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Furstenberg maps for CAT(0) targets of finite telescopic dimension

URI BADER†, BRUNO DUCHESNE‡ and JEAN LÉCUREUX§

† *Mathematics Department, Technion - Israel Institute of Technology, Haifa, 32000, Israel*
(e-mail: uri.bader@gmail.com)
‡ *Institut Élie Cartan de Lorraine, Université de Lorraine, B.P. 70239,*
F-54506 Vandoeuvre-lès-Nancy Cedex, France
(e-mail: bruno.duchesne@univ-lorraine.fr)
§ *Département de Mathématiques - Bâtiment 425, Faculté des Sciences d'Orsay,*
Université Paris-Sud 11, F-91405 Orsay, France
(e-mail: jean.lecureux@math.u-psud.fr)

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Abstract. We consider actions of locally compact groups G on certain CAT(0) spaces X by isometries. The CAT(0) spaces we consider have finite dimension at large scale. In case B is a G-boundary, that is a measurable G-space with some amenability and ergodicity properties, we prove the existence of equivariant maps from B to the visual boundary ∂X.

1. *Introduction*

Furstenberg maps, also called boundary maps or characteristic maps, first appeared in Furstenberg's work [**15**, **16**]. These maps proved to be powerful tools for rigidity results. Indeed, the existence of such Furstenberg maps is used in particular in order to prove commensurator rigidity [**1**] or superrigidity phenomena [**26**].

Our main topic of investigation in this paper is the existence of Furstenberg maps in the context of actions of groups on CAT(0) spaces. Recall that a CAT(0) space is a complete metric space that is non-positively curved in a way defined via the Bruhat–Tits inequality [**7**, p. 163].

We do not restrict ourselves to locally compact CAT(0) spaces but we replace this hypothesis by a condition at large scale. A CAT(0) space has finite telescopic dimension if any of its asymptotic cones has finite geometric dimension. Recall that a CAT(0) space has finite geometric dimension if there is a finite upper bound on the topological dimensions of its compact subspaces. Geometric dimension was introduced by Kleiner [**24**] and telescopic dimension by Caprace and Lytchack [**10**]. These two notions find their origins in Gromov's seminal work [**17**, 6.B2].

For example, CAT(0) cube complexes with an upper bound on the dimensions of cubes, Euclidean buildings (not necessarily locally compact nor with discrete affine Weyl groups) and (possibly infinite-dimensional) symmetric spaces with non-positive operator curvature and finite rank have finite telescopic dimension (see [**12**] for definitions and proof of this statement).

Our main result is the proof of existence of Furstenberg maps for such spaces.

THEOREM 1. *Let X be a CAT(0) space of finite telescopic dimension and let G be a locally compact second countable group acting continuously by isometries on X without invariant flats. If (B, ν) is a G-boundary then there exists a measurable G-map from B to ∂X.*

Roughly speaking, a G-boundary is a measurable space with an amenable G-action and strong ergodic properties (see §4 for a precise definition). In case G is amenable, B can be chosen to be a point and our theorem reduces to the following statement [**10**]: if G acts continuously by isometries on a CAT(0) space of finite telescopic dimension without invariant flat subspace then there is a fixed point at infinity. Our tools to pass from that last statement to Theorem 1 are *measurable fields of CAT(0) spaces* over B, see §3.

In this theorem, the G-boundary can be taken to be any Poisson boundary of G (for an admissible measure). However, the theorem can be proved for an *a priori* larger class of spaces B, with suitable ergodic properties. Our work relies heavily on a very strong ergodic property of the boundary, namely *relative metric ergodicity*. This property was introduced by the first author together with Furman in [**4**], where (generalizing results of [**8**, **20**]) it is proved that the Poisson boundary does indeed have this strong ergodic property. For the sake of completeness, we also include a proof in this paper.

We also prove a similar result for *boundary pairs*, see Theorem 34. For example, Poisson boundaries associated to forward and backward random walks form a boundary pair.

Another possible approach to prove Theorem 1, at least for the Poisson boundary associated to some random walk on G, would be to try to understand the behavior of the random walk itself. More precisely, if Z_n is the nth step of the random walk in G, and $o \in X$, one could hope that $Z_n \cdot o$ converges to a point in the visual boundary. This would give a measurable, G-equivariant map from B to ∂X.

It is known that this approach works in some cases by the work of Karlsson and Margulis [**22**, Theorem 2.1]. However, there is a crucial assumption in this theorem, namely, that the drift is positive. Our result doesn't rely on this hypothesis.

In general, it is a natural question to relate the map obtained by Theorem 1 and the random walk in the space X. For example, if the drift is positive, and with finite first moment, we know that the random walk converges to a boundary point, and one can wonder whether the boundary map can be far away (in the sense of the Tits metric) from this limit. In case X is Gromov hyperbolic, the two essentially coincide. This is not the case for a symmetric space, for example, as there are several different boundary maps.

In the same vein, one can ask whether the boundary map can be a (measurable) isomorphism, thus providing a geometric identification of the Poisson–Furstenberg boundary. The main tools for this possible application would be Kaimanovich's ray and strip criteria [**21**, Theorems 5.5 and 6.4].

Finally, it also turns out that the methods on which our paper relies are useful to answer a natural question raised by an implicit argument in [**10**]. Namely, in Appendix, we prove the following result of non-emptiness for boundaries of CAT(0) spaces of finite telescopic dimension. Quite surprisingly, its proof relies on the existence of (G, μ)-boundaries for countable groups G and the dichotomy given between existence of Furstenberg maps and existence of Euclidean subfields given by Theorem 33.

THEOREM 2. *Let X be a CAT(0) space of finite telescopic dimension. If* $\mathrm{Isom}(X)$ *has no fixed point then the visual boundary ∂X is not empty.*

A CAT(0) space X is called *boundary minimal* [**11**, §1.B] if there is no non-empty closed convex subspace $Y \subset X$ such that $\partial Y = \partial X$. For proper CAT(0) spaces with boundary of finite topological dimension, minimality of X (that is, no non-trivial closed convex subset is invariant under $\mathrm{Isom}(X)$) implies boundary minimality [**11**, Proposition 1.5]. The same holds for CAT(0) spaces of finite telescopic dimension.

COROLLARY 3. *Let X be a minimal CAT(0) space of finite telescopic dimension not reduced to a point. Then X is boundary minimal.*

2. *CAT(0) geometry*

2.1. *CAT(0) spaces of finite telescopic dimension.*

Throughout this text, we will deal with CAT(0) spaces *of finite telescopic dimension* as introduced in [**10**]. For the reader's convenience, we recall facts about these spaces which will be useful for us.

We start by defining dimension in a metric way, by Jung's theorem [**19**]. This theorem tells us that for any bounded subspace Y of a Euclidean space of dimension n

$$\mathrm{rad}(Y) \leq \sqrt{\frac{n}{2(n+1)}}\,\mathrm{diam}(Y) \tag{1}$$

where $\mathrm{rad}\, Y$ and $\mathrm{diam}\, Y$ are the radius and diameter of Y. Moreover, there is equality if and only if the closure of Y contains a regular n-simplex of diameter $\mathrm{diam}(Y)$.

Using this inequality, Caprace and Lytchack showed [**10**, Theorem 1.3] that a CAT(0) space Y has *geometric dimension* (as introduced by Kleiner [**24**]) at most n if and only if for any bounded subset $Y \subseteq X$, inequality (1) holds.

Telescopic dimension is a notion at large scale. By definition, a CAT(0) space X has telescopic dimension at most n if any asymptotic cone of X has geometric dimension at most n (see [**10**]). It can be expressed quantitatively: a CAT(0) space X has telescopic dimension at most n if and only if for any $\delta > 0$, there is $D > 0$ such that for any bounded subset $Y \subseteq X$ of diameter larger than D we have

$$\mathrm{rad}(Y) \leq \left(\delta + \sqrt{\frac{n}{2(n+1)}}\right)\mathrm{diam}(Y). \tag{2}$$

Note that a locally compact CAT(0) space may have infinite telescopic dimension.

One main feature of CAT(0) spaces of finite telescopic dimension is the following: if the intersection of a filtering family $\{X_\alpha\}$ of closed convex subsets is empty then the intersection of boundaries $\bigcap_\alpha \partial X_\alpha$ is not empty and there is a canonical point ξ in this intersection.

This canonical point is given by the fact that the boundary of a CAT(0) space of telescopic dimension at most n has geometric dimension at most $n-1$ and the fact that a CAT(1) space Ξ of finite geometric dimension and radius at most $\pi/2$ has such a canonical point. This point is defined as the unique circumcenter of the set of circumcenters of Ξ.

