GEOMETRIC DENSITY FOR INVARIANT RANDOM SUBGROUPS OF GROUPS ACTING ON CAT(0) SPACES

BRUNO DUCHESNE, YAIR GLASNER, NIR LAZAROVICH, JEAN LÉCUREUX

ABSTRACT. We prove that an IRS of a group with a geometrically dense action on a CAT(0) space also acts geometrically densely; assuming the space is either of finite telescopic dimension or locally compact with finite dimensional Tits boundary. This can be thought of as a Borel density theorem for IRSs.

1. INTRODUCTION

1.1. **Invariant random subgroups.** Let G be a locally compact second countable group. We denote by $\mathcal{S}(G)$ the space of closed subgroups of G. We endow it with the Chabauty topology. With this topology $\mathcal{S}(G)$ is a compact metrizable space [Pau07, Propositions 1.7&1.8]. Recently, a new and fruitful point of view about non-free probability measure preserving (shortly p.m.p) actions has appeared (see [AGV14]) and is currently a fast growing field of research.

Definition 1.1. An *invariant random subgroup* (shortly IRS) is a Borel probability measure on $\mathcal{S}(G)$ which is invariant under the adjoint action of G on $\mathcal{S}(G)$ by conjugations.

We say that an IRS is *not trivial*, if almost surely it is not the trivial group (i.e. $\mu(\{e\}) = 0)$). We emphasize that this convention is not widely used (for example it is different from the one in [TD12]) but it is more convenient for us and there will be no ambiguity in case of an ergodic IRS.

1.2. **Geometric density.** In [CM09a], P.-E. Caprace and N. Monod proved a density theorem in the spirit of Borel density theorem for groups acting on CAT(0) spaces. Let X be a CAT(0) space.

Definition 1.2. The action of a subgroup $H < \text{Isom}(X)$ is called *minimal* if it does not stabilize a strict closed convex subset of X. It is called *geometrically dense* if it is minimal and if H does not fix a point in ∂X.

Caprace and Monod proved that if G acts continuously and geometrically densely on a proper CAT(0) space with finite dimensional Tits boundary, then the same holds for closed subgroups of finite covolume [CM09a, Theorem 2.4]. This result implies Borel density in case G is a semi-simple algebraic group over a local field without anisotropic factor [CM09a, Proposition 2.8]. They also proved, under the same assumptions, that every action of a normal subgroup is, too, geometrically dense [CM09b, Theorem 1.10]. Since IRSs generalize both subgroups of finite covolume and normal subgroups, our theorem below subsumes both theorems of Caprace-Monod.

Theorem 1.3. *Assume X has finite telescopic dimension or is proper with finite dimensional Tits boundary, is irreducible and not the real line. If G acts faithfully, continuously and geometrically densely by isometries on X then every non trivial IRS of G is also geometrically dense.*

In case, the space is not irreducible, one may also obtain geometric density under the assumption that the IRS does not act trivially on any irreducible factor. This hypothesis is in particular weaker than the hypothesis of irreducibility (and non triviality) that already appeared in [ABB$^+$12, §4]. Observe that in case $n = 1$, Theorem 1.4 reduces to Theorem 1.3.

Theorem 1.4. *Let $G_1, \ldots, G_n$ be l.c.s.c groups such that each one acts continuously faithfully geometrically densely by isometries on some CAT(0) space X_i which is irreducible ($\neq \mathbb{R}$) with finite telescopic dimension or locally compact with finite dimensional Tits boundary. Let $G = G_1 \times \cdots \times G_n$ and $X = X_1 \times \cdots \times X_n$. Let μ be an IRS of G and μ_i be the pushforward of μ by the projection $G \to G_i$. Assume that for every i, the IRS μ_i of G_i is not trivial. Then μ-almost every $H \in \mathcal{S}(G)$ acts geometrically densely on X.*

In this particular situation (when G acts geometrically densely on some nice CAT(0) space), one recovers the result [BDL14] that an amenable IRS lies in the amenable radical, which is trivial in this situation. The paper [BDL14] was actually obtained after proving the following.

