y_3^2 + y_6) \\
P_{5,7} &= \frac{1}{60}(y_1^2 - y_2)\,(11\,y_1^5 + 5\,y_1^3 y_2 - 5\,y_1^2 y_3 - 5\,y_2 y_3 - 6\,y_5) \\
P_{5,8} &= \frac{1}{288}(y_1^2 - y_2)\begin{cases} +17\,y_1^6 - 9 y_1^4 y_2 - 16\,y_1^3 y_3 + 9\,y_1^2 y_2^2 - 18\,y_1^2 y_4 \\ +51\,y_2^3 - 18\,y_2 y_4 + 8\,y_3^2 - 24\,y_6 \end{cases} \\
P_{5,9} &= \frac{1}{144}(y_1^3 - y_3)\,(y_1^6 - 3\,y_1^4 y_2 + 9\,y_1^2 y_2^2 - 6\,y_1^2 y_4 - 7\,y_2^3 + 6\,y_2 y_4 + 8\,y_3^2 - 8\,y_6) \\
P_{5,10} &= \frac{1}{3840}\begin{cases} +y_1^{10} - 5\,y_1^8 y_2 + 30\,y_1^6 y_2^2 - 20\,y_1^6 y_4 - 70\,y_1^4\,y_2^3 + 60\,y_1^4 y_2 y_4 \\ +80\,y_1^4 y_3^2 + 125\,y_1^2 y_2^4 - 80\,y_1^4 * y_6 - 180\,y_1^2 y_2^2 y_4 - 160\,y_1^2 y_2 y_3^2 \\ -81\,y_2^5 + 160\,y_1^2 y_2 y_6 + 300\,y_1^2 y_4^2 + 140\,y_2^3 y_4 + 240\,y_2^2 y_3^2 \\ -240\,y_1^2 y_8 - 240\,y_2^2 y_6 - 300\,y_2 y_4^2 - 160\,y_3^2 y_4 + 240\,y_2 y_8 \\ +160\,y_4 y_6 + 384\,y_5^2 - 384\,y_{10} \end{cases}
\end{aligned}
$$

$$P_{6,7} = \frac{1}{7}(y_1^7 - y_7)$$

$$P_{6,8} = \frac{1}{480}\begin{cases} +87\,y_1^8 - 40\,y_1^6 y_2 - 32\,y_1^5 y_3 - 30\,y_1^4 y_2^2 - 40\,y_1^2 y_2^3 - 32\,y_1^3 y_5 - 40\,y_1^2 y_3^2 \\ +115\,y_2^4 + 40\,y_1^2 y_6 + 40\,y_2 y_3^2 - 40\,y_2 y_6 + 32\,y_3 y_5 - 60\,y_4^2 \end{cases}$$

$$P_{6,9} = \frac{1}{3240}\begin{cases} +236\,y_1^9 - 297\,y_1^7 y_2 - 195\,y_1^6 y_3 + 108\,y_1^5 y_2^2 - 162\,y_1^5 y_4 + 135\,y_1^4 y_2 y_3 \\ +315\,y_1^3 y_2^3 - 81\,y_1^4 y_5 + 60\,y_1^3 y_3^2 + 135\,y_1^2 y_2^2 y_3 - 180\,y_1^3 y_6 \\ +162\,y_1^2 y_2 y_5 - 315\,y_2^3 y_3 - 243\,y_2^2 y_5 + 340\,y_3^3 + 180\,y_3 y_6 \\ +162\,y_4 y_5 - 360\,y_9 \end{cases}$$

$$P_{6,10} = \frac{1}{576}\begin{cases} +7\,y_1^{10} - 17\,y_1^8 y_2 - 8\,y_1^7 y_3 + 36\,y_1^6 y_2^2 - 26\,y_1^6 y_4 + 16\,y_1^5 y_2 y_3 \\ +18\,y_1^4 y_2 y_4 + 28\,y_1^4 y_3^2 - 24\,y_1^3 y_2^2 y_3 - 51\,y_1^2 y_2^4 - 36\,y_1^4 y_6 + 16\,y_1^3 y_3 y_4 \\ +18\,y_1^2 y_2^2 y_4 - 8\,y_1^2 y_2 y_3^2 + 57\,y_2^5 + 24\,y_1^2 y_2 y_6 - 42\,y_2^3 y_4 - 12\,y_2^2 y_3^2 \\ -12\,y_2^2 y_6 - 8\,y_3^2 y_4 + 24\,y_4 y_6 \end{cases}$$