This property of filtering families of closed convex subspaces can be thought of as a compactness property for $\overline{X} = X \cup \partial X$. Let us define the topology $\mathscr{T}_c$ on $\overline{X}$ as the weakest topology such that $\overline{C}$ is $\mathscr{T}_c$-closed for any closed, for the usual topology, convex subset $C \subseteq X$ (see [**27**, §§3.7, 3.8]). Then for a CAT(0) space X of finite telescopic dimension $\overline{X}$ is compact, not necessarily Hausdorff.

2.2. *Geometry of flats in CAT(0) spaces.* In the heart of the proof of Theorem 1 we will deal with a Euclidean subfield of a CAT(0) field. In this subsection we gather useful facts about the geometry of Euclidean subspaces (also called flats subspaces) in CAT(0) spaces.

Let (X, d) be a CAT(0) space. If C is a closed convex subset of X, we denote by d_C the distance function to C, that is, $d_C(x) = \inf_{c\in C} d(c, x) = d(x, \pi_C(x))$, where $\pi_C(x)$ is the projection of x on C.

Let Y be a bounded subset of a metric space (Z, d). The *circumradius* (or simply *radius*) $\mathrm{rad}(Y)$ of Y is the non-negative number $\inf_{z\in Z} \sup_{y\in Y} d(y, z)$, its *intrinsic circumradius* is $\inf_{z\in Y} \sup_{y\in Y} d(y, z)$ and a *circumcenter* of Y is a point in Z minimizing $\sup_{y\in Y} d(y, \cdot)$.

More generally, we define a *center* of Y as a point of Z which is fixed by any isometry of Z stabilizing Y. The Bruhat–Tits fixed point lemma asserts that, in a CAT(0) space, a bounded set has a unique circumcenter, which is therefore a center.

If S_1 and S_2 are two subsets of Euclidean spheres, we denote by $S_1 * S_2$ their spherical join [**7**, Definition I.5.13]. This is the spherical analogue of Euclidean products. Such Euclidean products appear, for example, in the de Rham decomposition [**7**, Theorem II.6.15] of the CAT(0) space X: the space X is isometric to a product $H \times Y$ where H is a Hilbert space and Y does not split with a Euclidean factor. Isometries of X preserve this decomposition acting diagonally on $H \times Y$ and the choice of a base point in X allows us to identify H and Y with closed convex subspaces of X containing this point.

The following lemma details the possibilities for a convex function on a Euclidean space. It will be applied in two situations: the restriction of a Busemann function to a Euclidean subspace of X and the restriction to a Euclidean subspace of the distance function to another Euclidean subspace.

PROPOSITION 4. *Let E be a Euclidean space, f be a convex function on E and $m = \inf\{f(x) \mid x \in E\}$.*

(i) *If m is not attained, then the intersection $\bigcap_{\varepsilon>0} \partial(f^{-1}(]m, m+\varepsilon[))$ (respectively $\bigcap_{r\in\mathbb{R}} \partial(f^{-1}(]-\infty, r[))$ if $m = -\infty$) is not empty and has a center.*

If m is a minimum, let $E_m = f^{-1}(\{m\})$. Let $E_m = F_m \times T$ be its de Rham decomposition, where F_m is a maximal flat contained in E_m. Then exactly one of the

following possibilities happens:
(ii) *E_m is bounded and thus has a center (i.e. F is a point and T is compact);*
(iii) *T is bounded and $\partial E_m = \partial F_m$ is a (non-empty) sphere;*
(iv) *T is unbounded and its boundary ∂T has radius less than $\pi/2$ and thus has a center.*

Proof. If m is not a minimum then the net of closed convex subsets $(f^{-1}((m, m+\varepsilon)))_{\varepsilon>0}$ has empty intersection and the result follows from $\mathscr{T}_c$-compactness.

The other cases coincide with the join decomposition $\partial E_m = \partial F * \partial T_m$, using Lemma 5 below, where ∂E_m will correspond to C, ∂F_m to S_1 and ∂T to C_2. □

LEMMA 5. *Let (S, d) be a Euclidean sphere. Let C be a non-empty closed convex subset of S and let S_0 be the minimal subsphere of S containing C. Then there is a unique decomposition of S_0 as a spherical join $S_0 = S_1 * S_2$ where S_1 and S_2 are reduced to a point or are subspheres of S_0 such that*

$$C = S_1 * C_2,$$

where C_2 is a closed convex subset of S_2 with intrinsic radius $<\pi/2$.

In particular, any closed convex subset that is not a subsphere has a center (the one of C_2). Moreover, it coincides with the unique circumcenter of the set of circumcenters of C.

Proof. First observe that intersections of subspheres are empty or subspheres themselves. This yields the existence of S_0. In the same way, convex hulls of subspheres are subspheres and thus there exists a maximal subsphere S_1 contained in C. Let S_2 be the set of points of S_0 at distance $\pi/2$ from every point in S_1. Then $S_0 = S_1 * S_2$. Any point of C can be written (x_1, x_2, α) with $x_i \in S_i$ and $\alpha \in [0, \pi/2]$. Since C is convex and $S_1 \subseteq C$, there exists C_2, which is $S_2 \cap C$, such that $C = S_1 * C_2$. Observe that C_2 does not contain any sphere by maximality of S_1. Now C has diameter $<\pi$, otherwise, it contains at least a sphere of dimension 0 (that is, two antipodal points), and thus has also intrinsic circumradius $<\pi/2$.

Observe that C is not a sphere if and only if C_2 is not empty. In the case where S_1 and C_2 are not empty then any point of C_2 is a circumcenter since, in that case, the intrinsic circumradius is $\pi/2$. The fact that any convex subset of circumradius $<\pi/2$ has a unique circumcenter implies the last sentence of the lemma. □

There is a particular situation for the relative position of two Euclidean subspaces E, F in X: when the restriction on F of the distance to E is constant and vice-versa. In that situation E and F are said to be *parallel*. The Sandwich lemma [**7**, Exercise II.2.12(2)] implies that their convex hull splits isometrically as $\mathbb{R}^n \times [0, d]$. In particular, E and F are isometric and thus have the same dimension n.

LEMMA 6. *Let E be a Euclidean subspace of X and let Y be the union of subspaces parallel to E. Then Y is a closed convex subspace of X. A point $y \in X$ belongs to Y if and only if for any $x_1, \ldots, x_n \in E$,* $\mathrm{Conv}(y, x_1, \ldots, x_n)$ *is isometric to a convex subset of a Euclidean space.*

Let p be the restriction to Y of the projection to E. Fix some $x \in E$ and let Z be $p^{-1}(\{x\})$. Then Z is a closed convex subspace of X and Y decomposes isometrically as $E \times Z$.

Proof. The lemma is proved when E is a line in [**7**, Proposition II.2.14]. Let n be the dimension of E. We proceed by induction on n. Assume this is true for $n-1$ and choose an orthogonal splitting $F \times L$ of E where F has dimension $n-1$, L is a line and $F \cap L = \{x\}$. The induction assumption for F implies that the convex hull of subspaces parallel to F splits isometrically as $F \times Z_F$. Now let us apply the case $n = 1$ for the union of lines parallel to L in Z_0, which splits as $L \times Z_0 \subseteq Z_F$. It is clear that $E \times Z_0 \subset Y$.

If E' is a Euclidean subspace parallel to E then the restriction p' of p to E' is an isometry. Set $F' = p'^{-1}(F)$ and L' to be the orthogonal line to F' containing $x' = p'^{-1}(x)$. For any $y \in L'$, $F \times \{y\}$ is parallel to F and thus $L' \subset Z_F$. Since L' is parallel to L, $x' \in Z_0$. Finally, $Z_0 = Z$ and Y splits isometrically as $E \times Z$.

Assume that for any $x_1, \ldots, x_n \in E$, $\mathrm{Conv}(y, x_1, \ldots, x_n)$ is isometric to a convex subset of a Euclidean space. Then $\mathrm{Conv}(\{y\} \cup E)$, which is the union of such spaces, satisfies the following property: for any $z_1, \ldots, z_n \in \mathrm{Conv}(\{y\} \cup E)$, $\mathrm{Conv}(z_1, \ldots, z_n)$ is isometric to a convex subset of a Euclidean space. This property is a characterization of CAT(0) spaces that are isometric to a convex subset of a Hilbert space. In particular, in our case, $\mathrm{Conv}(\{y\} \cup E)$ is necessarily isometric to $\mathbb{R}^d \times d(y, E)$, where $d = \dim(E)$. □

LEMMA 7. *Let E be a Euclidean subspace of X and $\xi \in \partial X$ such that the Busemann function β_ξ associated to ξ (with respect to some fixed base point) is constant on E. If $x \in E$ and ρ is the ray from x to ξ then the convex hull of $E \cup \rho$ is isometric to $E \times [0, +\infty)$.*

In particular, with the notation of Lemma 6, $\xi \in \partial Z$.