Corollary 1.5. *With the same assumptions as in Theorem 1.4, any IRS of G with non trivial projections has a trivial amenable radical.*

Proof. Since X has a trivial Euclidean factor, there is no amenable group acting geometrically densely on X [CL10, Theorem 1.6]. □

1.3. **IRSs in linear groups and Borel density theorem.** The paper [Gla14] initiates a systemic study of IRSs in linear groups. In particular a Borel density theorem for IRS in countable linear groups (over any field) was obtained [Gla14, Theorem A.1]. As a corollary of Theorem 1.4, we get a Borel density theorem for IRSs in semisimple groups over local fields. Such a theorem was known by experts, but we include it since it is an easy applications of our previous theorem.

Let us begin with definitions about semisimple algebraic groups and their subgroups. Let k be a local field and assume G is the k-points of a connected semisimple k-group $\mathbf{G}$. There is an adjoint k-group $\overline{\mathbf{G}}$ which decomposes as $\mathbf{G}_1 \times \cdots \times \mathbf{G}_n$ where each $\mathbf{G}_i$ is an adjoint simple k-group and there is a k-isogeny $\overline{p}\colon \mathbf{G} \to \overline{\mathbf{G}}$. Moreover a subgroup of $H \leq G$ is Zariski-dense as soon as $\pi(H)$ is Zariski-dense.

Definition 1.6. Let H be a subgroup of G. We say that H *has non-trivial projections* if for all $i \in \{1, \ldots, n\}$, $\pi_i(\overline{p}(H)) \neq \{e\}$ where π_i is the projection $\mathbf{G} \to \mathbf{G}_i$. An IRS of G has non-trivial projections if almost surely, any $H \in \mathcal{S}(G)$ has non-trivial projections.

Theorem 1.7. *Let k be a local field and let G be the k-points of a k-isotropic semisimple algebraic k-group $\mathbf{G}$. Then any IRS of G with non-trivial projections is Zariski-dense.*

Proof. The group $\overline{G} = \overline{\mathbf{G}}(k)$ acts faithfully and geometrically densely on its Bruhat-Tits building (respectively symmetric space of non-compact type in the archimedean case) X which a product of the (irreducible $\neq \mathbb{R}$) Bruhat Tits buildings (respectively symmetric space) of the groups factors G_i's. The image in $\overline{G}$ of an IRS with non-trivial projections satisfies the hypothesis of Theorem 1.4. Thus almost surely $H \in \mathcal{S}(\overline{G})$ acts geometrically densely and [CM09a, Proposition 2.8] shows that H is Zariski-dense. □

Remark 1.8. Nevo-Stuck&Zimmer theorem (formulated in terms of IRSs see [ABB$^+$12, Theorem 4.1]) shows that non-atomic irreducible IRSs of higher rank in semi-simple Lie groups with property (T) and no center are irreducible lattices and thus are Zariski-dense (see merely [ABB$^+$12, Theorem 2.6] in the simple case). So Theorem 1.7 is mostly useful for non-archimedean fields with valuation.

2. Actions of IRSs on CAT(0) spaces

2.1. **Actions on CAT(0) spaces.** In this subsection, we recall the required facts about CAT(0) spaces.

We fix a CAT(0) space X. A large part of what will follow is inspired by [CM09a, CM09b], and we refer to these references for details. We assume either that X is proper with finite-dimensional boundary, or that X is of finite telescopic dimension (see [CL10] for the definition of the telescopic dimension). These geometric assumptions are used to guarantee the following fact: any decreasing sequence of closed convex subsets has an non-empty intersection or defines a canonical point at infinity. We also assume X to be separable. There is no loss of generality with this last assumption since a second countable group acting continuously by isometries on a CAT(0) space

has a separable closed subset which is invariant (for example, the closed convex hull of some orbit).