$$P_{6,11} = \frac{1}{1152}\,(y_1^3 - y_3)\begin{cases} +y_1^8 - 4\,y_1^6 y_2 + 18\,y_1^4 y_2^2 - 12\,y_1^4 y_4 - 28\,y_1^2 y_2^3 \\ +24\,y_1^2 y_2 y_4 + 32\,y_1^2 y_3^2 + 25\,y_2^4 - 32\,y_1^2 y_6 - 36\,y_2^2 y_4 \\ -32\,y_2 y_3^2 + 32\,y_2 y_6 + 60\,y_4^2 - 48\,y_8 \end{cases}$$

$$P_{6,12} = \frac{1}{46080}\begin{cases} +y_1^{12} - 6\,y_1^{10} y_2 + 45\,y_1^8 y_2^2 - 30\,y_1^8 y_4 - 140\,y_1^6 y_2^3 + 120\,y_1^6 y_2 y_4 \\ +160\,y_1^6 y_3^2 + 375\,y_1^4 y_2^4 - 160\,y_1^6 y_6 - 540\,y_1^4 y_2^2 y_4 - 480\,y_1^4 y_2 y_3^2 \\ -486\,y_1^2 y_2^5 + 480\,y_1^4 y_2 y_6 + 900\,y_1^4 y_4^2 + 840\,y_1^2 y_2^3 y_4 + 1440\,y_1^2 y_2^2 y_3^2 \\ +331\,y_2^6 - 720\,y_1^4 y_8 - 1440\,y_1^2 y_2^2 y_6 - 1800\,y_1^2 y_2 y_4^2 - 960\,y_1^2 y_3^2 y_4 \\ -750\,y_2^4 y_4 - 1120\,y_2^3 y_3^2 + 1440\,y_1^2 y_2 y_8 + 960\,y_1^2 y_4 y_6 + 2304\,y_1^2 y_5^2 \\ +1120\,y_2^3 y_6 + 2700\,y_2^2 y_4^2 + 960\,y_2 y_3^2 y_4 + 640\,y_3^4 - 2304\,y_1^2 y_{10} \\ -2160\,y_2^2 y_8 - 960\,y_2 y_4 y_6 - 2304\,y_2 y_5^2 - 1280\,y_3^2 y_6 - 1560\,y_4^3 \\ +2304\,y_{10} y_2 + 1440\,y_4 y_8 + 4480\,y_6^2 - 3840\,y_{12} \end{cases}$$

11.5 μ-generators and enumerating series.

Recall that $\gamma_\delta(t) := \sum t^r \dim\big(Flex^{root}_{r,1+\delta}(\mathfrak{E}^\bullet)\big)$ and $\xi_d(t) := \sum t^r \dim\big(Flex_{r,d}(\mathfrak{E}^\bullet)\big)$. Here are the first generating series γ_δ up to $\delta = 5$, calculated from formula (324). The same series assume distinct and more complex, though equivalent, expressions $\widetilde{\gamma}_\delta$ when calculated from formula (322). We mention the $\widetilde{\gamma}_\delta$'s up to $\delta = 3$, after which they become too clumsy, while the γ_δ's remain manageable. The two systems are seen to coincide only after each $\gamma_\delta(t)$ and each $\widetilde{\gamma}_\delta(t)$ gets expressed in terms of the sole series $\gamma_0(t)$ and its dilatees $\gamma_0(t^k)$.