Proof. Choose $x \in E$ and y on the geodesic ray from x to ξ. We claim that $\overline{\angle}_x(y, z) = \pi/2$ for any point $z \neq x$ in E. Assume $\overline{\angle}_x(y, z) < \pi/2$. Then there is $x' \in (x, z)$ such that $d(y, x') < d(y, x)$. This implies $\beta_\xi(x') < \beta_\xi(x)$, which is a contradiction. Arguing the same way with the symmetric of z with respect to x, we get the claim. In particular, for any $x, x' \in E$, $\angle_x(x', \xi) + \angle_{x'}(x, \xi) = \pi$ and [**7**, Proposition II.9.3] implies that the convex hull of x, x' and ξ is isometric to $[0, d(x, x')] \times [0, \infty)$.

This shows that the projection of E on any closed horoball is a flat parallel to E and the lemma is now a consequence of Lemma 6. □

3. *Measurable fields of complete separable metric spaces*

3.1. *Metric fields.* An important, although slightly technical, tool in our proof will be the notion of fields of metric spaces. Roughly speaking, a measurable field of metric spaces over a measurable space A is a way to attach measurably a metric space to any point in A. Thanks to [**28**] one can think of measurable fields of metric spaces over A in the following way: to each point of A one associates a closed subspace of a fixed metric space, namely the Urysohn space.

Let (A, η) be a Lebesgue space, that is, a standard Borel space with a Borel probability measure [**23**, §12]. All our definitions will depend only on the class of the measure η. In particular, measurability properties will refer to the completion of the Borel with respect to η.

Definition 8. Let $\{X_a\}_{a\in A}$ be a collection of complete separable metric spaces. The distance on X_a is denoted d_a, or simply d if there is no ambiguity. Measurability conditions are defined thanks to the notion of fundamental families. A *fundamental family* $\mathcal{F} = \{x^n\}_{n\in\mathbb{N}}$ is a countable family of elements of $\prod_{a\in A} X_a$ with the following properties:

- for all $n, m, a \mapsto d_a(x_a^n, x_a^m)$ is measurable;
- for almost every $a \in A$, $\{x_a^n\}_{n\in\mathbb{N}}$ is dense in X_a.

A *measurable field* $\mathbf{X}$ *of complete separable metric spaces*, or simply a *metric field* for short, is then the collection of data: (A, η), $\{X_a\}_{a\in A}$ and $\{x^n\}$.

A *section* of $\mathbf{X}$ is an element $x \in \prod_a X_a$ such that for all $y \in \mathcal{F}$, $a \mapsto d_a(x_a, y_a)$ is measurable. Two sections are identified if they agree almost everywhere. The set of all sections is the *measurable structure* $\mathcal{M}$ of $\mathbf{X}$. If x, y are two sections, the equality

$$d_a(x_a, y_a) = \sup_{z\in\mathcal{F}} |d_a(x_a, z_a) - d_a(z_a, y_a)|$$

shows that $a \mapsto d_a(x_a, y_a)$ is also measurable.

Let G be a second countable locally compact group. The Lebesgue space (A, η) is a *G-space* if G acts by measure class preserving automorphisms on A and the map $(g, a) \mapsto ga$ is measurable.

Definition 9. Let (A, η) be a G-space. A *cocycle* for G on $\mathbf{X}$ is a collection $\{\alpha(g, a)\}_{g\in G, a\in A}$ such that:

- for all g and almost every a, $\alpha(g, a) \in \mathrm{Isom}(X_a, X_{ga})$;
- for all g, g' and almost every a, $\alpha(gg', a) = \alpha(g, g'a)\alpha(g', a)$;
- for all $x, y \in \mathcal{F}$, the map $(g, a) \mapsto d_a(x_a, \alpha(g, g^{-1}a)y_{g^{-1}a})$ is measurable.

In that case we say that G *acts* on $\mathbf{X}$ via the cocycle α or that there is an *action* of G on $\mathbf{X}$. A section x is invariant if for all g and almost all a, $\alpha(g, g^{-1}a)x_{g^{-1}a} = x_a$.

3.2. *Fields of CAT(0) spaces.* A special case of metric fields is the case of CAT(0) fields. The theory of measurable fields of complete separable metric spaces and, more specifically, of CAT(0) spaces appeared in [**3**, **13**] and references therein.

Definition 10. Let $\mathbf{X}$ be a metric field and $\kappa \in \mathbb{R}$. We say that $\mathbf{X}$ is a CAT(κ) *field* if for almost every a, X_a is a CAT(κ) space.

A *subfield* $\mathbf{Y}$ of a CAT(0) field $\mathbf{X}$ is a collection $\{Y_a\}_{a\in A}$ of non-empty closed convex subsets such that for every section x of $\mathbf{X}$, the function $a \mapsto d(x_a, Y_a)$ is measurable. We identify subfields $\mathbf{Y}$ and $\mathbf{Y}'$ if $Y_a = Y'_a$ for almost every a.

Similarly, we speak of *Euclidean fields* and subfields of such fields. If a group G acts on $\mathbf{X}$, a subfield $\mathbf{Y}$ is *invariant* if for all g and almost all a, $\alpha(g, g^{-1}a)Y_{g^{-1}a} = Y_a$.

In CAT(1) spaces, subsets of circumradius less than $\pi/2$ are strictly convex and admit a unique circumcenter as well, see [**7**, Proposition II.2.7] or [**25**, Proposition 3.1] for quantitative statements. Those circumcenters can be defined canonically by means of the metric structure and, arguing as in [**13**, Lemma 8.7], we obtain the following lemmas.

LEMMA 11. *Let* $\mathbf{X}$ *be a CAT(1) field. If* $\mathbf{C}$ *is a subfield of* $\mathbf{X}$ *with fibers of radius less than* $\pi/2$ *then the family of circumcenters of* $\mathbf{C}$ *is a section of* $\mathbf{X}$.

LEMMA 12. *Let* $\mathbf{X}$ *be a metric field and* x *be a section of it. For* $a \in A$*, let* B_a^r *be the closed ball of radius* r *around* x_a*; then* $\mathbf{B}^r = \{B_a^r\}_{a \in A}$ *is a metric field. Moreover, if* G *acts on* $\mathbf{X}$ *and* x *is an invariant section then* G *acts on* $\mathbf{B}$ *as well.*

Proof. This statement is almost a straightforward verification of the definitions. The only fact that is not completely obvious is maybe the construction of a fundamental family. Fix a fundamental family $\{x^n\}_n$ of $\mathbf{X}$. For $n \in \mathbb{N}$, $a \in A$, set inductively k_a^n to be $\min\{k > k_a^{n-1};\ x_a^k \in \overline{B}(x_a, r)\}$. Let us denote by y_a^n the point $x_a^{k_a^n}$. The family $\{y^n\}_n$ is a fundamental family of $\mathbf{B}^r$. □

Building on Lemma 6, we obtain the following measurable decomposition of the union of flats parallel to a Euclidean subfield.

LEMMA 13. *Let* $\mathbf{X}$ *be a CAT(0) field and* $\mathbf{E}$ *be a Euclidean subfield. For* $a \in A$ *let* Y_a *be the union of flats parallel to* E_a*. Then* $\mathbf{Y} = (Y_a)$ *is a subfield of* $\mathbf{X}$ *which splits as a product of CAT(0) fields* $\mathbf{Y} = \mathbf{E} \times \mathbf{Z}$*. Moreover, if* G *acts on* $\mathbf{X}$ *and* $\mathbf{E}$ *is invariant then* $\mathbf{Y}$ *is invariant and* G *acts diagonally on* $\mathbf{E} \times \mathbf{Z}$ *with an invariant section in* $\mathbf{Z}$*.*

Proof. Fix $a \in A$. Thanks to Lemma 6, the condition $y \in Y_a$ can be checked using only distances $d(y, x_a^n)$, where $\{x^n\}_n$ is a fundamental family of the subfield $\mathbf{E}$. One can then readily check that $\mathbf{Y}$ is a subfield of $\mathbf{X}$. Fixing a section x of $\mathbf{E}$, one can recover $\mathbf{Z}$ as $p^{-1}\{x\}$, where p_a is the projection on E_a. □

LEMMA 14. *Let* $\mathbf{E}$ *be a Euclidean field. The map* $a \mapsto \dim(E_a)$ *is measurable.*

Proof. Fix a fundamental family $\{x^n\}_n$ for $\mathbf{E}$. Thanks to Jung's inequality (1) the dimension d of E_a can be obtained via the quantity $\sqrt{d/2(d+1)}$, which is the minimal non-negative number K such that $\mathrm{rad}(\{x_a^{n_1}, \ldots, x_a^{n_k}\}) \le K\, \mathrm{diam}(\{x_a^{n_1}, \ldots, x_a^{n_k}\})$, where $n_1, \ldots, n_k \in \mathbb{N}$. □

Any CAT(0) space X has a visual boundary ∂X (which may be empty). Making this construction pointwise, we can consider, at least in a set-theoretic way, the boundary field of a CAT(0) field $\mathbf{X}$.