We use *measurable fields of CAT(0) spaces* as a convenient tool to deal with measurability questions. We refer to [Duc13, AH14] for generalities about such fields. One does not need a deep knowledge of them to understand our proof. It suffices to have the following situation in mind. The group G acts continuously by isometries on X. To each subgroup $H \in \mathcal{S}(G)$ we associate a closed convex subset X_H. It will vary measurably with H in sense that for any $x \in X$, the distance function $H \mapsto d(x, X_H)$ is measurable. The collection $\mathbf{X} = \{X_H\}_{H \in \mathcal{S}(G)}$ is called a *subfield* of the constant field $\mathbf{X}_0$ with fiber X over the measurable space $\mathcal{S}(G)$. The group G acts on $\mathbf{X}$, meaning that for almost all $H \in \mathcal{S}(G)$ and for all $g \in G$, $gX_H = X_{gHg^{-1}}$. We also say that $\mathbf{X}$ is G-invariant. In particular, G-invariance implies that for almost all $H \in \mathcal{S}(G)$, X_H is a closed convex H-invariant subspace of X.

2.2. **Proof of Theorems 1.3 & 1.4.** For the remainder of this subsection, we assume that X has trivial Euclidean factor and that G acts faithfully, continuously and geometrically densely by isometries on X.

Let (S, μ) be a standard probability space with a p.m.p action of G. Let x_0 be an arbitrary point in X and let $\mathcal{C}_0$ be the space of 1-Lipschitz convex functions vanishing at x_0.

Proposition 2.1. *Let $s \mapsto f_s$ be a map from S to $\mathcal{C}_0$ such that for all $x \in X$, the map $s \mapsto f_s(x)$ is measurable and for all $x \in X$, $g \in G$ and almost all $s \in S$, $f_s(gx) = f_{g^{-1}s}(x) + f_s(gx_0)$. Then almost surely f_s is constant.*

Proof. Define

$$(1) \qquad f(x) = \int_S f_s(x) \, d\mu(s).$$

First observe that $|f_s(x)| \le d(x_0, x)$ which shows that the right-hand side of (1) is well-defined. The function f is a 1-Lipschitz convex function on X. It is moreover quasi-invariant: $f(gx) = f(x) + f(gx_0)$ for all $g \in G$, $x \in X$.

If f does not achieve its minimum then $\left(f^{-1}(-\infty, r]) \right)_{r > \inf f}$ yields filtering family of closed convex subsets with empty intersection. Thus [CL10, Lemma 5.5] or [CM09b, Proposition 3.2] yields a canonical point $\xi \in \partial X$ which is G-invariant since $gf^{-1}\left((-\infty, r]\right) = f^{-1}\left((-\infty, r + f(gx_0)]\right)$. Thus f achieves its minimum. Quasi-invariance implies that $g \mapsto f(gx_0)$ is a homomorphism, which is trivial since f has a minimum. Thus f is G-invariant and the set of points where this minimum is achieved is G-invariant. By minimality it is X. Thus f is constant. Since f_s is continuous, and X is separable, this implies that almost all f_s are affine but triviality of the Euclidean factor of X and [CL10, Proposition 4.8] (for the finite telescopic dimension case) or [LS07, Corollary 1.8] (for the proper case) imply almost all f_s are constant. □

Lemma 2.2. *Assume there is a map $s \mapsto X_s$ from S to the set of closed convex subsets of X which is G-equivariant and such that for any $x \in X$, $s \mapsto d(x, X_s)$ is measurable. Then for almost every s, $X_s = X$.*

Proof. It suffices to apply Proposition 2.1 to $f_s = d(x, X_s) - d(x_0, X_s)$. □

Lemma 2.3. *There is no G-equivariant measurable map $S \to \partial X$.*

Proof. Let $s \mapsto \xi_s$ be such a map. We denote by $f_s(x)$ the Busemmann function associated to ξ_s vanishing at x_0. Proposition 2.1 implies that almost all f_s are constant but constant Busemann functions do not exist. □

For $H \in \mathcal{S}(G)$ and $x \in X$, we denote by $\mathscr{C}_H(x)$ the closed convex hull of the H-orbit of x. A subset is H-*minimal* if it is closed, convex, H-invariant and minimal among closed convex (non empty) H-invariant subsets. Recall that any two H-minimal subsets are parallel, in particular isometric to some subspace M_H and the union of all such minimal subspaces Z_H splits as a product $Z_H \simeq M_H \times T_H$ [CM09b, Theorem 4.3].