$$\gamma_1(t) = \widetilde{\gamma}_1(t) = \frac{\gamma_0(t)}{1-\gamma_0(t)}\Big(\frac{1}{2}\gamma_0(t)^2 - \frac{1}{2}\gamma_0(t^2)\Big)$$

$$\gamma_2(t) \;=\; \frac{\gamma_0(t)}{1-\gamma_0(t)} \begin{cases} +\frac{1}{3}\left(\gamma_0(t)^3-\gamma_0(t^3)\right)+\frac{1}{4}\left(\gamma_0(t^2)^2-\gamma_0(t^4)\right) \\ +\gamma_0(t)\gamma_1(t) \\ +\frac{1}{2}\left(\gamma_1(t^2)+\gamma_0(t)^2\gamma_1(t)^2\right) \end{cases}$$

$$\widetilde{\gamma}_2(t) \;=\; \frac{\gamma_0(t)}{1-\gamma_0(t)} \begin{cases} +\frac{1}{2}\gamma_1(t)^2+\frac{1}{2}\gamma_1(t^2) \\ +\gamma_1(t)\left(\frac{1}{2}\gamma_0(t)^2-\frac{1}{2}\gamma_0(t^2)+\gamma_0(t)\right) \\ +\frac{1}{8}\gamma_0(t)^4-\frac{1}{4}\gamma_0(t^4)+\frac{3}{8}\gamma_0(t^2)^2-\frac{1}{4}\gamma_0(t)^2\gamma_0(t^2) \\ +\frac{1}{3}\gamma_0(t)^3-\frac{1}{3}\gamma_0(t^3) \end{cases}$$

$$\gamma_3(t) \;=\; \frac{\gamma_0(t)}{1-\gamma_0(t)} \begin{cases} +\frac{1}{4}(\gamma_0(t)^4-\gamma_0(t^2)^2)+\frac{1}{6}(\gamma_0(t^3)^2-\gamma_0(t^6)) \\ +\gamma_0(t)^2\gamma_1(t) \\ +\frac{1}{2}\left(\gamma_1(t)^2-\gamma_1(t^2)\right)+\gamma_0(t)\gamma_2(t) \\ +\frac{1}{3}\gamma_1(t^3)+\gamma_0(t)^{-2}\gamma_1(t)\gamma_2(t)-\frac{1}{3}\gamma_0(t)^{-3}\gamma_1(t)^3 \end{cases}$$

$$\widetilde{\gamma}_3(t) \;=\; \frac{\gamma_0(t)}{1-\gamma_0(t)} \begin{cases} +\frac{1}{3}\gamma_1(t^3)+\frac{1}{6}\gamma_1(t)^3+\frac{1}{2}\gamma_1(t)\gamma_1(t^2)+\gamma_1(t)\gamma_2(t) \\ +\gamma_2(t)\big(\frac{1}{2}\gamma_0(t)^2-\frac{1}{2}\gamma_0(t^2)+\gamma_0(t)\big) \\ +\gamma_1(t^2)\big(\frac{1}{4}\gamma_0(t)^2-\frac{1}{4}\gamma_0(t^2)-\frac{1}{2}\big) \\ +\gamma_1(t)^2\big(\frac{1}{4}\gamma_0(t)^2-\frac{1}{4}\gamma_0(t^2)+\gamma_0(t)+\frac{1}{2}\big) \\ +\gamma_1(t)\times \begin{cases} +\frac{1}{8}\gamma_0(t)^4-\frac{1}{4}\gamma_0(t^4)-\frac{1}{4}\gamma_0(t)^2\gamma_0(t^2) \\ +\frac{3}{8}\gamma_0(t^2)^2+\frac{5}{6}\gamma_0(t)^3-\frac{1}{3}\gamma_0(t^3) \\ -\frac{1}{2}\gamma_0(t)\gamma_0(t^2)+\gamma_0(t)^2 \end{cases} \\ +\frac{1}{4}\big(\gamma_0(t)^2-\gamma_0(t^2)\big)\big(\gamma_0(t)^2+\gamma_0(t^2)\big) \\ +\frac{1}{6}\big(\gamma_0(t)^3-\gamma_0(t^3)\big)\big(\gamma_0(t)^2-\gamma_0(t^2)\big) \\ +\frac{1}{48}\gamma_0(t)^6-\frac{1}{6}\gamma_0(t^6)-\frac{1}{16}\gamma_0(t)^4\gamma_0(t^2)+\frac{3}{16}\gamma_0(t)^2\gamma_0(t^2)^2 \\ -\frac{1}{8}\gamma_0(t)^2\gamma_0(t^4)-\frac{7}{48}\gamma_0(t^2)^3+\frac{1}{8}\gamma_0(t^2)\gamma_0(t^4)+\frac{1}{6}\gamma_0(t^3)^2 \end{cases}$$

