Boundary amenability of groups acting on buildings

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One of the main utility of buildings is to study and understand the structure of reductive groups over local fields and their lattices. However, there are also some other groups which act on buildings, such as for example Kac-Moody groups. We prove that all these groups have a topologically amenable action on a compact space. This compact space is described geometrically as a combinatorial boundary of a building.

1. Buildings

First, let me recall briefly the definition and the basic vocabulary about buildings. A *Coxeter group* is a group defined by a presentation of the form $\langle s \in S \mid (st)^{m_{st}} = 1\rangle$, where m_{st} are natural integers (or possibly infinite) such that $m_{ss} = 1$: the generators have order two.

For example, regular tilings of the euclidean plane or the hyperbolic plane give rise to Coxeter groups, generated by the reflections with respect to the lines (or geodesics) of the tiling. More generally, to every Coxeter group W is associated a metric simplicial complex, called the *Davis complex*, which is CAT(0), and on which W acts.

Let us introduce a few words of vocabulary. A *reflection* in W is a conjugate of an element of S. A *wall* is the set of fixed points of a reflection in Σ. A *chamber* is the closure of a connected component of Σ deprived of all its walls. A *panel* is an intersection of two adjacent chambers.

A building is a tessellation of such Davis complexes. More precisely:

Definition. Let (W, S) be a Coxeter group, and Σ its Davis complex.

A *building* of type (W, S) is a gluing of chambers along their panels, covered by subcomplexes called *apartments*, such that:

- Every apartment is isomorphic to Σ;
- Every two chambers are contained in some apartment;
- For every apartments A and A', there is an isomorphism from A to A' which fixes $A \cap A'$.

2. Combinatorial boundary of buildings

This section is a joint work with Pierre-Emmanuel Caprace [1].

Recall that there is a notion of *projection* in buildings : the projection of a chamber C on a panel σ is the unique chamber which is adjacent to σ and at minimal distance from C.

Let X a building. Denote $\mathrm{ch}(X)$ the set of chambers of X and, for any panel σ, $\mathrm{lk}(\sigma)$ the link of σ (*i.e.* the set of chambers containing σ). From this notion of projection, we get an injection

$$i_{\mathrm{ch}} : \mathrm{ch}(X) \to \prod \mathrm{lk}(\sigma),$$

the product being taken over all panels σ in X. Each of the links $\mathrm{lk}(\sigma)$ is endowed with the discrete topology, and the product is endowed with the product topology.

Definition. The *combinatorial compactification* $\mathscr{C}_{\mathrm{ch}}(X)$ of a building X is the closure of $i_{\mathrm{ch}}(\mathrm{ch}(X))$.

One of the interests of this compactification is that it parametrizes amenable subgroups of the automorphism group :

Theorem ([1])**.** *Let X be a locally finite building, and G a group acting properly on X. Then G is amenable if and only if it virtually fixes a point in $\mathscr{C}_{\mathrm{ch}}(X)$.*

Remark. It is also possible to define a compactification of the set of spherical residues instead of the set of chambers. In this case, the theorem above remains true even without the hypothesis that X is locally finite.

3. Boundary amenability

The notion of a topological amenable action is defined as follows:

Definition. Let G be a locally compact group acting on a locally compact space S. The action of G on S is *topologically amenable* if there exists a sequence of continuous maps $\mu_n : S \to \mathrm{Prob}(G)$ such that

$$\lim_{n\to+\infty} \|\mu_n(gs) - g\mu_n(s)\| = 0,$$

uniformly on every compact subset of $G \times S$.

This notion has many applications. In particular, the class of discrete groups that admit a topologically amenable action on a discrete space is a very interesting one (see for example [5]). It implies for example that the group can be coarsely embedded into a Hilbert space [3], which in turns implies that it satisfies the Novikov conjecture [6].

For groups acting on buildings, we prove the following theorem:

Theorem ([4])**.** *Let G be a group acting properly on a locally finite building X. Then the action of G on $\mathscr{C}_{\mathrm{ch}}(X)$ is topologically amenable.*

Let us give two important elements of the proof. The first one is the notion of *generalized sectors* in a building. Such a sector should be seen as kind of combinatorial convex hull of a chamber and a point at infinity. More precisely, the sector $Q(x,\xi)$ is defined as the pointwise limit of the convex hulls between x and C_n, where (C_n) is a sequence converging to ξ.

These sectors are very useful because of two points: a sector is contained in an apartment, and two sectors converging to the same point always intersect. These two features allow us, by constructing our measure $\mu_n(\xi)$ with support in a sector converging to ξ, to reduce the problem to the construction of $\mu_n(\xi)$ in an apartment.

In restriction to an apartment, we use a second idea, which was already used in a similar context in [2]: a Coxeter complex –and in fact its combinatorial compactification– can be embedded equivariantly into a finite product of trees.

References

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