However, the measurable structure is not so clear. We would need a separable metric on each fiber. One way to endow the boundary of a CAT(0) space with a metric is to consider the angle metric, or the Tits metric, which is the length metric associated to the previous one. These are invariant metrics, but then ∂X is not separable in general; for example, the boundary of the hyperbolic plane, endowed with the Tits metric, is an uncountable discrete space.

Actually, there is no way to construct a boundary field $\partial\mathbf{X}$ for a CAT(0) field $\mathbf{X}$ such that a group acting on $\mathbf{X}$ also acts (isometrically) on $\partial\mathbf{X}$. Otherwise, any Furstenberg map given by Theorem 1 would be constant because of double metric ergodicity.

One way to avoid these problems is to let down the desired invariance of the metric. Recall that a separable CAT(0) space X embeds continuously (not isometrically) to a subset of the Fréchet space $\mathcal{C}(X)$ of continuous functions on X endowed with the distance $d(f, g) = \sum_{n \in \mathbb{N}} 2^{-n}(|f(x_n) - g(x_n)|)/(1 + |f(x_n) - g(x_n)|)$, where $\{x_n\}$ is a dense subset of X. This metric topology coincides with the topology of pointwise convergence. More precisely, if x_0 is a base point in X, the embedding is given by $\iota : y \mapsto d(\cdot, y) - d(y, x_0)$. The closure of $\iota(X)$ is a compact metric space, which allows us to define a *bordification field* $\mathbf{K}$ for a CAT(0) field $\mathbf{X}$ (see [**13**, §9.2]). Sections of $\mathbf{K}$ correspond to some collections of Busemann functions. We define them as follows.

Definition 15. Let $\mathbf{X}$ be a CAT(0) field. We define its *boundary field* $\partial\mathbf{X}$ to be the set of sections f of its bordification field $\mathbf{K}$ such that for almost every $a \in A$ there is $\xi_a \in \partial X_a$ with $f_a = \beta_{\xi_a}(\cdot, x_0)$ where β_{ξ_a} is the Busemann function associated to $\xi_a \in \partial X_a$.

By an abuse of notation we will say that $\xi = (\xi_a)$ is a section of the boundary field. Observe in that case that for all x, y sections of $\mathbf{X}$, the function $a \mapsto \beta_{\xi_a}(x_a, y_a)$ is measurable. Even if we will not need it, we observe that one can decide if a section of $\mathbf{K}$ corresponds to a section of the boundary field.

LEMMA 16. *Let f be a section of $\mathbf{K}$. Let A' be the set of elements $a \in A$ such that there is $\xi_a \in \partial X_a$ with $f_a = \beta_{\xi_a}(\cdot, x_0)$. Then A' is a measurable subset of A.*

Proof. We use the fact that f_a coincides with a Busemann function if it is a limit of points in $\iota(X_a)$ for the topology of uniform convergence on bounded subsets [**7**, §II.8]. Fix x^n a fundamental family of $\mathbf{X}$ and $x^n(r)$ fundamental families of the fields $\mathbf{B}^r$ of closed balls around x^0. Now, $a \in A'$ if and only for any $r > 0$

$$\inf_{\{n\in\mathbb{N};\ d(x_a^n, x_a^0)>r\}} \sup_{m\in\mathbb{N}} |f_a(x_a^m(r)) - \iota(x_a^n)(x_a^m(r))| = 0. \qquad \square$$

The following lemma shows that our notion of boundary field gives something natural in the case of a constant field.

LEMMA 17. *Let (X, d) be a complete separable metric space and let us denote by $\mathbf{X}$ the measurable field over (A, η) with constant fibers equal to X. Sections of $\mathbf{X}$ and measurable maps $A \to X$ are in bijective correspondence. Moreover, if X is a CAT(0) space then sections of $\partial\mathbf{X}$ and measurable maps $A \to \partial X$ are in bijective correspondence.*

Proof. Recall that a fundamental family of $\mathbf{X}$ is given by constant maps $a \mapsto x_n$, where (x_n) is a countable dense subset of X. Now a map $f : A \to X$ is measurable if and only if, for any x, $a \mapsto d(x, f(a))$ is measurable if and only if, for any n, $a \mapsto d(x_n, f(a))$ is measurable.

Now consider the case where X is a CAT(0) space. We use the identification of the boundary of X with the set of Busemann functions vanishing at some base point x_0. In this identification, the cone topology corresponds to the topology of uniform convergence on closed balls around x_0. In particular, a map $f : A \to \partial X$ is measurable if and only if, for any $x, y \in X$, $a \mapsto \beta_{f(a)}(x_0, y)$ is measurable if and only if, for any $n, m \in \mathbb{N}$, $a \mapsto \beta_{f(a)}(x_n, x_m)$ is measurable. $\square$

Among CAT(0) spaces, Euclidean spaces have a special feature: the angle between two points at infinity is the same from any point from which you look at them. This gives a distance at infinity independent of the choice of a base point and this distance is invariant under the action of the isometry group of the Euclidean space. In the case of a field of Euclidean spaces we get the following fact.

LEMMA 18. *Let* $\mathbf{E}$ *be a measurable field of Euclidean spaces. The boundary field* $\partial\mathbf{E}$ *has a structure of a field of CAT(1) spaces for the Tits metric. If* G *acts isometrically on* $\mathbf{E}$ *then it also acts isometrically on* $\partial\mathbf{E}$.

Proof. Each ∂E_a is a complete separable metric space (isometric to a Euclidean sphere) for the Tits metric, and the action of G preserves this distance. What is left to do is to check the measurable structure.

We choose a fundamental family (x^n) of $\mathbf{E}$ such that there is a section x^0 such that, for any n and almost all a, $d(x_a^0, x_a^n) \neq 0$. Now define $\xi_a^n \in \partial E_a$ to be the end point of $[x_a^0, x_a^n)$. We claim that (ξ^n) is a fundamental family for $\partial\mathbf{E}$. We define

$$m(a, n, k) = \min\{m;\ d(x_a^0, x_a^m) > k, |d(x_a^0, x_a^n) + d(x_a^n, x_a^m) - d(x_a^0, x_a^m)| < 1\}$$

and

$$y_a^{(n,k)} = x_a^{m(a,n,k)}.$$

This way, for any (n, k), $y^{(n,k)}$ is a section and, for almost every a, $y_a^{(n,k)} \to \xi_a^n$ as $k \to \infty$. This shows that, for any n, m, $a \mapsto \angle(\xi_a^n, \xi_a^m) = \lim_{k\to\infty} \angle_{x_a^0}(y_a^{(n,k)}, y_a^{(m,k)})$ is measurable. □

Remark 19. Let E be a Euclidean space and ξ, η be two points at infinity. The very special geometry of E implies the following formula between Busemann functions β_ξ, β_η and the visual angle $\angle_{x_0}(\xi, \eta)$ (which does not depend on x_0 and is also the Tits angle):

$$\sup_{1/2<d(x,x_0)<1} \frac{|\beta_\xi(x, x_0) - \beta_\eta(x, x_0)|}{d(x, x_0)} = 2(1 - \cos(\angle(\xi, \eta))).$$

This formula shows that in the case of a measurable field of Euclidean spaces $\mathbf{E}$, the notion of section of $\partial\mathbf{E}$ defined in Definition 15 and the notion of section for the structure of a metric field introduced in Lemma 18 coincide.

Let E be a Euclidean space of dimension d_0. It is not hard to define a distance on the set S of subspheres of dimension $0 \leq d < d_0$ in ∂E turning S into a complete separable metric space. The following lemma does the same in a measurable context.

LEMMA 20. *Let* $\mathbf{E}$ *be a Euclidean field of constant dimension* d_0. *Let* d *be a positive integer less than* d_0. *For any* $a \in A$, *let* S_a *be the set of subspheres of dimension* d *of* ∂E_a. *The collection* $\mathbf{S} = (S_a)$ *has a structure of a metric field. If* G *acts on* $\mathbf{E}$ *then it also acts on* $\mathbf{S}$ *(isometrically).*

Let E_a^s *be the set of Euclidean subspaces* F *of* E_a *such that* $\partial F = s_a$. *Then* $\mathbf{E}^s = (E_a^s)$ *has a natural structure of Euclidean field such that any section of* $\mathbf{E}^s$ *corresponds to a Euclidean subfield of* $\mathbf{E}$.