$$\gamma_4(t) \;=\; \frac{\gamma_0(t)}{1-\gamma_0(t)} \begin{cases} +\frac{1}{5}\left(\gamma_0(t)^5-\gamma_0(t^5)\right)+\frac{1}{6}\left(\gamma_0(t^2)^3-\gamma_0(t^6)\right)+\frac{1}{8}\left(\gamma_0(t^4)^2-\gamma_0(t^8)\right) \\ +\gamma_0(t)^3\gamma_1(t) \\ +\gamma_0(t)^2\gamma_2(t)+\gamma_0(t)\gamma_1(t)^2+\frac{1}{2}\,\gamma_0(t^2)\gamma_1(t^2) \\ +\gamma_0(t)\gamma_3(t)+\gamma_1(t)\gamma_2(t) \\ +\frac{1}{4}\gamma_1(t^4)+\frac{1}{2}\gamma_2(t^2) \\ + \begin{cases} +\gamma_0(t)^{-2}\gamma_1(t)\gamma_3(t)+\frac{1}{2}\,\gamma_0(t)^{-2}\gamma_2(t)^2 \\ -\gamma_0(t)^{-3}\gamma_1(t)^2\gamma_2(t)+\frac{1}{4}\,\gamma_0(t)^{-4}\gamma_1(t)^4 \end{cases} \end{cases}$$

$$\gamma_5(t) \;=\; \frac{\gamma_0(t)}{1-\gamma_0(t)} \begin{cases} +\frac{1}{10}\,(\gamma_0(t^5)^2-\gamma_0(t^10))+\frac{1}{6}\,(\gamma_0(t^6)-\gamma_0(t^3)^2-\gamma_0(t^2)^3+\gamma_0(t)^6) \\ +\gamma_0(t)^4\gamma_1(t) \\ -\frac{1}{2}\,\gamma_0(t^2)\gamma_1(t^2)+\gamma_0(t)^3\gamma_2(t)+\frac{3}{2}\,\gamma_0(t)^2\gamma_1(t)^2 \\ +\frac{1}{3}\,(\gamma_1(t)^3-\gamma_1(t^3))+2\,\gamma_0(t)\gamma_1(t)\gamma_2(t)+\gamma_0(t)^2\gamma_3(t) \\ +\frac{1}{2}\,(\gamma_2(t)^2-\gamma_2(t^2))+\gamma_1(t)\gamma_3(t)+\gamma_0(t)\gamma_4(t) \\ + \begin{cases} +\frac{1}{5}\,\gamma_1(t^5)+\gamma_0(t)^{-2}\gamma_1(t)\gamma_4(t)+\gamma_0(t)^{-2}\gamma_2(t)\gamma_3(t) \\ -\gamma_0(t)^{-3}\gamma_1(t)^2\gamma_3(t)-\gamma_0(t)^{-3}\gamma_1(t)\gamma_2(t)^2 \\ +\gamma_0(t)^{-4}\gamma_1(t)^3\gamma_2(t)-\frac{1}{5}\,\gamma_0(t)^{-5}\gamma_1(t)^5 \end{cases} \end{cases}$$

There is no need to tabulate the series ξ_d since they relate quite elementarily to the series γ_δ:

$$1+\sum_{1\leqslant d} x^d\,\xi_d(t) = \Big(1-\sum_{0\leqslant\delta} x^{\delta+1}\,\gamma_\delta(t)\Big)^{-1} \tag{479}$$

With the complementary dimensions and their generating series

$$\gamma_d^{\mathrm{co}}(t) := \sum t^r\ \dim\big(Flex^{root}_{r,r-d}(\mathfrak{E}^\bullet)\big) \quad , \quad \xi_d^{\mathrm{co}}(t) := \sum t^r\ \dim\big(Flex_{r,r-d}(\mathfrak{E}^\bullet)\big)$$

the position is reversed: it is the $\xi_d^{co}(t)$'s that are now more basic and regular that the $\gamma_d^{co}(t)$'s. We have $\gamma_0^{co}(t)=\frac{1}{1-t}, \gamma_t^{co}(t)=\frac{t^2}{1-t^2}$ and for $2\leqslant d$:

$$\xi_d^{\mathrm{co}}(t) = \frac{t^{d+1}\ \ \widehat{\xi}_d^{\mathrm{co}}(t)}{(1-t)^2\prod_{k=1}^{d-1}(1-t^k)} \quad \Big(\widehat{\xi}_d^{\mathrm{co}}\ \textit{polynomial with positive coefficients.}\Big)$$

$$\widehat{\xi}^{\text{co}}_2(t) \;=\; 2$$
$$\widehat{\xi}^{\text{co}}_3(t) \;=\; 4+4\,t+t^2$$
$$\widehat{\xi}^{\text{co}}_4(t) \;=\; 9+18\,t+21\,t^2+13\,t^3+4\,t^4$$
$$\widehat{\xi}^{\text{co}}_5(t) \;=\; 20+63\,t+117\,t^2+150\,t^3+144\,t^4+99\,t^5+48\,t^6+12\,t^7+t^8$$
$$\widehat{\xi}^{\text{co}}_6(t) \;=\; \begin{cases} +48+200\,t+492\,t^2+874\,t^3+1250\,t^4+1470\,t^5+1454\,t^6+1200\,t^7\\ +823\,t^8+446\,t^9+179\,t^{10}+43\,t^{11}+5\,t^{12}\end{cases}$$
$$\widehat{\xi}^{\text{co}}_7(t) \;=\; \begin{cases} +115+612\,t+1856\,t^2+4092\,t^3+7338\,t^4+11188\,t^5+14952\,t^6\\ +17685\,t^7+18720\,t^8+17734\,t^9+15038\,t^{10}+11305\,t^{11}+7472\,t^{12}\\ +4220\,t^{13}+1959\,t^{14}+691\,t^{15}+164\,t^16+20\,t^{17}+t^{18}\end{cases}$$
$$\widehat{\xi}^{\text{co}}_8(t) \;=\; \begin{cases} +286+1829\,t+6579\,t^2+17158\,t^3+36312\,t^4+65747\,t^5+105393\,t^6\\ +152363\,t^7+201245\,t^8+244664\,t^9+275326\,t^{10}+287521\,t^{11}\\ +279084\,t^{12}+251511\,t^{13}+210011\,t^{14}+161704\,t^{15}+114080\,t^{16}\\ +72919\,t^{17}+41576\,t^{18}+20578\,t^{19}+8504\,t^{20}+2749\,t^{21}+631\,t^{22}\\ +86\,t^{23}+6\,t^{24}\end{cases}$$
$$\widehat{\xi}^{\text{co}}_9(t) \;=\; \begin{cases} +719+5400,t+22435\,t^2+67252\,t^3+162840\,t^4+337008\,t^5\\ +617895\,t^6+1026063\,t^7+1567380\,t^8+2224883\,t^9+2957005\,t^{10}\\ +3698414\,t^{11}+4370415\,t^{12}+4891788\,t^{13}+5195758\,t^{14}+5240862\,t^{15}\\ +5021676\,t^{16}+4567601\,t^{17}+3939032\,t^{18}+3213160\,t^{19}+2471544\,t^{20}\\ +1784369\,t^{21}+1201926\,t^{22}+748696\,t^{23}+426166\,t^{24}+217766\,t^{25}\\ +97470\,t^{26}+36824\,t^{27}+11143\,t^{28}+2485\,t^{29}+363\,t^{30}+29\,t^{31}+t^{32}\end{cases}$$

11.6 Generators.

We tabulate here the dimensions $dim\big(Flex^{root}_{r,d}(\mathfrak{E}^\bullet)\big)$. They coincide with the number of independent *dme*-generators of $Flex(\mathfrak{E}^\bullet)$ of depth r and alternality co-degree d; or again with the number of independent *mde*-generators of $Flex(\mathfrak{E}^\bullet)$ of depth r and alternality co-degree d.