Let $C \subseteq X$ be a closed, convex and H-invariant subspace. It minimal if and only if $C = \mathscr{C}_H(x)$ for all $x \in C$. Or in other words, it is minimal if and only if for all $x, y \in C$, $d(x, \mathscr{C}_H(y)) = 0$. Thus, in order to recover, measurably, H-minimal subspaces, we define

$$\varphi_H(x) = \sup_{y \in \mathscr{C}_H(x)} d(x, \mathscr{C}_H(y)).$$

It follows that $x \in X$ belongs to an H-minimal subspace if and only if $\varphi_H(x) = 0$. Moreover it will be shown in Lemma 2.6 that X admits an H-minimal subspace if $\varphi_H(x)$ is finite for some $x \in X$ (equivalently for all $x \in X$). Actually, we will see that $d(x, Z_H) \leq \varphi_H(x) \leq 2d(x, Z_H)$. Thus φ_H gives us a quantitative tool to express how far $\mathscr{C}_H(x)$ is from being H-minimal.

Remark 2.4. We will not use this fact, but one may observe that φ_H is actually a convex continuous function. Furthermore, in case $\mathscr{C}_H(x)$ contains a unique H-minimal subspace then $\varphi_H(x) = d(x, Z_H)$. This equality is not always satisfied, as the following example shows. Let T_3 be the regular trivalent tree and $X = T_3 \times T_3$ considered as a CAT(0) square complex. Let D be the diagonal of some square and H be the pointwise stabilizer of D. For $x \in X$, let $p(x)$ be its projection. One has $p(hx) = hp(x)$ for all $h \in H$. That is $p(hx) = p(x)$. Now if x and y have same projection on D, it does not mean that the midpoint m of $[x, y]$ projects on $p(x) = p(y)$ as the following drawing of a small part of X illustrates.
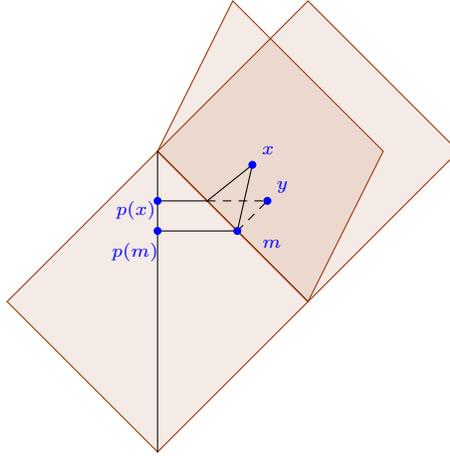
For any closed $Y, Z \subseteq X$ we denote by $D(Y, Z)$ the Hausdorff distance (possibly infinite) between Y and Z.

Lemma 2.5. *For any $x, y \in X$, $D(\mathscr{C}_H(x), \mathscr{C}_H(y)) \leq d(x, y)$.*

Proof. Since $d(hx, hy) = d(x, y)$ for any $h \in H$, the H-orbit of y lies in the closed $d(x, y)$-neighborhood of $\mathscr{C}_H(x)$. This neighborhood is convex and thus contains $\mathscr{C}_H(y)$. Exchanging x and y, one obtains the result. $\square$

Lemma 2.6. *For any $H \in \mathcal{S}(G)$ the following are equivalent:*
  (i) the subset Z_H is not empty,
  (ii) for any $x \in X$, $Z_H \cap \mathcal{C}_H(x) \neq \emptyset$,
  (iii) there is $x \in X$ such that $\varphi_H(x) < \infty$,
  (iv) for all $x \in X$, $\varphi_H(x) < \infty$.

Proof. Clearly (ii) implies (i) and (iv) implies (iii).

(**i**) $\Rightarrow$ (**iv**). Assume M is an H-minimal subset. Since $M = \mathcal{C}_H(m)$ for any $m \in M$, Lemma 2.5 implies that for all $x \in X$, $d(x, M) \geq d(y, M)$ for any $y \in \mathcal{C}_H(x)$.

$$(2) \qquad \varphi_H(x) = \sup_{y \in \mathscr{C}_H(x)} d(x, \mathscr{C}_H(y)) \leq \sup_{y \in \mathscr{C}_H(x)} d(x, M) + D(M, \mathcal{C}_H(y)) \leq 2d(x, M).$$