Moreover, if G *acts on* $\mathbf{E}$ *and* s *is an invariant section of* $\mathbf{S}$ *then* G *acts on* $\mathbf{E}^s$.

Proof. First, we claim that one can construct a fundamental family $\{x^n\}_n$ of $\mathbf{E}$ such that, for any choice $n_0, \ldots, n_d$, and almost all a, $x_a^{n_0}, \ldots, x_a^{n_d}$ is not included in a Euclidean subspace of dimension $<d$ since this condition can be checked only with distances.

For all a, we define S_a to be the set of subspheres of ∂E_a of dimension d. For $s^1, s^2 \in S_a$, we define $d(s^1, s^2)$ to be the Hausdorff distance between two compact subspaces associated to the Tits distance on ∂E_a. Now, for a choice of $N = \{n_0, \dots, n_d\}$ we define s_a^N to be the boundary of the affine span of $x_a^{n_0}, \dots, x_a^{n_{d_0}}$. There are countably many possibilities for N, and $\{(s^N)\}_N$ defines a fundamental family of $\mathbf{S}$. If G acts on $\mathbf{E}$ then it acts on $\partial\mathbf{E}$ and, since the Hausdorff distance is defined via the supremum of some Tits angles, G acts (measurably) on $\mathbf{S}$.

Let s be a section of $\mathbf{S}$ and for any x^n element of the fundamental family of $\mathbf{E}$ let F_a^n be the unique Euclidean subspace of E_a containing x_a^n such that $\partial F_a^n = s_a$.

The last statement comes from the fact that, in that case, for any $g \in G$ and all almost $a \in A$, $\alpha(g, a) E_a^s = E_{ga}^{\alpha(g,a)s_a} = E_{ga}^{s_{ga}}$. □

4. *Metric ergodicity and its relative version*

In [**6**], Bader and Furman introduced the notion of a boundary pair, which is further developed in their paper [**5**]. In this section we review this theory.

We fix a locally compact second countable group G for the rest of this section. All conditions of measurability in G will be relative to the Haar measure class.

Recall that a Lebesgue space is a pair (A, η) where A is a standard Borel space and η is a Borel probability measure on A. A subset of A is said to be measurable if it belongs to the completion of the Borel σ-algebra with respect to η. If X is a topological space, we say that a map $f : A \to X$ is *measurable* if it coincides with a Borel map on a full measure subset of A. Observe that in the case where X is second countable, f is measurable if and only if the preimage of a Borel subset of X is measurable in A. This equivalence still holds when X is metrizable [**14**, Theorem 2B].

Recall from the lines before Definition 9 that a G-space is a Lebesgue space with a measurable action of G which preserves the class of the measure.

Definition 21. [**4**, Definition 4.1] Let (A, η) be a G-space. The action $G \curvearrowright (A, \eta)$ is *metrically ergodic* if for any action of G by isometries on a complete separable metric space (X, d), any G-equivariant measurable map $A \to X$ is essentially constant.

If the diagonal action $G \curvearrowright A \times A$ is metrically ergodic, we say that $G \curvearrowright A$ is *doubly metrically ergodic*.

Remark 22. We will only use complete separable metric spaces. However, this is not an important restriction. About completeness, one may consider the extended action on the completion $\overline{X}$ and observe that $\overline{X} \setminus X$ has zero measure. Moreover, one may reduce to separable spaces, as the following argument shows.

Let f be a measurable map $A \to X$. Without loss of generality, we may assume f to be onto and, up to discarding a set of measure zero, we may assume f to be Borel. The cardinality of the set of Borel subsets of A (and of any of its subsets) is at most the continuum $\mathfrak{c}$. By contradiction, assume that X is not separable. So, there is $r > 0$ such that X is not covered by a countable number of balls of radius r.

For $Y \subseteq X$, let $U_Y = \bigcup_{x \in Y} B(x, r)$. By Zorn's lemma, one can find a maximal subset $X' \subseteq X$ such that for any $x \in X'$, $x \notin U_{X' \setminus \{x\}}$. In particular, the preimages $f^{-1}(U_Y)$ for $Y \subseteq X'$ are distinct Borel subsets of A. By the choice of r, X' is not countable and the set of all these preimages has cardinality larger than $\mathfrak{c}$, leading to the desired contradiction.

Below we present a *relative* notion of metric ergodicity as well [**6**]. The definition that we give here is not exactly the one given in the paper [**6**], but a version of it modified in order to fit in the context of measurable fields of complete separable metric spaces.

Definition 23. Let (A, η) and (B, ν) be two Lebesgue spaces. A measurable map $\pi : A \to B$ is a *factor map* if $\pi_*\eta$ and ν are in the same measure class. If A and B are G-spaces and π is G-equivariant then we say that π is a *G-factor*.

Definition 24. [**5**] Let (A, η) and (B, ν) be two Lebesgue spaces and $\pi : A \to B$ be a factor map. Let $\mathbf{X}$ be a metric field over (B, ν). A *relative section* is a map $\varphi : A \to \bigsqcup_{b\in B} X_b$ such that:

- for all $a \in A$, $\varphi(a) \in X_{\pi(a)}$;
- for any section x of $\mathbf{X}$, $a \mapsto d(x(\pi(a)), \varphi(a))$ is measurable.

If π is a G-factor and G acts on $\mathbf{X}$ via a cocycle α, such a relative section is said to be *invariant* if, for almost every a and all $g \in G$, $\varphi(ga) = \alpha(g, \pi(a))\varphi(a)$.

Definition 25. [**5**] We say that the G-map $\pi : A \to B$ is *relatively metrically ergodic* (or equivalently $G \curvearrowright A$ is *metrically ergodic relatively* to π) if any invariant relative section coincides with a section. In other words, for any G-metric field $\mathbf{X}$ and any invariant relative section φ, there is an invariant section x of $\mathbf{X}$ such that, for almost all $a \in A$, $\varphi(a) = x(\pi(a))$.

The following lemma shows how relative metric ergodicity implies metric ergodicity. Actually, metric ergodicity of $G \curvearrowright A$ is equivalent to metric ergodicity of $G \curvearrowright A$ relatively to the projection to a point.

LEMMA 26. [**5**] *Let (A, η), (B, μ) be two G-spaces. If $A \times B \to B$ is relatively metrically ergodic then $G \curvearrowright A$ is metrically ergodic.*

Proof. Let $\phi : A \to X$ be an equivariant G-map to some complete separable metric space. Consider $\mathbf{X}$ to be the trivial field $X \times B$ over B and define $\varphi(a, b) = \phi(a)$. The map φ is an invariant relative section and thus does not depend on a, that is, ϕ is essentially constant. □

Definition 27. [**5**] Let (B_-, ν_-), (B_+, ν_+) be G-spaces. We say that (B_-, B_+) is a *G-boundary pair* if:

- the actions $G \curvearrowright B_+$ and $G \curvearrowright B_-$ are amenable in Zimmer's sense [**30**];
- both first and second projections $B_- \times B_+ \to B_\pm$ are relatively ergodic.

A G-space (B, μ) is *G-boundary* if (B, B) is a G-boundary pair.

Let μ be a probability measure on G. Recall that a *(G, μ)-space* is a G-space (A, η) such that $\mu * \eta = \eta$, where the convolution measure $\mu * \eta$ is the pushforward measure of $\mu \times \eta$ under the map $(g, a) \mapsto ga$. Such a measure ν is called μ-harmonic or μ-stationary in the literature. Let $i : G \to G$ be the inversion given by $i(g) = g^{-1}$ for any $g \in G$. We denote by $\check{\mu}$ the probability measure $i_*\mu$. Recall that μ is *symmetric* if $\check{\mu} = \mu$.

For the remainder of this section, we fix a spread out non-degenerate probability measure μ on G, that is, μ is absolutely continuous with respect to a Haar measure and its support generates G as semigroup. Let (B, ν) be the Poisson boundary associated to μ.

We also denote by $(\check{B}, \check{\nu})$ the Poisson boundary of $(G, \check{\mu})$. We refer to [**15**, **20**] for notions and references about Poisson boundaries and related ergodicity properties. We emphasize that (B, ν) and $(\check{B}, \check{\nu})$ are respectively a (G, μ)-space and a $(G, \check{\mu})$-space.

THEOREM 28. [**5**] *The pair $(\check{B}, B)$ is a G-boundary pair.*

This will be deduced from the following statement.