$r\backslash d$	1	2	3	4	5	6	7
1	1	0	0	0	0	0	0
2	1	0	0	0	0	0	0
3	2	0	0	0	0	0	0
4	4	1	0	0	0	0	0
5	9	4	1	0	0	0	0
6	20	15	6	1	0	0	0
7	48	49	27	7	1	0	0
8	115	156	108	40	9	1	0
9	286	479	405	191	58	10	1
10	719	1452	1446	839	317	76	12
11	1842	4343	5013	3440	1568	476	100
12	4766	12908	16953	13475	7197	2654	693
13	12486	38146	56321	50889	31258	13539	4249
14	32973	112358	184385	186888	129898	64729	23749
15	87811	330064	596741	670807	521166	293759	123608
16	235381	967945	1912776	2363337	2031072	1278615	607456
17	634847	2834876	6082890	8197048	7726269	5375539	2848373
18	1721159	8295446	19214918	28057873	28793800	21951639	12842065
19	4688676	24258864	60352718	94957627	105438275	87443367	56007142
20	12826228	70912286	188635971	318236848	380265993	340952408	237391625
21	35221832	207230122	587096310	1057437216	1353335199	1304863162	981631597
22	97055181	605501661	1820465044	3487307579	4760371271	4912774608	3972200135
23	268282855	1769064947	5626509318	11424302201	16571839003	18230407812	15769569776
24	743724984	5168521107	17339703203	37203829560	57158419908	66782132346	61551392031
25	2067174645	15100910989	53300154409	120512266819	195516025077	241824272661	236627078432

$r\backslash d$	8	9	10	11	12	13	14
10	1	0	0	0	0	0	0
11	13	1	0	0	0	0	0
12	124	15	1	0	0	0	0
13	954	153	16	1	0	0	0
14	6433	1285	183	18	1	0	0
15	39183	9391	1672	218	19	1	0
16	220826	61791	13228	2143	253	21	1
17	1168622	374432	93614	18164	2679	294	22
18	5873298	2121851	607327	137419	24321	3314	335
19	28265358	11374830	3669736	949677	196038	31960	4025
20	131101055	58186393	20902760	6093480	1438399	273220	41221
21	589040325	285949923	113250510	36744581	9767987	2120267	372725
22	2574293016	1357342490	587795091	210179270	62167992	15190258	3051014
23	10980031682	6250542434	2939087278	1148763383	374370222	101746710	22998920
24	45834498690	28024212159	14223161327	6034760155	2149215465	643394656	161727336
25	187689037253	122697191276	66868523384	30616496083	11833911912	3870592511	1071442242

11.7 Dimensions.

We tabulate here the dimensions $dim\big(Flex_{r,d}(\mathfrak{E}^\bullet)\big)$, that is to say, the number of independent elements of $Flex(\mathfrak{E}^\bullet)$ with depth r and alternality co-degree d.

$r\backslash d$	1	2	3	4	5	6	7
1	1	0	0	0	0	0	0
2	1	1	0	0	0	0	0
3	2	2	1	0	0	0	0
4	4	6	3	1	0	0	0
5	9	16	12	4	1	0	0
6	20	45	41	20	5	1	0
7	48	123	138	83	30	6	1
8	115	344	446	328	147	42	7
9	286	957	1428	1222	667	237	56
10	719	2687	4497	4422	2815	1216	358
11	1842	7557	14068	15554	11364	5737	2049
12	4766	21358	43668	53702	44164	25586	10687
13	12486	60516	134911	182423	166881	108917	52194
14	32973	172034	414952	611986	615935	447319	241591
15	87811	490204	1272130	2031248	2230554	1783137	1071839
16	235381	1400182	3888611	6682780	7948687	6935568	4590562
17	634847	4007312	11858590	21819467	27942665	26418916	19091289
18	1721159	11490316	36088314	70777645	97080238	98857214	77431998
19	4688676	33000306	109629926	228277456	333854328	364239242	307360178
20	12826228	94919331	332512570	732566279	1137838606	1324025804	1197360112
21	35221832	273384776	1007132571	2340399407	3847252697	4755820093	4588243551
22	97055181	788366353	3046685364	7447310395	12916320185	16902453286	17327285879
23	268282855	2275974509	9206344974	23612856446	43088443273	59504811265	64589660866
24	743724984	6577376047	27791460920	74625349650	142917362794	207703204583	237970147740
25	2067174645	19025986499	83818587788	235147762795	471566018441	719406620114	867566561776