(**iv**) $\Rightarrow$ (**ii**)&(**iii**) $\Rightarrow$ (**i**). If $\varphi_H(x) < \infty$ for some $x \in X$ then any closed convex invariant subset of $\mathcal{C}_H(x)$ intersects the closed ball $\overline{B}(x, \varphi_H(x))$. By the analog of Banach-Alaoglu theorem for CAT(0) spaces [Mon06, Theorem 14], every filtering family of H-invariant closed convex subsets has a non-trivial intersection. Zorn's Lemma yields an H-minimal subset in $\mathcal{C}_H(x)$. $\qquad \square$

Lemma 2.7. *Let $x, y \in X$. The function $H \mapsto d(x, \mathscr{C}_H(y))$ is measurable. In particular $\mathscr{C}(\mathbf{x}) = \{\mathscr{C}_H(x)\}$ is a subfield.*

Proof. It suffices actually to show that for any $x, y \in X$, the map $H \mapsto d(y, \mathscr{C}_H(x))$ is upper semi-continuous. Let us introduce some notations. For $x_1, x_2 \in X$, let $m^1(x_1, x_2)$ be the midpoint of x_1 and x_2. We define by induction $m^k(x_1, \ldots, x_{2^k}) = m^1(m^{k-1}(x_1, \ldots, x_{2^{k-1}}), m^{k-1}(x_{2^{k-1}+1}, \ldots, x_{2^k}))$. For any subset $A \subseteq X$, we denote by $C^k(A)$ the set $\{m^k(x_1, \ldots, x_{2^k}); \ x_1, \ldots, x_{2^k} \in A\}$. Recall that the closed convex hull of A, $\overline{\mathrm{Conv}(A)}$ coincides with $\bigcup_{k \in \mathbb{N}} C^k(A)$.

Assume $H_n \to H$ in $\mathcal{S}(G)$. Fix $\varepsilon > 0$. There is $k \in \mathbb{N}$ such that $d(y, C^k(Hx)) - d(y, \mathscr{C}_H(x)) \le \varepsilon/2$. Thus there are $h_1, \ldots, h_{2^k} \in H$ such that $d(y, z) - d(y, \mathscr{C}_H(x)) \le \varepsilon/2$ where $z = m^k(h_1 x, \ldots, h_{2^k} x)$. For $n \in \mathbb{N}$, choose $h_i^n \in H_n$ converging to h_i and set $z^n = m^k(h_1^n x, \ldots, h_{2^k}^n x)$. Since $z^n \to z$, one has $\varlimsup_{n \to \infty} d(y, \mathscr{C}_{H_n}(x)) \le d(y, \mathscr{C}_H(x)) + \varepsilon$ for all $\varepsilon > 0$. This yields the desired semicontinuity. $\quad \square$

Lemma 2.8. *For every $x \in X$, the function $H \mapsto \varphi_H(x)$ is measurable.*

Proof. Let (x_n) be a dense countable family of X then $\varphi_H(x) = \sup_n d(x_H, \mathscr{C}_H(x_n))$. $\qquad \square$

Lemma 2.9. *Assume that for almost all $H \in \mathcal{S}(G)$, H has a minimal invariant closed convex subset. For $H \in \mathcal{S}(G)$, let Z_H be the union of closed convex minimal H-invariant subsets of X_H. Then $\mathbf{Z} = \{Z_H\}_{H \in \mathcal{S}(G)}$ is a G-invariant subfield of $\mathbf{X}_0$.*

Proof. It suffices to show that for any $x \in X$, the function $H \mapsto d(x, Z_H)$ is measurable. Fix a countable dense subset $\{x_n\}$ of X. This measurability is a consequence of the fact that

$$d(x, Z_H) = \lim_{k \to \infty} \inf\{d(x, x_n); \ \varphi_H(x_n) < 1/k\}.$$

To prove this equality, choose x_n such that $d(x_n, z) < 1/2k$ where z is the projection of x on Z_H. Since $\varphi_H(x_n) \le 2d(x_n, Z_H)$ (equation (2)) then $d(x, x_n) < 1/k + d(x, Z_H)$ and $d(x, Z_H) \ge \lim_{k \to \infty} \inf\{d(x, x_n); \ \varphi(x_n) < 1/k\}$.