THEOREM 29. [**5**] *Let (A, η) be a $(G, \check{\mu})$-space. The factor map $A \times B \to A$ is relatively ergodic.*

COROLLARY 30. [**5**] *The diagonal action of G on $(\check{B} \times B, \check{\nu} \times \nu)$ is metrically ergodic.*

Remark 31. The same argument as in Corollary 30 shows that if (B_-, B_+) is a G-boundary pair then $G \curvearrowright B_- \times B_+$ is metrically ergodic. In particular, if B is a G boundary then $G \curvearrowright B$ is doubly metrically ergodic.

Proof of Corollary 30. Let U be a metric separable space on which G acts continuously and by isometries. Assume $f : \check{B} \times B \to U$ is a G-equivariant measurable map. Take $A = \check{B}$. It follows from Theorem 29 that f depends only on the first coordinate.

Now consider the measure $\check{\mu}$. Then B is a $(G, \check{\check{\mu}})$-space (in other words, a (G, μ)-space), so we can apply Theorem 29 to $B \times \check{B}$. This implies that f does not depend on the first coordinate.

Putting together the two results, we see that f does not depend on the first coordinate, nor on the second. Hence, f is constant. □

The following proposition is a key tool in the proof of Theorem 28. It combines the Poincaré recurrence theorem for A and the SAT property for B. Recall that SAT, which means *strongly almost transitive*, is a weak mixing property introduced by Jarowski [**18**].

THEOREM 32. [**5**] *Let (A, η) be a $(G, \check{\mu})$-space and $Y \subset A \times B$ be a set of positive $\eta \times \nu$-measure. For $a \in A$, denote by Y_a the set $\{b \in B \mid (a, b) \in Y\}$. Then, for almost every $a \in \pi_1(Y)$, and for every $\varepsilon > 0$, there is a $g \in G$ such that:*

(i) *$\nu(gY_a) > 1 - \varepsilon$;*

(ii) *$ga \in \pi_1(Y)$ where π_1 is the first projection $A \times B \to A$.*

Proof. We will use the definition of the Poisson boundary as a space of ergodic components of the space of increments of the random walk $\Omega = G \times G^{\mathbb{N}}$ by the shift S.

We can define another shift T from $A \times \Omega$ to itself, defined by $T(a, g, \omega_1, \omega_2, \dots) = (g^{-1}a, \omega_1, \omega_2, \dots)$. Since A is a $(G, \check{\mu})$-space, we have $\check{\mu} * \eta = \eta$, and we conclude that T preserves the measure $m := \eta \times \mu \times \mu^{\mathbb{N}}$ on $A \times \Omega$.

Let us consider the fiber product $X = A \times_G \Omega$. This is defined as the quotient space of $Y \times \Omega$ by the relation $(a, hg, \omega_1, \dots) \sim (h^{-1}a, g, \omega_1, \dots)$, for all $h \in G$. It follows that the space X is isomorphic to $A \times G^{\mathbb{N}}$. The pushforward of the measure on $A \times \Omega$ to X is simply the measure $\eta \times \mu^{\mathbb{N}}$. Furthermore, the shift T preserves the equivalence relation, so it still acts on X, and also preserves the measure.

Let $Y \subset A \times B$ be such that $\eta \times \nu(Y) > 0$. We can consider the preimage of Y in $A \times \Omega$, and then push it forward to get a subset $\tilde{Y}$ of X. First we note that, by Poincaré's recurrence theorem, we have that, for almost every $x \in \tilde{Y}$, there are infinitely many $n \in \mathbb{N}$ such that $T^n x \in \tilde{Y}$.

If $a \in A$, define the set $\tilde{Y}_a = \{\omega \in G^{\mathbb{N}} \mid (a, \omega) \in \tilde{Y}\}$. By Fubini, we have $\mu^{\mathbb{N}}(\tilde{Y}_a) > 0$ for almost every $a \in \pi_1(\tilde{Y})$ (where $\pi_1 : A \times G^{\mathbb{N}} \to A$ denotes the first projection as well). In other words, for almost every $x \in \tilde{Y}$, we have $\nu(Y_{\pi_1(x)}) > 0$.

Let us fix $x \in \tilde{Y}$ in the intersection of the two conull sets defined above. Let $a = \pi_1(x)$. The set Y_a is of positive measure, so its characteristic function χ_{Y_a} is an element of $L^\infty(B)$ which is not zero. Let h be its Poisson transform. Recall that it is defined as

$$h(g) = \int_B \chi_{Y_a} dg_* \nu = \nu(g^{-1} Y_a).$$

This function is a non-zero bounded harmonic function on G.

By definition of the Poisson transform, we have, for almost every $\omega = (\omega_1, \omega_2, \dots) \in \tilde{Y}_a$,

$$\begin{aligned} \nu((\omega_1 \omega_2 \cdots \omega_n)^{-1} \tilde{Y}_a) &= h(\omega_1 \omega_2 \cdots \omega_n) \\ &\underset{n \to +\infty}{\longrightarrow} \chi_{Y_a}(\omega) = 1. \end{aligned}$$

In particular, we might pick n large enough so that $\nu((\omega_1 \omega_2 \cdots \omega_n)^{-1} \tilde{Y}_a) > 1 - \varepsilon$ and also satisfying $T^n(a, \omega) \in \tilde{Y}$. Setting $g = (\omega_1 \omega_2 \cdots \omega_n)^{-1}$, we have by definition $\nu(gY_a) > 1 - \varepsilon$. Furthermore, since $T^n(a, \omega) \in Y$, we have $(ga, \omega_{n+1}, \dots) \in Y$; hence $Y_{ga} \neq \emptyset$. □

Proof of Theorem 29. Let $\mathbf{X}$ be a metric field over (A, η) on which G acts via the cocycle α. Observe that we may assume that any fiber X_a has diameter at most one. Otherwise, we replace d_a by $\max(d_a, 1)$ and we obtain a new metric field over B on which G acts as well.

Let φ be an invariant relative section. Let us define

$$f(a) = \int_{B \times B} d_a(\varphi(a, b), \varphi(a, b')) d\nu(b) d\nu(b').$$

Our assumption implies that f is not essentially 0. In particular, there is an $r > 0$ such that $A(r) := f^{-1}([r, +\infty))$ is of positive measure.

Take a small $\delta > 0$. Let $\{x^n\}$ be a fundamental family of $\mathbf{X}$. Then there is $n \in \mathbb{N}$ such that $Y = \{(a, b) \in Y_r \times B \mid d(\varphi(a, b), x_a^n) \leq \delta\}$ has positive measure, say $> \varepsilon$.

By Theorem 32, this implies that there is an $a \in \pi_1(Y)$ and a $g \in G$ such that $ga \in \pi_1(Y)$ and $\nu(gY_a) > 1 - \varepsilon$.

Now, for $(ga, b) \in gY_a$ and $(ga, b') \in gY_a$ (in other words, $b, b' \in Y_{ga}$), we know that $d(\varphi(ga, b), \alpha(g, a)x_a^n) \leq \delta$ and $d(\varphi(ga, b'), \alpha(g, a)x_a^n) \leq \delta$. So these two points, $\varphi(ga, b)$ and $\varphi(ga, b')$, are in the same ball of radius δ. Therefore, we have $d(\varphi(ga, b), \varphi(ga, b')) \leq 2\delta$.

Let us decompose $B \times B$ as

$$(gY_a \times gY_a) \cup ((B \setminus gY_a) \times gY_a) \cup (gY_a \times (B \setminus gY_a)) \cup ((B \setminus gY_a \times B \setminus gY_a)).$$

We know that $\nu(B \setminus gY_a) < \varepsilon$. Hence, $\nu((B \times B) \setminus (gY_a \times gY_a)) < \varepsilon^2 + 2\varepsilon$.

Therefore, we have

$$f(ga) < \int_{gY_a \times gY_a} d(\varphi(ga, b), \varphi(ga, b')) d\nu(b) d\nu(b') + 2\varepsilon + \varepsilon^2.$$

It follows that $f(ga) < 2\delta + 2\varepsilon + \varepsilon^2$. Since we also assumed that $ga \in \pi_1(A)$, we have that $g \in A(r)$. Therefore, $f(ga) > r$. Since our choice of δ and ε is arbitrary, we get a contradiction. □

Proof of Theorem 28. The amenability of $G \curvearrowright B$ is due to Zimmer [**29**] and relative ergodicity for $\check{B} \times B \to \check{B}$ comes from Theorem 29. Relative ergodicity for $\check{B} \times B \to B$ is similar using that $\check{\check{\mu}} = \mu$. □

5. *Furstenberg maps*

The geometric part of Theorem 1 uses the following version of the Adams–Ballmann theorem [**2**, **10**] as a key tool.