$r\backslash d$	8	9	10	11	12	13	14
8	1	0	0	0	0	0	0
9	8	1	0	0	0	0	0
10	72	9	1	0	0	0	0
11	514	90	10	1	0	0	0
12	3249	710	110	11	1	0	0
13	18566	4911	950	132	12	1	0
14	98584	30517	7140	1239	156	13	1
15	493086	175067	47938	10053	1581	182	14
16	2350797	941072	295521	72526	13776	1981	210
17	10767971	4798866	1699270	478164	106285	18448	2443
18	47692835	23414204	9230508	2928874	746342	151574	24217
19	205246164	110053967	47799379	16885954	4851914	1129447	211113
20	861583377	500956305	237668718	92506872	29592819	7767218	1663843
21	3538999442	2217703307	1141051386	485173466	171010505	49970416	12068604
22	14260869394	9580881784	5313567180	2450468536	943597282	303804250	81682686
23	56496657745	40507559140	24088665924	11975644498	5001919605	1759411155	521214461
24	220440855233	168004013976	106637213867	56850890652	25599480499	9767448516	3161242858
25	848430232360	684889453403	462148297540	263008126630	127010918277	52248113104	18343321538

12 Conclusion.

12.1 Survey of the main results.

- We replace the usual pre-Lie formulae (for group composition, inversion, iteration; also for the group-to-algebra logarithm), which indiscriminately use all bracket combinations, by optimally economical formulae, which make do with only two types of bracketing: *backward* within *forward*. Beside using less summands, these formulae, unlike the old ones, affect their summands with quite explicit coefficients.

- We consider exotic pre-Lie products and connect them to exotic composition laws, which are rather trite in the identity-tangent case, but become interesting in the general transserial setting.

- We construct a pre-Lie product *dle* (pre-Lie to *lu*) which, unlike those previously in service, preserves alternality and generates the whole of $Flex^{al}(\mathfrak{E}^\bullet)$ from $\mathfrak{E}^\bullet$ alone. This naturally leads to another pre-Lie product *dari* (pre-Lie to *ari*), which also preserves alternality. To go from there to the whole of $Flex(\mathfrak{E}^\bullet)$, yet another operation is needed: either the pre-associative product *dme* or its twin *mde*.

- By suitably combining *dme, mde* and *dle*, we construct three rather complex, multivariate functions $\hat{c}alt$, $\breve{c}alt$, $\bar{c}alt$ of $Flex(\mathfrak{E})$ into itself. They are the *counter-alternators*, so-called because they combine and transform the alternality properties of their arguments in a counter-intuitive manner. For all their outward similarities, one of them, the counter-alternator $\hat{c}alt$, turns out to possess the nicest properties.

- We use $\hat{c}alt$ to construct on $Flex(\mathfrak{E}^\bullet)$ a basis $\{\hat{\mathfrak{o}}\mathfrak{te}^\bullet_{r,k}\}$, indexed by *ordered trees*, where the pre-Lie operations *dle* and *dari* assumes the simplest possible form: with *dle*, the second tree gets 'attached' to the first; with *dari* it gets 'inserted' into it. We then construct, again relying on $\hat{c}alt$, yet another basis $\{\mathfrak{ste}^\bullet_{r,d,k}\}$, indexed by *stacked trees* (appropriate tree superpositions), which faithfully reflects the stratification of $Flex(\mathfrak{E})$ by alternality co-degree d.

- The dimensions of the sub-spaces $Flex_{r,1}(\mathfrak{E}^\bullet) = Flex^{al}_r(\mathfrak{E}^\bullet)$ (alternal elements of depth r) are rather easy to determine. Not so the dimensions of the general sub-spaces $Flex_{r,d}(\mathfrak{E}^\bullet)$ (elements of depth r and alternality co-degree d). With the help of the basis $\{ste^\bullet_{r,d,k}\}$ and of special 'pilot' polynomials, we calculate the dimensions $dim(Flex_{r,d}(\mathfrak{E}^\bullet))$ as well as their more tractable generating series.

- We introduce a notion of pre-associative[68] algebra (– they 'enfold' pre-Lie algebras much like associative algebras 'envelope' Lie algebras –) with three aims in mind:
  (i) to detach the preceding construction from its origin in $Flex(\mathfrak{E})$ and the two rather dissimilar 'polar' models $Flex(Pa)$ and $Flex(Pi)$.
  (ii) to put notions like *counter-alternator* and *alternality co-degree* on a neat natural axiomatic basis
  (iii) to pave the way for possible extensions (for a start, we describe the structure of free pre-associative algebras with any number of generators).