Now, for any $n \in \mathbb{N}$, if $\varphi(x_n) < 1/k$ then $d(x_n, Z_H) < 1/k$. Let z be the projection of x_n on Z_H. We have $d(x, Z_H) \le d(x, x_n) + d(x_n, z) \le d(x, x_n) + 1/k$. This yields the reverse inequality. $\quad \square$

Proof of Theorem 1.3. Thanks to the ergodic decomposition, it suffices to deal with the ergodic case. Consider $\mathbf{X}_0$, the field of CAT(0) spaces with constant fiber X over $(\mathcal{S}(G), \mu)$ on which G acts. We apply [Duc13, Proposition 8.11] in case X has finite telescopic dimension or [AH14, Theorem 5.1] in case X is proper with Tits boundary of finite dimension[1] and get either a G-equivariant map $\mathcal{S}(G) \to \partial X$ or a minimal G-invariant subfield $\mathbf{X}$. The first possibility is ruled out by Lemma 2.3. Thus, we consider the second one.

Choose a point x of X. Since $\{H \in \mathcal{S}(G), \ \varphi_H(x) = +\infty\}$ is G-invariant and $G \curvearrowright \mathcal{S}(G)$ is ergodic the measurable function (Lemma 2.8) $H \mapsto \varphi_H(x)$ is essentially constant equal to $+\infty$ or almost surely H has a minimal closed convex subset in X_H. If $\varphi_H(x) = +\infty$, the intersection of boundaries of convex H-invariant subsets of X_H has a canonical center ξ_H, which yields an equivariant measurable map $\mathcal{S}(G) \to \partial X$ contradicting Lemma 2.3. Hence almost surely $X_H \cap Z_H \ne \emptyset$. Since the intersection of two subfields is still a subfield, Lemma 2.9 and the minimality assumption on $\mathbf{X}$ implies almost surely $X_H = X_H \cap Z_H$. Applying Lemma 2.2, we get $X_H = X$ almost surely. Thus $X = M_H \times T_H$ and this product has to be trivial since X is irreducible. That is M_H or T_H is reduced to a point. If M_H is point then all points are H-invariant and since

---

[1]The result is stated for proper CAT(0) spaces of finite dimension but the finite dimension assumption is used only for the boundary and not for the space itself.

$G \curvearrowright X$ is faithful, this means $H = \{e\}$ and $\mu = \delta_e$. This contradicts the fact that the IRS is not trivial. Thus T_H is point and this means $H \curvearrowright X$ is minimal.

It remains to show that almost surely H has no fixed point at infinity. Consider the push-forward of the measure μ under the map $H \mapsto H' = \overline{[H, H]}$. It yields a new IRS and one can apply what we did above and in particular, H' acts minimally. But if H has a fixed point at infinity, H' stabilizes any horoball centered at this point and this contradicts the minimality of the action of H' on X. □

Proof of Theorem 1.4. Decomposing $\mathcal{S}(G)$ in ergodic components, we may assume that μ is ergodic. We first prove that almost surely $H \in \mathcal{S}(G)$ does not fix a point in ∂X. Since ∂X is the spherical join $\partial X_1 * \cdots * \partial X_n$, if H fixes a point in ∂X then there is i such that H_i fixes a point in ∂X_i where H_i is the image of H under the projection $G \to G_i$. This contradicts Theorem 1.3 applied to the IRS μ_i on G_i.