THEOREM 33. (Equivariant Adams–Ballmann theorem [**13**, Theorem 1.8]) *Let (A, η) be an ergodic G-space such that $G \curvearrowright A$ is amenable. Let* $\mathbf{X}$ *be a CAT(0) field of finite telescopic dimension.*

If G acts on $\mathbf{X}$ *then there is an invariant section of the boundary field* $\partial\mathbf{X}$ *or there exists an invariant Euclidean subfield of* $\mathbf{X}$.

Before beginning the proof of Theorem 1 we can give a hint about its structure. Thanks to Theorem 33 used for a constant field, the only case where there is something to prove is the case where there is an invariant Euclidean subfield and no Furstenberg map. In that case, we choose a minimal such invariant Euclidean subfield. Then, we analyze the possible relative positions of two subflats of X by using Proposition 4. Using relative metric ergodicity we conclude that this subfield must actually be constant, equal to some fixed Euclidean subspace.

Theorem 1 will actually be deduced straightforwardly from this more general theorem for boundary pairs.

THEOREM 34. *Let X be a CAT(0) space of finite telescopic dimension and let G be a locally compact second countable group acting continuously by isometries on X without invariant flats. If (B_-, ν_-) and (B_+, ν_+) form a G-boundary pair then there exist measurable G-maps $\varphi_\pm : B_\pm \to \partial X$.*

Proof. First we do some reductions to prove the theorem. Clearly, it suffices to prove the theorem in the case where there is no fixed point of G in ∂X, since otherwise we have a trivial map $B_\pm \to \partial X$. Let us assume there is no fixed point at infinity. In that case, there is a G-invariant closed convex space with a minimal G-action. That is, there is no non-trivial G-invariant closed convex subset [**10**, Proposition 1.8(ii)]. Thus, we may assume that this space is actually X itself and X is the closed convex hull of any orbit. Orbits are separable since G acts continuously and G is second countable. Thus X is separable.

Let $X = H \times Y$ be the de Rham decomposition of X where H is a Hilbert space (of finite dimension since X has finite telescopic dimension) and Y is another CAT(0) space of finite telescopic dimension. The action of G on $X = H \times Y$ is diagonal and, since the action of G on X is minimal, both actions $G \curvearrowright H$ and $G \curvearrowright Y$ are minimal. Observe that it suffices to prove the result for $G \curvearrowright Y$, because $\partial Y \subseteq \partial X$ and if F is a flat of Y then

$H \times F$ is a flat of X. We may assume that $X = Y$, that is, X has no Euclidean de Rham factor of positive dimension.

Thus, we reduce the proof to the following case: X is separable, has trivial Euclidean de Rham factor and the action of G is minimal.

We consider the constant fields $\mathbf{X}^{\pm}$ over $B_{\pm}$ with fibers X. These fields are endowed naturally with actions of G and we can apply Theorem 33. For each $\mathbf{X}^{\pm}$ we get a map from $B_{\pm}$ to ∂X or an invariant Euclidean subfield of $\mathbf{X}^{\pm}$. In the first case, one gets two G-maps $B_{\pm} \to \partial X$, and we are done. Up to permuting B_+ and B_-, one may assume that either we get two invariant Euclidean subfields $\mathbf{E}^-$ and $\mathbf{E}^+$ of respectively $\mathbf{X}^-$ and $\mathbf{X}^+$ and there is no G-map $B_{\pm} \to \partial X$ (Case (I)) or we get a G-map $b \mapsto \xi_b$ from B_- to ∂X and a Euclidean subfield $\mathbf{E}$ of $\mathbf{X}^+$ (Case (II)).

Readers interested only in the case of a G-boundary B, that is, $B = B_- = B_+$, may read only case (I) with $\mathbf{X}^+ = \mathbf{X}^-$ and $\mathbf{E}^+ = \mathbf{E}^-$.

Case I: Thanks to Lemma 14, the map $b \mapsto \dim(E_b^-)$ is measurable and G-invariant. Thanks to ergodicity of $G \curvearrowright B_-$, this dimension is essentially constant. We may assume that this dimension is minimal among dimensions of invariant Euclidean subfields of $\mathbf{X}_-$. For simplicity, we note $E_b = E_b^-$ and $E_{b'} = E_{b'}^+$, for $b \in B_-$ and $b' \in B_+$.

We aim to apply Proposition 4 to each pair $(E_b, E_{b'})$ for $(b, b') \in B_- \times B_+$. First, we show that the four conditions in the proposition are measurable and G-invariant; thanks to ergodicity of $G \curvearrowright B_- \times B_+$, exactly one condition will be satisfied for almost every $(b, b') \in B_- \times B_+$.

Let $\{x_b^n\}_{b \in B_-}$ be a fundamental family of $\mathbf{E}^-$, and $\{x_{b'}^n\}_{b' \in B_+}$ be a fundamental family of $\mathbf{E}^+$. We define $d(b, b') = \inf_{n,m} d(x_b^n, x_{b'}^m)$. The function $d : B_- \times B_+ \to \mathbb{R}$ is measurable. The infimum distance between E_b and $E_{b'}$ is not achieved if and only if, for any $N \in \mathbb{N}$ and any n such that $d(x_b^0, x_{b'}^n) < N$, one has $\inf_m d(x_b^n, x_{b'}^m) > d(b, b')$. This condition is measurable and G-invariant.

If the minimal distance between E_b and $E_{b'}$ is achieved, then the subset of E_b where the distance is achieved is not bounded if and only if, for any $N \in \mathbb{N}$, there exist n_1, n_2 such that $d(x_b^{n_1}, x_b^{n_2}) > N$ and $\inf_m d(x_b^{n_i}, x_{b'}^m) \leq 1 + d(b, b')$.

Now, we know that exactly one of these four relative positions given in Proposition 4 between E_b and $E_{b'}$ happens for almost every (b, b'). We treat the four cases independently.

Case i. If the infimum distance is not achieved, choose a section y of $\mathbf{E}^-$ and consider (for b' in a set of full measure) the non-increasing sequence of subfields $\mathbf{X}^n$ where we define X_b^n as $X_b^n = \left\{x \in E_b,\ d(x, E_{b'}) \leq d(b, b') + 1/n\right\}$. We can now apply [**13**, Proposition 8.10] and obtain a section $\xi_b(b')$ of the boundary field $\partial\mathbf{E}^-$. This field is a metric field with a G-action by Lemma 18. By construction, ξ satisfies $\alpha(g, b)\xi_b(b') = \xi_{gb}(gb')$. This means that $\xi : B_- \times B_+ \to \bigsqcup \partial E_b$ is an invariant section relative to the first projection $B_- \times B_+ \to B_-$. Now, thanks to relative metric ergodicity, ξ does not depend on b'. So we can interpret $b \mapsto \xi_b$ as a measurable G-map from B_- to ∂X. This yields a contradiction.

If the minimal distance is achieved, we define $Y_b(b') = \{x \in X_b,\ d(x, E_{b'}) = d(b, b')\}$. Then the family $\mathbf{Y} = (Y_b(b'))_b$ is a subfield of $\mathbf{X}$ (which depends on b').

Case ii. Assume the minimal set $Y_b(b')$ is bounded. Let $x_b(b')$ be its circumcenter. Then for almost all b', the family of circumcenters $\{x_b(b')\}$ is a section of $\mathbf{E}^-$ (see [**13**, Lemma 8.7]) which satisfies $\alpha(g, b)x_b(b') = x_{gb}(gb')$, that is, an invariant relative section. Relative metric ergodicity shows that this map is essentially constant. So its essential image is a point which is fixed by G, which is excluded.

In the other case, thanks to [**13**, Proposition 9.2], we can write $Y_b(b') = F_b(b') \times T_b(b')$ where $\mathbf{F} = (F_b(b'))_b$ is a maximal Euclidean subfield of $\mathbf{Y}$ and $(T_b(b'))_b$ is a subfield of $\mathbf{Y}$; both subfields depending on b'.

Before attacking Case iii, which is harder, let us treat Case iv.

Case iv. If, for almost every (b, b'), $T_b(b')$ is not bounded then its boundary is not empty, and its circumradius $<\pi/2$ thanks to Proposition 4. Let $\xi_b(b')$ be the circumradius of $\partial T_b(b')$. Thanks to Lemma 11, we get a measurable G-map from $B_- \times B_+$ to the metric field $\partial\mathbf{E}^-$ over B_-. Once again, relative metric ergodicity implies that $\xi_b(b')$ coincides with a G-map $b \mapsto \xi_b$ and we are done.