[68]Quite distinct from what sometimes goes by this name.

- Using the correspondence $\mathfrak{\hat{o}te} \mapsto ot$ of $\mathit{Flex}(\mathfrak{E}^\bullet)$ onto $\mathbb{OT}$, we upload the whole of *flexion algebra* onto *tree algebra*. In the process, the inflected *ari* bracket receives as striking an interpretation as the uninflected *lu* bracket: *inserting* trees into one another or *attaching* them to one another. On the axiomatic side, this leads to the notion of Janus algebra,

12.2 Some open questions.

- $\mathbf{Q_1}$: Just as we have (two) natural Dynkin projectors from an associative algebra onto the Lie algebra it 'envelopes', it would be nice to have simple projectors dealing with the whole 'fourfold unfolding', that is to say:
  (i) from pre-associative onto pre-Lie
  (ii) from pre-associative onto associative
  (iii) (directly) from pre-associative onto Lie

- $\mathbf{Q_2}$: Are there *natural* incarnations (in flexion algebra or elsewhere) of the pre-associative algebras freely generated by more than one element?

- $\mathbf{Q_3}$: Is there a *simple* way of expressing the core involution *swap* and the related notions of bialternality, bisymmetrality etc[69] in any of the tree indexed bases?

- $\mathbf{Q_4}$: Are there simple Hermitian forms on $\mathit{Flex}_r(Pa^\bullet)$ or $\mathit{Flex}_r(Pi^\bullet)$ that make the bases $\{\mathfrak{bte}^\bullet_{r,k}\}$ or $\{\mathfrak{\hat{o}te}^\bullet_{r,k}\}$ or $\{\mathfrak{\hat{s}te}^\bullet_{r,d,k}\}$ orthonormal? Say, discrete Hermitian forms of type: $< A^\bullet, B^\bullet >= \sum_{\boldsymbol{w},\boldsymbol{w}'} h(\boldsymbol{w},\boldsymbol{w}')\ \ A^{\boldsymbol{w}}\ \bar{B}^{\boldsymbol{w}'}$ with the sequences $\boldsymbol{w}$ and $\boldsymbol{w}'$ consisting of finite sums of differences of unit roots of order $r+1$. Same question with $\mathit{Flex}_r(F^\bullet)$, where $F^\bullet$ denotes the *flat* flexion unit: $F^{w_1} = \frac{1}{2}\big(\mathrm{sgn}(u_1) + (\mathrm{sgn}(v_1)\big)$.

- $\mathbf{Q_5}$: Does there exist (say, for $\mathfrak{E}^\bullet = Pa^\bullet$ or $Pi^\bullet$) a simple, direct characterisation of the important and remarkably stable[70] subspace $\mathit{Flex}_{root}(\mathfrak{E}^\bullet)$ spanned by all elements of the form $\mathfrak{\hat{o}te}^\bullet_{r,k}$ with $1 \leqslant k \leqslant \kappa_{r-1}$, i.e. spanned by bimoulds attached to one-rooted trees?

12.3 Forthcoming: "the flexion structure and its plethora of dualities".

In this follow-up paper, we shall return to the flexion algebra *BIMU* of general bimoulds, and use $\mathit{Flex}(\mathfrak{E}^\bullet)$ to shed light on the many symmetries, involutions, and dualities that exist on *BIMU* and make it such a pliant tool for the study of *arithmetic dimorphy*, most glaringly manifest in the ring of multizetas. The centre-piece of the paper will be the bisymmetral bimoulds $tal^\bullet/til^\bullet$. They are the trigonometric counterpart of the polar bimoulds $pal^\bullet/pil^\bullet$, and an ideal key to the understanding of rational Drinfeld associators, of which they provide two completely distinct encodings.

[69] also the bialternality grid: see [E3].
[70] Notably, under the Lie products *lu*, *ari* and their pre-Lie companions *dle*, *dari*.

12.4 Index of terms.

12.5 Index of notations.

12.6 References.

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