Let Z_H be the closed convex hull of H-minimal subsets, which is almost surely not empty thanks to the previous paragraph. We claim that $Z_H = X$ and we prove it by an induction on n. The case $n = 1$ follows from Theorem 1.3. Assume the result holds for $n - 1 \geq 1$. Since $H_n \curvearrowright X_n$ is minimal, the projection of Z_H to X_n is X_n itself. Let $\widehat{X_n}$ be a notation for $X_1 \times \cdots \times X_{n-1}$. Fix $x \in X_n$ and denote by Z_H^x be the fiber over $\{x\}$ under the projection $X \to X_n$. This is a closed convex non-empty subspace of $\widehat{X_n}$. Observe that for any $g \in G$, $gZ_H^x = Z_{gHg^{-1}}^{g_n x}$ where g_n is the n-th coordinate of g. In particular for $g \in \widehat{G_n} = G_1 \times \cdots \times G_{n-1}$, $gZ_H^x = Z_{gHg^{-1}}^x$ and $H \mapsto Z_H^x$ is a $\widehat{G_n}$-equivariant map. As $\widehat{G_n}$ acts minimally on $\widehat{X_n}$, it follows from Lemma 2.2 that $Z_H^x = X_1 \times \cdots \times X_{n-1}$. Since it works for any $x \in \widehat{X_n}$, one has that $Z_H = \widehat{X_n} \times X_n$. That is $Z_H = X$. Now, thanks to the uniqueness of the decomposition of X has a product of irreducible spaces ([CL10, Proposition 6.1]&[CM09b, Theorem 5.1]) one has $Z_H = M_H \times T_H$ with $M_H = X_1 \times \cdots \times X_k$ and $T_H = X_{k+1} \times \cdots \times X_n$ after a possible reordering of the X_i's, where M_H is an H-minimal subset and the action of H on T_H is trivial. By ergodicity, the number k and the permutation are independent of H. Now, by definition, H acts trivially on T_H thus on $X_{k+1} \times \cdots \times X_n$. But our hypothesis that μ_i is not trivial and the faithfulness of $G_i \curvearrowright X_i$ imply that $k = n$. That is H acts minimally. □

REFERENCES

[ABB+12] Miklos Abert, Nicolas Bergeron, Ian Biringer, Tsachik Gelander, Nikolay Nikolov, Jean Raimbault, and Iddo Samet. On the growth of l^2-invariants for sequences of lattices in lie groups. 10 2012, 1210.2961. 1, 2

[AGV14]   Miklós Abért, Yair Glasner, and Bálint Virág. Kesten's theorem for invariant random subgroups. *Duke Math. J.*, 163(3):465–488, 2014. 1

[AH14]   Martin Anderegg and Philippe Henry. Actions of amenable equivalence relations on CAT(0) fields. *Ergodic Theory Dynam. Systems*, 34(1):21–54, 2014. 2, 5

[BDL14]   Uri Bader, Bruno Duchesne, and Jean Lécureux. Amenable invariant random subgroups. Preprint, 2014. 2

[CL10]   Pierre-Emmanuel Caprace and Alexander Lytchak. At infinity of finite-dimensional CAT(0) spaces. *Math. Ann.*, 346(1):1–21, 2010. 2, 3, 6

[CM09a]   Pierre-Emmanuel Caprace and Nicolas Monod. Isometry groups of non-positively curved spaces: discrete subgroups. *J. Topol.*, 2(4):701–746, 2009. 1, 2

[CM09b]   Pierre-Emmanuel Caprace and Nicolas Monod. Isometry groups of non-positively curved spaces: structure theory. *J. Topol.*, 2(4):661–700, 2009. 1, 2, 3, 6

[Duc13]   Bruno Duchesne. Infinite-dimensional nonpositively curved symmetric spaces of finite rank. *Int. Math. Res. Not. IMRN*, (7):1578–1627, 2013. 2, 5

[Gla14]   Yair Glasner. Invariant random subgroups of linear groups. 07 2014, 1407.2872. 2

[LS07]   Alexander Lytchak and Viktor Schroeder. Affine functions on CAT(κ)-spaces. *Math. Z.*, 255(2):231–244, 2007. 3

[Mon06]   Nicolas Monod. Superrigidity for irreducible lattices and geometric splitting. *J. Amer. Math. Soc.*, 19(4):781–814, 2006. 4

[Pau07]   Frédéric Paulin. De la géométrie et de la dynamique de $\mathrm{SL}_n(\mathbb{R})$ et $\mathrm{SL}_n(\mathbb{Z})$. In *Sur la dynamique des groupes de matrices et applications arithmétiques*, pages 47–110. Ed. Éc. Polytech., Palaiseau, 2007. 1

[TD12]   Robin D. Tucker-Drob. Shift-minimal groups, fixed price 1, and the unique trace property. 11 2012, 1211.6395. 1