Case iii. If $T_b(b')$ is bounded for almost every (b, b') then we set $t_b(b')$ to be its circumcenter and $E'_b(b') = F_b(b') \times \{t_b(b')\}$.

The dimension of $E'_b(b')$ is measurable (Lemma 14) in (b, b') and G-invariant. Thus, this dimension is essentially constant equal to some $k \in \mathbb{N}$.

Now $(b, b') \mapsto \partial E'_b(b')$ is a G-section of the metric field $\mathbf{S}$ of Euclidean subspheres of dimension $k - 1$ associated to $\mathbf{E}^-$ (see Lemma 20). Relative metric ergodicity implies that $\partial E'_b(b')$ is essentially equal to some s_b and $s = (s_b)$ is a G-invariant section of $\mathbf{S}$. Using the second part of Lemma 20, let us consider the field $\mathbf{E}^s$ such that for any $b \in B$, E^s_b is the set of Euclidean subspaces of E_b with boundary s_b. By definition, $(b, b') \mapsto E'_b(b')$ is a relative invariant section of $\mathbf{E}^s$ and relative metric ergodicity implies that there is a section of $\mathbf{E}^s$, that is, a Euclidean subfield $\mathbf{E}'$ of $\mathbf{E}^-$ such that for almost every (b, b'), $E'_b(b') = E'_b$.

Our assumption on the minimality of the dimension of $\mathbf{E}^-$ implies that $E'_b = E_b$ for almost every $b \in B_-$.

Going back to the definition of $E'_b = E_b(b')$, we see that the fact that $E'_b = E_b$ implies that the distance function $d(\cdot, E_{b'})$ must be constant in restriction to E_b. That is to say that E_b and $E_{b'}$ are parallel for almost all $b, b' \in B_- \times B_+$.

Fubini's theorem tells us that there is $b \in B_-$ such that for a set $B' \subseteq B_+$ of full measure and for any $b' \in B'$, $E_{b'}$ is parallel to E_b. Fix $\Gamma \leq G$ a dense countable subgroup of G and set $B_0 = \bigcap_{\gamma\in\Gamma} \gamma B'$, which is a Γ-invariant subset of full measure in B_+. Lemma 6 implies that the closed convex hull of the union of the $\{E_b\}_{b\in B_0}$ splits as some product $E \times T$, where each E_b is parallel to E. Observe that Γ preserves this convex set. By continuity of the action, the group G preserves this set as well and by minimality this set is X. Now the assumption that X has trivial Euclidean de Rham factor implies that E is a point and thus the dimension of E_b for almost every b is zero. Finally E_b, $E_{b'}$ are points of X and double metric ergodicity (Remark 31) implies that there is a fixed point.

Case II. Fix a point $x \in X$ and for $b \in B_-$ let β_b be the Busemann function associated to ξ_b vanishing at x. Now for $(b, b') \in B_- \times B_+$, look at the restriction $f_{b,b'}$ of β_b to $E_{b'}$. It is a convex function and, arguing as above, if it is not constant, one gets a relative section of $\mathbf{E}$ or of $\partial\mathbf{E}$. Once again, relative ergodicity gives an invariant Euclidean subspace of X or a fixed point at infinity.

If $f_{b,b'}$ is constant, the situation is different. Actually, since this situation is a measurable G-invariant condition on (b, b') thanks to double ergodicity it holds for almost all $(b, b') \in B_- \times B_+$. Applying Lemma 13 to $\mathbf{E}$, we get that the subfield $\mathbf{Y}$ of flats parallel to $\mathbf{E}$ splits as $\mathbf{E} \times \mathbf{Z}$ and G acts on $\mathbf{Z}$ with an invariant section z. Lemma 7 shows that ξ is actually a section of $\partial\mathbf{Z}$.

Let $\mathbf{B}^r$ be the subfield of $\mathbf{Z}$ consisting of closed balls of radius $r > 0$ around z. Thanks to Lemma 12, G acts on this CAT(0) field. Let $z^r_{b'}(b)$ be the point on the ray from $z_{b'}$ to ξ_b at distance r from $z_{b'}$. This is an invariant relative section of $\mathbf{B}^r$ and, thanks to relative ergodicity, there is an invariant section $z^r_{b'}$ of $\mathbf{B}^r$ such that $z^r_{b'}(b) = z^r_{b'}$ for almost every $(b, b') \in B_- \times B_+$.

Let us define $\eta_{b'}$ to be $\lim_{r\to\infty} z^r_{b'}$. By construction, this is an invariant section of $\partial\mathbf{X}_+$ that is a G-map $\varphi_+ : B_+ \to \partial X$ as desired. □

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A. *Appendix. Non-emptiness of the boundary of minimal CAT(0) spaces of finite telescopic dimension*

P. Py pointed out that the proof of [**10**, Proposition 1.8] uses implicitly the fact that the boundary of an unbounded CAT(0) space of finite telescopic dimension is non-empty as soon as its group of isometries acts minimally (meaning that is there is no non-trivial closed convex subset which is invariant under all isometries). The proof of [**10**, Proposition 1.8] was completed in [**9**] but the question about the non-emptiness of the boundary still remained.

We answer this question positively. The example of a rooted tree with an edge of length n attached to the root for any $n \in \mathbb{N}$ shows that the hypothesis of no fixed point for all isometries is essential. Observe that if the boundary is empty then any isometry is elliptic, since any hyperbolic or parabolic isometry has a fixed point at infinity. Moreover, [**10**, Theorem 1.6] shows that the isometry group of such a minimal space cannot be amenable.

We emphasize that Theorem 2 relies on Theorem 33, which only uses [**10**, Proposition 1.8(ii)] where there is no implicit assumption. So, the gap in the cited paper is filled in.

Proof of Theorem 2. First of all, we prove the result for finitely generated groups. Let G be a finitely generated group acting on some CAT(0) space of finite telescopic dimension X with empty boundary. This last assumption implies that there is a closed convex subset on which G acts minimally. We may assume that this subset is X itself. Since G is countable then the closed convex hull of any orbit is G-invariant and separable. The minimality assumption implies that X coincides with any such closed convex hull and thus X is separable. Since G is finitely generated then G has a symmetric probability measure μ uniformly supported on a symmetric finite set of generators and thus the Poisson boundary B associated to μ is a G-boundary.

We can apply Theorem 33 for the constant field of CAT(0) spaces X over B and we obtain a measurable map $B \to \partial X$ or an invariant subfield of flats. In any case, we get an non-empty boundary or an equivariant map $B \to X$. In this last case, double metric ergodicity for $G \curvearrowright B$ implies that this map is essentially constant and we get a G-fixed point.

We conclude in the general case in the following way: assume X has empty boundary and let G be the group of all isometries of X. Since $\partial X = \emptyset$, X is $\mathscr{T}_c$-compact (see §2.1). Let $\mathscr{F}$ be the collection of all finite subsets of G. For $F \in \mathscr{F}$ let G_F be the subgroup generated by F. We just proved that G_F has a non-empty, closed and convex set of fixed points that we denote by X_F. The collection $\{X_F\}_{F\in\mathscr{F}}$ is then a filtering family of $\mathscr{T}_c$-closed non-empty subspaces, by compactness $\bigcap_{F\in\mathscr{F}} X_F \neq \emptyset$ and thus one gets a G-fixed point. □

Proof of Corollary 3. The proof goes as in [**11**, Proposition 1.5]. We reproduce it for the reader's convenience. Theorem 2 shows that ∂X is not empty. Assume ∂X has radius at most $\pi/2$; then there is a canonical point $\xi \in \partial X$ fixed by all isometries and such that the closed ball of radius $\pi/2$, for the Tits distance, around ξ coincides with ∂X. If $g \in \mathrm{Isom}(X)$ does not make the Busemann function β_ξ invariant, then g is an hyperbolic isometry and translates along some geodesic line with extremities $\eta_-, \eta_+ \in \partial X$. One of these points does not belong to the closed ball of radius $\pi/2$ around ξ, yielding a contradiction.

Now, if ∂X has radius larger than $\pi/2$, there is a minimal closed convex subset $Y \subseteq X$ with $\partial Y = \partial X$. The union X_0 of all such minimal spaces is closed convex and can be decomposed as $X_0 = Y \times Z$. Moreover, X_0 is $\mathrm{Isom}(X)$-invariant with a diagonal action. Minimality implies that $X = X_0$. If $\partial Z \neq \emptyset$ one has $\partial Y = \partial X = \partial Y * \partial Z$, leading to a contradiction. Thus $\partial Z = \emptyset$ and Theorem 2 implies $\mathrm{Isom}(X) \curvearrowright Z$ has a fixed point. This fixed point is the whole of Z by minimality and thus $X = Y$.

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